Mathematisches Forschungsinstitut Oberwolfach

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Miniworkshop:

Discrete Mathematics and Proof in the High School

November 3rd – November 9th, 2002

Introduction

It is a pleasure to submit the following report on the Miniworkshop on Discrete Mathematics and Proof in the High School. Organizers of the workshop were Professors Gila Hanna (Toronto, Canada), Kristina Reiss (Augsburg, Germany), Jürgen Richter-Gebert (München, Germany) and Jacobus H. van Lint (Eindhoven, The Netherlands). Participants came in addition from Austria, Great Britain, Russia, and the United States, and included active researchers in several disciplines (mathematics, mathematics education, computer science, theoretical physics) who are concerned with the conference theme and related questions.

Due to unexpected illness, a few invited participants who planned to attend were unable to do so. The workshop participants found the Oberwolfach environment conducive to intense, stimulating, and productive interactions. This and similar workshops are seen as essential to thoughtful involvement by the mathematics community in helping to shape future directions in mathematics education.

Our week consisted of a program of prepared daytime presentations and structured discussions, with ample time between these for informal conversations. In addition, attention during the evenings was devoted to special topics and follow-up plans. Appended to this report are the abstracts of the prepared presentations. The balance of this overview is devoted to a summary of the key themes, and mention of future plans.

Discrete mathematics refers to a broad spectrum of mathematical topics. One theme of the workshop focused on specific topics within discrete mathematics which were deemed particularly appropriate for secondary school students (in some cases, for primary school students as well). These included combinatorics, graph theory, some aspects of algorithms, and coding theory. A distinction emerged between topics whose value to primary or secondary school students consisted partly or mainly in laying a foundation for more advanced mathematical study (e.g., certain areas of combinatorics), and topics whose value consisted mainly in providing a nonroutine, exploratory domain for students to develop skills of mathematical discovery, verification and proof, and problem solving (e.g., some introductory ideas of graph theory). We discussed problem solving heuristics, the construction of problem representations, and search strategies during nonroutine problem solving, including the motivational value of such activity. It was noted that there was a danger that, as with other area of mathematics, discrete mathematics could be both presented and assessed in ways that are quite contrary to the flavor of enquiry and problem solving described above. Work on projects implementing discrete mathematics in schools, ranging from modifications of core curriculum standards to professional development institutes for teachers, was described and discussed, and what was learned was summarized.

The theme of proof was discussed both in the discrete mathematics context, and in the more traditional contexts of high school algebra and geometry. We devoted attention to the level of mathematical reasoning (from informal and intuitive to formal reasoning) and the level of rigour appropriate for students at various levels and in various contexts. We discussed experimental mathematics and its relation to proof, and mathematical proofs that can be based in physics. Presentations of available geometry software (Cinderella) and algebraic problem solving software provided further concrete contexts for discussion, as well as fascinating explorations for the workshop participants.

An important concern that was not on the Miniworkshop agenda arose in the discussions – the sometimes undesirable influence of standardized assessments on the school curriculum, or on how the school curriculum is implemented. This concern arose in connection with discrete mathematics and proof, since those were the topics of the Miniworkshop, but relected the broader concern that assessments often tend to create a routinization of the curriculum that we viewed as highly undesirable.

Future plans developed during the workshop include follow-up communications among the participants, and preparation of a formal set of contributions that we expect to be able to publish electronically through Zentralblatt für Didaktik der Mathematik.

Abstracts

Barcodes - mathematics in the supermarket

Ian Anderson (Glasgow)

Using material that I prepared for masterclasses for 14-year old children, I describe how very simple divisibility ideas are used in the European Article Number system, and how EAN codewords are changed into barcodes. The key idea behind these codewords is the introduction of a check digit which will detect all single digit errors and most, if not all, of transpositional errors. As well as the EAN system, I describe the IBM scheme which is used by Visa and many libraries, and the International Standard Book Numbering system, used by book publishers worldwide. These applications of divisibility by 10 and 11 are good examples of how mathematics has surprising applications in everyday life.

Realistic Discrete Mathematics

Bram G. van Asch (Eindhoven)

In Holland the so called realistic mathematics has been introduced in the mathematics curriculum, and its role is so dominant that many fundamental mathematical concepts, for instance the concept of proof, have disappeared almost completely, Realistic means that all problems have to be embedded in a more or less realistic situation. In view of the theme this conference let us look at possibilities that discrete mathematics might offer, in particular with respect to the notion of proof. For ages geometry was considered to be the tool to teach the concept of proof in the mathematics curriculum. We all observe the many problems that students have in dealing with geometrical questions. Therefore we are looking for alternatives and discrete mathematics might be one. Suitable problems for children from grade 6 to 12 can be found in graph theory and combinatorics. Beside these two elementary number theory might yield opportunities to incorporate the concept of proof into the curriculum right from the start. For instance factorization of numbers, number of divisors, greatest common divisor, least common multiple, Euclidean algorithm, calculations modulo a natural number, simple diophantine equations, etc. At a later stage elements of coding theory and cryptography can also be introduced, and in this case we can introduce really realistic problems. The basic material is the set of natural numbers, something which might perhaps be more accessable for younger children than the rather abstract notions of lines and points.

Diagrams as means and objects of mathematical reasoning

Willi Dörfler (Klagenfurt)

According to Peirce a great part of mathematical thinking consists in observing or imagining the outcomes and regularities of manipulations of all sorts of diagrams. This tenet is explicated and substantiated by expounding the notion of diagram and by analyzing several specific cases from different parts of mathematics with a special emphasis on proofs. The starting point for that is the following statement by Peirce: It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. ... As for algebra, the very idea of the art is that it presents formulae, which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra.

Discrete mathematics: a cautionary tale

TONY GARDINER (BIRMINGHAM)

In the 1991 NCTM Yearbook, the two opening chapters, subtitled "Perspectives and issues", were "Discrete mathematics: the math for our time" (by John Dossey) and "A cautionary note" (by me) reflected the obvious upbeat and downbeat positions. The warning in the second chapter could be summarised by the quote: "School mathematics all too easily becomes a succession of meaningless routines. And discrete mathematics has characteristics which make it vulnerable to such degeneration."

Since 1991 "Discrete mathematics" has become a popular part of math courses for students aged 16-19 in English high schools. I will give examples of exam papers and textbooks to illustrate how the predicted degeneration has exceeded all expectations.

Problem Solving Heuristics and Discrete Mathematics

GERALD GOLDIN (PISCATAWAY)

It has been remarked that topics in discrete mathematics allow a kind of new beginning for students and teachers. Students who have been "turned off" by traditional school mathematics, and teachers who have long ago routinized their instruction, can find in the domain of discrete mathematics opportunities for mathematical discovery, and for interesting, nonroutine problem solving. Sometimes formerly low-achieving students demonstrate mathematical abilities their teachers did not know they had. To take maximum advantage of these opportunities, it is important to know what kinds of thinking during problem solving can be naturally evoked by discrete mathematical situations – so that in developing a curriculum, the objectives can include pathways to desired mathematical reasoning processes. I shall try to characterize some of these ways of thinking, with special attention to the idea of "modeling the general on the particular."

Geometrical proofs in the context of physics

Gila Hanna (Toronto)

This study investigated a novel approach to the effective teaching of proof: the use of concepts and models from physics. In this approach, proving is embedded in the context of physics. Ideas from physics that are already familiar to students, such as the concepts of the centre of gravity and of balance, serve as tools for proving geometrical theorems, and students are prompted to create mathematical proofs based upon physical considerations, by taking as given one or more principles of physics. A mathematical proof, by definition, can take a set of explicit givens and use them, applying the principles of logic, as the basis for a deductive argument. A deductive argument need be no less compelling if the givens happen to come from the science of statics, as they do in this study. In fact the physical context, appealing as it does to physical intuition, has the advantage of making the plausibility of the conclusion more readily apparent. In this study, the students were asked to base their proofs upon the following three principles of statics:

P1: The uniqueness of the centre of gravity (each system of masses has one and only one centre of gravity).

P2: The lever principle (the centre of gravity of any two masses lies on the straight line joining the masses, and its distances from the masses are inversely proportional to them).

P3: The principle of substitution (if any two individual masses are replaced by a single mass equal to the sum of the two masses and positioned at the centre of gravity of the two masses, then the location of the centre of gravity of the total system of masses remains unchanged).

The two theorems studied were the medians in a triangle meet at a single point and the Varignon theorem, which states that given an arbitrary quadrangle ABCD, the midpoints of its sides R, S, U, V form a parallelogram. The experiment carried out in three grade 12 classes showed that most students were successful in using arguments from statics in their proofs, and that they gained a better understanding of the theorems. From responses to questionnaire questions and from individual interviews, it was clear that on the whole the students seemed to find the physical arguments based on the three principles of statics interesting, convincing and explanatory. These findings lend support to the claim that the introduction of statics can make a significant contribution to creating the sort of rich context in which students can more effectively learn and understand proving.

The role of proof in the mathematics classroom

Aiso Heinze and Kristina Reiss (Augsburg)

The role of proof and argumentation in the mathematics classroom has been discussed intensively in the last few years. Proof and argumentation are considered to be important aspect of the curricula. Nonetheless, this topic is difficult for students. We reported in our lecture on students' abilities and their specific problems with respect to geometrical proofs. Data are presented which were collected during an empirical investigation in grade 13. These students were asked to solve elementary geometry items from the TIMSS item pool. Our research results suggests that the students declarative knowledge in elementary geometry was sufficent for solving the problems. They know for example that the sum of all angles in every triangle adds up to 180 degrees or that base angles in an isosceles triangle are identical. Nonetheless they failed in solving the tasks because they lacked procedural knowledge. In particular they were not able to combine these different concepts and ideas in a simple mathematical proof. The results support hypotheses which claim that mathematics education in German high schools fosters algorithmic activities but not problem solving processes. Accordingly, mathematics is regarded to be a collection of concepts and theorems which the indivual understand or may not understand. Mathematics as a developing and process-oriented discipline is not known to students even at the upper secondary level. Consequences for instruction were discussed in this talk.

Experimental Mathematics and Proofs – what is secure mathematical knowledge?

Ulrich Kortenkamp (Berlin)

The role of computers and discrete mathematics in mathematics research and education creates the need for a new (?) way of looking at proofs. Sometimes, instead of rigorous reasoning a strong believe can be enough for successfully following new ideas in research. In teaching, it can be more helpful to replace a proof by a reasonably convincing argument, not only if the other option would be to omit it completely, but also if the proof were easy enough to be followed with some effort. We want to investigate how we can ensure that these new approaches are used wisely and for the benefit of mathematical education, and how mathematics acts as an empirical science already today.

A Universal Mathematical Solver and its possible role in school education

Yuri Matiyasevich (St. Petersburg)

In my talk I presented and demonstrated educational mathematical program "Universal Mathematical Solver" (UMS for short). This is a long-term project developed in St.Petersburg, Russia.

The UMS was intended to play the following role in the process of education: when a student (of high school or first university years) cannot solve his/her homework problem (and has nobody who could come for help), the UMS should solve the problem and produce detailed explanations of every step in the solution. This is the main feature which distinguish the UMS from such professional systems as MATHEMATICA, Maple, Axiom and others. The explanations are given both in text and audio form, the latter being considered very important component because for some people oral channel for getting information is more important than the visual one.

In spite of the school level of mathematics, the development of the UMS required solving a number of non-trivial mathematical problems. Some of them are common to all programs dealing with mathematical formulas (for example, simplification), others are specific to the school (for example, classical Sturm method cannot be used for proving non-existence of real solutions of an algebraic equation and ad-hoc methods should be used). The UMS uses more than a hundred heuristics and elements of artificial intelligence in the search of a solution.

The main question suggested for discussion was: is such a program a good or a bad thing? Of course, for a student having motivation to master mathematics, the UMS could be of great help. On the other hand, non-motivated student might just copy solution (found by the program) without even understanding it.

During the discussion after the presentation, the UMS (it its present form) was criticized for the completely passive role of students. The following ways to overcome this shortcoming were suggested.

1) Instead of solving the hometask problem, the program could construct a similar problem and demonstrate its solution.

2) The program could produce several "solutions", only one among them being correct.

3) On each step the program could suggest several possible ways to continue and then follow the way chosen by the student.

Implementation of these suggestions is possible but would require much more "intelligence" from the program (what is a "similar problem" and how to construct it?; how to mask errors so that they would not be evident?; under the wrong guidance from the student no solution could be found).

Discrete Mathematics for Primary and Secondary Schools: An Overview

Joseph G. Rosenstein (Piscataway) and Valerie A. DeBellis (Greenville)

An important outcome of the Leadership Program in Discrete Mathematics at Rutgers University (1989 to present) has been the development of a comprehensive view of discrete mathematics for primary and secondary schools describing what topics and activities can be introduced at each grade level, under each of five major themes systematic listing and counting, modeling using discrete structures (graphs and trees), iteration and recursion, organizing and processing information, and algorithms and optimization. An extended document containing these recommendations was developed as part of the effort in 1994-1996 to establish standards that describe what New Jersey students should know and be able to do in mathematics at various grade levels. In addition to the above, this presentation will discuss discrete mathematics in the NCTM standards (1989 and 2000) and in New Jerseys revised standards (2002), and the question of why discrete mathematics should be introduced in schools.

Discrete Mathematics for Primary and Secondary Schools: Materials Produced and Lessons Learned

Joseph G. Rosenstein (Piscataway) and Valerie A. DeBellis (Greenville)

Since 1989, we have conducted programs in discrete mathematics for primary and secondary teachers at Rutgers University in New Jersey, and also in seven other states. Over 1000 teachers have attended programs that involved two to four weeks of summer institutes and follow-up sessions during the school year. Among the results of these programs have been a comprehensive description of discrete mathematics at each grade level that was developed for inclusion in the New Jersey Mathematics Curriculum Framework; a volume entitled "Discrete Mathematics in the Schools" that consists of 33 articles on various aspects of the topic; a set of workshop materials that enables the program for elementary and middle school teachers to be replicated at other sites; and, in preparation, a textbook for prospective elementary and middle school teachers (some of whose materials have been used in a college course).

The lessons that we have learned involve the appropriateness, value, and importance of discrete mathematics at all grade levels, and what teachers (and children) learn about mathematics and problem solving as a result of doing discrete mathematics. These programs have been sponsored by DIMACS – the Center for Discrete Mathematics and Theoretical Computer Science – and the Rutgers Center for Mathematics, Science, and Computer Education, and have been funded by the National Science Foundation.

Formal reasoning

Jürgen Richter-Gebert (München)

A first year student in mathematics might very well get the impression that "proving" of theorems is the ultimate goal of a mathematician. Thereby he will perhaps miss the fact that finding a mathematical proof requires an even more fundamental ability: the ability of "formal reasoning". In the presentation I will exemplify different scenarios, where formal reasoning is required to solve problems. The problems are taken such that different aspects of formal reasoning are emphasized: understanding, constructing, programming, proving, modeling, analyzing, experimenting. In particular it will be shown how the computer can help to make these different kinds of problems accessible to learners in all stages (primary school, high school, university).

About Traveling Salesmen and Telephone Networks: Problems from Combinatorial Optimization in High School

Andreas Schuster (Würzburg)

My presentation is about a project which integrated theTraveling-Salesman-Problem (TSP), the Minimum-Spanning-Tree-Problem (MST) and the Shortest-Path-Problem (SPP) into regular school lessons at a German High-School ("Gymnasium") and special workshops at the Technical University of Munich and at the University of Wuerzburg. TSP and MST was given to 9th and 10th graders, SPP to 11th and 12th graders of a German High-School ("Gymnasium"). A main focus of my studies dealt with the development of strategies and concepts of students while working on the problems. I was especially interested in how the students managed the construction of the algorithms and how they analysed the time complexity in the frame of these types of problems. The studies were recorded by videotapes which gave the basis for the evaluation of the project.

Sequences – Basic Elements of Discrete Mathematics

Hans–Georg Weigand (Würzburg)

Sequences are fundamental objects in mathematics with a long tradition in the history of mathematics. The way sequences are taught in school mathematics in Germany has changed the last years. Up to 1980 sequences were widely used in calculus lessons as a basis for the limit concept. Nowadays calculus starts immediately with continuous functions based on an intuitive limit concept, first introduced by Emil Artin 1957 and Serge Lang 19733. The result is, that freshmen at the university are quite confident with formal rules of calculus, but they have no idea of the sequence concept (and of the limit concept). I see three reasons for a revitalization of the sequences concept in school mathematics and these reasons should be seen in a close relationship to the ideas of discrete mathematics. Firstly, a lot of real life problems allow mathematical representations with sequences, e. g. growth processes or weight-cost-problems. Secondly, discrete aspects support the learning process of continuous concepts, e. g. the difference quotient is a basis for the understanding of the differential quotient. Thirdly, computers, especially spreadsheets, provide the possibility to generate sequences, to create symbolic, numerical and graphical representations and to work with sequences in different representations. My main thesis is, that new technologies may be a catalyst to revitalize the sequence concept in school mathematics. I will give some examples of empirical investigations, which show how students worked with sequences in a computer-supported environment.

Edited by Gerald Goldin and Kristina Reiss

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