Mathematisches Forschungsinstitut Oberwolfach

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Elementare und Analytische Zahlentheorie

March 9th – March 15th, 2003

The present conference was organized by

Jörg Brüdern (Stuttgart),
Hugh L. Montgomery (Ann Arbor),
Yuri V. Nesterenko (Moscow),
Robert C. Vaughan (State College).

Thirty mathematicians from twelve countries participated. The range of topics was exceptionally broad, and involved such things as the vector sieve, the asymptotic large sieve, ergodic theory, Smirnov–Pyke statistics, automorphic functions, attacks on the Riemann Hypothesis, consequences of the Alternative Hypothesis, the integer Chebyshev problem, values of quadratic forms, moments of the Riemann zeta function, sums of sets, divisors in short intervals, and small gaps between prime numbers.

Simultaneously, in a session organized by Prof. Nesterenko, twelve mathematicians from five countries studied polylogarithms from the standpoint of transcendence. The ultimate object of this line of work would be to show that $\zeta(2n+1)$ is transcendental for all positive integers n.

The stimulating atmosphere of the Institute was appreciated by all participants. The organizers and participants are grateful to the Land Baden-Württemberg, to the Director, Prof. Greuel, and to the staff of the Institute, for their support at this time when number theory is especially active.

Die Tagung fand unter der Leitung von

Jörg Brüdern (Stuttgart),
Hugh L. Montgomery (Ann Arbor),
Yuri V. Nesterenko (Moskau),
Robert C. Vaughan (State College)

statt, an der dreißig Mathematiker aus zwölf Ländern teilnahmen.

Das Spektrum der Vorträge war ausgesprochen breit und umfasste Themen wie das Vektorsieb, das asymptotische grosse Sieb, Ergodentheorie, automorphe Funktionen, das Čebyšev Problem mit ganzzahligen Koeffizienten, Funktionswerte quadratischer Formen, Momente der Riemannschen Zetafunktion, Summen von Mengen, Teiler in kurzen Intervallen und kurze Lücken zwischen Primzahlen.

Die besonders angenehme und anregende Atmosphäre am Institut wurde von allen Tagungsteilnehmern hervorgehoben. Die Tagungsleiter und die Teilnehmer danken hierfür dem Land Baden–Württemberg, Herrn Prof. Greuel, dem Direktor des Instituts, und dem gesamten Institutspersonal für Ihre Unterstützung im besonderen in einer Zeit in der Zahlentheorie ein besonders aktives Forschungsgebiet ist.

Abstracts

The Fractional Part Function and the Riemann Hypothesis

Michel Balazard, Université Bordeaux I
(joint work with Eric Saias)

The Nyman criterion, which is necessary and sufficient for the Riemann Hypothesis (RH) to hold, states that $\chi = \chi_{(1,\infty)}$ is arbitrarily well approximable by functions of the type $\sum c_\alpha \{x/\alpha\}$, $\alpha \geq 1$, in $\mathcal{H} = L^2(0,\infty; t^{-2}dt)$. To study this approximation problem Michel Balazard, Luis Báez-Duarte, Bernard Landreau, and Eric Saias studied the function

$$A(\lambda) = \int_0^\infty \{t\}\{\lambda t\}\, t^{-2}\, dt$$

with the help of Estermann's function. In particular, we know that A has a strict local maximum at each rational point. Let $e_\alpha(t) = \{t/\alpha\}$, and set

$$d_n^2 = \operatorname{dist}^2_{\mathcal{H}}\big(\chi,\, \operatorname{Vect}(e_1,\dots,e_n)\big) = \frac{\operatorname{Gram}(e_1,\dots,e_n,\chi)}{\operatorname{Gram}(e_1,\dots,e_n)}\,.$$

Báez-Duarte proved that Nyman's criterion can be strengthened. He proved that RH is equivalent to $d_n = o(1)$. Consider

$$\frac{\operatorname{Gram}(e_1,\dots,e_n)}{\operatorname{Gram}(e_1,\dots,e_{n-1})} = \operatorname{dist}^2_{\mathcal{H}}\big(e_n,\, \operatorname{Vect}(e_1,\dots,e_{n-1})\big) =: n^{-2}L_n\,.$$

Numerical estimates suggest that $L_n \asymp \log n$. We proved

$$6/5 + o(1) \leq L_n \leq \log n + O(1)\,.$$

Some Galois Theory Associated to Zudilin's Recurrences

Frits Beukers, Universiteit Utrecht

In this lecture we discuss some of the background underlying the recently discovered recurrence relations for $\zeta(n)$ and Catalan's constant. For example, the differential Galois group associated to the differential equation of the $\zeta(4)$–recurrence turns out to be $O(5)$. Also differential equations associated to such recurrences are G–operators with all due consequences for the arithmetic of the solutions.

Sums of Two Squareful Integers

Valentin Blomer, Universität Stuttgart

A positive integer n is called squarefree if $p \mid n$ implies $p^2 \mid n$ for all primes p. An old problem of Erdős asks for the number of integers not exceeding x that can be written as a sum of two squareful integers. Here we present an almost best possible answer.

Theorem 1. *Let $\mathcal{V}$ be the set of integers that can be written as a sum of two squareful numbers, and let $\alpha = 1 - 2^{-1/3} = 0.206\ldots$ Then*

$$\frac{x}{(\log x)^{\alpha+\varepsilon}} \ll \sum_{\substack{n\le x\\ n\in\mathcal{V}}} 1 \ll \frac{x}{(\log x)^{\alpha-\varepsilon}}\,.$$

Theorem 1 is derived from Theorem 2 which gives sharp uniform bounds for the number of integers represented by systems of quadratic forms.

Theorem 2. *Let x be large and $F_1,\ldots,F_m \in \mathbb{Z}[x,y]$ be positive primitive quadratic forms of discriminant $D_j = D_{j,0}f^2$ respectively ($D_{j,0}$ a fundamental discriminant) such that $(D_i,D_j)\,|\,4$, $(D_i,D_j)=1$ for $i\neq j$, and $D_j \sim (\log x)^{2\kappa_j\log 2}$ for some $\underline{\kappa}\in(0,1)^m$. Then*

$$\frac{x}{(\log x)^{f(\|\underline{\kappa}\|_\infty)+\varepsilon}} \ll \sum_{\substack{n\le x\\ n \text{ represented by}\\ F_1,\ldots,F_m}} 1 \ll \frac{x}{(\log x)^{f(\|\underline{\kappa}\|_1)-\varepsilon}}$$

where $f(u)=1/2^m$ if $u\le 1/2^m$ and $f(u)=1+u\big(\log(2^m u)-1\big)$ otherwise.

Single-Valued Multiple Polylogarithms

Francis Brown, École Normale Supérieure, Paris

Various single-valued versions of the classical polylogarithm functions $\mathrm{Li}_n(z)$ have been studied by Ramakrishnan, Wotjkowiak, Zagier, Deligne, and others. These functions generalise the Bloch-Wigner dilogarithm. They occur in the computation of volumes of hyperbolic manifolds, regulators in algebraic K–theory, and special values of L–functions.

Multiple polylogarithms in one variable $\mathrm{Li}_w(z)$ are multi-valued functions on the cut punctured plane $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$, indexed by a binary word w . They are natural generalisations of the ordinary polylogarithms $\mathrm{Li}_n(z)$.

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$$\mathrm{Li}_w(z)Li_{w'}(\overline{z}).$$

We prove that the functions thus obtained are linearly independent, and that every possible single-valued version of polylogarithms occurs in this way. The differential algebra they generate is canonically isomorphic to the algebra of multiple polylogarithms. In this manner we obtain a canonical realisation of a universal, abstract algebra of polylogarithms in terms of uniform functions on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$.

Equal Sums of Two Powers

Tim Browning, Oxford University
(joint work with Roger Heath–Brown)

A new paucity result is established for the equation $w^k + x^k = y^k + z^k$ for integers $k \geq 4$. This improves upon a long-standing result of Hooley for $k \geq 5$, and is the best result available for $k = 5$. The underlying tool, which this application can be seen as advertising, is a bound for the number of integer points which lie on plane curves in very lopsided regions.

Sums of Three Squares

Jörg Brüdern, Universität Stuttgart
(joint work with Valentin Blomer)

We consider the quantity

$$r(n;\gamma) = \#\big\{\underline{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3 \colon\, x_1^2 + x_2^2 + x_3^2 = n,\ p\,|\,n \Rightarrow p > n^\gamma\big\}$$

where p denotes a prime. The main result is as follows.

Theorem. *If $\gamma \leq 5/5218$, then*

$$r(n;\gamma) \gg \mathfrak{S}(n)\frac{\sqrt{n}}{(\log n)^3(\log\log n)^2}$$

for all $n \in \mathbb{N}$ with $5 \nmid n$, $n \equiv 3 (\mathrm{mod}\ 8)$ where $\mathfrak{S}(n)$ denotes the singular series for three squares.

The result is derived via the vector sieve from a result on the number of solutions of the diophantine equation $d_1^2y_1^2 + d_2^2y_2^2 + d_3^2y_3^2 = n$ in positive integers y_1, y_2, y_3. If one writes

$$\omega(\underline{d})\mathfrak{S}(n)\frac{\sqrt{n}}{d_1d_2d_3} + E(n, \underline{d})$$

with $\mathfrak{S}(n)$ from above and $\omega(\underline{d})$ multiplicative with respect to d_1, d_2, d_3, then the success of the method depends on the following bound.

Lemma. *If n satisfies the congruence conditions of the above theorem, $\mu(d_j)^2 = 1$ for $j = 1, 2, 3$, and $2 \nmid d_1d_2d_3$, then*

$$E(n, \underline{d}) \ll n^{13/28+\varepsilon}\big(d_1d_2d_3\big)^{45/14}.$$

An Algebraic Theory of Polylogarithms

Pierre Cartier, Inst. des Hautes Etudes Scientifiques

We consider the class of functions known as multiple polylogarithms in one variable. For completeness, we include the ordinary logarithm. These functions are multivalued holomorphic functions (defined on the universal covering space of the complex plane with 0 and 1 removed) satisfying a certain system of differential equations. I introduce an abstract algebra encompassing their differential properties, in the form of a certain differential ring extension of the field of rational functions in one variable. The main result is that this differential ring is differentially simple, and is the minimal differential ring solving all differential equations with rational coefficients regular except possibly at 0 and 1, which can be put into a triangular form. As a corollary, we identify the corresponding differential Galois group, and prove the linear independence of the analytic polylogarithmic functions. This lecture is also a preparation for the lecture of Francis Brown.

Asymptotic Large Sieve

Brian Conrey, American Institute of Mathematics and Oklahoma State University
(joint work with Henryk Iwaniec)

The large sieve inequality, character version, asserts that

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \bmod q}^{*}\Big|\sum_{n=1}^{N}a_n\chi(n)\Big|^2 \le (N+Q^2)\sum_{n=1}^{N}|a_n|^2. \tag{1}$$

This inequality allows for the estimation of character sums of length $N \ll Q^2$. It should be compared with the mean value theorem of Montgomery and Vaughan:

$$\int_0^T\Big|\sum_{n=1}^{N}a_n n^{it}\Big|^2 dt \;=\; \big(T+O(N)\big)\sum_{n=1}^{N}|a_n|^2\,,$$

which allows for the estimation of the mean of Dirichlet polynomials of length $N=o(T)$. Note that this formula gives an asymptotic, whereas (1) does not. In this work we prove an asymptotic formula for the left side of (1) for certain sequences (a_n) which satisfy the assumptions:

(i) $\displaystyle\sum_{\substack{n\le x\\(n,g)=1}} a_{n\ell}\,\chi(n) \ll (xgq\ell)^{\varepsilon}$ for all $\chi\ne\chi_0 \pmod q$

(ii) $\displaystyle\sum_{\substack{n\le x\\(n,g)=1}} a_{n\ell} = \mathcal{A}_{g,\ell}(x)+\alpha_{g\ell}(x)$

with $\frac{d}{dx}\mathcal{A}_{g,\ell}(x)\ll (g\ell x)^{\varepsilon}x^{-1/2}$ and $\alpha_{g\ell}(x)\ll (g\ell x)^{\varepsilon}$.

(iii) $\displaystyle M_{\ell_1,\ell_2,u,v}(s,w,z) = \sum_{1\le r_1,r_2,g<\infty}\frac{\mu(r_1r_2)\mathcal{A}'_{\ell_1,g}(r_1u)\mathcal{A}'_{\ell_2,g}(r_2v)}{r_1^{1/2+w}r_2^{1/2+z}g^{s+1}}$

is analytic for $\Re s,\Re z,\Re w>1/2+\varepsilon$ apart from a finite set of poles in $|s|,|z|,|w|<\delta$ for some small δ.

Applications include:

- Assume the Generalized Riemann Hypothesis (GRH). Then there are $\gg Q^2/\log^2 Q$ characters $\chi \bmod q$ with $q\sim Q$ and zeros $\frac12+i\gamma_\chi$, $\frac12+i\gamma'_\chi$ such that $|\gamma_\chi|,|\gamma'_\chi|\ll 1$ and $|\gamma_\chi-\gamma'_\chi|<(0.37)2\pi/\log Q$.

- Assume the Generalized Lindelöf Hypothesis (GLH). Then

$$\sum_{q\le Q}\frac{w(q/Q)}{\phi(q)}\sum_{\chi \bmod q}^{*}|L(1/2,\chi_q)|^6 \ \ge\ (41+o(1))\widehat{w}(1)b_3 Q\frac{\log^9 Q}{9!}\,,$$

$$\sum_{q\le Q}\frac{w(q/Q)}{\phi(q)}\sum_{\chi \bmod q}^{*}|L(1/2,\chi_q)|^6 \ \le\ (43+o(1))\widehat{w}(1)b_3 Q\frac{\log^9 Q}{9!}\,.$$

Lerch Generalized Functions and Diophantine Approximation

Jacky Cresson, Université de Franche–Comte
(joint work with Tanguy Rivoal)

Classical results about diophantine approximation of multiple zeta values (MZV) are obtained, following Beuker's proof of Apery's result on the irrationality of $\zeta(3)$, by considering multiple integrals with specific symmetry properties. A generalisation of Beuker's integrals is given by Sorokin. A first step, to understand these integrals is to give an algorithmic way to decompose it in linear combination of MZV. Such an algorithm can be developed by introducing a generalization of MZV called Lerch generalized functions (LGF). Moreover, by introducing multiple variables in Sorokin's integrals, we are able to explain why some MZV doesn't appear, using asymptotic properties of these functions. As a byproduct, we obtain precise decomposition theorem for Sorokin's integrals, a new proof of Vasilyev theorem and a proof of "non-enrichment arithmetic", which say that every LGF with negative exponents is a linear combination of LGF with positive exponents (generalizing Ecalle's result on MZV).

Polynomial Bounds for Equivalence of Quadratic Forms

Rainer Dietmann, Universität Stuttgart

Let $Q_1, Q_2 \in \mathbb{Z}[X_1,\dots,X_s]$ be nonsingular, classically integral quadratic forms given by

$$Q_1(X_1,\dots,X_s)=\sum_{i,j=1}^{s} a_{ij}X_iX_j,\quad Q_2(X_1,\dots,X_s)=\sum_{i,j=1}^{s} b_{ij}X_iX_j$$

for symmetric integer coefficient matrices $A=(a_{ij})_{1\le i,j\le s}$ and $B=(b_{ij})_{1\le i,j\le s}$ with $\det A\neq 0$ and $\det B\neq 0$. We call Q_1 and Q_2 equivalent if there is a unimodular transformation $R:\mathbb{Z}^s\to\mathbb{Z}^s$ with $Q_1(R(X_1,\dots,X_s))=Q_2(X_1,\dots,X_s)$, which can be expressed in the form $B=R^TAR$ for a unimodular integral $s\times s$ matrix R. By using special properties of ternary quadratic forms we recently established for $s=3$ a polynomial bound for equivalence of quadratic forms: If Q_1 and Q_2 are equivalent, which means $B=R^TAR$ for some unimodular R, then there is such an R with $\|R\|\ll H^{900}$ where $\|\cdot\|$ denotes the maximum norm and $H=\max\{\|A\|,\|B\|\}$. In our talk we indicated how this result can be extended inductively to forms in more than three variables satisfying a not too restrictive extra condition:

Theorem. *Suppose that* $\det A$ *is cubefree, not divisible by* 4, *and that* A *satisfies some further* 2*-adic condition. If* $B = R^T AR$ *for some unimodular* R, *then there is such* R *with*

$$\|R\| \ll_s \begin{cases} H^{9900}|\det A|^{19000} & \text{when } s=4\,, \\ H^{500000}|\det A|^{1250000} & \text{when } s=5\,, \\ H^{(4(s+5))^s}|\det A|^{30(1+4/(s-4))(4s+16)^{s-1}} & \text{when } s\geq 6\,. \end{cases}$$

This improves on earlier bounds which have been exponential in H for given s. The proof relies on local–global techniques from the theory of integral quadratic forms and bounds for the least solution of a quadratic diophantine equation.

Primes and Products

Peter D. T. A. Elliott, University of Colorado

A survey was given of results concerning product representations of positive rationals by integers of the form $(p{+}1)f(q)^{-1}$ and their reciprocals, where f is a polynomial with rational/integer coefficients, and p, q are primes. In particular results of Wirsing, Meyer and Tenenbaum, Wolke, Dress and Volkmann, Berrisbeitia and myself.

Conjecture 1 (Dickson). *Given* $r \in \mathbb{N}$, *then there are infinitely many representations* $r = (p{+}1)(q{+}1)^{-1}$ *with* p, q *prime.*

Conjecture 2. *If* $f \in \mathbb{Z}[x]$, $\deg f \geq 1$, $r \in \mathbb{Q}$, $r > 0$, *then there are infinitely many representations* $r = (p_1{+}1)f(q_1)^{-1}\big((p_2{+}1)f(q_2)^{-1}\big)^{-1}$ *with* $(p_i{+}1)f(q_i)^{-1} \in \mathbb{Z}$, p_i, q_i *prime.*

Theorem 1. *If* $f \in \mathbb{Q}[x]$, $\deg f \geq 1$, $c > 0$, *then at least one of the equations*

$$(x_1{+}1)f(x_2) - r^k(x_3{+}1)f(x_4) \;=\; 0 \qquad (k = 1, 2, 3)$$

for a given positive rational r, *has infinitely many solutions with* $x_2 > x_4^c$, *and all* x_i *prime.*

Doubtless this result holds with $k = 1$ (Conjecture 2). The condition $x_2 > x_4^c$ is of interest when $r = 1$.

Theorem 2. *Let* a *be a power of an odd prime. Then there are infinitely many representations*

$$a^k = \frac{p+1}{q+1} \qquad a^{2k} < q \leq a^{55k}\,,$$

with positive integers k, *and primes* p, q.

There is a similar result, valid for every $a \geq 2$, with another absolute constant in place of 55.

Hilbert Cubes in the Set of Primes

Christian Elsholtz, Universität Clausthal

Let $a_0, a_1, \ldots, a_d$ denote positive integers, and let $\mathcal{H} = \{a_0 + \sum_{i=1}^{d} \varepsilon_i a_i : \quad \varepsilon_i \in \{0,1\}\}$. $\mathcal{H}$ is called a Hilbert cube.

Theorem 1. *For $\mathcal{B} \subseteq [1, N]$ and $\varepsilon > 0$ there is a Hilbert cube $\mathcal{H} \subseteq \mathcal{B}$ of dimension d with*

$$d = \frac{1}{\log 2}\Big(\log\log N - \log\log\Big(\frac{(2+\varepsilon)N}{|\mathcal{B}|}\Big)\Big).$$

This improves the constant in Szemerédi's cube lemma. Hegyvári and Sárközy proved that a Hilbert cube in the set $\mathbb{P} \cap [1, N]$ where all base elements $a_1, \ldots, a_d$ are distinct has dimension $d < (16 + \varepsilon)\log N$. We improve this to

Theorem 2. *Let d be the maximal dimension of a Hilbert cube in $\mathbb{P} \cap [1, N]$ with distinct base elements. Then $d = O(\log N / \log\log N)$.*

The proof of Theorem 1 is a refinement of existing proofs. The proof of Theorem 2 uses results on the addition of distinct residue classes modulo primes.

We are able to show that in this situation the worst case distribution of the frequency of the occurring residue classes is *not* when all occurring residue classes modulo p occur with about the same frequency (as is usually the case). This asymmetric distribution can successfully be used in a sieve bound, based on Gallagher's larger sieve.

Padé Approximates and Poised Hypergeometric Series

Stephane Fischler, École Normale Supérieure
(joint work with Tanguy Rivoal)

We present and solve some very general new Padé approximate problems, whose solutions can be expressed with hypergeometric series. These series appear in the proof of the irrationality of $\zeta(3)$, of infinitely many $\zeta(2n+1)$, and in essentially all results of this kind in the literature. Let $\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} x^n n^{-s}$ be the s-th polylogarithm. Then

Theorem 1. *Let $\alpha \in \mathbb{Q}$, $0 < \alpha < 1$, and $\mu \in \mathbb{R}$. Then*

$$\{\mathrm{Li}_s(\alpha) + \mu \frac{\log^s \alpha}{(s-1)!},\ s \in \mathbb{N}\}$$

spans an infinite dimensional vector space over $\mathbb{Q}$.

Theorem 2. *Among the three numbers*

$$\mathrm{Li}_s(1/2) + \frac{\log^s(1/2)}{(s-1)!}, \qquad s \in \{2,3,4\},$$

at least one is irrational.

Distribution of Integers with a Divisor in a Given Interval

Kevin Ford, University of Illinois

Denote by $\varepsilon(y)$ the density of integers which have a divisor in $(y, 2y]$, and let $\varepsilon_r(y)$ be the density of integers which have exactly r divisors in $(y, 2y]$.

Theorem. *We have*

$$\varepsilon(y), \varepsilon_r(y) \asymp \frac{1}{(\log y)^{\delta} (\log\log y)^{3/2}}$$

where $\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.08607\ldots$

Corollary. *For each* $r \geq 1$

$$\liminf_{y\to\infty} \frac{\varepsilon_r(y)}{\varepsilon(y)} > 0\,.$$

The corollary answers in the negative a question of Erdős from 1960 (see the M.F.O book "Mathematical Problems", p. 3–4). Among the new tools are uniform upper bounds for the Smirnov–Pyke statistics from the theory of uniform statistics.

On Sign Changes of Klosterman Sums

Etienne Fouvry, Université Paris XI
(joint work with Philippe Michel)

Let $\mathrm{Kl}(a, b; n)$ be the usual Kloosterman sum

$$\mathrm{Kl}(a, b; n) = \sum_{\substack{x \,(\mathrm{mod}\, n)\\ (x,n)=1}} \exp\Big(2\pi i \,\frac{ax + b\overline{x}}{n}\Big)$$

where $\overline{x}$ is the multiplicative inverse of x modulo n. We are concerned with the sign changes of the Kloosterman sums $\mathrm{Kl}(1, 1; n)$ with n almost prime, and prove the

Theorem. *Let* $g \in \mathrm{C}_c^{\infty}\big([1, 2], \mathbb{R}\big)$, g *positive, and* $u \geq 23.9$. *Then there exist* $X_0(u, g)$ *and* $\delta_0(u, g) > 0$ *such that, for* $X > X_0$, *we have*

$$\Big| \sum_{\substack{n\\ p|n \Rightarrow p \geq X^{1/u}}} g(n/x)\, \frac{\mathrm{Kl}(1, 1; n)}{\sqrt{n}} \Big| \leq (1 - \delta_0) \sum_{\substack{n\\ p|n \Rightarrow p \geq X^{1/u}}} g(n/x)\, \frac{|\mathrm{Kl}(1, 1; n)|}{\sqrt{n}}\,.$$

This theorem implies the

Corollary. *There exists* $c_0 > 0$ *such that, for* $X > X_0$, *we have*

$$\#\big\{n\colon X < n \leq 2X,\, p|n \Rightarrow p \geq X^{1/23.9},\, \mathrm{Kl}(1, 1; n) > 0\big\} \geq c_0 \frac{X}{\log X}\,.$$

(The same corollary holds for negative Kloosterman sums.)

The proof uses sieve techniques, modular forms (sums of Kloosterman sums), Katz's result about the vertical Sato–Tate distribution of angles of Kloosterman sums, study of exponential sums via Deligne–Katz techniques.

A Theorem with Finite Shelf Life

JOHN B. FRIEDLANDER, UNIVERSITY OF TORONTO
(joint work with Henryk Iwaniec)

In this work we study the very strong statements (beyond even the capabilities of GRH) which can be deduced concerning the distribution of primes in arithmetic progressions under the (admittedly unlikely) assumption of the existence of exceptional zeros of Dirichlet L-functions.

Let χ be a real, primitive Dirichlet character of conductor D.

THEOREM 1. *Let $q \geq 1$, and let a be integers with $(a,q)=1$. Let $x \geq \max\{D^r, q^{462/233}\}$ where $r = 554{,}401$. Then*

$$\psi(x;q,a) \;=\; \frac{\psi(x)}{\phi(q)}\Big(1-\chi\Big(\frac{aD}{(q,D)}\Big)+O\big(L(1,\chi)(\log x)^{r^r}\big)\Big)\,.$$

Note that $462/233 < 2$. The theorem is only useful if $L(1,\chi)$ is very small.

Assume

$$\begin{aligned} (i) &\quad \log D \asymp \log x \\ (ii) &\quad q \not\equiv 0 \,(\mathrm{mod}\ D) \\ (iii) &\quad L(1,\chi) \leq (\log D)^{-(r^r+1)}. \end{aligned}$$

Then

$$\psi(x;q,a) = \frac{\psi(x)}{\phi(q)}\Big(1+O\Big(\frac{1}{\log x}\Big)\Big).$$

COROLLARY. *Let $p(q,a)$ denote the least prime $p \equiv a \,(\mathrm{mod}\ q)$. Let $D^r \leq q \leq \exp\big(L(1,\chi)^{-c}\big)$ where $c = r^r+1$. Then $p(q,a) \ll q^{2-1/59}$, under the assumptions $(i)-(iii)$.*

We also have a similar statement for primes in short intervals.

THEOREM 2. *Let $x \geq D^r$ with $r = 18{,}290$ and $x^{39/79} < y \leq x$. Then*

$$\psi(x)-\psi(x-y) \;=\; y\Big(1+O\big(L(1,\chi)(\log x)^{r^r}\big)\Big)\,.$$

Some day it will be proved that there are no exceptional characters. But perhaps the shelf life of these theorems will be rather long!

Lattice Point Problems and Distribution of Values of Quadratic Forms

FRIEDRICH GÖTZE, UNIVERSITÄT BIELEFELD

Let $Q(x)$ denote a positive definite quadratic form in $\mathbb{R}^d$. We prove that the number of lattice points in the ellipsoid $E_s = \{x \in \mathbb{R}^d : Q(x) \leq s\}$ for s large is equal to its volume up to an error of order $O\big(s^{d/2-1}\big)$ in general. For irrational ellipsoids the error is of order $o\big(s^{d/2-1}\big)$ depending on the diophantine properties of the coefficients. This result implies a conjecture of Davenport and Lewis that the gaps between consecutive values of $Q(m)$, $m \in \mathbb{Z}^d$, tend to zero at infinity for irrational positive definite forms in $d \geq 5$ variables. For indefinite forms recent joint work with G. Margulis provides effective bounds for dimensions $d \geq 5$ on the existence of lattice points $m \in \mathbb{Z}^d \setminus \{0\}$ such that $|Q(m)| < 1$ and $|Q_+(m)| \ll_{\delta,d} |\det Q|^{\gamma_d+\delta}$ for some not yet optimal constant $\gamma_d > 1$. These results are based on analytic bounds and methods developed in Eskin, Margulis, and Mozes.

Small Gaps Between Primes

Daniel A. Goldston, San Jose State University
(joint work with Cem Y. Yıldırım)

Let p_n denote the n-th prime. As a first step in the direction of the twin prime conjecture one would like to prove that

$$\liminf_{n\to\infty}\left(\frac{p_{n+1}-p_n}{\log p_n}\right) = 0 .$$

This conjecture itself may be very difficult to prove. The best result presently is due to Maier who proved the infimum above is $\le 0.24846\ldots$.

Over the last few years we have been developing a moment method based on approximating prime tuples by short divisor sums. Given a positive integer h, let $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$, with $1 \le h_1, h_2, \ldots, h_k \le h$ distinct integers, and let $\nu_p(\mathcal{H})$ denote the number of distinct residue classes modulo p the elements of $\mathcal{H}$ occupy. Define the singular series

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1-\frac{1}{p}\right)^{-k}\left(1-\frac{\nu_p(\mathcal{H})}{p}\right).$$

If $\mathfrak{S}(\mathcal{H}) \neq 0$ then $\mathcal{H}$ is called *admissible*. Thus $\mathcal{H}$ is admissible if and only if $\nu_p(\mathcal{H}) < p$ for all p. Letting

$$\Lambda(n;\mathcal{H}) = \Lambda(n+h_1)\Lambda(n+h_2)\cdots\Lambda(n+h_k),$$

with $\Lambda(n)$ the von Mangoldt function, the Hardy–Littlewood prime tuple conjecture states that for $\mathcal{H}$ admissible,

$$\sum_{n\le N}\Lambda(n;\mathcal{H}) = N\big(\mathfrak{S}(\mathcal{H}) + o(1)\big), \qquad \text{as} \quad N\to\infty .$$

(This is trivially true if $\mathcal{H}$ is not admissible.) We can approximate $\Lambda(n)$ by using the truncated divisor sum

$$\Lambda_R(n) = \sum_{\substack{d\mid n\\ d\le R}} \mu(d)\log\frac{R}{d},$$

and then approximate $\Lambda(n;\mathcal{H})$ by

$$\Lambda_R(n;\mathcal{H}) = \Lambda_R(n+h_1)\Lambda_R(n+h_2)\cdots\Lambda_R(n+h_k).$$

This approximation mimics the Hardy–Littlewood prime tuple conjecture if R is a small enough power of N. Suppose $R = o(N^{1/k})$. Then we have for $1 \le h \ll \log N$ and $R, N\to\infty$,

$$\sum_{n\le N}\Lambda_R(n;\mathcal{H}) = N\big(\mathfrak{S}(\mathcal{H}) + o(1)\big).$$

To obtain information about primes, we prove similar but more complicated asymptotic formulas for

$$\sum_{n\le N}\Lambda_R(n;\mathcal{H}_1)\Lambda_R(n;\mathcal{H}_2),$$

$$\sum_{n\le N}\Lambda_R(n;\mathcal{H})\Lambda(n+h_0),$$

and

$$\sum_{n\le N} \Lambda_R(n;\mathcal{H}_1)\Lambda_R(n;\mathcal{H}_2)\Lambda(n+h_0)\,.$$

Using these formulas we can improve on Maier's result somewhat. The method can also be substantially improved by more complicated choices for $\Lambda_R(n,\mathcal{H})$. At present we have not determined if the method is sufficiently strong to prove the conjecture or not.

Incomplete Sums of Multiplicative Functions

George Greaves, Cardiff University

Consider

$$G_z(x) \;=\; \sum_{\substack{n\le x\\ p\mid n\,\Rightarrow\, p<z}} g(n)\,,$$

where $g(n)\ge 0$, g is multiplicative, and there exist constants $\kappa>0$, $A\ge 1$, and $L\ge 1$ such that

$$-L \;\le\; \sum_{u\le p<v} g(p)-\kappa\log\frac{v}{u} \;\le\; A \tag{1}$$

when $2\le u\le v<z$. Largely for convenience, suppose that $g\big(p^\nu\big)=0$ when $\nu\ge 2$, although weaker assumptions such as $\sum_p\sum_{\nu\ge2} g\big(p^\nu\big)=O(1)$ would suffice. Observe that

$$G_z(x) \;=\; \prod_{p<z}\big(1+g(p)\big)=G_z(\infty)$$

as soon as x is large enough. The standard estimation in the literature is of the type

$$\frac{G_z(x)}{G_z(\infty)}=\sigma(s)+O\Big(\frac{E(s)}{\log z}\Big)\,,$$

where $x=z^s$, $\sigma=\sigma_\kappa$ a certain function, and $E(s)\ge 1$ is nondecreasing. For large s these entries $E(s)$ are somewhat unsatisfactory since $\sigma(s)=1+O\big(e^{-s\log s}\big)$, and an application of Rankin's trick shows that

$$\frac{G_z(x)}{G_z(\infty)} \;\ge\; 1-e^{-s\log s+O(s)}$$

for $x=z^s$. In the lecture a method was described which shows

$$\frac{G_z(x)}{G_z(\infty)} \;\ge\; \sigma(s)+\frac{e^{-s\log s+O(s)}}{\log z}$$

where in (1) we need assume only the right-hand inequality.

Linear Statistics of Zeros of Dirichlet L–Functions

Chris Hughes, American Institute of Mathematics

We consider the moments of the smooth counting function of low-lying scaled zeros of Dirichlet L–functions. If the test function f satisfies $\operatorname{supp} \hat{f} \subset [-\alpha, \alpha]$ then the first $\lfloor 2/\alpha \rfloor$ moments of the counting function are the Gaussian moments with mean $\int_{-\infty}^{\infty} f(x)\,dx$ and variance $\int_{-\infty}^{\infty} \min(|u|, 1) \hat{f}(u)^2 du$. But the overall distribution is not Gaussian. This same behaviour is seen in random matrix theory. As an application we show that there are an infinite number of L–functions whose lowest zero is less than $1/4$ times its expected height.

Integer and Rational Points Close to Curves

Martin N. Huxley, Cardiff University

Let $\mathcal{C}$ be a curve in the plane. Estimate the maximum number of possible integer points on the enlarged curve $M\mathcal{C}$ as a function of M. Bombieri and Pila showed that a constant $B(\mathcal{C}, d)$ exists, the maximum number of intersections of $\mathcal{C}$ with any algebraic curve of degree at most d; if $\mathcal{C}$ is itself an algebraic curve with no component of degree at most d, then $B(\mathcal{C}, d)$ is an intersection number. Bombieri and Pila estimated the number of integer points on $M\mathcal{C}$ when the curve $\mathcal{C}$ is algebraic or real-analytic.

Using conditions (bounds on determinants of derivatives) that ensure $B(\mathcal{C}, d)$ takes its minimum value $\frac{1}{2}d(d+3)$, we extend the result to smooth curves (in C^k) and to count integer points within a very small distance δ of the curve $M\mathcal{C}$. In analogous questions about rational points, the relevant algebraic curves are rational functions of x.

On the Moments of Hecke Series at Central Points

Aleksandar Ivić, University of Belgrade

Let $H_j(s)$ denote the Hecke series attached to the j–th Maass wave form $\psi_j(z)$, and let $\alpha_j = |\rho_j(1)|^2 (\cosh \pi\kappa_j)^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of $\psi_j(z)$, and $\{\kappa_j^2 + 1/4\} \cup \{0\}$ is the discrete spectrum of the hyperbolic Laplacian. It is proved that

$$\sum_{\kappa_j \le K} \alpha_j H_j^3(1/2) \;=\; K^2 P_3(\log K) + O\bigl(K^{5/4} (\log K)^{37/4}\bigr)$$

and

$$\sum_{\kappa_j \le K} \alpha_j H_j^4(1/2) \;=\; K^2 P_6(\log K) + O\bigl(K^{3/2} (\log K)^{25/2}\bigr)$$

where $P_3(z)$ is a cubic polynomial in z with leading coefficient $4/(3\pi^2)$, and $P_6(z)$ is a polynomial in z of degree six with leading coefficient $16/(15\pi^4)$.

It is conjectured that, for $k \in \mathbb{N}$ fixed, and suitable $0 \le c_k < 1$,

$$\sum_{\kappa_j \le K} \alpha_j H_j^k(1/2) + \frac{2}{\pi} \int_0^K \frac{\bigl|\zeta(\frac{1}{2} + it)\bigr|^{2k}}{\bigl|\zeta(1 + 2it)\bigr|^2}\, dt$$
$$= K^2 P_{(k^2-k)/2}(\log K) + O_{\varepsilon,k}\bigl(K^{1+c_k+\varepsilon}\bigr)$$

where $P_{(k^2-k)/2}(z)$ is a polynomial of degree $(k^2 - k)/2$ in z whose coefficients depend on k.

Spectral Averages of Hecke Series Attached to Maass Wave Forms

MATTI JUTILA, UNIVERSITY OF TURKU

Let

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s} \qquad (\sigma > 1)$$

be the Hecke series related to the j-th Maass wave form, where $t_j(n)$ are corresponding Hecke eigenvalues. Let the j-th eigenvalue of the hyperbolic Laplacian be written as $1/4 + \kappa_j^2$. We consider the following

CONJECTURE.

$$H_j(1/2 + it) \ll (|t| + \kappa_j)^{1/3+\varepsilon} .$$

This has been proved by Y. Motohashi and the author for $|t| \ll \kappa_j^{2/3-\varepsilon}$. On the other hand it follows from the estimate

$$\sum_{K \le \kappa_j \le K+G} \left| H_j(1/2+it) \right|^2 \ll \left(GK + t^{2/3}\right)(t+K)^{\varepsilon} \qquad (1 \ll G \ll K)$$

for $t \gg \kappa_j^{3/2}$. Also the estimate

$$\sum_{\kappa_j \le K} H_j(1/2)^{12} \ll K^{4+\varepsilon}$$

was discussed. This implies the bound

$$H_j(1/2) \ll \kappa_j^{1/3+\varepsilon}$$

due to A. Ivić for individual Hecke L-functions.

On Zeros off the Critical Line

JERZY KACZOROWSKI, A. MICKIEWICZ UNIVERSITY, POZNAN
(joint work with M. Kulas)

PROBLEM. Suppose an L-function from the extended Selberg class $S^{\#}$ satisfies GRH. Is it true that then the L-function has an Euler product expansion, up to a finite number of primes?

We say that $F \in S^{\#}$ has the *density property* if for every $1/2 < \sigma < 1$ we have $N_F(\sigma, T) = o(T)$ as $T \to \infty$.

We prove the following theorem solving the above problem in degree 1.

THEOREM. *Let $F \in S^{\#}$ have degree 1. Then F has the density property if and only if $F(s+i\theta) = P(s)L(s,\chi)$ for certain real θ, a Dirichlet character χ, and a Dirichlet polynomial $P(s)$ satisfying RH.*

COROLLARY. *If $F \in S^{\#}$ of degree 1 satisfies GRH then it has an Euler product expansion apart from at most a finite number of primes.*

The proof is based on the general converse theorem for degree 1 L-functions belonging to the extended Selberg class [J. Kaczorowski and A.Perelli, *Acta Math.* **182** (1999), 207-241] and on a suitably generalized Voronin type universality theorem for Dirichlet L-functions.

An Estimate for a Cubic Weyl Sum over Primes

Koichi Kawada, Iwate University, Morioka

Let k be a natural number greater than 1, and let the letter p denote a prime number. We introduce the multiplicative function $w_k(q)$ by defining its value for prime powers as follows; $w_k(p^{uk+1}) = kp^{-u-1/2}$ for $u \geq 0$, and $w_k(p^{uk+v}) = p^{-u-1}$ for $u \geq 0$ and $2 \leq v \leq k$. Recently Kawada and Wooley proved that if $k \geq 4$, $X \geq 2$, α is a real number, and q and a are coprime integers satisfying $1 \leq q \leq X^{k/2}$ and $|q\alpha - a| \leq X^{-k/2}$, then for each $\varepsilon > 0$ one has

$$\sum_{X<p\leq 2X} e(p^k\alpha) \ll X^{1-2^{-k-1}+\varepsilon} + \frac{q^\varepsilon w_k(q)^{1/2} X(\log X)^4}{(1+P^k|\alpha - a/q|)^{1/2}},$$

where $e(\alpha) = e^{2\pi i\alpha}$ as usual. Our aim here is to obtain the corresponding bound for the case $k = 3$.

The argument of Kawada and Wooley faces a substantial obstacle when $k = 3$, concerning the treatment of sums of the form

$$\sum_{M<m\leq 2M} a_m \sum_{X/m<n\leq 2X/m} b_n e((mn)^3\alpha),$$

with certain sequences $\{a_m\}$ and $\{b_n\}$. Actually their method works satisfactorily when $X^{4/7} \ll M \ll X$, but our goal requires to deal with the cases where $X^{1/2} \ll M \ll X^{4/7}$, too. By a different approach, it becomes possible to get a desired bound for the latter cases, if we impose an additional condition upon the sequence $\{b_n\}$ that $b_n = 0$ whenever n is not a prime power. In fact, this is a key lemma to our purpose.

Now set $z = 2X^{1/8}$, let $\Pi(z)$ be the product of all primes less than z, and for a natural number x, write $\nu(x)$ for the number of distinct primes p such that $p|x$ and $z \leq p < 2X^{3/7}$. We use the identity

$$\sum_{X<p\leq 2X} e(p^3\alpha) = \sum_{\substack{X<x\leq 2X\\(x,\Pi(z))=1}} e(x^3\alpha) - \sum_{\substack{X<x\leq 2X\\(x,\Pi(z))=1\\ \nu(x)>0}} e(x^3\alpha) - \sum_{\substack{X<p_1p_2\leq 2X\\ 2X^{3/7}\leq p_1\leq p_2\leq X^{4/7}}} e((p_1p_2)^3\alpha).$$

As for the first sum on the right hand side, one may express the condition $(x,\Pi(z)) = 1$ by means of the Möbius function in the familiar way. As for the second sum on the right hand side, one may rewrite it using the fact that x has a prime factor of appropriate size. In any case, the first and second sums may be estimated by known techniques. And the last sum on the right hand side takes a shape to which we can apply the above key lemma. In this way we may show that the above result of Kawada and Wooley is valid for $k = 3$, as well.

Combinatorial Problems with Squarefree Numbers

Sergei V. Konyagin, Moscow State University

Let S_0 be the set of positive squarefree integers, $N \in \mathbb{Z}$. Denote

$$ES_N = \max\{\#A : A \subset \{1, 2, \ldots, N\}, a, a' \in A \Rightarrow a + a' \in S_0\},$$

and

$$BR_N = \max_P \left(\frac{\max_{u \in [0,1]} |P(u)|}{\int_0^1 |P(u)|\, du} \right)$$

where the maximum is taken over nonzero polynomials

$$P(u) = \sum_{\substack{n \in S_0 \\ n \le N}} a_n e(nu)$$

with $e(z) = e^{2\pi i z}$. It is easy to see that $ES_N \le BR_{2N}$. For $N \ge 3$ we prove the following estimates:

$$(\log N)^2 \log\log N \ll ES_N \ll_\varepsilon N^{11/15+\varepsilon},$$
$$BR_N \ll_\varepsilon N^{11/15+\varepsilon}.$$

The upper estimate uses an improvement of the large sieve over prime squares, based on the recent result of Bombieri and Zannier: For every $\varepsilon > 0$, for every $X \ge 1$ and for every $Y \ge 1$, the number of quadruples $(a_1/p_1^2, a_2/p_2^2, a_3/p_3^2, a_4/p_4^2)$ where the p_j are primes, $p_j \le X$, $1 \le a_j < p_j^2$, $a_1/p_1^2 < a_2/p_2^2 < a_3/p_3^2 < a_4/p_4^2$, $a_4/p_4^2 - a_1/p_1^2 \le Y/X^4$ is $\ll_\varepsilon X^\varepsilon Y^4$.

Beurling Primes with Large Oscillations

Hugh L. Montgomery, University of Michigan

(joint work with Harold G. Diamond and Ulrike M. A. Vorhauer)

In 1903, Landau gave the first proof of the Prime Number Theorem that did not use the analytic continuation of the Riemann zeta function. We now show that Landau's method is best possible.

Let $\lambda_1 \le \lambda_2 \le \ldots$ be a sequence of real numbers, with $\lambda_1 > 1$ and $\lambda_i \to \infty$, taken to be a set of generalized primes (Beurling primes), and the finite products $\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_r^{k_r}$ are considered to be the generalized integers arising from these primes. Let $N_B(x)$ denote the number of such products not exceeding x (counted with appropriate multiplicity in case some real numbers are represented as Beurling integers in more than one way). Let $\pi_B(x)$ denote the number of Beurling primes λ_i not exceeding x, and let

$$\zeta_B(s) = \sum_{\boldsymbol{k}} \frac{1}{\left(\lambda_1^{k_1}\lambda_2^{k_2}\cdots\right)^s} = \int_{1^-}^{\infty} x^{-s}\, dN_B(x) = \prod_{i=1}^{\infty} \left(1 - \lambda_i^{-s}\right)^{-1} \qquad (\sigma > 1)$$

be the associated generalized zeta function, where $\boldsymbol{k} = (k_1, k_2, \ldots)$ is a vector with non-negative integer components, all but a finite number of which are zero.

Theorem. *Suppose that $1/2 < \theta < 1$ and $a > (4/e)(1-\theta)$ are fixed. Then there is a system of Beurling primes such that* (*i*) $N_B(x) = \kappa x + O\big(x^\theta\big)$ *with* $\kappa > 0$;

(*ii*) $\zeta_B(s)$ *is analytic for* $\sigma \geq \theta$, *apart from a simple pole at* $s = 1$ *with residue* κ;

(*iii*) $\zeta_B(s)$ *has infinitely many zeros on the curve* $\sigma = 1 - a/\log t$, $2 \leq t < \infty$, *and no zero to the right of this curve*;

(*iv*) $\psi_B(x) - x \;=\; \Omega_\pm\big(x \exp(-2\sqrt{a \log x}\,)\big)$;

(*v*) $\psi_B(x) \;=\; x + O\big(x \exp(-2\sqrt{a \log x}\,)\big)$.

A New Way to Treat the Fourth Moment of the Riemann Zeta–Function

Yoichi Motohashi, Nihon University, Tokyo

(joint work with Roelof W. Bruggeman)

We proved the following

Theorem. *Let* $\mathit{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$, $G = \mathrm{PSL}_2(\mathbb{R})$, *and let*

$$M(\zeta^2, g) \;=\; \int_{-\infty}^{\infty} \big|\zeta(\tfrac{1}{2} + it)\big|^4 g(t)\, dt\,.$$

Then there exists a family $\{\psi\} \subset L^2(\mathit{\Gamma}\backslash G)$ *such that* $\psi(1)$ *tends to the nondiagonal part of* $M(\zeta^2, g)$.

The argument works with a Pointcaré series on G. Thus its spectral decomposition gives that of $M(\zeta^2, g)$. The spectral decomposition or the orthogonal projection to each irreducible subspace of $L^2(\mathit{\Gamma} \backslash G)$ is computed with the use of the Kirillov map. No Kloosterman sums are involved. This solves the basic problem posed in my book, namely to find a way to prove the explicit formula for $M(\zeta^2, g)$ without using the spectral theory of sums of Kloosterman sums.

A peculiar role of the Bessel function of representations of G is stressed as a mean to understand the geometric structure of mean values of automorphic L-functions in general.

Linear Forms in Values of Polylogarithms

Yuri V. Nesterenko, Moscow State University

The talk presents a review of results about arithmetic properties of values of polylogarithmic functions $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} z^k/n^k$, $k \geq 1$, at rational points, and also it presents methods used for their proof. In particular, results about values of the Riemann zeta function are described. Criteria of irrationality and linear independence of numbers that are known in the theory of transcendental numbers are based on constructions of linear forms from examined numbers. Linear forms should have integer coefficients not so big in size and to be small enough. For values of polylogarithms such constructions may be realized as follows. For each rational function $R(s)$ having poles only at negative integers and zero at infinity, the following identity holds:

$$G(z) = \sum_{\nu=1}^{\infty} R(\nu) z^\nu = A_0(z^{-1}) + \sum_{k=1}^{q} A_k(z^{-1})\, \mathrm{Li}_k(z)$$

where $A_k(x)$ are polynomials. Then the value $G(a/b)$ is a linear form in $\mathrm{Li}_k(a/b)$ having rational coefficients. All applications of this construction use rational functions

$$R(s) = \gamma \sum_{j=1}^{n} \frac{\Gamma(s+a_j)}{\Gamma(s+b_j)}$$

where a_j, b_j are positive integers, and γ is a rational number, or derivatives of such functions. The correct choice of the paramaters a_j, b_j enables one to obtain arithmetic results. Some theorems derived in such a way by Nikishin in 1979, Gutnik in 1979 and 1982, Kessanii Pilerud in 1999, Rivoal in 2000, and Zudilin in 2001 are discussed.

Gelfond–Schnirelman Method in Prime Number Theory

Igor Pritsker, Oklahoma State University

The original Gelfond–Schnirelman method, proposed in 1936, uses polynomials with integer coefficients and small norms on $[0,1]$ to give a Chebyshev–type lower bound in the Prime Number Theorem. We study a generalization of this method for polynomials in many variables. Our main result is a lower bound for the integral of Chebyshev's ψ–function, expressed in terms of the weighted capacity. This extends previous work of Nair and Chudnovsky, and connects the subject to the potential theory with external fields generated by polynomial–type weights.

Approximation of Values of Zeta Functions at Integers

George Rhin, Université de Metz

In the first part I explain how Apéry's method of 'accelerating the convergence of series of rational numbers' had been used 20 years ago by H. Cohen and myself to construct a sequence of good rational approximations to $\zeta(4)$. This sequence does not give a new proof of the irrationality of $\zeta(4)$. But surprisingly it is the same sequence as the one given by W. Zudilin (with a different normalization) and also, independently, by V. N. Sorokin. The numerators and the denominators of these approximations are both proved to be rational numbers which satisfy a nice linear recurrence relation of order 2 with coefficients which are polynomials in n of degree 5. We explain why Apéry's method does not provide a proof that the denominators of the approximations are integers, as conjectured by W. Zudilin after numerical computations. In the second part we give (for the moment at least experimentally by use of Mathematica) for all $k \geq 2$ a sequence of rational approximations to $\zeta(k)$ by means of integrals of dimension 1, involving classical Padé approximations of $\log(1-z)$. Here the denominators are clearly integers and generalize the classical Apéry sequence for $\zeta(3)$:

$$q_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}^2.$$

Padé Approximants and Catalan's Constant

Tanguy Rivoal, Université de Caen

In this lecture we discuss some recent (unsuccessful) attempts in order to prove the irrationality of Catalan's constant $G = \sum_{n\geq 0}(-1)^n/(2n+1)^2$.

One approach uses the hypergeometric method to produce linear forms $S_n = a_n G + b_n$ with $2^{4n} d_{2n} a_n \in \mathbb{Z}$, $2^{4n} d_{2n}^3 b_n \in \mathbb{Z}$, but unfortunately $2^{4n} d_{2n}^3 S_n$ does not tend to zero.

The second approach uses a method first used by Prévost in 1994 to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. It is based on a Padé approximate to the remainder term ψ of the expansion

$$G = \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^2} + (-1)^n \psi\big(\tfrac{1}{2n}\big).$$

This produces another linear form $\widetilde{S}_n = \widetilde{a}_n G + \widetilde{b}_n$ with $2^{4n}\widetilde{a}_n \in \mathbb{Z}$, $2^{4n} d_{2n}^2 \widetilde{b}_n \in \mathbb{Z}$, but again $2^{4n} d_{2n}^2 \widetilde{S}_n$ does not tend to zero. As a by-product we see that $a_n = \widetilde{a}_n$, and $b_n = \widetilde{b}_n$, which improves the denominator bounds for a_n and b_n above.

Central Limit Theorems for Lattice Point Counts and for Eigenvalues of Hyperbolic Laplacians

Zeev Rudnick, Tel Aviv University

We consider the number $N(t,\varrho)$ of eigenvalues of $\sqrt{\Delta}$ which lie in an interval $[t, t+\varrho]$, $\varrho = \varrho(t)$, and Δ is the Laplacian on either the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ or the modular surface $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$. If $\varrho(t)$ tends to 0 sufficiently slowly, we show that, suitably normalized, $N(t,\varrho)$ has a Gaussian distribution (in the modular case, one takes a 'smooth' counting function).

A Three Primes Theorem with Congruence Conditions

Jan-Christoph Schlage–Puchta, Universität Freiburg

Let n be sufficiently large, $Q < n^{1/2}\log^{-A} n$. We give a simple proof for the fact that for almost all $q \leq Q$ and all a with $(a,q) = 1$ the number $r_q(n)$ of representations of n as a sum of three primes $n = p_1 + p_2 + p_3$ with $p_1 \equiv a \pmod{q}$ satisfies

$$r_q(n) = \mathfrak{S}(n,q)\frac{n^2}{\phi(q)\log^3 n} + O\left(\frac{n^2}{\phi(q)\log^B n}\right),$$

proving a conjecture of Tolev. Moreover, we show that congruential restrictions may be imposed on p_2 and p_3 as well.

A Turán–Kubilius Inequality for Friable Integers, with Applications

Gerald Tenenbaum, Université de Nancy
(joint work with Régis de la Bretèche)

This talk was devoted to provide an account on two recent joint papers, in which we investigate the local behaviour of the counting function of friable (i. e. without large prime factors) integers that are coprime to a given integer and applied the resulting estimates to a general form of the Turán–Kubilius inequality over friable integers that is valid without any restriction on the friability parameter.

Several consequences of the above described estimates have been presented, including: friable extensions of the Erdős–Wintner theorem and of Daboussi's theorem and an effective theorem for describing the structure of the set of prime factors of a stochastic friable integer.

Mean Value Theorems for Primes in Arithmetic Progression

Robert C. Vaughan, Pennsylvania State University

Let

$$\psi(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a (\mathrm{mod}\ q)}} \Lambda(n)$$

where Λ is von Mangoldt's function. Let

$$F_R(n) = \sum_{r\le R} \frac{\mu(r)}{\phi(r)} \sum_{\substack{b=1\\ (b,r)=1}}^{r} e(bn/r)$$

and

$$\varrho_1(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a (\mathrm{mod}\ q)}} F_R(n)$$

and let

$$M_k(x,Q,\omega,\varrho_1) = \sum_{q\le Q} \omega(q) \sum_{a=1}^{q} \big(\psi(x;q,a) - \varrho_1(x;q,a)\big)^k .$$

Then the following theorems are established.

Theorem 1. *Suppose that* $\omega_0(q) = 1$, $Q \le x$, *and* $R \le (\log x)^A$ *where* A *is a fixed positive real number. Then*

$$M_2(x,Q,\omega_0,\varrho_1) = Qx\log(x/R) - cQx + O\big(QxR^{-1/2} + x^2(\log x)^2R^{-1}\big)$$

where $c = 1 + \gamma + \sum_p \frac{\log p}{p(p-1)}$.

Theorem 2. *Suppose that* $\omega_1(q) = q$, $Q \le x$, *and* $R \le (\log x)^A$ *where* A *is a fixed positive real number. Then*

$$\begin{aligned} M_3(x,Q,\omega_1,\varrho_1) =& \tfrac{1}{2}Q^2x(\log x)^2 - \tfrac{3}{2}Q^2x\log x(\log R + c)\\ &+ O\big(x^3(\log x)^5R^{-1} + Q^2x(\log R)^3 + Q^2x(\log x)R^{-1/2}\big)\end{aligned}$$

where $c = \gamma + \frac{2}{3} + \sum_p \frac{\log p}{p(p-1)}$.

The Group Method for the Dilogarithm

Carlo Viola, Universita di Pisa
(joint work with G. Rhin)

Let

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} n^{-2} z^n = \int_0^z \frac{-\log(1-t)}{t} dt$$

denote the dilogarithm. M. Hata [*Trans. Amer. Math. Soc.* **336** (1993), 363–387] proved that $\mathrm{Li}_2(1/q)$ is irrational for all integers $q \leq -5$ or $q \geq 7$ and gave irrational measures of these numbers. For example, $\mu(L_2(1/7)) < 95.0605\ldots$, $\mu(L_2(-1/5)) < 228.612\ldots$. Hata's method is a technically difficult modification of his own method for the irrationality of $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$. We extend our algebraic method for $\zeta(2)$ and $\zeta(3)$ [*Acta Arith.* **77 (1996)**, 23–56; *Acta Arith.* **97 (2001)**, 269–293], based on the study of the actions of suitable permutation groups on double or triple integrals of Euler's type. The corresponding group action on the double integrals related to $\mathrm{Li}_2(1/q)$ is generated by a unidimensional birational transformation, as well as by suitable hypergeometric integral transformations obtained through Euler's integral representation of Gauß's hypergeometric function. In this way we extend Hata's results. We prove that $\mathrm{Li}_2(1/6)$ is irrational, and that $\mu(L_2(1/6)) < 997.882\ldots$. We also improve upon Hata's irrationality measures for $\mathrm{Li}_2(1/q)$ for any $q \leq -5$ or $q \geq 7$. For instance, we get $\mu(L_2(1/7)) < 69.688\ldots$, and $\mu(L_2(-1/5)) < 158.5$.

Twisted Hoffman Algebras

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Let $H = K\langle x_0, x_1\rangle$ be the free algebra on the alphabet $\{x_0, x_1\}$. For $\lambda \in K$, denote by ψ_λ the endomorphism of H which fixes x_0 and maps x_1 onto $x_1 + \lambda x_0$. We study the endomorphism φ_λ on $H^1 = Ke + Hx_1$ such that $\varphi_\lambda(wx_1) = \psi_\lambda(w)x_1$ and $\varphi_\lambda(e) = e$.

We extend φ_λ to an automorphism of H by $\varphi_\lambda(wx_0) = \varphi_\lambda(w)x_0$ for $w \in H$. Next we transport the shuffle and stuffle structures from H by means of this automorphism. For $\lambda = -1$ this yields the shuffle and stuffle products related to functions

$$\sum_{n_1 \geq \ldots n_k \geq 1} z^{n_1} n_1^{-s_1} \cdots n_k^{-s_k}$$

and their values at $z = 1$.

Mean Value Estimates for Thin Sets

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(joint work with Jörg Brüdern)

We investigate mean values of the shape $\sum_{n\in\mathcal{B}} c(n)^2$ where $c(n)$ is an arithmetically interesting Fourier coefficient, and $\mathcal{B}$ is a thin subset of $\mathbb{Z}$.

Examples: With

$$f(\alpha) = \sum_{1\leq x\leq P} e(\alpha x^3), \qquad h(\alpha) = \sum_{x\in\mathcal{A}(P,P^\eta)} e(\alpha x^3),$$

wherein $\mathcal{A}(X,Y)$ denotes the set of all integers up to X all of whose prime divisors are at most Y, consider

$$c_1(n) = \int_0^1 |f(\alpha)|^6 e(-n\alpha)\,d\alpha\,, \qquad c_2(n) = \int_0^1 |h(\alpha)|^5 e(-n\alpha)\,d\alpha\,,$$

and the mean values

$$\sum_{1\le x,y\le P} c_1(x^3-y^3)^2\,, \qquad \sum_{x,y\in\mathcal{A}(P,P^\eta)} c_2(x^3-y^3)^2\,.$$

We discuss three methods for estimating these mean values: (1) use of underlying arithmetic structure, (2) trivial use of Bessel's inequality, and (3) none of the above.

One can show using (1):

$$\sum_{1\le x,y\le P} c_1(x^3-y^3)^2 = \int_0^1\int_0^1 \left|f(\alpha)^6 f(\beta)^6 f(\alpha+\beta)^2\right| d\alpha\,d\beta\,,$$

$$\sum_{x,y\in\mathcal{A}(P,P^\eta)} c_2(x^3-y^3)^2 = \int_0^1\int_0^1 \left|h(\alpha)^5 h(\beta)^5 h(\alpha+\beta)^2\right| d\alpha\,d\beta\,.$$

Then applying the ideas of (2) one obtains

$$\sum_{1\le x,y\le P} c_1(x^3-y^3)^2 \ll P^8\,,$$

and a minor arc analogue with $P^8(\log P)^{\varepsilon-2}$ in place of P^8. This permits the proof of

THEOREM 1. *Let $s\ge 14$ and b_i, d_i $(1\le i\le s)$ be fixed integers. Then the number of solutions $N(P)$ of*

$$\sum_{i=1}^s b_i x_i^3 = \sum_{i=1}^s d_i x_i^3 = 0 \tag{1}$$

with $|x_i|\le P$ $(1\le i\le s)$ satisfies

$$N(P) \sim C\mathfrak{S}P^{s-6}$$

where C is the volume of the manifold (1) with $|x_i|\le 1$ $(1\le i\le s)$ and $\mathfrak{S}=\prod_p v_p$ with

$$v_p = \lim_{h\to\infty} p^{h(2-s)}\,\#\Big\{\underline{x}\in(\mathbb{Z}/p^h\mathbb{Z})^s:\ \sum_{i=1}^s b_i x_i^3 \equiv \sum_{i=1}^s d_i x_i^3 \equiv 0 \pmod{p^h}\Big\}\,.$$

Finally, using estimates for the frequency of large values (method (3)), one obtains the estimate

$$\sum_{x,y\in\mathcal{A}(P,P^\eta)} c_2(x^3-y^3)^2 \ll P^{6+\xi+\varepsilon}$$

where $\xi = (\sqrt{2833}-43)/41 = 0.24941\ldots$. This permits the proof of

THEOREM 2. *Let $s\ge 13$. Then the Hasse principle holds for (1), and in particular, whenever (1) possesses a nontrivial 7-adic solution, then (1) possesses a nontrivial integral solution.*

Euler–Type Multiple Integrals as Linear Forms in Zeta Values

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In 1978 Apéry showed that $\zeta(3)$ is irrational. A few months later, Beukers gave another proof by means of multiple integrals

$$\iiint\limits_{[0,1]^3} \frac{\prod_{j=1}^{3} x_j^n(1-x_j)^n}{\big(1-x_1(1-x_2x_3)\big)^{n+1}}\,dx_1dx_2dx_3 \in \mathbb{Q}\,\zeta(3)+\mathbb{Q},$$

and independently another hypergeometric series approach was put forward for showing $\zeta(3) \notin \mathbb{Q}$ in the works of Gutnik and Nesterenko. Recently, a very simple observation of Ball together with the generalization of the Gutnik–Nesterenko construction allowed Rivoal to prove that infinitely many numbers in the list $\zeta(3), \zeta(5), \zeta(7), \dots$ are irrational. Rivoal's construction is based on the well-poised property of hypergeometric series. This property originated by classical works of Barnes, Whipple, and many others. We prove the general theorem that states a relationship between the Rivoal-type well-poised hypergeometric series and the Euler-type multiple integrals that generalize those of Beukers. By means of this theorem we give an answer to Vasilyev's question:

$$\int\!\cdots\!\int\limits_{[0,1]^k} \frac{\prod_{j=1}^{k} x_j^n(1-x_j)^n}{\big(1-x_1(1-x_2(1-\dots(1-x_{k-1}(1-x_k))\dots))\big)^{n+1}}\,dx_1\dots dx_k$$
$$\in \mathbb{Q}\,\zeta(k)+\mathbb{Q}\,\zeta(k-2)+\cdots+\mathbb{Q}\,\zeta(3)+\mathbb{Q}$$

for odd integers k, and similarly for even k. Another consequence of the theorem is the connection with the new arithmetic group-structure approach introduced by Rhin and Viola. And the theorem gives a link between classical analysis of hypergeometric series and diophantine problems of $\zeta(s)$ for integers $s > 1$.

Edited by Ulrike M. A. Vorhauer

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