Mathematisches Forschungsinstitut Oberwolfach

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Mini-Workshop: Classification of Surfaces of General Type with Small Invariants

Organised by
Fabrizio Catanese (Bayreuth)
Ciro Ciliberto (Roma)

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Introduction by the Organisers

This mini-workshop has been organized by Fabrizio Catanese and Ciro Ciliberto. Unfortunately Catanese was unable to participate.

The classification of algebraic surfaces is a long-standing research subject in algebraic geometry, started by Castelnuovo and Enriques more than one hundred years ago, and continued by the Italian school (Severi, de Franchis, etc.) until about 1950.

In more recent times, fundamental contributions have been given by Kodaira in the 1950's and later in the 1970's by Bombieri, whose works on pluricanonical maps gave a strong impulse in studying surfaces of general type, and Mumford.

Adding important information to classical results by Noether and Castelnuovo, sharp bounds on the invariants have been given by Miyaoka and Bogomolov-Yau, allowing many authors to develop a systematic study of the "geography" of surfaces of general type.

Interesting investigations about the moduli space of surfaces of general type have been worked out in the last twenty years by Catanese, Manetti, and others.

Despite the intensive effort made in the last decades in order to make more precise our knowledge about surfaces of general type, their fine classification is still an open problem, even for small invariants. It is actually rather embarassing that, after more than one century of research on the subject, a complete classification of surfaces with geometric genus zero or one is still lacking.

This mini-workshop carried together 14 mathematicians actively working on this subject, and related arguments, with the idea of updating the state-of-the-art, exchanging information, discussing interesting open problems and stimulating collaborations. In this respect, the workshop has been very successful.

The atmosphere has been lively and very collaborative. During every talk, several questions have been posed and interesting problems pointed out. It has been especially remarkable the active presence of young participants.

During the week, 16 formal lectures have been given by the participants. This report contains extended abstracts of all the talks and also a contribution by Catanese, in collaboration with Pignatelli, about the lecture he was supposed to give.

The topics include: pluri-canonical maps for surfaces of general type (M. Mendes Lopes), canonical rings, projective embeddings and birational techniques (C. Böhning, F. Catanese, S. Papadakis, U. Persson, R. Pignatelli), irregular surfaces with low invariants (F. Polizzi, F. Zucconi), surfaces with $p_g = 0$ (A. Calabri, C. Ciliberto, K. Keum, M. Mendes Lopes, C. Werner), general techniques (V. Brînzănescu, K. Konno). Ulf Persson chaired an "open problem and discussions" session, which especially concerned surfaces with $p_g = 0$.

The organizers thank the Institute staff for providing a comfortable environment to the participants.

Mini–Workshop on Classification of Surfaces of General Type with Small Invariants

Table of Contents

Abstracts

Canonical surfaces in $\mathbb{P}^4$ and Gorenstein algebras in codimension 2

Christian Böhning

Consider minimal surfaces of general type S with $p_g = 5$, $q = 0$ such that the 1-canonical map π is a birational morphism onto a surface $Y \subset \mathbb{P}^4$, the latter being referred to as a canonical surface in $\mathbb{P}^4$. The canonical ring $\mathcal{R} := \bigoplus_{n\geq 0} H^0(S, \mathcal{O}_S(nK))$ is then a Gorenstein algebra of codimension 2 with twist -6 over $\mathcal{A} := \mathbb{C}[x_0, \dots, x_4]$, the homogeneous coordinate ring of $\mathbb{P}^4$. In general I make the

Definition. Let $S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$ be a positively graded ring with S_0 a field, S finitely generated over S_0 as an algebra; a finite graded perfect S-algebra B is called a *Gorenstein S-algebra of codimension c* (and *with twist $t \in \mathbb{Z}$*) if $B \cong \operatorname{Ext}^c_S(B, S(t))$ as B-modules where $c = \dim S - \dim_S B$.

By Castelnuovo's second inequality and Bogomolov-Miyaoka-Yau $8 \leq K^2 \leq 54$ for the above surfaces, the complete intersections of type $(2,4)$ resp. $(3,3)$ being the only solutions for $K^2 = 8$ resp. $= 9$. Moreover (cf. [Cil], [Cat4], [Böh1])

Theorem 1. *For a canonical surface in $\mathbb{P}^4$ with $q = 0$, $p_g = 5$, $K^2 \geq 10$ one has a resolution of the canonical ring $\mathcal{R}$*

$$(1)\quad \mathbf{R}_\bullet :\ 0 \to \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n \xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^{2n+2} \xrightarrow{(\alpha\ \beta)} \mathcal{A} \oplus \mathcal{A}(-2)^n \to \mathcal{R} \to 0,$$

where $n := K^2 - 9$.

Resolution (1) displays the symmetry of a "generalized" Koszul complex (cf. [Gra]). The important point, however, is that knowledge of the resolution (1) easily allows us to reconstruct our entire geometric set-up; more precisely (cf. [Böh1], [Böh2])

Theorem 2. *Let $\mathcal{R}$ be some finite $\mathcal{A}$-module with minimal graded free resolution as in (1). Write $A := (\alpha\ \beta)$, $A' := A$ with first row erased, $I_n(A') =$ Fitting ideal of $n \times n$ minors of A', and assume* $\operatorname{depth} I_n(A') \geq 4$.

Then $\mathcal{R}$ is a Gorenstein algebra, and if one assumes that $\operatorname{Ann}_{\mathcal{A}}(\mathcal{R})$ is a prime ideal, then $Y := \operatorname{Supp}(\mathcal{R}) \subseteq \mathbb{P}^4$ with its reduced induced subscheme structure (thus the ideal of polynomials vanishing on Y is $\mathcal{I}_Y = \operatorname{Ann}_{\mathcal{A}} \mathcal{R}$) is an irreducible surface, and if furthermore one assumes $X := \operatorname{Proj}(\mathcal{R})$ has only rational double points as singularities, then X is the canonical model of a surface S of general type with $q = 0$, $p_g = 5$, $K^2 = n + 9$. More precisely, writing $\mathcal{A}_Y$ for the homogeneous coordinate ring of Y, one has that the morphism $\psi : X \to Y \subset \mathbb{P}^4$ induced by the *inclusion* $\mathcal{A}_Y \subset \mathcal{R}$ *is a finite birational morphism, and is part of a diagram*

$$\begin{array}{ccc} S & \xrightarrow{\ \pi\ } & Y \subset \mathbb{P}^4 \\ & {\scriptstyle\kappa}\searrow \quad \nearrow{\scriptstyle\psi} & \\ & X & \end{array}$$

where S *is the minimal desingularization of* X*,* κ *is the contraction morphism contracting exactly the (-2)-curves of* S *to rational double points on* X*, and the composite* $\pi := \psi \circ \kappa$ *is a birational morphism with* $\pi^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_S(K_S)$ *(i.e. is 1-canonical for* S*).*

In some sense the most delicate part of the above theorem consists in recovering the ring structure of $\mathcal{R}$ from the resolution (1), cf. [Böh2], thm. 1.3 and 2.5.
To see how the above theorem may be applied, take $K^2 = 11$ as sample case: here the symmetry condition $\alpha\beta^t = \beta\alpha^t$ can be explicitly solved (cf. [Böh1], section 2) in order to re-prove by this method a result previously obtained by D. Roßberg (cf. [Roß]) with different techniques:

Theorem 3. *There is a unique irreducible component of the moduli space of regular surfaces of general type with* $p_g = 5$*,* $K^2 = 11$ *containing points corresponding to surfaces with canonical map a birational morphism onto a surface* $Y \subset \mathbb{P}^4$ *with only isolated singularities, which is unirational and of dimension 38.*

It may be hoped that this method will facilitate the study of canonical surfaces with higher K^2, the first unsolved case being $K^2 = 13$.

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On the classification of surfaces of general type with non birational bicanonical map and Du Val double planes

Giuseppe Borrelli

Let S be a minimal surface of general type and consider the bicanonical map φ_{2K} associated to the linear system $|2K_S|$. If there exists a rational map $S \to B$ onto a curve B with the general fiber a smooth irreducible curve of genus 2 then φ_{2K} is not birational, and in this situation one says that S presents the *standard case* (for the non birationality of φ_{2K}). By a theorem of I. Reider the standard case is the only possible exception to φ_{2K} being birational when $K_S^2 \geq 10$. In the 1950's P. Du Val [6] considered the problem for regular surfaces ($q = h^1(S, \mathcal{O}_S) = 0$), he obtained a list of possible surfaces with non birational bicanonical map and do not presenting the standard case. The examples of Du Val are as follows. Let X be a smooth surface and $G \subset X$ a reduced curve such that

$\mathcal{B}$) *either $X = \mathbb{F}_2$ and $G = C_0 + G'$, where $G' \in |7C_0 + 14\Gamma|$ and G' has at most non essential singularities;*

$\mathcal{D}$) *or $X = \mathbb{P}^2$ and G is a smooth curve of degree 8;*

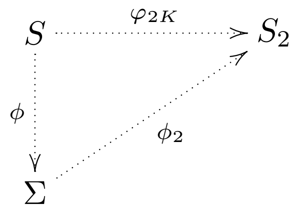
$\mathcal{D}_n$) *or $X = \mathbb{P}^2$ and $G = G' + L_1 + \cdots + L_n$, with $n \in \{0, 1, \ldots, 6\}$ ($G = G'$ if $n = 0$), where $L_1, \ldots, L_n$ are distinct lines meeting at a point γ and G' is a curve of degree $10 + n$. The singularities of G, besides the non essential ones, are a $(2n+2)$-tuple point at γ, a $[5,5]$-point lying on L_i, $i = 1, \ldots, n$, possibly some 4-tuple points or $[3,3]$-points;*

then S is the smooth minimal model of the double cover $X' \to X$ branched along G. Here $\mathbb{F}_2$ is the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ and Γ, C_0 its fibre and negative section with $C_0^2 = -2$. We will refer to X' as a *Du Val double plane* (of type $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{D}_n$).

The exceptions to the standard case have been classified for surfaces with $p_g \geq 4$ by C. Ciliberto, P. Francia and M. Mendes Lopes [4]; F. Catanese, C. Ciliberto and M. Mendes Lopes classified those with $p_g = 3, q > 0$ [3], and C. Ciliberto and M. Mendes Lopes worked out the regular case with $p_g = 3$ [5]. Finally, I classified the regular case with $p_g = 2$ under the assumption that the canonical system has no fixed part [1]. It follows from [5, 4], that if $q = 0, p_g \geq 3$ and φ_{2K}

is non birational then S either presents the standard case or is one of the Du Val examples.

It is easy to see that if φ_{2K} is non birational, the conditions $p_g \geq 2, q = 0$ force φ_{2K} to be a map of degree 2 (generically) onto a rational ruled surface. Hence, it is natural to consider more in general a surface whose bicanonical map factors through a rational map of degree 2 onto a rational or ruled surface, that is if there exists the following commutative diagram



where ϕ is a (generically finite) rational map of degree two and Σ is a rational or ruled surface. The result is the following,

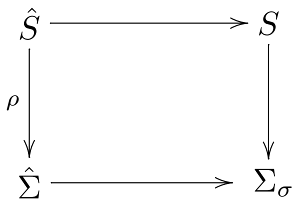
Theorem 1. *Let S be a smooth minimal surface of general type which does not present the standard case. Then the following three conditions are equivalent:*

a) the bicanonical map of S factors through a rational map of degree 2 onto a rational or ruled surface
b) the bicanonical map of S factors through a rational map of degree 2 onto a rational surface
c) S is the smooth minimal model of a Du Val double plane.

Moreover, let S be as in (c) (resp. (a) or (b)) then:

d) $q(S) = 0$ unless $p_g(S) = q(S) = 1$;
e) unless K_S is ample and $p_g(S) = 6, K_S^2 = 8$ or $p_g(S) = 3, K_S^2 = 2$, there is a rational pencil whose general member is a smooth hyperelliptic curve of genus 3 such that the bicanonical map of S induces the hyperelliptic involution on it.

Sketch of the proof of Theorem 1, (a) $\Rightarrow$ (b), (c). (See [2] for the complete proof.) Consider the quotient Σ_σ of S by the involution σ induced by ϕ. Then Σ_σ is a rational or ruled surface birational equivalent to Σ whose only singularities are the k nodes, which corresponds to the isolated fixed points of σ. Let $\hat{\Sigma} \to \Sigma_\sigma$ be the minimal resolution, then we have the commutative diagram



where $\hat{S}$ is the blow up of S at the isolated fixed points of σ and ρ is a finite double cover branched along a smooth curve B. Since $\hat{\Sigma}$ is smooth it is either $\mathbb{P}^2$ or ruled. When $\hat{\Sigma} \cong \mathbb{P}^2$ one has that $\hat{S} = S$ and B has degree 8 or 10. Otherwise we have that

i) $\hat{\Sigma}$ is rational,
ii) there exists a suitable birational morphism $\psi : \hat{\Sigma} \to X$ such that $G := \psi_*(B)$ and X are as in $\mathcal{B}$ or $\mathcal{D}_n$,
iii) $\hat{S}$ is the *canonical resolution* of the double cover $X' \to X$ branched along G.

For the proof of $i), ii)$ one uses a result of Xiao [9] who studied the possible images of the bicanonical map. □

As we remarked, the result for regular surfaces with $p_g \geq 3$ was already known and Theorem 1 extends the classification to regular surfaces with $p_g = 2$,

Theorem 2. *Let S be a regular surface of general type with $p_g \geq 2$ and non birational bicanonical map. Then either S presents the standard case or it is the smooth minimal model of a Du Val double plane.*

For $p_g = 0, 1$ we get some corollaries of Theorem 1.

Theorem 3 ([2, 9])**.** *Let S be a regular surface of general type with $p_g = 1$ and bicanonical map of degree 2. Then,*
i) *either S presents the standard case*
ii) *or S is the smooth minimal model of a Du Val double plane of type $\mathcal{D}_n$,*
iii) *or S_2 is a $K3$ surface.*

Theorem 4 ([2, 7, 9])**.** *Let S be a minimal surface of general type with $p_g = 0, K_S^2 \geq 2$ and bicanonical map of degree 2. Then,*
i) *either S presents the standard case*
ii) *or $K_S^2 = 3$ and $\varphi_{2K}(S)$ is an Enriques surface,*
iii) *or S is the smooth minimal model of a Du Val double plane of type $\mathcal{D}_n$ with K_S^2 and n as in the following table*

K_S^2	2	3	4	5	6	7	8
n	$0,1,2,3$	$1,2,3$	$2,3,4$	$3,4$	$4,5$	5	6

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Twisted Fourier-Mukai transforms on some elliptic surfaces

Vasile Brînzănescu
(joint work with Ruxandra Moraru)

Let X be a non-singular projective variety. The derived category $D(X)$ of X is a triangulated category whose objects are complexes of sheaves on X with bounded and coherent cohomology sheaves. In general, there exist pairs of non-singular projective varieties (X, Y) for which there are triangle-preserving equivalences $\Phi : D(Y) \to D(X)$. Such equivalences are called Fourier-Mukai transforms. In some cases, Φ takes sheaves to sheaves (not complexes) and this fact is used to study moduli spaces of some sheaves (for example, vector bundles). Sometimes, Fourier-Mukai transforms can be constructed on non-projective complex varieties.

Let $\pi : X \to B$ be a minimal non-Kähler elliptic surface (B a smooth compact connected curve). It is well-known that $X \to B$ is a quasi-bundle over B, i.e. all the smooth fibres are pairwise isomorphic and the singular fibres are multiples of elliptic curves (see [24], [8]). Let T denote the general fibre of π, which is an elliptic curve and let T^* denote the dual of T (i.e. $T^* := \mathrm{Pic}^0(T) \cong T$ non-canonically). It is known that the Jacobian surface associated to $\pi : X \to B$, in this case, is simply $J(X) = B \times T^* \to B$ and the surface $X \to B$ is obtained from its Jacobian surface $B \times T^*$ by a finite number of logarithmic transformations.

Now, we shall define a twisted Fourier-Mukai transform on non-Kähler elliptic surfaces. For simplicity, we shall consider that $\pi : X \to B$ has no multiple fibres, i.e. X is a principal elliptic bundle over B. Then, $X = \Theta^*/ < \tau >$, where Θ is a line bundle over B with positive Chern class l, Θ^* is the complement of the zero section in the total space of Θ, and $< \tau >$ is the multiplicative cyclic group generated by a fixed complex number τ with $| \tau |> 1$. The standard fibre of this bundle is $T \cong \mathbb{C}^*/ < \tau >$. Multiplication by τ defines a natural $\mathbb{Z}$-action on $X \times \mathbb{C}^*$ that is trivial on X, inducing the quotient $(X \times \mathbb{C}^*)/\mathbb{Z} = X \times T^* = X \times_B J(X)$.

Since X does not have multiple fibres, then the set of all holomorphic line bundles on X with trivial Chern class is given by the zero component of the Picard group $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(B) \times \mathbb{C}^*$. In this case, any line bundle in $\mathrm{Pic}^0(X)$ is therefore of the form $H \otimes L_\alpha$, where H is the pullback to X of an element of $\mathrm{Pic}^0(B)$ and L_α is the line bundle corresponding to the constant automorphy factor $\alpha \in \mathbb{C}^*$; in particular, there exists a universal (Poincaré) line bundle $\mathcal{U}$ on $X \times \mathrm{Pic}^0(X)$ whose restriction to $X \times \mathbb{C}^* := X \times \{0\} \times \mathbb{C}^*$ is constructed in terms of constant automorphy factors (for details, see [10]).

Given a rank two vector bundle over X, its restriction to a generic fibre of π is semistable. More precisely, its restriction to a fibre $\pi^{-1}(b)$ is unstable on at most an isolated set of points $b \in B$; these isolated points are called the *jumps* of the bundle. Furthermore, there exists a divisor in the relative Jacobian $J(X) = B \times T^*$ of X, called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle over each fibre of π. The spectral curve can be constructed as follows: we associate to the rank-2 vector bundle E the sheaf on $B \times \mathbb{C}^*$ defined by

$$\widetilde{\mathcal{L}} := R^1\pi_*(s^*E \otimes \mathcal{U}),$$

where $s : X \times \mathbb{C}^* \to X$ is the projection onto the first factor, id is the identity map, and π also denotes the projection $\pi := \pi \times id : X \times \mathbb{C}^* \to B \times \mathbb{C}^*$. This sheaf is supported on a divisor $\widetilde{S_E}$, defined with multiplicity, that descends to a divisor S_E in $J(X)$ of the form

$$S_E := \left(\sum_{i=1}^{k} \{x_i\} \times T^* \right) + \overline{C},$$

where $\overline{C}$ is a bisection of $J(X)$ and $x_1, \cdots, x_k$ are points in B that correspond to the jumps of E. The spectral curve of E is defined to be the divisor S_E. Note that there is also a natural $\mathbb{Z}$-action on $B \times \mathbb{C}^*$ defined as multiplication by τ on the second factor and $(B \times \mathbb{C}^*)/\mathbb{Z} \cong J(X)$. Moreover, this action extends to the torsion sheaf $\widetilde{\mathcal{L}} := R^1\pi_*(s^*E \otimes \mathcal{U})$, taking the stalk $\widetilde{\mathcal{L}}_{(x,\alpha)}$ to $\widetilde{\mathcal{L}}_{(x,\tau\alpha)} \otimes L_{\tau^{-1},x}$. Therefore, $\widetilde{\mathcal{L}}$ cannot descend to $J(X)$ because it is not invariant with respect to this action. To fix this problem, we construct a sheaf $\mathcal{N}$ on $B \times \mathbb{C}^*$ and a $\mathbb{Z}$-action that leaves the tensor product $\widetilde{\mathcal{L}} \otimes \mathcal{N}$ invariant (see [10], [11]). We denote the quotient sheaf

$$\mathcal{L} := (\widetilde{\mathcal{L}} \otimes \mathcal{N})/\sim .$$

Note that the support of $\mathcal{L}$ is S_E; moreover, if we take the pull back of $\mathcal{L}$ to $B \times \mathbb{C}^*$ and tensor it by $\mathcal{N}^*$, then we recover $\widetilde{\mathcal{L}}$ (we also denote $\mathcal{N}^*$ the sheaf on $B \times \mathbb{C}^*$ obtained by extending the line bundle $\mathcal{N}^*$ on $\widetilde{S_E}$ by zero outside $\widetilde{S_E}$).

Given a locally free sheaf E on X, we define the twisted Fourier-Mukai transform to be the complex of sheaves $\Phi(E)$ on $J(X)$ given by

$$\Phi(E) := \left(R\pi_* \left(s^*E \otimes \mathcal{U}\right) \otimes \mathcal{N}\right) / \sim .$$

Conversely, if $\mathcal{L}$ is a sheaf on $J(X)$, we define the "inverse" twisted Fourier-Mukai transform as the complex of sheaves $\hat{\Phi}(\mathcal{L})$ on X given by

$$\hat{\Phi}(\mathcal{L}) := R\underline{s}_* \left(\left(\pi^* \left((\rho^*\mathcal{L}) \otimes \mathcal{N}^* \right) \otimes \mathcal{U}^* \right) / \sim \right),$$

where $\underline{s} : X \times_B J(X) \to X$ is projection onto the first factor, $q : X \times \mathbb{C}^* \to X \times T^* = X \times_B J(X)$ and $\rho : B \times \mathbb{C}^* \to B \times T^* = J(X)$ are the natural quotient maps induced by the $\mathbb{Z}$-actions and π and s are the projections defined above.

We state some of their properties in:

Theorem 1. *(i) Suppose that E is a rank-2 vector bundle on X without jumps. Then, $\Phi^0(E) = 0$ and $\hat{\Phi}^0(\Phi^1(E)) = E$.*

(ii) If $\mathcal{L}$ is a torsion sheaf on $J(X)$, supported on a bisection $C \subset J(X)$, that has rank 1 on the smooth points of C and rank at most 2 on the singular ones, then $\hat{\Phi}^1(\mathcal{L}) = 0$ and $\Phi^1(\hat{\Phi}^0(\mathcal{L})) = \mathcal{L}$.

For the proof, see [11].

We use this result in the classification of rank two vector bundles over non-Kähler elliptic surfaces, including the study of moduli spaces of stable vector bundles (see [11], [12]).

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On the classification of numerical Godeaux surfaces with an involution

Alberto Calabri and Ciro Ciliberto
(joint work with Margarida Mendes Lopes)

In the one-century-and-a-half history of algebraic geometry in dimension two, projective surfaces with geometric genus $p_g = 0$ and irregularity $q = 0$ have been studied from the very beginning. They were supposed to be rational by Max Noether, until Enriques suggested the existence of the surfaces with $p_g = q = 0$ and bi-genus $P_2 = 1$ which now bear his name, and Castelnuovo proved in 1896 his celebrated *rationality criterion*, which states that a surface X is rational if and only if $P_2(X) = q(X) = 0$.

In 1931–32, Godeaux and Campedelli gave the first two examples of minimal surfaces of general type with $p_g = 0$ and $K^2 = 1, 2$, respectively. Godeaux considered a quotient of a quintic surface in $\mathbb{P}^3$ by a $\mathbb{Z}/5\mathbb{Z}$-action, whereas Campedelli constructed a double plane, i.e. a double cover of $\mathbb{P}^2$, branched along a degree 10 curve with six points of type $[3, 3]$, that is a triple point with another infinitely near triple point, not lying on a conic.

Campedelli also suggested the construction of a minimal surface of general type with $p_g = 0$ and $K^2 = 1$ as the smooth minimal model of a double plane branched along a curve C of degree 10 with a 4-tuple point and five points of type $[3, 3]$, not lying on a conic. The existence of a curve like C was proved only 50 years later by Kulikov, Oort and Peters. We will say that a double plane is *of Campedelli type* if the branch curve is of this type.

Minimal surfaces of general type with $p_g = 0$ and $K^2 = 1$, nowadays called *numerical Godeaux surfaces*, have been studied by several authors in the last 30 years: Miyaoka (1976), Dolgachev (1977), Reid (1978, 1988), Barlow (1984–85), which gave the first example of a simply connected one, Werner, Craighero-Gattazzo, Naie (1994), Stagnaro (1997), Dolgachev-Werner (1999), Catanese-Pignatelli, Keum-Lee (2000), and others (cf. e.g. [CP]).

Miyaoka proved that the subgroup Tors(S) of torsion elements of the Picard group of a numerical Godeaux surface S is a cyclic group of order strictly less than 6. He classified those with Tors(S) $= \mathbb{Z}/5\mathbb{Z}$ by describing the canonical ring of the 5-tuple covering given by the torsion, and similarly Miles Reid classified those with Tors(S) $= \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$.

Some examples of those with Tors(S) $= \mathbb{Z}/2\mathbb{Z}$ or Tors(S) $= 0$ have been found by Barlow, Werner and Craighero-Gattazzo (as shown by Dolgachev and Werner); nonetheless the classification problem is still open.

Note that these surfaces are interesting also because of Bloch's conjecture, which states that the Chow group of degree zero 0-cycles on a surface with $p_g = q = 0$ is trivial.

Here we report on a work in progress about the classification of numerical Godeaux surfaces S with an *involution*, i.e. with an automorphism $\sigma : S \to S$ of order 2. A first investigation of this subject has been done by J. Keum and Y. Lee in [KL]: under the assumption that the bicanonical system has no fixed components, they described all the possibilities for the fixed locus of the involution.

We make no assumption on fixed components of the bicanonical system $|2K_S|$ and we follow the ideas contained in joint works of the third author and Rita Pardini, namely we combine the topological and the holomorphic fixed point formulas for involutions on surfaces and the Kawamata-Viehweg vanishing theorem, in order to prove the following:

Theorem 1. *Let S be a minimal surface of general type with $p_g(S) = q(S) = 0$ and an involution $\sigma : S \to S$. The fixed locus of σ is composed of a smooth curve R and k isolated fixed points. Then:*

- $4 \leq k \leq K_S^2 + 4$;
- $k \equiv K_S^2 \pmod 2$;
- $K_S \cdot R \leq K_S^2$ *and equality holds if and only if* $k = K_S^2 + 4$;
- *if* $k = K_S^2 + 4$, *then the bicanonical map* $\phi : S \dashrightarrow \mathbb{P}^{K_S^2}$ *is composed with* σ;
- *if* $|2K_S|$ *has no fixed component, then* ϕ *is composed with* σ *if and only if* $k = K_S^2 + 4$.

In particular, if S is a numerical Godeaux surface, i.e. $K_S^2 = 1$, then $k = 5$, ϕ is composed with σ, $K_S \cdot R = 1$ and $R = \Gamma + Z$, where Z are disjoint (-2)-curves, and $0 \leq p_a(\Gamma) \leq 2$ (cf. also [KL]*).*

Then we study the quotient surface S/σ, and, by a fine use of adjunction on S/σ and a deep analysis of some Del Pezzo surfaces, we prove the following:

Theorem 2. *A numerical Godeaux surface S with an involution σ is birationally equivalent to one of the following:*

1. *a double plane of Campedelli type;*
2. *a double plane branched along the union of two distinct lines r_1, r_2 and a curve B of degree 12 with the following singularities:*
   - *the point $p_0 = r_1 \cap r_2$ of multiplicity 4;*

- *a point* $p_i \in r_i$, $i = 1, 2$, *of type* $[4, 4]$, *where the tangent line is* r_i;
- *further three points* p_3, p_4, p_5 *of multiplicity 4 and a point* p_6 *of type* $[3, 3]$, *such that there is no conic through* $p_1, \dots, p_6$;

(3) *a double cover of an Enriques surface.*

In case (3), $\mathrm{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$, *whilst in case* (2), $\mathrm{Tors}(S)$ *is either* $\mathbb{Z}/2\mathbb{Z}$ *or* $\mathbb{Z}/4\mathbb{Z}$.

Moreover if the fixed locus R *of* σ *has an irreducible component* Γ *of genus* 2, *then* S *belongs to case* (3).

All the previously known constructions of numerical Godeaux surfaces as double planes belong to case (1). Examples of case (3) have been given by Keum and Naie.

We show the existence of examples of case (2) by constructing degree 12 curves with the required singularities: we found out some examples with $\mathrm{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$ and some with $\mathrm{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$. Let us say that a double plane as in case (2) is *of Du Val type*, because it is the degeneration of a double plane, described by Du Val, whose smooth minimal model has $p_g = 4$ and $K^2 = 8$, with non-birational bicanonical map (see [Ci], [Bo]).

In both cases (1) and (2), it is possible to determine the possible configurations of components of the branch curve of the double planes.

In case (3) we prove that the double cover of the Enriques surface is branched along a curve which moves in a pencil whose general member is an irreducible curve of genus 2.

Theorem 1 suggests that it is possible to study in a similar way minimal surfaces of general type with an involution, $p_g = 0$ and $K^2 > 1$, in particular with $K^2 = 2$, i.e. *numerical Campedelli* surfaces.

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On pencils of small genus

Fabrizio Catanese and Roberto Pignatelli

1. The relative canonical algebra

Throughout this abstract X will be a projective surface, $f : X \to B$ a morphism onto a smooth curve of genus b. Without loss of generality, we may assume that f has connected fibres F of genus g. These maps are studied (see, e.g., [Fuj1], [Fuj2], [Xia]) analyzing their relative canonical algebra.

Definition 1. Consider the relative dualizing sheaf

$$\omega_{X|B} := \omega_X(-f^*K_B).$$

Then the relative canonical algebra $\mathcal{R}(f)$ is the commutative graded algebra $\oplus_0^\infty V_n$, where V_n is the vector bundle on B given as the direct image sheaf $f_*(\omega^n_{X|B})$

Definition 2. The multiplication maps $\mu_{n,m} : V_n \otimes V_m \to V_{n+m}$ yield natural sheaf homomorphisms

$$S^n(V_1) = S^n(f_*(\omega_{X|B})) \xrightarrow{\sigma_n} V_n = f_*(\omega^n_{X|B}),$$

and we define $\mathcal{T}_n =$ coker σ_n.

Remark 1. By Noether's theorem on canonical curves, $\mathcal{T}_n$ is a torsion sheaf if the general fibre of f is non-hyperelliptic.

Remark 1 shows that the hyperelliptic and the non hyperelliptic case should be treated separately; assume in fact for the time being that a general fibre is hyperelliptic. Then there is a birational involution σ on X, and σ acts linearly on the space of sections $\mathcal{O}_X(U, \omega^n_{X/B})$, which splits as the direct sum of the $(+1)$-eigenspace and the (-1)-eigenspace. Accordingly, we get direct sums $V_n = V_n^+ \oplus V_n^-$: therefore, in the hyperelliptic case, where obviously $V_1 = V_1^-$, the cokernels $\mathcal{T}_n$ will be bigger than in the non hyperelliptic case.

2. The structure theorems

Let $f : X \to B$ be a genus 2 fibration. The rank 2 vector bundle $V_1 := f_*\omega_{X|B}$ induces a natural factorization of f as $\pi \circ \varphi$, where $\varphi : X \dashrightarrow \mathbb{P}(V_1)$ is a rational map of degree 2, and $\pi : \mathbb{P}(V_1) \to B$ is the natural projection.

The indeterminacy locus of φ is contained in the fibres of f which are not 2-connected, i.e., which split as $\mathcal{E}_1 + \mathcal{E}_2$ with $\mathcal{E}_1\mathcal{E}_2 = 1$. Then $\mathcal{E}_i^2 = -1$, $\mathcal{E}_i$ has arithmetic genus 1 and is called an elliptic cycle. These fibres are recognizable through $\mathcal{T}_2$ as follows.

Lemma 1. *Let $f : X \to B$ be a genus 2 fibration. Then $\mathcal{T}_2$ is the structure sheaf of an effective divisor $\tau \in Div_{\geq 0}(B)$, whose support is given by the points whose corresponding fibres of f are not 2-connected.*

The typical example is given by a fibre consisting of two smooth elliptic curves $\mathcal{E}_1, \mathcal{E}_2$ meeting transversally in a point P'. The blow-up of the point P' maps isomorphically to the fibre F'' of $\mathbb{P}$ over the point $P \in B$, while the elliptic curves $\mathcal{E}_1, \mathcal{E}_2$ are contracted to two distinct points of the fibre F''.

The resolution $\tilde{\varphi}$ of φ is the composition of the contraction of $\mathcal{E}_1, \mathcal{E}_2$ to two simple -2-elliptic singularities, with a finite double cover where the branch curve Δ in $\mathbb{P}$ contains the fibre and has two distinct 4-tuple points on it. More complicated fibres containing elliptic tails can produce different configurations of singularities of the branching divisor of φ: a complete list is the one given by Ogg and by Horikawa in [Ogg],[Hor]. This approach is widely used to construct genus 2 fibrations; the main difficulty is in the construction of Δ, often very singular.

Definition 3. We denote by $\mathcal{A}$ the graded subalgebra of $\mathcal{R}$ generated by V_1 and V_2; let $\mathcal{A}_n$ be its graded part of degree n, $\mathcal{A}_{even} = \oplus_k \mathcal{A}_{2k}$.

It is easy to see that the natural map $Sym(V_2) \to \mathcal{A}_{even}$ is surjective with kernel generated by the image of the map $i_2 : \det V_1^2 \hookrightarrow S^2(V_2)$ defined locally by $i_2(x_0 \wedge x_1)^2 = \sigma_2(x_0)^2\sigma_2(x_1)^2 - \sigma_2(x_0x_1)^2$.

Concretely, this gives explicit equations for $\mathbf{Proj}(\mathcal{A})$ as conic subbundle of the $\mathbb{P}^2$-bundle $\mathbb{P}(V_2)$. $\mathbf{Proj}(\mathcal{A})$ and $\mathbb{P}(V_1)$ are clearly birationally equivalent and biregularly equivalent outside the fibers over $\mathrm{supp}(\mathcal{T}_2)$. One can check that the fibres of $\mathrm{supp}(\mathcal{T}_2)$ are in fact the reducible fibres of the conic bundle.

If we consider the natural morphism $\varphi_{\mathcal{A}} : X \to \mathbf{Proj}(\mathcal{A})$ induced by the inclusion $\mathcal{A} \subset \mathcal{R}$ and the natural projection morphism $\pi_{\mathcal{A}} : \mathbf{Proj}(\mathcal{A}) \to B$ we get a new factorization of the fibration ('birational' to the previous one): $f = \pi_{\mathcal{A}} \circ \varphi_{\mathcal{A}}$. The advantage in considering $\varphi_{\mathcal{A}}$ instead of φ is that the branch curve $\Delta_{\mathcal{A}}$ has only simple singularities. In the typical example above described, the elliptic curves $\mathcal{E}_i$ will not be contracted by $\varphi_{\mathcal{A}}$ but they will be double covers of the two lines of the corresponding fibre of the conic bundle.

Lemma 2. *$\mathcal{A}_6$ is the cokernel of the map $\det V_1^2 \otimes V_2 \to S^3(V_2)$ naturally induced by the map i_2 above; note that $\mathcal{A}_6$ depends only on B, V_1 and σ_2. The branch curve $\Delta_{\mathcal{A}}$ is induced by a map $(\det(V_1) \otimes \mathcal{O}_B(\tau))^{\otimes 2} \to \mathcal{A}_6$.*

We can now introduce the building package of a genus 2 fibration:

Definition 4. Define the **associated 5-tuple** (B, V_1, τ, ξ, w) of a genus 2 fibration $f : X \to B$ as follows:

- B is the base curve;
- $V_1 = f_*(\omega_{X|B})$;
- τ is the effective divisor of B with $\mathcal{O}_\tau \cong \mathcal{T}_2$;
- $\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ the class induced by σ_2;
- $w \in \mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det(V_1) \otimes \mathcal{O}_B(\tau))^{\otimes -2}))$ inducing $\Delta_{\mathcal{A}}$ on $\mathbf{Proj}(\mathcal{A})$.

Definition 5. We will say that a a 5-tuple (B, V_1, τ, ξ, w) is **admissible** if

- B is a smooth curve;
- V_1 is a vector bundle on B of rank 2;

- $\tau \in Div^+(B)$;
- $\xi \in Ext^1_{\mathcal{O}_B}(\mathcal{O}_\tau, S^2(V_1))/Aut_{\mathcal{O}_B}(\mathcal{O}_\tau)$ yields a vector bundle V_2;
- $w \in \mathbb{P}(H^0(B, \mathcal{A}_6 \otimes (\det(V_1) \otimes \mathcal{O}_B(\tau))^{\otimes -2}))$ inducing $\Delta_{\mathcal{A}}$ on $\mathbf{Proj}(\mathcal{A})$, where $\mathcal{A}_6$ is the vector bundle induced by ξ;

and if moreover they satisfy some open conditions ensuring that the associated double cover has Rational Double Points as singularities.

We do not specify here the open conditions in detail for lack of space. The vector bundle $\mathcal{A}_6$ is 'induced' taking the map σ_2 induced by ξ and defining $\mathcal{A}_6$ as the cokernel of the map in lemma 2.

Theorem 1. *Let f be a relatively minimal genus 2 fibration. Then its associated 5-tuple is admissible. Viceversa, every admissible 5-tuple is the associated 5-tuple of a genus 2 fibration $f : X \to B$, and the surface X has invariants $\chi(\mathcal{O}_X) = \deg(V_1) + (b-1), K^2 = 2\deg V_1 + \deg \tau + 8(b-1)$. Two relatively minimal genus 2 fibration having the same associated 5-tuple are isomorphic.*

We can prove a very similar statement for a genus 3 fibrations f with non hyperelliptic general fibre, under the assumption that every fibre of f is 2-connected.

3. Applications

The first application of theorem 1 is a short proof of the following theorem (already proved by Bombieri ([Bom]) using Ogg's list of genus 2 fibres (cf. [Ogg])).

Theorem 2. *Let S be a Godeaux surface, and let $f : S \to \mathbb{P}^1$ be the fibration induced by the bicanonical pencil of S. Then the genus of the fibre can only be 3 or 4.*

We have an interesting application of theorem 1 to minimal surfaces of general type with $p_g = q = 1$. In this case $2 \le K_S^2 \le 9$ and the Albanese map is a morphism $f : S \to B$ where B is a smooth elliptic curve.

The case $K_S^2 = 2$ is completely described in [Cat1] where it is proved (among other things) that the moduli space is generically smooth, unirational of dimension 7.

The class of surfaces of general type with $K^2 = 3$, $p_g = q = 1$ is studied in [CC1], [CC2]. In [CC1] it is proved that for this class of surfaces the genus of the Albanese fibre is 2 or 3. The second case is completely classified in [CC2], where it is shown that the corresponding moduli space is generically smooth, unirational of dimension 5.

In [CC1] all surfaces with $p_g = q = 1$, $K^2 = 3$ and genus 2 of the Albanese fibre are described as double covers of $B^{(2)}$. It was conjectured there (see problem 5.5) that this family of surfaces should form an irreducible family of the moduli space. We can disprove this conjecture. More precisely (considering also the family in [CC2])

Theorem 3. *The family, in the moduli space of the minimal surfaces of general type, corresponding to the surfaces S with $p_g(S) = q(S) = 1$, $K_S^2 = 3$ has at least 4 connected components and at most 5 irreducible components, all of dimension 5.*

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Numerical Godeaux surfaces with an involution

JongHae Keum

A minimal surface of general type with $p_g = 0$ and $K^2 = 1$ is called a numerical Godeaux surface, or simply a Godeaux surface. A joint work with Y. Lee [2] describes all possible fixed loci of an involution acting on a numerical Godeaux surface, under an assumption that the bicanonical system has no base components. Recently M. Mendes Lopes, R. Pardini [3] have proved the same result without the assumption.

Let X be a numerical Godeaux surface and σ be an involution acting on it. Its fixed locus consists of 5 isolated points, a curve l with $K_X l = 1$, and at most $g(l) + 2$ nodal curves. The genus $g(l)$ can take values 0, 1 and 2.

Let h denote the number of nodal curves. All known examples of Godeaux surfaces have an involution, and the corresponding $(g(l), h)$ is as follows:

a classical Godeaux surface from D_{10}-invariant quintic, Beauville's example, Barlow surface, and Craighero-Gattazzo-Dolgachev-Werner surface have $(0, 0)$*; Werner's example with Tors = $\mathbb{Z}/2$,* $(1, 1)$*; Stagnaro's example,* $(1, 2)$*; Oort-Peters' example,* $(1, 3)$.

In [2], two families of Godeaux surfaces with Tors = $\mathbb{Z}/4$ were constructed via canonical ring method due to M. Reid. These have involutions with $(g(l), h) = (1, 0)$, $(2, 0)$, respectively.

In this talk, I give an improvement as follows:

Theorem 1. *If $g(l) = 2$, then $h = 0$.*

Sketch of the proof of Theorem 1. If $g(l) = 2$, then the quotient surface X/σ is birational to an Enriques surface. This was one of the result presented by C. Ciliberto and A. Calabri [1] during this workshop. Let $W \to X/\sigma$ be a resolution of the five nodes. Then the branch $B \subset W$ is of the form $B = B_0 + N_1 + \cdots + N_5$, where N_i are nodal curves coming from the resolution. From the double covering formulas, we see that B_0 is a smooth curve of genus 2 with $B_0^2 = 2$. We also see that B_0 is disjoint from the exceptional curves on W which are to be blown down to an Enriques surface W'. On W', the branch consists of a genus 2 curve and 5 nodal curves. This means that no components other than l arise by the double covering process. □

I also suggest a way of constructing examples of Godeaux surfaces as double Enriques surfaces, whose covering involutions have $(g(l), h) = (0,1)$, $(0,2)$, the only missing cases.

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On fibred rational surfaces

Kazuhiro Konno

(joint work with Shinya Kitagawa)

Let X be a non-singular projective surface with $p_g = q = 0$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of curves of genus $g \geq 2$. We denote by F a general fibre of f. Then $K_X + F$ is nef and the restriction map $H^0(X, K_X + F) \to H^0(F, \omega_F)$ is an isomorphism, because $p_g = q = 0$. In particular, $h^0(X, K_X + F) = g$. If $(K_X + F)^2 < 2g - 2$, then X is automatically a rational surface. Assume that the rational map defined by $|K_X + F|$ is generically finite onto the image W. Then,

Theorem 1. *$|K_X + F|$ is free from base points if $(K_X + F)^2 \leq 2g - 4$. Furthermore, the ring $\oplus_{n \geq 0} H^0(X, n(K_X + F))$ is generated in degree 1 if $(K_X + F)^2 \leq 2g - 5$.*

Such an analysis is carried out by passing through the reduction (Y, G) obtained from (X, F) by blowing down all the (-1)-curves E satisfying $(K_X + F)E = 0$, where G is the image of F by the natural map $\mu : X \to Y$. The original fibration f is obtained from a pencil $\Lambda_f \subset |G|$ by blowing up the base points.

When X is a rational surface which is not $\mathbb{P}^2$, we can find a base point free pencil $|D|$ of rational curves on Y such that $c = (K_Y + G)D$ is minimal among such pencils. Then going down further to its #-minimal model $(Y^\#, G^\#)$, we get

$$(K_X + F)^2 = \frac{2c}{c+1}(g - c - 1) + \frac{1}{c+1}\sum_{i=1}^{N}(c + 1 - m_i)(m_i - 1),$$

where the m_i denotes the multiplicity of a singular point of $G^{\#}$, $m_i \leq c/2+1$. Furthermore, we can show the following by Serrano's theorem [5]:

Theorem 2. *Assume that* $c \geq 2$ *and* $G^2 > (c+2)^2$. *Then every morphism* $\phi : G \to \mathbb{P}^1$ *of degree at most* $c+2$ *can be extended to a morphism* $\tilde{\phi} : Y \to \mathbb{P}^1$. *Furthermore,*

(1) $\mathrm{gon}(F) = c+2$, *and*

(2) *the number of* g^1_{c+2}*'s on* G *is finite. In particular,* $\mathrm{Cliff}(F) = c$.

We use these results to study the Mordell-Weil lattice MWL(f) of f. Recall that the Mordell-Weil lattice is the group of sections of f endowed with a symmetric bilinear form coming from the intersection pairing on X. Put $r = \mathrm{rank}\ \mathrm{MWL}(f)$. Then Shioda [6] shows

$$r = \rho(X) - 2 - \sum_{P \in \mathbb{P}^1} (v_P - 1),$$

where $\rho(X)$ denotes the Picard number and v_P the number of irreducible components of the fibre $f^{-1}(P)$. In particular, we have $r = \rho(X) - 2$ if f has irreducible fibres only.

MWL(f) of maximal rank for fibred rational surfaces is determined so far by Saito-Sakakibara when f is hyperelliptic [3], by Saito-Nguyen Khac when f is of Clifford index one [4] and by Kitagawa when f is bi-elliptic [1]. As to the general fibrations of Clifford index two, we have the following:

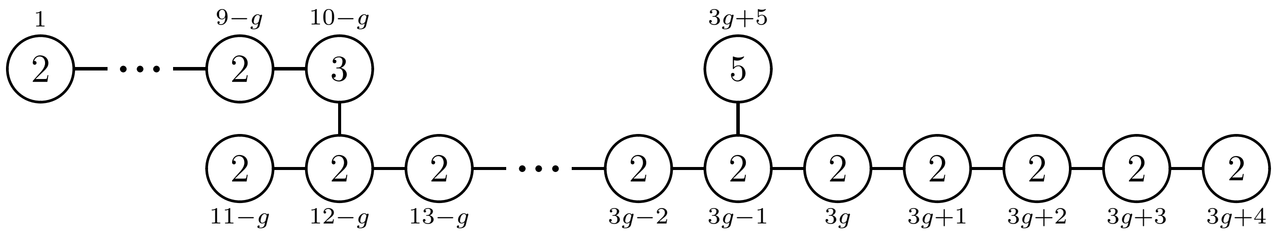
Theorem 3. *Let* X *be a non-singular rational surface,* $f : X \to \mathbb{P}^1$ *a relatively minimal fibration of genus* g *and of Clifford index* 2. *Let* r *be the Mordell-Weil rank of* f.

(1) *If* $5 \leq g \leq 10$, *then* $r \leq 3g+5$.

(2) *If* $g \geq 11$, *then* $r \leq 3g + 8 - (g+\epsilon)/3$, *where* ϵ *is the smallest non-negative integer with* $g + \epsilon \equiv 0$ *modulo* 3.

Assume that r *attains the maximum. Then all the fibres of* f *are irreducible and the reduction* Y *is obtained as the image of* $\Phi_{|K_X+F|}$. *Furthermore,* Y *is a del Pezzo surface and* $\Lambda_f \subset |-2K_Y|$ *when* $5 \leq g \leq 10$; *it is a Hirzebruch surface blown up* ϵ *points and* Λ_f *comes from a linear system of quadruple sections when* $g \geq 11$.

We can completely determine MWL(f) when the rank is maximum. For example, when $5 \leq g \leq 10$ and Y is obtained from $\mathbb{P}^2$ by blowing up $10-g$ points in general position, we get the following Dynkin diagram:



where the numbers in circles are self-pairing numbers of elements of a suitably fixed basis whose numbering is given near the circles. Therefore, it is an odd unimodular lattice of rank $3g+5$.

For $g \geq 11$, the maximal MWL(f) depends not only on g but also on ϵ and is much more complicated. We have four different types when $\epsilon = 0$, two types for each when $\epsilon = 1, 2$. The most interesting phenomena can be observed when $\epsilon = 0$, because the degree d of the Hirzebruch surface Y is an invariant of the fibration in this case. The parity of the lattice is the same as that of $g - d + 1$ and the structure of MWL(f) depends on the combination of g mod 4 and the parity of d. In particular, even and odd lattices both occur for a fixed g. See [2] for the detail.

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The bicanonical map of surfaces of general type with $p_g = 0$ and $K^2 = 6$

Margarida Mendes Lopes
(joint work with Rita Pardini)

Many examples of complex surfaces of general type with $p_g = 0$ are known, but a detailed classification is still lacking, despite much progress in the theory of algebraic surfaces. Surfaces of general type are often studied using properties of their canonical curves. If a surface has $p_g = 0$, then there are of course no such curves, and it is natural to look instead at the bicanonical system, which is not empty.

Let S be a minimal surface of general type with $p_g = 0$. It is well known that $1 \leq K_S^2 \leq 9$. By a theorem of Xiao Gang [12], for $K_S^2 \geq 2$ the image of the bicanonical map of S is a surface Σ and, by Reider's theorem [11], the bicanonical map φ is a morphism if $K_S^2 \geq 5$.

Assume that $K_S^2 \geq 3$. Since $h^0(S, 2K_S) = K_S^2 + 1$, the bicanonical image of S is a surface of degree $m \geq K_S^2 - 1$ in $\mathbb{P}^{K_S^2}$. If, in addition, φ is a morphism (so, in particular, if $K_S^2 \geq 5$), one has $dm = (2K_S)^2 = 4K_S^2$, where d is the degree of φ. It is known that, if $K_S^2 \geq 3$ and φ is a morphism, then $d \leq 4$ [4]. Furthermore if

$K_S^2 = 9$, φ is birational [3], whilst if $K_S^2 = 7, 8$ the degree of φ is at most 2 and this bound is effective [6, 7, 9].

In the case $K_S^2 = 6$ one has the following numerical possibilities for the pair (d, m): $(1, 24), (2, 12), (3, 8), (6, 4)$.

The latter possibility occurs and in fact it can be completely characterized. Such surfaces turn out to be Burniat surfaces (see [2, 10]). More precisely one has the following:

Theorem 1. [5] *Let S be a minimal complex surface of general type such that $p_g(S) = 0$ and $K_S^2 = 6$ and let $\varphi\colon S \to \mathbb{P}^{K_S^2}$ the bicanonical map of S. Then* $\deg \varphi = 4$ *if and only if S is a Burniat surface.*

In particular, K_S is ample.

Theorem 2. [5] *Smooth minimal surfaces of general type S with $K_S^2 = 6$, $p_g(S) = 0$ and bicanonical map of degree 4 form an unirational 4-dimensional irreducible connected component of the moduli space of surfaces of general type.*

In this talk we discuss the other possible cases of non birationality of the bicanonical map, i.e., degrees 2 and 3. The results are the following:

Theorem 3. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 6$ for which the bicanonical map φ is not birational. Then the degree of φ is either 2 or 4 and the image of φ is a rational surface.*

Theorem 4. *Let S be a minimal surface of general type with $p_g(S) = 0$ and $K_S^2 = 6$ for which the bicanonical map φ has degree 2. Then there is a fibration $f\colon S \to \mathbb{P}^1$ such that the general fibre F of f is hyperelliptic of genus 3 and f has 4 or 5 double fibres. Furthermore the bicanonical involution of S induces the hyperelliptic involution on F.*

Idea of the proof of Theorem 3. It is necessary to exclude the possibility that $d = 3$ occurs. For $d = 3$ the bicanonical image would be a rational surface of degree 8 in $\mathbb{P}^6$. By using repeated adjunction (an idea which dates back to Enriques), such surfaces are studied and their geometry is used to show that $d = 3$ does not occur. For details see [8]. □

Idea of the proof of Theorem 4. Let σ be the bicanonical involution. The quotient surface $T := S/\sigma$ is a rational surface whose only singularities are nodes (corresponding to the isolated fixed points of σ). Since the bicanonical map factors through T it is possible to show that T has exactly 10 nodes. The statement of the theorem is obtained by a careful analysis of the binary linear code associated to the nodes. For details see again [8]. □

Remark. Note that Theorem 4 is not a mere list of possibilities because there are examples of both situations (see again [8]). G. Borrelli (see [1]) has obtained recently with different methods the same list of possibilities and a description of them as double planes.

Remark. It would be very interesting to describe the moduli space of the surfaces appearing in Theorem 4 and in particular to find whether these surfaces deform to surfaces with birational bicanonical map (no such example is known for $K_S^2 = 6$).

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A new family of surfaces with $p_g = 0$ and $K^2 = 3$

Margarida Mendes Lopes
(joint work with Rita Pardini)

The starting point of the subject of this talk is the following Theorem:

Theorem 1 (Xiao Gang, [5])**.** *Let S be a minimal complex surface of general type with $p_g(S) = 0$ such that the bicanonical map φ is not birational and let T be the bicanonical image. If T is not a rational surface, then T is birational to an Enriques surface and φ is a degree 2 morphism.*

Furthermore $K_S^2 = 3$ or $K_S^2 = 4$.

This theorem lists possibilities and a natural question is whether it is sharp.

Both J. Keum, [1], and D. Naie, [4], constructed examples of surfaces S with $p_g(S) = 0$ and $K_S^2 = 3$ or $K_S^2 = 4$ as double covers of nodal Enriques surfaces. For these surfaces the bicanonical map, although it factorizes through the covering map, has degree 4 and the bicanonical image is a rational surface.

In [2], it is shown that, in fact, if the bicanonical image of a surface S with $p_g(S) = 0$ is birationally an Enriques surface then, necessarily, $K_S^2 = 3$. So the case with $K_S^2 = 4$ of Theorem 1 does not occur. Furthermore it is shown that the minimal surfaces S with $p_g(S) = 0$ and $K^2 = 4$ having an involution σ such that S/σ is birational to an Enriques surface and such that the bicanonical map is composed with σ are precisely the Keum-Naie examples.

No example of a surface S with $p_g(S) = 0$ and $K_S^2 = 3$, with bicanonical image birational to an Enriques surface appears in the literature, and so the question is whether it can occur at all. It turns out such surfaces exist.

The subject of this talk is not only showing the existence of surfaces S with $p_g(S) = 0$ and $K_S^2 = 3$, with bicanonical image birational to an Enriques surface, but also explaining an explicit construction of all such surfaces. This explicit construction enables us to show that the corresponding subset of the moduli space of surfaces of general type is irreducible and uniruled of dimension 6. Since the closure of this subset contains the Keum-Naie surfaces, whose fundamental group is isomorphic to $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ (cf. [4]), also the fundamental group of all these surfaces is $\mathbb{Z}_2^2 \times \mathbb{Z}_4$.

The description of these surfaces is based on a very detailed study of the normalization of their bicanonical images. These are polarized Enriques surfaces of degree 6 with 7 nodes, satisfying some additional conditions.

For the proofs and details we refer to [3].

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Kustin–Miller unprojections

Stavros Papadakis

Kustin-Miller unprojection is a method that constructs more complicated Gorenstein rings from simpler data. Geometrically it corresponds to the inverse of the classical method of projection. The first talk was about the scheme–theoretic foundations of the simplest type of Kustin–Miller unprojection called Type I, which is joint work with M. Reid [3], and algebraically corresponds to the unprojection of a codimension one ideal I of a Gorenstein ring R with the quotient R/I being Gorenstein. In addition, I gave examples and mentioned a method, essentially due to A. Kustin and M. Miller [1], which calculates type I unprojection in the relative setting using projective resolutions and maps between complexes.

The second talk was about Tom and Jerry. They are two families of codimension four Gorenstein rings defined by M. Reid and studied by me at [2], which are constructed as Type I unprojection and appear in a variety of examples coming from Algebraic Geometry. Moreover, I talked about Type II unprojection, which is work in progress, and constructs a codimension $n+2$ conjecturally Gorenstein ring, starting from a codimension n complete intersection containing a certain codimension $n+1$ subscheme.

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Surfaces in your backyard

Ulf Persson

How do you give elementary examples of surfaces? Hypersurfaces in $\mathbb{P}^3$ are obvious candidates, but of course they are far too restrictive to present a wide variety of phenomena. It is e.g. impossible to give an example of a so called honestly elliptic surface (i.e. $\kappa = 1$ in the Kodaira classification). A natural thing is to consider imposing singularities. Just imposing ordinary double points (or more generally simple-singularities i.e. A-D-E singularities) does not give you anything new, although it certainly gives you interesting ecxamples with high Picard numbers. The next step is to consider ordinary triple points, i.e. points whose resolutions give you smooth elliptic curves with self-intersection -3. It is an elementary but instructive exercise to present the following list of quintics with ordinary triple points

Theorem 1. *If Q is a quintic with k ordinary triple points then $0 \leq k \leq 5$ and its resolution $\tilde{Q}$ satisfies*

$k=0$, $c_1^2 = 5$, $\chi = 5$ *(minimal of general type)*

$k=1$, $c_1^2 = 2$, $\chi = 4$ *(minimal of general type, a double octic)*

$k=2$ $c_1^2 = -1$, $\chi = 3$ *(an elliptic surface blown up once)*

$k=3$ $c_1^2 = -4$, $\chi = 2$ *(a K-3 surface blown up four times)*

$k=4$ $c_1^2 = -7$, $\chi = 1$ *(a rational surface)*

$k=5$ $c_1^2 = -10$, $\chi = 0$ *(a ruled surface over an elliptic curve blown up ten times)*

The proof is completely elementary. The interesting feature is the way those surfaces are geometrically realised. To take the example of $k = 2$. The line joining the two triple points becomes exceptional, and the elliptic fibration is given by the pencil of planes through it, intersecting the quintic residually in quartics with two double points. Those planes incidentally cut out the canonical divisors. In the case of $k = 3$ the canonical divisor consists of the plane through the three triple points, whose intersection is a triangle of lines and a circumscribed conic, all four easily seen to be exceptional. And finally the case of $k = 5$ the ruling consists of twisted cubics passing through the five triple points. By Bezout, any such twisted cubic having an additional intersection will be contained, and clearly through any six points, there is a twisted cubic. The degenerate fibers will be ten by choosing two points out of the five, defining a line and a residual conic through the remaining three. This distinction between the reducible components allow a canonical minimal model, which turns out to be a ruled surface over an elliptic curve defined by a stable rank-two bundle.

Now with my co-workers Endrass and Stevens I considered whether a similar classification can be effected for degree six, and the surprising answer is yes! However, the situation becomes more complicated. For one thing one can now no longer in general choose the locations of the triple points arbitrarily (there will be two many conditions). E.g. there will be no examples of eight generic triple points, but if the triple points happen to form the base points of a net of quadrics one can write down a simple example $C(Q_1, Q_2, Q_3)$ where Q_i span the net, and C is a plane cubic. This will actually be an honestly elliptic surface fibered over an elliptic curve (given by $C = 0$). Other special choices of eight points will also yield examples. In the case of nine triple points we get examples of non-minimal K-3 surfaces, as well as non-minimal fake K-3 surfces, namely honestly elliptic surfaces gotten from elliptic K-3 surfaces through logarithmic transforms. One may also find rational sextics with ten triple points, but ten is the upper limit.

For the complete classification I refer to the paper below. Let me only note that a typical construction is to consider a linear space made up by highly reducible, often not even reduced, hypersurfaces, such that the base points are of multiplicity three. (As a simple example consider a quintic Qu with five nodes on a conic $C = H \cap Q$, where H is a plane and Q a quadric. Then consider the generic member of the pencil spanned by HQu and Q^3).

One may wonder where to go from here? One may note that we prove that for degree seven or higher only minimal surfaces of general type occur in this way. Thus one should either consider other elementary constructions of low degree, like complete intersections in $\mathbb{P}^4, \mathbb{P}^5$ and maybe $\mathbb{P}^6$. The same thing for multi-projective spaces. In short, I suspect that there will be no more than perhaps a dozen different cases, similar to the ones I have refered to above. To be more specific, try to do a similar analysis for hypersurfaces of low degree in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The case of tri-degree $(3, 3, 3)$ is analogous to the case of quintics, (but of course more involved). It turns out that its chern-invariants are given by $c_1^2 = 18, \chi = 9$.

So I would like to point this out by describing an analogy to the Godeaux quotient, which although elementary, has never been written down and published to my knowledge[1]. The key point is an action of $\mathbb{Z}_9$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ inducing an action on the monomials $x_i y_j z_k$ involving an amalgation of the cyclic permutation of the co-ordinates and the action of a primitive 9-th root of unity. More precisely letting a generator of $\mathbb{Z}_9$ act accordingly

$$(x, y, z) \mapsto (\rho z, \rho x, \rho y)$$

It is easy to find the fixed points of the actions, and just like in the quintic case, avoid those by a judicious inclusion of certain extreme monomials. Once we have a fixed point free action the quotient will have $c_1^2 = 2, \chi = 1$. As the quotient is regular, we conclude that $p_g = 0$.

Finally instead of considering just triple points, one may take into account four-tuple points, or other more subtle singularities, one thinks of elliptic singularities with $E^2 = -2, -1$. Those two types are easily exhibited on double covers, by considering four-tuple points or so called infinitely close triple points.

All of those obviously are directed to the main question

Question. *Is it possible to classify all surfaces of small invariants?*

One first attempt would be to classify all such surfaces which can be deformed into double coverings, especially double planes.

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Extrasymmetric matrices and surfaces with $p_g = 4$ and $K^2 = 6$

Roberto Pignatelli
(joint work with Ingrid Claudia Bauer and Fabrizio Catanese)

Minimal surfaces with $p_g = 4$ have been studied by several mathematicians since the publication of the famous book of Enriques [Enr]. By the standard inequalities of Noether and Bogomolov-Miyaoka-Yau, for these surfaces it holds $4 \leq K^2 \leq 45$.

The case $K^2 = 4$ is completely described in [Hor2]. All these surfaces are double covers of an irreducible quadric in $\mathbb{P}^3$. Their moduli space is generically smooth, unirational, of dimension 42; its singular locus has codimension 1, and it is exactly the locus corresponding to the double covers of the quadric cone.

In [Hor1] (see also [Rei2], [Gri]) the case $K^2 = 5$ is completely described: the canonical map is either a birational morphism to a quintic in $\mathbb{P}^3$, or a rational map of degree 2 onto an irreducible quadric. Their moduli space has two irreducible unirational components of dimension 40 whose general point corresponds to surfaces with canonical image respectively a quintic or a smooth quadric. The

[1] I thought of it some twenty years ago, and may have circulated it around privately.

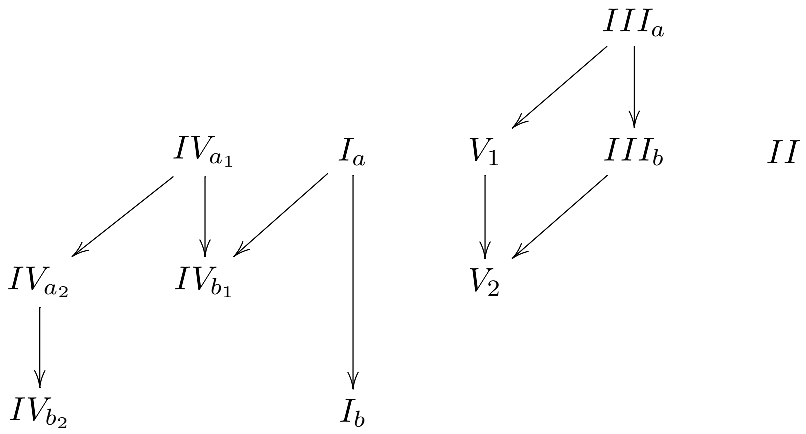
surfaces whose canonical image is a quadric cone form a 39-dimensional subvariety of this moduli space, the intersection of the two irreducible components.

The case $K^2 = 6$ is the first case not completely solved. In [Hor3] Horikawa listed all possibilities for the canonical map, dividing these surfaces in 11 classes (and therefore their moduli space in 11 strata). He proved that each of these cases occurs, and studying the local deformations of these surfaces (to understand how these strata can 'glue'), Horikawa proved that their moduli space has 4 irreducible components (one of dimension 39, the other three of dimension 38), and at most 3 connected components.

More precisely, Horikawa named the 11 classes as I_a, I_b, II, III_a, III_b, IV_{a_1}, IV_{a_2}, IV_{b_1}, IV_{b_2}, V_1, V_2 (see [Hor3] for precise definitions of each class). According to Horikawa's notation we define

Definition. Let A and B be two of the above introduced classes. If we write "$A \to B$", it means that there is a flat family with base a small disc $\Delta_\varepsilon \subset \mathbb{C}$ whose central fibre is of type B and whose general fibre is of type A.

With this notation Horikawa summarized its results in the following picture



He could disprove many other degenerations, but he could neither prove nor disprove the specializations $II \to III_b$, $II \to V$ and $I_a \to V$; we have shown that the degeneration $II \to III_b$ occurs.

Definition. A minimal surfaces of general type with $p_g = 4$ and $K^2 = 6$ is of type II if the canonical map has degree 3.

Horikawa proved that in this case the canonical image is a quadric cone.

Surfaces of type III_b are described by Horikawa as follows:

Theorem (5.2 in [Hor3]**).** *Let S be a surface of type III_b. Then S is birationally equivalent to a double covering of $\mathbb{F}_2$ whose branch locus B consists of the 0-section Δ_0 and $B_0 \in |7\Delta_0 + 14\Gamma|$ which has a quadruple point at $x \in \Gamma$ and a 2-fold triple point at $y \in \Gamma$ on a fibre Γ, with x and y being possibly infinitely near.*

The canonical ring of these surfaces is very complicated: it is a quotient of a polynomial ring of big (at least 6, maybe more) codimension. We do not know how to investigate the flat deformations of rings of high codimension. We look then for a 'bigger' and easier ring, a ring containing the canonical ring and of smaller codimension.

By standard computations one can show that the canonical system of S is $|2L| + Z$ where L is the genus 3 pencil pull-back of the ruling of $\mathbb{F}_2$, and Z is a fundamental cycle. Therefore, even if K_S is not 2-divisible in the Picard group, it can be divided by 2 when considered only as a Weil divisor on the canonical model.

Definition. Let S be a surface of type III_b, let Z be the fixed part of its canonical system, and let δ be a generator of $H^0(Z)$.

Let R be the graded ring whose homogeneous components are the spaces $R_d := H^0(dL + \lfloor \frac{d}{2} Z \rfloor)$, $d \in \mathbb{N}$, with product defined on the homogeneous elements as $ab = a \otimes b$ or $a \otimes b \otimes \delta$ according if the product of the degrees of a and b is even or odd.

Note that enlarging the ring 'restricts' the possible deformations. In fact, if the canonical rings induce, given a flat family of surfaces, a flat family of rings, the same does not hold for these 'half-canonical' rings, since the 2-divisibility of the canonical divisor (as a Weil divisor on the canonical model) is not necessarily preserved by a deformation.

As proved in [MP] (where these surfaces are studied in detail) the canonical system of a surface of type II, can be written again as $2L+Z$ with L genus 3 pencils and Z fundamental cycle. It is then natural to expect, if a family "$II \to III_b$" exists, that this family preserves the genus 3 pencils and the 'half-canonical' rings.

Theorem 1. *We have* $R \cong \mathbb{C}[x_0, x_1, y, z, w, v, u]/I$ *with* $\deg(x_0, x_1, y, z, w, v, u) = (1, 1, 2, 3, 4, 5, 6)$*, where* I *has codimension* 4*, generated by* 9 *equations yoked by* 16 *syzygies; the* 9 *generators of* I *are homogeneous polynomial of respective degrees* $(4, 5, 6, 7, 8, 9, 10, 11, 12)$.

Miles Reid and Duncan Dicks introduced in [Rei1] (see also [Rei2], [Rei3], [BCP]) the 'extrasymmetric format', for some Gorenstein rings of codimension 4 with 9 relations and 16 syzygies.

Roughly speaking, they noticed that the ideal generated by the pfaffians of order 4 of a 6×6 skewsymmetric matrix is, if the matrix has some further symmetry (it is 'extrasymmetric') of codimension 4 with 9 generators and 16 syzygies. This format is flexible, *i.e.* every deformation of the matrix preserving the symmetries induces a flat deformation of the ideal. This property allowed us to prove our main result.

Theorem 2. *Let* $(x_0, x_1, y, z, w, v, u)$ *variables of degrees* $(1, 1, 2, 3, 4, 5, 6)$*, Let* M *be the* 6×6 *skewsymmetric matrix*

$$M = \begin{pmatrix} 0 & t & z & v & y & x_1 \\ & 0 & w & u & P_3 & y \\ & & 0 & P_9 & u & v \\ & & & 0 & wP_4 & zP_4 \\ & & & & 0 & tP_4 \\ -sym & & & & & 0 \end{pmatrix}.$$

where the P_i*'s are homogeneous of degree* i *in the above introduced variables and* t *is the parameter on a small disc* $\Delta_\varepsilon \subset \mathbb{C}$.

For general choice of P_3, P_4 *and* P_9 *the* 4×4 *pfaffians of* M *define a variety* $X \subset \Delta_\varepsilon \times \mathbb{P}(1, 1, 2, 3, 4, 5, 6)$ *whose projection on* Δ_ε *is flat, with central fibre a surface of type* III_b *and with general fibre a surface of type* II.

Sketch of the proof of Theorem 2. The flatness of the above family (for general entries) follows directly from the flexibility of the format. One can check that for general choice of the polynomials P_i and for t small the above equations define a surface with only rational double points as singularities: the invariants can be easily computed.

Note that the pfaffians Pf_{1235} and Pf_{1236} are of the form $tu - \cdots$ and $tv - \cdots$, and that the pfaffian Pf_{1256} can, for general choice of P_4, be written as $t^2w - ....$ Therefore, for $t \neq 0$, we can 'eliminate' the variables u, v, w, and $R \cong \mathbb{C}[x_0, x_1, y, z]/J$ for some ideal J: a straightforward computation shows that J is a principal ideal generated by the equation obtained by Pf_{1234} after 'eliminating' u, v, w using Pf_{1235}, Pf_{1236} and Pf_{1256}.

We get then an hypersurface of degree 9 in $\mathbb{P}(1, 1, 2, 3)$, whose canonical system is induced by $\mathcal{O}(2)$: since for general entries of M the coefficient of the monomial z^3 in its equation does not vanish, we see that the canonical map has degree 3 (and image $\mathbb{P}(1, 1, 2)$, a quadric cone). This shows that the surface is of type II.

If $t = 0$, the canonical map is given again by the projection on $\mathbb{P}(1, 1, 2)$, but the surface meets the center of the projection in a point (if $P_4 = w + \cdots$, the point (0,0,0,0,1,0,1)), therefore the projection has only degree 2; one can easily check that the branch locus has the behavior described by Horikawa. □

As a corollary, we can improve Horikawa's bound on the deformation types

Corollary. *The number of deformation types of minimal surfaces of general type with* $p_g = 4$ *and* $K^2 = 6$ *is at most* 2.

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Surfaces of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2

Francesco Polizzi

In [Par03] R. Pardini classified the minimal surfaces S of general type with $p_g = q = 0$, $K_S^2 = 8$ and a rational involution, i.e. an involution $\sigma : S \longrightarrow S$ such that the quotient $T := S/\sigma$ is a rational surface. All the examples constructed by Pardini are *isogenous to a product*, i.e. there exist two smooth curves C, F and a finite group G acting faithfully on C, F and whose diagonal action is free on the product $C \times F$, in such a way that $S = (C \times F)/G$. Pardini's classification contains five families of such surfaces; in particular, four of them are irreducible components of the moduli space of surfaces with $p_g = q = 0$, $K_S^2 = 8$, and represent the surfaces with the above invariants and non- birational bicanonical map.

In this paper we deal with the irregular case, in fact we study the case $p_g = q = 1$, $K_S^2 = 8$. Surfaces with $p_g = q = 1$ are the minimal irregular surfaces of general type with the lowest geometric genus, therefore they are natural candidates to starting the investigation of irregular surfaces with $q = 1$ or, more generally, with an irrational pencil. However, such surfaces are still quite mysterious, and only a few families have been hitherto discovered. If S is a surface with $p_g = q = 1$, then $2 \leq K_S^2 \leq 9$; the case $K_S^2 = 2$ is studied in [Ca81], whereas [CaCi91] and [CaCi93] deal with the case $K_S^2 = 3$. For higher values of K_S^2 only some sporadic examples were so far known; see [Ca99], where a surface with $K_S^2 = 4$ and one with $K_S^2 = 5$ are constructed.

When $p_g = q = 1$, there are two basic tools that one can use in order to study the geometry of S: the *Albanese fibration* and the *paracanonical system*. First of all, $q = 1$ implies that the Albanese variety of S is an elliptic curve E, hence the Albanese map $\alpha : S \longrightarrow E$ is a connected fibration; we denote by F the general fibre of α and by $g = g(F)$ its genus. Let us fix a zero point $0 \in E$, and for any $t \in E$ let us write $K_S + t$ for the line bundle $K_S + F_t - F_0$. By Riemann-Roch and semicontinuity theorem we have $h^0(S,\ K_S + t) = 1$ for general $t \in E$, hence denoting by C_t the only element in the complete linear system $|K_S + t|$ we obtain a 1-dimensional algebraic family $\{K\} = \{C_t\}_{t\in E}$ parametrized by the elliptic curve E. We will call it the paracanonical system of S; according to [Be88], it is the irreducible component of the Hilbert scheme of curves on S algebraically equivalent to K_S which dominates E. The *index* $\iota = \iota(K)$ of the paracanonical system $\{K\}$ is the number of distinct curves of $\{K\}$ through a general point of S. The *paracanonical map* $\omega : S \longrightarrow E(\iota)$, where $E(\iota) := \mathrm{Sym}^{\iota} E$, is defined in the following way: if $x \in S$ is a general point, then $\omega(x) = t_1 + \cdots + t_\iota$, where $C_{t_1}, \dots, C_{t_\iota}$ are the paracanonical curves containing x. The best result that one might obtain would be to classify the triples (K^2, g, ι) such that there exists a minimal surface of general type S with $p_g = q = 1$ and these invariants. Since by the results of Gieseker the moduli space $\mathcal{M}_{\chi,\, K^2}$ of surfaces of general type with fixed $\chi(\mathcal{O}_S)$, K_S^2 is a quasiprojective variety, it turns out that there exist only finitely many such triples, but a complete classification is still missing.

By the results of [Re88], [Fr91] and [CaCi91] it follows that the bicanonical system $|2K_S|$ of a minimal surface of general type with $p_g = q = 1$ is base-point free, whence the bicanonical map $\phi := \phi_{|2K|} : S \longrightarrow \mathbb{P}^{K_S^2}$ of S is a morphism. Moreover such a morphism is generically finite by [Xi85], so $\phi(S)$ is a surface Σ. We will say that a surface S contains a *genus 2 pencil* if there is a morphism $f : S \longrightarrow B$, where B is a smooth curve and the general fibre Φ of f is a smooth curve of genus 2. Notice that in this case the bicanonical map ϕ of S is not birational, since $|2K_S|$ cuts out on Φ a subseries of the bicanonical series of Φ which is composed with the hyperelliptic involution. In this case we say that S presents the *standard case* for the non-birationality of the bicanonical map; otherwise, namely if ϕ is not birational but S does not contain any genus 2 pencils, we say that S presents the *non-standard* case. By the results of Bombieri (later improved by Reider, see [Bo73] and [Re88]) it follows that, if $K_S^2 \geq 10$ and the bicanonical map is not birational, then S contains a genus 2 pencil. Whence there exist only finitely many families of surfaces of general type presenting the non-standard case, and one would like to classify all of them; however, this problem is still open, although many examples are known. In the paper [Xi90] G. Xiao gave two list of possibilities for the bicanonical image of such a surface; later on several authors investigated about their real occurrence. For more details about this argument, we refer the reader to the paper [Ci97].

No examples of surfaces with $p_g = q = 1$ and presenting the non-standard case were hitherto known; if S is such a surface and $K_S^2 \geq 5$, then a result of Xiao ([see Xi90, Proposition 5]) implies that the degree of ϕ is either 2 or 4. In this work we describe the surfaces of general type with $p_g = q = 1$, $K_S^2 = 8$ and such that the degree of ϕ is 2. It will turn out that they belong to three distinct families, which provide as well the first known examples of surfaces which such invariants. None of these surfaces contains a genus 2 pencil, thus they are three substantially new pieces in the classification of surfaces presenting the non-standard case.

What we show is that, as in the case $p_g = q = 0$, the surfaces with $p_g = q = 1$, $K_S^2 = 8$ and bicanonical map of degree 2 are isogenous to a product. More precisely, our result is the following:

Theorem 1. *Let S be a minimal surface of general type with $p_g = q = 1$, $K_S^2 = 8$ and such that its bicanonical map has degree 2. Then S is a quotient of type $S = (C \times F)/G$, where C, F are smooth curves and G is a finite group acting faithfully on C, F and freely on $C \times F$. Moreover C is a curve of genus 3 which is both hyperelliptic and bielliptic, $E := C/G$ is an elliptic curve isomorphic to the Albanese variety of S and $F/G \cong \mathbb{P}^1$. The bicanonical map ϕ of S factors through the involution σ of S induced by the involution $\tau \times id$ on $C \times F$, where τ is the hyperelliptic involution of C. The occurrences for $g(F)$ and G are the following three:*

I. $g(F) = 3$, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
II. $g(F) = 4$, $G \cong S_3$;
III. $g(F) = 5$, $G \cong D_4$.

The curve F is hyperelliptic in case I, whereas it is not hyperelliptic in cases II and III.

Surfaces of type I, II, III do exist and they form three generically smooth, irreducible component $\mathcal{S}_I$, $\mathcal{S}_{II}$, $\mathcal{S}_{III}$ of the moduli space $\mathcal{M}$ of surfaces with $p_g = q = 1$, $K_S^2 = 8$, whose respective dimensions are:

$$dim\ \mathcal{S}_I = 5, \quad dim\ \mathcal{S}_{II} = 4, \quad dim\ \mathcal{S}_{III} = 4.$$

The proof of Theorem 1 is somewhat involved as it requires the understanding of many different techniques.

Sketch of the proof of Theorem 1. **Step 1**. We analyze the *bicanonical involution* σ of S, following [Xi90] and [CM02]. It turns out that σ has 12 isolated fixed points and that the divisorial fixed locus of σ is contained in fibres of the Albanese pencil.

Step 2. Using the results obtained in Step 1 we prove that if S is a minimal surface of general type with $p_g = q = 1$, $K_S^2 = 8$ and bicanonical map of degree 2, then S contains a rational pencil of hyperelliptic curves of genus 3 with six double fibres. This in turn implies, by the results of Serrano contained in [Se90] and [Se96], that S is isogenous to a product, i.e. $S = (C \times F)/G$. We show moreover that there are at most three families of such surfaces, and we describe them.

Step 3. We show that the three families described in Step 2 actually exist, by constructing the two curves C, F and by exhibiting explicitly the actions of G on them.

Step 4. We study the moduli space of the surfaces S constructed in Step 3. This is not difficult because the group G acts separately on C and F, hence the Kuranishi family of S turns out to be smooth. □

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On numerical Godeaux surfaces constructed as double planes

Caryn Werner

Let S be a minimal surface of general type with $p_g = q = 0, K_S^2 = 1$. The torsion of S, Tors(S), is cyclic of order at most five, and Reid has shown that in the cases of torsion $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$ the moduli spaces are smooth and irreducible of dimension eight. In comparison, in the cases of $\mathrm{Tors}(S) = 0$ and $\mathrm{Tors}(S) = \mathbb{Z}_2$, little is known about the moduli space; while several examples of these surfaces have been found a more general classification is still unknown.

Surfaces with these invariants are called numerical Godeaux surfaces, after Godeaux who provided the first example, as the $\mathbb{Z}_5$-quotient of a quintic hypersurface in $\mathbb{P}^3$. Most known constructions of numerical Godeaux surfaces have an involution. One particular method for constructing these surfaces was proposed by Campedelli: as the minimal resolution of the double cover of the plane, branched along a degree ten curve with one quadruple point, five infinitely near triple points, such that these six singular points do not lie on a conic. In this talk we survey the known numerical Godeaux surfaces constructed as double planes; the cases of torsion equal to $0, \mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$ all occur.

The first construction of a numerical Godeaux as a double plane is due to Oort and Peters, whose resulting surface has order four torsion. Reid proved that the classical Godeaux construction can also be realized as a Campedelli double plane; this construction has torsion of order five. As both the numerical Godeaux surfaces with torsion group $\mathbb{Z}_4$ and $\mathbb{Z}_5$ have irreducible moduli spaces, and constructions as double planes, one can ask if the same will be true for the other three cases.

For trivial torsion, a surface constructed as the resolution of a singular quintic in $\mathbb{P}^3$ by Craighero and Gattazzo has been shown to be a double plane. In the case of order two torsion there is a four dimensional family of double plane Godeaux surfaces.

After cataloguing these known double plane Godeaux surfaces, we classify the possible degree ten branch curves that are invariant under an involution of the plane. The idea of looking for branch curves with this additional symmetry was proposed by Stagnaro; following this idea one can prove

Theorem 1. *Let C be a degree ten plane curve with the singularities required for a numerical Godeaux double plane, and suppose C is invariant under involution. Then the resulting double cover branched along C has torsion group $\mathbb{Z}_4$.*

Moreover one can determine all possible decompositions of the branch curve; the example of Oort and Peters belongs to this class of constructions.

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A new proof for the adjoint theorem and a Castelnuovo's conjecture

Francesco Zucconi

Let $\xi \in H^1(X, \mathcal{T}_X)$ be the class of a first order deformation $\pi : \mathcal{X} \to \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$ being X a n-dimensional projective variety. Let $\langle \eta_1, \ldots, \eta_{n+1}\rangle$ be an ordered set of $n+1$ linearly independent sections of $\text{Ker}(\delta_\xi : H^0(X, \Omega^1_X) \to H^1(X, \mathcal{O}_X))$ where δ_ξ is the coboundary map associated to the sequence: $0 \to \mathcal{O}_X \to \Omega^1_{\mathcal{X}|X} \to \Omega^1_X \to 0$. If $s_1, \ldots, s_{n+1}$ are liftings in $H^0(X, \Omega^1_{\mathcal{X}|X})$ of respectively $\eta_1, \ldots, \eta_{n+1}$ and $\Omega \in H^0(X, \wedge^{n+1}\Omega^1_{\mathcal{X}|X})$ is the form corresponding to $s_1 \wedge \ldots \wedge s_{n+1} \in \wedge^{n+1} H^0(X, \Omega^1_{\mathcal{X}|X})$ then via the isomorphism $L_\xi : H^0(X, \wedge^{n+1}\Omega^1_{\mathcal{X}|X}) \to H^0(X, \wedge^n \Omega^1_X)$ we obtain a top form $\omega_{\xi, \langle \eta_1, \ldots, \eta_{n+1}\rangle} = L_\xi(\Omega)$. This form is called adjoint form of ξ and $\langle \eta_1, \ldots, \eta_{n+1}\rangle$. If W is the subvector space generated by $\langle \eta_1, \ldots, \eta_{n+1}\rangle$ and $\wedge^n W$

is the subvector space of $H^0(X, \wedge^n \Omega^1_X)$ given by $\langle \eta_1 \wedge \ldots \wedge \hat{\eta}_i \ldots \wedge \eta_{n+1} \rangle$ the adjoint theorem states that: if $\omega_{\xi, \langle \eta_1, \ldots, \eta_{n+1} \rangle} \in \wedge^n W$ then $\xi \in \mathrm{Ker}(H^1(X, \mathcal{T}_X) \to H^1(X, \mathcal{T}_X(D)))$ where D is the fixed component of the sublinear system given by $| \wedge^n W |$.

In this talk we present a new proof of this theorem based on the natural interpretation of the condition $s_1 \wedge \ldots \wedge s_{n+1} = 0$ as integrability condition for the system $s_1 \wedge \ldots \wedge \widehat{s_i *} \wedge \cdots \wedge s_{n+1} = 0$, $i = 1, \ldots, n+1$. We explain the relations between the solution of this system and the geometry of the natural map $\pi : \mathbb{P}(\Omega^n_{\mathcal{X}|X}) \to X$. In the second part of the talk we show the proof of the Castelnuovo conjecture stating that the number m of moduli of an irregular surfaces with $q \geq 4$ and Albanese map of degree 1 is less or equal to $p_g + 2q - 3$. In the final part we discuss some possible applications to surfaces with $q = 4$.

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Reporter: Alberto Calabri

Participants

Christian Böhning
boehning@btm8x5.mat.uni-bayreuth.de
Lehrstuhl für Mathematik VIII
Universität Bayreuth
NW - II
95440 Bayreuth

Prof. Dr. Giuseppe Borrelli
borrelli@mat.uniroma3.it
c/o Ciro Ciliberto
Dipartimento di Matematica
Universita di Roma Tor Vergata
via della Ricerca Scientifica
I-000133 Rome

Prof. Dr. Vasile Brinzanescu
Vasile.Brinzanescu@imar.ro
Institute of Mathematics of the
Romanian Academy
P.O. Box 1-764
70700 Bucharest
ROMANIA

Dr. Alberto Calabri
calabri@dm.unibo.it
Dipartimento di Matematica
Universita degli Studi di Bologna
Piazza di Porta S. Donato, 5
I-40126 Bologna

Prof. Ciro Ciliberto
cilibert@mat.uniroma2.it
Dipartimento di Matematica
Universita di Roma "Tor Vergata"
V.della Ricerca Scientifica, 1
I-00133 Roma

Prof. Dr. Jonghae Keum
jhkeum@kias.re.kr
School of Mathematics
Korea Inst. for Advanced Study
207-43 Cheongnyangri-dong.
Dongdaemun-gu
Seoul 130-722
Korea

Prof. Dr. Kazuhiro Konno
konno@math.wani.osaka-u.ac.jp
Department of Mathematics
Graduate School of Science
Osaka-University Machikaneyama 1-16
Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Margarida Mendes Lopes
mmlopes@math.ist.utl.pt
Departamento de Matematica
Instituto Superior Tecnico
Avenida Rovisco Pais, 1
P-Lisboa 1049-001

Prof. Dr. Stavros Papadakis
spapad@maths.warwick.ac.uk
stavrospapadakis@hotmail.com
papadakis@math.uni-s
FB Mathematik und Informatik
Geb. 27
Universität des Saarlandes
66123 Saarbrücken

Prof. Dr. Ulf Persson
ulfp@math.chalmers.se
Department of Mathematics
Chalmers University of Technology
S-412 96 Göteborg

Roberto Pignatelli
Roberto.Pignatelli@uni-bayreuth.de
pignatel@science.unitn.it
Dipartimento di Matematica
Universita di Trento
Via Sommarive 14
I-38050 Povo (Trento)

Prof. Dr. Francesco Polizzi
polizzi@mat.uniroma2.it
Dipartimento di Matematica
Universita di Roma "Tor Vergata"
V.della Ricerca Scientifica, 1
I-00133 Roma

Prof. Dr. Caryn Werner
cwerner@allegheny.edu
Mathematics Department
Allegheny College
520 North Main St.
Meadville PA 16335
USA

Dr. Francesco Zucconi
zucconi@dimi.uniud.it
Dipartimento di Matematica e
Informatica
Universita di Udine
Via delle Scienze 206
I-33100 Udine