Mathematisches Forschungsinstitut Oberwolfach

Report No. 26/2004

Local Time-Space Calculus with Applications

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May 16th – May 22nd, 2004

Introduction by the Organisers

Itô's formula has been celebrated as a fundamental extension of the classical Leibnitz-Newton formula and forms a cornerstone to what is known today as stochastic calculus. In recent years, there have been further extensions of this formula which essentially weaken further the smoothness of the function to which the formula is applied at the cost of an additional term involving integration over a two dimensional field which has been labeled local time-space. As of yet the full potential of local time-space calculus and its applications has neither been realized nor properly understood. The current proposal aimed at bringing together researchers who in some way have touched on this currently unchartered territory in order to examine its importance as both a pure and applied field of study as well as to establish the state of the art in the current literature and identify new road map of research.

The appearance of local time-space as a natural analytical development

The fundamental result of stochastic calculus is Itô's formula firstly established by Itô [11] for a standard Brownian motion and then later extended to continuous semimartingales by Kunita and Watanabe [12]. The function F appearing in Itô's formula is C^2 in the space variable, and the correction to the classic Leibnitz-Newton formula is expressed by means of the quadratic variation.

Various extensions of the Itô formula have been established for functions F which are not C^2 in the space variable. The best know of these extensions is the Itô-Tanaka formula firstly derived by Tanaka [20] for $F(x) = |x|$ and then extended to absolutely continuous F with F' of bounded variation by Meyer [17] and Wang [21]. The correction term appearing in this formula is expressed by means of the local time which goes back to Lévy [14]. A different extension to absolutely continuous F with locally bounded F' is due to Bouleau and Yor [4]. The correction term appearing in this formula is also expressed by means of the local time, however, in a different manner which suggests a formal integration by parts. Both formulas are derived only in dimension one.

In the attempts to understand a more general rule unifying the various correction terms reviewed above it was noticed that Eisenbaum [6] made an interesting contribution in the case of standard Brownian motion by deriving an extension of Itô's formula where the correction term is expressed as an area integral with respect to both the time variable t and the space variable x of the local time ℓ_t^x. The arguments of Eisenbaum rely on combining the Bouleau-Yor extension with the Föllmer-Protter-Shiryaev extension [9] and thus strongly depend on the time-reversal property of standard Brownian motion.

The main aim of the article [10] is to make use of the formula [18] and show that the representation of the correction term in the Eisenbaum's extension [6] as an area integral with respect to the local time-space (as we call it in short) is not only restricted to a time-invariant Brownian motion process but extends quite generally to all continuous semimartingales. This is firstly done for C^1 functions F and then extended to absolutely continuous functions F with F_t and F_x of bounded variation. The proposed extension is still incomplete in many ways.

The appearance of local time-space as an applied tool

Applications in free-boundary problems of optimal stopping related to pricing of American-type options, where one needs to consider functions of time and space that are not C^2, have recently led to a new extension of Itô's formula [18]. Already the formula has found a home in other papers related to financial mathematics such as [5] and [13]. The most interesting development in this formula, in comparison with the older extensions above, is its final term where the possible jump of $x \mapsto F_x(t,x)$ along the curve $t \mapsto b(t)$ is integrated with respect to the *time variable* t of the local time ℓ_t^b. It is obvious that the appearance of this latter local time term has an intimate relationship to the smooth fit principle in optimal stopping. Studies such as [19] and [2], where optimal stopping problems involving non-diffusive processes, have shown that the smooth fit principle can break down. The relationship with local time-space is thus begging.

Most notably, the same formula (in a somewhat disguised form) appears independently in [8] where it is used to study solutions of the heat equation in the presence of caustics (i.e. when the corresponding Burgers equations have shock waves). It is also clear that the formula with its ramifications is generally useful in any two-phase problem such as the obstacle problems or the problems of melting and solidification including e.g. models of tumor growth etc.

Open questions and new directions

No attempt has been made to specify, for the case of continuous semimartingales, the most general class of functions G for which the double integral $\int_0^t \int_{\mathbb{R}} G(s,x)\, d\ell_s^x$ makes sense. Instead it has been shown (cf. [10]) how a number of known extensions of the Itô formula can be obtained by formal manipulations of the $d\ell_t^x$ integral. This formalism (or formal $d\ell_t^x$ calculus as we call it) appears to be useful for at least two reasons. Firstly, it helps to develop intuition needed to understand and compare known formulas. Secondly, if a new function F is given and one needs to write down a change-of-variable formula for $F(t, X_t)$, then such a formalism can be helpful in guessing a candidate formula before a rigorous proof is known or given. It remains a challenging task, however, to carry out this programme on firm mathematical grounds to a more satisfactory completion.

As far as the new developments for continuous semimartingales and the natural questions that follow thereof are concerned, when considering the case of general semimartingales, nothing has as of yet been done. A particular case of interest is that of Lévy processes being a class of semimartingales which has some degree of tractability and spans processes of both bounded and unbounded variation. When one takes account of the relevance of local time-space calculus in optimal stopping problems that appear frequently in financial mathematics and further that financial mathematics itself has turned more and more to models based on Lévy processes in recent years, there is a clear case for developing the theory at least in this direction.

Precise formalization of the integration of local time-space calculus and the applied problems mentioned in the previous section is awaiting attention. Exactly what can one achieve with current knowledge of local time-space calculus and what kind of results would push the applications further? Specific points of interest are elaborated on below in the goals of the workshop. For other related work with references see [7], [16], [3], [1].

Goals

Some of the main goals of the workshop were:

- To summarize what is exactly the state of the art in current literature with respect to the appearance and manipulation of local time-space in Itô-type formulas.
- To identify concrete mathematical problems which will make the formal manipulation rules of the local-time space calculus rigorous.
- To identify concrete connections between local time-space and phenomena appearing in related areas. In particular, this includes:
  - the failure of the smooth-fit principle in optimal stopping problems with underlying jump processes;

– analysis and stochastic analysis of variational inequality problems related to optimal stopping;
– applications of the previous two points to American-type option pricing in mathematical finance;
– applications in partial differential equations of two-phase problems (such as the heat equation in the presence of caustics and similar);
– applications in optimal stochastic control (the Hamilton-Jacobi-Bellman equation);
– to identify further areas of research where the appearance of local time-space calculus is of relevance and needs development.

Format of workshop

The workshop consisted of a number of invited speakers who highlighted their recent work in the context of the above bullet point list. Speakers were asked to put their perspective on new possible directions and any proposals or ideas for future efforts. There has also been time set aside for discussion and formation of new collaborations. Further, the workshop rounded off with a summary of the week's discussions and the formulation of some new directions. Finally, the organizers are working on publishing proceedings from the workshop (in the *Seminaire de Probabilités* by Springer-Verlag).

Organizers

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- Andreas Kyprianou (Utrecht)
- Ana-Maria Matache (Zürich)
- Goran Peskir (Aarhus)

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Workshop on Local Time-Space Calculus with Applications

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Abstracts

Optimal stopping of linear diffusions

Savas Dayanik
(joint work with I. Karatzas)

It is well-known that excessive functions of one-dimensional Brownian motion coincide with the collection of concave functions on the real line. In this work, we show that excessive functions for general one-dimensional regular diffusions admit similarly an equivalent concave characterization in some generalized sense. This observation leads us to a new methodology to calculate *directly* the value function of optimal stopping problems for aforementioned processes: the new approach does not require an a-priori decomposition of the state-space into stopping and continuation regions. The intrinsic properties of concave functions also let us draw deeper conclusions about the regularity of the value function and the smooth-fit principle in general.

We also mention an interesting connection of our results to *Martin boundary theory* of Markov processes. We point out some possible generalizations of our results to higher dimensional diffusions processes, and suggest some open problems.

Monetary utility functions and financial markets

Freddy Delbaen

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will denote by L^∞ the set of bounded random variables. The predual is denoted by $L^1(\Omega, \mathcal{F}, \mathbb{P})$. The dual space of L^∞ is $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$. It is the space of bounded finitely additive measures μ such that $\mathbb{P}[A] = 0$ implies $\mu(A) = 0$. The utility function $u : L^\infty \to \mathbb{R}$ is called coherent if it satisfies the following properties (1) $u(X) \geq 0$ if $X \geq 0$, (2) $u(X + Y) \geq u(X) + u(Y)$, (3) for $\lambda \in \mathbb{R}, \lambda \geq 0$ we have $u(\lambda X) = \lambda u(X)$, and (4) for $\alpha \in \mathbb{R}$ we have $u(X + \alpha) = u(X) + \alpha$, this means that u is monetary. If $u \colon L^\infty \to \mathbb{R}$ is a coherent utility function, then the set, called the acceptance cone, $\mathcal{A} = \{X \mid X \in L^\infty, u(X) \geq 0\}$ is a convex norm-closed cone that contains L^∞_+. The utility function satisfies $u(X) = \max\{\alpha \in \mathbb{R} \mid X - \alpha \in \mathcal{A}\}$. Conversely if $\mathcal{A}$ is a convex norm-closed cone containing the set of nonnegative bounded random variables, then $u(X) = \max\{\alpha \mid X - \alpha \in \mathcal{A}\}$ defines a coherent utility function. With each coherent utility function u we can associate a convex $\sigma(\mathbf{ba}, L^\infty)$ closed convex set, $\mathcal{P}^{\mathbf{ba}}$ of normalised finitely additive, nonnegative measures (also called finitely additive probability measures), such that $u(X) = \inf\{\mu(X) \mid \mu \in \mathcal{P}^{\mathbf{ba}}\}$. Conversely a set of finitely additive probability measures $\mathcal{P}^{\mathbf{ba}}$ defines via the relation $u(X) = \inf\{\mu(X) \mid \mu \in \mathcal{P}^{\mathbf{ba}}\}$ a coherent utility function. To make things more constructive, we add a continuity axiom. We say that the coherent utility function $u \colon L^\infty \to \mathbb{R}$ satisfies the Fatou property if, given a sequence $(X_n)_{n\geq 1}$, such that $\|X_n\|_\infty \leq 1$, then $X_n \xrightarrow{\mathbb{P}} X$ implies $u(X) \geq \limsup u(X_n)$ If u satisfies the Fatou property, then $\mathcal{A}$ is closed for the weak* topology $\sigma(L^\infty, L^1)$ and conversely. Of course we must have that $\mathcal{P} = \mathcal{P}^{\mathbf{ba}} \cap L^1$ is $\sigma(\mathbf{ba}, L^\infty)$ dense in $\mathcal{P}^{\mathbf{ba}}$.

With the obvious notation, if u_1 and u_2 are given coherent utility functions, both having the Fatou property and with their corresponding sets: $\mathcal{A}_1$, $\mathcal{P}_1$, $\mathcal{P}_1^{\mathbf{ba}}$, $\mathcal{P}_1 = \mathcal{P}_1^{\mathbf{ba}} \cap L^1$ and $\mathcal{A}_2$, $\mathcal{P}_2$, $\mathcal{P}_1^{\mathbf{ba}}$, $\mathcal{P}_2 = \mathcal{P}_2^{\mathbf{ba}} \cap L^2$, we can construct other utility functions by taking $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$, $\mathcal{P}_0^{\mathbf{ba}} = \mathcal{P}_1^{\mathbf{ba}} \cap \mathcal{P}_2^{\mathbf{ba}}$ or by taking $\mathcal{A} = \overline{conv\,(\mathcal{A}_1, \mathcal{A}_2)}^{\sigma(L^\infty, L^1)} = \overline{\mathcal{A}_1 + \mathcal{A}_2}^{\sigma(L^\infty, L^1)}$ or even $\mathcal{A}_0 = \overline{conv\,(\mathcal{A}_1, \mathcal{A}_2)}^{\|\ \|_\infty} = \overline{\mathcal{A}_1 + \mathcal{A}_2}^{\|\ \|_\infty}$. The closure is either taken in the norm topology or in the weak* topology $\sigma(L^\infty, L^1)$. If we take the closure in the norm topology we only get a coherent utility function. If we take the closure in the weak* topology we get a utility function with the Fatou property. Obviously $\mathcal{A}$ and $\mathcal{P}$ correspond. Its coherent utility function is denoted by u and it satisfies the Fatou property. In the same way we state: $\mathcal{A}_0$ and $\mathcal{P}_0^{\mathbf{ba}}$ correspond. Its coherent utility function is denoted by u_0. It is the smallest coherent utility function greater than u_1 and u_2. It is usually denoted as $u_1 \Box u_2$ and it is called the convex convolution of u_1 and u_2. $u_1 \Box u_2$ has the Fatou property if and only if (the bar indicates $\sigma(\mathbf{ba}, L^\infty)$ closure): $\overline{\mathcal{P}_1^{\mathbf{ba}} \cap \mathcal{P}_2^{\mathbf{ba}} \cap L^1} = \mathcal{P}_1^{\mathbf{ba}} \cap \mathcal{P}_2^{\mathbf{ba}}$. This is equivalent to: $\overline{\mathcal{P}_1 \cap \mathcal{P}_2} = \mathcal{P}_1^{\mathbf{ba}} \cap \mathcal{P}_2^{\mathbf{ba}}$, where again, the bar indicates $\sigma(\mathbf{ba}, L^\infty)$ closure.

We now give the relation with arbitrage theory. We follow the notation of Delbaen and Schachermayer [3]. So let $(\Omega, (\mathcal{F}_t)_{0 \leq t}, \mathbb{P})$ be a filtered probability space, satisfying the usual assumptions, and let $S : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a càdlàg locally bounded, adapted process. We suppose that the set $\mathbb{M}^e = \{\mathbb{Q} \mid \mathbb{Q}$ probability $\mathbb{Q} \sim \mathbb{P}$, S is a $\mathbb{Q}$ local-martingale$\}$ is non-empty. This is equivalent to the condition "NFLVR". Since S is locally bounded, the closure of the set $\mathbb{M}^e$ is the closed convex set of absolutely continuous local martingale measures for S. The space $W = \{(H \cdot S)_\infty \mid H \cdot S$ bounded$\}$. is a weak* closed subspace of L^∞. The analysis in [3] also shows that the set $\mathcal{A} = \{f + h \mid f \in W, h \geq 0\}$ is a weak* closed convex cone and that $f \in \mathcal{A}$ if and only if for all $\mathbb{Q} \in \mathbb{M}^a$ we have $\mathbb{E}_\mathbb{Q}[f] \geq 0$. In other words $m(X) = \inf\{\mathbb{E}_\mathbb{Q}[X] \mid \mathbb{Q} \in \mathbb{M}^a\}$.

Now we will use two coherent utility functions. One is defined through a convex closed set of probability measures $\mathcal{P}$ and is denoted by u. The other one is defined by the set $\mathbb{M}^a$ of absolutely continuous risk neutral measures of a locally bounded d-dimensional price process S. The economic agent is interested in the quantity $\tilde{u}(X) = \sup\{u(X + Y) \mid Y \in W\} = u_0(X) = u \Box m(X)$. It has the Fatou property as soon as $\mathcal{P}$ is weakly compact. In general this is not true as we now describe. There are two independent Brownian motions describing the source of uncertainty. The filtration is the natural filtration coming from $B = (B^1, B^2)$, where B is a standard 2-dimensional Brownian motion. The time interval is restricted to $[0, 1]$ and the measure $\mathbb{P}$ is risk neutral (this to simplify notation). The movement of $S = (S^1, S^2)$ is given by $dS_t^1 = dB_t^1$ and $dS_t^2 = dB_t^1 + \epsilon_t\, dB_t^2$, where ϵ is a deterministic function, rapidly decreasing to zero as $t \to 1$, e.g. $\epsilon_t = \exp(-\frac{1}{1-t})$. We denote by $\mathbb{M}_1^a$ and $\mathbb{M}_2^a$ the absolutely continuous probability measures that turn resp. S^1 and S^2 into a local martingale. The utility functions are denoted by resp. m_1 and m_2. Both have the Fatou property. The closures of the sets $\mathbb{M}_1^a$ and $\mathbb{M}_2^a$ in **ba** are denoted by resp. $\mathcal{P}_1^{\mathbf{ba}}$ and $\mathcal{P}_2^{\mathbf{ba}}$. We can show that $\mathbb{M}_1^a \cap \mathbb{M}_2^a = \{\mathbb{P}\}$ but that $\mathcal{P}_1^{\mathbf{ba}} \cap \mathcal{P}_2^{\mathbf{ba}} \neq \{\mathbb{P}\}$. This means that $m_1 \Box m_2$ does not have the Fatou property.

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Local time-space stochastic calculus for reversible semi-martingales

Nathalie Eisenbaum

Consider a semimartingale $(X_t, t \geq 0)$ and a deterministic function $F(x,t)$ from $R \times [0,\infty)$ to R. To develop $F(X_t,t)$ according the classical Itô formula, we would need to assume that F is in C^2. Here we are looking for minimum assumptions on F to develop $F(X_t,t)$.

We start by assuming that F admits Radon-Nicodym derivatives with respect to x and t. Under the required integrability conditions on $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial t}$, we know the existence of the expression

$$F(X_t,t) - F(X_0,0) - \int_0^t \frac{\partial F}{\partial t}(X_s,s)ds - \int_0^t \frac{\partial F}{\partial x}(X_s,s)dX_s.$$

Now assume that the semi-martingale X satisfies the two following conditions:

(1) the process $(X_{(1-t)-}, 0 \leq t < 1)$ is a semi-martingale too

(2) $\sum_{0 \leq t \leq 1} |\Delta X_t| < \infty$

we can then construct a stochastic integration of deterministic functions with respect to $(L_t^x, x \in R, 0 \leq t \leq 1)$ the local time process of X and show that for $t \leq 1$ the above expression actually coincides with

$$-\frac{1}{2}\int_0^t \int_R \frac{\partial F}{\partial x}(x,s)dL_s^x.$$

That way we obtain an Itô formula from which many known Itô formulas (such as Bouleau and Yor's formula, Föllmer Protter and Shiryaev's formula, Peskir's formula, Russo and Valois formula, ...) can be derived at least in the case of a Lévy process satisfying condition (2).

In view of the recent works of Elworthy, Truman and Zhao, and Ghomrasni and Peskir, it seems natural to conjecture that this generalized Itô formula involving local time-space stochastic integrals should remain true without the two conditions (1) and (2).

A useful extension to Itô's formula with applications to optimal stopping

Markus Jaeger

Given a continuous semimartingale $M = (M_t)_{t\geq 0}$ and a continuous process of locally bounded variation $V = (V_t)_{t\geq 0}$, the Itô formula states that

$$f(M_t, V_t) - f(M_0, V_0) = \int_0^t \frac{\partial}{\partial x} f(M_s, V_s)\, dM_s + \int_0^t \frac{\partial}{\partial y} f(M_s, V_s)\, dV_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(M_s, V_s)\, d\langle M\rangle_s$$

for any function $f \in \mathcal{C}^{2,1}(\mathbb{R}^2)$. This formula is very useful when solving various optimal stopping problems arising, for instance, from Mathematical Finance, but typically with a function f whose first partial derivative is only absolutely continuous with respect to the first component. We prove that this formula remains true for such functions and give an application by the following example.

Let $S = (S_t)_{t\geq 0}$ be a price process of an asset given by $S_t = e^{X_t}$, $t \geq 0$, where $X = (X_t)_{t\geq 0}$ is a spectrally negative Lévy process. Consider the stopping problem

$$v^*(x) = \sup_\tau E_x e^{-r\tau} \max\{K, S_\tau\},$$

$r > 0$, $K > 0$. The supremum is taken over all a.s. finite stopping times τ. This problem was considered in [3] where $(S_t)_{t\geq 0}$ is a geometric Brownian motion. We obtain the optimal stopping strategy and determine an explicit formula for v^* by combining the extended Itô formula with some properties of spectrally negative Lévy processes which can be found in [2].

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Some remarks on first passage of Lévy processes, the American put and pasting principles

A. E. Kyprianou
(joint work with L. Alili)

Let $X = \{X_t : t \geq 0\}$ be a Lévy process defined on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions. For $x \in \mathbb{R}$ denote by $\mathbb{P}_x(\cdot)$ the law of X when it is started at x and write simply $\mathbb{P}_0 = \mathbb{P}$. We denote its Lévy-Khintchine exponent by ψ i.e. $\mathbb{E}[e^{i\theta X_1}] = \exp\{-\Psi(\theta)\}$, $\theta \in \mathbb{R}$. Now, consider the following optimal stopping problem

$$v(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_x \left[e^{-r\tau}(K - e^{X_\tau})^+ \right] \tag{1}$$

where $K > 0, r \geq 0$ and $\mathcal{T}_{0,\infty}$ is the family of stopping times with respect to $\{\mathcal{F}_t : t \geq 0\}$. Establishing the optimal value and strategy in (1) is closely related to the pricing and exercise strategy of an American put option in an incomplete Black-Scholes type markets driven by Lévy processes (see Schoutens (2003) or Cont and Tankov (2004)). For this reason we refer to (1) as the *American put optimal stopping problem.*

In a number of numerical simulations and theoretical calculations for specific choices of Lévy processes, various authors have found that the American put optimal stopping problem is solved in the same way as for the case that X is a scaled Brownian motion with drift (the Black-Scholes market). Namely by a strategy of the form

$$\tau^* = \inf\{t \geq 0 : X_t < x^*\}$$

for a specific value $x^* < \log K$ so that

$$v(x) = K\mathbb{E}_x \left[e^{-r\tau^*} \right] - \mathbb{E}_x \left[e^{-r\tau^* + X_{\tau^*}} \right]$$

thus linking the American perpetual put optimal stopping problem to the first passage problem of a Lévy process. See Gerber and Shiu (1994), Chan (1999, 2003), Mordecki (1999, 2002), Avram et al. (2002), Asmussen et al. (2003), Boyarchenko and Levendorskii (2002), Hirsa and Madan (2002), Matache et al. (2003) and Almendral and Oosterlee (2003). Notably Mordecki (2002) handles the case when X is a general Lévy process.

A simple identity concerning a general first passage time of the form

$$\tau_y^- = \inf\{t \geq 0 : X_t < y\}$$

where $y \in \mathbb{R}$ is as follows. For all $\alpha, \beta \geq 0$ and $x \geq 0$ we have

$$\mathbb{E}\left[e^{-\alpha\tau_{-x}^- + \beta X_{\tau_{-x}^-}} \mathbf{1}_{(\tau_{-x}^- < \infty)} \right] = \frac{\mathbb{E}\left[e^{\beta \underline{X}_{\mathbf{e}_\alpha}} \mathbf{1}_{(-\underline{X}_{\mathbf{e}_\alpha} > x)} \right]}{\mathbb{E}\left[e^{\beta \underline{X}_{\mathbf{e}_\alpha}} \right]}.$$

We show how this identity can be used to show using a principle of variation that the optimal stopping problem (1) is solved in such a way that there is smooth pasting at the optimal threshold x^* if and only if the point 0 is regular for $(-\infty, 0)$ for X.

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Integro–partial differential equations in a market driven by geometric Lévy processes: theoretical aspects and numerical approximation

Claudia La Chioma

The aim of this lecture is to present some analytical and numerical results, recently obtained in the framework of my Ph.D. thesis [9] which concerns viscosity solutions to integro-differential problems arising in Mathematical Finance when derivatives are valuated in a market driven by general jump-diffusion processes.

When analyzing financial data it comes out that the prices process are not continuous, as it is commonly assumed, see [6], but they can jump: in particular jumps become more visible as one samples the path more frequently, making the assumption of high or infinite jump frequencies plausible. This approach is based upon a Lévy modeling of the prices of the asset [11, 12] and gives a better fit to real-life data, see [10]. In view of these considerations, the prices of the stocks are modeled in terms of exponential Lévy models, the choice of a particular Lévy process standing in the choice of its distribution. The discontinuous feature of Lévy processes has important consequences on the description of financial markets for what concerns the assumption of completeness of the market itself, see [5]. Using Ito's calculus [8, 13] we can derive a nonlinear integro–partial differential problem to get the price of a prescribed financial product.

In this lecture we shall discuss a new comparison principle for unbounded semicontinuous viscosity sub- and supersolutions for this kind of equations, in the case of *geometric* Lévy process, see [1, 2, 4]. As a consequence of the "geometric form" of the underlying processes, the comparison principle holds without assigning spatial boundary data. Applications of this result will be presented for: (i) backward stochastic differential equations, (ii) Merton problem, (iii) pricing of European and American derivatives via backward stochastic differential equations.

Despite presenting a great resemblance to real markets, which is appealing for practitioners, this problem is nonlinear and does not have a closed form solution. To overcome this difficulty a useful tool is given by numerical approximations, which makes possible to deal with more complicated nonlinear problems.

Starting from the fundamental result by Barles and Souganidis [3] we shall show convergence for monotone, stable, consistent schemes approximating integro-differential parabolic problems with bounded and unbounded Lévy measures, see [7].

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Pricing American options under Lévy driven assets

A.-M. Matache

In asset pricing, models beyond the classical Black-Scholes (B-S) have been proposed for the stochastic dynamics of the risky asset: we mention only stochastic volatility models and ‘stochastic clocks’. The former lead to multivariate generalizations of the B-S equation with stochastic volatility, whereas the latter lead to so-called jump-diffusion price processes: the Wiener process in the B-S model is replaced by a Lévy process (see e.g. [2, 1]). Originally, jumps in risky assets’ log-returns have been modeled as finite intensity processes, i.e in any finite time interval only a finite number of large jumps occur. In the early 90ies, however, processes with infinite jump intensity and no diffusion component have been proposed as models for the log-returns. We mention here the Variance Gamma (VG) and the extended Koponen family (also referred to as KoBoL [1], temperated or truncated tempered stable processes and later used in [2] under the name of CGMY-model). All these processes, together with the B-S model, are Markov processes of Lévy type, or Lévy processes for short. Since their introduction, empirical evidence for their superiority over B-S in modelling observed returns has been gathered (e.g. [2]).

For pricing European Vanilla contracts on assets with Lévy price processes, the translation invariance of the process' infinitesimal generator implied by stationarity and explicitly available characteristic functions allow to apply Fourier-Laplace transformations for the numerical pricing. For American style contracts on assets with Lévy price processes, the analytical tool of Wiener-Hopf factorization allows, at least for infinite horizon problems, to derive semi-analytical solutions. These approximate, analytical methods, however, are not directly applicable to time-dependent or local volatility models where stationarity and, hence, translation invariance are absent.

Here we discuss the analysis and implementation of fast, convergent deterministic pricing schemes for American style contracts on assets driven by a class of Markov processes which contains, in particular, Lévy processes. Our approach presented in [3] is based on a multilevel finite element solution of the parabolic variational inequality formally associated with the optimal stopping problem for these processes. This inequality involves the Dynkin operator of the semigroup generated by the price process which, for the processes under consideration, is an integro-differential operator with possibly nonintegrable kernel stemming from the process' jump measure.

We discretize the variational integro-differential inequality by 'Canadization', or backward Euler, in time and by a piecewise linear, continuous wavelet Finite Element basis in the (logarithmic) price variable. This basis has two advantages: i) it allows to 'compress' the dense and ill-conditioned moment matrices due to the nonlocal infinitesimal generator of the process to sparse, well-conditioned ones while not affecting the accuracy of the computed prices and ii) the wavelet basis allows to precondition the iterative solver for the associated Linear Complementarity Problems (LCPs) in each time step.

The resulting algorithm allows the deterministic pricing of American style contracts on assets for which the log price process is a general Markovian jump-diffusion price process that may exhibit infinite jump activity and has possibly nonstationary increments. It moreover allows for general, non-monotonic and non-smooth pay-off functions, in particular for 'butterfly' and compound options with an American style early exercise feature.

Our approach does not impose any pasting condition a-priori and, indeed, our numerical experiments demonstrate failure of the smooth pasting condition for certain pure jump processes of bounded variation.

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The uniqueness question

Carl Christian Kjelgaard Mikkelsen

Settling the uniqueness question for a particular non-linear equation can be as troublesome as proving the existence of a solution. The aim of this note is demonstrate that the uniqueness question can occasionally be settled by studying a *linear* equation which can be derived from the original one.

We give three examples. Consider first the ordinary differential equation

$$x'(t) = f(x(t)), \tag{1}$$

where $f \in C^1(I, \mathbb{R})$. Suppose that $x, y : J \to I$ are solutions to this equation with the same initial condition: $x(t_0) = y(t_0), t_0 \in J$. Then $z = x - y$ satisfies

$$z'(t) = g(t)z(t) \tag{2}$$

where

$$g(t) = \begin{cases} \frac{f(x(t))-f(y(t))}{x(t)-y(t)} & \text{for } t \in J : x(t) \neq y(t) \\ f'(x(t)) & \text{for } t \in J : x(t) = y(t) \end{cases}$$

Since f is continuous differentiable and x, y are continuous, g is continuous and

$$z(t) = \exp\left(\int_{t_0}^{t} g(s)ds\right) z(t_0)$$

But $z(t_0) = 0$ so $x(t) = y(t)$ for all $t \in J$.

Next consider the non-linear heat equation

$$u_t = (a(u)u_x)_x = a(u)u_{xx} + a'(u)u_x^2 \tag{3}$$

where $a \in C^2(I, \mathbb{R})$,. Assume that $u, v : \Omega \to [\alpha, \beta] \subset I$ are two solutions then it can be shown that $w = u - v$ satisfies the equation

$$w_t = a(u)w_{xx} + a'(u)(u_x + v_x)w_x + \left(P(x,t)v_{xx} + Q(x,t)v_x^2\right) w \tag{4}$$

where the key functions $P, Q : \Omega \to \mathbb{R}$ are defined as

$$P(x,t) = \begin{cases} \frac{a(u)-a(v)}{u-v} & \forall (x,t) \ : \ u(x,t) \neq v(x,t) \\ a'(u) & \forall (x,t) \ : \ u(x,t) = v(x,t) \end{cases}$$

and

$$Q(x,t) = \begin{cases} \frac{a'(u)-a'(v)}{u-v} & \forall (x,t) \ : \ u(x,t) \neq v(x,t) \\ a''(u) & \forall (x,t) \ : \ u(x,t) = v(x,t) \end{cases}$$

Now it possible to derive an uniqueness theorem for the non-linear heat equation. It is a matter of finding a space of functions V that is so small that the linear equation (4) has at most one solution w for all pairs $u, v \in V$.

Finally consider the optimal stopping boundary $b : [0, T] \to \mathbb{R}$ for the American Put Option. It has been shown by various authors that $\gamma(t) = K - b(t)$ satisfies an integral equation of the form

$$\gamma(t) = \int_0^K F(\gamma(t), t, z)dz + \int_0^{T-t} G(\gamma(t), \gamma(t+u), u)du \tag{5}$$

for a specific pair functions F and G. If F, G did *not* have singularities, but were smooth and bounded with bounded derivatives we could proceed as before, letting u, v be two solutions and defining $w = u - v$. Then w would satisfy the equation

$$\left(\int_0^K f(t,z)dz - \int_0^{T-t} g_1(t,u)du\right) w(t) = \int_0^{T-t} g_2(t,u)w(t+u)du \tag{6}$$

for appropriately chosen f, g_1, g_2. It is now possible to place conditions on f, g_1, and g_2 that will guarantee that $w \equiv 0$, at least in some small interval $[T - \delta, T]$. My efforts to extend this procedure to the full problem have not been successful.

Integration with respect to local time and self-intersection local time of a one-dimensional Brownian motion

Joseph Najnudel

If we denote by L_t^a the local time at a of B, a one-dimensional Brownian motion on $[0, t]$, it is possible to give a meaning to $\int f(a)d_a L_t^a$ for any locally square-integrable function f.

During the workshop, we proved that it is possible to do approximatively the same thing with the self-intersection local time. For any continuous function h, the following equality holds:

$$\int_0^t \int_0^u h(B_s - B_u)dsdu = \int h(a)\alpha_t^a da$$

which defines the self-intersection local time of B. We can prove that $a \to \alpha_t^a + 2ta^-$ is derivable, with derivative denoted by β_t^a. Moreover, we have, for any step function f (with the natural definition of integration with respect to $d_a\beta_t^a$) :

$$\int f(a)d_a\beta_t^a = 2\int_0^t f(B_s - B_t)ds + 4\int_0^t \left[\int_0^{-X_u^{(u)}} f + \int_0^u f(-X_s^{(u)})dX_s^{(u)}\right] dB_u$$

where $X_s^{(u)} = B_u - B_{u-s}$ ($X^{(u)}$ is a Brownian motion).

This formula can be used to define $\int f(a)d_a\beta_t^a$ for any locally square-integrable function f. If $f(0)$ is well-defined, it is also possible to give some meaning to the expression $\int_0^t du \int_0^u dsg(B_s - B_u)$, where g is the second derivative of f in the sense of the distributions.

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On local time-space calculus and applications

Goran Peskir

Recent progress in free-boundary problems of optimal stopping (cf. [4] and [5]) has been based on the following extension of Itô's formula (cf. [3]):

$$F(t,X_t) = F(0,X_0) + \int_0^t F_t(s,X_s\pm)\,ds + \int_0^t F_x(s,X_s\pm)\,dX_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s\pm)\,d\langle X,X\rangle_s + \frac{1}{2}\int_0^t \big(F_x(s,b(s)+) - F_x(s,b(s)-)\big)\,d\ell_s^{b\mp}(X) \tag{1}$$

where $X = (X_t)_{t\geq 0}$ is a continuous semimartingale and $F = F(t,x)$ is a continuous function that is smooth off the given curve $b = b(t)$. (The extension to general semimartingales X and continuous functions F that are smooth off the given surface b is given in [6].)

The formula (1) can be obtained by formal manipulations of the $d\ell_t^x$ integral in the general formula:

$$F(t,X_t) = F(0,X_0) + \int_0^t D_tF(s,X_s)\,ds + \int_0^t D_xF(s,X_s)\,dX_s - \frac{1}{2}\int_0^t \int_{\mathbf{R}} D_x(s,x)\,d\ell_s^x \tag{2}$$

which is established by Eisenbaum [1] in the case when X is a standard Brownian motion. Similar formulae are derived independently by Elworthy, Truman and Zhao in [2].

The talk reviews the relevant history and explains the need for further development of the 'local time-space calculus'.

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A problem of sequential optimal stopping times

Huyên Pham

We consider a stochastic system that may operate in different modes or regimes. In a given mode, it yields a stream of profit depending on the state of the system. The regimes can be switched at a sequence of stopping times decided by the operator (individual, firm, ...). The problem is to find the optimal switching strategies that maximize the expected net profit over time. This is formulated as a sequential optimal stopping times problem and studied by dynamic programming principle and viscosity solutions theory. The resulting dynamic programming equations involve embedded variational inequalities on the value functions. We prove smooth-fit pasting conditions on the value functions and analyse carefully the regime switching regions. Some explicit solutions are provided.

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Calculus via regularization, weak Dirichlet processes and time-inhomogeneous stochastic differential equations

Francesco Russo

The first part of the talk consisted in the introduction of the 'so called *stochastic calculus via regularization* started by F. Russo and P. Vallois in 1993, see [6]. Given two processes X and Y, there was defined a forward stochastic integral $\int_0^t Y d^- X$ and a covariation $[X,Y]$ by means of regularization procedures. This notion of integral extends the notion of Itô integral in the non-causal classical case and when the processes are semimartingales, $[X,Y]$ is the classical bracket.

Later, further developments concerning different types of Itô formula for C^1-functions f of a semimartingale S have been stated, see [7] for the continuous case and [2] for the jump case. Clearly $f(S)$ is not generally a semimartingale but only a *Dirichlet process*, meant as the sum of a local martingale M plus a zero quadratic variation process A, i.e. $[A,A] = 0$.

A useful generalization of Dirichlet processes was defined in [1]. A process X is called *weak Dirichlet process* if it is the sum of a local martingale plus a process A such that $[A, N] = 0$ for any local martingale related to the underlying filtration. Of course a Dirichlet process is also a weak Dirichlet process. However, the converse is not true: typical examples for this are the following.

- A Volterra type process given by $X_t = \int_0^t G(t,s)dM_s$, where M is a local martingale and G is a continuous kernel, is a weak Dirichlet process.
- Given a continuous function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, and semimartingale S, then $X_t = u(t, S_t)$ defines a weak Dirichlet process.

The third part of the talk is devoted to present a suitable framework for studying time-inhomogeneous stochastic differential equations where a distributional drift in space appears. In some cases, the drift is even allowed to be a Radon measure in time. Examples arise from irregular medium equations and Bessel processes. Solutions are neither semimartingales nor Dirichlet processes but only weak Dirichlet processes. When the coefficients are time-homogeneous, the solutions are true Dirichlet processes and the complete study was done in [3, 4].

This presentation concerns a joint work with G. Trutnau, [8]. Recent papers can be dowloaded from http://zeus.math.univ-paris13.fr/∼russo/

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On one effective case of solution of an optimal stopping problem for random walk

Albert N. Shiryaev

(joint work with A. A. Novikov)

We consider a class of the following optimal stopping problems:

To find the value functions

$$V^*(x) = \sup_\tau \mathbf{E}\left[g(X_\tau)I(\tau < \infty)\right]$$

and optimal stopping times for functions $g(x) = (x^+)^n$, $n \geq 1$, where $X_k = x + \xi_1 + \cdots + \xi_k$ and $(\xi_1, \xi_2, \ldots)$ are i.i.d. random variables with $\mathbf{E}\xi_1 < 0$.

The main result: *if* $\mathbf{E}(\xi^+)^{n+1} < \infty$, *then the stopping time*

$$\tau_n^* = \inf\{k \geq 0 : X_k \geq a_n^*\}$$

is optimal, a_n^* *is the maximal root of the equation* $Q_n(y) = 0$, *where* $Q_n(y)$ *is Appell's polynomial defined from decomposition*

$$\frac{e^{uy}}{\mathbf{E}\, e^{u\mu}} = \sum_{k=0}^{\infty} \frac{u^k}{k!}\, Q_k(y)$$

with $\mu = \max_{k\geq 0} S_k$, $S_0 = 0$, $S_k = \xi_1 + \cdots + \xi_k$.

For case $n = 1$ the corresponding result with $a_1^* = \mathbf{E}\mu$ was obtained by Darling, Liggett and Taylor in [1]. We give a new proof which works for any $n \geq 1$, also for gain functions $g(x)$ of type $1 - e^{-x^+}$, $(e^x - 1)^+$. For case $n = 1$ $\quad Q_1(y) = y - \mathbf{E}\mu$ what explains the answer $a_1^* = \mathbf{E}\mu$ for this case. For $n = 2, 3, \ldots$

$$Q_2(y) = (y - x_1)^2 - \kappa_2,$$

$$Q_3(y) = (y - x_1)^3 - 3\kappa_2(y - \kappa_1) - \kappa_3, \ldots$$

where $\kappa_1, \kappa_2, \ldots$ are semiinvariants of the random variable μ. Generally,

$$\frac{d}{dy}\, Q_k(y) = kQ_{k-1}(y), \qquad k \leq n,$$

if $\mathbf{E}|\mu|^n < \infty$ and $Q_0(y) = 1$.

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On stochastic heat and Burgers equations

Aubrey Truman

The inviscid limit of Burgers equation, with body forces white noise in time, is discusses in terms of the level surfaces of the minimising Hamilton-Jacobi function and the classical mechanical caustic. Prelevel surfaces and precaustics are introduced by using the classical mechanical flow map. When the prelevel surface touches the precaustic, the geometry (number of cusps) on the level surface changes infinitely rapidly causing what we call 'real turbulence' (Davies, Truman, Zhao). Using an idea of Felix Klein, it is shown that the geometry (number of swallow tails) on the caustic allow changes infinitely rapidly when the real part of the precaustic touches its complex counterpart at a limit point of a complex sequence mapped into the real configuration space by classical flow map. This we refer to as 'complex turbulence'. These two new kinds of turbulence are inherently stochastic in nature. A complex analysis of this problem is given in terms of a reduced (one dimensional) action function. This characterizes which parts of the original caustic are singular (cool) – an old problem in applied mathematics relevant for our 'elementary formula' with Elworthy and Zhao. It also determines when this turbulence is intermittent in terms of recurrence or transience of two stochastic processes. Infinitely many examples are given where recurrence and intermittence follow from Strassen's law.

Limiting laws associated with Brownian motion perturbed by weights involving the local time process

Pierre Vallois

(joint work with B. Roynette and M. Yor)

Let $(\Omega, (\mathcal{F}_t), (X_t))$ be the canonical probability space equipped with the Wiener measure P_0. Let $(L_t^0; t \geq 0)$ be the local time process at 0, $\varphi : [0, \infty[\mapsto]0, \infty[$ such that $\int_0^\infty \varphi(x)dx = 1$.

We prove that the family of p.m. $Q_{0,t}^\varphi$ on Ω, weakly converges as $t \to \infty$ to Q_0^M, where:

$$Q_{0,t}^\varphi(A) = \frac{E_0[1_A \varphi(L_t^0)]}{E_0[\varphi(L_t^0)]}, \quad Q_0^M(A) = E_0[1_A(|X_t|\varphi(L_t^0) + 1 - \Phi(L_t^0))], \ \forall A \in \mathcal{F}_t,$$

and $\Phi(l) = \int_0^l \varphi(x)dx$.

We are able to describe the law of (X_t) under Q_0^M. More precisely, $Q_0^M(L_\infty^0 < \infty) = 1$, $Q_0^M(0 < g < \infty) = 1$, with $g := \sup\{t \geq 0; X_t = 0\}$. Moreover:

- the processes $(X_t; t \leq g)$ and $(X_{t+g}; t \geq 0)$ are independent,
- $(|X_{t+g}|; t \geq 0)$ is distributed as a three dimensional Bessel process, started at 0,
- conditionally to $L_\infty^0 = l$, $(X_t; t \leq g)$ is distributed as a Brownian motion started at 0, and stopped when its local time at 0 equals l.

On the Itô-Tanaka formula for strictly local martingales: does it need a correction term?

Marc Yor

(joint work with D. Madan)

D. Madan suggested recently that the price of an European option, for $(S_t\,,\, t \geq 0)$ the price process which is only assumed to be a strictly local martingale (i.e. a local martingale which is not a martingale) should be modified as follows: instead of the customary $C(T; K) = E[(S_T - K)^+]$ take

(1) $$C^*(T; K) \stackrel{\text{def}}{=} \lim_{n\to\infty} E[(S_{T\wedge T_n} - K)^+]$$

where $(T_n\,,\,n\to\infty)$ is a reducing sequence of the local martingale.

In my lecture, I showed that: (a) (1) is well defined; (b) there is in general a correction term $c_S(\sigma)=E[S_0-S_\sigma]$ (σ is any stopping time) such that $E[(S_\sigma-K)^+]=(S_0-K)^+ + \frac{1}{2}E[L_\sigma^K]-c_S(\sigma)$. Integrating with respect to K, one gets a "corrected" integrated Itô-Tanaka formula.

Generalized Itô formulae using local time and applications in analysing asymptotics of heat equations in the presence of caustics

Huaizhong Zhao

Local time is a very useful tool in analysing partial differential equations, especially when some singularities appear. I first presented the following generalized Itô's formula

$$\begin{aligned} f(t,X(t)) &= f(0,z)+\int_0^t \frac{\partial^-}{\partial s}f(s,X(s))\mathrm{d}s+\int_0^t \nabla^- f(s,X(s))\mathrm{d}X_s \\ &\quad +\int_{-\infty}^{\infty}\wedge_t(x)\mathrm{d}_x\nabla^- f(t,x)-\int_{-\infty}^{+\infty}\int_0^t \wedge_s(x)\mathrm{d}_{s,x}\nabla^- f(s,x)\quad a.s. \end{aligned}$$

Here $f(t,x)$ is a continuous function such that its left derivatives $\frac{\partial^-}{\partial t}f(t,x)$ and $\nabla^- f(t,x)$ exist for all (t,x) and $\nabla^- f(t,x)$ is of locally bounded variation in (t,x), $X(t)$ is a continuous semimartingale with $X(0)=z$, $\wedge_t(x)$ is its local time and $\mathrm{d}_{s,x}\nabla^- f(s,x)$ is the two-dimensional Lebesgue-Stieltjes measure. When there exists a continuous curve $x=l(t)$ of locally bounded variation such that for each $t\geq 0$, $f(t,x)$ is C^1 in t and C^2 in x for $x\in(-\infty,l(t))$ and $x\in(l(t),\infty)$,and and $\Delta^- f$ is well defined, then

$$\begin{aligned} f(t,X(t)) &= f(0,z)+\int_0^t \frac{\partial^-}{\partial s}f(s,X(s))\mathrm{d}s+\int_0^t \nabla^- f(s,X(s))\mathrm{d}X_s \\ &\quad +\frac{1}{2}\int_0^t \Delta^- f(s,X(s))\mathrm{d}<X,X>_s \\ &\quad +\int_0^t (\nabla^+ f(s,l(s))-\nabla^- f(s,l(s)-))\mathrm{d}_s\wedge_s^*(0)\quad a.s. \end{aligned}$$

The latter formula was also observed by Peskir (2003) independently in studying the smooth fitting problem in American put option. We then established the stochastic elementary formula and asymptotics of heat equation in the presence of caustic. The asymptotics was obtained by estimating $\mathrm{E}e^{\frac{1}{\epsilon^2}\int_0^t\int_{-\infty}^{\infty}\wedge_s^\epsilon(a)\mathrm{d}_{s,a}\nabla S(t-s,a)}$ when $\epsilon\to 0$ using the law of local times, where $\wedge_s^\epsilon(a)$ is the local time of diffusion $dX_s=\epsilon dB_s-\nabla S(t-s,X_s)ds$. Here $S(t,x)$ is the Hamilton Jacobi function which has a jump gradient due to appearance of caustics. This problem had remained open since the work of Elworthy and Truman (1982) under a no-caustics assumption. This formula and asymptotics has now gone beyond large deviation theory which gives first term in the asymptotics.

The generalized Itô formulae have now been extended to two-dimensions. This is not a trivial extension and the stochastic Lebesgue-Stieltjes integral

$$\int_{-\infty}^{\infty}\int_0^t f(s,a)d_{s,a}h(s,a)$$

has been defined. Here $s \mapsto h(s,a)$ is a continuous martingale and $< h(a), h(b) >_s$ is of locally bounded variation in (a,b).

This talk is based on the articles [1] and [2] below.

References

[1] K.D. Elworthy, A. Truman and H.Z. Zhao, *A generalized Itô formula and asymptotics of heat equations with caustics in one-dimension*, Preprint, 2003.

[2] C.R. Feng and H.Z. Zhao, *A generalized Itô formula in two-dimensions and stochastic Lebesgue-Stieltjes integrals*, Preprint, 2004.

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