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Mini-Workshop: Compactness Problems in Interpolation Theory and Function Spaces

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Introduction by the Organisers

Compactness is undoubtedly one of the central and most relevant notions in mathematics. The present mini-workshop was centered around some important compactness problems connected to interpolation theory and to the theory of function spaces, two very closely related areas. Our aim was to bring together leading experts and active younger researchers in these fields. The mini-workshop was attended by 16 participants from Germany (5), Spain (5), Israel (2), Poland (2), Sweden (1) and United States (1), and the main problems we dealt with can, more specifically, be grouped as follows.

1. Interpolation of compactness and related properties
   1.1. Complex interpolation of compact operators
   1.2. Real interpolation of compactness and similar properties
2. Compact embeddings in function spaces
   2.1. Entropy numbers of such embeddings and applications to spectral theory of differential operators
   2.2. Entropy techniques in sequence spaces

Let us shortly describe these topics.

1.1. An outstanding problem in interpolation theory is the question, whether the complex interpolation method preserves compactness of operators. This is open since 40 years, and by now only partial answers are known. Recently a new and promising general approach has been proposed. We discussed this approach.

1.2. For the real interpolation method, however, it is well-known that it does preserve compactness. So it is quite natural to ask for *quantitative* versions of this purely *qualitative* result, for instance in terms of the measure of non-compactness, or in terms of entropy numbers. Another natural question is to study whether similar properties, like weak compactness, for example, are stable under real interpolation as well. These problems were addressed in some talks.

2.1. The sequence of entropy numbers $(e_k(T))_{k=1}^{\infty}$ of a bounded linear operator T between quasi-Banach spaces can be considered as a quantification of compactness, since T is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$. The basis for applications to spectral theory is the famous Carl-Triebel inequality, which relates entropy numbers of Riesz operators to its eigenvalues. Many concrete problems lead to the investigation of compact embeddings of certain function spaces, e.g. Sobolev or Besov spaces. In the talks both a survey on the general framework as well as new entropy estimates for specific embeddings were given.

2.2. Using various methods, for instance wavelet or atomic decompositions, the function space embeddings can very often be reduced to embeddings of (fairly complicated) sequence spaces. For the estimation of their entropy numbers one needs many different techniques, some of them quite new. Such techniques also were the subject of talks.

Finally, several other aspects of interpolation were treated in talks, e.g. approximation spaces, bilinear interpolation, relation to eigenvalues and operator ideals. We list the abstracts of all talks in chronological order.

The scientific program started with two survey lectures by Triebel and Cwikel, leading experts in their fields, followed by two more survey-style talks of the organisers. Then all other participants reported on own recent research results. In addition to this "official" program, which was already scheduled in advance, there was a number of further activities. Several participants offered a second talk, on another topic of common interest, or continued their respective talks in order to explain some technicalities in greater detail. The remaining time was used for many intensive discussions in smaller groups, and on Friday a problem session was held. The aim was to summarize the results of the mini-workshop and to discuss and collect several relevant problems, thus pointing out possible directions for further research in our field.

Concerning social activities, one should mention the traditional hiking tour to St. Roman on Wednesday afternoon and the, maybe less traditional, joint session of all three parallel mini-workshops. The aim of this informal interdisciplinary session was to explain very briefly the kind of problems and ideas of our respective areas. Before this meeting there were serious doubts whether the intended goal would be achievable in only a few minutes, but afterwards it was general opinion that we have had a surprisingly inspiring and interesting evening, giving in fact a rough impression of the other two research areas.

Last but not least, the organisers would like to express their gratitude to the director and the authorities of the Mathematisches Forschungsinstitut Oberwolfach for making this mini-workshop possible and for the constant support in its organisation and preparation. On behalf of all participants we thank all members of the staff for creating the unique working atmosphere, which made our stay so pleasant and which contributed substantially to the success of our mini-workshop.

Mini-Workshop: Compactness Problems in Interpolation Theory and Function Spaces

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Abstracts

Compact embeddings in function spaces

Hans Triebel

We outline the symbiotic relationship between

1. spectral theory of compact operators in abstract quasi-Banach spaces,
2. function spaces on $\mathbb{R}^n$ and on bounded domains Ω in $\mathbb{R}^n$,
3. compact embeddings in weighted sequence spaces,
4. representations of function spaces in terms of elementary building blocks.

1. The well-known Riesz theory for compact operators T in complex Banach spaces can be extended to quasi-Banach spaces B and

$$|\lambda_k(T)| \le \left(\prod_{j=1}^{k} |\lambda_j(T)| \right)^{1/k} \le \sqrt{2} e_k(T), \qquad k \in \mathbb{N}, \tag{1}$$

where $\lambda_k(T)$ are the eigenvalues of T (counted to their algebraic multiplicity and ordered by decreasing magnitude) and $e_k(T)$ are the entropy numbers of T. Recall that $e_k(T)$ is the infimum of all $\varepsilon > 0$ such that the image TU_B of the unit ball U_B in B can be covered by 2^{k-1} balls of radius ε. Details, proofs and references may be found in [ET].

2. Let $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ in the F-case), $0 < q \le \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are the two well-known scales of function spaces covering (fractional) Sobolev spaces, classical Besov spaces and Hölder-Zygmund spaces. The spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ for bounded domains Ω in $\mathbb{R}^n$ are defined by restriction of the respective spaces on $\mathbb{R}^n$ to Ω. Then the embedding

$$\mathrm{id}_B \;:\; B^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2 q_2}(\Omega) \tag{2}$$

is compact if, and only if, $s_2 < s_1$ and $s_2 - \frac{n}{p_2} < s_1 - \frac{n}{p_1}$ (similar for the F-spaces) and in this case

$$e_k(\mathrm{id}_B) \sim k^{-\frac{s_1 - s_2}{n}}, \qquad k \in \mathbb{N}. \tag{3}$$

The spectral theory of (regular, singular, fractal) elliptic (pseudo-)differential operators can often be reduced (via suitable factorisations) to (2),(3), and then to (1). As for the theory of function spaces we refer to [T1,T2] and, in connection with (2),(3), to [ET,T3].

3. Let $\delta > 0$, $M_j \sim 2^{jd}$ for some $d > 0$, $0 < p \le \infty$ and

$$\left\| \lambda \,|\ell_p \left(2^{j\delta} \ell_p^{M_j} \right) \right\| = \left(\sum_{j=0}^{\infty} 2^{j\delta p} \sum_{r=1}^{M_j} |\lambda_{jr}|^p \right)^{1/p},$$

$\lambda = \{\lambda_{jr}\} \subset \mathbb{C}$. Let

$$0 < p_1 \leq \infty, \qquad \frac{1}{p*} = \frac{1}{p_1} + \frac{\delta}{d}, \qquad p* < p_2 \leq \infty.$$

Then

$$\mathrm{id}_\ell \ : \quad \ell_{p_1}\left(2^{j\delta}\ell_{p_1}^{M_j}\right) \hookrightarrow \ell_{p_2}\left(\ell_{p_2}^{M_j}\right) \tag{4}$$

is compact and

$$e_k(\mathrm{id}_\ell) \sim k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \qquad k \in \mathbb{N}, \tag{5}$$

[T3,HT], and the references given there.

4. Elements f of $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ can be represented in terms of wavelet bases and wavelet frames, typically of type

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}(\cdot), \tag{6}$$

where a_{jm} are elementary building blocks (for example Daubechies wavelets) related to balls of radius $\sim 2^{-j}$ and centred at $2^{-j}m$, and $\{\lambda_{jm}\}$ are elements of some sequence spaces. When reduced to bounded domains Ω one arrives at sequence spaces as considered in **3**. We refer to [T3-T5,HT].

Then (2),(3) *can be reduced to* (4),(5) *via* (6),
which in turn results in a far-reaching spectral theory via (1).

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Some thoughts about complex interpolation of compact operators

Michael Cwikel
(joint work with Fedor Nazarov)

The following question was initially considered by Alberto Calderón [1] some forty years ago, when he developed his powerful and beautiful theory of complex interpolation spaces.

Question 1: Suppose that A_0 and A_1 are compatible Banach spaces, i.e., they form a Banach pair, and that so are B_0 and B_1. Suppose that $T : A_0 + A_1 \to B_0 + B_1$ is a linear operator such that $T : A_0 \to B_0$ compactly and $T : A_1 \to B_1$ boundedly. Does it follow that $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ compactly for each $\theta \in (0, 1)$?

Calderón [1] solved Question 1 in the affirmative, in the case where the spaces B_0 and B_1 satisfy a certain approximation property. Since then affirmative answers have also been obtained under quite a number of other conditions on the spaces B_0 and B_1 and/or the spaces A_0 and A_1. For such partial results and other related material we refer, for example, to [10], [12], [4], [5], [2], [6], [7], [8], [11], [13] and [1]. However, in its general form as stated above, Question 1 remains open. We do not even know the answer to the following seemingly simpler variant of it:

Question 2: Same as Question 1, but under the stronger hypothesis that $T : A_1 \to B_1$ is also compact.

This talk briefly surveys some of the above–mentioned previous partial results and then describes some recently observed reductions of the problem, which in the future may perhaps lead to positive answers to Question 1 and/or Question 2.

We also make some remarks about Calderón's compactness result, which is formulated in [1] without explicitly mentioning any operators. Its statement is made in terms of the space $\mathcal{F}(B_0, B_1)$ of analytic vector valued functions used by Calderón in the definition of his spaces $[B_0, B_1]_\theta$. Let K_0 be some compact subset of B_0, let E be some measurable subset of the real line with positive measure, and let $\mathcal{K}$ be the subset of the unit ball of $\mathcal{F}(B_0, B_1)$ consisting of those functions f for which $f(it) \in K_0$ for all $t \in E$. Then Calderón's theorem states, subject to the above–mentioned approximation hypothesis on the spaces B_0 and B_1, that the set $\{f(\theta) : f \in \mathcal{K}\}$ is a compact subset of $[B_0, B_1]_\theta$.

We show that certain compactness results like this one, even though they do not mention a second couple (A_0, A_1) nor operators T mapping from A_j to B_j, are in fact equivalent to results expressed in terms of (A_0, A_1) and T. We have to assume that $E = \mathbb{R}$ and use a slight variant, due to Jaak Peetre, of Svante Janson's orbital characterization of the complex method [9].

We consider some new questions, expressed in terms of the following notation and terminology.

$\mathbb{D}$ *is the closed unit disk.* N *is a (usually very big) positive integer. Let* $H^\infty(\mathbb{D}, \mathbb{C}^N)$ *be the space of all* $\mathbb{C}^N$ *valued functions* $f(z) = (\phi_1(z), \phi_2(z), ..., \phi_N(z))$ *where each* ϕ_j *is in* $H^\infty(\mathbb{D})$*. Let* $\|\cdot\|_p$ *denote the* ℓ^p *norm on* $\mathbb{C}^N$*.*

Question 3: Let $f_1, f_2, ..., f_k$ be k functions in $H^\infty(\mathbb{D}, \mathbb{C}^N)$. For each fixed $z \in \mathbb{D}$, let $M(z)$ be the subspace of $\mathbb{C}^N$ defined by $M(z) = \text{span}\{f_1(z), f_2(z), .., f_k(z)\}$. Suppose that g is another function in $H^\infty(\mathbb{D}, \mathbb{C}^N)$ which satisfies $\|g(z)\|_p \le 1$ for all $z \in \mathbb{D}$ and also

$$\text{dist}_{\ell^p}(g(z), M(z)) \le \varepsilon \text{ for almost every } z \in \mathbb{T}.$$

Does it follow that $\text{dist}_{\ell^p}(g(0), M(0)) \le \varepsilon$?

At first sight Question 3 looks very reasonable. However there is an almost embarrassingly simple example which shows that the answer to it is no. This is unfortunate, since a positive answer to it would have implied a positive answer to Question 1. The ill–fated Question 3 is nevertheless a good motivating model for the following more elaborate question:

Question 4: Let $f_1, f_2, ..., f_k$ be k functions in $H^\infty(\mathbb{D}, \mathbb{C}^N)$. For each fixed $z \in \mathbb{D}$, let $M(z)$ be the subspace of $\mathbb{C}^N$ defined by $M(z) = \text{span}\{f_1(z), f_2(z), .., f_k(z)\}$. For each positive ε, does there exist some subspace S of $\mathbb{C}^N$, whose dimension depends only on k and ε, with the following property: Whenever g is a function in $H^\infty(\mathbb{D}, \mathbb{C}^N)$ which satisfies $\|g(z)\|_p \leq 1$ for all $z \in \mathbb{D}$ and also

$$\text{dist}_{\ell^p}(g(z), M(z)) \leq \varepsilon \text{ for almost every } z \in \mathbb{T},$$

then $\text{dist}_{\ell^p}(g(0), S) \leq \psi(\varepsilon)$?

Here $\psi : (0, \infty) \to (0, \infty)$ is a fixed function of one variable (i.e., it does ***not*** depend on k or N) which satisfies $\lim_{\varepsilon \to 0} \psi(\varepsilon) = 0$?

A positive answer to Question 4 for $p = \infty$, would imply a positive answer to Question 2. If, furthermore, such an answer can be obtained with $\psi(\varepsilon) = \varepsilon^r$ for some positive constant r, then this would imply a positive answer to Question 1.

So far we have positive answers only to certain variants of Question 4. Some of them apply when g is merely harmonic, rather than analytic. On the other hand, if $p = \infty$, and if we allow g to be merely harmonic, there are examples which strongly suggest that the answer to the harmonic version of Questions 4 is no, even when $k = 1$.

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Interpolation and operator ideals

Fernando Cobos

In 1964, Lions and Peetre [6] proved a compactness result that applies to the real interpolation method and to the complex method. This result turned out to be an important tool for establishing a number of other compactness theorems in interpolation theory and it has been used as a model to investigate the behaviour of other ideal properties. An important contribution in this direction is due to Heinrich [5] and refers to closed injective and surjective operator ideals. A quantitative version of Heinrich's theorem was established by Cobos, Manzano and Martínez [2] by using the functionals $\gamma_{\mathcal{J}}(T)$, $\beta_{\mathcal{J}}(T)$ which measure how far is an operator $T \in \mathcal{L}(A,B)$ from a given operator ideal $\mathcal{J}$. Recall that the outer measure $\gamma_{\mathcal{J}}(T)$ is defined as the infimum of all $\sigma > 0$ such that for some Banach space E and some operator $R \in \mathcal{J}(E,B)$ we have $T(U_A) \subseteq \sigma U_B + R(U_E)$. The inner measure $\beta_{\mathcal{J}}(T)$ is the infimum of all positive numbers σ such that for some Banach space F and some $S \in \mathcal{J}(A,F)$ the inequality $\|Tx\|_B \leq \sigma \|x\|_A + \|Sx\|_F$ holds for all $x \in A$.

Another quantitative approach to Heinrich's result has been developed by Cobos, Cwikel and Matos [1] and applies to general intermediate spaces A with respect to a Banach couple $\overline{A} = (A_0, A_1)$. For this, they also need the functions

$$\rho_A(t) = \inf\{J(t,a) : a \in A_0 \cap A_1, \|a\|_A = 1\}, \quad \psi_A(t) = \sup\{K(t,a) : \|a\|_A = 1\}.$$

In particular, they proved that if $T \in \mathcal{L}(B, \overline{A})$ with $\beta_{\mathcal{J}}(T_{B,A_0}) = 0$, then $\beta_{\mathcal{J}}(T_{B,A}) \leq \beta_{\mathcal{J}}(T_{B,A_1}) \lim_{t\to 0}(t/\rho_A(t))$. If $T \in \mathcal{L}(\overline{A}, B)$ with $\gamma_{\mathcal{J}}(T_{A_0,B}) = 0$, then $\gamma_{\mathcal{J}}(T_{A,B}) \leq \gamma_{\mathcal{J}}(T_{A_1,B}) \lim_{t\to\infty}(\psi_A(t)/t)$. This investigation has been continued by Cobos, Manzano, Martínez and Matos [3], who gave also applications to embeddings between Banach function lattices.

We describe some of their results, as well as, the following one due to Cobos and Pustylnik [4]: Let F be a Banach function lattice intermediate with respect to the couple $(L_1[0,1], L_\infty[0,1])$. Then a necessary and sufficient condition for the inclusion $L_\infty[0,1] \hookrightarrow F$ to be strictly singular is that $\lim_{t\to 0} t/\rho_F(t) = 0$. We finish the talk with some results for other measure spaces.

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Old and new entropy techniques

Thomas Kühn

The concept of metric entropy of sets was already introduced in 1932 in a paper on metric and topological dimension theory by Pontrjagin and Schnirelman [PS]. But then it took more than three decades until mathematicians began to understand the far reaching importance of metric entropy. So it was only in the late 1950s - mainly initiated by the famous 1959 survey paper by Kolmogorov and Tihimirov [KT] - when attention was focused again on this notion, and when it became a subject of intensive research. Later on the concept was modified and adapted to operators, see e.g. [P]. Since then metric entropy has found numerous applications in such diverse branches of mathematics as, for instance, analysis (operator theory, spectral theory), approximation theory, probability theory, coding theory, computational complexity, mathematical theory of learning.

Let K be a subset of a metric space (X, d). Recall that, given $\epsilon > 0$, the *covering number* $N(\epsilon, K)$ is defined as the minimal number n such that K can be covered by n balls in X of radius ϵ. Often it is enough to consider the quantity $H(\epsilon, K) := \log_2 N(\epsilon, K)$, which is usually referred to as *Kolmogorov's ϵ-entropy*. Obviously, K is relatively compact if and only if $N(\epsilon, K)$ (or, equivalently, $H(\epsilon, K)$) is finite for every $\epsilon > 0$.

Sometimes, especially in connection with operators, it is more convenient to change the point of view, that means to fix the number of balls and to ask for the minimal radius. For a (bounded linear) operator $T : X \to Y$ between two (quasi) Banach spaces the *k-th dyadic entropy number* $e_k(T)$ is the infimum of all $\epsilon > 0$ such that $T(B_X)$ can be covered by 2^{k-1} balls in Y of radius ϵ, where B_X denotes the unit ball in X. Clearly, T is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$, thus the rate of decay of $e_k(T)$ as $k \to \infty$ describes the "degree" of compactness of T. Via the Carl-Triebel inequality (see formula (1) in Hans Triebel's lecture) entropy numbers are intimately related to eigenvalues, a fact on which all applications to spectral theory of differential operators are based.

Typically in such applications, one has to estimate entropy numbers of certain embeddings in function spaces. In many cases, using splines, wavelets, atomic or subatomic decompositions, the problem can be shifted to a sequence space setting. Although this simplifies the original problem already considerably, the resulting sequence spaces are usually quite complicated (multi-indexed, vector-valued, with weights). Therefore, in order to find asymptotically sharp entropy estimates, it is essential to have a full arsenal of entropy techniques at hand.

The main aim of this talk is to give an overview over these techniques. Some of them are by now almost standard (at least to specialists), but still very powerful, some others are quite new and rather elaborate or tricky. Every technique will be illustrated by a typical example/application, explaining in this way the main underlying ideas. Here is a list of the techniques that will be discussed:

- Factorization methods
- Volume estimates
- Combinatorial arguments
- Interpolation methods
- Operator ideal techniques
- Random techniques
- Tensor product techniques

The applications are:

(1) the Carl-Triebel inequality (factorization and volume estimates)
as well as entropy estimates for
(2) identities $\ell_p^n \to \ell_q^n$ (combinatorics and interpolation),
(3) diagonal operators with polynomial diagonal (operator ideal techniques)
(4) diagonal operators with logarithmic diagonal (random techniques)
(5) certain tensor product operators from $\ell_p(X)$ to $\ell_q(Y)$ (tensor techniques)

Most of these results are well known. For (1)–(3) we refer to the monographs by Pietsch [P], König [Kö], and Carl and Stephani [CS]; the extension to operators in quasi-Banach spaces was mainly done by Edmunds and Triebel [ET]. Concerning random techniques (Dudley's and Sudakov's inequalities, and their refinements) see Pisier's book [Pi]. Result (4) is contained in the article [K1], while (5) is very recent work [K2]. To conclude with, we state one typical result of [K2].

Theorem. *Let $T : X \to Y$ be an operator between quasi-Banach spaces and let $D : \ell_p \to \ell_q$, $0 < p, q \le \infty$, be a diagonal operator. Assume that*

$$e_k(T) \sim k^{-\alpha} \qquad \text{and} \qquad e_k(D) \sim k^{-\beta}$$

for some $\alpha > 0$ and $\beta > \max(0, 1/p - 1/q)$. Then, if $\alpha \neq \beta$,

$$e_k(D \otimes T : \ell_p(X) \to \ell_q(Y)) \sim k^{-\min(\alpha,\beta)} .$$

In the limiting case $\alpha = \beta$ similar two-sided estimates can be shown, which are optimal up to logarithmic factors. Using this result, one can improve some entropy estimates for embeddings of Besov spaces in spaces of Lipschitz type.

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Closed operator ideals and indices

Luz M. Fernández-Cabrera

Let E and F be rearrangement invariant spaces on $[0,1]$ with fundamental functions having regular variation at 0 and the embedding $E \hookrightarrow F$ failing to be disjointly strictly singular. Then, spaces E and F cannot be distinguish by means of the spaces $L_p[0,1]$. This result was proved by García del Amo, Hernández, Sánchez and Semenov [3]. To understand it fully, it is useful to consider the notion of inclusion indices. Recall that if $L_\infty[0,1] \hookrightarrow E \hookrightarrow L_1[0,1]$ then the inclusion indices of E are defined by $\overline{\delta}_E = \sup\{p \geq 1 : E \hookrightarrow L_p[0,1]\}$, $\overline{\gamma}_E = \inf\{p \leq \infty : L_p[0,1] \hookrightarrow E\}$. The conclusion of the result means that $\overline{\delta}_E = \overline{\gamma}_E = \overline{\delta}_F = \overline{\gamma}_F$. Since L_p-spaces can be obtained applying the complex interpolation method to the couple $(L_\infty[0,1], L_1[0,1])$, it is natural to wonder whether inclusion indices are only a special case of an abstract notion, with more ample applications. This investigation has been carried out in a joint paper with Cobos, Hernández and Sánchez [1]. In this talk I will describe some of our results.

Let A_0 and A_1 be Banach spaces with $A_0 \hookrightarrow A_1$. We assume that A_0 is not closed in A_1. For $0 < \theta < 1$, we put $A_\theta = (A_0, A_1)_{[\theta]}$. Given any Banach space A with $A_0 \hookrightarrow A \hookrightarrow A_1$, the indices of A relative to the scale $\{A_\theta\}_{0\leq\theta\leq 1}$ are defined by

$$\delta_A = \sup\{\theta : A_\theta \hookrightarrow A\}\ ,\ \gamma_A = \inf\{\theta : A \hookrightarrow A_\theta\}.$$

Our first result shows that the interpolation theorems for closed operator ideals can be used to compare the indices of two spaces. We have also computed the indices analytically by using the functions $\psi_A(t) = \sup\{K(t,a) : \|a\|_A = 1\}$ and $\rho_A(t) = \inf\{J(t,a) : a \in A_0, \|a\|_A = 1\}$. It turns out that

$$\gamma_A = \limsup_{t\to\infty} \frac{\log \psi_A(t)}{\log t}\ \ ,\ \ \delta_A = \liminf_{t\to\infty} \frac{\log \rho_A(t)}{\log t}.$$

As a consequence we can estimate the grade of proximity between two intermediate spaces $A \hookrightarrow B$ when the embedding fails an ideal property that the inclusion $A_\theta \hookrightarrow A$ has. Here $\gamma_A < \theta$. Applying these results to function spaces on $[0,1]$ we recover the result of [3] and the extension established in [2] to spaces not necessarily rearrangement invariant. Results proved in [2] for Banach spaces of sequences can be recovered as well. Our approach also applies to $[0, \infty)$ with the Lebesgue measure. Furthermore, we can derive results for function spaces near to $L_\infty[0, 1]$ and spaces close to $L_1[0, 1]$.

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Measures of non-compactness and real interpolation

Pedro Fernández-Martínez

(joint work with F. Cobos and A. Martínez)

Let A and B be Banach spaces and let $T \in \mathcal{L}(A, B)$ be a bounded linear operator acting from A into B. The (ball) measure of non-compactness of T is defined by

$$\beta(T) = \inf \Big\{ r > 0 \quad \text{s.t.} \quad T(\mathcal{U}_A) \subseteq \bigcup_{\text{finite}} \{b_j + r\mathcal{U}_B\} \Big\}$$

where $\mathcal{U}_A$ (resp. $\mathcal{U}_B$) stands for the closed unit ball of A (resp. B).

Clearly $\beta(T) = 0$ if and only if T is compact. Otherwise, $\beta(T)$ measures how far the operator T is from being compact.

In this talk we study the behaviour under real interpolation of the measure of non-compactness (see [1]). Previous result on this question have been obtained by Edmunds and Teixeira [4] (see also the monograph by A. Pietsch [9], Prop.12.1.11 and 12.1.12 and that of Triebel [6] § 1.16). These results require the assumption that one of the Banach couples degenerates into a Banach space, i.e. $A_0 = A_1$ or $B_0 = B_1$, or that they are different but the image couple (B_0, B_1) satisfies a certain approximation condition.

We consider here the case of general couples without assuming any approximation hypothesis on them, and we show (using the techniques in [2]) that the following logarithmically convex inequality holds:

$$\beta\left(T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}\right) \leq c\beta(T : A_0 \longrightarrow B_0)^{1-\theta} \beta(T : A_1 \longrightarrow B_1)^{\theta}$$

In the special case where one restriction of T is compact we recover Cwikel's compactness theorem.

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Entropy numbers of embeddings of weighted Besov spaces. I. Polynomial weights with perturbations II. Weights of subpolynomial growth

Hans-Gerd Leopold and Leszek Skrzypczak

(joint work with T. Kühn and W. Sickel)

We investigate the asymptotic behaviour of the entropy numbers of the compact embedding

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$$

of the weighted Besov space $B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w)$ into the unweighted space $B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$. Here w is a smooth, strictly positive function on $\mathbb{R}^d$ and a distribution f belongs to the weighted Besov space $B^{s}_{p,q}(\mathbb{R}^d, w)$ if the product wf belongs to the unweighted space $B^{s}_{p,q}(\mathbb{R}^d)$.

The problem of estimation of entropy numbers of the above compact embedding was first treated by D.Haroske and H.Triebel in [HT 1], where polynomial weights were considered. Possible applications of the estimates to spectral theory of differential and pseudo-differential operators were described in a second paper, cf. [HT 2].

To estimate the asymptotic behaviour of the entropy number for a larger range of weights and the full range of parameters (s, p, q) we use the following strategy. First of all we reduce the problem to the level of sequence spaces via wavelet bases of weighted Besov spaces. This concept was developed in [KLSS1] and [HT 3]. To estimate the entropy numbers of the corresponding sequence space embeddings we use the known estimates for embeddings of finite dimensional ℓ_p spaces as well as the estimates of entropy numbers of diagonal operators acting between sequence ℓ_p spaces together with the proper splitting of the original embedding operators. The technique of the operator ideals is often used to glue the estimates of different parts of the splitting together. We consider the following types of weights.

(1) polynomial weights $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha > 0$
(2) logarithmic weights $w_{0,\beta}(x) = (1 + \log(1 + |x|))^\beta$, $\beta > 0$

(3) polynomial weights perturbed (for $|x| \geq 1$) by a slowly varying function φ, $\quad w_{\alpha,\varphi}(x) = w_\alpha(x)\varphi(|x|)$.

Slowly varying (in the sense of Kohlbecker) roughly means that $\varphi(|x|)$ is small in comparison with $w_\alpha(x)$ for large $|x|$. Any power of $(1 + \log(1 + |x|))$ is an example of a slowly varying function, but the class is of course much wider.

The symbol $a_n \preceq b_n$ means that there exist a constant $c > 0$ such that $a_n \leq c\,b_n$ for all $n \in \mathbb{N}$, while $a_n \sim b_n$ stands for $a_n \preceq b_n \preceq a_n$. Furthermore let $\frac{1}{p^*} = (\frac{1}{p_1} - \frac{1}{p_2})_+$ and $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2})$.

A previous version of the following results is contained in [KLSS1].

Theorem 1. *Let $0 < p_1, p_2, q_1, q_2 \leq \infty$, $s_1 > s_2$, and let φ be a slowly varying function. Then for the entropy numbers of the embedding*

$$\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_{\alpha,\varphi}) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$$

it holds

$$e_n(\mathrm{id}) \sim \begin{cases} n^{-\frac{\delta}{d} - (\frac{1}{p_1} - \frac{1}{p_2})} & \text{if} \quad d/p^* < \delta < \alpha \\ \\ n^{-\frac{\alpha}{d} - (\frac{1}{p_1} - \frac{1}{p_2})}(\varphi(n^{1/d}))^{-1} & \text{if} \quad d/p^* < \alpha < \delta\,. \end{cases}$$

Clearly, the theorem covers also polynomial weights ($\varphi \equiv 1$), but the most difficult case $\alpha = \delta$ is not contained. In that so-called limiting case we gave in [KLSS2] a new and simpler proof of the following result for polynomial weights.

Theorem 2. *Let $0 < p_1, p_2, q_1, q_2 \leq \infty$ and $s_1 > s_2$. Suppose*

$$\alpha = \delta > d/p^* \qquad \text{and set} \quad \tau := \frac{s_1 - s_2}{d} + \frac{1}{q_2} - \frac{1}{q_1}\,.$$

Consider the embedding $\quad \mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$.

(i) *If $\tau \neq 0$, then* $\quad e_n(\mathrm{id}) \sim n^{-\frac{s_1 - s_2}{d}}\,(1 + \log n)^{\max(\tau,0)}\,.$

(ii) *If $\tau = 0$, then* $\quad n^{-\frac{s_1 - s_2}{d}} \preceq e_n(\mathrm{id}) \preceq n^{-\frac{s_1 - s_2}{d}}\,(1 + \log\log n)^{\frac{1}{q_1}}\,.$

The corresponding estimates for logarithmic weights look as follows.

Theorem 3. *Let $0 < q_1, q_2 \leq \infty$, $0 < p_1 \leq p_2 \leq \infty$, $\delta > 0$ and $\beta > 0$. Then for the embedding*

$$\mathrm{id} : B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_{0,\beta}) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$$

one has

$$e_n(\mathrm{id}) \sim \begin{cases} n^{-(\frac{1}{p_1} - \frac{1}{p_2})}\big(\log n\big)^{\frac{1}{p_1} - \frac{1}{p_2} - \beta} & \text{if} \quad \beta \geq \frac{1}{p_1} - \frac{1}{p_2}\,, \\ \\ n^{-\beta} & \text{if} \quad \beta < \frac{1}{p_1} - \frac{1}{p_2}\,. \end{cases}$$

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A new class of approximation spaces and operator ideals

Evgeniy Pustylnik

Let X be a quasi-Banach space and let G_n, $n \in \mathbb{N}_0$, be a sequence of subsets of X with the following properties:

$$G_0 = \{0\}, \quad G_n \subset G_{n+1}, \quad \lambda G_n \subset G_n, \quad G_n + G_m \subset G_{n+m}\,,$$

so that, for any $f \in X$, we can define its approximation numbers

$$a_n(f, X) = \inf_{g \in G_{n-1}} \|f - g\|_X.$$

Let $E = \mathcal{F}(l_1, l_\infty)$ be arbitrary symmetric sequence space, defined by some interpolation functor $\mathcal{F}$. Denote by $\widetilde{E}$ the corresponding sequence space $\widetilde{E} = \mathcal{F}(\widetilde{l}_1, l_\infty)$, where

$$\widetilde{l}_1 = \{a = (a_n) : \ \|a\|_{\widetilde{l}_1} = \sum_{n=1}^{\infty} |a_n|/n < \infty\}\,,$$

(for more details see [1]). Now, for arbitrary positive increasing function $\varphi(t)$ with positive extension indices, we define an approximation space $X_{\varphi,E}$ as the set of all $f \in X$ such that

$$\|f\|_{X_{\varphi,E}} = \|\big(\varphi(n) a_n(f, X)\big)\|_{\widetilde{E}} < \infty.$$

For approximation spaces thus obtained, we prove the standard statements of approximation theory analogous to [2]:

a) equivalent formula of norms $\|f\|_{X_{\varphi,E}} = \|\big(\varphi(2^n) a_{2^n}(f, X)\big)\|_E$;

b) representation theorem: *an element $f \in X$ belongs to $X_{\varphi,E}$ if and only if it can be represented in a form $f = \sum_1^\infty g_n$ such that any $g_n \in G_{2^n}$ and*

$$\|f\|^{repr}_{X_{\varphi,E}} = \inf \|\varphi(2^n) g_n\|_E < \infty.$$

c) interpolation theorem: a space $X_{\varphi,E}$ is interpolation between spaces X_{φ,E_0} and X_{φ,E_1} if and only if the space E is interpolation between spaces E_0 and E_1 and in both cases one can use the same interpolation functor.

Some other properties and interrelations with interpolation theory also are found. At last, all results are applied to operator spaces $X = \mathcal{L}(A, B)$, where the set G_n is defined as the set of all operators with the rank no greater than n.

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Distance of L_1-functions to the ball of radius $t$in the space $Lip\ \alpha$

Natan Kruglјak

If the function $f \in L_1 = L_1(R^n)$ is given, it is easy to calculate the distance from it (in the metric of L_1) to the ball of radius t of the space L_∞ as follows:

$$\rho(f, B_t(L_\infty))_{L_1} = \int (|f| - t)_+ \, dx.$$

However, if instead of the space L_∞ we consider another space, for example the space $Lip\,\alpha$, the situation becomes much more complicated. From the point of view of real interpolation it is enough to have a formula up to the following equivalence, which we will call *the radial equivalence.*

Definition. *We will call two non-negative functions on R_+ radial equivalent (notation $f \approx_R g$) if it is possible to find positive constants $c_1, c_2, c_3,$ and c_4 such that*

$$f(t) \le c_1 g(c_2 t) \quad \text{and } g(t) \le c_3 g(c_4 t)$$

for all $t \in R_+$.

From [C] it follows that the usual seminorm in the space $Lip\ \alpha$

$$\|f\|_{Lip\ \alpha} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is equivalent to

$$\|f\|_{Lip\ \alpha} \approx \sup_Q \frac{\int_Q |f(x) - f_Q|}{|Q|^{1+\frac{\alpha}{n}}} = \sup_Q \frac{\frac{1}{|Q|}\int_Q |f(x) - f_Q|}{(|Q|^{\frac{1}{n}})^\alpha}, \quad f_Q = \frac{\int_Q f}{|Q|},$$

where sup is taken over all cubes Q with sides parallel to the coordinate hyperplanes. In particular, for $\alpha = 0$ instead of the space $Lip\ \alpha$ we obtain the space BMO. Using the technique developed in the paper by Bennett-Sharpley [BS] it is possible to obtain the following formula for the distance to the ball of radius t of the space BMO :

$$\rho(f, B_t(BMO))_{L_1} \approx_R t \cdot \left|\left\{x \in R^n \mid f^\natural(x) > t\right\}\right|,$$

where $f^\natural(x) = \sup_{Q \ni x}(\frac{1}{|Q|}\int_Q |f(x) - f_Q|)$ is the Fefferman-Stein maximal function. Therefore, it looks natural to try to obtain an analogous formula for the distance $\rho(f, B_t(Lip\ \alpha))_{L_1}$. This hope is justified; however, instead of the measure of the set $\left\{x \in R^n \vdots f^\natural(x) > t\right\}$ we need to take another quantative. To formulate the result let us consider the set of cubes Ω_t defined by the formula

$$\Omega_t = \left\{Q \mid \frac{\int_Q |f(x) - f_Q|}{|Q|^{1+\frac{\alpha}{n}}} > t\right\}$$

and denote by $|\Omega_t|_\alpha$ the $\sup(\sum |Q_i|^{1+\frac{\alpha}{n}})$, where sup is taken over all families of pairwise disjoint cubes from Ω_t .

Theorem. *Let $f \in L_1$ be given. Then the distance from f to the ball of radius t of $Lip\alpha$ is given by the formula*

$$\rho(f, B_t(Lip\ \alpha))_{L_1} \approx_R t \cdot |\Omega_t|_\alpha$$

with the constants of equivalence independent of f and t.

Remark. *In the case $\alpha = 0$ the formula is equivalent to the formula of distance to the ball of radius t of the space BMO (the proof is based on the Besicovitch covering theorem). Therefore, the theorem provides an extension of the previous formula to the case of the space Lip α.*

The proof of the Theorem is based on a technique (covering lemma and smooth analogs of Calderon-Zygmund decomposition) developed in [K].

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Compact operators between real interpolation spaces

Antón Martínez

(joint work with F. Cobos and L.M. Fernández-Cabrera)

The behaviour of compact operators under interpolation is a question that has received much attention during the last few years. Talking only about the real method, it was shown in the joint papers of one of the present authors with Edmunds and Potter [2], with Fernandez [3] and with Peetre [6], that properties of vector valued sequence spaces related to the real interpolation space $(A_0, A_1)_{\theta,q}$ are very useful to study the behaviour of compact operators under interpolation. These efforts were culminated by Cwikel [7] who proved that if $T : \bar{A} \longrightarrow \bar{B}$ with $T : A_0 \longrightarrow B_0$ compact then $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ is also compact. Later, approach developed in [2, 3, 6] was used by Cobos, Kühn and Schonbek [5] to give a broad generalization of Cwikel's result, including a function parameter version and even compactness theorems for other interpolation methods.

Another side of these ideas has been shown very recently by other of the present authors in [9], where she has characterized compact operators between real interpolation spaces in terms of weaker compactness conditions and convergence of certain sequences of operators involving projections on the vector valued sequence spaces.

In this talk we show some results of the paper [4], where we continue the research of [9] working now with general K-functors and general J-functors. The interest of these interpolation methods has been pointed out by many authors. See, for example, the books by Brudnyĭ and Krugljak [1], as well as the papers by Cwikel and Peetre [8], Janson [10] and Nilsson [11].

We establish necessary and sufficient conditions for compactness of operators acting between K-spaces, between J-spaces and from a J-space into a K-space. Moreover, we show by means of examples that conditions required on the sequence space that define the K-functor (respectively, the J-functor) are essential for the result.

When we specialize the results we recover the theorems of [9], but we also obtain new results. In particular, we get a characterization of compact operators between real interpolation spaces that blends conditions found in [9]. As another application of our characterizations we obtain extended versions of compactness results of [2, 3, 5, 6, 7].

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Weak compactness and general J- and K- methods

Antonio Manzano
(joint work with F. Cobos, L.M. Fernández-Cabrera, and A. Martínez)

The behaviour of weak compactness under interpolation has attracted considerable attention since Davis, Figiel, Johnson and Pelczyński [7] established their celebrated result on factorization of weakly compact operators. For the classical real method $(A_0, A_1)_{\theta,q}$, relevant contributions on this problem are due to Beauzamy [1] and Heinrich [H] (other related results can be found in [9], [6] and [4]).

But the real method is not enough to describe all interpolation spaces with respect to many important couples. For example, applying this method to (L_1, L_∞) we only obtain L_p and $L_{p,q}$ spaces, while Lorentz spaces, Marcinkiewicz spaces and the majority of symmetric spaces are interpolation spaces with respect to (L_1, L_∞). However, as a famous result of Calderón and Mitjagin says, any interpolation space with respect to the couple (L_1, L_∞) is K-monotone and so it can be obtained by the general K-method (see [2]).

The general K-method has been studied widely, as well as the general J-method, and the interest of both interpolation methods has been pointed out by many authors (we refer to the book by Brudnyĭ and Krugljak [2] for wide information and relevant references about these methods).

Aizenstein and Brudnyi [2], Section 4.6, and Mastyło [10] have investigated the interpolation of weakly compact operators by the general K-method. In all cases, the techniques used by these authors are based on specific properties of the K-space.

We develop a new approach that allows us to establish interpolation results for general K- and J- spaces at the same time, as well as to apply them to other closed operator ideals different from weakly compact operators. In particular, we cover the cases of Rosenthal operators and Banach-Saks operators. The new approach is based on ideas of Heinrich [H] and our previous results in [3] and [5].

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Entropy numbers in vector valued sequence spaces

Tomas Schonbek
(joint work with P. Fernández-Martínez)

In this talk we consider the following question. Assume T is a bounded linear operator from the quasi-Banach space X to the quasi-Banach space Y and for $N \in \mathbb{N}$ let $\tilde{T} = \tilde{T}_N$ be the operator $(x_n)_{n=1}^N \mapsto (Tx_n)_{n=1}^N$ from X^N to Y^N. We consider X^N, Y^N as quasi-Banach spaces normed with (weighted) l^p, l^q norms, respectively; $0 < p \leq q \leq \infty$. How do the entropy numbers $\tilde{e}_k = e_k(\tilde{T} : \ell_p^N(X) \to \ell_q^N(Y))$ of $\tilde{T}$ relate to the entropy numbers $e_k = e_k(T : X \to Y)$ of T? Apart from its intrinsic merit, solving this problem would allow one to extend the methods used by Cobos, Fernández and Martínez in [1] to estimate the measure of non-compactness of an operator under real interpolation, to estimating the entropy numbers of such an operator.

We further simplify the problem assuming $q = \infty$ and use a combinatorial approach to estimating the entropy numbers of $\tilde{T}$ based on the following idea. For each $\rho > 0$ we construct an approximation to a set $S_N(\rho)$ of vectors contained in the closed ball of radius ρ of $\ell_p^N(X)$ such that if $x, y \in S_N(\rho)$ and $x \neq y$, then $\|Tx - Ty\|_Y > 1$; if $x \in X$ and $\|x\|_X \leq \rho$, then there exists $y \in S_N(\rho)$ such that $\|Tx - Ty\|_Y \leq 1$. Let $K_N(\rho)$ be the cardinality of the set $S_N(\rho)$. Because $K_N(\rho) \leq 2^{k-1}$ implies $\tilde{e}_k \leq 1/\rho$, it suffices to estimate $K_N(\rho)$ to get an estimate for the entropy numbers. To achieve this estimate, the following simple recurrence formula for K_N is established, where $0 = \rho_0 < \rho_1 < \cdots < \rho_n = \rho$ is a partition of the interval $[0, \rho]$ and $\lambda(\rho) = K_1(\rho)$:

$$K_{N+1}(\rho) \leq \lambda(\rho_1) K_N(\rho) + \sum_{k=2}^{n} \left(\lambda(\rho_k) - \lambda(\rho_{k-1})\right) K_N\left((\rho^p - \rho_{k-1}^p)^{1/p}\right)$$

Assuming a moderate growth of λ as a function of ρ; specifically that there exists $\gamma > 0$ such that $\lambda((n+1)\gamma) \leq \kappa\lambda(n\gamma)$ for some constant $\kappa > 0$, $n = 1, 2, \ldots$, we obtain that there exist positive constants C, a, b, d, r (not depending on k or N) such that

$$(1) \qquad \tilde{e}_k \leq C e_{\log(\frac{k}{b}) - \log\log(\frac{abN}{k}+1)} \qquad \text{for} \quad k < Nd + 1, \qquad \text{and}$$

$$(2) \qquad \tilde{e}_k \leq C e_{\frac{k-1}{N} + \log\frac{N}{r}} \qquad \text{if} \quad \lambda(\gamma) = 1 \text{ and } k \geq dN + 1.$$

The growth condition on λ is satisfied if the entropy numbers of T decay sufficiently fast: $e_k \sim k^{-\alpha}$ for $\alpha \geq 1$. For details, see [2].

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Optimal approximation of elliptic problems by linear and nonlinear mappings

Winfried Sickel

(joint work with Stephan Dahlke and Erich Novak)

Let H be a Hilbert space and let $\mathcal{B} = \{g_1, g_2, \ldots\}$ be a countable subset of H. Then the best n-term approximation of $g \in H$ with respect to $\mathcal{B}$ is defined to be

$$\sigma_n(g, \mathcal{B})_H := \inf_{i_1,\ldots,i_n} \inf_{c_1,\ldots,c_n} \| \, g - \sum_{k=1}^{n} c_k \, g_{i_k} \, \|_H \, .$$

For $C \geq 1$ let

$$\mathcal{B}_C := \{\mathcal{B} : \, \mathcal{B} \subset H \, , \, \mathcal{B} \text{ is a Riesz basis of } H \text{ with constants } A, B > 0 \, , B/A \leq C\} \, .$$

Here the constants A, B, related to the Riesz basis $\mathcal{B} = \{g_1, g_2, \ldots\}$, are the optimal positive numbers in the inequality

$$A \Big(\sum_k |c_k|^2 \Big)^{1/2} \leq \Big\| \sum_k c_k \, g_k \Big\|_H \leq B \Big(\sum_k |c_k|^2 \Big)^{1/2}$$

which has to be valid for any set of coefficients $c_k \in \mathbb{C}$, $k = 1, 2, \ldots$. Let G be a Hilbert space and let $S : G \to H$ be an isomorphism. Let F be a quasi-normed subspace of G. Then, parallel to the approximation numbers of S, we introduce the quantities:

$$e_n^{non,C}(S, F, H) := \inf_{\mathcal{B} \in \mathcal{B}_C} \sup_{\|f\|_F \leq 1} \sigma_n(Sf, \mathcal{B})_H \, , \qquad n = 1, 2, \ldots \, .$$

Similar widths, even in a more general context, have been considered by Temlyakov [T]. In the particular situation, where $H = H^s(Q)$ is a fractional order Sobolev space on $Q = [0,1]^d$, $F = B^{s+t}_{p,q}(Q)$ is a Besov space and S is the identity operator, $S = I : B^{s+t}_{p,q}(Q) \to H^s(Q)$ we have obtained the following.

Theorem. Let $-\infty < s < \infty$, $0 < p, q \leq \infty$ and

$$t > d \, \max \Big(0, \frac{1}{p} - \frac{1}{2}\Big) \, .$$

Let $C \geq 1$. Then there exists a constant c_1 such that

$$e_n^{non,C}(I, B^{s+t}_{p,q}(Q), H^s(Q)) \leq c_1 \, n^{-t/d}$$

holds for all natural numbers n. In case $0 < p \leq 2$ there exists a positive constant c_2 such that

$$e_n^{non,C}(I, B_{p,q}^{s+t}(Q), H^s(Q)) \geq c_2\, n^{-t/d}$$

holds.

Remark. In the preprint [DNS] we discuss consequences of this Theorem for our model problem, the Poisson equation on a general Lipschitz domain. In particular, we compare nonlinear methods of approximation of the solution with linear methods.

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On interpolation of bilinear operators

Mieczysław Mastyło

Multilinear operators arise naturally in many areas of classical, harmonic analysis as well as functional analysis, including the theory of Banach operator ideals. The latest progress in study of the bilinear Hilbert transform and bilinear multipliers of Marcinkiewicz type has stimulated the need of development of a systematic analysis of bilinear operators. Interpolation of bilinear operators is a classical problem in interpolation theory. The situation for the real and complex method of interpolation is well understood however few results are known for other interpolation methods. We present some new results on interpolation of multilinear operators between products of Banach spaces generated by abstract methods of interpolation in the sense of Aronszajn and Gagliardo. A variant of bilinear interpolation is proved for bilinear operators from corresponding weighted c_0-spaces into Banach spaces of non-trivial periodic Fourier cotype. This result is then extended to the spaces generated by minimal and maximal methods of interpolation determined by quasi-concave functions. In the case when a maximal costruction is generated by Hilbert spaces, we obtain a general variant of bilinear interpolation theorem. Combining this result with the abstract Grothendieck theorem of Pisier yields further results. The results are applied in deriving a bilinear interpolation theorem for Calderón-Lozanovsky, for Orlicz spaces and an embedding formula for (E, p)-summing operators.

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Continuity envelopes in spaces of generalised smoothness, and applications

Dorothee D. Haroske
(joint work with A.M. Caetano and S.D. Moura)

Continuity envelopes provide a new tool to characterise various types of function spaces by their smoothness properties (with respect to Lipschitz continuity); in particular, we consider

$$\mathcal{E}_{\mathsf{C}}^X(t) = \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}, \quad t>0,$$

for function spaces X, and claim that $\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^X, u_{\mathsf{C}}^X\right)$ gives precise information about X; here u_{C}^X is an additional fine index. We continue and extend recent first results ([H], [T]) now in the context of spaces of generalised smoothness, say, $X = B_{p,q}^{(s,\Psi)}(\mathbb{R}^n)$, where Ψ might be an 'admissible' or, more general, a slowly varying function.
In joint work with A.M. Caetano (Aveiro) and S.D. Moura (Coimbra), [CH], [HM], we proved, for instance, that

$$\mathfrak{E}_{\mathsf{C}}\left(B_{p,q}^{(s,\Psi)}\right) = \left(\left(\int_t^1 \Psi(y)^{-q'} \frac{\mathrm{d}y}{y}\right)^{1/q'}, \ q\right), \quad \frac{n}{p} < s \le \frac{n}{p}+1, \quad 0<p,q\le\infty,$$

assuming $\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}} \notin \ell_{q'}$ when $s=\frac{n}{p}+1$; for $0<q\le\infty$ the number q' is given by $\frac{1}{q'} = \max\left(0, 1-\frac{1}{q}\right)$.
From results like this we can derive Hardy-type inequalities, sharp assertions on limiting embeddings, and estimates for the asymptotic behaviour of approximation numbers of related compact embeddings. The proofs involve different characterisations of the underlying spaces like the Fourier-analytically based definition, atomic decompositions, and equivalent norms involving differences.

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Banach operator ideals and eigenvalue estimates

Mieczysław Mastyło

We surevy some recent results from [1] and [2]. Let E, F and G be Banach sequence spaces, (e_n) be the standard unit vector basis in c_0 and $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices. Our aim is to give estimate $\Lambda_E(T) \le f(n)\,\|T\|_{F[G]}$, where

$$\Lambda_E(T) = \Big\| \sum_{i=1}^{n} \lambda_i(T) e_i \Big\|_E, \quad \|T\|_{F[G]} = \Big\| \sum_{j=1}^{n} \Big\| \sum_{i=1}^{n} \tau_{ij} e_i \Big\|_G e_j \Big\|_F$$

for all $T = (\tau_{ij}) \in \mathcal{M}_n$ and $f : \mathbb{N} \to \mathbb{R}_+$ is a function which does not depend on T. It is shown (see [1]) that under some geometrical conditions the above mentioned estimate holds for a large class of sequence spaces with $f(n) \simeq \log(1+n)$. Combining this estimate with some furher geometrical estimates for Kronecker's matrices, we obtain that if ℓ_φ is an Orlicz sequence space such that φ is a supermultiplicative Orlicz function such that $t \mapsto \varphi(\sqrt{t})$ is equivalent to a convex function, then $\Lambda_{\ell_\varphi}(T) \le C\,\|T\|_{\ell_{\varphi_*}[\ell_{\overline{\varphi}}]}$ where φ_* is a Young's conjugate function of φ, $\overline{\varphi}$ is the minimal submultiplicative function dominating φ and $C = C(\varphi) > 0$. For the power function $\varphi(t) = t^p$, $2 \le p < \infty$, we obtain $C = 1$, and this gives a celebrated result of Jonhson, König, Maurey and Retherford [3].

For a Banach sequence space E containing ℓ_2, the Banach operator ideal of $(E,2)$-summing operators consists of all operators T between Banach spaces for which $\{\|T(x_n)\|\} \in E$ for all weakly 2-summable sequences $\{x_n\}$. Based on interpolation theory, recently several key results within the theory of $(q,2)$-summing operators and its applications have been extended to the more general case of $(E,2)$-summing operators. We only present a variant for $(E,2)$-summing operators of a striking composition formula due to H. König, J.R. Retherford and N. Tomczak-Jaegermann [4], which in its original formulation says that the composition $T_N \circ ... \circ T_1$ of N operators T_k between Banach spaces which are $(q_k,2)$-summing, is 2-summing and compact provided that $1/q_1 + ... + 1/q_N > 1/2$.

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