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Nonlinear Waves and Dispersive Equations

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Introduction by the Organisers

Nonlinear dispersive equations are models for nonlinear waves in a wide range of situations. Mathematically they display an interplay between linear dispersion and nonlinear focusing or defocusing effects. They are linked to diverse areas of mathematics and physics, ranging from nonlinear optics over oscillatory integrals and integrable systems to algebraic geometry. The conference did focus on the analytic (PDE) aspects with a view towards applications.

Major results and areas are:

(1) The introduction of Bourgain-type spaces, which are a tool to describe the L^2 information of the linear equation and link it to geometry of the characteristic set and bilinear/multilinear estimates.
(2) The study of problems with rough initial data. This includes the work on semilinear problems as well as substantial progress in understanding the interaction of nonlinear waves. Also one should add the recent ill-posedness results above the scaling exponents.
(3) There has been considerable improvement in our understanding of the blow up of solutions to critical dispersive equations.
(4) The long time and global behavior of solutions to dispersive equations. There are two aspects of this; for small data the dispersive efffects are dominant, while for large data a (largely missing) understanding of the nonlinear focusing/defocusing effects becomes essential.

(5) Dispersive equations occur in some formal expansion. Typically the first terms of the expansion lead to some universal problems like third order nonlinear Schrödinger or Korteweg-de-Vries equations. There has been progress in the understanding of 'better' approximations and higher dimensional waves.
(6) There has been important work in bridging the gap between the asymptotic dispersive equations and the full problem.

Motivated by recent developments there has been a series of lectures by T. Kappeler and P. Topalov on their application of the inverse scattering transform to rough initial data for the Korteweg-de-Vries equation.

This meeting was attended by 45 participants. The organizers made an effort to include young mathematicians and to give them the opportunity of a shorter talk.

Workshop: Nonlinear Waves and Dispersive Equations

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Abstracts

Some Dispersive Estimates for Schrödinger Equations with Repulsive Potentials

Luis Vega
(joint work with Juan A. Barceló and Alberto Ruiz)

We consider for V a real potential the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u + V(x)u = 0 & (x,t) \in \mathbb{R}^n \times \mathbb{R} \qquad n \geq 3, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

We suppose that $W(r)$ is a radial majorant of the negative part of the radial derivative of $V(x)$, that is

$$(\partial_r V)_-(r) := \sup_{x\in\mathbb{R}^n \ : \ |x|=r} (\partial_r V)_-(x) \leq W(r). \tag{2}$$

Our interest is to revisit the work of M. Arai [1] with a double purpose. Firstly to obtain exact decay conditions on W, which turn out to be different either one considers dimension three or higher. And secondly to prove the so-called local smoothing effect of $1/2$ gain derivative -see [3], [5], and [6]. We use pure integration by parts techniques which allows us to give precise constants on the assumptions of the potential -see (8) and (12). These constants are sharp at least for $n > 3$. Also we obtain an estimate for the full gradient what allows us to give non-radial perturbations using weighted Sobolev inequalities -see Theorem 3. We assume that V satisfies the following conditions:

- Hypothesis 1: Problem (1) has a unique solution $u(x,t)$ for $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^n)$ which satisfies the a priori estimate

$$||u(.,t)||_{H^{\frac{1}{2}}(\mathbb{R}^n)} \leq C(V)||u_0||_{H^{\frac{1}{2}}(\mathbb{R}^n)}. \tag{3}$$

- Hypothesis 2: There exists a class of data u_0, which is dense in $H^{\frac{1}{2}}(\mathbb{R}^n)$, such that the solution of (1) is in $\mathcal{C}(\mathbb{R};(H^s(\mathbb{R}^n))$ for some $s \geq 3/2$ and $\int_{\mathbb{R}^n} |u|^2|V| < \infty$.

For example we can consider real potentials V such that

$$\|Vf\|_{L^2} \leq a\|\Delta f\|_{L^2} + b\|f\|_{L^2}$$

holds with some $a < 1$.

Let us assume that the function

$$H(r) = r\int_r^\infty tW(t)dt < \infty, \tag{4}$$

and we suppose that

$$H(0) = \liminf_{r \longrightarrow 0} r \int_r^\infty tW(t)dt < \infty, \tag{5}$$

$$H(\infty) = \limsup_{r \longrightarrow \infty} r \int_r^\infty tW(t)dt < \infty. \tag{6}$$

We prove for $n > 3$

Theorem 1. *Let $V(x)$ be a potential in $\mathbb{R}^n$, $n > 3$ satisfying Hypothesis 1 and 2. Let $W(r)$ be such that satisfies (2), (5), (6) and*

$$\frac{n-3}{2(n-2)r^{n-2}} \int_0^r t^{n-1} W(t)dt \leq \tag{7}$$

$$\leq \frac{n-1}{2nr^n} \int_0^r t^{n+1} W(t)dt + \frac{1}{n(n-2)} \int_r^\infty tW(t)dt.$$

Let $\epsilon > 0$ be such that

$$\epsilon + H(\infty) < \frac{(n-1)(n-3)}{2}, \tag{8}$$

then the unique solution of (1) satisfies

$$\epsilon \sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_0^\infty |\nabla u(x,t)|^2 dtdx + \epsilon \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_0^\infty |u(x,t)|^2 dtdx +$$

$$H(0) \left(\int_{\mathbb{R}^n} \int_0^\infty \frac{|\partial_\tau u(x,t)|^2}{|x|} dtdx + \int_{\mathbb{R}^n} \int_0^\infty |u(x,t)|^2 (\partial_r V)_+(x) dtdx \right) \tag{9}$$

$$\leq C(n,V,W) ||u_0||^2_{H^{\frac{1}{2}}(\mathbb{R}^n)},$$

where ∂_τ denotes the spherical tangential component of the gradient.

Remarks

- The condition that naturally appears in the integration by parts is the usual radiation term

$$\Im \left(\int_{\mathbb{R}^n} \bar{u}(x,T) \nabla u(x,T) \nabla \Phi_R(x) dx \right),$$

for a family of test functions Φ_R. If we know a priori that this term is bounded, then the left hand side of (9) is also bounded. As a consequence we rule out the existence of 0-resonances.
- The standard function, see the corollary, satisfying (7) and (8) is

$$W(r) = \frac{c}{|x|^3} \tag{10}$$

when

$$c < \frac{(n-1)(n-3)}{2}. \tag{11}$$

-see [2].

Theorem 2. *Let $V(x)$ be a potential in $\mathbb{R}^3$ satisfying Hypothesis 1 and 2. Let $W(r)$ satisfy (2) and*

$$\eta + \int_0^\infty t^2 W(t) dt < 1 \tag{12}$$

for $\eta > 0$.

Then the unique solution of (1) satisfies

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_0^\infty \frac{|\partial_\tau u(x,t)|^2}{|x|} dt dx + \int_{\mathbb{R}^3} \int_0^\infty |u(x,t)|^2 (partial_r V)_+(x) dt dx \\ &+ \sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_0^\infty |\nabla u(x,t)|^2 dt dx \\ &+ \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_0^\infty |u(x,t)|^2 dt dx \leq \frac{C(V,W)}{\eta} ||u_0||^2_{H^{\frac{1}{2}}(\mathbb{R}^3)}. \end{aligned} \tag{13}$$

The following theorem relaxes the radiality in the assumptions on the potential.

Theorem 3. *Let $V(x)$ be a potential in $\mathbb{R}^n$ $n \geq 3$ satisfying Hypothesis 1 and 2. Let $W(x)$ be such that satisfies*

$$(\partial_r V)_-(x) \leq W(x) \tag{14}$$

for some nonnegative function $W(x)$ which can be written as

$$W(x) = \sum_{j=0}^\infty w_j(x) \tag{15}$$

with

$$supp\ w_j \subset \{x \in \mathbb{R}^n : \quad 2^{j-2} < |x| \leq 2^{j-1}\} = \Omega_j, \qquad j \geq 1;$$

$$supp\ w_0 \subset \{x \in \mathbb{R}^n : \quad |x| \leq 2^{-1}\} = \Omega_0.$$

Assume that the a priori estimate

$$\int_{\mathbb{R}^n} w_j(x) |u(x)|^2 dx \leq c(w_j) \int_{\mathbb{R}^n} |\nabla u|^2 dx \tag{16}$$

holds and that for $0 < \gamma < 1$

$$\sum_{j=0}^\infty 2^j c(w_j) < \frac{(n-1)(n-3)}{4}(1-\gamma) \qquad n > 3, \tag{17}$$

$$\sum_{j=0}^\infty 2^j c(w_j) < \frac{3}{7}(1-\gamma) \qquad n = 3. \tag{18}$$

Then the unique solution of (1) *satisfies*

$$\begin{aligned}(19)\quad \gamma \sup_{R>0} \frac{1}{R} \int_{B(0,R)} \int_0^\infty |\nabla u(x,t)|^2 dtdx \\ + \gamma(n-3) \int_{\mathbb{R}^n} \int_0^\infty \frac{|u(x,t)|^2}{|x|^3} dtdx \\ + \gamma \sup_{R>0} \frac{1}{R^3} \int_{B(0,R)} \int_0^\infty |u(x,t)|^2 dtdx \\ + \int_{\mathbb{R}^n} \int_0^\infty \frac{|\partial_\tau u(x,t)|^2}{|x|} dtdx \\ + \int_{\mathbb{R}^n} \int_0^\infty |u(x,t)|^2 (\partial_r V)_+(x) dtdx \\ \leq C(n,V,W) ||u_0||^2_{H^{\frac{1}{2}}(\mathbb{R}^n)}.\end{aligned}$$

Remark

- We could allow a slightly more general situation. Namely to write $W = W_1 + \delta W_2$ with W_1 as in either in Theorem 1 or Theorem 2, and W_2 as in Theorem 3, choosing δ small enough depending either on (8) or on (12).

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Well-posedness of the Cauchy problem for the modified KdV equation

Yoshio Tsutsumi

(joint work with Hideo Takaoka)

We consider the time local well-posedness of the Cauchy problem for the modified Korteweg-de Vries equation on the one-dimensional torus $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$.

$$(1)\qquad \partial_t u + \partial_x^3 u + u^2 \partial_x u = 0, \qquad t \in [-T,T], \quad x \in \mathbf{T},$$

$$(2)\qquad u(0,x) = u_0(x), \qquad x \in \mathbf{T},$$

where T is a positive constant and unknown function u is real-valued. If u is a "nice" solution of (1)-(2), then we have the conservation law of L^2 norm, that is, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Therefore, when we change the spatial variable x to $x + ct$ with $c = \|u_0\|_{L^2}^2$, equation (1) can be rewritten as follows.

$$(3)\ \partial_t u + \partial_x^3 u + \big(u^2 - \frac{1}{2\pi}\int_{\mathbf{T}} u^2(t,x)\ dx\big)\partial_x u = 0, \quad t \in [-T,T], \quad x \in \mathbf{T}.$$

Hereafter, we consider equation (3) instead of (1), since equation (3) is equivalent to (1) under suitable assumptions.

In [1], Bourgain proved that (2)-(3) is locally well-posed in H^s, $s \geq 1/2$. His proof is based on the trilinear estimate in terms of the Fourier restriction norms relevant to the linear KdV equation. Bourgain's trilinear estimate is sharp, because it fails for $s < 1/2$ (see Kenig, Ponce and Vega [7]). In [4], Christ, Colliander and Tao showed that if $1/2 > s > -1$, then (2)-(3) is ill-posed in H^s in such a sense that the uniformly continuous dependence of solution on initial data breaks down. The uniformly continuous dependence means the modulus of continuity for the solution map depends only on the size of initial data (see, e.g., [2] and [8] for related results on the ill-posedness). However, it is known that while the uniformly continuous dependence fails, the continuous dependence holds for some nonlinear evolution equations. In this respect, the requirement of uniformly continuous dependence for the well-posedness conception seems slightly too strong. In fact, the authors [9] have proved that when $1/2 > s > 3/8$, (2)-(3) is time locally well-posed in H^s, though the dependence of solutions on initial data is not uniformly continuous. In this talk, we describe an improvement on the result in [9].

Before we state our theorem, we define the function spaces, which we work with. For b, $s \in \mathbf{R}$, we put

$$\|v\|_{Z_{b,s}} = \Big(\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle k\rangle^{2s}\langle \tau - k^3 - k|\hat{u}_0(k)|^2\rangle^{2b}|\tilde{v}(\tau,k)|^2\ d\tau\Big)^{1/2},$$

$$Z_{b,s} = \{v \in \mathcal{S}'(\mathbf{R}\times\mathbf{R});\ v(t,x+2\pi) = v(t,x),\ \|v\|_{Z_{b,s}} < +\infty\},$$

where u_0 is the initial data given in (2), $\langle a\rangle = (1+|a|^2)^{1/2}$, and $\hat{u}$ and $\tilde{u}$ denote the Fourier transforms of u with respect to x only and with respect to both t and x, respectively.

We have the following theorem.

Theorem. *Let $1/2 > s > 1/3$. For any $u_0 \in H^s$, there exists $T = T(\|u_0\|_{H^s}) > 0$ such that (2)-(3) has a unique solution on $[-T,T]$ satisfying*

$$(4) \qquad u \in C([-T,T];H^s), \qquad \varphi u \in Z^T_{1/2,s},$$

where φ is a C^∞ function on $\mathbf{R}$ with its support included in $[-T,T]$. Moreover, let $\{u_{0n}\}$ be a sequence in H^s such that $u_{0n} \to u_0$ in H^s and let u_n be solutions of (3) with $u_n(0) = u_{0n}$. Then, $u_n \to u$ in $C([-T',T'];H^s)$ for any T' with $0 < T' < T$ as $n \to \infty$.

Remark. In [6], by the inverse scattering method, Kappeler and Topalov have proved the global well-posedness in H^s, $s \geq -1$ for the KdV equation on $\mathbf{T}$. In [5], they have also showed the global well-posedness in H^s, $s \geq 0$ for the modified KdV equation on $\mathbf{T}$ by converting the results of the KdV to the case of the modified KdV with the Miura transformation. Their results are better than our theorem in such two respects that they can cover weaker spaces and that they can show the global existence. On the other hand, the uniqueness in [5] and [6] means that a solution constructed through a limiting procedure of smooth solutions is unique, but our theorem gives function spaces, where a solution is unique. Actually, our theorem implies that all solutions constructed through a limiting procedure of smooth solutions belong to class (4) as well as solutions in class (4) constructed by other methods, for example, by the Galerkin method, are unique.

In order to prove Theorem, we use the $Z_{b,s}$ norm, which is a variant of the Fourier restriction norm. Furthermore, we take advantage of the cancellation satisfied by the solution for the estimate of low-high frequency interaction.

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Well-posedness of KdV on $H^{-1}(\mathbb{T})$

Thomas Kappeler and Peter Topalov

Let us consider the Initial Value Problem (IVP) for the Korteweg-deVries equation on the circle

$$v_t = -v_{xxx} + 6vv_x \qquad t \in \mathbb{R},\ x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$$
$$v\big|_{t=0} = q \in H^{\alpha}(\mathbb{T}).$$

This problem has been studied extensively. In particular it is known that for $q \in C^{\infty}(\mathbb{T})$, the (IVP) admits a unique solution $\mathcal{S}(t,q)$ which exists for all times (see [1]). Our aim is to solve the (IVP) for very rough initial data such as distributions in the Sobolev space $H^{-1}(\mathbb{T})$.

We say that a continuous curve $\gamma : [T_1, T_2] \to H^\alpha(\mathbb{T})$ with $T_1 < 0 < T_2, \gamma(0) = q$ and $\alpha \in \mathbb{R}$ is a solution of (IVP) if for any $T_1 < t < T_2$ and for any sequence $(q_k)_{k\geq 1} \subseteq C^\infty(\mathbb{T})$ with $q = \lim_{k\to\infty} q_k$ in $H^\alpha(\mathbb{T})$, the solutions $\mathcal{S}(\cdot, q_k)$ have the property that $\gamma(t) = \lim_{k\to\infty} \mathcal{S}(t, q_k)$ in $H^\alpha(\mathbb{T})$. It then follows from the definition of a solution of (IVP) that it is unique whenever it exists. If the solution of (IVP) exists, we denote it by $\mathcal{S}(t, q)$.

The above (IVP) is said to be globally [uniformly] C^0-wellposed on $H^\alpha(\mathbb{T})$ if for any $q \in H^\alpha(\mathbb{T})$ the solution $\mathcal{S}(t, q)$ exists globally in time and the solution map $\mathcal{S}$ is continuous [uniformly continuous on bounded sets] as a map $\mathcal{S} : H^\alpha(\mathbb{T}) \to C^0(\mathbb{R}, H^\alpha(\mathbb{T}))$.

Theorem 1. *([7]) KdV is globally C^0-wellposed on $H^\alpha(\mathbb{T})$ for any $-1 \leq \alpha \leq 0$.*

Remarks: (1) Theorem 1 improves in particular on earlier results of [2], [3], [12], [5]. Using earlier results, it is proved in [5] that KdV is globally uniformly C^0-wellposed on $H_0^\alpha(\mathbb{T})$ for any $\alpha \geq -1/2$.

(2) In [4] it is shown that KdV is *not* uniformly C^0-wellposed on $H_0^\alpha(\mathbb{T})$ for $-2 < \alpha < -1/2$ where $H_0^\alpha(\mathbb{T}) = \{q \in H^\alpha(\mathbb{T}) \mid \int_{\mathbb{T}} q = 0\}$. See also [3].

The following theorem states that well known features [15] of solutions of (IVP) for smooth initial data continue to hold for rough initial data.

Theorem 2. *([7]) For any $q \in H^\alpha(\mathbb{T})$ with $-1 \leq \alpha \leq 0$, the solution of (IVP) has the following properties:*

(i) *the orbit $t \mapsto \mathcal{S}(t, q)$ is relatively compact.*

(ii) *$t \mapsto \mathcal{S}(t, q)$ is almost periodic.*

Theorem 1 and Theorem 2 can be applied to obtain corresponding results for the IVP of the modified KdV (mKdV)

$$u_t = -u_{xxx} + 6u^2 u_x \qquad t \in \mathbb{R},\ x \in \mathbb{T}$$
$$u\Big|_{t=0} = r \in H^\alpha(\mathbb{T}).$$

Theorem 3. *([8]) mKdV is globally C^0-wellposed on $H^\alpha(\mathbb{T})$ for $0 \leq \alpha \leq 1$.*

Remarks: (1) Theorem 3 improves on earlier results of [2], [12], [5]. Using earlier results it is proved in [5] that mKdV is globally uniformly C^0-wellposed on $H^\alpha(\mathbb{T})$ for any $\alpha \geq 1/2$.

(2) In [4] it is shown that mKdV is *not* uniformly C^0-wellposed on $H_0^\alpha(\mathbb{T})$ for $-1 < \alpha < 1/2$. See also [3].

Besides Theorem 1, the main ingredient of the proof of Theorem 3 is the following result on the Miura map, $B : L^2(\mathbb{T}) \to H^{-1}(\mathbb{T}), r \mapsto r_x + r^2$, first introduced by Miura [16] and proved to be a Bäcklund transformation, mapping solutions of mKdV to solutions of KdV.

Theorem 4. *([10])*

(i) *For any $\alpha \geq 0$, the Miura map $B : H^\alpha(\mathbb{T}) \to H^{\alpha-1}(\mathbb{T})$ is a global fold.*

(ii) *Restricted to $H_0^\alpha(\mathbb{T})$, B is a real analytic isomorphism onto the real analytic submanifold $H_0^{\alpha-1}(\mathbb{T}) := \{q \in H^{\alpha-1}(\mathbb{T}) \mid \lambda_0(q) = 0\}$ where $\lambda_0(q)$*

denotes the lowest eigenvalue in the periodic spectrum of the operator $-d^2/dx^2 + q$.

Remark: Theorem 4 is based on earlier results on the Riccati map [9] which used as one of the ingredients estimates on the gaps of the periodic spectrum of impedance operators of [13]. Some of the results in [9] have been obtained independently by [14].

The main ingredient in the proof of Theorem 1 is a result on the normal form of the Korteweg-deVries equation considered as an integrable Hamiltonian system. To formulate it, introduce the following model spaces ($\alpha \in \mathbb{R}$)

$$h^\alpha := \{(x_k, y_k)_{k\geq 1} \,\big|\, x_k, y_k \in \mathbb{R};\ \sum_{k\geq 1} k^{2\alpha}(x_k^2 + y_k^2) < \infty\}$$

with the standard Poisson bracket where $\{x_k, y_k\} = 1 = -\{y_k, x_k\}$ and all other brackets between the coordinate functions vanish.

On the space $H_0^\alpha(\mathbb{T}) := \{q = \sum_{k\neq 0} \hat{q}_k e^{2\pi i k x} \,\big|\, q \in H^\alpha(\mathbb{T})\}$ we consider the Poisson bracket introduced by Gardner and, independently, by Faddeev and Zakharov

$$\{F, G\} = \int_{\mathbb{T}} \frac{\partial F}{\partial q(x)} \frac{d}{dx} \frac{\partial G}{\partial q(x)} dx.$$

Theorem 5. *(*[11], [6]*) There exists a real analytic diffeomorphism* $\Omega : H_0^{-1}(\mathbb{T}) \to h^{-1/2}$ *so that*

- (i) Ω *preserves the Poisson bracket;*
- (ii) *for any* $-1 \leq \alpha \leq 0$, *the restriction* Ω_α *of* Ω *to* $H_0^\alpha(\mathbb{T})$ *is a real analytic isomorphism,* $\Omega_\alpha : H_0^\alpha(\mathbb{T}) \to h^{\alpha+1/2}$*;*
- (iii) *on* $H_0^1(\mathbb{T})$, *the KdV Hamiltonian* $\mathcal{H}(q) = \int_{\mathbb{T}} (\frac{1}{2} q_x^2 + q^3) dx$, *when expressed in the new coordinates* $(x_k, y_k)_{k\geq 1}$, *is a real analytic function of the actions* $I_k := (x_k^2 + y_k^2)/2$ $(k \geq 1)$ *alone.*

Remark: In [11] it is shown that $\Omega_0 : L_0^2 \to h^{1/2}$ is a real analytic isomorphism with properties (i) and (iii). Moreover it is proved that for any $\alpha \in \mathbb{N}$, the restriction Ω_α of Ω to $H_0^\alpha(\mathbb{T})$ is a real analytic isomorphism, $\Omega_\alpha : H_0^\alpha(\mathbb{T}) \to h^{\alpha+1/2}$. This result has been extended in [6] as formulated in Theorem 5.

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Global Regularity for the Yang–Mills Equations on High Dimensional Minkowski Space

Joachim Krieger and Jacob Sterbenz

Let G be a compact semisimple Lie group, and denote $M = \mathbf{R}^{n+1}$, the $(n+1)$-dimensional Minkowski space, equipped with the Lorentzian metric

$$m_{\alpha\beta} = \mathrm{diag}(-1, 1, \dots, 1)$$

Throughout $n \geq 6$. We consider connections $(A_\alpha), \alpha = 0, \dots, n$, on the bundle $V = M \times \mathfrak{g}$, and their associated curvature components

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

The Yang-Mills(YM) equations describe those connections which are extrema with respect to the functional

$$L(F) = -\frac{1}{4}\int_M < F_{\alpha\beta}, F^{\alpha\beta} > dV_M$$

where

$$< A, B >= tr(AB^*),\ F^{\alpha\beta} = m^{\alpha\gamma} m^{\beta\delta} F_{\gamma\delta}$$

Thus they satisfy the corresponding Euler-Lagrange equations, which read

$$D_\beta F^{\alpha\beta} = 0,\ D_\alpha = \partial_\alpha + A_\alpha \tag{1}$$

Writing these out in terms of the A_α, one deduces a system of nonlinear wave equations describing the evolution of the connection components. Specifically, we get

$$\Box A_\alpha = \partial_\alpha(\partial_\beta A^\beta) + \partial^\beta[A_\alpha, A_\beta] + [A^\beta, F_{\alpha\beta}]$$

Correspondingly, it is natural to pose the Cauchy problem, which we formulate as follows:
given initial data $(F_{\alpha\beta}(0), A_\alpha(0))$ satisfying the natural compatibility relations, construct at least locally a solution pair $(F_{\alpha\beta}(t), A_\alpha(t))$. In particular, we are interested in the question of global-in-time solutions, and more specifically the regularity properties of the solutions. Numerical evidence suggests that large smooth data may result in singular solutions, so it is necessary to impose a smallness condition on the initial data to prevent singularities. To motivate this condition, note that the equations (1) are invariant under the scaling transformation

$$F_{\alpha\beta}(t,x) \to \lambda^2 F_{\alpha\beta}(\lambda t, \lambda x)$$

At the level of the connection form, this corresponds to the transformation

$$A_\alpha(t,x) \to \lambda A_\alpha(\lambda t, \lambda x)$$

Thus the following norm of the initial data is left invariant under this scaling:

$$||F||_{\dot{H}^{\frac{n-4}{2}}} := \sum_{|I|=\frac{n-4}{2}} ||D^I F||^2_{L^2_x},\ D^I = D_{i_1} D_{i_2} \dots D_{i_n}$$

In particular, local well-posedness with respect to this norm implies global well-posedness. While we cannot show well-posedness, we can prove the following result:

Theorem 1. *There exists a number $\epsilon > 0$ such that for all smooth initial data $(F_{\alpha\beta}(0), A_\alpha(0))$, satisfying*

$$||F||_{\dot{H}^{\frac{n-4}{2}}} < \epsilon,$$

there exists a unique smooth solution $(F_{\alpha\beta}(t), A_\alpha(t))$ for (1).

This result is similar to earlier results on the Wave Maps(WM) equation [5], [2], as well as the Maxwell-Klein-Gordon(MKG) equation [4]. It shares with these the crucial property of an intrinsic Gauge invariance of the equations. Specifically, the equations (1) are carried into themselves upon performing a change of connection form as follows

$$\tilde{A}_\alpha = g\partial_\alpha(g^{-1}) + gA_\alpha g^{-1}$$

where g is a section of the bundle $M \times G$. However, while judicious choice of a Gauge renders the nonlinearity of WM amenable to estimation by means of Strichartz type norms (for spatial dimensions ≥ 4) and use of the standard Duhamel parametrix, this is not possible for either the MKG or the YM equations. The reason for this is the fact that the curvature of the connection A_α is not a priori small in a suitable norm, whence formulating everything in terms of the Coulomb Gauge won't eliminate the bad terms in the nonlinearity. Nevertheless, working in the Coulomb Gauge $\sum_{i=1}^n \partial_i A_i = 0$ is still useful, and we do so (using a version of Uhlenbeck's fundamental result on the existence of these Gauges). One can then work with the wave equations satisfied by the curvature components, which have roughly the following form:

$$\Box F_{\alpha\beta} = 2[\partial^\nu F_{\alpha\beta}, A_\nu] + \text{error}$$

We then use a strategy inspired by earlier work of Rodnianski-Tao on the MKG equation. What complicates our situation is that we are in a non-abelian context. More precisely, our strategy consists in building the bad terms in the nonlinearity(which are the bilinear terms depicted above)[1] into the wave operator, thus introducing a covariant wave operator[2] of the form

$$\Box_A u = \Box u + [A_\alpha, \partial^\alpha u]$$

for $\mathfrak{g}$-valued functions on M. We then construct an approximate parametrix for the equation $\Box_A u = 0$, of the form

$$u(t,x) = \int_{\mathbf{R}^n} e^{i(t|\xi|+x\cdot\xi)} g_+(\xi,t,x)^{-1}\hat{f}(\xi) g_+(\xi,t,x)d\xi \\ + \int_{\mathbf{R}^n} e^{i(t|\xi|-x\cdot\xi)} g_-(\xi,t,x)^{-1}\hat{f}(\xi) g_-(\xi,t,x)d\xi,$$

where $g_\pm(\xi,t,x)$ are suitable G-valued phase functions. These phase functions have the property of transforming the connection into an approximate Cronstrom-Gauge in direction $(\pm 1,\omega)$, where $\omega = \frac{\xi}{|\xi|} \in S^{n-1}$. More precisely, they satisfy the property

$$g_\pm(\xi,t,x) L^\pm_\omega (g_\pm^{-1})(\xi,t,x) + g_\pm(\xi,t,x) A\cdot(\pm 1,\omega) g_\pm(\xi,t,x)^{-1} \approx 0,$$

where we employ the notation

$$L^\pm_\omega = \partial_t \pm \nabla_x \cdot \omega$$

The gist of the work then consists in identifying the right function spaces to work with (they are refinements of the Strichartz type spaces customarily used; observe that for MKG the standard function spaces suffice, since the Gauge potential there is of a better nature than for YM.) and show that the parametrix is compatible with these function spaces, i. e. satisfies appropriate estimates when $f(x)$ is a frequency localized L^2 function. The method for establishing these estimates is a variant of the TT^* method used to prove ordinary Strichartz estimates for the (flat) Duhamel parametrix, although the details are much more involved due to the rough dependence of the phase functions $g_\pm$ on ξ. The reason for the restriction $n \geq 6$ is the fact that in lower dimensions the available Strichartz type estimates are no longer enough to bound the error terms.

[1] We call those terms 'bad' which cannot be estimated with respect to the Duhamel space $L^1_t \dot{H}^{\frac{n-6}{2}}$ by means of Strichartz type estimates on A_α.

[2] More precisely, we will not use A_α but a suitably microlocalized version thereof

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On almost parallel vortex filaments

Gustavo Ponce

(joint work with C. E. Kenig and L. Vega)

In a 2-D incompressible, homogeneous, ideal fluid the vorticity is an scalar function $w : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, whose time evolution is modeled by the transport equation

$$\partial_t w + u^j \partial_{x_j} w = 0,$$

where $u = (u^1, u^2)$ is the velocity. From this equation , and the Biot-Savart law, one has that an initial N-point vortices with strength Γ_k, $k = 1, .., N$, and initial position $(X_k(0))_{k=1}^N$, evolves in time to $(X_k(t))_{k=1}^N$ portrayed by the dynamical system (in complex notation)

$$\frac{d}{dt} X_k = i \sum_{j \neq k} \Gamma_j \frac{X_k - X_j}{|X_k - X_j|^2} = -\nabla^{\perp}(-\Delta)^{-1} w(X_k(t), t), \quad k = 1, .., N, \tag{1}$$

with $X_k(t) = (x_k(t) + i y_k(t))$, and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. This system can be rewritten in the Hamiltonian form

$$\Gamma_k \frac{dx_k}{dt} = \partial_{y_k} H, \qquad \Gamma_k \frac{dy_k}{dt} = -\partial_{x_k} H, \qquad k = 1, .., N,$$

where $H = \sum_{j \neq k} 2\Gamma_j \Gamma_k \log |X_k - Xj|$.

In addition to the Hamiltonian function solutions of (1) also preserve :

$$I = \sum_{j=1}^{N} \Gamma_j ((x_j(t))^2 + (y_j(t))^2), \qquad \overline{x}_0 = \sum_{j=1}^{N} \Gamma_j x_j(t), \quad \overline{y}_0 = \sum_{j=1}^{N} \Gamma_j y_j(t).$$

When $\Gamma^* = \sum_{j=1}^N \Gamma_j \neq 0$ the point $(x_0, y_0) = (\overline{x}_0, \overline{y}_0)/\Gamma^*$, is called "the center of vorticity". We recall some special solution of the system (1), see [4].

Case $N = 2$: All solutions are relative equilibrium ones since $d = |X_1(t) - X_2(t)|$ is a constant of their motion. If $\Gamma^* \neq 0$ the pair of points rotates about the center of vorticity with angular velocity $\omega = \Gamma^*/(2\pi d^2)$. If $\Gamma^* = 0$ they translate with speed $((\Gamma_1^2 + \Gamma_2^2)/2)^{1/2}/2\pi d$.

Case $N = 3$: If $(X_1(0), X_2(0), X_3(0))$ is an equilateral triangle of side d one has a relative equilibrium solution. For $\Gamma^* \neq 0$ it rotates about the center of vorticity with angular velocity $\omega = \Gamma^*/(2\pi d^2)$. If $\Gamma^* = 0$ it translates with speed $((\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2)/2)^{1/2}/2\pi d$. This equilateral triangle is unstable if $\Gamma_3 < 0$ and $\Gamma_1^{-1} + \Gamma_2^{-1} + \Gamma_3^{-1} > 0$. Otherwise it is neutrally stable.

Case $N > 3$: N point vortices of identical strength Γ placed at the vertex of a regular N-polygon of radius R form a relative equilibrium, rotating about the center of vorticity with angular frequency $\omega = \Gamma(N-1)/4\pi R^2$. This configuration is stable if and only if $N \leq 7$.

Also, $X_j(t) = x_j(t) + iy_j(t)$ describes the time evolution of N-perfect parallel vortex filaments perpendicular to the plane containing the points $X_j(t)$'s.

The equation modeling the motion of a self-induced vortex filament $Z = Z(\sigma, t)$ (da Rios, Hama, and Arms, see[1]) is

$$\partial_t Z = c\hat{t} \times \partial_\sigma^2 Z, \tag{2}$$

where $\hat{t}$ is the unit tangent vector, $\times$ represents the cross product of vectors, and c is a constant depending on the circulation and the reference time chosen. For an almost parallel filament to the z-axis of the form $Z(\sigma, t) = (0, 0, \sigma) + \epsilon\,(x(\sigma, t), y(\sigma, t), 0)$, $\epsilon \ll 1$, the leading order in ϵ of the equation (2) is $\partial_t Z = i\,\Gamma \partial_\sigma^2 Z$.

In [3], Klein, Majda, and Damodaran proposed the following model to describe the time evolution of N-vortex filaments nearly parallel to the z-axis

$$\begin{cases} \partial_t \Psi_j = i\,\Gamma_j \partial_\sigma^2 \Psi_j + i \displaystyle\sum_{k \neq j} \Gamma_k \frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2}, \quad j = 1, .., N, \\ \Psi_j(\sigma, 0) = \Psi_{0,j}(\sigma), \end{cases} \tag{3}$$

$\Psi_j(\sigma, t) = x_j(\sigma, t) + iy_j(\sigma, t)$, with $\sigma, t \in \mathbb{R}$, Ψ_j is the position of the j-th filament, σ parametrizes the z-axis, t is the time, and Γ_j the circulation of the j-th vortex filament.

Observe that the solutions of the ode system (1) $(X_k(t))_{k=1}^N$ are also solutions of the pde system (3).

In [2] we proved the following results.

We define the perturbation $u_k(\sigma, t)$ as $\Psi_k(\sigma, t) = X_k(t) - u_k(\sigma, t)$, $k = 1, 2, 3$.

Theorem (Global Existence)

Assume $N = 3$ (or $N = 2$), $\Gamma_j = \Gamma$, $j = 1, .., N$. If $(X_1, X_2, X_3)(0)$ forms an equilateral triangle of side $d > 0$, then there exists $\delta = \delta(d; \Gamma) > 0$ such that for any $(u_{0_1}, u_{0_2}, u_{0_3}) \in (H^1(\mathbb{R}))^3$ with

$$\mu = \sum_{j=1}^{3} \|u_{0_j}\|_{1,2} \leq \delta,$$

the IVP (3) has a unique global solution

$$\Psi_k(\sigma, t) = X_k(t) - u_k(\sigma, t), \qquad u_k \in C([0, T] : H^1(\mathbb{R})), \quad k = 1, .., 3,$$

and

$$d/4 \leq \inf_{t\in[0,\infty)} \inf_{k\neq j} \|(X_j - X_k)(t) - (u_j - u_k)(\cdot,t))\|_\infty \leq 4d.$$

Special solution for the system (3) for arbitrary N

For $\Gamma_j = \Gamma$, $j = 1,..,N$, system (3) is Galilean invariant, i.e. if $(\Psi_k(\sigma,t))_{K=1}^N$ is a solution of the system (5), then

$$\tilde{\Psi}_{k,\nu}(\sigma,t) = e^{-i\Gamma\nu^2 t} e^{i\nu\sigma} \Psi_k(\sigma - 2\Gamma\nu t, t), \quad k = 1,..,N,$$

is also a solution of the system (3) with data

$$\tilde{\Psi}_{k,\nu}(\sigma,0) = e^{i\nu\sigma}\Psi_k(\sigma,0), \quad k = 1,..,N, \quad \nu \in \mathbb{R}.$$

Consider the solution of the system (1) (and (3)) getting by placing at each vortex of a regular N-polygon of radius R of strength Γ

$$X_j(t) = R\, e^{2\pi i(j-1)/N} e^{2\pi i\omega t}, \quad j = 1,..,N,$$

where $\omega = \Gamma(N-1)/4\pi R^2$ is the angular velocity. So

$$\tilde{\Psi}_{k,\nu}(\sigma,t) = R\, e^{i\nu\sigma} e^{2\pi i(j-1)/N} e^{i(2\pi\omega - \Gamma\nu^2)t}, \quad j = 1,..,N,$$

is a solution of (3) for any $\nu \in \mathbb{R}$. Taking $\nu = \nu^*$ with

$$\nu^* = \left(\frac{\Gamma(N-1)}{2\Gamma R^2}\right)^2 = \nu^*(N,R,\Gamma),$$

we obtain an stationary solution of (3) in the form of a N-helix

$$\tilde{\Psi}_{k,\nu^*}(\sigma,t) = R\, e^{i\nu\sigma} e^{2\pi i(j-1)/N}.$$

For $N = 2,3$ these stationary 3-D solutions are globally stable under small perturbation.

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Smoothing and dispersive estimates for 1D Schrödinger equations with BV coefficients and applications

Nicolas Burq

(joint work with Fabrice Planchon)

Let us consider

$$(1) \qquad i\partial_t u + \partial_x(a(x)\partial_x u) = 0, \quad u(x,t=0) = u_0(x).$$

We take $a \in \mathrm{BV}$, the space of bounded functions whose derivatives are Radon measures. Moreover, we assume a to be real-valued and bounded from below: $0 < m \leq a(x)(\leq M)$. We are interested in proving smoothing and dispersive estimates for the function u. This type of equations has been recently studied by Banica [3] who considered the case where the metric a is piecewise constant (with a finite number of discontinuities). In [3], Banica proved that the solutions of the Schrödinger equation associated to such a metric enjoy the same dispersion estimates (implying Strichartz) as in the case of the constant metric, and conjectured it would hold true for general $a \in \mathrm{BV}$ as well. Unfortunately, her method of proof (which consists in writing a complete description for the evolution problem) leads to constants depending upon the *number* of discontinuities rather than on the norm in BV of the metric and consequently does not extend to more general settings. On the other hand, Castro and Zuazua [5] show that the space BV is more or less optimal: they construct metrics $a \in C^{0,\beta}$ for all $\beta \in [0, 1[$ (but not in BV) and solutions of the corresponding Schrödinger equation for which any *local* dispersive estimate of the type

$$\|u(t,x)\|_{L^1_{\mathrm{loc},t}(L^q_{\mathrm{loc},x})} \leq C\|u_0\|_{H^s}$$

fail if $1/p < 1/2 - s$ (otherwise, the estimate is a trivial consequence of Sobolev embeddings). We can prove that the BV regularity threshold is optimal in a different direction from [5]: there exist a metric $a(x)$ which is in $L^\infty \cap W^{s,1}$ for any $0 \leq s < 1$, bounded from below by $c > 0$, and such that no smoothing effect nor (non trivial) Strichartz estimates are true (even with derivatives loss).

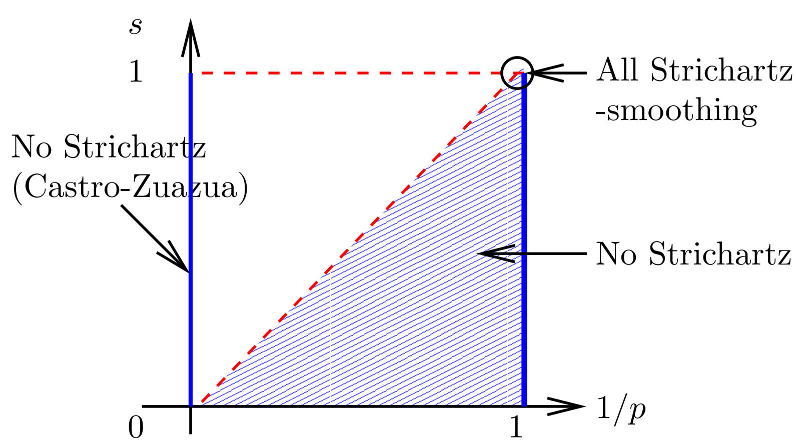

FIGURE 1. Range of regularity $a \in W^{s,p}$

In this talk, we present the natural conjecture, namely that for BV metrics, the Schrödinger equation enjoys the same smoothing, Strichartz and maximal function estimates as for the constant coefficient case, globally in time. In the context of variable coefficients, this appears to be the first case where such a low regularity (including discontinuous functions) is allowed, together with a translation invariant formulation of the decay at infinity (no pointwise decay). Previous works on dispersive estimates, while applying equally to any dimension, dealt with C^2 compact pertubations of the Laplacian ([14]) or short range pertubations with symbol-like decay at infinity ([11]). The idea to use local smoothing to derive Strichartz, however, goes back to Staffilani-Tataru ([14]) in the context of variable coefficients, and was used earlier to obtain full dispersion in [10] where a potential pertubation was treated. All recent works on this topic make definitive use of resolvent estimates for the elliptic operator, see e.g. [12]. Finally, it has to be noticed that Salort [13] recently obtained dispersion (hence, Strichartz) (locally in time) for 1D Schrödinger equations with C^2 coefficients through a completely different approach involving commuting vector fields.

We now say a word on the relevance of non-trapping conditions. In higher dimension, it has been known since the works of Doï [8, 9] and the first author [4] that the non trapping assumption is necessary for the optimal smoothing effect to hold and the study of eigenfunctions on compact manifolds somewhat shows that a non trapping condition is also necessary for Strichartz estimates. In the one dimensional case, a smooth metric is always *non trapping* as can be easily seen by a simple change of variables. However, some trapping-related behaviours (namely the existence of waves localized at a point) appear for metric with regularity below BV (see the work by Castro-Zuazua [5]). In fact the assumption $a \in \mathrm{BV}$ ensures some kind of non trappingness and this fact has been known for a while in the different context of control theory [7]. Let us picture this on the model case of piecewise constant metrics : consider a wave coming from minus infinity. Then the wave propagates freely (at a constant speed) until it reaches the first discontinuity. At this point some part of the wave is reflected whereas some part is transmitted. It is easy to see that a fixed amount of the energy (depending on the size of the jump of the velocities) is transmitted. Then the transmitted wave propagate freely until it reaches the second discontinuity, and so on and so forth... Finally, we get that a fixed part of the energy of the incoming wave is transmitted at the other end and propagates freely to plus infinity. Whereas some part of the energy can remained trapped by multiple reflections, this shows that some part is not trapped. As a consequence, our geometry is (at least weakly) non trapping. This phenomenon is clearly specific to the one dimensional case as can be easily seen (using Snell law of refraction) with simple models involving only two speeds. Finally let us say a word about the method of proof:

- We first prove a smoothing estimate which is the key to all subsequent results, by an elementary integration by parts argument
- We deduce Strichartz and maximal function estimates by combining our smoothing estimate with known estimates for the flat case (and a suitably modified version, with reversed norms, of Christ-Kiselev Lemma [6]).
- To perform spectral localizations, we use some results of Auscher-Tchamitchian [2] and Auscher-MacIntosh-Tchamitchian [1] which imply that the spectral localization with respect to the operators $\partial_x a(x)\partial_x$ and ∂_x^2 are reasonably equivalent.

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On the singularity formation for the L^2 critical non linear Schrödinger equation

Pierre Raphael
(joint work with Frank Merle)

We present a series of results obtained in collaboration with Frank Merle concerning the singularity formation for the L^2 critical non linear Schrödinger equation

$$(1) \qquad (NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}} u, & (t,x) \in [0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbb{R}^N \to \mathbb{C} \end{cases}$$

with $u_0 \in H^1 = H^1(\mathbb{R}^N)$ in dimension $N \geq 1$. Local well posedness in the energy space is a standard result from Ginibre, Velo, [3], and from energy type of arguments, it is well known that the power non linearity is the smallest one for which blow up may occur. Our aim in this work is to provide some insight into the singularity formation.

All symmetries of the linear Schrödinger equation are symmetries of (1) and are L^2 isometries. These invariances imply in particular the conservation of the L^2 norm and the energy of the solution which is $E(u(t)) = \frac{1}{2}\int|\nabla u(t)|^2 - \frac{1}{2+\frac{4}{N}}\int|u(t)|^{2+\frac{4}{N}} = E(u_0)$. The variational structure of the ground state Q which is the unique positive solution going to zero at infinity of $\Delta Q - Q^{1+\frac{4}{N}} = Q$, see [1] and [4], and the Hamiltonian structure of (1) yield a sharp criterion for global wellposedness, see [14]: for $|u_0|_{L^2} < |Q|_{L^2}$, the solution is global in H^1. In addition, this condition is sharp: the solitary wave $u(t,x) = Q(x)e^{it}$ is a global solution to (1) while the pseudo-conformal symmetry applied to it yields an explicit blow up solution $S(t,x) = \frac{1}{|t|^{\frac{N}{2}}}Q(\frac{x}{t})e^{-i\frac{|x|^2}{4t}+\frac{i}{t}}$ which blows up at $T=0$ with $|S(t)|_{L^2} = |Q|_{L^2}$. From [6], $S(t)$ is the unique minimal mass blow up solution up to the symmetries.

We now focus onto the perturbative situation when

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \ \text{ with } \ \int Q^2 \leq \int |u_0|^2 < \int Q^2 + \alpha^*\}$$

for some small constant $\alpha^* > 0$. At least two different blow up behaviors are known to possibly occur. There exist in dimension $N = 1,2$ a family of solutions of type $S(t)$ by a result of Bourgain, Wang, [2], that is solutions with $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$ near blow up time. On the other hand, numerical simulations and formal arguments, [5], suggest the existence of solutions blowing up like $|\nabla u(t)|_{L^2} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$ in dimension $N=2$. This behavior should be interpreted as a slow correction to a self similar type of blow up. Perelman proves in [12] in dimension $N=1$ the existence of a solution of this type.

The situation has been clarified in a sequel of papers [7], [8], [9], [10], [11], [13]. All the results assume a positivity property of some explicit quadratic form based on the ground state Q which has been proved in dimension $N=1$ where Q is explicit, and checked numerically in dimension $N = 2,3,4$.

We first have the following theorem which proves the existence of a universal singular structure in space.

Theorem 1 (Existence of a L^2 profile at blow up time, [11]). *Let $N = 1,2,3,4$. There exists a universal constant $\alpha^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$, then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}^*_+ \times \mathbb{R}^N \times \mathbb{R}$ and an asymptotic profile $u^* \in L^2$ such that*

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}}Q\left(\frac{x-x(t)}{\lambda(t)}\right)e^{i\gamma(t)} \to u^* \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T.$$

Moreover, blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbb{R}^N \quad as \quad t \to T.$$

This results implies in particular that the mass that is brought into the singularity is *universal and quantized.* A different type of results allows one to separate within the different blow up regimes.

Theorem 2 (Dynamics of (1), [7], [8], [9], [10], [13]). *Let $N = 1, 2, 3, 4$. There exists a universal constant $\alpha^* > 0$ such that the following holds true. For $u_0 \in H^1$, let $u(t)$ the corresponding solution to (1) with $[0, T)$ its maximum time interval existence on the right in H^1. Let the set*

$$\mathcal{O} = \{u_0 \in \mathcal{B}_{\alpha^*} \ \ with \ T < +\infty \ \ and \ \ \lim_{t\to T} \frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left(\frac{T-t}{\log|\log(T-t)|}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}\},$$

then:

(1) *Dynamic of non positive energy solutions:*
$$\{u_0 \in \mathcal{B}_{\alpha^*} \ \ with \ \ E_0 \le 0 \ \ and \ \ \int |u_0|^2 > \int Q^2\} \subset \mathcal{O}.$$
(2) *Stability of the log-log regime: $\mathcal{O}$ is open in H^1. Moreover, for $u_0 \in \mathcal{O}$, let u^* given by Theorem 1, then $u^* \notin H^1$.*
(3) *If $0 < T < +\infty$ and $u_0 \in \mathcal{B}_{\alpha^*}$ does not belong to $\mathcal{O}$, then $E_0 > 0$. Moreover, we have $|\nabla u(t)|_{L^2} \ge \frac{C(u_0)}{(T-t)}$ for t close to T and $u^* \in H^1$.*

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Sobolev regularity for scalar conservation laws

Benoît Perthame

We present a proof of Sobolev regularity for scalar conservation laws in the framework of entropic or quasi-entropic solutions. It also applies to variant problems as systems like isentropic gas dynamics with $\gamma = 3$, or to some variational problems arising in thin micromagnetic films. The steps for the derivation are firstly the kinetic formulation which allows to linearize the PDE to the expense of a new variable and a nonlinear unknown. The second ingredient is to use velocity regularity for the solution to the transport equation under consideration.

1. Kinetic formulation

Kinetic formulations allow to consider nonlinear problems (balance laws or variational problems) and, using a nonlinear function f of the unknown, to transform these problems in a singular linear transport equation on f. The simplest example is that of the entropy solution $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ to a multidimensional scalar conservation law

$$\begin{aligned} &\partial_t u(t,x) + \mathrm{div} A(u) = 0, \qquad t > 0,\ x \in \mathbb{R}^d, \\ &\partial_t S(u(t,x)) + \mathrm{div}\, \eta^S(u) \le 0, \end{aligned} \tag{1}$$

for all convex function $S(\cdot)$ with $S(0) = 0$ and using the notations $\eta^S(u) = \int_0^u S'(\cdot)a(\cdot)$, $a = A' : \mathbb{R} \to \mathbb{R}^d$. Then, we define, for $v \in \mathbb{R}$, the 'equilibrium' function $f(t,x,v)$ thanks to

$$f(t,x,v) = \begin{cases} +1, & \text{for } \ 0 < v < u(t,x), \\ -1, & \text{for } \ u(t,x) < v < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The theory of kinetic formulations ([7, 8]) states that (1) is equivalent to write the kinetic equation on f

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v m(t,x,v), \tag{3}$$

for some unknown nonnegative bounded measure m such that

$$\int_0^\infty \int_{\mathbb{R}\times\mathbb{R}^d} m(t,x,v)\, dt\, dv\, dx \le \frac{1}{2} \|u^0\|^2_{L^2(\mathbb{R}^d)}.$$

The case of dispersive/diffusive limits for scalar conservation laws leads to the same kinetic formulation except that $m(t,x,v)$ is a bounded measure without sign (see [5]). This is also the case of thin micromagnetic films (see [2], [6] and the references therein). We refer to the case where m is a (unsigned) bounded measure as the *quasi-entropic* case.

Theorem 1. *Let $u(t,x) \in C(\mathbb{R}^+; L^1(\mathbb{R}^d))$ an entropy solution to a nondegenerate (in the sense of (9)) multidimensional scalar conservation law (1), with $u(t=0) \in L^1 \cap L^\infty(\mathbb{R}^d)$, then locally we have*

$$u \in W^{s,r}_{t,x} \qquad \text{for all } s < \frac{1}{3},\ r < \frac{3}{2}.$$

This regularity was obtained in [7] with a more complicate argument which involves a bootstrap of averaging lemmas combined with the L^1 contraction property. Notice that for the *entropy solution* in 1D, the optimal regularizing effect, from $u^0 \in L^\infty$ to $u(t) \in BV$ is a classical result of Oleinik and multidimensional cases are not known. For quasi-entropic solutions, the exponents in theorem 1 are sharp as proved in [1].

2. The averaging lemmas

We consider the following equation

$$v \cdot \nabla_x f = \Delta_x^{\alpha/2}\, g, \quad x \in \mathbb{R}^d,\ v \in \mathbb{R}^d. \tag{4}$$

Notice that the steady case contains the evolution case up to changing (t,x) in x and $(1,v)$ in v. Now we choose any $\phi \in C_c^\infty(\mathbb{R}^d)$ and define

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v)\phi(v)dv. \tag{5}$$

Assume that

$$\begin{aligned} &g \in L^p(\mathbb{R}^d,\ W_v^{\beta,p}(\mathbb{R}^d)), \quad 1 < p \le 2,\ \beta \le \frac{1}{2},\\ &f \in L^q(\mathbb{R}^d,\ W_v^{\gamma,q}(\mathbb{R}^d)), \quad 1 < q \le 2,\ 1 - \frac{1}{q} < \gamma \le \frac{1}{2}. \end{aligned} \tag{6}$$

We also point out that the results extend to exponents p or q larger than 2. Then, we have to replace p and q by $\min(p,\bar{p})$ and $\min(q,\bar{q})$ in formula (7) below ($\bar{p}$ denotes here the conjugate exponent to p).

As usual for averaging lemmas ([4, 3, 8]), we state that the average ρ is in fact more regular than f itself. This can be quantified as follows (see [6])

Theorem 2. *(Case $0 \le \alpha < 1$) Let f, g satisfy (4) and (6), then we have for $s' < s = \theta(1-\alpha)$ and $r' < r$ with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$,*

$$\|\rho\|_{W^{s',r'}_{loc}} \le C\left(\|g\|_{L^p_x W_v^{\beta,p}} + \|f\|_{L^q_x W_v^{\gamma,q}}\right),$$

$$\text{with} \quad \theta = \frac{1+\gamma-1/q}{1+\gamma-\beta+1/p-1/q}. \tag{7}$$

For $\gamma = 0$, $\beta \le 0$, we are in a case included in standard averaging lemmas (see in particular [3]). However our result is a bit weaker since it is known in this case that $\rho \in W^{s,r}$ with s and r given by the formulas of Theorem 2.

To conclude let us state that these results are also true for the evolution equation

$$\partial_t f + a(v) \cdot \nabla_x f = \Delta_x^{\alpha/2} g, \tag{8}$$

when the field $v \to a(v)$ satisfies the strongest *non degeneracy condition*, namely: for all $R > 0$, there is a constant $C(R)$ such that for $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$ with $|\xi|+|\tau| \le 1$, then

$$\text{meas}\{v \text{ s. th. } |v| \le R, \text{ and } |a(v).\xi - \tau| \le \varepsilon\} \le C\varepsilon. \tag{9}$$

The regularity on the average ρ is then a regularity in time and space but all the formulas given above for the exponents are exactly the same.

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On Multilinear Oscillatory Integrals

Michael Christ

(joint work with Xiaochun Li, Terence Tao, Christoph Thiele)

Consider multilinear integral operators of the form

$$T_\lambda(f_1, \cdots, f_n) = \int_{\mathbb{R}^d} e^{i\lambda P(y)} \prod_{j=1}^{n} f_j \circ \ell_j(y)\, \eta(y)\, dy$$

where P is a real-valued polynomial, $\lambda \in \mathbb{R}$ is a large parameter, η is a smooth compactly supported cutoff function, and $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$ are surjective linear transformations. Is

$$|T_\lambda(\{f_j\})| \le C|\lambda|^{-\delta} \prod_j \|f_j\|_{L^\infty}$$

uniformly for all functions f_j as $|\lambda| \to \infty$?

The most fundamental example is the inequality

$$\Big| \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{-i\lambda x \cdot y} f(x) g(y) \eta(x,y)\, dx\, dy \Big| \le C|\lambda|^{-d/2} \|f\|_2 \|g\|_2,$$

which implies the L^2 boundedness of the Fourier transform. Here every point $x \in \mathbb{R}^d$ interacts with every point $y \in \mathbb{R}^d$. This talk, in contrast, is concerned with generalizations where the integral is taken over a d-dimensional linear subspace of $\prod_j \mathbb{R}^{d_j}$; most n-tuples of points $(x_1, \cdots, x_n) \in \prod_j \mathbb{R}^{d_j}$ do not interact.

In the linear/bilinear case $n = 2$ this problem has been studied intensively, in particular by Stein and by Phong-Stein but also by many others. For bilinear expressions $\iint_{\mathbb{R}^{d+d}} e^{i\lambda P(x,y)} f(x) g(y) \eta(x,y)\, dx\, dy$ with P polynomial, a power decay bound holds if and only if P is not of the form $p(x) + q(y)$. In the truly multilinear case quite little is known. The focus here is on the basic question of whether there is any decay at all.

From linear experience we expect the case of polynomial phases P to be fundamental. We're putting the strongest norm on the functions f_j not involving any smoothness, and aren't trying to quantify δ.

There is an obvious necessary condition: If $P = \sum_j q_j \circ \ell_j$ for some functions q_j then there's no decay (take $f_j = e^{-i\lambda q_j}$ to cancel out all the apparent oscillation).

Definition. P is nondegenerate relative to $\{\ell_j\}$ if P can not be represented as $\sum_j q_j \circ \ell_j$ for any functions q_j.

Question. Does power decay always hold for nondegenerate polynomial phase functions P? This remains open, even for quadratic polynomials in three variables.

Lemma. (Suppose P homogeneous, to simplify statements.) The following are equivalent:

(1) $P \ne \sum_j q_j \circ \ell_j$ for polynomials q_j of degrees $\le$ degree(P).

(2) $P \ne \sum_j h_j \circ \ell_j$ for any distributions h_j.

(3) There exists a constant-coefficient homogeneous linear partial differential operator $\mathcal{L}$ satisfying $\mathcal{L}(f_j \circ \ell_j) \equiv 0$ for all functions f_j, for all j and $\mathcal{L}(P) \ne 0$.

Warning: Nondegeneracy of P relative to $\{\ell_j : 1 \le j \le n\}$ imposes no bound whatsoever on n in terms of the degree of P and the ambient dimension d.

Definition. P is *simply nondegenerate* if there exists $\mathcal{L}$ of the form $\mathcal{L} = \prod_j (v_j \cdot \nabla)$ which kills all functions $f_j \circ \ell_j$, yet $\mathcal{L}(P)$ does not vanish identically.

Theorem. If P is simply nondegenerate then it satisfies a power decay bound.

Proposition. When each $d_j = d-1$, simple nondegeneracy is equivalent to nondegeneracy. Consequently nondegeneracy is equivalent to the power decay property in the codimension one case $d_j = d - 1$.

Theorem. If each $d_j = 1$ and if the number of functions n satisfies $n < 2d$ then any nondegenerate polynomial P satisfies a power decay bound (under an auxiliary general position hypothesis on $\{\ell_j\}$).

A more elementary question arises in several different ways in the discussion: For what exponents $p_j \in [1,\infty]$ does the multilinear expression make sense for all $f_j \in L^{p_j}$? Bennett, Carbery, and Tao analyzed the global version (for different reasons) and obtained a nice characterization:

Theorem. Let $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$ be surjective linear transformations. Then

$$\int_{\mathbb{R}^d} \prod_j |f_j \circ \ell_j| \, dy \leq C \prod_j \|f_j\|_{L^{p_j}}$$

if and only if $\sum_j p_j^{-1} d_j = d$ and $\sum_j p_j^{-1} \dim(\ell_j(V)) \geq \dim(V)$ for every subspace $V \subset \mathbb{R}^d$.

I've given an alternative proof which also establishes the following generalization:

Theorem.

$$\int_{\mathbb{R}^d \cap \{y : |\ell_0(y)| \leq 1\}} \prod_{j=1}^n |f_j \circ \ell_j(y)| \, dy \leq C \prod_{j=1}^n \|f_j\|_{L^{p_j}}$$

for all measurable f_j if and only if every subspace $V \subset \mathbb{R}^d$ satisfies $d - \dim(V) \geq \sum_j p_j^{-1}\big(d_j - \dim(\ell_j(V))\big)$ and furthermore $\sum_j p_j^{-1} \dim(\ell_j(V)) \geq \dim(V)$ if $V \subset \mathrm{kernel}(\ell_0)$.

The results stated above for multilinear oscillatory integrals fail to cover a well-known example, and the techniques don't yield optimal decay exponents δ. The twisted convolution inequality is $\big|\iint_{\mathbb{C}^n \times \mathbb{C}^n} e^{i\lambda \Im(z \cdot \bar{w})} f_1(z) f_2(w) f_3(z-w)\, dz\, dw\big| \leq C|\lambda|^{-n/2} \prod_j \|f_j\|_2$. This inequality is self-dual in sense that when it is rewritten as a trilinear expression in the three Fourier transforms $\widehat{f_j}$, precisely the same expression is obtained, except for changes in various constants.

The last part of the talk is a preliminary report on joint work with Justin Holmer. We've analyzed the inequality

$$\Big| \int_{\mathbb{R}^d} e^{i\lambda Q(y)} \prod_{j=1}^n f_j \circ \ell_j(y) \eta(y) \, dy \Big| \leq C|\lambda|^{-\delta_0} \prod_j \|f_j\|_{L^2}$$

where Q is a homogeneous quadratic polynomial, all $d_j = D$, all norms on the right-hand side are L^2 norms, and $\delta_0 = \frac{d}{2} - \frac{nD}{4}$ is the largest exponent for which such an estimate isn't ruled out by scaling considerations. Thus we're trying to characterize the maximally nondegenerate quadratic phase functions. We've established a sufficient condition which we believe is also necessary. Unfortunately, we don't yet have a palatable formulation of our sufficient condition, so I discuss only the method of proof without formulating the result.

Our analysis uses an FBI transform. Define $\mathcal{F}(f)(x,\xi) = \langle f, \varphi_{(x,\xi)} \rangle$ where $\varphi_{(x,\xi)}(y) = e^{iy\cdot\xi} e^{-|x-y|^2/2}$. There are a Plancherel identity and inversion formula analogous to those for the Fourier transform. Proving the desired multilinear L^2

bound is equivalent to proving a global inequality without any large parameter, of the form $\big|\int_{\mathbb{R}^d} e^{iQ}\prod_j f_j\circ \ell_j\big| \le C\prod_j \|f_j\|_{L^2}$. Here there is a preferred unit scale. With respect to the FBI transform there is no longer any self-duality.

Expressing each f_j in terms of $\mathcal{F}(f_j)$ yields

$$\int_{\oplus_j T^*(\mathbb{R}^D)} a(x,\xi)\prod_j \mathcal{F}(f_j)(x_j,\xi_j)\,dx\,d\xi$$

where $(x,\xi)=(x_1,\xi_1,\cdots,x_n,\xi_n)\in(\mathbb{R}^{2D})^n$ and $|a(x,\xi)|\le Ce^{-c\,\mathrm{distance}((x,\xi),\Sigma)^2}$ where the linear subspace Σ equals the set of all (x,ξ) for which there exists $y\in\mathbb{R}^d$, necessarily unique, such that $\ell_j(y)=x_j$ for all j and $\nabla Q(y)+\sum_j \ell_j^*(\xi_j)=0$. Moreover a exhibits no useful cancellation or decay on Σ. Thus a good model for this expression is $\int_\Sigma \prod_j \mathcal{F}(f_j)(x_j,\xi_j)\,d\sigma$ where σ is Lebesgue measure on Σ. This is a nonoscillatory multilinear integral operator of precisely the type discussed in the middle portion of this talk.

Under certain hypotheses of general position on $\{\ell_j\}$, the dimension of Σ is always half of the dimension of the ambient space $\oplus_j T^*(\mathbb{R}^{d_j})$. Thus scaling considerations are consistent with a bound $\big|\int_\Sigma \prod_j F_j(x_j,\xi_j)\,d\sigma\big| \le C\prod_j \|F_j\|_{L^2(T^*(\mathbb{R}^{d_j}))}$, and we have $F_j=\mathcal{F}(f_j)\in L^2$ if $f_j\in L^2$ by the Plancherel identity for the FBI transform.

Our preliminary theorem says that the original multilinear oscillatory integral operator satisfies the strongest possible L^2 decay estimate provided that Σ (that is, Σ together with the collection of mappings $\pi_j|_\Sigma$ where $\pi_j:\oplus_i T^*(\mathbb{R}^{d_i})\mapsto T^*(\mathbb{R}^{d_j})$ is the canonical projection) satisfies the hypothesis of the theorem of Bennett, Carbery, and Tao with all exponents $p_j=2$. Special cases include the inequality for twisted convolution, and Plancherel's inequality itself.

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Mass Concentration Properties of Rough Blowup Solutions of Cubic NLS on $\mathbb{R}^2$

James Colliander

This talk described two new results concerning blowup solutions of the initial value problem for the two-dimensional, cubic, focusing nonlinear Schrödinger (NLS) equation:

$$\begin{cases} iu_t+\Delta u+|u|^2u=0, \\ u(0,x)=u_0(x), \qquad x\in\mathbb{R}^2. \end{cases} \tag{1}$$

This problem is L^2-*critical* in the sense that the rescaling $u(t,x)\to\rho[u](t,x)=\rho^{-1}u(\rho^{-2}t,\rho^{-1}x)$ maps solutions to solutions and is an isometry in L^2_x. If global well-posedness fails to hold then there is a finite T^* such that for all $\delta > 0$

$$\|u\|_{L^4_{t\in[0,T^*-\delta],x\in\mathbb{R}^2}} < +\infty, \tag{2}$$

but diverges to infinity for $\delta = 0$.

In the setting of merely L^2_x initial data, if global well-posedness fails to hold for (1) then a nontrivial parabolic concentration of L^2-mass occurs [1] as $t \uparrow T^*$:

$$\limsup_{t\uparrow T^*} \sup_{\substack{\text{cubes } I\subset\mathbb{R}^2 \\ \text{side}(I)<(T^*-t)^{\frac{1}{2}}}} \left(\int_I |u(t,x)|^2 dx\right)^{\frac{1}{2}} \gtrsim \|u_0\|_{L^2}^{-M}. \tag{3}$$

Finite time blowup solutions of (1) with initial data in H^1 are known [9], [7], [8] to satisfy

$$\liminf_{t\uparrow T^*} \sup_{\substack{\text{cubes } I\subset\mathbb{R}^2 \\ \text{side}(I)<(T^*-t)^{\frac{1}{2}-}}} \left(\int_I |u(t,x)|^2 dx\right)^{\frac{1}{2}} \geq \eta(\|u_0\|_{L^2}) \geq \|Q\|_{L^2} \tag{4}$$

where $\frac{1}{2}-$ denotes $\frac{1}{2}-\epsilon$ for any fixed $\epsilon > 0$ and where Q is the *ground state*: the unique positive (up to translations) solution of

$$\Delta w - w + |w|^2 w = 0. \tag{5}$$

A natural question, stated in [6], is to determine whether small L^2-mass concentrations take place when $u_0 \in L^2$. The conjectured answer is no: Solutions of (1) with L^2 initial data and with a finite maximal (forward) existence interval are conjectured to concentrate at least the L^2-mass of the ground state, as is known to hold for blowup solutions with H^1 initial data. Two recent results corroborate this expectation.

The following theorem was proved in joint work with S. Raynor, C. Sulem and J. D. Wright.

Theorem 1. [3] *There exists $s_Q \leq \frac{1}{5} + \frac{1}{5}\sqrt{11}$ such that the following is true for any $s > s_Q$. Suppose $H^s \cap \{radial\}^1 \ni u_0 \longmapsto u(t)$ solves* (1) *on the maximal (forward) time interval $[0, T^*)$, with $T^* < \infty$. Then*

$$\limsup_{t\uparrow T^*} \|u\|_{L^2_{\{|x|<(T^*-t)^{s/2-}\}}} \geq \|Q\|_{L^2}. \tag{6}$$

The next result[2], obtained in joint work with W. Staubach, establishes a different intermediate result.

[1] The non-radial case is amenable to treatment by employing the methods of compensated compactness as used in [10], [8].

[2] S. Keraani [5] has proved a similar result and the corresponding result in one space dimension.

Theorem 2. [4] *Let* $L^2(\mathbb{R}^2) \ni u_0 \longmapsto u$ *solve* (1) *on a maximal (forward) time interval* $[0, T^*)$, $T^* < \infty$. *Then there exits a fixed constant* $\mu_0 > 0$ *such that*

$$\limsup_{t \uparrow T^*} \sup_{\substack{\text{cubes } I \subset \mathbb{R}^2 \\ \text{side}(I) < (T^* - t)^{\frac{1}{2}}}} \left(\int_I |u(t,x)|^2 dx \right)^{\frac{1}{2}} \geq \mu_0. \tag{7}$$

Remark 3. The value of μ_0 may be taken to be the "scattering threshold mass" defined by the property: If $\|v_0\|_{L^2} < \mu_0$ then the (1) evolution $v_0 \longmapsto v$ is global-in-time and satisfies $\|v\|_{L^4_{xt}} < \infty$. There exists a nonzero lower bound on the scattering threshold mass in terms of constants in the Strichartz inequalities. It is conjectured that all L^2 solutions of (1) with initial mass less than $\|Q\|_{L^2}$ are global-in-time and bounded in L^4_{xt}.

The proof of Theorem 1 revisits the proof of the corresponding H^1 result from [7]. The role played by the conserved energy in the H^1 setting is played instead by an H^s-based almost conserved quantity introduced in [2].

The proof of Theorem 2 uses an asymptotic representation formula with asymptotic orthogonality properties proven in [6]. The proof proceeds by contradiction. If the bump functions which eventually concentrate in the [6] formula have mass smaller than μ_0, it is shown by a perturbation argument that the solution is bounded in $L^4_{t \in [0,T^*],x}$ and thus $[0, T^*)$ is not the maximal interval of existence, which is a contradiction.

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Mathematical Analysis of Vortex Sheet

Sijue Wu

We investigate questions related to the vortex sheet problem. This problem serves as a prototype for the evolution of vorticity in fluid flows. One can think for example of the wake of an airfoil as a typical problem of this type. The problem can be described by equations

$$\begin{aligned} &v_t + v\cdot\nabla v + \nabla p = 0 \\ &\operatorname{div} v = 0, \qquad\qquad (x,y)\in\mathbb{R}^2, t\geq 0 \\ &v(x,y,0) = v_0(x,y) \end{aligned}$$

Here v is the fluid velocity, p is the pressure and $v_0(x,y)$ is the initial data. The basic assumption is that the initial vorticity $w_0 = \mathrm{curl} v_0$ is ideally only a finite Radon measure. The vortex sheet problem assumes that the vorticity is a measure supported on a curve. The issue is to determine the evolution of this curve.

The evolution of the vortex sheet curve is described by the equivalent Birkhoff-Rott equation. If $z(\alpha,t)$ denotes the vortex sheet curve in complex variables at time t, then the evolution equation can be written in the following manner:

$$\partial_t \bar{z}(\alpha,t) = \frac{1}{2\pi i} p.v. \int \frac{1}{z(\alpha,t)-z(\beta,t)}\, d\beta.$$

Here $\gamma := 1/|z_\alpha|$ is the vortex strength and α is the circulation variable. I consider the question of the weakest possible assumptions such that an equation of the above type makes sense. This led me to introduce Chord-Arc curves to this problem. A chord-arc curve is a curve in which its arc-length between any two points is always no larger than a fixed multiple of the chord length between the same points. I proved three main results concerning this problem. The first can be stated as the following: Assume that the Birkhoff-Rott equation has a solution in a weak sense and the vortex strength is bounded away from zero and infinity. Moreover assume that the solution gives rise to a vortex sheet curve that is Chord-Arc. Then the curve is automatically smooth, in fact analytic for fixed time. The second and third results demonstrates that the Birkhoff-Rott equation is in fact not well-posed. That is one can solve the equation if and only if *only* half the initial data is given.

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On the limit from the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation

Kenji Nakanishi

This is a report on recent joint work with Nader Masmoudi (Courant Institute). The nonlinear Schrödinger equation is known to arise in various physics phenomena as an approximating equation. The approximation is usually carried out by taking several formal limits on the level of equations. Its mathematical justification, namely proving convergence on the level of actual solutions, often turns out to be highly non-trivial, even if we have satisfactory knowledge for both equations before and after the limits. Such an example is in the context of the plasma physics, for which I will describe the main difficulties and our idea to resolve them.

In the plasma physics, the nonlinear Schrödinger equation

$$2i\dot{u} - \Delta u = |u|^2 u, \tag{NLS}$$

is used as a model for describing the so-called Langmuir turbulence. It is derived from the Maxwell equation coupled with the fluid equations of the electrons and ions, via the Klein-Gordon-Zakharov system:

$$\begin{cases} c^{-2}\ddot{E} - \Delta E + c^2 E = nE, \\ \alpha^{-2}\ddot{n} - \Delta n = -\Delta |E|^2, \end{cases} \tag{KGZ}$$

and the Zakharov system:

$$\begin{cases} 2i\dot{u} - \Delta u = nu, \\ \alpha^{-2}\ddot{n} - \Delta n = -\Delta |u|^2, \end{cases} \tag{Z}$$

where $E : \mathbb{R}^{1+3} \to \mathbb{R}^3$ and $n : \mathbb{R}^{1+3} \to \mathbb{R}$ approximately describes the electric field and the ion density respectively, c^2 is called the plasma frequency and α is the ion sound speed. Physically we have $c \gg \alpha \gg 1$ and formally, (Z) is obtained by putting $E = e^{ic^2 t}u$ in (KGZ) and letting $c \to \infty$, and then (NLS) is derived by $\alpha \to \infty$. We are interested in convergence of solutions in these limits in terms of the Cauchy problem. In other words, we want to show convergence of solutions assuming that of initial data.

As for the solution class, the energy space is the most natural and important, from both physical and mathematical view points. It is defined by

$$\begin{aligned} &(E(t), \dot{E}(t), n(t), \dot{n}(t)) \in H^1 \times L^2 \times L^2 \times \dot{H}^{-1}, && \text{for (KGZ)}, \\ &(u(t), n(t), \dot{n}(t)) \in H^1 \times L^2 \times \dot{H}^{-1}, && \text{for (Z)}, \\ &u(t) \in H^1 && \text{for (NLS)}, \end{aligned}$$

where the following energy is well-defined and conserved for each equation respectively:

$$\int_{\mathbb{R}^3} |c^{-1}\dot{E}|^2 + |\nabla E|^2 + |cE|^2 + \frac{|\alpha^{-1}|\nabla|^{-1}\dot{n}|^2 + |n|^2}{2} - n|E|^2 dx, \quad \text{(KGZ)}$$

$$\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|\alpha^{-1}|\nabla|^{-1}\dot{n}|^2 + |n|^2}{2} - n|u|^2 dx, \quad \text{(Z)}$$

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{|u|^4}{2} dx, \quad \text{(NLS)}$$

H^s and $\dot{H}^s$ respectively denote the inhomogeneous and the homogeneous Sobolev spaces on L^2, and $f(\nabla) = \mathcal{F}^{-1} f(i\xi)\mathcal{F}$ denotes the Fourier multiplier for any function f.

Local wellposedness in the energy space for each equation is well known. Indeed, $u \in H^{1/2}$ is sufficient for (Z) and (NLS). See [9, 4, 3]. Blow-up solutions are known to exist for (KGZ) and (NLS) [8, 5] and also expected for (Z) [7].

On the other hand, convergence of solutions are known only in function spaces with much more regularity. [11] proved the convergence from (Z) to (NLS) in $H^5 \times H^4 \times H^3$ assuming $n = |E|^2$ at the initial time, while [1] proved the convergence in $H^6 \times W^{1,6} \times H^3$ without assuming $n = |E|^2$ but for E small in H^1. In [10, 6], convergence is given together with optimal convergence rate, assuming initial convergence in function spaces with even more regularity and decay at spatial infinity. The convergence from (KGZ) to (Z) was proved in [2] in $H^s \times H^{s-1} \times H^{s-1} \times H^{s-2}$ for $s > 7/2$.

The essential reason for this regularity gap between local wellposedness and convergence is that the usual iteration scheme for solving the Cauchy problem does not work uniformly for the limit $\alpha \to \infty$ in any Sobolev space H^s. In fact, we can prove that for any $s \in \mathbb{R}$, there exists $\varphi \in H^s$ such that the second iteration for

$$\begin{cases} 2i\dot{u}_k - \Delta u_k = n_{k-1}u_{k-1}, \\ \alpha^{-2}\ddot{n}_k - \Delta n_k = -\Delta|u_{k-1}|^2, \\ u_k(0) = \varphi, \quad n_k(0) = \dot{n}_k(0) = 0 \end{cases}$$

is not bounded, i.e.,

$$\limsup_{\alpha\to\infty} \|u_2(t)\|_{H^{s-1+\varepsilon}} = \infty$$

for any $0 < t \ll 1$ and $\varepsilon > 0$. (The uniform iteration is still possible in some weighted spaces [6].)

The above blow-up phenomenon can be easily observed in the space-time Fourier space. Free solutions for the Schrödinger and the wave equations are respectively supported on a paraboloid and light cones, and at their intersecting frequency $M = 2\alpha$, those free solutions oscillate in the same way. The contribution of the quadratic nonlinearities can be estimated in oscillatory integrals outside of this frequency M, since the difference of oscillation brings a lot of cancellation by integration. But around the frequency M, linear analysis implies resonance of n and u, and so no gain can be expected. We have the same problem for the limit from (KGZ) to (NLS). However, this frequency becomes a trouble only in the limit $M \to \infty$, since otherwise it is a bounded frequency portion, which may be estimated with arbitrary regularity.

Because of the above difficulty, the previous results on the convergence from (Z) to (NLS) are based essentially on the energy conservation, except for [6]. As for the limit from (KGZ) to (NLS), there is also a difficulty in this type of argument, namely the energy is diverging in the order $O(c^2)$. Although the energy for $u = e^{-ic^2t}E$ might appear to be bounded, it is actually the same quantity as the energy of E, provided that E is real-valued.

Our idea to resolve these difficulty is quite simple: we decompose the solution into the frequency around the resonance M and outside of it. For the non-resonant part, we apply bilinear estimates which derives regularity gain from the non-resonant property, and for the resonant part, we use a frequency-localized, modified nonlinear energy. Combining those estimates, we obtain

Theorem 1. *Let $s > 3/2$ and $0 < \gamma < 1$. Let (E, n) be the time-local solution of (KGZ) with the maximal existence time T. Consider the limit $(c, \alpha) \to \infty$ under the restriction $\alpha < \gamma c$. Assume the following initial convergence:*

$$(E(0), c^{-2}I_c\dot{E}(0)) \to (E_0, E_1) \text{ in } H^s,$$
$$(n(0), \alpha^{-1}|\nabla|^{-1}\dot{n}(0)) \to (n_0, n_1) \text{ in } H^{s-1},$$

where $I_c = (1 + |\nabla/c|^2)^{-1/2}$. Let $u = (u_+, u_-)$ be the time-local solution of the nonlinear Schrödinger equation

$$2i\dot{u} - \Delta u = |u|^2 u, \quad u(0) = \frac{1}{2}(E_0 - iE_1, \overline{E_0} - i\overline{E_1}).$$

with the maximal existence time T^∞. Then we have $\liminf T \geq T^\infty$, *and*

$$E(t) - (e^{ic^2t}u_+ + e^{-ic^2t}\overline{u_-}) \to 0, \quad \text{in } C([0, T^\infty); H^s),$$
$$ic^{-2}I_c\dot{E}(t) + (e^{ic^2t}u_+ - e^{-ic^2t}\overline{u_-}) \to 0, \quad \text{in } C([0, T^\infty); H^s),$$
$$n(t) - |u(t)|^2 - n_f(t) \to 0 \quad \text{in } C([0, T^\infty); H^{s-1}),$$
$$\alpha^{-1}|\nabla|^{-1}(\dot{n}(t) - \dot{n}_f(t)) \to 0 \text{ in } C([0, T^\infty); H^{s-1}),$$

where n_f is the free wave solution defined by

$$\alpha^{-2}\ddot{n}_f - \Delta n_f = 0, \quad n_f(0) = n(0) - |u(0)|^2, \quad \dot{n}_f(0) = \dot{n}(0).$$

We have a similar result for the limit from (Z) to (NLS). The main reason for the lower bound 3/2 for the regularity is coming from the error estimate for the modified energy at M, where we are forced to bound n in L^∞_x by using the Strichartz estimate.

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An improved local well-posedness result for the derivative nonlinear Schrödinger equation

Axel Grünrock

The Cauchy problem for the derivative nonlinear Schrödinger equation (DNLS) in one space dimension

$$iu_t + u_{xx} = i(|u|^2u)_x, \qquad u(0) = u_0 \tag{1}$$

has been shown to be locally well posed for H^s-data, $s \geq \frac{1}{2}$, by Takaoka in 1999 [5], where he improved earlier results of Hayashi and Ozawa [4]. On the H^s-scale, the $H^{\frac{1}{2}}$- result is optimal; in fact, ill-posedness in the C^0-uniform sense has been demonstrated by Biagioni and Linares in 2001 [1] using an appropriate counterexample. On the other hand the scaling argument suggests local well-posedness for $s \geq 0$ or at least $s > 0$, which is the critical scaling exponent in this case.

Inspired by works of Vargas and Vega [6] and of Cazenave, Vega and Vilela [2] we extend the $H^{\frac{1}{2}}$-result to a larger class of data in the two parameter scale of spaces $\widehat{H^r_s}(\mathbb{R})$, defined by the norm

$$\|u_0\|_{\widehat{H^r_s}} = \|\widehat{J^s u_0}\|_{L^{r'}},$$

here J^s is the Bessel potential operator of order $-s$. (Furthermore we use $\widehat{L^r_x}$ instead of $\widehat{H^r_0}$). The main result is:

Theorem 1. *The Cauchy problem (1) is locally well posed for data $u_0 \in \widehat{H^r_s}(\mathbb{R})$, provided $2 \geq r > 1$ and $s \geq \frac{1}{2}$.*

This theorem contains Takaoka's result in the special case $r = 2$. Since $\widehat{H^r_s}$ scales like H^σ, if $s - \frac{1}{r} = \sigma - \frac{1}{2}$, we can compare the results from a scaling point of view: By pushing down r from 2 to 1+, we can almost reach the scaling line $s - \frac{1}{r} = -\frac{1}{2}$; the case $s = \frac{1}{2}$ and $r = 1$ becomes critical in this setting and must be left as an open question. On the other hand the result is sharp, since, for given $r \in (1,2)$ and $s < \frac{1}{2}$, we have local ill-posedness in the C^0-uniform sense, which can be seen by using the counterexample of Biagioni and Linares.

In order to prove the theorem we rely on a variant of Bourgain's Fourier restriction norm method, especially we use the function spaces $X^r_{s,b}$, defined by the norms

$$\|f\|_{X^r_{s,b}} := \left(\int d\xi d\tau \langle \xi \rangle^{sr'} \langle \tau + \xi^2 \rangle^{br'} |\hat{f}(\xi,\tau)|^{r'} \right)^{\frac{1}{r'}},$$

and the corresponding restriction spaces

$$X^r_{s,b}(\delta) := \{ f = \tilde{f}|_{[-\delta,\delta]\times\mathbb{R}} : \tilde{f} \in X^r_{s,b} \},$$

endowed with the restriction norm. These are our solution spaces. This variant of Bourgain's method has been described in detail by the author in section 2 of [3], where a general theorem is shown, that reduces the question of local well-posedness completely to nonlinear estimates in the $X^r_{s,b}$-norms.

Other key tools in the proof of Theorem 1 are a gauge transform, which has already been used before by Takaoka and Hayashi/Ozawa, and certain bi- and trilinear estimates for solutions of the free Schrödinger equation, which imply corresponding $X^r_{s,b}$-estimates:

Theorem 2. *Let $u = e^{it\partial^2}u_0$, $v = e^{it\partial^2}v_0$ and $w = e^{-it\partial^2}w_0$. Then, with $\|f\|_{\widehat{L^r_{xt}}} = \|\hat{f}\|_{L^{r'}_{\xi\tau}}$, we have*

$$\begin{aligned} \|I^{\frac{1}{r}}(vw)\|_{\widehat{L^r_{xt}}} &= c\|v_0\|_{\widehat{L^r_x}}\|w_0\|_{\widehat{L^r_x}}, \\ \|uvw\|_{\widehat{L^r_{xt}}} &\leq c\|u_0\|_{\widehat{L^r_x}}\|v_0\|_{\widehat{L^r_x}}\|w_0\|_{\widehat{L^r_x}}, \end{aligned}$$

provided $1 < r < \infty$. The corresponding $X^r_{s,b}$-estimates are

$$\begin{aligned} \|I^{\frac{1}{r}}(u_1\overline{u}_2)\|_{\widehat{L^r_{xt}}} &\leq c\|u_1\|_{X^r_{0,b}}\|u_2\|_{X^r_{0,b}}, \\ \|u_1u_2\overline{u}_3\|_{\widehat{L^r_{xt}}} &\leq c\prod_{i=1}^{3}\|u_i\|_{X^r_{0,b}}, \end{aligned}$$

whenever $1 < r < \infty$, $b > \frac{1}{r}$.

Here I denotes the Riesz potential operator of order -1. Observe that the first bilinear estimate is in fact an equality. Moreover, the $X^r_{s,b}$-version of the trilinear estimate leads directly to local well-posedness for

$$(2) \qquad iu_t + u_{xx} = |u|^2u, \qquad u(0) = u_0 \in \widehat{L^r_x},$$

provided $2 \geq r > 1$.

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Bilinear restriction Theorems: The Hyperbolic Case

Ana Vargas

The solutions of the Schrödinger equation $2\pi i\partial_t u - \Delta u(x,t) = 0$, $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$, can be written in terms of its initial value $u_0(x) = u(x,0)$ as

$$u(x,t) = \int_{\mathbb{R}^2} \widehat{u_0}(\xi) e^{2\pi i(x\xi + t|\xi|^2)}\, d\xi = \big(\widehat{u_0}\, d\sigma\big)\check{}(x,t), \tag{1}$$

where $d\sigma$ is the measure on $S = \{(\xi, |\xi|^2) : \ \xi \in \mathbb{R}^2\}$ given as $d\sigma(\xi, |\xi|^2) = d\xi$.

Given S a surface in $\mathbb{R}^3$ and a measure $d\sigma$ on S, we define the operator

$$f \ \text{ defined on } \ S \longrightarrow \widehat{f\, d\sigma}(x,t) \ \text{ defined on } \ \mathbb{R}^3.$$

Formally, this is the adjoint of the operator

$$g \ \text{ defined on } \ \mathbb{R}^3 \longrightarrow \widehat{g}_{|S} \ \text{ defined on } \ S.$$

The estimates of the type

$$\|\widehat{f d\sigma}\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^p(S)} \tag{2}$$

receive the name of (linear) **restriction estimates.**

The classical results consider the case $p = 2$. For S a relatively compact surface with non-vanishing Gaussian curvature (2) holds for $p = 2$ and $q \geq 4$ (see P. Tomas [20], P. Sjölin [6], R. S. Strichartz [14] and E. M. Stein [13]).

The case $p > 2$ and $q < 4$ is still open. There are partial results by J. Bourgain [2], [3], [5], T. Wolff [23],[24], for non vanishing curvature and by A. Moyua, L. Vega, A. Vargas [9],[10], T. Tao, L. Vega, A. Vargas [17], T. Tao, A. Vargas [18] and T. Tao [16] for surfaces of positive curvature.

This is related to the pointwise convergence (a.e.) of the solutions (1) of the Schrödinger equation to the initial value, under regularity assumptions. The problem was posed by L. Carleson [7]. Dahlberg and Kenig [8] proved the convergence for $u_0 \in H^{1/2}(\mathbb{R}^2)$. J. Bourgain [5] showed that there is some $s_0 < 1/2$, such that, for any $s > s_0$ and any $u_0 \in H^s$, the convergence holds. The regularity, s_0, was improved by A. Moyua, T. Tao, L. Vega, A. Vargas [9],[17],[19], [16]. The best result known is $s_0 = 2/5$. In higher dimensions, $s_0 = 1/2$, is due to P. Sjölin [12] and L. Vega [22].

The most recent improvements on these problems involve a bilinear version of the restriction estimates. Given two surfaces S_1, S_2, and measures $d\sigma_1$, and $d\sigma_2$, a bilinear restriction estimate has the form

$$\|\widehat{f_1\, d\sigma_1}\,\widehat{f_2\, d\sigma_2}\|_{L^r(\mathbb{R}^3)} \le C\|f_1\|_{L^p(S_1)}\|f_2\|_{L^p(S_2)} \tag{3}$$

Note that if (2) is true and S_1, $S_2 \subset S$, Cauchy–Schwarz inequality shows that we have (3) for all $q = r/2$. But in general, inequalities as (3) have a wider range under appropriate assumptions on S_1 and S_2. For the elliptic (positive curvature) case, the natural assumption is that S_1 and S_2 are separated compact subsets of S. There are some results on this direction by A. Moyua, T. Tao, L. Vega and A. Vargas. The best result so far is

Theorem 1. *(T. Tao [16]). If S_1 and S_2 are separated compact subsets of S, a surface with positive Gaussian curvature, then (3) holds for*

$$r > \frac{5}{3}, \qquad p = 2. \tag{4}$$

From this, linear restriction estimates can be obtained, by using

Theorem 2. *(T. Tao, A. Vargas, L. Vega [17]) Assume that (3) holds for any S with positive curvature, any S_1 and S_2 separated subsets of S and for all (p, r) in a neighborhood of some (p_0, r_0) with $p_0' \ge r_0$. Then (2) holds for $(\hat{p}_0, 2r_0)$, where $\hat{p}_0' = r_0$.*

There are higher dimensional versions of these theorems ([16], [17]). There is an application to the problem of pointwise convergence

Theorem 3. *(T. Tao, A. Vargas [19]) Assume that (3) holds for all separated subsets of the paraboloid, $p = 2$ and some $r < 2$. Then, for all $s > 1 - \frac{1}{r}$, all $u_0 \in H^s(\mathbb{R}^2)$, and u as in (1),*

$$\|\sup_t |u(\cdot, t)|\|_{L^{2r}(\mathbb{R}^2)} \le C_s\|u_0\|_{H^s(\mathbb{R}^2)}.$$

Therefore, $u(x, t) \to u_0(x)$ a.e. as $t \to 0$.

For the case of cones, some theorems were proven by B. Barceló [1], Bourgain [4], T. Tao and A. Vargas [18]. The problem was completely solved (in all dimensions) by T. Wolff [25] and T. Tao [15].

We consider here the case of negative Gaussian curvature. The model surface is the saddle, $\tau = \xi\eta$, in $\mathbb{R}^3$. The hypothesis on S_1 and S_2 have to be different from the ones that we had in the elliptic case. This is shown by the following

Remark 4. ([21]) Consider the hyperbolic paraboloid $S = \{(\xi, \eta, \tau) /\ \ \tau = \xi\eta\} \subset \mathbb{R}^3$. Define the subsets of S, $S_1 = S \cap \{(\xi, \eta, \tau) /\ \ 1/2 < \xi < 1,\ -1 < \eta < 1\}$ and $S_2 = S \cap \{(\xi, \eta, \tau) /\ \ -1 < \xi < -1/2,\ -1 < \eta < 1\}$. Then, (3) is false for any $r < 2$ when $p = 2$.

Our result is the following,

Theorem 5. *([21]) Consider the surface*

$$S = \{(\xi, \eta, \tau) /\ \ \tau = \xi\eta, \quad |\xi|, |\eta| \leq 1\} \subset \mathbb{R}^3.$$

Consider compact subsets of S, S_1 and S_2 satisfying: for all $(\xi_1, \eta_1, \xi_1\eta_1) \in S_1$ and $(\xi_2, \eta_2, \xi_2\eta_2) \in S_2$ we have $|\xi_1 - \xi_2| \geq 1$ and $|\eta_1 - \eta_2| \geq 1$.

Then, (3) holds for any $r > 5/3$, $p = 2$.

A refinement of the proof of Theorem 2 gives us the following.

Theorem 6. *([21]) For the saddle, (2) holds for any $q > 10/3$, $p' < q/2$.*

These theorems (with a general version of Theorem 5 in higher dimensions) and the example were independently obtained by Sanghyuk Lee [11].

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Construction of asymptotic N–soliton–like solutions of the generalized Korteweg–de Vries equations

Yvan Martel

We consider the generalized Korteweg–de Vries equations

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x \in \mathbf{R}, \tag{1}$$

in the subcritical cases $p = 2, 3, 4$. The following quantities are formally conserved for solutions of (1):

$$\int u^2(t) = \int u^2(0) \quad (L^2 \text{ mass}) \tag{2}$$

(3)
$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = \frac{1}{2}\int u_x^2(0) - \frac{1}{p+1}\int u^{p+1}(0) \quad \text{(energy)}.$$

The Cauchy problem for (1) is globally well posed in the energy space H^1 by results of Kenig, Ponce and Vega [3]. Moreover, all solutions in H^1 are global and uniformly bounded.

A fundamental property of equations (1) is the existence of a family of explicit traveling wave solutions. Let Q be the only solution (up to translation) of
(4)

$$Q > 0, \quad Q \in H^1(\mathbf{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}\, x\right)} \right)^{\frac{1}{p-1}}.$$

Then, for all $c_0 > 0$ and $x_0 \in \mathbf{R}$,

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - x_0 - c_0 t) \quad \text{is solution of (1), where} \quad Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x).$$

We call *solitons* these solutions.

One of the most striking feature of the original KdV equation ($p = 2$) is the inverse scattering method which, among other things, leads to the construction of explicit solutions called N–soliton solutions, which generalize the soliton solutions (see e.g. [9], Section 6). These solutions are remarkable in two ways. First, they describe the interaction between several solitons with different speeds. One can observe from the expression of an N–soliton solution the stability of one soliton under interaction by other solitons. Second, as $t \to +\infty$, these N–soliton solutions decompose exactly as sum of N solitons: for any given $0 < c_1 < \ldots < c_N$, $x_1, \ldots, x_N$, there exists an explicit N-soliton solution $U(t)$ such that

$$\left\| U(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t) \right\|_{H^1(\mathbf{R})} \longrightarrow 0 \quad \text{as } t \to +\infty.$$

Moreover, there exist $y_1, \ldots, y_N$ such that

$$\left\| U(t) - \sum_{j=1}^{N} Q_{c_j}(. - y_j - c_j t) \right\|_{H^1(\mathbf{R})} \longrightarrow 0 \quad \text{as } t \to -\infty.$$

In general, y_j is different from x_j, but observe that the speeds of the solitons as $t \to +\infty$ or $t \to -\infty$ are the same.

Results on the qualitative behavior of smooth and decaying solutions of the KdV equation are known by the integrability theory, whereas known results for the nonintegrable cases concern solutions that are initially close to solitons:

It is well-known that for $p = 2, 3, 4$, the solitons are H^1 stable, in the following sense:

Stability of the 1*–soliton. For all* $\epsilon > 0$*, there exists* $\delta > 0$*, such that if* $\|u(0) - Q\|_{H^1} \leq \delta$*, then* $\forall t \in \mathbf{R}$*, there exists* $x(t) \in \mathbf{R}$*, such that*

$$\|u(t, . + x(t)) - Q\|_{H^1} \leq \epsilon.$$

See Benjamin [1], Bona [2] and Weinstein [10]. This result follows from the conservation of energy and mass, and the variational characterization of Q.

Next, Martel and Merle [6] proved the asymptotic completeness of the family of solitons in the H^1 setting.

Asymptotic completeness. *There exists $\delta_0 > 0$ such that if $\|u(0)-Q\|_{H^1} = \delta \leq \delta_0$ then there exist $c_{+\infty} > 0$ and $x(t) \in \mathbf{R}$, such that*

$$u(t, . + x(t)) \rightharpoonup Q_{c_{+\infty}} \quad \text{in } H^1(\mathbf{R}) \text{ weak as } t \to +\infty.$$

Moreover, $|c_{+\infty} - 1| + |x'(t) - 1| \leq f(\delta)$, where $f(\delta) \to 0$ as $\delta \to 0$.

We now turn to general results involving multi–solitons. Let us recall the main result obtained for $p = 2$, 3 and 4 by Martel, Merle and Tai-Peng Tsai [7].

Stability of the sum of N solitons. *Let $0 < c_1 < \ldots < c_N$. There exist γ_0, A, L_0, $\alpha_0 > 0$ such that the following is true. Assume that there exist $L > L_0$, $\alpha < \alpha_0$, and $x_1^0 < \ldots < x_N^0$, such that*

$$\Big\|u(0) - \sum_{j=1}^{N} Q_{c_j}(. - x_j^0)\Big\|_{H^1} \leq \alpha, \quad \text{with } x_j^0 > x_{j-1}^0 + L, \text{ for all } j = 2, \ldots, N.$$

Then, there exist $x_1(t), \ldots, x_N(t) \in \mathbf{R}$ such that

$$\forall t \geq 0, \quad \Big\|u(t) - \sum_{j=1}^{N} Q_{c_j}(x - x_j(t))\Big\|_{H^1} \leq A\left(\alpha + e^{-\gamma_0 L}\right).$$

Moreover, the asymptotic completeness of the sum of N solitons is also true, see [7].

Using refinements of tools developed for the proof of the previous results and an idea of Merle [8] for the critical nonlinear Schrödinger equation (see remark below the next theorem), we could answer two natural questions related to N–soliton solutions, see [5].

Theorem 1. *Let $p = 2, 3, 4$. Let $N \in \mathbf{N}$, $0 < c_1 < c_2 < \ldots < c_N$, and $x_1, \ldots, x_N \in \mathbf{R}$. There exists one and only one function $\varphi \in C(\mathbf{R}, H^1(\mathbf{R}))$, which is an H^1 solution of (1) in the sense of* [3] *and such that*

$$\lim_{t\to+\infty}\Big\|\varphi(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t)\Big\|_{H^1(\mathbf{R})} = 0. \tag{5}$$

Moreover, $\varphi \in C(\mathbf{R}, H^s(\mathbf{R}))$ for all $s \geq 0$, and there exist constants $A_s > 0$ such that for all $s \geq 0$, for all $t \geq 0$,

$$\Big\|\varphi(t) - \sum_{j=1}^{N} Q_{c_j}(. - x_j - c_j t)\Big\|_{H^s(\mathbf{R})} \leq A_s e^{-\gamma t}, \tag{6}$$

where $\gamma > 0$.

Remark 2. For the critical nonlinear Schrödinger equation

$$\mathrm{i}\, u_t = -\Delta u - |u|^{\frac{4}{d}} u, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^d, \tag{7}$$

Merle [8] proved a result similar to the existence part of Theorem 1, see [8], Corollary 2. The main objective in [8] was to prove the existence of a solution of (7) that blows up in finite time at exactly k given points $x_1, \ldots, x_k$. Recall that the critical Schrödinger equation has explicit minimal mass solutions that blow up in finite time at one point (they are built using Q and the conformal invariance). In [8], a solution that blows up at k given points is found close in H^1 to the sum of k explicit blow up solutions, by proving uniform estimates on the interaction of the different solutions.

The proof of existence in the present paper follows the same key starting idea. However, the proof of the uniform estimates is different, see complete proofs in [5].

Remark 3. The methods of this paper apply equally well to some other generalizations of the KdV equation:

$$u_t + (u_{xx} + f(u))_x = 0,$$

for suitable subcritical f (see [10] for suitable conditions on f).

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Well-posedness of the KdV equations with low regularity forcing terms

Kotaro Tsugawa

We consider the initial value problem of the forced KdV equations as follows:

$$\partial_t u + \partial_x^3 u + u\partial_x u = f, \quad (x,t) \in \mathbf{R} \times [0,T], \tag{1}$$

$$u(x,0) = u_0(x) \in H^s(\mathbf{R}), \tag{2}$$

where $u(x,t)$ and $f(x)$ are real valued functions. The KdV equation (without forcing term) has been studied by many people. However, in many real situations, one can not neglect external excitation mechanism. For example, the case $f = p\delta'(x)$ appears in the study of the excitation of long nonlinear water waves by a moving pressure distribution, where $\delta'(x)$ is the first derivative of the Dirac delta function and p is a constant (see [1]).

Bona and Zhang proved that (1)–(2) is time locally well-posed with $\sigma + 3/2 > s > -5/8$, $\chi(t)f(x,t) \in H^{1/2}(\mathbf{R} : H^\sigma(\mathbf{R}))$ and the unique solution u is in $C([0,t_0] : H^s(\mathbf{R}))$ (see [2]), where $\chi(t)$ is a smooth cut-off function. In the same manner as the L^2 conservation law for the KdV equation (without forcing term), by calculating

$$\int_0^T \int (1) \times u \, dxdt,$$

we have

$$\|u(T)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 = \int_0^T \int f u \, dxdt.$$

By Schwartz' inequality, we obtain the following L^2 a priori estimate:

$$\sup_{0 \le t \le T} \|u(t)\|_{L^2} \le C\Big(\|u(0)\|_{L^2} + \int_0^T \|f\|_{L^2} \, dt\Big).$$

Therefore, combining this estimate and Bona and Zhang's results, we obtain the time global well-posedness of (1)–(2) with $f \in L^2$, $s = 0$. However, we can not apply this estimate to (1)–(2) with $f \in H^\sigma, \sigma < 0$.

To overcome this difficulty, we divide the forcing term into the low frequency part and the high frequency part as follows:

$$f = f_1 + f_2, \qquad \text{where} \qquad \widehat{f_1} = \widehat{f}\,\Big|_{|\xi|<\lambda}, \qquad \widehat{f_2} = \widehat{f}\,\Big|_{|\xi|>\lambda}.$$

Put $\widehat{v} = \widehat{f_2}/(i\xi)^3$ and $w = u - v$, Then, (1)–(2) is rewritten into

$$\partial_t w + \partial_x^3 w + w\partial_x w = f_1 - \partial_x(vw) - v\partial_x v, \tag{3}$$

$$w(x,0) = w_0(x) = u_0 - v. \tag{4}$$

We note that the right-hand side of (3) is sufficiently smooth and that the last two terms go to 0 when $\lambda \to \infty$. Applying the Fourier restriction norm method (See [3]) to (3)–(4), we obtain the following results.

Theorem 1 Let $-3/4 < s \le \sigma + 3$, $-3 < \sigma$, $f \in H^\sigma(\mathbf{R})$. Then, (1)–(2) is time locally well-posed and the solution u is in $C([0,t_0] : H^s(\mathbf{R}))$.

The condition $s \le \sigma + 3$ is optimal. However, we have more smoothing effect as follows.

Corollary Let $\sigma + 3 < s < \sigma + 7/2$, $\sigma > -5/2$, $f \in H^\sigma(\mathbf{R})$. Assume that $u(x,0) - v$ is in H^s. Then, (1) is time locally well-posed in $H^{\sigma+3}$ and $u - v$ is in $C([0,t_0] : H^s(\mathbf{R}))$.

Applying the I-method (See [4]) to (3)–(4), we obtain the following a priori estimates.

Proposition Let $0 > s > -3/4$, $-3 > \beta > -15/4$, $\theta = (6s - \beta(2s+3)) > 0$ and u satisfy (1)–(2). Then, for $f \in H^\sigma$, $0 \geq \sigma > -3/2$, $T \geq 1$, we have

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \leq C\Big\{ \big(T^{-s}(\|u_0\|_{H^s}^{-\beta} + 1)\big)^{3/\theta} + \big(T\|f\|_{H^\sigma}\big)^{3/(2\sigma+3)} \Big\}. \tag{5}$$

We also have a priori estimates for $f \in H^\sigma$, $\sigma = -3/2$ or $f = \delta'(x)$, which grow up exponentially. From these a priori estimates and Theorem 1, we obtain the following theorem.

Theorem 2 Assume that (Case 1) $f \in H^\sigma(\mathbf{R})$, $-3/4 < s \leq \sigma + 3$, $\sigma \geq -3/2$ or (Case 2) $f = p\delta'(x)$, $-3/4 < s < 3/2$. Then, (1)–(2) is time globally well-posed and the solution u is in $C([0,\infty] : H^s(\mathbf{R}))$.

We finally mention a related problem. The existence of global attractor of the weakly damped, forced KdV equations has been studied by many people (See e.g. [6]). By the L^2 a priori estimate, Goubet and Rosa proved the existence of global attractor on the real line with $u_0(x)$, $f(x) \in L^2(\mathbf{R})$ in [5]. We can apply our technique to this equations and we have time global well-posedness for weakly damped, forced KdV equations with $f \in H^\sigma(\mathbf{R})$, $0 > \sigma \geq -3/2$. However, the existence of global attractor for that is still open.

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Uniform estimates for the Zakharov system

Justin Holmer

Consider the Cauchy problem for the (scalar) Zakharov system

$$\mathrm{ZS}_\epsilon = \begin{cases} i\partial_t u + \Delta u = nu \\ \epsilon^2\partial_t^2 n - \Delta n = \Delta|u|^2 \\ u\big|_{t=0} = u_0 \\ n\big|_{t=0} = n_0 \\ \partial_t n\big|_{t=0} = n_1 \end{cases}$$

where $u : \mathbb{R}^d \times [0,T] \to \mathbb{C}$, $n : \mathbb{R}^d \times [0,T] \to \mathbb{R}$. The initial data are assumed to belong to Sobolev spaces $u_0 \in H^k(\mathbb{R}^d)$, $n_0 \in H^l(\mathbb{R}^d)$, $n_1 \in H^{l-1}(\mathbb{R}^d)$. ZS_ϵ is a (simplified version of) a model for Langmuir turbulence in a plasma, and was introduced by Zakharov (1970's).

Formally, as $\epsilon \downarrow 0$, if we assume $u \to v$, then we expect (from the second equation) that $n \to -|v|^2$ and (from the first equation) that v solves

$$\mathrm{NLS}_3 = \begin{cases} i\partial_t v + \Delta v = -v|v|^2 \\ v|_{t=0} = u_0 \end{cases}$$

Our goal is to obtain rigorous results on the convergence $u \to v$ as $\epsilon \to 0$ for generalized systems at high regularity (k, l large).

Now I give a brief overview of earlier work in this direction for ZS_ϵ. Let $T^* > 0$ be the maximal time of existence of the solution to NLS_3, and let $T < T^*$. Added-Added [1] obtain convergence at rate $\epsilon^{1/2}$ in H^k on $[0,T]$, i.e. $\|u-v\|_{L^\infty_{[0,T]}H^k} \le c\epsilon^{1/2}$, with no weights on the initial data, for dimensions $d = 1,2,3$. Ozawa-Tsutsumi [6] obtain convergence at rate ϵ on $[0,T]$ if $n_0 + |u_0|^2 \neq 0$ (noncompatible case) and rate ϵ^2 on $[0,T]$ if $n_0 + |u_0|^2 = 0$ (compatible case) and $n_1 \in \dot{H}^{-1}$, but require weights on the initial data, in dimensions $d = 1,2,3$. The limitation of these results (from our perspective) is that the method of proof uses energy-type identities that are too sensitive to the form of the nonlinearity. For example, it does not work when nu is changed to $-nu$ because this changes a sum of squares to a difference of squares in these identities.

Another method that applies to more general nonlinearities is that of Kenig-Ponce-Vega [4] who address ZS_ϵ with techniques previously developed for the derivative NLS equation

$$\mathrm{DNLS} = \begin{cases} i\partial_t u + \Delta u = p(u,\bar{u},\nabla u,\nabla\bar{u}) \\ u|_{t=0} = u_0 \end{cases}$$

where $\deg p \ge 2$. They prove estimates that enable one to view the composition $\Box_\epsilon^{-1}\Delta$ as behaving like one spatial derivative and obtain, with the additional help of local smoothing estimates for the Schrödinger operator, the uniform in ϵ bound

$$\|u\|_{L^\infty_{[0,T]}H^k_x} + \sup_\alpha \|D^{k+\frac{1}{2}}u\|_{L^2(Q_\alpha)L^2_{[0,T]}} \le c$$

where Q_α is the unit cube in $\mathbb{R}^d$ at the lattice point $\alpha \in \mathbb{Z}^d$, provided the initial data is small in a weighted sense: $\|\langle x\rangle^m u_0\|_{H^{k_0}} < \delta$, where $k_0 << k$. Their method is modeled on that used to obtain local well-posedness of DNLS for small initial data in [3]. Chihara [2] removed the smallness assumption in [3] for DNLS by using a pseudo-differential operator change of variable, and this technique was further developed by Kenig-Ponce-Vega [5]. By similarly introducing a pseudo-differential operator change of variable to ZS_ϵ, we obtain (thus far only in $d = 1$)

Theorem 1. *Let $k \geq 4$. Let*

$$M = \|u_0\|_{H^k} + \|\langle x\rangle^2 u_0\|_{H^1} + \|n_0\|_{H^{k-\frac{1}{2}}} + \|n_1\|_{H^{k-\frac{3}{2}}}$$

Then $\forall\, T > 0$, $\exists\, \epsilon_0 = \epsilon_0(T, M) > 0$ and $\exists\, c = c(M) > 0$ such that we have

$$\|u\|_{L^\infty_{[0,T]} H^k_x} + \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k u\|_{L^2_x L^2_{[0,T]}} \leq c \quad 0 < \epsilon \leq \epsilon_0$$

and

$$\lim_{\epsilon \downarrow 0} \left(\|u - v\|_{L^\infty_{[0,T]} H^k_x} + \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k (u - v)\|_{L^2_x L^2_{[0,T]}} \right) = 0$$

and

$$\|u - v\|_{L^\infty_{[0,T]} H^{k-1}_x} + \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k-1} (u - v)\|_{L^2_x L^2_{[0,T]}} \leq c\epsilon \quad 0 < \epsilon \leq \epsilon_0$$

and if we further require $n_0 + u_0 \bar{u}_0 = 0$ and assume that $\exists\, \nu \in L^1$ such that $\partial_x \nu = n$, then

$$\|u - v\|_{L^\infty_{[0,T]} H^{k-2}_x} + \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k-2} (u - v)\|_{L^2_x L^2_{[0,T]}} \leq c\epsilon^2 \quad 0 < \epsilon \leq \epsilon_0$$

The primary difficulty in treating ZS_ϵ, as opposed to DNLS, by this method is that the pseudo-differential operator does not commute with the inverse wave-operator.

Remark 1. Because we have restricted to one dimension ($d = 1$), the result is global (T can be taken arbitrarily large).

Remark 2. The local smoothing estimate (on $\|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k u\|_{L^2_x L^2_{[0,T]}}$) is needed to obtain the convergence at rate ϵ with only one derivative of separation between the space in which convergence is obtained and the space in which the initial data is assumed to belong. A gap of this order cannot, it appears, be achieved with energy identities alone.

Remark 3. The method is flexible with regard to the nonlinearity, and there is no smallness assumption. The same proof written in full generality should enable us to treat dimension-one systems like

$$\begin{cases} i\partial_t u + \partial_x^2 u = p(u, \bar{u}, \partial_x u, \partial_x \bar{u}, n_+, n_-) \\ \epsilon \partial_t n_+ + \partial_x n_+ = p_+(u, \bar{u}, \partial_x u, \partial_x \bar{u}) \\ \epsilon \partial_t n_- - \partial_x n_- = p_-(u, \bar{u}, \partial_x u, \partial_x \bar{u}) \\ u\big|_{t=0} = u_0 \\ n_+\big|_{t=0} = n_{+0} \\ n_-\big|_{t=0} = n_{-0} \end{cases}$$

with $(u_0, n_{+0}, n_{-0}) \in H^k \cap H^1(\langle x\rangle^2 dx) \times H^{k-\frac{1}{2}} \times H^{k-\frac{1}{2}}$, $k \geq 4$, with $\deg p, p_+, p_- \geq 2$ and every monomial in p contains at least one factor of u, $\bar{u}$. One should also be able to provide a convergence result, where the limiting form of the equation is obtained by replacing n_+ in the first equation by

$$-\int_{s=0}^{+\infty} p_+(u, \bar{u}, \partial_x u, \partial_x \bar{u})(x-s, t)\, ds$$

and $n_-(x,t)$ by

$$+\int_{s=0}^{+\infty} p_-(u, \bar{u}, \partial_x u, \partial_x \bar{u})(x+s, t)$$

ZS_ϵ for dimension $d = 1$ is a special case of this with $p_+ = -\frac{1}{2}\partial_x |u|^2$, $p_- = \frac{1}{2}\partial_x |u|^2$

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Global Solutions for a Semi-Linear 2D Klein-Gordon Equation with Exponential Type Nonlinearity

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(joint work with M. Majdoub and N. Masmoudi)

1. Introduction

Let $d \geq 3$ and consider the d-dimensional defocusing semi-linear wave equation of the type:

$$\Box u + |u|^{p-1} u = 0, \tag{1}$$

where p is a real number $p > 1$, $\Box = \partial_t^2 - \Delta$ and $u = u(t,x)$ is a real-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$.

The Cauchy problem associated to (1) has been widely investigated and there is a large literature dealing with the local and global solvability in the scale of the Sobolev spaces H^s, the uniqueness in suitable subspaces of the energy space and the asymptotic of the solutions as t goes to infinity (scattering theory). For a beautiful survey of the results, see [8].

For the global solvability in the energy space $H^1 \times L^2$ there are mainly three cases namely; $p < p^*-1$ where the critical exponent is given by $p^* = \frac{2d}{d-2}$, $p = p^*-1$ and $p > p^* - 1$.
Global solutions exist when $p \leq p^* - 1$ and are unique in the class of finite energy solutions only if $p < p^* - 1$. This is an open question when $p = p^* - 1$. In the case $p > p^* - 1$, the well-posedness is still an open problem except for some partial results (see for example [4]).
Note that the difference of the conditions for the solvability in the above three cases basically comes from the Sobolev embedding $H^1 \hookrightarrow L^p$ for all $2 \leq p \leq p^*$.

In dimension $d = 2$, we have $H^1 \hookrightarrow L^p$ for any $2 \leq p < \infty$. So heuristically, every power nonlinearity is "sub-critical" and an exponential term seems to be a natural critical nonlinearity. So in this work, we consider the following equation:

$$(\partial_t^2 - \Delta + 1)u + u\big(\exp(4\pi u^2) - 1\big) = 0. \tag{2}$$

The initial data $u(0,x) = f(x)$ and $\partial_t u(0,x) = g(x)$ are in the energy space $H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. Define the total energy

$$E_0 := \int_{\mathbb{R}^2} |\nabla f|^2(x) + g^2(x) + \frac{\exp(4\pi f^2(x)) - 1}{4\pi} dx.$$

In particular, note that the nonlinear term of the energy is finite because of Trudinger-Moser inequality and that the exponent 4π is sharp (see [5]).
The Cauchy problem is said to be sub-critical, when $E_0 < 1$. It is called critical when $E_0 = 1$ and finally super-critical if $E_0 > 1$.

2. Results and Ideas of the Proofs

Every local well-posedness result is based on an estimation $L_t^1(L_x^2)$ of the nonlinear term. But the problem when taking $\|u\big(\exp(4\pi u^2) - 1\big)\|_{L_x^2}$ is to double the exponent 4π and therefore we loose any control of that term using only Moser-Trudinger inequality.
Our first result is the following local (in time) existence theorem.

Theorem 1. *Assume that* $\|\nabla f\|_{L^2(\mathbb{R}^2)} < 1$. *Then, a unique time* $T^* > 0$ *and a unique function* u *solution of the equation (2) exist such that*

$$u \in \mathcal{C}([0,T^*);\ H^1(\mathbb{R}^2)) \cap \mathcal{C}^1([0,T^*);\ L^2(\mathbb{R}^2)) \cap L^4([0,T^*);\ \mathcal{C}^{1/4}(\mathbb{R}^2)).$$

Moreover, if $E_0 \leq 1$ *then one of the following two alternatives occurs:*

- $T^* = +\infty$ *or*
- $T^* < \infty$ *and,*

$$\overline{\lim_{t \to T^*}}\ \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^2)} = 1. \tag{3}$$

The proof of this Theorem is based on the combination of the Strichartz estimates for the linear Klein-Gordon equation (see [2]), the Moser-Trudinger inequality and the following logarithmic inequality.

Lemma 2. *For any real $\lambda > \frac{2}{\pi}$ there exists a constant C_λ such that for any function $u \in \mathcal{C}^{\frac{1}{4}}(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, one has*

$$(4) \quad \|u\|^2_{L^\infty(\mathbb{R}^2)} \leq \lambda\big(\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + 1/4\|u\|^2_{L^2(\mathbb{R}^2)}\big) \cdot \log\left(e + C_\lambda \frac{\|u\|_{\mathcal{C}^{1/4}(\mathbb{R}^2)}}{\sqrt{\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + 1/4\|u\|^2_{L^2(\mathbb{R}^2)}}}\right),$$

where, for any $\alpha \in]0,1[$, $\mathcal{C}^\alpha(\mathbb{R}^2)$ denotes the space of α-Hölder continuous functions. The local wellposedness is then derived via a classical fixed point argument.

Remark 3. The constant $\lambda = \frac{2}{\pi}$ in (4) is "almost" sharp. The proof of Lemma 2 and further related inequalities are discussed with details in [3].

The assumption $E_0 \leq 1$ in particular implies that $\|\nabla f\|_{L^2(\mathbb{R}^2)} < 1$ and consequently we have short time existence of solutions in both sub-critical and critical cases. So it makes sense to deal with global existence in these situations.
Let u be the solution given by Theorem 1, with $T^* < \infty$ is the largest time of existence. Then in the sub-critical case, the conservation of the total energy shows us that $\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^2)}$ is uniformly below 1, so (3) is not satisfied and therefore the solution can be continued in time. Precisely we have the following corollary

Corollary 4 (Sub-critical case)**.**
Assume that $E_0 < 1$, then the problem (2) has a unique global solution u satisfying the energy identity and

$$u \in \mathcal{C}(\mathbb{R},\ H^1(\mathbb{R}^2)) \cap \mathcal{C}^1(\mathbb{R},\ L^2(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2))$.

Note that the uniqueness is obtained in the class of finite energy solutions. This result is based on a classical boot-strap argument.

In the critical case, we loose this uniform control and therefore the total mass of the energy can be concentrated in the $\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^2)}$ part. However, establishing some localized (in space-time) identities in the spirit of Shatah-Struwe's result [7], we show that such concentration cannot hold in the critical case and therefore we have the following result.

Theorem 5 (Critical case)**.**
Assume that $E_0 = 1$, then the problem (2) has a unique global solution u satisfying the energy identity and

$$u \in \mathcal{C}(\mathbb{R},\ H^1(\mathbb{R}^2)) \cap L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/4}(\mathbb{R}^2)) \quad ; \quad \partial_t u \in \mathcal{C}(\mathbb{R},\ L^2(\mathbb{R}^2)).$$

Remark 6. To the best of the author's knowledge, Corollary 4 and Theorem 5 are the only results for global solutions of such 2D problems with exponential growth nonlinearity. In [6], Nakamura and Ozawa proved, under an assumption of smallness of the initial data, the existence of global solutions.
More recently, A. Attallah [1] proved a local existence result for solution of (2) assuming that the first initial data $f = 0$, and the second one is radially symmetric and with compact support.

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Unique continuation for the wave equation with time independent L^p potential

David Dos Santos Ferreira
(joint work with Alberto Ruiz)

The aim of this joint work with Alberto Ruiz is to investigate the unique continuation of a solution of the wave equation $\partial_t^2 u - \Delta_x u + V(x)u = 0$ (where $(t, x) \in \mathbb{R}^{d+1}_{t,x}$) with time independent potential $V \in L^{\frac{d}{2}}(\mathbb{R}^d_x)$ across a non-characteristic hypersurface. More generally, we consider the following unique continuation problem: we suppose that there is a splitting $x = (x', x'') \in \mathbb{R}^{n-d} \times \mathbb{R}^d$ of the coordinates and we consider a solution of the partial differential equation

$$P(x', D_{x'}, D_{x''})u + V(x')u = 0 \tag{1}$$

where the second order differential operator P has real $\mathcal{C}^\infty$ coefficients independent of x'' and is *partially elliptic* in the sense that if p denotes the principal symbol of P then $p(x', \xi', 0)$ is elliptic. The potential $V(x')$ belongs to $L^{\frac{d}{2}}(\mathbb{R}^d_x)$. The question of unique continuation is to see whether if u vanishes below a non-characteristic hypersurface in the neighbourhood of a point x_0 then it vanishes on a full neighbourhood of x_0.

When the potential depends on all the variables and P is an elliptic operator, unique continuation is known to hold for potentials in $L^{\frac{n}{2}}$ and to fail when $V \in L^p$ with $p < n/2$. This result was obtained by Jerison and Kenig [7] (see also [13] chapter 5) in the case of the Laplace operator and by Sogge [14] in the case of elliptic operators with variable coefficients. Further improvements on the smoothness of the coefficients were obtained by Wolff [21] and on the addition of a gradient term by Koch and Tataru [8]. These results rely on Carleman estimates based on L^p spaces. In this work, the goal is to go below the critical index $n/2$ provided the potential (and the coefficients of the differential operator) does not depend on one part of the variables.

In the case of partially elliptic operator, the case where V belongs to, say, L^∞ is fully understood. There is a series of papers dealing with cases increasing in generality beginning with Robbiano [11] followed by Tataru [17], [19], Hörmander [5], [6] and by Robbiano-Zuily [12] which rely strongly on the use of L^2 Carleman estimates. An idea of Tataru has proved to be particularly efficient to tackle the problem of partially elliptic operators: in [17] he remarked that a modified Carleman estimate involving a Gaussian transform of the form $e^{-|D''|^2/2\lambda}$ (where λ is the large parameter of the usual Carleman estimates) still implied unique continuation.

The virtue of this transform is to microlocalise the Carleman inequality near $\xi'' = 0$ where the partially elliptic operator behaves like an elliptic operator. The modified Carleman estimates are then obtained by a modification of the arguments used to prove standard Carleman estimates in the case of elliptic operators. We wish here to use the same idea but in the setting of L^p Carleman estimates. More accurately, we develop Carleman estimates involving the transform $e^{-|D''|^2/2\lambda}$ and set in mixed $L^p_{x'}L^2_{x''}$ norms.

The theorem that we obtain is as follows:

Theorem 1. *Let $P(x', D_{x'}, D_{x''})$ be a second order differential operator with real symbol defined on a neighbourhood $\Omega \subset \mathbb{R}^n$ of x_0 with $\mathcal{C}^\infty$ coefficients independent of $x'' \in \mathbb{R}^{n-d}$. Assume furthermore that $p(x', \xi', 0)$ is elliptic (where p denotes the principal symbol of P). Let S be a C^2 non-characteristic hypersurface in the neighbourhood Ω of x_0. If $u \in H^1$ satisfies the equation* (1) *with $V(x') \in L^{\frac{d}{2}}_{\mathrm{loc}}(\mathbb{R}^d)$ and u vanishes below S then $x_0 \notin \operatorname{supp} u$.*

In particular, this gives unique continuation for the wave operator with time independent potential $V(x)$ in $L^{\frac{d}{2}}(\mathbb{R}^d)$.

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L^p estimates for eigenfunctions in planar domains

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(joint work with Hart Smith)

Let M be an n-dimensional C^∞ open manifold with compact closure and boundary ∂M. Consider a Riemannian metric $g = g_{jk}dx^j dx^k$ on M and the associated Dirichlet-Laplacian $\Delta = \Delta_{g,\mathcal{D}}$. We shall then be concerned with estimates for the eigenfunctions,

$$\begin{cases} -\Delta\phi_\lambda(x) = \lambda^2\phi_\lambda(x), & x \in M \\ \phi_\lambda(x) = 0, & x \in \partial M. \end{cases}$$

The eigenvalues are discrete and tend to $+\infty$. We count them with respect to multiplicity and order them as $0 < \lambda_1^2 \le \lambda_2^2 \le \lambda_3^2 \le \cdots$. Also, as before, we wish to study the behavior of the L^p norms as the eigenvalue goes to infinity. We are also interested in stronger estimates for the spectral projection operator,

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1]} e_j(f),$$

where $e_j(f)$ is the projection of f onto the eigenspace with eigenvalue λ_j.

In the case of compact manifolds without boundary, the author [9] established the following bounds

$$\|\chi_\lambda f\|_q \le C\lambda^{\sigma(q)}\|f\|_2, \quad \lambda > 1, \tag{1}$$

where

$$\sigma(q) = \begin{cases} n(1/2 - 1/q) - 1/2, & q \ge \frac{2(n+1)}{n-1} \\ \frac{n-1}{2}(1/2 - 1/q), & 2 \le q \le \frac{2(n+1)}{n-1} \end{cases} \tag{2}$$

These estimates were proved using the Hadamard parametrix.

Thus, a goal would be try to extend (1)-(2) to the setting of compact Riemannian manifolds with boundary. We immediately encounter two difficulties:

- It is much harder to use the Hadamard parametrix and the wave equation in manifolds with boundary.
- Less is true: Rayleigh whispering gallery modes say that the above bounds cannot hold for all values of $2 \le p \le \infty$. Specifically in every dimension $n \ge 2$ the favorable bounds with exponent $\sigma(q) = n(1/2 - 1/q) - 1/2$ can only hold for a smaller range of exponents.

Let us give a brief explanation of these two facts. We start with the first one. In order to explain the complicated nature of parametrices for wave operators in manifolds with boundary, we start by reviewing what happens for the very simple case of the Dirichlet-wave equation for the half-plane $(t, x) \in \mathbb{R} \times (\mathbb{R}^{n-1} \times \mathbb{R}_+)$, i.e.,

$$\begin{cases} \Box u = 0 \\ u(t, x', 0) = 0 \\ u(0, x) = f(x), & \partial_t u(0, x) = 0. \end{cases}$$

Here we are writing $x = (x', x_n)$, with $x' = (x_1, \dots, x_{n-1})$. The kernel for the solution operator is given by the formula

$$U_{\mathcal{D}}(t; x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \cos t|\xi|\, d\xi - (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x_r - y)\cdot\xi} \cos t|\xi|\, d\xi, \tag{3}$$

where x_r is the reflection of x across the boundary.

In a manifold with boundary it is difficult to construct the second term in the parametrix for the solution kernel. Here, we work in geodesic normal coordinates about y and then think of $(x_r - y)$ as the reflected geodesic normal coordinates of x about y. Simple examples where M is either the interior or exterior of a euclidean ball show that these coordinates become degenerate as x approaches the boundary in a tangential direction from y. Indeed, these two examples tell us that if y is a geodesic distance $d = d(y) << 1$ from the boundary ∂M, then we cannot hope to construct the phase functions corresponding to the second term in (3) for all x if the time variable satisfies $t > cd^{1/2}$ for some fixed constant c.

Let us address the other difficulty that arises in trying to extend (1)-(2). This is an observation of D. Grieser [2]. We shall focus on the case where $n = 2$, but similar considerations show that the above bounds cannot hold for all $2 \le q \le \infty$ for higher dimensions as well. Grieser observed that if (M, g) is the interior of the unit disk with the euclidean metric and if ϕ_λ is a so-called whispering gallery mode with eigenvalue λ^2 then, for $q \le 8$, f_λ has most of its L^q mass in a $\lambda^{-2/3}$ neighborhood of the boundary. Hence

$$\frac{\|f_\lambda\|_2}{\|f_\lambda\|_q} \ge c\lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{q})}, \quad q \le 8.$$

Since f_λ is an eigenvalue, we conclude that, for such exponents, we can have the favorable 2-dimensional estimates

$$\|\chi_\lambda f\|_q \le C\lambda^{2(1/2-1/q)-1/2}\|f\|_2,$$

only when

$$2(1/2 - 1/q) - 1/2 \ge \tfrac{2}{3}(1/2 - 1/q),$$

which means that the best possible analog of (1)-(2) for manifolds with boundary when $n = 2$ would be

$$\|\chi_\lambda f\|_q \le C\lambda^{2(1/2-1/q)-1/2}\|f\|_2, \quad q \ge 8, \tag{4}$$

instead of $q \ge 6$ as in the boundaryless case when $n = 2$. For the other range, by interpolation, we have

$$\|\chi_\lambda f\|_q \le C\lambda^{\frac{2}{3}(1/2-1/q)}\|f\|_2, \quad 2 \le q \le 8. \tag{5}$$

Let us turn to positive results now. We shall first indicate how one can obtain sharp pointwise estimates for eigenfunctions and then discuss recent joint work with H. Smith [7] that shows that when $n = 2$ the optimal estimates, i.e., (4)-(5), are valid.

The pointwise estimate says that for compact Riemannian manifolds with boundary we have

$$\|\chi_\lambda f\|_\infty \le C(1+\lambda)^{\frac{n-1}{2}}\|f\|_2. \tag{6}$$

It is well known and straightforward to see that this estimate is valid if and only if the kernels of the spectral projections have the following bounds when evaluated along the diagonal

$$\chi_\lambda(x,x) = O(\lambda^{n-1}). \tag{7}$$

Next, Tauberian arguments show that this holds if

$$\tilde\chi_\lambda(x,x) = O(\lambda^{n-1}), \tag{8}$$

where

$$\tilde\chi_\lambda(x,x) = \int \rho(t)\cos t\lambda\, U_{\mathcal{D}}(t;x,x)\,dt,$$

with $\rho \in C_0^\infty(\mathbb{R})$ being a fixed function supported in a small neighborhood of the origin.

It turns out that the parametrix for the wave kernel at the diagonal, $U_{\mathcal{D}}(t;x,x)$ only allows one to show (8) when $d(x) \geq c\lambda^{-1}$ for some fixed constant $c > 0$, where $d(x)$ is the geodesic distance of x from the boundary. Thus, by using wave equation techniques that are more technical but similar to the ones for the boundaryless case, one can show the following special case of (6)

$$\chi_\lambda(x,x) \leq C\lambda^{n-1}, \quad \text{if } d(x) \geq c\lambda^{-1}, \tag{9}$$

where c and C are uniform constants which are independent of λ.

To prove the bounds for the missing case, a $O(\lambda^{-1})$ neighborhood of ∂M, it turns out that one can use a maximum principle argument. This observation goes back to Grieser [3] for the case of eigenfunctions, and Grieser's argument can be modified to handle the case of functions whose spectrum lies in unit bands $[\lambda, \lambda+1]$. In the latter case, one can use a variant of the maximum principle to see that if c is fixed then

$$\sup_{\{x:\, d(x)\leq c\lambda^{-1}\}} \chi_\lambda(x,x) \leq C \sup_{\{x:\, d(x)=c\lambda^{-1}\}} \chi_\lambda(x,x),$$

which, by (9) means that (7) must hold and thus completes the proof of (6).

The pointwise bounds (6) can be used to show that the operators S_λ^δ are uniformly bounded on $L^1(M)$ and $L^\infty(M)$ when $\delta > (n-1)/2$. Recently, X. Xu [17] has proved more refined pointwise estimates that include sharp pointwise bounds for the gradient of eigenfunctions on compact Riemannian manifolds with boundary. Using these estimates, he was able to show that the Hörmander multiplier theorem extends to this setting.

Let us turn to the other estimate, (4). In a work in progress, Smith and Sogge [7] have shown that the estimate (4) holds for general two-dimensional Riemannian manifolds with boundary on the range $q \geq 8$. Interpolation with the trivial boundedness of χ_λ on $L^2(M)$ then yields L^q estimates on spectral clusters which are the best possible, as shown by Grieser's observation. The proof depends on the fact that, for functions with microlocal support disjoint from a thin set in phase space consisting of geodesics tangent to the boundary, the full spectral estimates hold. This is because the wave group for transverse reflections has the same essential properties as the free wave group.

To handle the contribution of directions in phase space that are nearly tangent to the boundary, Smith and Sogge exploited ideas of Smith [6] and Tataru [16] developed to handle wave speed metrics of low regularity. The latter work involved a combination of paradifferential and frequency dependent scaling arguments to show that functions similar to the Rayleigh whispering gallery modes are the worst case. Interpolating between the tangent and transverse reflection cases yields the desired L^8 estimates for all functions.

One can obtain estimates for eigenfunctions on manifolds with boundary from appropriate estimates for Lipschitz metrics since one can reflect the eigenfunctions and metric normally across the boundary to obtain equivalent L^p estimates for the resulting Lipschitz metrics. Fortunately, the problems are tractable, at least in two-dimensions, since the metrics one obtains by doubling are piecewise smooth with special types of Lipschitz singularities contained in the image of the boundary.

To motivate this proof, let us see what happens for the analogous result in $\mathbb{R}^2$, which of course is much simpler. We note that the euclidean analog of the dual form of the $L^2(M) \to L^8(M)$ estimates for the χ_λ operators would be the $L^{8/7}(\mathbb{R}^2) \to L^2(S^1)$ restriction theorem for the Fourier transform,

$$\|\hat{f}\|_{L^2(S^1)} \le C\|f\|_{L^{8/7}(\mathbb{R}^2)}, \quad f \in \mathcal{S}(\mathbb{R}^2). \tag{10}$$

Let us see how one can give a simple proof of this estimate. If we square the left side, and use Hölder's inequality, we get

$$\begin{aligned}\|\hat{f}\|^2_{L^2(S^1)} = \int_{S^1} \hat{f}\overline{\hat{f}d\theta} &= \int f(x)\,\overline{(f * \widehat{d\theta})}(x)\,dx \\ &\le \|f\|_{L^{8/7}(\mathbb{R}^2)}\|\, f * \widehat{d\theta}\|_{L^8(\mathbb{R}^2)}.\end{aligned}$$

Thus, (10) would hold if

$$\|\, f * \widehat{d\theta}\|_{L^8(\mathbb{R}^2)} \le C\|f\|_{L^{8/7}(\mathbb{R}^2)}.$$

But, $\widehat{d\theta} \approx \cos|x|/|x|^{1/2}$, and so this estimate would be a consequence of

$$\|f * |x|^{-1/2}\,\|_{L^8(\mathbb{R}^2)} \le C\|f\|_{L^{8/7}(\mathbb{R}^2)}, \tag{11}$$

which follows from the classical Hardy-Littlewood-Sobolev theorem for fractional integrals.

Estimate (10) and the above proof is due to Stein (unpublished), and this was the first restriction theorem for the Fourier transform. Earlier Schwartz had noticed that the restriction to the circle of the Fourier transform of an $L^p(\mathbb{R}^2)$, $p < 4/3$, makes sense as a distribution, and K. DeLeeuw raised the question of whether this distribution was actually a function. Stein's result of course answered this in the affirmative when $n = 2$ for exponents $1 \le p \le 8/7$. Stein's $L^{8/7}(\mathbb{R}^2)$ theorem was followed by much activity, including the optimal L^2 restriction theorems of C. Fefferman, P. Tomas, and Strichartz, and the flurry of activity in the 1990's on trying to sharpen these results and prove the higher dimensional versions of the sharp two-dimensional restriction theorem, which is due to Zygmund [18] and says that, for $p < 4/3$, $f \in L^p(\mathbb{R}^2)$ has Fourier transform which restricts as a function to $L^q(S^1)$, $q = p'/3$.

The situation for compact manifolds with boundary studied by Smith and the author [7] is much more technical. However, the fact that the estimate (10) follows from estimate (11) which does not involve oscillation implicitly carries over to this setting. Indeed a key fact in the proof of (4) is that after microlocally breaking up the operators that arise according to the angle from tangency to the boundary one can add up the contributions of the various pieces and still get (4). Heuristically, this works for the same reason that in two-dimensions it is a special property of $L^{8/7}$ that the estimate (10), which seems to be an estimate involving oscillatory integrals, actually follows from estimate (11) which of course does not.

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Long range scattering for the Maxwell-Schrödinger system

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(joint work with Giorgio Velo)

We study the theory of scattering for the Maxwell-Schrödinger system $(\text{MS})_3$ in space dimension 3 in the Coulomb gauge, namely,

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + g(|u|^2)u \ , \\ \\ \Box A = P \ \text{Im} \ \overline{u}\nabla_A u \ , \quad \nabla \cdot A = 0 \ . \end{cases} \tag{MS$_3$}$$

Here u is a complex valued function defined in space time $\mathbb{R}^{3+1}$, A (the magnetic vector potential) is an $\mathbb{R}^3$ vector valued divergence free function defined in $\mathbb{R}^{3+1}$, $\nabla_A = \nabla - iA$ is the covariant gradient and $\Delta_A = (\nabla - iA)^2$ is the covariant Laplacian, $\Box = \partial_t^2 - \Delta$ is the d'Alembertian in $\mathbb{R}^{3+1}$, $P = \mathbb{1} - \nabla\Delta^{-1}\nabla$ is the projector on divergence free vector fields, and $g(|u|^2) = (4\pi|x|)^{-1} * |u|^2$.

We regard scattering theory as a method to study the asymptotic behaviour in time of the solutions of that system and hopefully to classify those solutions by their asymptotic behaviour. The first step is the construction of the wave operators and for that purpose one has to solve the local Cauchy problem at infinity in time, namely to construct solutions with prescribed asymptotic behaviour (u_a, A_a) parametrized by asymptotic data $(u_+, A_+, \dot{A}_+)$. We concentrate on that problem. In the theory of scattering for such systems as $(\text{MS})_3$, one has to distinguish the short range case, where (u_a, A_a) can be taken as a solution of the underlying linear system (here the free Schrödinger and the free wave equations) from the long range case, where that choice is inadequate and has to be modified, thereby leading to so called modified wave operators. In that respect, the $(\text{MS})_3$ system is in the limiting long range case. The previous problem in the long range case has been treated by two methods, of which we consider only the first one. That method has been applied to various equations and systems in space dimension n for suitable n, namely to the nonlinear Schrödinger equation $(\text{NLS})_n$, to the Hartree equation $(\text{R3})_n$ and to the Klein-Gordon-Schrödinger $(\text{KGS})_2$, the Wave-Schrödinger $(\text{WS})_3$ and the Zakharov $(\text{Z})_n$ systems (see the references). It is intrinsically restricted to the case of small Schrödinger data and to the limiting long range case (to which $(\text{MS})_3$ belongs). We have applied that method to the $(\text{MS})_3$ system and improved previous results on that problem by (i) eliminating an additional smallness condition on the magnetic potential A and (ii) using larger function spaces and refined estimates resulting in a significant weakening of the assumptions on the asymptotic state $(u_+, A_+, \dot{A}_+)$.

The method proceeds in two steps.

Step 1. One looks for the solution (u, A) of $(\text{MS})_3$ with prescribed asymptotics (u_a, A_a) in the form $(u, A) = (u_a + v, A_a + B)$ and one solves the system for the difference variables (v, B) under suitable assumptions of regularity and of decay at infinity of (u_a, A_a). The method proceeds by a partial linearization followed by a contraction argument in a suitable function space $X(I)$ where $I = [T, \infty)$ for suitably large T. The choice of $X(I)$ is dictated by the available estimates, namely L^2 (or energy) estimates, and Strichartz inequalities for the wave and Schrödinger equation. We take

$$X(I) = \Big\{(v, B) : v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2), \| (v, B); X(I) \|$$

$$\equiv \operatorname*{Sup}_{t\in I} \ h(t)^{-1}\Big(\| v(t); H^2 \| \ + \ \| \partial_t v(t) \|_2 + \ \| v; L^{8/3}([t, \infty), W_4^1) \|$$

$$+ \ \| B; L^4([t, \infty), W_4^1) \| \ + \ \| \partial_t B; L^4([t, \infty), L^4) \| \Big) < \infty \Big\} .$$

where $h \in \mathcal{C}([1, \infty), \mathbb{R}^+)$ is such that $t^{3/8}h(t)$ is non increasing and tends to zero at infinity. That choice allows to complete step (1).

Step 2 consists in constructing asymptotic (u_a, A_a) satisfying the assumptions needed for step 1. Using the decomposition

$$U(t) = \exp\left(i(t/2)\Delta\right) = M \ D \ F \ M \ ,$$

$$M = \exp(ix^2/2t) \ , \ D \equiv D(t) = (it)^{-3/2}D_0(t) \ , \ (D_0(t)f)(x) = f(x/t) \ ,$$

where F is the Fourier transform, we choose

$$u_a = M \ D \exp(-i\varphi)Fu_+ \quad , \quad A_a = A_0 + A_1 \ ,$$

$$A_0 = \cos\omega t \ A_+ + \omega^{-1} \sin\omega t \ \dot{A}_+ \ ,$$

$$A_1(t) = t^{-1}D_0(t)\widetilde{A}_1 \ ,$$

$$\widetilde{A}_1 = \int_1^\infty d\nu \ \nu^{-3} \ \omega^{-1} \sin(\omega(\nu - 1))D_0(\nu)P \ x|Fu_+|^2 \ ,$$

$$\varphi = (\ell n \ t) \left(g(|Fu_+|^2) - x \cdot \widetilde{A}_1\right) \ ,$$

where $\omega = (-\Delta)^{1/2}$, thereby ensuring that $\Box A_a = \Box A_1 = P(x/t)|u_a|^2$. We define the spaces

$$H^{k,s} = \Big\{u : \| u; H^{k,s} \| = \| (1 + x^2)^{s/2}(1 - \Delta)^{k/2}u \|_2 < \infty\Big\} \ .$$

The final result can then be stated as follows.

Proposition. *Let $h(t) = t^{-1}(2 + \ell n \ t)^2$. Let (u_a, A_a) be defined as above. Let $u_+ \in H^{3,1} \cap H^{1,3}$ with $\| xFu_+ \|_4$ and $\| Fu_+ \|_3$ sufficiently small. Let $\nabla^2 A_+$, $\nabla\dot{A}_+$, $\nabla^2(x \cdot A_+)$ and $\nabla(x \cdot \dot{A}_+) \in W_1^1$ with A_+, $x \cdot A_+ \in L^3$ and $\dot{A}_+$, $x \cdot \dot{A}_+ \in L^{3/2}$ and let $\nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0$.*

Then there exists T, $1 \le T < \infty$ and there exists a unique solution (u, A) of the system (MS)$_3$ such that $(v, B) = (u - u_a, A - A_a) \in X([T, \infty))$. Furthermore ∇B, $\partial_t B \in \mathcal{C}([T, \infty), L^2)$ and B satisfies the estimate

$$\| \nabla B(t) \|_2 \ \vee \ \| \partial_t B(t) \|_2 \le C \ t^{-3/2}(2 + \ell n \ t)^2$$

for some constant C depending on $(u_+, A_+, \dot{A}_+)$ and for all $t \ge T$.

Remark. The only smallness conditions bear on $\| xFu_+ \|_4$ and on $\| Fu_+ \|_3$ and are required by the magnetic interaction and the Hartree interaction respectively. In particular there is no smallness condition on $(A_+, \dot{A}_+)$.

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Blow up for the semilinear Wave Equation in Schwarzschild metric

Vladimir Georgiev

(joint work with Davide Catania)

Consider the manifold

$$M = \mathbf{R} \times \mathbf{\Omega},\ \mathbf{\Omega} = \{(r,\omega) : r > 2M, \quad \omega \in \mathbf{S}^2\} = (2M,\infty) \times \mathbf{S}^2,$$

equipped with the Schwarzschild metric having the form (see chapter V in [1] or chapter 31 in [7]):

$$g = F(r)\, dt^2 - F(r)^{-1}\, dr^2 - r^2\, d\omega^2. \tag{1}$$

Here

$$F(r) = 1 - \frac{2M}{r},$$

the constant $M > 0$ has the interpretation of mass and $d\omega^2$ is the standard metric on the unit sphere $\mathbf{S}^2$. The D'Alembert operator associated with the metric g is

$$\square_g = \frac{1}{F}\left(\partial_t^2 - \frac{F}{r^2}\partial_r(r^2F)\partial_r - \frac{F}{r^2}\Delta_{\mathbf{S}^2}\right),$$

where $\Delta_{\mathbf{S}^2}$ denotes the standard Laplace–Beltrami operator on $\mathbf{S}^2$.

Our goal is to study the existence of global solution to the corresponding Cauchy problem for the semilinear wave equation

$$\square_g u = |u|^p \qquad \text{in } [0,\infty[\times\mathbf{\Omega}. \tag{2}$$

It is well–known (see [5], [6], [3], [9], [10], [11], [2], [4] for a more complete list of references on the subject) that for any space dimension $n \geq 2$, there exists a critical value $p_0 = p_0(n) > 1$ such that the Cauchy problem for the semilinear wave equation in flat metric admits a global small data solution provided $p > p_0(n)$.

For subcritical values of $p \leq p_0(n)$, a blow–up phenomenon is manifested. In the case of space dimension $n = 3$, the critical exponent is $p_0(3) = 1 + \sqrt{2}$, while in the general case of space dimension $n \geq 2$, the critical exponent is defined as the positive solution to

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The blow up results in [5], [6], [3], [9], [10] require a suitable comparison principle for the free wave equation. One further remark is connected with the fact that the critical exponent $p_0(n)$ is the same for the smaller class of radially symmetric solutions.

Our main goal in this work is to study the semilinear wave equation in the presence of Schwarzschild metric and to show a blow - up result for $1 < p < 1+\sqrt{2}$.

Introducing the Regge–Wheeler coordinate

$$s(r) = r + 2M \log(r - 2M), \tag{3}$$

we can rewrite equation (2) as

$$\partial_t^2 u - \partial_s^2 u - \frac{2F}{r(s)}\partial_s u - \frac{F}{r(s)^2}\Delta_{\mathbf{S}^2} u = F|u|^p, \tag{4}$$

where

$$F = F(s) = 1 - \frac{2M}{r(s)}$$

and $r(s)$ is the function inverse to (3).

For simplicity (and with no loss of generality), we shall restrict our considerations to the case of solutions of the form $u = u(t,s)$. Then (4) is simplified to the following equation:

$$\partial_t^2 u - \partial_s^2 u - \frac{2F}{r(s)}\partial_s u = F|u|^p. \tag{5}$$

Making further the substitution $u(t,s) = \frac{v(t,s)}{r(s)}$, we obtain the semilinear problem

$$\partial_t^2 v + Gv = F r^{1-p}|v|^p, \tag{6}$$

where

$$G = -\partial_s^2 + \frac{2MF}{r^3}. \tag{7}$$

Our goal is to treat the subcritical case $1 < p < 1 + \sqrt{2}$ and to show that some solutions with arbitrarily small initial data blow up in finite time.

To study the maximal time interval of existence of solutions to the wave equation in Schwarzschild metric

$$\begin{cases} \Box_g u = |u|^p & \text{in } [0,\infty[\times\mathbf{\Omega}, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \mathbf{\Omega}, \end{cases} \tag{8}$$

we suppose that our initial data are radial

$$u_0 = u_0(r), u_1 = u_1(r),\ (u_0,u_1) \in \mathbf{H}^2((2M,\infty)) \times \mathbf{H}^1((2M,\infty))$$

and that there exists a compact interval $\mathbf{B} \doteq \overline{\mathbf{B}(r_0, R)} \doteq \{|r-r_0| \leq R\} \subset (2M, \infty)$, so that

$$
(9) \qquad \begin{cases} u_0(r), \ u_1(r) \geq 0 & \text{almost everywhere,} \\ u_0(r) = u_1(r) = 0 & \text{for } \ |r - r_0| \geq R, \\ \int_{2M}^{\infty} u_j(r)\, dr \geq \epsilon & j = 0, 1 \end{cases}
$$

for a positive constant ϵ, $R > 0$ and $r_0 = r_0(\epsilon, p) \in \mathbf{\Omega}$. We also assume that r_0 is near $2M$ for $p \in]2, 1+\sqrt{2}[$, far from it for $p \in]1, 2[$ (we make no assumption in the case $p = 2$).

Now we can state the main result.

Theorem. *For any p, $1 < p < 1 + \sqrt{2}$ there exists a positive number ε_0 so that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $r_0 = r_0(p, \varepsilon)$ and $R = R(p, \varepsilon)$ so that for any initial data*

$$u_0 = u_0(r), u_1 = u_1(r), \ (u_0, u_1) \in \mathbf{H}^2((2M, \infty)) \times \mathbf{H}^1((2M, \infty))$$

satisfying (9) in $\mathbf{B} \doteq \overline{\mathbf{B}(r_0, R)}$, there exists a positive number $T = T(\varepsilon) < \infty$ and a solution

$$u \in \cap_{k=0}^{2} \mathcal{C}^k([0, T[; \mathbf{H}^{2-k}((2M, \infty))$$

of (8) such that

$$\lim_{t \nearrow T} ||u(t)||_{\mathbf{L}^2((2M,\infty))} = \infty.$$

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The hyperbolic-elliptic Ishimori system

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(joint work with Carlos E. Kenig)

The hyperbolic-elliptic Ishimori system,

$$(1)\qquad \begin{aligned} &\partial_t s = s \times \Box_{xy} s + \kappa(\zeta_x s_y + \zeta_y s_x) \\ &\Delta\zeta = 2s\cdot(s_x \times s_y) \end{aligned}$$

with $s : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$, $\lim_{|x|,|y|\to\infty} s(x, y, t) = (0, 0, -1)$ and κ a real constant was proposed in 1984 by Y. Ishimori [1]. In his paper, Ishimori -seeking to show that the dynamics of topological vortices need not generally be non-integrable-introduced the system (1) in analogy with the 2d CCIHS chain [15], as a model having the same topological properties as the latter yet permitting topological vortices whose dynamics are integrable; the system (1) is completely integrable when $\kappa = 1$. The linearized equation of (1) in this case is the same as that of the hyperbolic-elliptic Davey-Stewartson system and has been tackled by inverse scattering methods. On the other hand, system (1) with Δ_{xy} in lieu of $\Box_{xy}$ and $\kappa = 0$ reduces to the 2d CCIHS or Schrödinger map system. The Ishimori system (1) describes the time evolution of a system of static spin vortices in the plane. The right hand side of the equation for the scalar potential function $\zeta(x, y, t)$ is the topological charge density of the system. The integer values of the topological charge $Q := \frac{1}{4\pi}\int_{\mathbb{R}^2} s\cdot(s_x\times s_y)\,dxdy$, classify the static spin vortices. Geometrically $\zeta(x, y, t)$ is a multiple of the curvature tensor and the second equation can be viewed as describing the pull back of a piece of the volume in the target $\mathbb{S}^2$.

In our talk we described recent joint work with C. Kenig [5] proving that both the $2d$ Schrödinger map equation into $\mathbb{S}^2$ and the Ishimori system (1) admit a local in time solution for the Cauchy initial value problem with large data in $H^\gamma(\mathbb{R}^2)$, $\gamma > 3/2$ suitably avoiding the north pole.[1] Uniqueness holds in $H^2(\mathbb{R}^2)$. We also described the main difficulties that need to be addressed to extend our results to data at or close to the energy critical level.

In the context of Schrödinger maps it was shown in [10] that one can find an appropriate frame on $s^{-1}\mathcal{T}(\mathbb{S}^2)$ so that the derivatives of the solution satisfy a certain nonlinear Schrödinger system, referred to as the 'modified Schrödinger map' system (MSM). The same ideas transform very similarly the Ishimori system into a nonlinear hyperbolic Schrödinger equation. In analogy with Schrödinger maps, we will refer to it as the 'modified Ishimori system' (MIS).

[1]Because the solution of the Schrödinger equation depends on the initial data at every point and since no chart on the sphere can cover the entire complex plane, we ask the data to vanish in a small neighborhood of the north pole. This is a stronger than needed technicality that simplifies the translation back and forth between the Ishimori or Schrödinger map system and the Modified Ishimori or Schrödinger map equation. The weaker assumption of the map having degree zero should suffice; but it alone makes the translation back to the map more involved. For nearly parallel spins -i.e. sufficiently small data – initially close to the south pole; none of these requirements are needed since for a short time the map will stay close to the south pole.

To prove our main result we rely on the latter transformation in combination with energy estimates as in [11], Strichartz estimates and a formulation devised by Kenig [3], [4] of the ideas in Koch-Tzvetkov's work [8], to obtain a priori estimates for classical smooth solutions to the MIS. Standard approximation methods then give local in time existence in $H^s(\mathbb{R}^2)$, $s > 1/2$ for the MIS. Uniqueness for these solutions are shown in $H^1(\mathbb{R}^2)$. Local existence in $H^\gamma(\mathbb{R}^2)$, $\gamma > 3/2$ and uniqueness in $H^2(\mathbb{R}^2)$ for the original Ishimori map can then be established [11] [5].

The hyperbolic-elliptic Ishimori system was studied by A. Soyeur [14]. He proved local and global existence for sufficiently small data in H^3; and uniqueness of large data solutions in $H^4(\mathbb{R}^2)$.

It should be noted that local existence for large smooth data does not follow from the results in [7] because $\nabla\zeta$ is not necessarily in L^1. Thus in [5] we also need to do a parabolic regularization in the covariant derivative equation and prove energy estimates in H^k, k large.

Consider (1) with $\kappa \neq 0$ any real constant. The precise value of κ does not enter but in the constants bounding the estimates. We write

$$\begin{aligned} \partial_t s &= s \times \Box_{xy} s + (\phi_x s_y + \phi_y s_x) \\ \Delta\phi &= 2\kappa s \cdot (s_x \times s_y) \end{aligned} \tag{2}$$

Relabel (t, x, y) as (t, x_1, x_2). Following [11] we start with a description in terms of the stereographic projection of $\mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ where N is the north pole and rewrite system (2) using covariant derivatives. Assuming finite energy of the map s, one can apply the 'good gauge' theorem of Uhlenbeck -existence of a global Coulomb gauge- (Theorem 1 in [11]; [16]) to obtain a system of the form

(3)
$$\begin{aligned} \frac{\partial u_1}{\partial t} &= i\Box u_1 + \gamma_1(a_1\frac{\partial u_1}{\partial x_2} - a_2\frac{\partial u_1}{\partial x_1}) + \gamma_2\alpha_1 u_1 + \gamma_3\alpha_2 u_2 + \gamma_4(a_1^2 - a_2^2)u_1 + \gamma_5 a_0 u_1 \\ \frac{\partial u_2}{\partial t} &= i\Box u_2 + \gamma_1(a_1\frac{\partial u_2}{\partial x_2} - a_2\frac{\partial u_2}{\partial x_1}) + \gamma_2\alpha_1 u_2 + \gamma_3\alpha_2 u_1 + \gamma_4(a_1^2 - a_2^2)u_2 + \gamma_5 a_0 u_2 \end{aligned}$$

with γ_m, $m = 1, \dots 5$ some constants that may depend on κ but $\gamma_1 \in \mathbb{R}$. We have denoted by α_1, α_2 quadratic terms in u_1, u_2 of the form $\alpha_1 = \mathcal{R}(\text{ Im } (u_1\overline{u_2}))$, $\alpha_2 = \text{ Im } (u_1\overline{u_2})$ where by $\mathcal{R}$ we generically represent an appropriate linear combination of Riesz transforms. This is the system we call the Modified Ishimori system [5]. Just as above

$$a = (a_1, a_2) = (-\frac{\partial\beta}{\partial x_2}, \frac{\partial\beta}{\partial x_1}) \qquad \Delta\beta = \pm 4Im(u_1\overline{u_2}) \quad \text{and} \quad a_0 \sim \mathcal{R}(u_k\overline{u_j}) \tag{4}$$

Note that (3) can be viewed as a *linear* system with a priori given time dependent coefficients, the coupling of the systems occurs through one of the cubic nonlinearities. Schematically we write the Cauchy problem for either system as

$$u_t - i\Box u + \gamma\,\delta(a\,u) \;=\; F(u, \overline{u}) \tag{5}$$

where, u is either u_1 or u_2, a is as above, γ is a real nonzero constant and (i) δw represents $\tilde{\text{div}}\, w = \frac{\partial w}{\partial x_1} - \frac{\partial w}{\partial x_2}$ (ii) $\tilde{\nabla} = (\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2})$. Finally, $F(u, \overline{u})$ are all the cubic-type and 'quintic' terms appearing in (3)-(4).

Remark. The derivative term in the MIS (3) has the form $a \cdot \tilde{\nabla} u$. Thus at the expense of adding an extra cubic term of the form $(\frac{\partial a_1}{\partial x_1} - \frac{\partial a_2}{\partial x_2})u$ we can view the derivative term as $\tilde{\text{div}}\,(au)$. Unlike the case for the MSM, the derivative term in MIS does not have a null form structure.

Main Theorem [5]**.** *The Cauchy initial value problem associated to* (5) *admits a local in time solution in* H^s, $s > 1/2$. *More precisely, given data* $u_0 \in H^s(\mathbb{R}^2)$, $s > 1/2$ *there exists a time* $0 < \mathbf{T} = \mathbf{T}(\|u_0\|_{H^s_x})$ *and a solution to* (5) *such that*

$$u \in C([0, \mathbf{T}]; H^s) \quad \textit{and} \quad u \in L^2_t([0, \mathbf{T}]; L^\infty_x).$$

Furthermore, for data in H^1, *the* $L^\infty_{\mathbf{T}} H^1_x$*-solution can be shown to be unique and the mapping* $u_0 \to u \in C([0, \mathbf{T}]; H^1)$ *is continuous.*

The key to prove the theorem is the control of an $L^2_T L^\infty$ norm of a smooth solution to the MIS in terms of the $H^{1/2+}$ norm of the initial data. Uniqueness in $H^1(\mathbb{R}^2)$ follows as in [2].

In general, it is not a simple issue to go from solutions of the MIS and MSM systems to the full Ishimori and Schrödinger map systems directly. The transformation formulas between a solution u and the map s are quite complex. The well-posedness result on the modified system MIS apply to a larger class of formal solutions to the equation than those which come from the Ishimori map system. Our method of using the results on the modified map equations to show existence of the Ishimori maps is the same idea used in [10], and in [12] and [13] for wave maps. The main purpose behind the idea of fixing a particular gauge and passing to the modified systems is that of obtaining a priori estimates for smooth solutions and -when possible their differences- in 'rougher' norms and using them to pass to a weak limit in the full original map system in an appropriate lower regularity space. Smooth solutions transform over to solutions of the complete (overdetermined) system. Our results show that the time of existence depends only on $\|u_0\|_{H^{1/2+}}$. So given an initial data $H^{3/2+}$, we approximate it by smooth data in H^m, whose solutions satisfy the full set of equations and consistency conditions and the a priori estimates satisfied by the solution to the MIS system. These a priori estimates are now used to pass to a weak limit. The solution produced by the well-posedness result in our Main Theorem will be a weak limit of a subsequence of the smooth solutions in $C([0, T); H^{1/2+}) \cap L^2([0, T); L^\infty_x)$ and thus will also satisfy the entire set of consistency conditions as desired. A key step in the argument is to derive a bound on the extrinsic $H^\alpha(\mathbb{R}^2), \alpha > 1/2$ norm of ds. Details on how carry such an argument out appear in [5].

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Global well-posedness and scattering for the energy-critical nonlinear Schroedinger equation in $\mathbb{R}^3$

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(joint work with James Colliander, Gigliola Staffilani, Hideo Takaoka, Terence Tao)

We consider the Cauchy problem for the quintic defocusing Schrödinger equation in $\mathbb{R}^{1+3}$,

$$\begin{cases} iu_t + \Delta u = |u|^4 u \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where $u(t,x)$ is a complex-valued field in spacetime $\mathbb{R}_t \times \mathbb{R}^3_x$. This equation has an energy,

$$E(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \; dx \tag{2}$$

which is preserved by the flow.

Semilinear Schrödinger equations - with and without potentials, and with various nonlinearities - arise as models for diverse physical phenomena. Our interest here in the defocusing quintic equation (1) is motivated mainly though by the fact that the problem is critical with respect to the energy norm: we map a solution to another solution through the scaling $u \mapsto u^\lambda$ defined by $u^\lambda(t,x) := \frac{1}{\lambda^{1/2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$, and this scaling leaves the energy invariant. Our main result is global well-posedness for (1) in the energy class.

Theorem 1. *For any u_0 with finite energy, $E(u_0) < \infty$, there exists a unique*[1] *global solution $u \in C_t^0(\dot{H}_x^1) \cap L_{t,x}^{10}$ to (1) such that*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t,x)|^{10}\, dxdt \le C(E(u_0)). \tag{3}$$

for some constant $C(E(u_0))$ that depends only on the energy. In addition, there exists finite energy solutions $u_\pm(t,x)$ to the free Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$ such that

$$\|u_\pm(t) - u(t)\|_{\dot{H}^1} \to 0 \text{ as } t \to \pm\infty.$$

Finally, if $u_0 \in H^s$ for some $s > 1$, then $u(t) \in H^s$ for all time t, and one has the uniform bounds

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^s} \le C(E(u_0), s)\|u_0\|_{H^s}.$$

For powers less than five in the nonlinearity on the right side of (1), large data global well-posedness and scattering was shown in [6]. For large finite energy data which is assumed to be in addition radially symmetric, Bourgain [1] proved global existence and scattering for the quintic problem (1) in $\dot{H}^1(\mathbb{R}^3)$. Subsequently Grillakis [7] gave a different argument which recovered part of [1] - namely, global existence from smooth, radial, finite energy data. Our goal in this work was then to remove the radial assumption on the data. (Results previous to [1], [7] established global well-posedness from small data, and local well posedness from large data (see [3, 2]).)

We now sketch very briefly two of the ideas involved in the proof of the above Theorem: a suitable modification of the Morawetz inequality for (1), along with the frequency localized L^2 almost-conservation law that we ultimately use to prohibit energy concentration.

Building on work of Lin-Strauss [9] (who cite [10] as their motivation), we obtained in [4], [5] the following *interaction Morawetz* estimate for solutions of (1) [2]

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^4\, dxdt \lesssim \|u(0)\|_{L^2}^2 (\sup_{t \in I} \|u(t)\|_{\dot{H}^{1/2}})^2. \tag{4}$$

[1] In fact, uniqueness actually holds in the larger space $C_t^0(\dot{H}_x^1)$ (thus eliminating the constraint that $u \in L_{t,x}^{10}$), as one can show by adapting the arguments of e.g. [8].

[2] Strictly speaking, in [4], [5] this estimate was obtained for the *cubic* defocusing nonlinear Schrödinger equation instead of the quintic, but the argument in fact works for all nonlinear Schrödinger equations with a pure power defocusing nonlinearity.

However this estimate is not suitable for the critical problem because the right-hand side is not controlled by the energy $E(u)$: it increases without bound when we simply scale given finite energy initial data as above with λ large.

Our way around this is to localize the estimate in frequency space. We work in the context of an induction-on-energy argument as in [1]: assume for contradiction that Theorem 1 is false, and consider a solution of minimal energy among all those solutions with $L^{10}_{x,t}$ norm above some threshhold. We first show that such a *minimal energy blowup solution* would have to be localized in both frequency and in space at all times. Second, we prove that this localized blowup solution satisfies a *frequency localized Morawetz inequality* which states that after throwing away some low frequency portions of the blow-up solution, the remainder obeys good $L^4_{t,x}$ estimates. In principle, this estimate should follow simply by repeating the proof of (4) with u replaced by the high frequency portion of the solution, and then controlling error terms. Some of the error terms can only be controlled by using the fact that the solution under consideration is frequency and spatially localized. Hence the frequency-localized Morawetz inequality is not an *a priori* estimate valid for all solutions of (1), but instead is proven valid only for minimal energy blowup solutions.

The strategy is then to try to use Sobolev embedding to boost this $L^4_{t,x}$ control to $L^{10}_{t,x}$ control which would contradict the existence of the blow-up solution. The remaining worry is that the solution may shift its energy from low frequencies to high, possibly causing the $L^{10}_{t,x}$ norm to blow up while the $L^4_{t,x}$ norm stays bounded. To prevent this we look at what such a frequency evacuation would imply for the location -in frequency space - of the blow-up solution's L^2 mass. Specifically, we prove a frequency localized L^2 mass estimate that gives us information for longer time intervals than seems to be available from the *spatially* localized mass conservation laws used in the previous radial work ([1, 7]). By combining this frequency localized mass estimate with the $L^4_{t,x}$ bound and plenty of Strichartz estimate analysis, we can control the movement of energy and mass from one frequency range to another, and prevent the low-to-high cascade from occurring. The argument here is motivated by our previous low-regularity work involving almost conservation laws (e.g. [5]).

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