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# Free Probability Theory

Organised by
Philippe Biane (Paris)
Roland Speicher (Kingston)
Dan Voiculescu (Berkeley)

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Abstract. Free probability theory is a line of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. The theory grew out of attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost twenty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability and the theory of large deviations, algebraic combinatorics, topology. Free probability also has connections with applied mathematics (wireless communication) and some mathematical models in theoretical physics.

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## Introduction by the Organisers

The workshop *Free Probability Theory*, organised by Philippe Biane (Paris), Roland Speicher (Kingston), and Dan Voiculescu (Berkeley) was held March 27th–April 2nd, 2005. This meeting was well attended with over 50 participants with broad geographic representation from Austria, Canada, Denmark, France, Germany, Hungary, Japan, Netherlands, Poland, USA.

This workshop was sponsored by a project of the European Union which allowed us to invite in addition to established researchers also a couple of young people who were interested in learning about free probability.

Free probability theory is a line of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out from attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost twenty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability and the theory of large deviations, algebraic combinatorics, topology. Free probability also has connections with applied mathematics (wireless communication) and some mathematical models in theoretical physics.

The Oberwolfach workshop on free probability brought together a very strong group of mathematicians representing the current directions of development in the area. The diversity of the particants and the ample free time left in the programme stimulated a lot of fruitful discussions, laying the seed for many new collaborations.

The programme consisted of 13 lectures of 50 minutes, supplemented by 13 shorter contributions of 30 minutes. Because of the various backgrounds of the participants much emphasis was put on making the lectures accessible to a broad audience; most of them provided a survey on the background as well as highlighting some recent developments in connection with free probability. Instead of trying to summarize all these developments we will let the following abstracts speak for themselves.

## Workshop: Free Probability Theory

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# Abstracts

## Multilinear Function Series and Transforms in Free Probability Theory

Ken Dykema

In this talk, the R– and S–transforms of random variables over a general Banach algebra $B$ were described. Given a random variable $a$ in a $B$–valued Banach noncommutative probability space $(A, E)$, consider the $B$–valued analytic function

$$\Phi_a(b) = \sum_{n=0}^{\infty} E(a(ba)^n) = E(a(1-ba)^{-1}), \qquad (b \in B,\ \|b\| < \|a\|^{-1}).$$

Then the R–transform of $\Phi_a$ is the $B$–valued analytic function $R_{\Phi_a}$ defined by

$$C_{\Phi_a}^{\langle -1\rangle}(b) = (1 + b\,R_{\Phi_a}(b))^{-1} b$$

where $C_{\Phi_a}^{\langle -1\rangle}$ is the inverse with respect to composition of the function

$$C_{\Phi_a}(b) = b + b\,\Phi_a(b)\,b.$$

Then for $x, y \in A$ free with respect to $B$, it was proved by Voiculescu [9] that the R–transform linearizes additive free convolution, namely,

$$R_{\Phi_{x+y}} = R_{\Phi_x} + R_{\Phi_y}.$$

The scalar–valued case (when $B = \mathbf{C}$) was earlier proved by Voiculescu in [7]. The case of general $B$ (not necessarily a Banach algebra) was treated also by Voiculescu in [9], and was given a beautiful combinatorial description by Speicher in [5] and [6].

The situation for multiplicative free convolution was also described. For $a$ a random variable in a $B$–valued Banach noncommutative probability space $(A, E)$, with $B$ a general banach algebra, and assuming $E(a)$ is invertible, we define the S–transform of $\Phi_a$ to be the $B$–valued analytic function $S_{\Phi_a}$ defined by

$$D_{\Phi_a}^{\langle -1\rangle}(b) = b(1+b)^{-1}\,S_{\Phi_a}(b),$$

where $D_{\Phi_a}^{\langle -1\rangle}$ is the inverse with respect to composition of the function

$$D_{\Phi_a}(b) = bF(b).$$

We described the new result [2], that the S–transform satisfies a twisted multiplicativity property with respect to freeness, namely

$$S_{\Phi_{xy}}(b) = S_{\Phi_y}(b)\,S_{\Phi_x}(\tilde{b}) \tag{1}$$

where

$$\tilde{b} = (S_{\Phi_y}(b))^{-1}\,b\,S_{\Phi_y}(b)$$

is $b$ conjugated by the inverse of $S_{\Phi_y}(b)$, assuming $x$ and $y$ are free in $(A, E)$ and $E(x)$ and $E(y)$ are invertible. The proof of this result was outlined, based on certain annihilation and creation operators on an analogue of full Fock space over $B$. The scalar–valued case (when $B = \mathbf{C}$) was treated by Voiculescu in [8], and the case for $B$ a commutative Banach algebra was treated by Aagaard [1]; in both these cases, twisted mutliplicativity (1) reduces to plain multiplicativity. Also, Voiculescu's result treats random variables in any noncommutative probability space over $\mathbf{C}$, which is not necessarily Banach. In [4], Uffe Haagerup gave alternative proofs of Voiculescu's result [8] on the scalar–valued S–transform. Both Aagaard's proof [1] for $B$ commutative and our proof [2] for a general Banach algebra $B$ are inspired by Haagerup's proofs, but by different ones.

By the T–transform, we mean the inverse with respect to multiplication of the S–transform. The T–transform turns out to be more cannonical than the S–transform.

In order to handle general algebras $B$ (not necessarily Banach algebras) over an arbitrary field $K$ and in order to treat all moments of $B$–valued random variables (not only the symmetric ones), we describe the algebra Mul[[B]] of formal multilinear function series over $B$, which was introduced in [3]. Elements of Mul[[B]] are sequences $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ where $\alpha_0$ belongs to $B$ (or to the unitization of $B$, if $B$ lacks an identity element) and where $\alpha_n : B \times \cdots \times B \to B$ is a multilinear function from the $n$–fold product of $B$ into $B$, where linearity is over $K$. The operation of formal multiplication makes Mul[[B]] into an algebra, and we define the operation of formal composition, which behaves analogously to usual composition. If $B = K = \mathbf{C}$, then Mul[[B]] is the algebra of formal power series in one variable with complex coefficients, with the familiar operations.

Given a random variable $a$ in a noncommutative probability space $(A, E)$ over $B$, the distribution series of $a$ is defined to be the formal multilinear funcition series $\widetilde{\Phi}_a \in$ Mul[[B]] given by $\widetilde{\Phi}_{a,0} = E(a)$, and $\widetilde{\Phi}_{a,n}(b_1, \dots, b_n) = E(ab_1ab_2a \cdots b_na)$. The unsymmetrized R– and T–transforms of $a$ are the elements of Mul[[B]] defined using formulas entirely analogous to those defining the usual R– and T–transforms, but in the context of Mul[[B]]. We describe additivity and twisted multiplicativity results [3] for the unsymmetrized R– and T–transforms, that are entirely analogous to the corresponding properties of the usual R– and T–transforms. The main difference is that while the usual R– and T–transforms capture information about only the symmetric moments of a random variable, the unsymmetrized versions capture all moments.

The final topic covered in the talk was the partially ordered set of noncrossing linked partitions, which was introduced in [3]. This plays a role for multiplication of noncommutative random variables similar to the role of the lattice of noncrossing partitions in Speicher's treatment of the R–transform [5], [6]. It was used in order to prove the twisted multiplicativity property of the unsymmetrized T–transform.

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## Quantum Permutation Groups and Free Probability

Teodor Banica

(joint work with Julien Bichon)

This is a presentation of joint work with Julien Bichon [3].

A quantum group is an abstract object, dual to a Hopf algebra. Finite quantum groups are those which are dual to finite dimensional Hopf algebras.

A surprising fact, first noticed by Wang in [13], is that the quantum group corresponding to the Hopf algebra $\mathbb{C}^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ has a faithful action on the set $\{1, 2, 3, 4\}$. This quantum group, which is of course not finite, is a so-called quantum permutation group.

In general, a quantum permutation group $G$ is described by a special type of Hopf $\mathbb{C}^*$-algebra $A$, according to the heuristic formula $A = \mathbb{C}(G)$. See [2], [13].

The simplest case is when $A$ is commutative. Here $G$ is a subgroup of the symmetric group $S_n$. This situation is studied by using finite group techniques.

In general $A$ is not commutative, and infinite dimensional. In this case $G$ is a non-classical, non-finite compact quantum group. There is no analogue of a Lie algebra in this situation, but several representation theory methods, due to Woronowicz, are available ([14], [15]).

A useful point of view comes from the heuristic formula $A = \mathbb{C}^*(\Gamma)$. Here $\Gamma$ is a discrete quantum group, obtained as a kind of Pontrjagin dual of $G$. Number of discrete group techniques are known to apply to this situation. See e.g. [10], [11].

The aim of this work is to bring into the picture some free probability techniques.

The starting point is the classical formula $G(X \dots X) = G(X) \times_w G(X_n)$ for usual symmetry groups. Here $X$ is a finite connected graph, $X_n$ is a set having $n$ elements, $X \dots X$ is the disjoint union of $n$ copies of $X$, and $\times_w$ is a wreath product. A series of free quantum analogues and generalisations, started in [6] and continued here, leads to a general formula of type $A(X * Y) = A(X) *_w A(Y)$. Here $X, Y$ are colored graphs, and $*_w$ is a free wreath product.

The corepresentation theory of free wreath products is worked out in two particular situations in [2], [6]. Our key remark here is that a formula of type

$$\mu(A *_w B) = \mu(A) \boxtimes \mu(B)$$

holds in both cases, where $\mu$ is the associated spectral measure. We conjecture that this formula holds in general, and under mild assumptions on $A$ and $B$.

This is to be related to a planar algebra formula of type $\mu(P*Q) = \mu(P)\boxtimes\mu(Q)$, known to Bisch and Jones ([7]). In fact, a general formula of type $A(X * Y) = A(X)*_w A(Y)$, with colored graphs replaced by planar algebras, would be equivalent to the conjecture.

Of particular interest is the case $B = A(X_n)$. Here the conjecture, together with Voiculescu's $S$-transform technique ([12]) reduces computation of $\mu(X)$ with $X$ homogeneous to that of $\mu(X)$ with $X$ connected and homogeneous. For $n = 2$ the conjecture is actually a theorem, and, as an application, we compute $\mu$ for the graph which looks like 2 rectangles. This completes previous classification work for graphs having at most $8$ vertices ([1], [2]).

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## Orthogonal Polynomials and Fluctuations of Random Matrices

James Mingo
(joint work with Timothy Kusalik, Roland Speicher)

Wishart matrices arose in the estimation of covariance matrices in multivariate statistics. Besides the Gaussian random matrices they constitute the most important random matrix ensemble. They can be described as follows. Let $G_{M,N}$ be a $M \times N$ matrix whose entries are independent complex Gaussian random variables with mean 0 and complex variance $1/N$. Let $X_N = G^*_{M,N} G_{M,N}$; $X_N$ is a complex Wishart matrix (of parameter $c = M/N$).

The fundamental quantities of interest for random matrix ensembles are the asymptotic eigenvalue distribution and the fluctuations around this asymptotics. In the case of Wishart matrices, the large $N$ limit of the eigenvalue distribution was found in 1967 by Marchenko and Pastur and is now named after them. The question of fluctuations was addressed for the first time by Jonsson [JON] in 1982 and recently by Cabanal-Duvillard [CD] in 2001 in a more detailed manner.

For many random matrices $Y$ the family of random variables $\{\mathrm{Tr}(Y^n)\}_n$ becomes asymptotically Gaussian, as the size $N$ of the matrices goes to infinity. The fundamental quantities mentioned above consist then in understanding the limit of the expectation and of the covariance of these Gaussian random variables. For the latter one would in particular like to diagonalize it. Whereas the expectation (i.e., the eigenvalue distribution) depends on the considered ensemble, the covariance (i.e., the fluctuations) seem to be much more universal. There are quite large classes of random matrices which show the same fluctuations. The most important class is the one which is represented by the Gaussian random matrices. Its fluctuations are diagonalized by the Chebyshev polynomials, see Johansson [JOH].

In the case of the Wishart matrices the asymptotic Gaussianity of the traces was shown by Jonsson; the explicit form of the covariance, however, was revealed only recently by Cabanal-Duvillard [CD]. He found polynomials $\{\Gamma_n\}_n$, which were shown to be shifted Chebyshev polynomials, such that the random variables $\{\mathrm{Tr}(\Gamma_n(X)\}_n$ are asymptotically Gaussian and independent in the large $N$ limit; that is the polynomials $\{\Gamma_n\}_n$ diagonalize asymptotically the covariance. Cabanal-Duvillard's approach relies heavily on stochastic calculus. In [KMS] we gave a combinatorial proof of his results which rested on a combinatorial interpretation of the polynomials $\Gamma_n$. This combinatorial approach allowed very canonically an extension of Cabanal-Duvillard's results to a family of independent Wishart matrices, yielding our main result.

**Theorem** *Let $\{\Gamma_n\}_n$ be the shifted Chebyshev polynomials of the first kind as considered by Cabanal-Duvillard and let $\{\Pi_n\}_n$ be the orthogonal polynomials of the Marchenko-Pastur distribution (which are shifted Chebyshev polynomials of the second kind). Let $X_1, \dots, X_p$ be independent Wishart matrices and consider in addition to $\mathrm{Tr}(\Gamma_n(X_i))$ also, for $k \geq 2$, the collection of random variables $\mathrm{Tr}(\Pi_{m_1}(X_{i_1}) \cdots \Pi_{m_k}(X_{i_k}))$, where the Wishart matrices which appear must be* *cyclically alternating, i.e.,* $i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_k \neq i_1$. *These latter traces depend only on the equivalence class of* $(i_1, \ldots, i_k)$ *and* $(m_1, \ldots m_k)$ *under cyclic permutation. Assuming that we have chosen one representative from each equivalence class, the random variables*

$$\{\mathrm{Tr}(\Gamma_n(X_i))\} \cup \{\mathrm{Tr}(\Pi_{m_1}(X_{i_1}) \cdots \Pi_{m_k}(X_{i_k}))\}$$

*are asymptotically independent and Gaussian.*

The main motivation for our investigations comes from our belief that the theory of free probability provides the right tools and concepts for attacking questions on fluctuations of random matrices – in particular, for multi-matrix models. Even though "freeness" did not appear explicitly in [KMS], our methods and results are very much related to our investigations around "second order freeness" in [MS]. The present paper can, in particular, be seen as a complementary treatment of some of the questions treated in [MS].

Our starting point is the paper of Mingo-Nica [MN], where a genus expansion in terms of permutations was provided for the cumulants of the random variables $\mathrm{Tr}(X^n)$. Since cumulants of different orders have different leading contributions in $N$, this has as a direct consequence the asymptotic Gaussianity of these traces. The main problem left is to understand and diagonalize the covariance. Also in [MN], it was shown that the covariance of the random variables $\{\mathrm{Tr}(X^n)\}_n$ has asymptotically a very nice combinatorial interpretation, namely it is given by counting a class of planar diagrams which were called non-crossing annular permutations. More precisely, if we denote by $\kappa_2(A, B)$ the covariance of two random variables $A$ and $B$ and if $c$ is the asymptotic ratio of $M$ and $N$ for our Wishart matrices, we have

$$\lim_{N\to\infty} \kappa_2(\mathrm{Tr}(X^m), \mathrm{Tr}(X^n)) = \sum_{\pi \in S_{NC}(m,n)} c^{\#(\pi)},$$

where $S_{NC}(m, n)$ denotes the set of non-crossing $(m, n)$-annular permutations, i.e., permutations on $m + n$ points which connect $m$ points on one circle with $n$ points on another circle in a planar or non-crossing way. In the above formula we are summing over all non-crossing $(m, n)$-annular permutations and each block of such a permutation contributes a multiplicative factor $c$.

In this pictorial description $\mathrm{Tr}(X^m)$ corresponds to the sum over non-crossing half-permutations on one circle with $m$ points and $\mathrm{Tr}(X^n)$ corresponds to a sum over non-crossing half-permutations on another circle with $n$ points. The limit of $\kappa_2(\mathrm{Tr}(X^m), \mathrm{Tr}(X^n))$ corresponds to pairing the half-permutations for $\mathrm{Tr}(X^m)$ with the half-permutations for $\mathrm{Tr}(X^n)$. The pairing between two half-permutations is given by glueing them together in all possible planar ways.

Finally, we would like to point out that our circular half-permutations are, after a small redrawing, the diagrams used by V. F. R. Jones [J, §5] to create a basis for the irreducible representations of the annular Temperly-Lieb algebras.

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## Combinatorial Aspects of Matrix Models

Alice Guionnet

In this talk, we discuss the approach of physicists to matrix models and related combinatorics and apply it to free probability, summarizing a joint work with E. Maurel. More precisely, let $\mu_N$ be the law of the GUE, that is the law of an $N \times N$ Hermitian matrix with entries which are independent centered complex Gaussian variables with covariance $(N)^{-1}$ above the diagonal and centered real Gaussian variables with covariance $N^{-1}$ on the diagonal;

$$\mu_N(dA) = \frac{1}{Z_N} 1_{A \in \mathcal{H}_N} e^{-\frac{N}{2}\operatorname{tr}(A^2)} dA.$$

Let $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ be the set of polynomials in $m$ non commutative variables. We equip $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ with the involution $*$ so that for all choices of $n \in \mathbb{N}$, $i_k \in \{1, \cdots, m\}$, $k \in \{1, \cdots, n\}$, all $z \in \mathbb{C}$,

$$(zX_{i_1} \cdots X_{i_n}) = \bar{z} X_{i_n} \cdots X_{i_1}$$

Let $V = V_{\bar{t}} = \sum_{i=1}^{n} t_i(q_i + q_i^*)$ for monomial functions $q_i \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ and real parameters $\bar{t} = (t_1, \cdots, t_n)$. Then, it is widely used in physics (see e.g. [2] for a nice review) that

$$F_N(\bar{t}) = \frac{1}{N^2} \log \int e^{-N \operatorname{tr}(V_{\bar{t}}(A_1, \cdots, A_m))} d\mu_N(A_1) \cdots d\mu_N(B)$$

expands, when the real parameters $t_i$ are small enough, into an enumeration of colored maps. This expansion is formal; for instance, it can be proved by Wick formula that

$$(1) \qquad (-1)^{k_1 + \cdots + k_n} D(k_1, \cdots, k_n, 0) = \lim_{N \to \infty} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} F_N(V)|_{\bar{t}=0}$$

enumerates some planar maps.

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## Rectangular Random Matrices, Freeness with Amalgamation, and Free Entropy

Florent Benaych-Georges

We characterize asymptotic collective behaviour of rectangular random matrices, the sizes of which tend to infinity at different rates: we prove that independent rectangular random matrices, when embedded in a space of larger square matrices, are asymptoticaly free with amalgamation over a commutative finite dimensional subalgebra $\mathcal{D}$ (under an hypothesis of unitary invariance).

It allows us to define a "rectangular free convolution" $\boxplus_\lambda$: for every$\lambda \geq 0$, for all $\mu, \nu$ symetric probability measures, $\mu \boxplus_\lambda \nu$ is the symetrization of the limit, when the dimensions go to infinity in a ratio $\lambda$, of spectral distribution of the absolute value of $M + N$, where $M, N$ are independant random matrices, whose distributions are invariant under the action of unitary groups, and such that the symetrization of the limit spectral distribution of the absolute value of $M$ (resp. $N$) is to $\mu$ (resp. $\nu$).

This convonlution is linearized by cumulants. It allows us to investigate the related notion of infinite divisiblity, which appears to be closely related the classical infinite divisibility.

Then we consider elements of a finite von Neumann algebra containing $\mathcal{D}$, which have kernel and range projection in $\mathcal{D}$. We associate them a free entropy with the microstates approach, and a free Fisher's information with the conjugate variables approach. Both give rise to optimization problems whose solutions involve freeness with amalgamation over $\mathcal{D}$.

It could be a first proposition for the study of operators between different Hilbert spaces with the tools of free probability. As an application, we prove a result of freeness with amalgamation between the two parts of the polar decomposition of $R$-diagonal elements with non trivial kernel.


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## Applications of $L^2$ Cohomology to Free Entropy Dimension

Dimitri Shlyakhtenko

*We use connections between free entropy theory and the theory of $L^2$ cohomology to derive two statements in free probability theory. First, we give a counterexample to the semi-continuity question of free entropy dimension, posed by Voiculescu in* [Voi94]*. We give a reformulation of the question, which avoids the counterexample, while (if true) still implying the non-isomorphism of free group factors. Second, we prove that whenever an self-adjoint $n$-tuple $X_1, \dots, X_n$, $n \geq 2$ generates a diffuse hyperfinite $II_1$ factor, then there is no dual system to $X_1, \dots, X_n$.*

In [CS], we found a connection between $L^2$ cohomology [Ati76, CG86, Lüc02] and free probability theory (more precisely, free entropy theory)

[VDN92, Voi98, Voi94, Voi96]. This connection is exemplified by the inequality $\delta^*(X_1, \dots, X_n) \leq \Delta(X_1, \dots, X_n)$, where $X_1, \dots, X_n$ is an $n$ tuple of self-adjoint elements in a tracial von Neumann algebra $(M, \tau)$, $\delta^*$ is the non-microstates free entropy dimension [Voi98, Shl04], while $\Delta$ is a quantity that appears in $L^2$ cohomology. More precisely,

$$\Delta(X_1, \dots, X_n) = n - \dim_{M \bar{\otimes} M} \mathrm{cl}\ V(X_1, \dots, X_n), \qquad \text{where}$$
$$V(X_1, \dots, X_n) = \{(T_1, \dots, T_n) \in FR^n : \sum [T_i, X_i] = 0\} \subset HS^n,$$

cl refers to the closure in the Hilbert-Schmidt topology, and $FR$ and $HS$ stand respectively for the $M, M$-bimodules of finite-rank and Hilbert-Schmidt operators on $L^2(M, \tau)$ with the bimodule action given by $(m \otimes n) \cdot T = Jm^*J\ T\ Jn^*J$.

In the case that $X_1, \ldots, X_n$ generate the group algebra of a discrete group $\Gamma$ (which we take with its natural trace), $\Delta(X_1, \ldots, X_n) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$, where $\beta_j^{(2)}$ are the $L^2$ Betti numbers of $\Gamma$, i.e. dimensions of its $L^2$ cohomology groups.

In the case of groups, the connection with free entropy dimension is much tighter; in [MS], we proved (under the above assumptions that $X_1, \ldots, X_n$ generate $\mathbb{C}\Gamma$) that

$$\delta^*(X_1, \ldots, X_n) = \Delta(X_1, \ldots, X_n) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)} + 1.$$

The connection between $\Delta$ and $\delta^*$ can be direct (as we have just seen), but it can also be used as an inspiration to transport results from the theory of $L^2$ cohomology to that of free entropy dimension.

Using the well-known properties of Murray-von Neumann dimension, it is easy to construct a counterexample to semicontinuity of $\Delta(X_1^{(k)}, \ldots, X_n^{(k)})$, where $X_j^{(k)} \in M$ and $X_j^{(k)} \to X_j$ strongly. The same counterexample works for free entropy dimension $\delta_0$ and its variants $\delta$, $\delta^*$ [Shl05]:

Let $a, b$ be the canonical generators of a free group on two generators. Let $X_1^{(k)} = a^2$, $X_2^{(k)} = b^2$, $X_3^{(k)} = ab$, $X_4^{(k)} = a$. Then $X_j^{(k)} \to X_j$ where $X_1 = a^2$, $X_2 = b^2$, $X_3 = ab$ and $X_4 = 0$. Since $X_1, \ldots, X_4$ generate a free group on 3 generators, while $X_1^{(k)}, \ldots, X_4^{(k)}$ generate a free group on two generators, we conclude that $2 = \liminf_k \delta(X_1^{(k)}, \ldots, X_4^{(k)}) \not\geq \delta(X_1, \ldots, X_4) = 3$.

Our second application concerns the question of existence of "dual systems" in the sense of Voiculescu [Voi98]. If $X_1, \ldots, X_n$ are as above, one says that $D_1, \ldots, D_n \in B(L^2(M))$ are a dual system to $X_1, \ldots, X_n$, if one has

$$[D_j, X_k] = \delta_{jk} P_1, \quad 1 \leq j, k \leq n,$$

where $P_1$ denotes the rank-one projection onto $1 \in L^2(M)$. One has the following implications:

$$\begin{aligned} \text{dual system exists} \quad &\Rightarrow \quad \Phi^*(X_1, \ldots, X_n) < \infty \;\Rightarrow\; \chi^*(X_1, \ldots, X_n) \\ &\Rightarrow \quad \delta^*(X_1, \ldots, X_n) = n \;\Rightarrow\; \Delta(X_1, \ldots, X_n) = n. \end{aligned}$$

By analogy with the situation for groups [Lüc98], one expects that

$$\Delta(X_1, \ldots, X_n) = 1$$

under some amenability assumptions. This would have as consequence the non-existence of a dual system in the amenable case. While the bound on $\Delta$ is out of reach at present, we prove that if $X_1, \ldots, X_n$ generate the hyperfinite $\mathrm{II}_1$ factor, then a dual system cannot exist.

We sketch the argument (which goes through under an additional technical assumption).

Step 1. *There exists operators* $T_1, \ldots, T_n \in M\bar{\otimes}M^o$ *so that* $\sum [T_i, X_i] = 0$ *while* $\mathrm{Tr}(T_1 P_1) \neq 0$. Indeed, assume that this is not possible. Then if we let $W = \{(T_1, \ldots, T_n) \in M\bar{\otimes}M^o : \sum [T_i, X_i] = 0\}$, we obtain that $(P_1, 0, \ldots, 0) \perp W$.

On the other hand, since for any $H \in HS$, and any $(T_1, \dots, T_n) \in HS^n$ satisfying $\sum [T_i, X_i] = 0$,

$$\sum_i \mathrm{Tr}([H, X_i] T_i) = \mathrm{Tr}(H \sum [X_i, T_i]) = 0,$$

we find that $W^\perp = \mathrm{cl}\{([H, X_1], \dots, [H, X_n]) : H \in HS\}$. Thus its Murray-von Neumann dimension over $M \bar{\otimes} \bar{M}^o$ equal to 1 (since the map $H \mapsto ([H, X_1], \dots, [H, X_n])$ has no kernel). But if $(P_1, 0, \dots, 0) \in W^\perp$, it follows that $W^\perp = HS \oplus 0 \oplus \cdots \oplus 0$, which is clearly not possible (as it would imply that an arbitrary $H \in HS$ commutes with $X_2, \dots, X_n$, which implies that they are scalar, contradicting the assumption that $X_1, \dots, X_n$ generate a $\mathrm{II}_1$ factor).

Step 2. *There exists a non-normal state* $\Upsilon : M \bar{\otimes} M^o \to \mathbb{C}$ *satisfying* $\Upsilon((x \otimes 1 - 1 \otimes x^o) T) = 0$ *for all* $x \in M$ *and* $T \in M \bar{\otimes} M^o$*, and such that for any finite tensor* $\sum a_i \otimes b_i \in M \otimes M^o, \Upsilon(\sum a_i \otimes b_i) = \mathrm{Tr}(\sum \theta_{a_i, b_i})$*, where* $\theta_{a_i, b_i}$ *is the finite-rank operator corresponding to* $a_i \otimes b_i \in M \otimes M^o \subset L^2(M) \otimes L^2(M)^o = FR(L^2(M))$. Indeed, the state $\Upsilon_0(\sum a_i \otimes b_i) = \langle \sum a_i J b_i^o J \ 1, 1 \rangle$ satisfies the necessary assumptions on $C^*(M, JMJ) \subset B(L^2(M))$. Since $M$ is hyperfinite, $C^*(M, JMJ) = M \otimes_{\min} M^o \subset M \bar{\otimes} M^o$ [Con76]. Thus the state $\Upsilon_0$ is defined on $M \otimes_{\min} M^o$ and admits by the Hahn-Banach theorem a (non-normal) extension $\Upsilon$ to $M \bar{\otimes} M^o$, satisfying the desired properties.

Step 3. Assume now that a dual system $D_1, \dots, D_n$ exists. Let $T_1, \dots, T_n \in M \bar{\otimes} M^o \subset HS$ be as in Step 1, so that $\sum [T_i, X_i] = 0$, $\mathrm{Tr}(T_1 P_1) \neq 0$. We make the technical assumption that $T_i D_1 \in M \bar{\otimes} M^o$, $i = 1, \dots, n$. Then

$$\begin{aligned} \mathrm{Tr}(T_1 P_1) &= \mathrm{Tr}(\sum T_i [D_1, X_i]) = \Upsilon(\sum T_i [D_1, X_i]) \\ &= \Upsilon(\sum T_i D_1 X_i - T_i X_i D_1) \end{aligned}$$

(here we identify $M \bar{\otimes} M^o$ with a subset of $HS$). Note that since $T_1 P_1 = (1 \otimes \tau)(T_1) \in M \bar{\otimes} M^o$ and $[D_1, X_i] = 0$ for $i \geq 2$, we have that $T_i [D_1, X_i] \in M \bar{\otimes} M^o$ for each $i$, so that we can indeed apply $\Upsilon$. Since $T_i D_1 \in M \bar{\otimes} M^o$ by assumption, so is $T_i D_1 X_i$ and hence also $T_i X_i D_1$. Thus

$$\begin{aligned} 0 &\neq \sum \Upsilon(T_i D_1 X_i) - \Upsilon(T_i X_i D_1) \\ &= \sum \Upsilon(X_i T_i D_1) - \Upsilon(T_i X_i D_1) - \Upsilon(X_i T_i D_1 - T_i D_1 X_i) \\ &= \Upsilon(\sum [X_i, T_i] D) - \sum \Upsilon((X_i \otimes 1 - 1 \otimes X_i^o) \cdot \xi_i), \end{aligned}$$

where $\xi_i = T_i D_1$ regarded as an element of $M \bar{\otimes} M^o$, and $\cdot$ denotes the multiplication of $M \bar{\otimes} M^o$. Since $\sum [T_i, X_i] = 0$, the term $\Upsilon(\sum [X_i, T_i] D)$ is zero. But the properties of $\Upsilon$ imply that $\Upsilon((X_i \otimes 1 - 1 \mathrm{otimes} X_i^o) \xi_i) = 0$. We have thus arrived at a contradiction.

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## The Microstates Free Entropy Dimension of a DT-Operator is 2

Kenley Jung
(joint work with Ken Dykema, Dimitri Shlyakhtenko)

We show that any one of the DT-operators introduced by Ken Dykema and Uffe Haagerup has microstates free entropy dimension exactly equal to 2. This is joint work with Ken Dykema and Dimitri Shlyakhtenko.

## Finite Free Entropy and Free Group Factors

Nathaniel Brown

My talk concerned the resolution of a specific question raised by Dimitri Shlyakhtenko: Is it true that a set of noncommutative random variables with finite free entropy necessarily generate a free group factor? (For an excellent survey of Voiculescu's free entropy theory we refer to [4].) This question was motivated by the seminal work of Gaboriau who showed that an equivalence relation is the free product of hyperfinite equivalence relations if and only if there exists a generating set of minimal support. (See [2] for a precise statement.)

Though there was hope for an affirmative answer to Shlyakhtenko's question it turns out that counterexamples exist and the obstruction lies in some technical approximation properties of operator algebras. The main result is as follows.

**Theorem:** There exist noncommutative random variables $X_1, \ldots, X_n$ with the property that $\chi(X_1, \ldots, X_n) > -\infty$ but $M = W^*(X_1, \ldots, X_n)$ is not isomorphic to any (not necessarily unital) subalgebra of a free group factor.

The construction of the counterexamples is fairly easy to describe. We begin with a discrete, residually finite group $\Gamma$ which has Kazhdan's property T (e.g. $SL(3, \mathbb{Z})$). Let $Y_1, \ldots, Y_n$ be a set of self-adjoint generators of $N_0 = L(\Gamma)$ (the von Neumann algebra generated by the left regular representation). Inside $N_0 * L(\mathbb{F}_n)$ consider the n-tuple of self-adjoints

$$Y_\epsilon = \{Y_1 + \epsilon S_1, \ldots, Y_n + \epsilon S_n\},$$

where $S_i \in L(\mathbb{F}_n)$ are free semicircular elements, and let $N_\epsilon = W^*(Y_\epsilon)$ be the von Neumann algebra generated by these elements. A result of Voiculescu (cf. [5, Theorem 3.9]) says that the free entropy of $Y_\epsilon$ is finite for all $\epsilon > 0$. However, it turns out that for all sufficiently small $\epsilon$ the von Neumann algebras $N_\epsilon$ are not embeddable into free group factors.

An outline of the reasoning is as follows. Assume the contrary. Since free group factors enjoy the *Haagerup approximation property* (cf. [3]) it would follow that each $N_\epsilon$ has this property. One then shows that this property passes to limits, in a suitable sense, and hence $N_0 = L(\Gamma)$ would also have the Haagerup approximation property. But a result of Connes-Jones asserts that this is impossible for property T groups (cf. [1]) and hence we get our contradiction. The details can be found in our preprint entitled "Finite free entropy and free group factors."

After my lecture, Shlyakhtenko asked whether finite free entropy plus the Haagerup approximation property would be enough to ensure that one was looking at a free group factor. This question remains open and would be a nice project for future research.

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## Integration on Compact Groups and Applications

Benoit Collins

We explain how to solve the problem of computing arbitrary moments of Haar measures on compact goups and mention some applications. This note is based on papers [5].

Let $G \subset \mathcal{M}d$ be a compact Lie group viewed as a group of matrices. The matrix structure provides a natural coordinate system on $G$ and we are interested in the family of functions $e_{ij} : G \to \mathbb{C}$ defined by $e_{ij} : \mathcal{M}d \ni m \mapsto m_{ij}$ which to a matrix assign one of its entries. We call polynomials in $(e_{ij})$ polynomial functions on $G$. We wish to compute the integrals of polynomial functions in $(e_{ij}, \overline{e_{ij}})$ on compact Lie groups with respect to the Haar measure $\mu_G$ on $G$, i.e. the integrals of the form

$$\int_G U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_{n'}, j'_{n'}}} \, d\mu_G(U). \tag{1}$$

We call such integrals *moments* of the group $G$.

Let $S_n$ be the permutation group over the integers $[1,n]$. For $\sigma \in S_n$, let $|\sigma| := n - \#\mathsf{cycles}(\sigma)$ in in its product cycle decomposition. The map $(\sigma,\tau) \to |\tau\sigma^{-1}|$ is a distance on $S_n$, invariant by left and right translation.

We denote by $\mathbb{C}[S_n]$ the algebra of the symmetric group. Its canonical basis is $\{\delta_\sigma, \sigma \in S_n\}$ and the multiplication is $\delta_\sigma \delta_\tau = \delta_{\sigma\tau}$. The neutral element of $S_n$ (the identity permutation) is denoted by $e$.

**Theorem.** *[2]*

- *Let* $n \leq d$. *One has, for the unitary group:*

$$\int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_n j_n} \overline{U_{i'_1 j'_1}} \cdots \overline{U_{i'_n j'_n}} d\mu_U(U) =$$
$$\sum_{\sigma,\tau \in \mathcal{S}_n} \delta_{i_1 i'_{\sigma(1)}} \ldots \delta_{i_n i'_{\sigma(n)}} \delta_{j_1 j'_{\tau(1)}} \ldots \delta_{j_n j'_{\tau(n)}} Wg_u(d, \sigma\tau^{-1})$$

  *where* $Wg_u(d,\sigma) = \frac{1}{n!^2} \sum_{\lambda \vdash n} \frac{\chi_\lambda(e)^2 \chi_\lambda(\sigma)}{s_{\lambda,d}(1)}$*;* $\chi_\lambda$ *are the characters of the symmetric group, and* $s_{\lambda,d}$ *are the Schur polynomials. In particular,* $Wg_u(d,\cdot)$ *is central.*
- *Let* $\Phi_u = \sum_{\sigma \in \mathcal{S}_n} \delta_\sigma d^{n-|\sigma|}$ *and* $Wg_u = \sum_{\sigma \in \mathcal{S}_n} \delta_\sigma Wg(d,\sigma)$*. One has, in* $\mathbb{C}[\mathcal{S}_n]$*:* $\Phi Wg = \delta_e$*.*

Let $B_{2n}$ be the set of pairings of the set $[1,2q]$ into $q$ pairs. Let $p, p'$ be two pairs of $B_{2n}$. We call $\mathsf{loops}(p,p')$ the number of blocks of the partition generated by $p$ and $p'$. It is less than $n$ and equals $n$ iff $p = p'$. The function $(p,p') \to n - \mathsf{loops}(p_1,p_2)$ is a distance on $B_{2n}$.

Consider the endomorphism $\Phi_o$ of $\mathbb{C}[B_{2n}]$ given by $p \to \sum_{p' \in B_{2n}} p' d^{-n-d(p,p')}$. For given $n$, this map is invertible for $d$ large enough.

Let $Wg_o : p \to \sum_{p' \in} p' Wg_o(d,p,p')$ be the peudo-inverse of $\Phi_o$. In the orthogonal group case, Theorem becomes

**Theorem.** [5] *One has*

$$\int_{\mathbb{O}_d} U_{i_1 j_1} \cdots U_{i_{2n} j_{2n}} d\mu_O(U) = \sum_{p,p' \in B_{2n}} \delta_{p,i}\delta_{p',j} Wg_o(d,p,p')$$

*where* $\delta_{p,i} = \prod_k \delta_{i_k, p(i_k)}$ *and* $Wg_o : p \to \sum_{p' \in B_{2n}} p' Wg_o(d,p,p')$ *is the pseudo-inverse of* $\Phi_o : p \to \sum_{p' \in B_{2n}} p' d^{-n-d(p,p')}$. *The endomorphism* $\Phi_o$ *is invertible iff* $d \geq n$.

Let $\sigma, \tau \in \mathcal{S}_n$. A *path* $P$ between $\sigma$ and $\tau$ is a finite sequence $\sigma = \sigma_0 \neq \sigma_1 \neq \ldots \neq \sigma_k = \tau$. The (infinite) family of such paths is denoted by $\mathcal{P}_S(\sigma, \tau)$. By definition, the *length* of $P$ is $l(P) = |\sigma_0 \sigma_1^{-1}| + \ldots + |\sigma_{k-1}\sigma_k^{-1}|$. For $p, p' \in B_{2n}$ it is possible similarly to define $\mathcal{P}_B(p, p')$ and a length function.

**Theorem.** *For* $d \geq n$*, one has:* $Wg_u(d, \sigma\tau^{-1}) = \sum_{P \in \mathcal{P}_S(\sigma,\tau)} (-d)^{-l(P)}$*, and* $Wg_u(d,p,p') = \sum_{P \in \mathcal{P}_B(p,p')} (-d)^{-l(P)}$.

## 1. Applications

The main application of this result is to the asymptotic behaviour of random matrices. The following theorem is the key tool.

**Theorem.**

- *For any* $n$*, let* $c_n = \binom{2n}{n}/n$. *One has*

(2) $$Wg_u((1,\ldots,n), d) =_{d\to\infty} \frac{(-1)^{n-1} c_n}{(d-n+1)(d-n+2)\ldots(d+n-1)}$$

- *Let* $\sigma = \sigma_1 \sqcup \sigma_2$. *Then* $Wg_u(d,\sigma) =_{d\to\infty} Wg(\sigma_1)Wg(d,\sigma_2)(1+0(d^{-2}))$. *Let* $p, p' \in B_{2q}$ *and assume that they can be written as* $p = p_1 \sqcup p_2$ *and* $p' = p_1' \sqcup p_2'$*, where* $p_1, p_1'$ *are pairings of* $B_{2k}, k < q$. *Then*

  $$Wg_o(d,p,p') =_{d\to\infty} Wg_o(d,p_1,p_1')Wg_o(d,p_2,p_2')(1+0(d^{-2})).$$

- *Let* $p, p' \in B_{2n}$ *and assume that they generate a partition with blocks of length* $2k_1 \geq 2k_2 \geq \ldots$. *On the other hand, let* $\sigma \in \mathcal{S}_n$ *be a permutation with cycles of length* $k_1 \geq k_2 \geq \ldots$. *Then* $Wg_o =_{d\to\infty} Wg_u \to (1 + o(d^{-1}))$

With theorems 1 and , it is possible to give a mathematical meaning to theoretical physics assertions that large families of matrix integrals and partitions functions converge. In addition, one can reprove under weaker hypotheses almost sure convergence results of Voiculescu (cf for example [1]).

Trying to understand better the signed enumeration of paths of Theorem , and in particular being able to interpret combinatorially the coefficients of the development in $d^{-1}$ of functions $Wg$ (as a number enumerating a combinatorial structure related to the endpoints of the path) is a challenging problem that would have many important applications if a nice interpretation was found.

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## The Polynomial Method: From Theory to a "Free Calculator"

Raj Rao
(joint work with Alan Edelman)

In infinite random matrix theory (RMT), the limiting level density of a large class of random matrices can, in principle, be obtained by applying either resolvent or free probability based theorems. These theorems are often formulated *explicitly* in terms of either the limiting level density [1, 2] or the R and S transforms [3, 4] of free probability. This has hindered the applicability of these theorems in many practical situations where the limiting level density or the R and S transforms cannot be explicitly determined. The free commutator of Nica and Speicher [5] is a prominent example in free probability where this observation is particularly true. We propose a method that overcomes these hurdles.

We introduce bivariate polynomials of the form $L_{uv} = \sum_j \sum_k c_{jk} u^j v^k$, where $u$ and $v$ are an appropriately chosen pair of variables *implicitly* defined such that $L_{uv}(u, v) = 0$. We demonstrate that resolvent and free probability based theorems can be interpreted as simple transformations of these bivariate polynomials. We use this observation to argue that these polynomials are a more natural mathematical object to work with than the explicit transforms or densities.

We then combine known theorems with new random matrix transformations, derived using the bivariate polynomial framework, to extend the class of random matrices for which the limiting level density and the limiting moments can be determined analytically. The mathematical principles that lie at the core of this *polynomial method* [6] lend themselves to a surprisingly simple computational realization as well. We use this to implement a 'free calculator' which allows researchers to truly begin to harness the power of infinite RMT and obtain concrete answers to their random matrix questions.

This is joint work with Alan Edelman (M.I.T.)

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## Finite Free Cumulants and Moments of Unitary/Orthogonal Matrices


Alan Edelman

(joint work with N. Raj Rao, Plamen Koev)


Free probability has concentrated on the $n \to \infty$ limit of random matrix theory. The current viewpoint is that this is an alternative to the $n = 1$ classical probability. Finite random matrix theory, however, allows for finite $n$ as well, but with the addition of the parameter $\beta$ which takes on the values $1, 2, 4$ for real, complex, quaternion respectively, but can take on a continuum of values without serious difficulty.

In finite random matrix theory, a matrix model has joint eigenvalue density

$$c \prod_{i<j} |x_i - x_j|^\beta \prod_i (w(x_i))$$

where $w(x)$ is a weight function, typically a classical weight function such as Hermite ($\exp(-x^2/2)$), Laguerre ($x^\alpha \exp(-x)$), Jacobi ($(1-x)_1^\alpha (1+x)_2^\alpha$), or Fourier ( $1$ on $|z| = 1$).

For the Hermite, Laguerre, Jacobi cases, the univariate densities are the normal distribution, chi-distribution, and beta distributions of classical probability [3]. In linear algebra, the matrix problems are connected with the symmetric eigenvalue problem, the singular value decomposition (svd), and the generalized singular value decomposition (gsvd). The names that we prefer are the ones based on orthogonal polynomial theory, as they seem to be a more consistent naming convention [4]. In free probability we have the free semi-circle, free Poisson, and free products of projections.

In the finite case, if we seek a computational procedure for finite free cumulants such as described by Capitaine and Casalis [1] ultimately we need to compute moments of unitary or orthogonal matrices under Haar measure.

For $\beta = 1, 2$ this may be expressed with the Weingarten formula as in the work of Collins[2].

For general $\beta$ this is possible using the property of Jack polynomials

$$E\left[J_\kappa(Q'AQB)\right] = \frac{J_\kappa(A)J_\kappa(B)}{J_\kappa(I)}.$$

This begs the question of what we mean by this measure for general $\beta$. One feels it is the wrong question to look to extend real, complex, quaternion because these are the only three division algebras. Rather it is right to extend "RANDOM" real, complex, and quaternion. We propose one approach is to use axiomatically the properties of $(x + x')/2$ (always a real Gaussian), $x * x'$ (a $\chi^2_\beta$ variable) and the Pythagorean addition formula $ax + by \approx \sqrt{a^2 + b^2} * z$.

*This is joint work with N. Raj Rao (M.I.T.) and Plamen Koev (M.I.T.)*

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## A Free Analogue of Brillinger's Formula

Franz Lehner

Brillinger's formula expresses classical cumulants in terms of cumulants of conditioned cumulants:

$$\kappa_n(X_1, X_2, \dots, X_n) = \sum_{\pi \in \Pi_n} \kappa_{|\pi|}(\kappa_{|\pi_j|}(X_i : i \in \pi_j | \mathcal{F}) : j = 1, \dots, |\pi|)$$

where the summation runs over the lattice $\Pi_n$ of set partitions and the conditioned cumulants $\kappa_n(X_1, \dots, X_n | \mathcal{F})$ are random variables measurable on some sub-σ-algebra $\mathcal{F}$ which are defined analogously to the usual cumulants by Möbius inversion on the partition lattice

$$\kappa_\pi(X_1, X_2, \ldots, X_n|\mathcal{F}) = \sum_{\sigma\leq\pi} E_\sigma(X_1, X_2, \ldots, X_n|\mathcal{F})\, \mu(\pi, \hat{1}_n)$$

replacing expectations by conditional expectations. We propose a free analog of this formula. As expected, the lattice of all set partitions is replaced by the lattice of noncrossing partitions, however the notion of "cumulants of cumulants" needs to be defined appropriately: Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $\psi : \mathcal{A} \to \mathcal{B}$ a conditional expectation onto some subalgebra such that $\phi = \phi \circ \psi$. Then the free analog of conditioned cumulants are Speicher's $\mathcal{B}$-valued noncrossing cumulants

$$C_n^\psi(X_1, X_2, \ldots, X_n) = \sum_{\pi\in NC_n} \psi_\pi(X_1, X_2, \ldots, X_n)\, \mu_{NC}(\pi, \hat{1}_n)$$

where now $\mu_{NC}$ is the Möbius function on the lattice of noncrossing partitions. If we define for noncrossing partitions $\sigma \geq \pi$ the partitioned expectations of cumulants as

$$\phi_\sigma \circ C_\pi^\psi(X_1, \ldots, X_n) = \prod_{S\in\sigma} \phi\Big(\prod_{\substack{B\in\pi\\ B\subseteq S}} C^\psi(X_i : i \in B)\Big)$$

and

$$C_\sigma \circ C_\pi^\psi(X_1, \ldots, X_n) = \sum_{\pi\leq\rho\leq\sigma} \phi_\rho \circ C_\pi^\psi(X_1, \ldots, X_n)\, \mu_{NC}(\rho, \sigma)$$

then we have the following analog of Brillinger's formula

$$C_n(X_1, X_2, \ldots, X_n) = \sum_{\pi\in NC_n} C_{\hat{1}_n} \circ C_\pi^\psi(X_1, X_2, \ldots, X_n)$$

As an application one can give a purely combinatorial proof of a recent characterization of freeness due to Nica, Shlyakhtenko and Speicher.

## Invariant Subspaces for Operators in a General $II_1$-Factor

Uffe Haagerup

(joint work with Hanne Schultz)

The main result is, that if $T$ is an operator in a general $II_1$-factor $M$, then for every Borel set $B \subset \mathbb{C}$, there is a unique closed $T$-invariant subspace $K = K(T, B)$ affiliated with $M$, such that with respect to the decomposition $H = K \oplus K^\perp$, $T$ has the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

where the Brown measures of $T_{11}$ and $T_{12}$ are concentrated on $B$ and $\mathbb{C}\backslash B$ respectively. Moreover, $K$ is $T$-hyperinvariant. In particular if the Brown measure of $T$ is not a Dirac measure, then $T$ has a non-trivial hyperinvariant closed subspace.

The results extend the results in my unpublished MSRI notes from 2001, where Connes' embedding property was assumed.

## Free Levy Processes

Marek Bozejko
(joint work with Wlotek Bryc)

The talk presented is on joint work with Wlotek Bryc. The free Meixner-Levy laws arise as the distributions of orthogonal polynomials with constant coefficient recursions. We show that these are the laws of the free pairs of random variables which have linear regressions and quadratic conditional variances when conditioned with respect to their sum. We apply this result to describe free Levy processes with quadratic conditional variences, and to prove a converse implication related to asymptotic freeness of random Wishart matrices.

## Brown Measures of Sets of Commuting Operators in a $\mathrm{II}_1$ Factor

Hanne Schultz
(joint work with Uffe Haagerup)

In recent work by U. Haagerup and the speaker it was shown that for any operator $T$ in a $\mathrm{II}_1$ factor $M$ and for any Borel set $B$ in the complex plane there is a maximal $T$-invariant socalled spectral subspace $K_T(B)$ affiliated with M such that the Brown measure of $T|_{K_T(B)}$ is concentrated on $B$. Moreover, $K_T(B)$ is $T$-hyperinvariant (i.e. $S$-invariant for every $S$ in $T'$). This enables us to prove the existence of a Brown measure for any finite set $T = (T_1, ..., T_n)$ of commuting operators in $M$. The Brown measure $\mu_T$ is a probability measure on $C^n$ with certain nice properties. We also show that exactly as in the case of a single operator, one can associate to every Borel set in $C^n$ a $T_1-, ..., T_n$-invariant spectral subspace.

## Interval Partitions, Hopf Algebras, and the Inversion of Power Series

Michael Anshelevich
(joint work with Edward G. Effros, Mihai Popa)

For power series with non-commuting coefficients (which commute with the variables), the composition operation is not associative. Moreover, if one looks at $n$-tuples of power series in $n$ non-commuting variables, the left and the right compositional inverses exist but are different. It is not hard to prove recursively that they can be written as sums over labeled trees. The vertices of trees are labeled by elements of $\{1, 2, \ldots, n\}$, and for each vertex $x$, $Y(x) = f^i_{u_1,\ldots,u_p}$, where $i$ is the label of $x$ and $u_1, \ldots, u_p$ are the labels of its progeny. Then the coefficient of $z_{u_1} \ldots z_{u_p}$ in the $i$'th component of the left inverse to $\mathbf{F}$ is

$$\sum_{T \in \mathbf{T}^i_{u_1,\ldots,u_p}} (-1)^{\ell(T)} Y(x_1) Y(x_2) \ldots Y(x_k).$$

Here, $\mathbf{T}^i_{u_1,\ldots,u_p}$ are the proper trees with the root label ed $i$ and the leaves labeled $u_1, \ldots, u_p$, $\ell(T)$ is the number of levels of the tree, and

$$x_1 \gg x_2 \gg \ldots \gg x_k$$

are the vertices of the tree $T$ ordered according to the breadth-first ordering. It is similarly easy to show recursively that for the right inverse, the coefficients are

$$\sum_{T \in \mathbf{R}^i_{u_1,\ldots,u_p}} (-1)^{\mathbf{v}(T)} Y(y_1) Y(y_2) \ldots Y(y_k),$$

where $\mathbf{R}^i_{u_1,\ldots,u_p}$ are the reduced trees, $\mathbf{v}(T)$ is the number of non-leaf vertices of the tree, and

$$y_1 \Rsh y_2 \Rsh \ldots \Rsh y_k$$

are the vertices of the tree $T$ ordered according to the depth-first ordering.

A systematic way to prove these formulas, remove the redundancies they contain, and exhibit the relationship between them, is to use Hopf algebras. Specifically, the algebra of such power series under composition is closely related to the incidence algebra of colored interval partitions, which we define and investigate. We show that the left and right compositional inverses for power series can be obtained from the antipode $S$ of this Hopf algebra, and its inverse $S^{-1}$ (note that $S^2 \neq I$). More surprisingly, the inverses are directly related to the antipodes of two transformations of the interval partitions Hopf algebra, which we call the left and the right Lagrange Hopf algebras. As a consequence, we obtain the (reduced trees, depth-first ordering) and (all trees, breadth-first ordering) expansions for *both* inverses, thus explicitly describing the relationship between them. Moreover, the breadth-first expansion is a particular case of the general Hopf algebra "geometric series" expansion for the antipode.

For commutative power series, a direct combinatorial argument of Haiman and Schmitt shows that in the sum over all trees, the contributions of non-reduced trees cancel. In our non-commutative case, a more complicated argument is necessary. First, we show that the expansion over all trees can be replaced with the expansion over only order-reduced simple trees. Second, we show that such trees are in fact in one-to-one correspondence with the reduced trees, and the correspondence moreover transforms the breadth-first ordering into the reverse depth-first ordering. In this way, the desired reduced tree expansion is obtained from the general Hopf algebra antipode formalism.

The cancelation result above is reminiscent of Zimmermann's formula in perturbative quantum field theory, and of the results of Connes and Kreimer on the

Hopf algebra of rooted trees. Our results on the "free Faà di Bruno algebra" suggest that similar simplifications may occur in the non-commutative version of the Connes-Kreimer algebra.

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## Introduction to $L^2$-Betti Numbers and Their Relation to Free Probability

Thomas Schick

In this talk we present an introduction to the theory of $L^2$-Betti numbers. This theory has been developed over the last 30 years by a number of mathematicians. In the talk we put a particular emphasis on the connections with free probability theory

A basic definition of $L^2$-Betti numbers is given for a finite CW-complex $X$, using the cellular chain complex of the universal covering and the group von Neumann algebra $L\pi$ of the fundamental group $\pi$ together with its canonical trace. In this approach, one constructs the combinatorial Laplacians $\Delta_p$, and the von Neumann dimensions of their kernels are the $L^2$-Betti numbers.

This displays the first connection to free probability: both use as underlying theory tracial von Neumann algebras and their properties in a crucial way.

In particular, the construction can be applied to the classifying space $BG$ of a discrete group $G$, provided it has a model which is a finite CW-complex, this way defining the $L^2$-Betti numbers of a group $b_p^{(2)}(G) := b_p^{(2)}(BG)$. An obvious drawback here is that not all groups admit such a classifying space. The definition of $L^2$-Betti numbers can be extended to arbitrary spaces, either by an approximation method due to Cheeger-Gromov [2] or by Lück's systematic extension of the dimension function coming from the trace to arbitrary modules (in the algebraic sense) over $L\pi$ [10].

The latter approach is used by Connes and Shlyakhtenko [3] to define $L^2$-Betti numbers for tracial algebras, in particular von Neumann algebras. For the group algebra, they give back the $L^2$-Betti numbers of the group. In general, the calculation of these invariants remains one of the big problems. If the $L^2$-Betti numbers of $L\pi$ coincide with the ones of $\pi$, it would follow in particular that the different free group factors are all non-isomorphic.

The construction of [3] is inspired by Gaboriau's definition of $L^2$-Betti numbers for measurable equivalence relations [6], where Gaboriau proved in particular that the $L^2$-Betti numbers of a group depend only on the measure equivalence class of the group. Continuing the development started in [3], Mineyev and Shlyakhtenko establish in [12] the following deep connection between free probability and $L^2$-Betti numbers. For self adjoint generators $X_1, \ldots, X_n$ of the group algebra $\mathbb{C}[\pi]$, Voiculescu's non-microstates free entropy $\delta^*(X_1, \ldots, X_n)$ coincides with $b_1^{(2)}(\pi) - b_0^{(2)}(\pi) + 1$.

Explicit calculations of $L^2$-invariants are usually quite hard, because they require a detailed understanding of the spectrum of the combinatorial Laplacians. However, using the methods of free probability and e.g. the R-transform, in special situations, in particular for free groups, such calculations can be carried out; compare e.g. [17] or [13].

The talk also addresses applications of $L^2$-Betti numbers to algebra. In particular, we have the following conjecture (often called the Atiyah conjecture about $L^2$-Betti numbers): If $\Gamma$ is a torsion-free group, then all $L^2$-Betti numbers of finite CW-complexes with fundamental group $\Gamma$ are integers. This conjecture is known to be true for large classes of groups, compare e.g. [5, 8, 9, 14, 15, 16]. It implies the zero divisor conjecture for group rings: if $\Gamma$ is torsion-free and satisfies the Atiyah conjecture, then $\mathbb{Q}[\Gamma]$ does not contain non-trivial zero divisors. For some groups, the route via $L^2$-Betti numbers provides the only known way to prove this conjecture.

**Open questions**

The study of $L^2$-Betti numbers is still a wide open field with many interesting and important questions. In relation to the chosen subjects of the talk, let me mention only the following:

(1) A very strong generalization of the Atiyah conjecture (to groups with torsion) has been disproved [7]. Find an example of an $L^2$-Betti number of a finite CW-complex which is not rational (compare the candidate constructed in [4]). Prove or disprove the Atiyah conjecture for torsion-free groups.

(2) Find ways to calculate the $L^2$-Betti numbers of Connes and Shlyakhtenko for von Neumann algebras.
(3) The calculations of Mineyev and Shlyakhtenko [12] depend on certain approximation properties of $L^2$ 1-coboundaries by $L^\infty$-coboundaries. A finer understanding of these approximation properties, and more precise estimates in this contexts, should be established at least under additional geometric conditions on the group (like negative curvature). This should allow to carry out further calculations of $L^2$-Betti numbers of groups, in particular in relation to invariants coming from free probability.
(4) In particular, it would be important to extend such approximation results to higher degrees (from degree zero and one). Unfortunately, it seems to be completely open how this could be achieved.
(5) Extend the relation between $L^2$-invariants and free probability to other (refined) invariants, in particular to the Novikov-Shubin invariants of a group.

The literature on the subject is vast. Instead of listing the earlier literature (before 2002) we refer to the extensive bibliography of the monograph [11], which also gives a detailed introduction and a comprehensive account of the status of the theory until then.

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## Combinatorial Fock Spaces and Non-Commutative Gaussian Processes

Hans Maassen
(joint work with Mădălin Guţă)

This talk reports on collaboration with Mădălin Guţă, forming part of the latter's Ph. D. work [1, 2, 3, 4].
It concerns the use of Joyal's notion [5, 6] of a *combinatorial species* in order to construct Fock spaces and Gaussian processes (or 'generalised white noise'). A combinatorial species $F$ sends any finite set $U$ to a set $F[U]$ of '$F$-structures on $U$' and a bijection $\sigma : U \to V$ to another bijection $F[\sigma] : F[U] \to F[V]$, transporting the $F$-structure from $U$ to $V$. It leads to an endofunctor $\mathcal{F}_F$ of the category of Hilbert spaces and contractions by the definitions

$$\begin{aligned} \mathcal{F}_F(\mathcal{H}) \quad &: \quad = \bigoplus_{n=0}^{\infty} \frac{1}{n!}\, l^2_{\text{sym}}\big(F[n] \to \mathcal{H}^{\otimes n}\big)\ , \\ \mathcal{F}_F(T) \quad &: \quad \psi \mapsto (T \otimes T \otimes \cdots \otimes T)\psi\ , \end{aligned}$$

where $\mathcal{H}$ is any complex Hilbert space, and $T$ any contraction between Hilbert spaces. In the first line the symmetrisation is with respect to the double action of $S_n$ on the right: on $\mathcal{H}^{\otimes n}$ in the natural way and on $F[n]$ via $F[\sigma]$, $\sigma \in S_n$. The functor $\mathcal{F}_F$ is second quantisation on the Hilbert space level.
The introduction of a 'trimming rule' on the structures of $F$ leads to an annihilation operator $a$ on $\mathcal{F}_F(\mathcal{H})$. The Gaussian process to be constructed is then

$$\omega(f) := a(f) + a(f)^*\ , \qquad (f \in \mathcal{H}).$$

If $F[\emptyset]$ is a singleton, and $l^2(F[\emptyset]) \subset \mathcal{F}_F(\mathcal{H})$ is spanned by the unit vector $\Omega$, then

$$\langle \Omega, \omega(f_1)\cdots\omega(f_n)\Omega\rangle = \sum_{\pi\in\mathcal{P}_2[n]} t(\pi) \prod_{(i,j)\in\pi} \langle f_i, f_j\rangle$$

for some function $t$ on the pair partitions of the set $n := \{0, 1, \ldots, n-1\}$.
By a suitable generalisation of the notions of 'species' and 'trimming rule' all positive definite functions $t$, i.e. all Gaussian processes, can be obtained.

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## Deformation of $C^*$-algebras on Generators and Relations


Palle Jorgensen

(joint work with Daniil P. Proskurin and Yuriĭ S. Samoĭlenko)


In his talk, Jorgensen summarized a recent joint result [JPS05] with Daniil P. Proskurin and Yuriĭ S. Samoĭlenko in which they consider $C^*$-algebras on generators and relations, and their $q$-deformations. For a particular case of the Bożejko–Speicher $q$-relations, they show that the isomorphism interval $J$ is maximal, but only in the case of two generators.

By an isomorphism interval $J$ we mean an interval $J$ of real values $q$ centered at $q = 0$ in which we have $C^*$-isomorphism of all the $C^*$-algebras $A(q)$ z on the $q$-commutation relations.

For fixed $q$ in the open interval $(-1, 1)$, the $C^*$-algebra $A(q)$ on the $q$-commutation relations was introduced by Bożejko and Speicher in [BoSp94] where they also studied the Fock representation of $A(q)$. Their results on the Fock representation were motivated by free probability, and they were extended by a number of authors, among them Dykema and Nica [DyNi93].

Jorgensen started to work on the $C^*$-algebras $A(q)$ with Werner and Schmitt [JoWe94, JSW94a, JSW94b] in the early 1990's, at which time they also introduced a number of related and more general $C^*$-algebras on generators and relations. These classes of $C^*$-algebras include, among others, the $q$-deformations of Woronowicz and other authors; see, e.g., [PuWo89]. In his joint work with Werner and Schmitt, Jorgensen showed that for the various classes of $A(q)$-$C^*$-algebras, i.e., when the relations are fixed, there is an open interval $J$ of positive length, centered at $q = 0$, for which the $C^*$-algebras $A(q)$ are all isomorphic to the case $q = 0$. For the Bożejko–Speicher $q$-relations, we showed that this interval is of the form $(-a, a)$ with $a = \sqrt{2} - 1$. In this case for $q = 0$, $A(0)$ is in fact

the familiar Cuntz–Toeplitz $C^*$-algebra. The Jorgensen–Schmitt–Werner isomorphism theorems for a variety of more general $C^*$-algebras are based on a Banach fixed-point principle. That is, our isomorphism is obtained as an application of the Banach fixed-point principle to a certain non-linear contractive transformation. The contractivity here refers to a $C^*$-norm. The size of our isomorphism interval for a particular application $A(q)$ depends on some suitable *a priori* estimate for the resulting Banach-contractivity constant. Hence, the number $a = \sqrt{2} - 1$.

In our new result [JPS05] with Daniil P. Proskurin and Yuriĭ S. Samoĭlenko, we extend this: We show that if there are two generators then $C^*$-isomorphism holds in the whole interval $(-1, 1)$, i.e., $a = 1$. Or stated differently, the isomorphism interval is maximal. Unfortunately, we have not been able to extend our result for the interval $(-1, 1)$ to more than two generators. The two endpoints of the maximal interval may be shown to correspond to the relations of the fermions and respectively the bosons from particle physics.

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# Free Talagrand Inequality

Yoshimichi Ueda
(joint work with Fumio Hiai)

This is a brief summary with additional comment of the talk I gave in the conference “Free Probability Theory,” Mar.27–Apr.2, 2005, at Oberwolfach. The materials are mainly taken from a recent joint work with Fumio Hiai [9].

## 1. Transportation Cost Inequalities in Free Probability Theory

Transportation cost inequalities estimate the 2-Wasserstein metric $W_2$ by the square root of relative entropy $H$ for a given pair of probabilistic distributions, and it was Talagrand [11] who first obtained such a kind of inequality in 1996. In free probability theory, Biane and Voiculescu [2] introduced the free analog of Wasserstein metrics $W_p$, $1 \le p < \infty$, and obtained a natural free analog of Talagrand's inequality in the 1-dimensional case, that is,

$$\text{(BV)} \qquad W_2(X,S) \le \sqrt{2\left(-\chi(X) + \frac{1}{2}\tau(X^2) + \frac{1}{2}\log 2\pi\right)}$$

for any (bounded) self-adjoint random variable $X$, where $S$ is a standard semicircular element with distribution $\frac{1}{2\pi}\sqrt{4-\lambda^2}d\lambda$ supported on $[-2,2]$. Then, Hiai, Petz and I [7] strengthened it to an expected setup but still in the 1-dimensional case. Quite recently, Hiai and I [9] took up the first step towards the desired multivariate case, and obtained a natural multivariate free analog of Talagrand's inequality or other words a multivariate generalization of the inequality (BV), that is,

$$\text{(HU)} \quad W_2\left((X_1,\dots,X_n),(S_1,\dots,S_n)\right) \le \sqrt{2\left(-\chi(X_1,\dots,X_n) + \frac{1}{2}\sum_{k=1}^n \tau(X_k^2) + \frac{n}{2}\log 2\pi\right)}$$

for any $n$-tuple $(X_1,\dots,X_n)$ of (bounded) self-adjoint random variables, where $(S_1,\dots,S_n)$ is the standard semicirclar system, i.e, the $n$-tuple of freely independent standard semicircular elements. The inequality we actually obtained in [9] is slightly more general than the above (HU), that is, the above inequality (HU) is still valid with replacing the multiple constant 2 appeared in the square root by a suitable one even when the standard semicircular system $(S_1,\dots,S_n)$ is replaced by any $n$-tuple of freely independent self-adjoint random variables with suitable convexity condition. We also obtained its unitary version. I refer the interested reader to the original article [9] for those details.

## 2. Method – Random Matrix Approximation

Unlike in [2] the main technical ingredient in both [7] and [9] is the use of so-called random matrix approximation, which means the following pattern: For a given non-commutative random variable, a suitable sequence of random matrices indexed by their matrix sizes is chosen in such a way that its scaling limit realizes the given non-commutative random variable in distribution, and then a result in usual (i.e., classical or commutative) probability theory is shown to "converge" to its right free analog. This pattern was initially from Voiculescu's asymptotic freeness result for several independent self-adjoint Gaussian random matrices (see [12]), and first used by Voiculescu himself [13] to seek for a free analog of Shannon's entropy for single random variables. Then, Biane [1] used the pattern to obtain a free analog of logarithmic Sobolev inequality for single self-adjoint random variables or measures, and slightly after that Hiai, Petz and I [7][8] (also see [6]) systematically used it to strengthen Biane-Voiculecu's free transportation cost inequality (BV) and obtain the unitary versions of free transportation cost and free logarithmic Sobolev inequalities. Here, I should emphasize that Hiai, Mizuo and Petz's previous work [5] on perturbation theory for free entropy in the 1-dimensional case was of considerable importance behind the works [7][8]. Finally, Ledoux [10] combined the pattern with the so-called Hamilton-Jacobi technique and unified free transportation cost and free logarithmic Sobolev inequalities from free Brunn-Minkowski inequality in the 1-dimensional case. Those works we mentioned so far all treat only the 1-dimensional case, and if the pattern was applied to the multivariate case, one would need to handle matrix integrals with general interaction potentials. Matrix integrals with general intraction potentials are quite difficult objects and there are very few results known at the present moment so that it is natural to think that the pattern cannot be easily applied to the multivariate case. However, we found that the pattern is indeed applicable to getting free transportation cost inequalities with respect to freely independent $n$-tuples of random variables, see the original article [9] for details.

## 3. Results related to Free Entropy-like Quantity

Hiai [4] introduced a free analog of pressure function as a certain scaling limit of matrix integrals with multivariable interaction potentials, whose definition apparently came from his joint work [5] with Mizuo and Petz. Following an idea in statistical mechanics Hiai also introduced a free entropy-like quantity for non-commutative distributions as the Legendre transform of the free analog of pressure function, which is different from Voiculescu's free entropy in general, but they coincide for single random variables, freely independent families and $R$-diagonal pairs. In [9], we also obtained the same free Talagrand's inequality with replacing Voiculescu's (microstates) free entropy $\chi$ by the free entropy-like quantity under an additional "equilibrium" condition so that it is far from the expected one. However, the inequality implies, for example, a phase transition result for the free entropy-like quantity, which is non-trivial because the free entropy-like quantity involves matrix integrals with multivariate interaction potentials. There are many questions about the quantity, but all of those seem to be difficult to fix at the present moment. I refer the interested reader to the original article for the precise statements as well as the detailed proofs.

## 4. Additional Remark

This section is devoted to part of works in progress with Hiai. In the conference, Ledoux, Biane and some others asked me whether or not our proof of multivariate free Talagrand's inequality can be applied even when the standard semicircular system $(S_1, \dots, S_n)$ is replaced by a more general non-commutative distribution like those treated in [3]. Concerning it, I would like to give the following comment: Let $Q$ be a "potential" polynomial in self-adjoint indeterminates $X_1, \dots, X_n$, and assume that $Q$ gives the well-defined probability measure on $(M_N(\mathbf{C})^{sa})^n$

$$\lambda_N^Q(dA_1, \dots, dA_n) := \frac{1}{Z_N(Q)} \exp\left(-\mathrm{Tr}_N(Q(A_1, \dots, A_n))\right) dA_1 \cdots dA_n$$

for each dimension $N$. Then, define the tracial distribution $\widehat{\lambda_N^Q}$ on the non-commutative polynomials $\mathbf{C}\langle X_1, \dots, X_n\rangle$ in self-adjoint indeterminates $X_1, \dots, X_n$ by

$$\widehat{\lambda_N^Q}(P) := \int_{(M_N(\mathbf{C})^{sa})^n} \frac{1}{N}\mathrm{Tr}_N\left(P(A_1, \dots, A_n)\right)\lambda_N^Q(dA_1, \dots, dA_n)$$

for $P \in \mathbf{C}\langle X_1, \dots, X_n\rangle$. Also, we define the restricted probability measure $\lambda_{N,R}^Q$ on $(M_N(\mathbf{C})_R^{sa})^n$ associated with cut-off constant $R > 0$ by

$$\lambda_{N,R}^Q(dA_1, \dots, dA_n) := \frac{1}{Z_{N,R}(Q)} \exp\left(-N\mathrm{Tr}_N(Q(A_1, \dots, A_n)\right) dA_1 \cdots dA_n,$$

and the corresponding tracial distribution $\widehat{\lambda_{N,R}^Q}$ on $\mathbf{C}\langle X_1, \dots, X_n\rangle$ (extended to that on the $C^*$-algebra $\mathcal{A}_R^{(n)} := C[-R, R]^{\star n}$ with $X_k(t) = t$ in the kth free component $C[-R, R]$) by

$$\widehat{\lambda_{N,R}^Q}(P) := \int_{\left(M_N(\mathbf{C})_R^{sa}\right)^n} \frac{1}{N}\mathrm{Tr}_N(P(A_1, \dots, A_n))\lambda_{N,R}^Q(dA_1, \dots, dA_n)$$

for $P \in \mathbb{C}\langle X_1, \dots, X_n\rangle$. We now suppose that

(1) $\tau_Q(P) := \lim_{N\to\infty} \widehat{\lambda_N^Q}(P)$ exists and is finite for each $P \in \mathbf{C}\langle X_1, \dots, X_n\rangle$;
(2) $Z(Q) := \lim_{N\to\infty} \frac{1}{N^2}\log Z_N(Q) + \frac{n}{2}\log N$ exists and is finite;
(3) $(A_1, \dots, A_n) \mapsto \mathrm{Tr}_N\left(Q(A_1, \dots, A_n)\right) - \frac{\rho}{2}\|(A_1, \dots, A_n)\|_2^2$ is convex for all dimensions $N$ with a fixed constant $\rho > 0$;
(4) (a "compact support" condition) there is a $R_Q > 0$ so that every $R \geq R_Q$ satisfies that

$$\lim_{n\to\infty} \lambda_N^Q\left((M_N(\mathbf{C})_R^{sa})^n\right) = 1, \quad \tau_Q(P) = \lim_{N\to\infty} \widehat{\lambda_{N,R}^Q}(P),\ P \in \mathbb{C}\langle X_1, \dots, X_n\rangle.$$

(Remark that $\lambda_N^Q((M_N(\mathbf{C})_R^{sa})^n) = Z_{N,R}(Q)/Z_N(Q)$, which implies that $\lim_{N\to\infty} \frac{1}{N^2} Z_{N,R}(Q) + \frac{n}{2}\log N = Z(Q)$.) The potential polynomials $Q$ treated by Guionnet and Maurel-Segala [3] seem to satisfy those properties (1),(2) and (4). Those four properties gurantee that the method in [9] works for $\tau_Q$, and we can indeed prove that

$$W_2(\tau, \tau_Q) \leq \sqrt{\frac{2}{\rho}\left(-\chi(\tau) + \tau(Q) + Z(Q)\right)}$$

for every tracial state $\tau$ on $\mathcal{A}_R^{(n)}$ with $R \geq R_Q$, where $\chi(\tau)$ is defined via the GNS representation of $\mathcal{A}_R^{(n)}$ associated with $\tau$. More on this will be discussed elsewhere. In closing, I thank Professor Michael Ledoux for useful discussions, and also thank Professor Alice Guionnet for her wonderful talk on [3] in the conference, both of which gave a motivation to us.

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## A Log-Fourier Interpretation of the R-Transform and Related Assymptotics of the Spherical Integrals

Mylene Maida
(joint work with Alice Guionnet)

If we want to understand the interaction between random matrices, a fundamental object to look at is the spherical integral

$$I_N(A_N, B_N) = \mathbb{E}_U \ e^{N\,\mathrm{tr}(A_N U B_N U^*)},$$

where $A_N$ and $B_N$ are two diagonal matrices and $\mathbb{E}_U$ denotes the expectation under the Haar measure on the orthogonal group $\mathcal{O}_N$ or the unitary group $\mathcal{U}_N$. In other words, we take two matrices with specified spectral measures that we put in generic position with respect to each other.
In our work, we got interested in the regime when one of the matrices, say $A_N$ is of fixed rank, independent of $N$. In the rank one case, for example, $I_N$ can be viewed as the Laplace (or Fourier) transform of the upper left corner of $B_N$ in generic position. In [5], we got the following result

**Theorem** *If $\theta$, the unique non zero eigenvalue of $A_N$, is small enough, for $B_N$ whose spectral radius is uniformly bounded and with spectral measure converging to $\mu$,*

$$\frac{1}{N}\log I_N(A_N, B_N) = \frac{1}{N}\log \mathbb{E}_U\, e^{N\theta(UB_N U^*)_{11}}$$

$$\xrightarrow[N\to\infty]{} \frac{1}{2}\int_0^{2\theta} R_\mu(u)du =: I_\mu(\theta),$$

*where $\mathbb{E}_U$ is the Haar measure on $\mathcal{O}_N$*[1] *and $R_\mu$ denotes the R-transform of the limiting measure $\mu$.*

If the result above was not a surprise to us (it was conjectured by physicists in [4] and B. Collins in [2] could show the convergence of the coefficients of the series, so that we knew that the primitive of the R-transform was a good candidate to be the limit), more surprising was the fact that, as $\theta$ becomes large enough, the limit involves not only the limiting mesure $\mu$ but also the limit $\lambda_{\max}$ of the largest eigenvalue of $B_N$ (the complete asymptotics in rank one are given by Theorem 6 in [5]). Heuristically, this can be justified as follows : $e^{N\theta(UB_N U^*)_{11}}$ can be expressed in terms of the first column vector $U_1$ of the orthogonal (or unitary) matrix $U$. Under the Haar measure, all components o f this vector like to be of the same order $1/\sqrt{N}$ whereas, if $\theta$ is positive, it tries to put more weight on components corresponding to larger eigenvalues of $B_N$. If $\theta$ is small, its attraction is not so strong and all eigenvalues, that is the limiting measure $\mu$, are involved in the limit whereas when $\theta$ becomes larger, the column vector $U_1$ tends to align with the eigenvector corresponding to $\lambda_{\max}$, which now appears at the limit.

Going on with this heuristics, we know that a finite number of column vectors of a Haar distributed orthogonal (or unitary) matrix in generic position are “almost” independent, what allowed us to show that as long as the eigenvalues of $A_N$, that we denote $\theta_i$, are small enough and the rank of $A_N$ remains small in comparison with $\sqrt{N}$, the limit of the spherical integral in higher rank behaves like a sum of $I_\mu(\theta_i)$.

Of course, if these eigenvalues $\theta_i$ become larger, the column vectors ten d to align with the eigenvectors corresponding to large eigenvalues of $B_N$ and can no longer be considered in generic position so that additivity no longer holds. We are now working with J. Najim and S. Péché to establish complete asymptotics in higher rank and to show the following

[1] Similar resu lts, with different constants, hold in the unitary case

**Conjecture** *If* $A_N$ *is of rank* $M$*, the limit of* $1/N \log I_N(A_N, B_N)$ *involves* $\mu$ *the limiting spectral measure of* $B_N$ *and (at mos t) its* $M$ *largest limiting eigenvalues.*
Partial results in this direction can be found in [3]. Note that this work was motivated by the problem of finding the deviations of the largest eigenvalues of a Wigner matrix perturbed by a finite rank deterministic matrix (cf. [9]).

Other natural questions arising from the above result is to wonder whether we can find similar matrix models for other interesting functionals such that the R-transform with several variables (in particular for R-diagonal elements) or the S-transform. What would be the suitable matrix integrals to consider ?

On the other side, if instead of considering (see [7] for notations)

$$ {}_0F_0(A_N U B_N U^*) = e^{\operatorname{tr}(A_N U B_N U^*)}, $$

we would consider integrals of the full family of hypergeometric functions ${}_pF_q$, would it give (and under which assumptions) interesting functionals as a limit ?

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## Analysis and Arithmetic of Free Convolutions

Friedrich Götze
(joint work with Gennadii Chistyakov)

**Definition of free convolutions.** Free convolution of probability measures (p-measures) has been introduced by D. Voiculescu [7], [8] (for compactly supported measures) by means of the algebraic concept of freeness of subalgebras of von Neumann algebras. Denote by $\mathcal{M}$ the family of all Borel p-measures on the real line $\mathbb{R}$. Let $\mu_1 * \mu_2$ denote the classical convolution of $\mu_1, \mu_2 \in \mathcal{M}$. By $\mu_1 \boxplus \mu_2$ we denote the free (additive) convolution of $\mu_1$ and $\mu_2$ introduced by Voiculescu and extended by Maassen [6] and finally by Bercovici and Voiculescu [1] to all measures in $\mathcal{M}$ as described as follows.

For $\mu \in \mathcal{M}$ let $F_\mu$ denote the reciprocal Cauchy transform $1/(\int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}$, $z \in \mathbb{C}^+$, which maps the the open complex upper plane $\mathbb{C}^+$ into itself, hence it is a function of Nevanlinna type. The class $\mathcal{F}$ of such reciprocal Cauchy transforms coincides with the subclass of Nevanlinna functions such that $F(z)/z \to 1$ as $z \to \infty$ nontangentially to $\infty$ (i.e., such that $|\Re z|/\Im z$ stays bounded). As a consequence the inverse function of $F_\mu(z)$ exists, and is well defined on a subset of $\mathbb{C}^+$, which *depends* on $\mu$. The socalled Voiculescu transform of $\mu$, that is $\varphi_\mu(z) = F_\mu^{(-1)}(z) - z$ characterizes $\mu$ uniquely. The free convolution may be thus analytically defined via the equation

$$\varphi_{\mu_1 \boxplus \mu_2}(z) = \varphi_{\mu_1}(z) + \varphi_{\mu_2}(z),$$

valid on the *common* domain of definition of $\varphi_{\mu_j}(z), j = 1, 2$ only. Obviously this definition may restrict the domain of for $n$-fold convolutions of *non identical* measures $\mu_j, j = 1, \dots, n$ very sharply, which leads to serious problems in investigating classical limit theorems using this definition.

We propose the following alternative analytic approach to the definition of additive free convolutions $\mu_1 \boxplus \mu_2$ based on properties of the Nevanlinna functions $F_{\mu_1}(z)$ and $F_{\mu_2}(z)$ and the following result:

**Theorem.** There exist unique Nevanlinna functions $Z_1(z)$ and $Z_2(z)$ of class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)), \tag{1}$$

where $F_{\mu_1}(Z_1(z))$ is in $\mathcal{F}$ again. Thus there exists a p-measure $\mu_1 \boxplus \mu_2$ with $F_{\mu_1}(Z_1(z)) = F_{\mu_1 \boxplus \mu_2}(z)$.

The definition (1) coincides with previous definitions, but does not restrict the class of p-measures, nor restricts the domain of its characterizing functions in a measure dependent way. It allows for an obvious extension to the case of *multiplicative* convolutions on $\mathbb{R}_+$ and the circle (see [4]).

**Khintchine theorems for free convolutions.**

We shall call $\mu_1 \in \mathcal{M}$ a *factor* of $\mu \in \mathcal{M}$ if there exists $\mu_2 \in \mathcal{M}$ such that $\mu = \mu_1 \boxplus \mu_2$. The obvious Dirac factors $\delta_a, a \in \mathbb{R}$ and $\mu \boxplus \delta_{-a}$ are called *improper*, and a p-measure $\mu$ which is not a Dirac measure is called *indecomposable* if it has improper factors only otherwise *decomposable*. With infinite divisibility defined as for the classical convolution $*$ we prove the following result.

**Theorem.** Any $\mu$ of $(\mathcal{M}, \boxplus)$ may be classified as follows. Either

- $\mu$ is indecomposable,
- $\mu$ is decomposable (possibly infinitely divisible) and has an indecomposable factor,
- $\mu$ is infinitely divisible and has no indecomposable factors. (This class will be denoted by $I_0$.)

Here, any $\mu \in \mathcal{M}$ with indecomposable factors may be decomposed (non uniquely) as $\mu = \mu_0 \boxplus \mu_1 \boxplus \mu_2 \boxplus \ldots$, where $\mu_0 \in I_0$ and $\mu_1, \mu_2, \ldots$ denotes a finite or denumerable sequence of indecomposabl $p$-measures. Furthermore,

- the class $I_0$ consists of Dirac-measures only
- measures with finite number of support points are indecomposable
- idecomposable measures are dense in the weak topology in $\mathcal{M}$.

These results extend to multiplicative convolutions, where measures with a prime number of support points are indecomposable.

Furthermore, we extend limit results by Bercovici and Pata [2] from the case of identical measures to the non identical case. Let $\mu_{nk}$ be a triangular scheme of infinitesimal $p$-measures and denote shifted measure by

$$\widehat{\mu}_{nk}((-\infty, u)) := \mu_{nk}((-\infty, u + a_{nk})),$$

where $a_{nk} := \int_{(-\tau,\tau)} u\, \mu_{nk}(du)$ with finite arbitrary but fixed $\tau > 0$. Then

**Theorem.**

i) The family of limit measures of sequences $\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ coincides with the family of $\boxplus$-infinitely divisible measures.

ii) There exist constants $a_n$ such that $\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converges weakly if, and only if, $\nu_n$ converges weakly to some finite nonnegative measure $\nu$, where $\nu_n$, for any Borel set $S$, $\nu_n(S) := \sum_{k=1}^{k_n} \int\limits_S \frac{u^2}{1+u^2}\, \widehat{\mu}_{nk}(du)$. iii) All admissible $a_n$ are of the form $a_n = \alpha_n - \alpha + o(1)$, where $\alpha$ is an arbitrary finite number and $\alpha_n = \sum_{k=1}^{k_n} \Big( a_{nk} + \int\limits_{\mathbb{R}} \frac{u}{1+u^2}\, \widehat{\mu}_{nk}(du) \Big)$.

iv) Furthermore, all possible limit measures $\mu \in \mathcal{M}$ have a Voiculescu transform of type $\phi_\mu = (\alpha, \nu)$, that means

$$\phi_\mu(z) = \alpha + \int\limits_{\mathbb{R}} \frac{1 + uz}{z - u}\, \nu(du), \quad z \in \mathbb{C}^+,$$

where $\alpha$ is a real number and $\nu$ is a finite nonnegative measure $\nu$, on $\mathbb{R}$.

Note that statement i) of the theorem is due to Bercovici and Pata [3].

In view of the complete analogue to the classical limit theorems, we extend this socalled as Bercovici-Pata bijection [2] to the case of non identical measures $\mu_{nj}$, $j = 1, \ldots, k_n$. Recall that the Lévy-Khintchine formula for characteristic functions $\varphi(t; \mu) = \int_{\mathbb{R}} e^{itu}\, \mu(du)$, $t \in \mathbb{R}$, of an $*$-infinitely divisible measure

$\mu \in \mathcal{M}$ has the form

$$\varphi(t;\mu) = \exp\{f_\mu(t)\} = \exp\Big\{i\alpha t + \int\limits_{\mathbb{R}} \Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big)\frac{1+u^2}{u^2}\,\nu(du)\Big\}, \quad t \in \mathbb{R},$$

where $\alpha$ is a real number, $\nu$ is a finite nonnegative Borel measure on $\mathbb{R}$, and $(e^{itu} - 1 - itu/(1+u^2))(1+u^2)/u^2$ is defined as $-t^2/2$ when $u = 0$. Since there is a one-to-one correspondence between the functions $f_\mu(t)$ and $(\alpha, \nu)$, we shall write $f_\mu = (\alpha, \nu)$.

**Theorem.** Let $\mu_{nk}$ be as above. There exist constants $a_n$ such that the sequence $\delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converges weakly to $\mu^{\boxplus} \in \mathcal{M}$ such that $\phi_{\mu^{\boxplus}} = (\alpha, \nu)$ if and only if the sequence $\delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ converges weakly to $\mu^* \in \mathcal{M}$ such that $f_{\mu^*} = (\alpha, \nu)$.

**Rate of Convergence in the CLT.** Let $m_k(\mu) := \int u^k \mu(du)$ denote moments and let $\mu_n((-\infty, x]) := \mu((-\infty, x\sqrt{n}])$ denote the rescaled measure $\mu$. Assume $m_1(\mu) = 0, m_2(\mu) = 1$ and $m_4(\mu) < \infty$. Denote $\mu_n^{n\boxplus} = \mu_n \boxplus \cdots \boxplus \mu_n$ ($n$ times). Our analytic approach to free convolution allows us to show the following bound.

**Theorem.** The Kolmogrov distance beween $\mu_n^{n\boxplus}$ to Wigner semicircle distribution $w$ (with density $\frac{1}{2\pi}\sqrt{(4-x^2)_+}$, where $a_+ = \max\{a, 0\}$) is bounded as follows

$$\Delta(\mu_n^{n\boxplus}, w) \le c\,\frac{|m_3(\mu)| + (m_4(\mu))^{1/2}}{n^{1/2}},$$

where $c > 0$ is an absolute constant. The rate $n^{-1/2}$ is sharp.

For additional results about free convolutions and limit theorems for non identical measures we refer to the preprints [4],[5].

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## The limit shape of Young diagrams for Weyl groups of type B

Akihito Hora

Let $S(n)$ and $\mathbb{Y}_n$ denote the symmetric group of degree $n$ and the Young diagrams of size $n$ respectively. Considering irreducible decomposition of a representation of $S(n)$, we get a probability on $\mathbb{Y}_n$. This gives rise to a statistical ensemble of Young diagrams. Its asymptotic behavior as $n \to \infty$ under appropriate scaling has good analogy and connection to the similar problems in random matrix theory, which lead us to the free probability world.

Let us recall the simplest case of the regular representation of $S(n)$. $\chi^\lambda$ denotes the irreducible character corresponding to $\lambda \in \mathbb{Y}_n$. Set $\tilde{\chi}^\lambda = \chi^\lambda / \dim \lambda$. The decomposition of the normalized regular character

$$\delta_e = \sum_{\lambda \in \mathbb{Y}_n} \mathfrak{P}_n(\lambda) \tilde{\chi}^\lambda \ , \qquad \mathfrak{P}_n(\lambda) = \frac{\dim^2 \lambda}{n!}$$

yields the Plancherel measure $\mathfrak{P}_n$ on $\mathbb{Y}_n$.

An analytic description of Young diagrams due to Vershik-Kerov is quite useful for discussing their scaling limit. Regarded as functions on $\mathbb{R}$, Young diagrams are embedded into the space of continuous diagrams. Furthermore a probability $m_\omega$ on $\mathbb{R}$ is assigned to any continuous diagram $\omega$ and called the transition measure of $\omega$. Vershik-Kerov [8] and Logan-Shepp [6] showed that, if we look at irreducible decomposition of the regular representation of $S(n)$ in $1/\sqrt{n}$-scaling, we see concentration at the limit shape $\Omega$:

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x\arcsin\frac{x}{2} + \sqrt{4-x^2}) & (|x| \le 2) \\ |x| & (|x| > 2). \end{cases}$$

As a weak law of large numbers, the result is stated as

$$\lim_{n\to\infty} \mathfrak{P}_n\big(\big\{\lambda \in \mathbb{Y}_n \,\big|\, \sup_{x\in\mathbb{R}} |\lambda^{\sqrt{n}}(x) - \Omega(x)| \ge \epsilon\big\}\big) = 0 \qquad (\epsilon > 0),$$

where we set $\lambda^{\sqrt{n}}(x) = \lambda(\sqrt{n}x)/\sqrt{n}$ for $\lambda = \lambda(x)$. Note that the transition measure $m_\Omega$ is the standard semicircle distribution.

Such a concentration phenomenon in other representations of $S(n)$ is extensively studied by Biane [1], [2]. Analysis around the limit shape with a smaller scale can be formulated as appropriate central limit theorem. Kerov [5], Hora [3] and Ivanov-Olshanski [4] showed Gaussian fluctuation for the Plancherel measure of $S(n)$. Recently Śniady found that Gaussian fluctuation is valid for a wide variety of representations of $S(n)$. See Śniady's report in the present volume.

In this report, we discuss the above concentration phenomenon for Weyl groups of type B. Set $W_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S(n)$. The irreducible representations of $W_n$ are parametrized by pairs of Young diagrams: $\{(\lambda,\mu) | \lambda \in \mathbb{Y}_m, \mu \in \mathbb{Y}_{n-m}; m = 0, 1, \dots, n\}$. More precisely, setting $H_{n,m} = (\mathbb{Z}/2\mathbb{Z})^n \rtimes (S(m) \times S(n-m))$ and $\chi_m = (0, \cdots, 0, 1, \cdots, 1) \in \widehat{(\mathbb{Z}/2\mathbb{Z})^n}$ ($m$ 0's and $n-m$ 1's), we have the corresponding irreducible representation $U^{(\lambda,\mu)}$ to $(\lambda,\mu)$ as $U^{(\lambda,\mu)} = \mathrm{Ind}_{H_{n,m}}^{W_n} \chi_m U^\lambda \boxtimes U^\mu$. Then

$\mathrm{Res}^{W_n}_{W_{n-1}} U^{(\lambda,\mu)}$ is decomposed in a multiplicity-free way. Analogously to $S(n)$, the Plancherel measure of $W_n$ is defined by

$$\delta_e = \sum_{(\lambda,\mu)} \mathfrak{P}^B_n(\lambda,\mu)\tilde{\chi}^{(\lambda,\mu)} \ , \qquad \mathfrak{P}^B_n(\lambda,\mu) = \frac{\dim^2(\lambda,\mu)}{2^n n!} \ .$$

We can claim that $(\lambda^{\sqrt{n/2}}, \mu^{\sqrt{n/2}})$ concentrates at $(\Omega,\Omega)$ as $n \to \infty$ in irreducible decomposition of the regular representation of $W_n$. Namely we see the following law of large numbers. $M_k(\mathfrak{m}.)$ denotes the $k$th moment of the transition measure $\mathfrak{m}.$ $(k \in \mathbb{N})$.

**Theorem** For any $\epsilon > 0$ and $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathfrak{P}^B_n\big(\{(\lambda,\mu) \mid {}^M|M_k(\mathfrak{m}_{\lambda^{\sqrt{n/2}}}) - M_k(\mathfrak{m}_\Omega)| \vee |M_k(\mathfrak{m}_{\mu^{\sqrt{n/2}}}) - M_k(\mathfrak{m}_\Omega)| \geq \epsilon\}\big) = 0$$

holds.

The proof is based on a modification of asymptotic factorization argument due to Biane. A central part consists of analysis on the moments of Jucys-Murphy elements. Set $\delta_j = (0,\cdots,0,1,0,\cdots,0) \in (\mathbb{Z}/2\mathbb{Z})^{n+1}$ (1 at the $j$th coordinate). Following Ram [7], we consider the Jucys-Murphy element (of type B) in $\mathbb{C}[W_{n+1}]$:

$$J_n = (1\,n+1) + \cdots + (n\,n+1) + \delta_1\delta_{n+1}(1\,n+1) + \cdots + \delta_n\delta_{n+1}(n\,n+1) \ .$$

Since $J_n$ commutes with $\delta_{n+1}$, we can consider 'joint distribution' of $J_n$ and $\delta_{n+1}$. $\mathbb{E}_n : \mathbb{C}[W_{n+1}] \longrightarrow \mathbb{C}[W_n]$ denotes the canonical conditional expectation. Looking at the action of $J_n$ and $\delta_{n+1}$ onto the seminormal basis for each irreducible component, we have

$$\tilde{\chi}^{(\lambda,\mu)}\big(\mathbb{E}_n J_n^k \delta_{n+1}^l\big) = \frac{1}{2} M_k(\mathfrak{m}_{\lambda^{1/2}}) + \frac{(-1)^l}{2} M_k(\mathfrak{m}_{\mu^{1/2}}) \qquad (k,l \in \mathbb{N}).$$

This equality plays a key role in our discussion.

Actually, as readers readily see, this report gives just a beginning part from a viewpoint of vast extension of concentration and fluctuation to various wreath product groups and their representations, which are to be rich fields in asymptotic representation theory.

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## Gaussian Fluctuations of Yound Diagrams: Connection to Random Matrices

Piotr Sniady

### 1. What is the asymptotic theory of the representations of $S_n$?

**Irreducible representations.** Irreducible representations $\rho^\lambda$ of the symmetric group $S_n$ are in a one-to-one correspondence with *Young diagrams* $\lambda$ *having* $n$ *boxes*. An example of a Young diagram is presented on Figure 1.

**Reducible representations.** Every (reducible) representation $\rho$ of $S_n$ defines *the canonical probability measure on Young diagrams with* $n$ *boxes*, given as follows. We decompose $\rho$ as a direct sum of irreducible representations and the probability of a Young diagram $\lambda$ should be proportional to the total dimension of the irreducible components of type $[\lambda]$ in this decomposition. We are interested in the statistical properties of a randomly chosen Young diagram.

**Example of a problem.** For an integer $n \geq 1$ we consider a Young diagram $\nu$ with a shape of a $n \times n$ square. A *Young tableaux* is a filling of this Young diagram with numbers $1, \dots, n^2$ such that the numbers increase along the diagonals $\nearrow$, $\nwarrow$ from the bottom to the top, cf Figure 2 (left). We can think that a Young diagram is a *pile of stones* and the Young tableau is the order in which the stones are placed.

Let $0 < \alpha < 1$ be fixed; we remove from a randomly chosen Young tableaux all boxes with numbers bigger than $\alpha n^2$, cf Figure 2 (right). What is the shape of the resulting Young diagram $\lambda$ with $\alpha n^2$ boxes, when $n \to \infty$? In other words: *What was the shape of this pile of stones in the past* [PR04]*?* This problem is equivalent to the study of *the restriction of representations*: the random Young diagram $\lambda$ described above is distributed according to the canonical probability

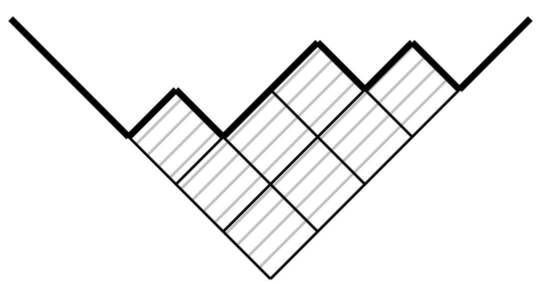

Figure 1. Graphical representation (Russian style) of a Young diagram $\lambda = (4, 3, 1)$.

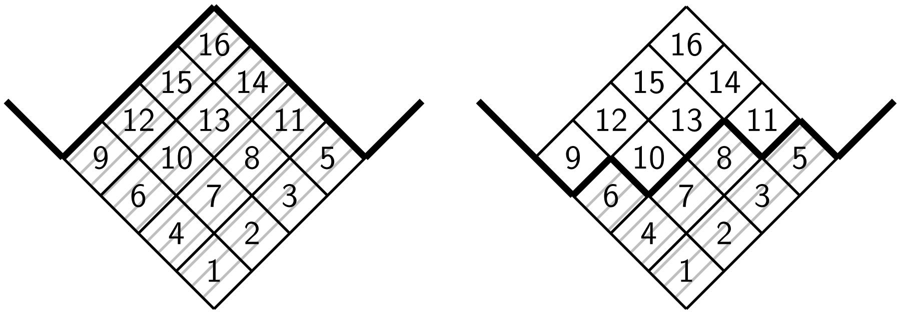


Figure 2. On the left: example of a Young tableaux. On the right: the Young diagram resulting from this tableaux by removing the half of the boxes with the biggest numbers.

measure associated to the restricted representation $\rho = \rho^{\nu}\big\downarrow^{S_{n^2}}_{S_{\alpha n^2}}$. We will give an answer to this problem (and to the lots of other problems) in Section 2.

**Conclusions from the above example.** In principle, for any question concerning representations of $S_n$ there is a well-known answer given by some *combinatorial* algorithm. However, when $n \to \infty$, such *combinatorial* answers are too complicated to be useful. We need more analytic methods! It was an idea of Kerov [Ker93a] to associate to a Young diagram $\lambda$ its *transition measure* $\mu_\lambda$ which is a certain probability measure on $\mathbb{R}$. When $\lambda$ is random, $\mu_\lambda$ is a random probability measure on $\mathbb{R}$. The transition measure encodes the information about the shape of the Young diagram in a very compact and efficient way and it can be defined in many equivalent ways [Bia98].

## 2. The main result: Representations with approximate factorization of characters

Below I will define a very large class of representations for which a lot of questions can be explicitly answered [Śni05].

**Informal definition:** We say that a sequence of representations $(\rho_n)$ has the property of *approximate factorization of characters* if for any permutations $\pi_1, \dots, \pi_l$ with disjoint supports the (normalized) character $\chi_{\rho_n}$ fulfills

$$\chi_{\rho_n}(\pi_1 \cdots \pi_l) \approx \chi_{\rho_n}(\pi_1) \cdots \chi_{\rho_n}(\pi_l),$$

where the approximate equality should hold for $n \to \infty$ [Bia98].

**More formal definition:** permutations $\pi_1, \dots, \pi_l$ commute hence we can treat them as classical random variables and as the expected value we take the normalized character $\chi_{\rho_n}$. We require that the classical cumulant $k(\pi_1, \dots, \pi_l)$ converges quickly enough to zero [Śni05].

**Law of large numbers** [Bia98]: Let the sequence $(\rho_n)$ be as above and let $(\lambda_n)$ be the corresponding sequence of random Young diagrams. Then the sequence of rescaled random Young diagrams $(\frac{1}{\sqrt{n}}\lambda_n)$ converges in probability to some (generalized) Young diagram $\lambda$. The shape of this limit can be described by the free probability theory.

**Central Limit Theorem**: [Śni05]] The sequence of the fluctuations $(\frac{1}{\sqrt{n}}\lambda_n - \lambda)$, after some additional rescaling, converges in distribution to a Gaussian process.

**Lots of examples.** In each of the cases below the sequence $(\rho_n)$ has the *characters factorization property*: when $\rho_n$ is the *left regular representation* (the corresponding measure on Young diagrams is the famous *Plancherel measure*; the Gaussianity of fluctuations was proved for this case by Kerov [Ker93b, IO02]); when $\rho_n$ is the representation such that $S_n$ is acting on $(\mathbb{C}^{d_n})^{\otimes n}$ by permuting the factors (this representation appears in the Schur-Weyl duality); when $\rho_n$ is an irreducible representation. Many natural operations on representations preserve the character factorization property, for example: tensor product, outer product, induction and restriction.

We leave it as a simple exercise to the Reader to check that from the above properties it follows that the example from Section 1 has the property of approximate factorization of characters and hence the fluctuations of the shape of the Young diagrams are Gaussian.

## 3. Analogy to random matrices

There are mysterious and deep connections between the random matrix theory and the theory of representations of the symmetric groups. One of them is the following one: if $M$ is a hermitian matrix, we can encode its eigenvalues in its spectral measure $\mu_M$ which is a probability measure on $\mathbb{R}$. When $M$ is a random matrix, $\mu_M$ is a random probability measure on $\mathbb{R}$. For many representations of $S_n$ one can find a random matrix $M$ such that the properties of the transition measure $\mu^\lambda$ of the corresponding random Young diagram are analogous to the properties of the spectral measure $\mu_M$.

For random matrices results concerning Gaussian fluctuations are proved by the *genus expansion*: we express moments of traces of the random matrix in terms of the cumulants of the entries which involves summation over certain partitions and permutations. To each such summand we associate a two-dimensional surface. The asymptotic behavior of a summand depends only on its topology.

We prove that the *genus expansion* can be applied for representations of $S_n$ as well [Śni04] (a different result concerning genus expansion was obtained by Okounkov [Oko00]). It follows that the proof of Gaussian fluctuations for random matrices works for Young diagrams as well [Śni05] .

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## Cumulants for Random Matrices as Convolutions on the Symmetric Group


Mireille Capitaine

(joint work with M. Casalis)


Voiculescu ([11]) and after that several authors (see [7] and references therein) showed that large independent matrices provide an asymptotic model for free random variables. Our intention is to show that free cumulants can be naturally seen as the limiting value of scalar "cumulants of matrices", which actually already mostly satisfy some classical properties of free cumulants.

Let us introduce briefly some notations. Let $\mathcal{S}_n$ be the symmetric group on $\{1, \ldots, n\}$ and $\pi$ be a permutation in $\mathcal{S}_n$; denoting by $\mathcal{C}(\pi)$ the set of all the disjoint cycles of $\pi$ and by $\gamma_n(\pi)$ the number of cycles of $\pi$, we set for any $n$-tuple $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ of $N \times N$ complex matrices

$$r_\pi(\mathbf{B}) = r_\pi(B_1, \ldots, B_n) := \prod_{C \in \mathcal{C}(\pi)} \mathrm{Tr}\left(\prod_{j \in C} B_j\right)$$

We call *generalized moments* with order $n$ of a set $\mathcal{X}$ of random matrices any expression $\mathbb{E}(r_\pi(X_1, \ldots, X_n))$ where $X_i \in \mathcal{X}$ and $\pi \in \mathcal{S}_n$. The definition of our cumulant functions naturally arises from the writing of any mixed generalized moment $\mathbb{E}(r_\pi(B_1X_1, \ldots, B_nX_n))$, where $\mathbf{X}$ and $\mathbf{B}$ are independent, as a convolution of the generalized moments of $\mathbf{B}$ by one function of $\mathbf{X}$ (in the spirit of the results of [9] about the multiplication of free $n$-tuples). In one hand, when the distribution of one tuple, $\mathbf{X}$ for example, is unitarily invariant (that is, for any unitary matrix $U$, $(X_1, \ldots, X_n)$ and $(UX_1U^*, \ldots, UX_nU^*)$ are identically distributed), the convolution occurs on the symmetric group $\mathcal{S}_n$ and involves the $\mathbb{U}$-*cumulant function* $C^{U}_{\mathbf{X}}$ defined on $\mathcal{S}_n$. In the other hand, when the distribution of one tuple is orthogonally invariant (that is, for any orthogonal matrix $O$, $(X_1, \ldots, X_n)$ and $(OX_1\,{}^tO, \ldots, OX_n\,{}^tO)$ are identically distributed), we establish such convolution formulas but on the symmetric group $\mathcal{S}_{2n}$ and it requires an other cumulant function, the $\mathbb{O}$-*cumulant function* $C^{O}_{\mathbf{X}}$, defined on $\mathcal{S}_{2n}$. Roughly speaking, our cumulant f unctions $C^{U}_{\mathbf{X}}$ as well as $C^{O}_{\mathbf{X}}$ appear as the convolution of the generalized moments of $\mathbf{X}$ and the Weingarten function (defined in [4]). We make use of integration formulas on the unitary, respectively orthogonal, group given in [3] and [4]. Note that, according to section 4 in [4], the same exposition could be carried out if the distribution of one tuple is invariant under the action of the symplectic group but we do not develop it there. Our work in the orthogonally invariance case has been amply inspired by [5]. We will call *cumulants* of $\mathbf{X}$, the collection $\{C^U_{\mathbf{X}}(\pi)$ (resp $C^O_{\mathbf{X}}(\pi)); \pi$ single cycle of $\mathcal{S}_n, n \leq N\}$. The most interesting properties of these cumulants is that they do vanish as soon as the involved matrices are taken in two independent sets and therefore they do linearize the convolution. These properties together with the convergence towards the free cumulants lead us to adopt this terminology. Nevertheless, our cumulants fall outside the very general setting of [6].

We point out the analogues in our matricial context of some results of A. Nica and R. Speicher in [9] concerning conjugation with a circular element or compression of a family of random variables by a projection which is free with the family. We also explain how one can deduce asymptotic freeness from our matricial convol ution relations.

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## Monomorphisms of the Class of Infinitely Divisible Laws

Steen Thorbjornsen
(joint work with Ole Barndorff-Nielsen)

We introduce a mapping $\Gamma$ from the class $\mathrm{ID}(*)$ of infinitely divisble probability laws on $\mathbb{R}$ into itself. The defining property of $\Gamma$ is the relation

$$C_{\Gamma(\mu)}(u) = \mathcal{C}_{\Lambda(\mu)}(iu), \; u \in (-\infty, 0)$$

where $C$ and $\mathcal{C}$ denote, respectively, the classical and free cumulant transforms, and where $\Lambda$ is the Bercovici-Pata bijection between $\mathrm{ID}(*)$ and its free counterpart $\mathrm{ID}(\boxplus)$. Via the above defining property, the mapping $\Gamma$ inherits a number of properties from $\Lambda$. Thus, $\Gamma$ preserves the affine structure of $\mathrm{ID}(*)$ (convolution, dilation by constants and Dirac measure), and $\Gamma$ is a homeomorphis with respect to weak convergence. We present furthermore a stochastic representation of $\Gamma$, namely

$$\Gamma(\mu) = L\left\{ \int_0^1 -\log(1-t)\,dX_t \right\},$$

where $X_t$ is a Levy process corresponding to $\mu$. Finally we introduce a one-parameter family of mappings $\Gamma^\alpha : \mathrm{ID}(*) \to \mathrm{ID}(*), \; \alpha \in [0,1]$, which interpolates between $\Gamma$ and the identity mapping on $\mathrm{ID}(*)$.

*Reporter: Jonathan Novak*

# Participants

**Prof. Dr. Michael Anshelevich**

Dept. of Mathematics
University of California
Riverside, CA 92521-0135
USA

**Dr. Teodor Banica**

Laboratoire de Mathematiques
E. Picard
Universite Paul Sabatier
118 route de Narbonne
F-31062 Toulouse Cedex 4

**Prof. Dr. Florent Benaych-Georges**

Departement de Mathematiques et
Applications
Ecole Normale Superieure
45, rue d'Ulm
F-75230 Paris Cedex 05

**Prof. Dr. Philippe Biane**

CNRS, DMA
Ecole Normale Superieure
45, rue d'Ulm
F-75005 Paris

**Prof. Dr. Luc Bouten**

Dept. of Mathematics
Radboud Universiteit Nijmegen
Postbus 9010
NL-6500 GL Nijmegen

**Prof. Dr. Marek Bozejko**

Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

**Dr. Nathaniel Brown**

Department of Mathematics
Pennsylvania State University
University Park, PA 16802
USA

**Prof. Dr. Thierry Cabanal-Duvillard**
Universite Rene Descartes
UFR Mathematiques et Informatique
45, rue des Saints-Peres
F-75270 Paris Cedex 6

**Prof. Dr. Mireille Capitaine**

Laboratoire de Statistique et
Probabilites
Universite Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 4

**Prof. Dr. Benoit Collins**
Dept. of Mathematics
Faculty of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

**Prof. Dr. Catherine Donati-Martin**

Laboratoire de Probabilités et
Modèles Aléatoires
Université Paris VI, Case 188
4, Place Jussieu
F-75252 Paris Cedex

**Prof. Dr. Ken Dykema**

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

**Dr. Alan Edelman**

Department of Mathematics
MIT
77 Massachusetts Avenue
Cambridge, MA 02139-4307
USA

**Prof. Dr. Emmanuel Germain**

Institut de Mathematiques
"Theorie de Nombres"
Case 247
175, rue du Chevaleret
F-75013 Paris

**Prof. Dr. Friedrich Götze**

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld

**Prof. Dr. Frederick Goodman**

Department of Mathematics
15 MLH
The University of Iowa
Iowa City, IA 52242-1419
USA

**Prof. Dr. Alice Guionnet**

Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

**Prof. Dr. Uffe Haagerup**

Department of Mathematics and
Computer Science
University of Southern Denmark
Campusvej 55
DK-5230 Odense M

**Prof. Dr. Akihito Hora**

Dept. of Environmental and Math.
Sciences
Faculty of Env. Sciences and Techn.
Okayama University
Okayama 700-8530
JAPAN

**Artur Jez**

ul. Gajowicka 131/7
53-407 Wroclaw
Poland

**Prof. Dr. Palle E.T. Jorgensen**

Dept. of Mathematics
University of Iowa
Iowa City, IA 52242-1466
USA

**Prof. Dr. Kenley Jung**

Department of Mathematics
UCLA
Box 95f1361
405 Hilgard Avenue
Los Angeles CA 90095-1361
USA

**Prof. Dr. Claus Koestler**

Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston ONT K7L 3N6
Canada

**Anna Krystek**

Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

**Prof. Dr. Michel Ledoux**

Institut de Mathematiques
Universite de Toulouse
118, route de Narbonne
F-31062 Toulouse Cedex 4

**Dr. Franz Lehner**

Institut für Mathematik C
Technische Universität Graz
Steyrergasse 30
A-8010 Graz

**Dr. Hans Maassen**

Subfaculteit Wiskunde
University of Nijmegen
Toernooiveld 1
NL-6525 ED Nijmegen

**Prof. Dr. Mylene Maida**

Mathematiques
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

**Edouard Maurel-Segala**

Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

**Prof. Dr. James Mingo**

Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston ONT K7L 3N6
Canada

**Dr. Paul Mitchener**

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

**Behnam Norouzizadeh**

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

**Jonathan Novak**

Department of Mathematics
Queen's University
Jeffery Hall
99 University Avenue
Kingston ONT K7L 3N6
Canada

**Prof. Dr. Denes Petz**

Department for Mathematic. Analysis
Mathematical Institute
University of Technology & Economic
Muegyetem rakpart 3-9
H-1111 Budapest

**Prof. Dr. Gilles Pisier**

Equipe d'Analyse; Boite 186
Universite Pierre et Marie Curie
(Universite Paris VI)
4, Place Jussieu
F-75252 Paris Cedex 05

**Raj(Nadakuditi) Rao**

Department of Mathematics
Computer Science AI Laboratories
Massachusetts Institute of Technol.
77 Massachusetts Avenue
Cambridge, MA 02139
USA

**Philipp Rumpf**

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

**Dr. Thomas Schick**

Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

**Dr. Hanne Schultz**

Department of Mathematics and
Computer Science
University of Southern Denmark
Campusvej 55
DK-5230 Odense M

**Prof. Dr. Hermann Schulz-Baldes**

Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. 1 1/2
91054 Erlangen

**Prof. Dr. Dimitri Shlyakhtenko**

UCLA
Mathematics Department
BOX 951555
Los Angeles CA 90095-1555
USA

**Dr. Piotr Sniady**

Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

**Prof. Dr. Roland Speicher**

Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston, Ontario ONT K7L 3N6
CANADA

**Prof. Dr. Marius B. Stefan**

Dept. of Mathematics
University of California
405 Hilgard Avenue
Los Angeles, CA 90095-1555
USA

**Dr. Michael Stolz**

Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

**Prof. Dr. Stanislaw Jerzy Szarek**

Dept. of Mathematics and Statistics
Case Western Reserve University
10900 Euclid Avenue
Cleveland, OH 44106-7058
USA

**Georg Tauböck**

ftw. Telecommunications Research
Center Vienna
Donaustadtstrasse 1
A-1220 Wien

**Prof. Dr. Steen Thorbjornsen**

Department of Mathematics and
Computer Science
University of Southern Denmark
Campusvej 55
DK-5230 Odense M

**Prof. Dr. Yoshimichi Ueda**

Graduate School of Mathematics
Kyushu University
Ropponmatsu, Chuo-ku
Fukuoka 810-8560
Japan

**Dr. Cedric Villani**

Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

**Dr. Lukasz Wojakowski**

Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND