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Dynamical Systems

Organised by
Helmut W. Hofer (New York)
Jean-Christophe Yoccoz (Paris)
Eduard Zehnder (Zürich)

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Abstract. This workshop, continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser–Zehnder meeting" in 1981.

The main theme of the workshop were the new results and developments in the area of classical dynamical systems, in particular in celestial mechanics and Hamiltonian systems. Among the main topics were new global results on the Reeb dynamics on 3-manifolds, KAM theory in finite and infinite dimensions, as well as new developments in Floer homology and its applications. High points were the first complete existence proof of quasiperiodic solutions in the planetarian N-body problem, and the solution of a long-standing conjecture of Anosov about the number of closed geodesics on Finsler 2-spheres.

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Introduction by the Organisers

This workshop, organised by Helmut Hofer (New York), Jean-Christophe Yoccoz (Paris), and Eduard Zehnder (Zürich), continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser–Zehnder meeting" in 1981. The workshop was attended by more than 50 participants from 12 countries.

The main theme of the workshop were the new results and developments in the area of classical dynamical systems, in particular in celestial mechanics and Hamiltonian systems. Among the main topics were new global results on the Reeb dynamics on 3-manifolds, KAM theory in finite and infinite dimensions, as well as new developments in Floer homology and its applications. High points were the first complete existence proof of quasiperiodic solutions in the planetarian N-body problem, and the solution of a long-standing conjecture of Anosov about the number of closed geodesics on Finsler 2-spheres.

The meeting was held in a very informal and stimulating atmosphere.

Workshop: Dynamical Systems

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Abstracts

A new approach to the Weinstein conjecture in dimension three

Casim Abbas
(joint work with Kai Cieliebak and Helmut Hofer)

Let (W, ω) be a symplectic manifold, and let $H : W \to \mathbb{R}$ be a smooth function ('Hamiltonian function'). We associate to H a vector field X_H via the equation $dH = i_{X_H}\omega$, and we are interested in the dynamics of the vector field X_H. In classical mechanics the function H represents the total energy of the mechanical system, and W is the phase space of the system. Trajectories $x(t)$ of the system $\dot{x}(t) = X_H(x(t))$ lie on hypersurfaces of constant energy $S = \{H = c\}$. A fundamental question to ask is whether a given energy hypersurface S carries periodic trajectories. Using variational methods, Paul Rabinowitz proved the following result [7].

Theorem 1. *Let $W = \mathbb{R}^{2n}$, $n \geq 1$, with the standard symplectic structure $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$. Moreover, let S be a star–shaped compact regular energy hypersurface of some Hamiltonian function H (regular meaning that $\nabla H(x) \neq 0$ if $x \in S$). Then S contains a periodic trajectory of the system $\dot{x} = X_H(x)$.*

Alan Weinstein then tried to find a general geometric condition for a hypersurface S which might guarantee the existence of periodic trajectories. In his paper [10] he introduced the notion of contact type.

Definition 1. *A hypersurface S in a $2n$-dimensional symplectic manifold (W, ω) is of contact type if there is a 1–form λ on S such that $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on S and $d\lambda = \omega|_S$.*

A. Weinstein observed that all known examples of hypersurfaces with periodic orbits on it were of contact type, and he formulated the following conjecture in his 1979 paper [10].

Conjecture 1. *Assume S is a compact regular energy hypersurface in a symplectic manifold. If S is of contact type and $H^1(S, \mathbb{R}) = 0$ then S carries a periodic trajectory.*

This conjecture was resolved by Claude Viterbo in the case where $W = \mathbb{R}^{2n}$ without the condition on the cohomology of the hypersurface [9]. In [6] Helmut Hofer and Eduard Zehnder proved their famous 'almost existence result' for a family of compact regular energy hypersurfaces. On the other hand, without the contact type condition, one cannot guarantee the existence of periodic trajectories since there are compact regular energy hypersurfaces without periodic orbits (see the article by Viktor Ginzburg [3] for a smooth counterexample in dimension 6, and [4] for a counterexample in dimension 4 with regularity C^2). The general Weinstein conjecture can be formulated without the ambient symplectic manifold as follows: If M is a $(2n-1)$–dimensional manifold then a contact form on M is a 1–form λ

such that $\lambda \wedge (d\lambda)^{n-1}$ is a volume form. The subbundle $TM \supset \xi := \ker \lambda \to M$ is called the contact structure associated to λ, and $(\xi, d\lambda|_{\xi\oplus\xi})$ is a symplectic vector bundle. We define the so–called Reeb vector field X_λ by the equations

$$i_{X_\lambda} d\lambda \equiv 0 \ , \ i_{X_\lambda}\lambda \equiv 1.$$

If $M = \{H = c\} \subset (W, \omega)$ is a contact type hypersurface then the trajectories of $X_H|_M$ and X_λ coincide up to parametrization. The first result in the general context was established by Helmut Hofer in 1993 [5] using pseudoholomorphic curve techniques:

Theorem 2. *Let (M, λ) be a closed three dimensional contact manifold. Then X_λ has a periodic trajectory in the following cases:*

(1) $M = S^3$
(2) $\pi_2(M) \neq 0$
(3) $\xi = \ker \lambda$ *is an overtwisted contact structure*

The methods used in [5] are not adequate for the general case. In the paper [1] we prove the Weinstein conjecture for an interesting class of contact manifolds in dimension three, the ones which admit a so–called planar open book decomposition. Modifying the pseudoholomorphic curve equation in a suitable way, the same program as in the planar case would prove the general conjecture in dimension three. Here is a very brief outline:
Let (S, j) be a closed Riemann surface, let $\Gamma \subset S$ be a finite set, denote by π_λ the projection onto the first factor in the splitting $TM = \xi \oplus \mathbf{R} \cdot X_\lambda$, and let $J : \xi \to \xi$ be a complex structure compatible with the symplectic form $d\lambda$. The crucial partial differential equation is the following:

$$(*) \begin{cases} \pi_\lambda Du(z) \circ j(z) \ = \ J(u(z)) \circ \pi_\lambda Du(z) \text{ if } z \in S\backslash\Gamma \\ u^*\lambda \circ j \ = \ da + \gamma \text{ on } S\backslash\Gamma \\ d\gamma \ = \ d(\gamma \circ j) \ = \ 0 \text{ on } S \\ 0 < \sup_{\phi\in\Sigma} \int_S \tilde{u}^* d(\phi\lambda) < \infty \text{ with } \Sigma := \{\phi \in C^\infty(\mathbf{R}, [0,1]) \,|\, \phi' \geq 0\} \end{cases}$$

where $\tilde{u} = (a, u) : S\backslash\Gamma \to \mathbf{R} \times M$ and γ is a suitable harmonic 1–form on the closed surface S. It can be shown that the map u must approach the set of periodic trajectories of X_λ if restricted to smaller and smaller circles around each puncture $z \in \Gamma$. Hence existence of a solution to the problem $(*)$ would confirm the Weinstein conjecture. Without the harmonic 1–form γ, equation $(*)$ is just the usual pseudoholomorphic curve equation in the symplectization $\mathbf{R} \times M$ of the contact manifold M [5]. The so–called planar case discussed in the paper [1] permits us to choose the Riemann surface S equal to the two–sphere. In this case γ is equal to zero. The main idea of the proof is a cobordism argument. It is possible to modify the contact form λ to another one of the form $f\,\lambda$ with a positive function f such that the corresponding problem $(*)$ with λ replaced by $f\lambda$ has solutions. We then pick a positive smooth function $F(t, x)$ on $\mathbf{R}\times M$ such that $F(t, x) = f(x)$ for $t >> 0$ and $F(t, x) \equiv 1$ for $t << 0$. We define an $\mathbf{R}$–dependent contact form $\lambda_a(x) := F(a, x)\lambda(x)$. Similarly, we replace $J(x)$ by an almost complex structure $J(a, x)$ such that $J(a, \,.\,)$ is compatible with $d\lambda_a$ and a–independent if $|a| >> 0$, and we consider the PDE $(*)$ with λ, J, π_λ replaced with the corresponding **R**-dependent objects. For $a >> 0$ there are solutions, and the objective is to show that there is a family (a_τ, u_τ) of solutions with $\inf_\tau \inf_z \{a_\tau(z) \,|\, z \in S\backslash\Gamma\} = -\infty$. As a consequence of the Symplectic Field Theory compactness result [2] the family must decompose into pieces (a so–called holomorphic building), and one of them must lie in the part where $\lambda_a \equiv \lambda$, proving the existence of a solution to the original problem $(*)$. Why is the harmonic form in the equation necessary if $S \neq S^2$? The reason is that otherwise the index of the Fredholm operator corresponding to the linearization of $(*)$ would be negative. On the other hand, existence theory of solutions for a modified contact form does not always yield spheres, but also curves with genus. Without the harmonic forms present in the equation, the compactness result from Symplectic Field Theory is available. A more general compactness result for $(*)$ is work in progress.

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On the Floer homology of cotangent bundles

Alberto Abbondandolo

(joint work with Matthias Schwarz)

Theorem 1. *Let M be a compact orientable manifold.*

(a) *The Floer homology of T^*M is isomorphic to the singular homology of $\Lambda(M)$, the free loop space of M (with integer coefficients).*

(b) *This isomorphism is a ring isomorphism from the pair-of-pants product on the Floer homology of T^*M to the Chas-Sullivan loop product on $H_*(\Lambda(M))$.*

The first statement was first proved by Viterbo in [4]. Salamon and Weber have presented a different proof in [3]. After recalling the definition of Floer homology, we shall present a third proof of statement (a), which makes use only of very classical tools, namely classical Morse theory and the Legendre transform. The isomorphism constructed in this way has allowed us to prove statement (b), which will not be discussed here.

The cotangent bundle of M carries the canonical exact symplectic form $\omega = dp \wedge dq$. A Hamiltonian $H \in C^\infty(\mathbb{T} \times T^*M)$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) determines a 1-periodic vector field X_H by the formula $\omega(X_H, \cdot) = -dH$. We are interested in the set $\mathcal{P}(H)$ of 1-periodic orbits of X_H (contractible or not). We shall assume that every 1-periodic orbit x is non-degenerate. This implies that $\mathcal{P}(H)$ is at most countable.

The elements of $\mathcal{P}(H)$ are critical points of the Hamiltonian action functional

$$\mathcal{A}_H(x) = \int_{\mathbb{T}} (x^*(p\,dq) - H(t, x(t))\,dt)\,.$$

Floer's approach to develop a Morse theory for $\mathcal{A}_H$ was to study its L^2-gradient equation. More precisely, fixing an ω-compatible almost complex structure J on T^*M, the associated metric $\omega(J\cdot, \cdot)$ induces an L^2-metric on the space of loops on T^*M, and the corresponding negative gradient equation $\frac{d}{ds}u = -\nabla_{L^2}\mathcal{A}_H(u)$ is

$$\partial_s u + J(t,u)(\partial_t u - X_H(t,u)) = 0, \tag{1}$$

where $u \colon \mathbb{R} \times \mathbb{T} \to T^*M$. This is a Cauchy-Riemann-type equation, so it does not generate a well-posed Cauchy problem. However, its stationary solutions are critical points, that is elements of $\mathcal{P}(H)$, and the action functional $\mathcal{A}_H$ strictly decreases along each non-stationary solution. Fix two orbits $x^-, x^+ \in \mathcal{P}(H)$, and consider the set of solutions of (1) connecting them,

$$\mathcal{M}(x^-, x^+) = \left\{ u \in C^\infty(\mathbb{R} \times \mathbb{T}, T^*M) \mid u \text{ solves (1)},\ \lim_{s \to \pm\infty} u(s, \cdot) = x^\pm \right\}.$$

For a generic choice of J, $\mathcal{M}(x^-, x^+)$ is an oriented manifold of dimension $\mu(x^-) - \mu(x^+)$, $\mu(x)$ denoting the Conley-Zehnder index of the periodic orbit x. The Conley-Zehnder index is indeed well-defined because on cotangent bundles there is a preferred set of trivializations, namely those mapping $(0) \times \mathbb{R}^n$ into the vertical subbundle T^vT^*M. For the same reason, the question of coherent orientations of moduli-spaces is simpler than on a general symplectic manifold.

A crucial issue is the question of a priori bounds for spaces of solutions of (1). Standard elliptic estimates show that C^1 bounds imply C^k bounds for every $k \in \mathbb{N}$. Since ω is exact, C^0 bounds imply C^1 bounds. C^0 bounds cannot be expected to hold for every Hamiltonian. If H is assumed to have quadratic growth in p for $|p|$ large, it can be proved that for every $a \in \mathbb{R}$, $\mathcal{P}(H) \cap \{\mathcal{A}_H \le a\}$ is finite, and the set of solutions of (1) such that $|\mathcal{A}_H(u(s,\cdot))| \le a$ for every $s \in \mathbb{R}$ is C^0-bounded.

If $F_k(H)$ denotes the free Abelian group generated by the elements $x \in \mathcal{P}(H)$ with $\mu(x) = k$, we can define a boundary homomorphism $\partial : F_k(H) \to F_{k-1}(H)$, $k \in \mathbb{Z}$, by counting the elements of the finite sets $\mathcal{M}(x^-, x^+)/\mathbb{R}$ with appropriate orientation signs, for $\mu(x^-) - \mu(x^+) = 1$. The resulting complex is known as the

Floer complex of (H, J). Changing J one obtains isomorphic chain complexes. Changing also H one obtains homotopically equivalent chain complexes. In particular, the homology $H_*F(H, J)$ of the Floer complex is independent of J and H, and it is called the *Floer homology of* T^*M.

While the Floer homology of a compact manifold P is just the singular homology of P (over appropriate coefficient rings), the Floer homology of T^*M is isomorphic to the singular homology of $\Lambda(M)$. Viterbo's original proof makes use of generating function homology. Salamon and Weber's approach consists in relating the Floer equation (1) to a heat equation for loops on M. Here we will compare the Floer complex of (H, J) to the Morse complex of the Lagrangian action functional $\mathcal{S}_L$, H and L being related by the Legendre transform. More precisely, if a Lagrangian $L \in C^\infty(\mathbb{T} \times TM)$ is fiberwise strongly convex and has quadratic growth in v, the action functional

$$\mathcal{S}_L(\gamma) = \int_{\mathbb{T}} L(t, \gamma(t), \dot{\gamma}(t))\, dt$$

is smooth on $W^{1,2}(\mathbb{T}, M)$ the Hilbert manifold of loops on M of Sobolev class $W^{1,2}$, it is bounded below, and it satisfies the Palais-Smale condition. Its critical set is the set $\mathcal{P}(L)$ of 1-periodic orbits of the corresponding Lagrangian system, and each critical point γ has finite Morse index $m(\gamma)$. Under these assumptions, classical infinite dimensional Morse theory as developed by Palais and Smale in the sixties applies. Actually, it is convenient to use the Morse complex approach (see [1] for full details). If $M_k(\mathcal{S}_L)$ denotes the free Abelian group generated by the elements of $\mathcal{P}(L)$ of Morse index k, the $W^{1,2}$ negative gradient flow of $\mathcal{S}_L$ allows to define a boundary operator $\partial : M_*(\mathcal{S}_L) \to M_{*-1}(\mathcal{S}_L)$. This time we are dealing with a true flow, so this is just the chain complex associated to a suitable cellular filtration of $W^{1,2}(\mathbb{T}, M)$, hence its homology is the singular homology of this space, or - by homotopy equivalence - of the space $\Lambda(M)$.

Assume now that the Hamiltonian H is the Legendre transform of a Lagrangian L satisfying the above conditions,

$$H(t, q, p) = \max_{v \in T_q M} \left(p[v] - L(t, q, v)\right),$$

so that there is a one-to-one correspondence between $\mathcal{P}(H)$ and $\mathcal{P}(L)$ (given by the projection $\pi : T^*M \to M$) and the Conley-Zehnder index equals the Morse index, $\mu(x) = m(\pi(x))$. These observations show that the chain complexes $F_*(H, J)$ and $M_*(\mathcal{S}_L)$ have the same set of generators, with the same grading. However, there is no reason why the two boundary homomorphisms should coincide, the first one involving a Cauchy-Riemann-type PDE on T^*M, the second one an abstract ODE on $W^{1,2}(\mathbb{T}, M)$. Nevertheless, we can prove the following:

Theorem 2 ([2])**.** *Let $L \in C^\infty(\mathbb{T} \times TM)$ be a fiberwise strongly convex Lagrangian growing quadratically in v, and let $H \in C^\infty(\mathbb{T} \times T^*M)$ be its Legendre transform. Then there is a chain complex isomorphism $\Phi : M_*(\mathcal{S}_L) \to F_*(H, J)$.*

In particular, the homology of the Floer complex is isomorphic to the singular homology of $\Lambda(M)$, as claimed. Actually, the above result says that Morse theory for the Hamiltonian action functional and for the Lagrangian action functional, although constructed in a completely different way, agree up to the chain level.

The isomorphism Φ is defined by looking at the moduli-spaces

$$\mathcal{M}^+(\gamma,x) = \Big\{ u \in C^\infty([0,+\infty)\times\mathbb{T}, T^*M) \,\Big|\, u \text{ solves (1)}, \lim_{s\to+\infty} u(s,\cdot) = x,\ \pi(u(0,\cdot)) \in W^u(\gamma) \Big\},$$

for every pair $\gamma \in \mathcal{P}(L)$, $x \in \mathcal{P}(H)$. Here $W^u(\gamma) \subset W^{1,2}(\mathbb{T},M)$ is the unstable manifold of the critical point γ with respect to the negative $W^{1,2}$ gradient flow of $\mathcal{S}_L$. The fact that we are dealing with a Cauchy-Riemann-type equation with a finite dimensional family of Lagrangian boundary conditions implies that, for a generic choice of J, $\mathcal{M}^+(\gamma,x)$ is an oriented manifold of dimension $m(\gamma)-\mu(x)$. A priori bounds for the solutions in $\mathcal{M}^+(\gamma,x)$ are easily proved by noticing that for every loop z in T^*M, $\mathcal{A}_H(z) \leq \mathcal{S}_L(\pi\circ z)$, the equality holding when z is a periodic orbit of X_H. This indeed yields to the estimate

$$\mathcal{A}_H(x) \leq \mathcal{A}_H(u(s,\cdot)) \leq \mathcal{A}_H(u(0,\cdot)) \leq \mathcal{S}_L(\pi\circ u(0,\cdot)) \leq \mathcal{S}_L(\gamma), \tag{2}$$

the starting point to get a priori bounds for all derivatives. The same estimate also shows that $\mathcal{M}^+(\gamma,x)$ is empty whenever $\mathcal{S}_L(\gamma) \leq \mathcal{A}_H(x)$ and $\pi\circ x \neq \gamma$. If we define the homomorphism Φ by counting the elements of the zero-dimensional moduli-spaces $\mathcal{M}^+(\gamma,x)$, for $m(\gamma)=\mu(x)$, we easily get that Φ is a chain map from $M_*(\mathcal{S}_L)$ to $F_*(H,J)$. The above facts imply that if we order the periodic orbits by increasing action level, this homomorphism is represented by a (possibly infinite dimensional) upper triangular square matrix. When γ and x correspond to the same periodic solution, i.e. $\pi\circ x=\gamma$, the differential version of the inequality (2) implies that the coefficient of x in the expression of $\Phi\gamma$ is ±1. Therefore the upper triangular matrix has entries ±1 on its diagonal, hence it is an isomorphism of free chain complexes.

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Shadowing chains of collision orbits in the elliptic 3 body problem

Sergey Bolotin

Suppose Sun of mass 1 and Jupiter of mass $\mu \ll 1$ move in $\mathbb{R}^2$ along ellipses with foci at 0 and eccentricity $\epsilon \in (0,1)$. Jupiter's position $u(t,\mu,\epsilon)$ is a function of time $t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ depending on the parameters μ, ϵ. The *elliptic 3-body problem* describing the motion of the Asteroid of negligible mass has the Hamiltonian

$$H_{\mu,\epsilon}(p,q,t) = |p|^2/2 - |q + \mu u(t,\mu,\epsilon)|^{-1} - \mu|q - u(t,\mu,\epsilon)|^{-1}, \qquad p,q \in \mathbb{R}^2.$$

For $\mu = 0$ Jupiter disappears and $H_{0,\epsilon} = H_0(p,q)$ is the Hamiltonian of the Kepler problem Sun–Asteroid. We say that $\sigma = (\gamma_i)_{i\in\mathbb{Z}}$ is a collision chain of the system $(H_{0,\epsilon})$ if (for simplicity we write $u(t) = u(t,0,\epsilon)$)

- $\gamma_i : [t_{i-1}, t_i] \to \mathbb{R}^2 \setminus \{0\}$ is a solution of Kepler's problem;
- $\gamma_i(t_{i-1}) = u(t_{i-1})$, $\gamma(t_i) = u(t_i)$ and $\gamma_i(t) \neq u(t)$ for $t_{i-1} < t < t_i$;
- relative collision velocities change: $v_i^+ \wedge v_i^- \neq 0$, where $v_i^\pm = p_i^\pm - \dot u(t_i)$, $p_i^+ = \dot\gamma_i(t_i)$, $p_i^- = \dot\gamma_{i+1}(t_i)$;
- relative collision energies are preserved: $h_i^+ = h_i^- = h_i$, where $h_i^\pm = H_0(p_i^\pm, u(t_i)) - p_i^\pm \cdot \dot u(t_i)$.

We are interested in almost collision orbits of the system $(H_{\mu,\epsilon})$ which $O(\mu)$-shadow such chains for small $\mu > 0$. Periodic orbits of this type were first considered by Poincaré who named them *second species solutions*. Poincaré claimed the existence of such solutions shadowing a 2-chain of elliptic collision orbits for the general 3 body problem, but did not provide a complete proof. For $\epsilon = 0$ the Hamiltonian $H_{\mu,0}$ of the *circular 3 body problem* is autonomous in the rotating coordinate frame, and hence has Jacobi's integral $H_{\mu,0} - G = h$, where G is the angular momentum. Second species periodic solutions with given h were studied in [4], see also the references there. In [3] the existence of symbolic dynamics of almost collision orbits was proved. In this talk we prove the existence of chaotic second species orbits for the elliptic 3 body problem $(H_{\mu,\epsilon})$ with small eccentricity ϵ.

We say that a collision chain $\sigma = (\gamma_i)_{i\in\mathbb{Z}}$ of the system $(H_{0,\epsilon})$ is μ_0-*shadowed* if for any $\mu \in (0,\mu_0)$ there exists an orbit of the system $(H_{\mu,\epsilon})$ which is $O(\mu)$-shadowing σ.

Theorem 1. *For any $h \in (-3/2, \sqrt{2})$ there is a dense set $\{G_k\}_{k\in L}$ in $(2 - \sqrt{4h+6}, -h)$ such that for any finite set $K \subset L$ there exist $\epsilon_0 > 0, a > 0$ such that for any $\epsilon \in (0,\epsilon_0)$ there exists $\mu_0 > 0$ such that for any sequence $(k_i \in K)_{i\in\mathbb{Z}}$, $k_{i-1} \neq k_i$, and any sequence $(l_i \in \mathbb{N})_{i\in\mathbb{Z}}$, $l_i \geq a$, there exists a μ_0-shadowed collision chain*

$$\cdots \underbrace{\gamma_{11}\cdots\gamma_{1m_1}}_{m_1}\underbrace{\gamma_{21}\cdots\gamma_{2m_2}}_{m_2}\cdots\underbrace{\gamma_{i1}\cdots\gamma_{im_i}}_{m_i}\cdots, \qquad |m_i - l_i| \leq 1,$$

of the system $(H_{0,\epsilon})$, where γ_{ij} is an elliptic collision orbit with angular momentum and energy which are $O(\epsilon)$-close to G_{k_i} and $E_{k_i} = h - G_{k_i}$, respectively.

Hence the system $(H_{\mu,\epsilon})$ has a chaotic invariant set in $|H_{\mu,\epsilon} - G - h| < C\epsilon$. The proof of Theorem 1 is based on Theorems 2–3 below.

Let Σ be the set of all collision orbits $\gamma\colon [t_0,t_1] \to \mathbb{R}^2$ of the system $(H_{0,\epsilon})$ with non-conjugate $t_0 < t_1$. Then Σ is a 2-dimensional manifold and the projection $\pi : \Sigma \to \mathbb{R}^2$, $\pi(\gamma) = (t_0,t_1)$, is a local diffeomorphism. Hence Σ has an open covering $\{\Sigma_k\}_{k\in L}$ such that $\pi_k = \pi|_{\Sigma_k} : \Sigma_k \to U_k \subset \mathbb{R}^2$ is a diffeomorphism. Let

$$S_k(t_0,t_1) = \int_{t_0}^{t_1} (|\dot\gamma(t)|^2/2 + |\gamma(t)|^{-1})\, dt$$

be the action of $\gamma = \pi_k^{-1}(t_0,t_1)$. Fix a finite set $K \subset L$ such that $D_{12}S_k(t_0,t_1) \neq 0$ in U_k. For $\mathbf{k} = (k_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}}$ and $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$ such that $(t_{i-1},t_i) \in U_{k_i}$ let

$$A_{\mathbf{k}}(\mathbf{t}) = \sum S_{k_i}(t_{i-1},t_i).$$

The functional is formal but its componentwise derivative $A_{\mathbf{k}}'(\mathbf{t}) \in l_\infty$ is well defined. If $\mathbf{t}$ is a critical point of $A_{\mathbf{k}}$, then $\sigma = (\gamma_i)_{i\in\mathbb{Z}}$, $\gamma_i = \pi_{k_i}^{-1}(t_{i-1},t_i)$, is a collision chain of the system $(H_{0,\epsilon})$ with relative collision energies

$$h_i = -D_2S_{k_i}(t_{i-1},t_i) = D_1S_{k_{i+1}}(t_i,t_{i+1}).$$

We say that a critical point $\mathbf{t}$ is *c-nondegenerate* if the Hessian matrix $A_{\mathbf{k}}''(\mathbf{t})$ satisfies $\|(A_{\mathbf{k}}''(\mathbf{t}))^{-1}\|_\infty \le c < \infty$. If the collision velocities $v_i^\pm$ satisfy $|v_i^+ \wedge v_i^-| > d > 0$ for all $i \in \mathbb{Z}$, then $\mathbf{t}$ will be called (c,d)-*nondegenerate*.

Theorem 2 ([1])**.** *There exists $\mu_0 = \mu_0(c,d,K) > 0$ such that for any $\mu \in (0,\mu_0)$, any $\mathbf{k} \in K^{\mathbb{Z}}$, and any (c,d)-nondegenerate critical point $\mathbf{t}$ of $A_{\mathbf{k}}$, the system $(H_{\mu,\epsilon})$ has a unique orbit which is $O(\mu)$-shadowing the collision chain corresponding to $\mathbf{t}$.*

To prove Theorem 1 it is enough to find many nondegenerate critical points of $A_{\mathbf{k}}$. We give a dynamical reformulation of the nondegeneracy condition. Let $f_k : V_k \subset \mathbb{A} \to \mathbb{A} = \mathbb{T}\times\mathbb{R}$ be a symplectic map with generating function S_k. Then

$$f_k(t_0,h_0) = (t_1,h_1), \qquad h_0 = D_1S_k(t_0,t_1), \quad h_1 = -D_2S_k(t_0,t_1).$$

A critical point $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$ of $A_{\mathbf{k}}$ defines an orbit $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$, $x_i = (t_i,h_i) \in V_{k_i}$, of a sequence $(f_{k_i})_{i\in\mathbb{Z}}$ of symplectic maps: $x_{i+1} = f_{k_i}(x_i)$. For $\mathbf{k} = (k_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}}$ and $x \in V_{k_1}$ set $\mathcal{F}(\mathbf{k},x) = (\mathcal{S}(\mathbf{k}), f_{k_1}(x))$, where $\mathcal{S}\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is the Bernoulli shift. The dynamics of random compositions $f_{k_n} \circ \cdots \circ f_{k_1}$ can be viewed as the dynamics of a single skew product map $\mathcal{F}$ of a subset of $K^{\mathbb{Z}} \times \mathbb{A}$. If $\mathcal{F}$ has a hyperbolic compact invariant set Λ such that $x \in V_{k_1}$ for all $(\mathbf{k},x) \in \Lambda$, then there exists $c > 0$ such that for any $(\mathbf{k},x) \in \Lambda$ the corresponding $\mathbf{t}$ is a c-nondegenerate critical point of $A_{\mathbf{k}}$.

Suppose that the symplectic maps $\{f_k\}_{k\in K}$ are almost integrable:

$$S_k(t_0,t_1) = \Psi_k(t_1 - t_0) + \epsilon\psi_k(t_0, t_1 - t_0) + O(\epsilon^2), \qquad \epsilon \ll 1.$$

Let $\Phi_k : J_k \to \mathbb{R}$ be the Legendre transform of $-\Psi_k$ and $\rho_k(h) = \Phi_k'(h)$. Then

$$f_k(t,h) = (t + \rho_k(h) + O(\epsilon), h + \epsilon D_1\psi_k(t,\rho_k(h)) + O(\epsilon^2)).$$

To prove the existence of a nontrivial hyperbolic invariant set of f_k is a hard problem related to exponentially small splitting of separatrices. This difficulty disappears if we take random compositions of different maps $\{f_k\}_{k\in K}$. Suppose for simplicity (this holds in our application) that

$$\psi_k(t,\rho_k(h)) = \sum a_{kn}(h)e^{int}$$

is a Fourier polynomial and there are no resonances, i.e. if $a_{kn}(h) \neq 0$, then $n\rho_k(h) \notin 2\pi\mathbb{Z}$. Then f_k has an approximate first integral $h + \epsilon\chi_k(t,h)$, where

$$\chi_k(t,h) = \sum ina_{kn}(h)(e^{in\rho_k(h)} - 1)^{-1}e^{int}.$$

Theorem 3. *There exist constants $a, b, c, d, \epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, any sequence $(k_i)_{i\in\mathbb{Z}}$ such that $h \in J_{k_i}$, $\rho'_{k_i}(h) < 0$ and $\chi_{k_i}(t,h) \not\equiv \chi_{k_{i+1}}(t,h)$ for all i, and any sequence $(l_i \in \mathbb{N})_{i\in\mathbb{Z}}$, $l_i \geq a$, there exists a sequence*

$$\mathbf{k} = \dots, \underbrace{k_1, \dots, k_1}_{m_1}, \dots, \underbrace{k_i, \dots, k_i}_{m_i}, \dots, \qquad |l_i - m_i| \leq d,$$

such that $A_{\mathbf{k}}$ has a $c\epsilon^{-1/2}$-nondegenerate critical point $\mathbf{t}$ with $h_i \in (h - b\epsilon, h + b\epsilon)$.

Then $\mathcal{F}$ has a hyperbolic invariant set in $K^{\mathbb{Z}} \times \mathbb{T} \times (h - b\epsilon, h + b\epsilon)$ with Lyapunov exponents of order $\sqrt{\epsilon}$.

For the system $(H_{0,\epsilon})$, any $h \in (-3/2, \sqrt{2})$ is contained in an infinite number of intervals J_k and the functions ψ_k have only 2 harmonics [2]. We may set $d = 1$. Then Theorem 1 follows.

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Central configurations with vanishing total mass in the four-body problem

Martin Celli

All the results of the abstract are proved in [2]. Newton's equations define the motion of a system of N point particles with positions $\vec{r}_1, \dots, \vec{r}_N$ (elements of a Euclidean space) and constant strictly positive or negative masses $m_1, \dots, m_N$ which interact through gravitation:

$$\ddot{\vec{r}}_i = \vec{\gamma}_i(\vec{r}_1, \dots, \vec{r}_N) = \sum_{j\neq i} m_j \frac{\vec{r}_j - \vec{r}_i}{||\vec{r}_j - \vec{r}_i||^3}.$$

We are interested in the particular case $M = m_1 + \cdots + m_N = 0$, where computations become easier.

In the two-body problem, let us write: $\vec{R} = \vec{r}_1 - \vec{r}_2$ and $\vec{P} = \dot{\vec{r}}_1 - \dot{\vec{r}}_2$. We only consider the case where $\vec{P}$ and $\vec{R}$ are not collinear. Let us suppose for instance: $\det(\vec{P}, \vec{R}) > 0$. Let us denote by $\vec{P}'$ the element of $\mathrm{vect}(\vec{P}, \vec{R})$ such that $\left(\frac{\vec{P}}{||\vec{P}||}, \vec{P}'\right)$ is a direct orthonormal basis. Let us define $s(t)$ by

$$\mathrm{sh}(s(t)) = -\frac{(\vec{P}|t\vec{P} + \vec{R})}{\det(\vec{P}, \vec{R})}\cdot$$

For $m_1 = 1$, $m_2 = -1$, the solutions of the two-body problem have the following expression ($\vec{a}_i$, $\vec{b}_i$ are constant vectors):

$$\vec{r}_i(t) = \frac{1}{||\vec{P}||^2}\left(s(t)\frac{\vec{P}}{||\vec{P}||} + \mathrm{ch}(s(t))\vec{P}'\right) + t\vec{a}_i + \vec{b}_i\cdot$$

In a same translating frame whose origin describes a chain curve, the two bodies have uniform rectilinear motions.

When $M \neq 0$, we can define a center of inertia:

$$\vec{G} = \vec{\Omega} + \frac{1}{M}\sum_{i=1}^{N} m_i(\vec{r}_i - \vec{\Omega})$$

(the equality does not depend on the origin $\vec{\Omega}$). When $M = 0$, the center of inertia is no more defined, but we can define a vector of inertia:

$$\vec{\lambda} = \sum_{i=1}^{N} m_i(\vec{r}_i - \vec{\Omega})\cdot$$

For $M \neq 0$, the principle of inertia expresses that there are two constant vectors $\vec{u}$ and $\vec{v}$ such that $\vec{G}(t) = t\vec{u} + \vec{v}$. In an analogous way, for $M = 0$, this principle expresses that there are two constant vectors $\vec{u}$ and $\vec{v}$ such that $\vec{\lambda}(t) = t\vec{u} + \vec{v}$. For $M \neq 0$, the knowledge of $\vec{G}$ provides information on the absolute motion of the N-body system. But for $M = 0$, the vector of inertia $\vec{\lambda}$ is invariant under translations. Thus its knowledge gives information on the motion of the bodies after reduction of the translations. For instance, the norm of $\vec{\lambda}$ only depends on the mutual distances, and we have

$$-\sum_{1 \leq i < j \leq N} m_i m_j ||\vec{r}_j(t) - \vec{r}_i(t)||^2 = ||\vec{\lambda}(t)||^2 = ||t\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 t^2 + 2(\vec{u}\cdot\vec{v})t + ||\vec{v}||^2\cdot$$

This equality already appeared in [6] as a consequence of the equality of Lagrange-Jacobi.

The degeneracy when $M = 0$ provides a case of integrability in the collinear three-body problem: under the assumption $\dot{\vec{\lambda}}(0) = \vec{0}$, the equations become integrable and the vector $\vec{r}_2 - \vec{r}_1$ is a solution of a three fixed center problem.

A configuration is said to be central if, and only if, with vanishing initial velocities, it generates a homothetical motion (collapse or repulsion). This is equivalent to write that there exists ξ such that, for every i, j:

$$\vec{\gamma}_j(\vec{r}_1, \ldots, \vec{r}_N) - \vec{\gamma}_i(\vec{r}_1, \ldots, \vec{r}_N) = \xi(\vec{r}_j - \vec{r}_i)\cdot$$

If $\xi \neq 0$, there is a fixed center for the homothety, and each $\vec{r}_j - \vec{r}_i$ is the solution of a two-body problem with $M \neq 0$. If $\xi = 0$, there is no fixed center for the homothety, the $\vec{\gamma}_i(\vec{r}_1, \ldots, \vec{r}_N)$ do not depend on i, and each $\vec{r}_j - \vec{r}_i$ is the solution of a two-body problem with $M \neq 0$. Central configurations are also linked to rigid motions. They can be seen as singularities in the configuration space.

In the case $N = 3$, for any set of positive masses, it is known that there is exactly one collinear central configuration and that the only non collinear central configuration is the equilateral triangle. This is also true for $N = 3$ and $M = 0$. The collinear central configurations are the configurations with $\vec{\lambda} = \vec{0}$. For these configurations, we have $\xi \neq 0$, whereas for the equilateral triangle, we have: $\xi = 0$. As a problem for the 21st century, S. Smale asked whether the number of central configurations (up to isometries and homotheties) was finite ([10]). The problem is already difficult with $N = 4$. The central configurations with $N = 4$ and positive masses were studied numerically in [9]. We can enumerate them when the masses are equal thanks to [1]. Thanks to a numerically assisted proof, it has recently been proved that the answer was positive for $N = 4$ ([7]).

The problem of central configurations is linked to algebraic equations with the masses as real parameters. Thus it can be raised with positive and negative masses. The proof in [7] also works with positive and negative masses, but unfortunately, its present form does not include the case $M = 0$! On the other hand, it has been proved that, by allowing one negative mass, one could obtain continua of five-body central configurations ([8]). But there is no example of such a continuum with $M = 0$.

For a central configuration with $M = 0$ and $\xi \neq 0$, we have $\vec{\lambda} = \vec{0}$. This property makes computations easier. It is linked to the better "integrability" with $M = 0$. The equality $\vec{\lambda} = \vec{0}$ enables to express the position of one body as a function of the others. Thus we have to deal with a $(N-1)$-body problem. The $\vec{\gamma}_i$ are replaced with $\vec{\gamma}'_i$ which are also invariant under translations and rotations and are homogeneous. Thanks to this, I could prove that a four-body central configuration with $\xi \neq 0$ is not cocircular. The most important result is the following:

For any set of masses $(x, -x, y, -y)$, where x, $y \neq 0$, there are exactly two non collinear central configurations with $\xi \neq 0$. They are trapezia.

Central configurations with $M = 0$ are involved in the study of choreographies. A choreography ([4]) is a solution of Newton's equations such that the bodies chase each other on the same curve with the same phase shifts between two bodies. A. Chenciner and R. Montgomery recently proved the existence of the "eight" orbit ([5]). It is the first non trivial choreography to be discovered, and it had been found numerically by C. Moore in 1993. Since this discovery, many choreographies have been found, but they all require equal masses. It has been proved that planar choreographies with distinct masses do not exist for $N \leq 5$ ([3]). Thanks to the properties of equilibria with $M = 0$, I proved that they do not exist for any N if we replace the Newtonian forces with forces associated with a logarithmic potential. This result can be applied to choreographies of N vortices in a planar fluid.

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Arnold diffusion in Hamiltonian Systems: *A priori* Unstable Case

Chong-Qing Cheng

In this talk, we study *a priori* hyperbolic and time-periodic Hamiltonian systems with arbitrary $n+1$ degrees of freedom. The Hamiltonian has the form

$$H(u, v, t) = h_1(p) + h_2(x, y) + P(u, v, t)$$

where $u = (q, x)$, $v = (p, y)$, $(p, q) \in \mathbb{R} \times \mathbb{T}$, $(x, y) \in \mathbb{T}^n \times \mathbb{R}^n$, P is a time-1-periodic small perturbation. $H \in C^r$ $(r \geq 3)$ is assumed to satisfy the following hypothesis:

H1: $h_1 + h_2$ is a convex function in v, i.e., the Hessian matrix $\partial^2_{vv}(h_1 + h_2)$ is positive definite. It is finite everywhere and has superlinear growth in v, i.e., $(h_1 + h_2)/\|v\| \to \infty$ as $\|v\| \to \infty$.

H2: it is *a priori* hyperbolic in the sense that the Hamiltonian flow $\Phi^t_{h_2}$ determined by h_2 has a non-degenerate hyperbolic fixed point $(x, y) = (0, 0)$, the function $h_2(x, 0) : \mathbb{T}^n \to \mathbb{R}$ attains its strict maximum at $x = 0 \mod 2\pi$. We set $h_2(0, 0) = 0$.

Here, we do not assume that the hyperbolic fixed point $(x, y) = (0, 0)$ is connected to itself by its stable and unstable manifold, i.e., $W^s(0, 0) \equiv W^u(0, 0)$. Such a condition appears unnatural when $n > 1$.

Let $\mathcal{B}_{\epsilon,K}$ denote a ball in the function space

$$C^r(\{(u, v, t) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{T} : \|v\| \leq K\} \to \mathbb{R}),$$

centered at the origin with radius ϵ. Now we can state our main result, which is a higher-dimensional version of the theorem formulated by Arnold where it was assumed that $n = 1$.

Theorem 1. *Let $A < B$ be two arbitrarily given numbers and assume H satisfies the above hypotheses* **H1** *and* **H2**. *There exist a small number $\epsilon > 0$, a large number $K > 0$ and an open and dense set in $\mathcal{S}_{\epsilon,K} \subset \mathcal{B}_{\epsilon,K}$ such that for each $P \in \mathcal{S}_{\epsilon,K}$ there exists an orbit of the Hamiltonian flow which connects the region with $p < A$ to the region with $p > B$.*

We use variational arguments to construct diffusion orbits. In order to use a variational method, we put the problem of consideration into Lagrangian formalism. Using Legendre transforme $\mathcal{L}^* : H \to L$ we obtain the Lagrangian

$$L(u, \dot{u}, t) = \max_v \{\langle v, \dot{u}\rangle - H(u, v, t)\}.$$

Here $\dot{u} = \dot{u}(u, v, t)$ is implicitly determined by $\dot{u} = \frac{\partial H}{\partial v}$. We denote by $\mathcal{L} : (u, v, t) \to (u, \dot{u}, t)$ the coordinate transformation determined by the Hamiltonian H.

Roughly speaking, we construct diffusion orbits by connecting different Mañé sets, along which the Lagrange action attains its local minimum. To construct local connecting orbits between different Mañé sets, we introduce so-called pseudo connecting orbit sets. These sets contain the minimal configurations of some modified Lagrangian which do not necessarily generate orbits determined by the Lagrangian L. Based on the upper semi-continuity of the set functions, from Lagrangian to Mañé set and to pseudo connecting orbit set, and on the understanding of these sets with respect to the configuration manifold and its finite covering, we show that each configuration in the pseudo connecting orbit set generates a real orbit of the Lagrangian L which connects some Mañé set to another Mañé set nearby if this Mañé set has some kind of topological triviality. Such construction does not need the manifold structure of the Mather sets, and is applicable to systems with arbitrary degrees of freedom. Thus, some global connecting orbits can be constructed if some so-called generalized transition chain is established. Such a chain does exist in the system we study in this talk.

KAM theory for partial differential equations

Walter Craig

§1. The setting is that of Hamiltonian partial differential equations, which are evolution equations of the form

$$\partial_t v = J\mathrm{grad}_v H(v) \ , \quad v(0) = v^0 \in \mathcal{H} \ , \tag{1}$$

where the phase space $\mathcal{H}$ is a Hilbert space, typically infinite dimensional. The Hamiltonian vector field

$$X_H = J\mathrm{grad}_v H(v)$$

is given in terms of the symplectic form

$$\omega(X,Y) = \langle X, J^{-1}Y\rangle_{\mathcal{H}} \ , \quad J^T = -J \ ,$$

and the Hamiltonian is a real valued functional $H(v) : \mathcal{H} \mapsto \mathbb{R}$. Denoting the flow by $\varphi_t(v) = v(t,\cdot)$, we are interested in orbits of (1) which are stable motions of the system, which means in this problem that the orbit has the property that

$$\overline{\{\varphi_t(v) : t \in \mathbb{R}\}} = \mathbb{T}^m \ ,$$

where the torus $\mathbb{T}^m$ is of dimension m, $1 \le m \le \infty$.

§2. The question of the existence of such orbits can be posed as a variational problem, on a formal level at least. Consider mappings of tori into phase space; $S : \mathbb{T}^m \mapsto \mathcal{H}$ which satisfy the property of flow invariance,

$$S(\xi + t\Omega) = \varphi_t(S(\xi)) \ , \tag{2}$$

with frequency vector $\Omega \in \mathbb{R}^m$. This implies that both

$$\Omega \cdot \partial_\xi S = \partial_t S \ , \qquad \partial_t S = J\mathrm{grad}_v H(S) \ .$$

The problem is thus to solve

$$\mathrm{grad}_v H(S) - J^{-1}\Omega \cdot \partial_\xi S = 0 \tag{3}$$

for the mapping $S(\xi)$ and the frequency vector Ω.

On the space of mappings $S \in X := \{S(\xi) : \mathbb{T}^m \mapsto \mathcal{H}\}$ define the averaged Hamiltonian $\overline{H}(S)$ and the action functionals $I_j(S)$, $j = 1, \dots m$, by

$$\overline{H}(S) = \int_{\mathbb{T}^m} H(S(\xi))\, d\xi \ , \qquad I_j(S) = \tfrac{1}{2}\int_{\mathbb{T}^m} \langle S, J^{-1}\partial_{\xi_j} S\rangle_{\mathcal{H}}\, d\xi \ ,$$

The variations of these functionals are

$$\delta_S I_j(S) = J^{-1}\partial_{\xi_j} S \ , \qquad \delta_S \overline{H}(S) = \mathrm{grad}_v H(S) \ ,$$

therefore (3) can be rewritten as

$$\delta_S \overline{H}(S) = \Omega \cdot \delta_S I(S) \ . \tag{4}$$

Consider the codimension m subvariety M_a of the space of mappings X defined by fixing the actions;

$$M_a := \{S \in X : I_1(S) = a_1, \dots, I_m(S) = a_m\} \ .$$

Then (4) corresponds to the following Lagrange multiplier rule.

Principle. *Critical points of $\overline{H}(S)$ on M_a correspond to solutions of* (4) *with the actions $I(S) = a$, with frequency vector given by the Lagrange multipliers Ω.*

The functionals $I_j(S)$ and $\overline{H}(S)$ are invariant under a group action of the torus $\mathbb{T}^m$, namely the transformations $\tau_\alpha : S(\xi) \mapsto S(\xi+\alpha)$, $\alpha \in \mathbb{T}^m$, preserve their level sets. Therefore the variety M_a is invariant under this group action. The problem thus consists in finding critical $\mathbb{T}^m$ orbits of $\overline{H}(S)$ on M_a, for whenever S_* is a critical point of $\overline{H}(S)$ on M_a then the entire orbit $\{\tau_\alpha(S_*) : \alpha \in \mathbb{T}^m\}$ consists of critical points.

§3. It is a basic question as to whether such critical points exist. The first difficulty is that in general the PDE (3) is a small divisor problem, its linearized equation is degenerate in the space of torus mappings and its inverse typically exhibits a loss of derivatives. It is therefore not clear whether direct methods in the calculus of variations can play a rôle. Secondly, the torus action by $\mathbb{T}^m$ gives rise to the question of multiplicity of solutions. Even if m is infinite this can be a problem of lower dimensional tori, and it is a question as to which out of a continuous family of tori will survive in the perturbation theory. If the first issue can be addressed, this still gives rise to a problem of counting, or at least providing a lower bound on the number of critical orbits of $\overline{H}(S)$ for fixed actions $I(S) = a$.

There are at least several cases in which both these hurdles can be overcome. Addressing the nonlinear wave equation, the approach of C.E. Wayne and myself [CW93] handles the small divisor problem using Fröhlich – Spencer resolvant estimates [FS83] for the linearized operators in a Nash – Moser iteration scheme.

The second difficulty, that of multiplicity of critical orbits in the presence of a group action, can be addressed by versions of Morse – Bott theory. I will describe the basic outline of this in the context of an example.

We will take the nonlinear Schrödinger equation as an illustrative case, namely;

$$\partial_t u = i(-\tfrac{1}{2}\Delta u + Q(x, u, \overline{u})) \ , \tag{5}$$

posed on the spatial domain $\mathbb{T}^d = \mathbb{R}^d/\Gamma$ where Γ is a lattice of full rank. This is a Hamiltonian system with Hamiltonian

$$H(u) = \int_{\mathbb{T}^d} \tfrac{1}{2}|\nabla u|^2 + G(x, u, \overline{u}) \ dx \ , \quad \partial_{\overline{u}} G = Q \ .$$

Suppose that $G = g_1(x)|u|^2 + \dots$, then in Taylor expansion $H(v) = H^{(2)}(v) + R(v)$, for R vanishing up to at least second order at $v = 0$. The Hamiltonian $H(v)$ has an elliptic stationary point at $v = 0$, meaning that the linearized equation around zero is given by the quadratic Hamiltonian

$$H^{(2)}(u) = \int_{\mathbb{T}^d} \tfrac{1}{2}|\nabla u|^2 + g_1(x)|u|^2 \, dx = \textstyle\sum_{k \in \Gamma'} \omega_k |\hat{u}(k)|^2 \ .$$

Here we have expanded the functions u in their generalized Fourier series in eigenfunction/eigenvalue pairs $(\psi_k(x), \omega_k)$ with respect to the operator $L(g_1)\psi = -\frac{1}{2}\Delta\psi + g_1(x)\psi$. This is a harmonic oscillator with frequencies $\{\omega_k\}$, $k \in \Gamma'$. In a similar spirit consider the generalized Fourier expansion of torus mappings

$$S(x, \xi) = \sum_{(j,k) \in \mathbb{Z}^m \times \Gamma'} s(j,k)\psi_k(x) e^{ij \cdot \xi} \ .$$

In these coordinates the linearized equations about the solution $S = 0$ take the form

$$(\delta_S^2 H^{(2)}(0) - \Omega \cdot \delta_S^2 I(0)) s(x, \xi) = \sum_{(j,k)} (\omega_k - \Omega \cdot j) s(j,k) \psi_k(x) e^{ij \cdot \xi} = F(x, \xi) \ . \tag{6}$$

The eigenvalues of the linearized operator are $\{\omega_k - \Omega \cdot j\}_{(j,k) \in \mathbb{Z}^m \times \Gamma'}$ which typically forms a dense set in $\mathbb{R}$. Choosing frequencies $\omega_{k_1}, \dots \omega_{k_m}$ and then a frequency vector $\Omega^0 = (\Omega_1^0, \dots, \Omega_m^1)$ satisfying the resonance relations $\omega_{k_\ell} - \Omega^0 \cdot j_\ell$, define $N := \{(j,k) : \omega_k - \Omega^0 \cdot j = 0\}$ a subset of the lattice $\mathbb{Z}^m \times \Gamma'$. The set N has cardinality $2M \geq 2m$, which is always even, and it may be infinite. The linearized operator in (6) has a null space $X_1 = span \{\psi_{k_1}(x) \exp(ij \cdot \xi) : (j,k) \in N\} \subseteq X$. The torus we seek is *nonresonant* if $m = M$, and *resonant* if $m < M$.

The Morse – Bott theory as conceived in this context is based on the character of the intersection $M_a \cap X_1$. The result, which is still conjectural in part, is that for a given, there are integers $p_1, \dots, p_m$ such that $\sum_{\ell=1,\dots,m} p_\ell = M$, and

$$M_a \cap X_1 = \times_{\ell=1}^m S^{2p_\ell - 1} \ , \tag{7}$$

a product of odd dimensional spheres. The torus group acts on this set, and through $\mathbb{T}^m$ equivariant cohomology we find that functions invariant under this torus action and Morse – Bott with respect to it must have a minimum number β of critical $\mathbb{T}^m$-orbits. This lower bound $\beta = \beta(p)$ depends in particular upon the dimensions given by $p_1, \dots, p_m$, but in any case it is bounded below independently of p by

$$\beta \geq M - m + 1 \ . \tag{8}$$

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KAM for the non-linear Schrödinger equation

L. H. Eliasson

(joint work with S. B. Kuksin)

We shall discuss the perturbation theory of lower-dimensional tori (KAM) for the non-linear Schrödinger equation with perodic boundary conditions in dimension d. The difficulties in applying KAM in infinite dimensions are substantial and become larger with increasing d. This is a report on a recent work (with S. Kuksin) that aims to solve the problem for any d.

We consider the d-dimensional nonlinear Schrödinger equation (NLS)

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon |u|^2 u; \quad u = u(t,x)$$

under the periodic boundary conditions $x \in \mathbb{T}^d$. The convolution potential $V : \mathbb{T}^d \to \mathbb{C}$ must have real [1] Fourier coefficients $\hat{V}(a),\ a \in \mathbb{Z}^d$, and we shall suppose it is analytic,

$$|\hat{V}(a)| \le C_1 e^{-C_2|a|} \quad \forall\, a \in \mathbb{Z}^d .$$

If we write

$$\begin{cases} u(x) = \sum_{a\in\mathbb{Z}^d} (\xi_a + i\eta_a) e^{ila,x} \\ \overline{u(x)} = \sum_{a\in\mathbb{Z}^d} (\xi_a - i\eta_a) e^{ila,x}, \end{cases}$$

then, in the symplectic space

$$\begin{cases} \{(\xi_a,\eta_a) : a \in \mathbb{Z}^d\} = \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d} \\ \sum_{a\in\mathbb{Z}^d} d\xi_a \wedge d\eta_a, \end{cases}$$

the equation becomes a Hamiltonian system with Hamiltonian

$$H = \frac{1}{2} \sum_{a\in\mathbb{Z}^d} (|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2) \quad + \varepsilon h(\xi,\eta),$$

where h is a quartic form in (ξ,η).

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$ and fix

$$0 < r_a(0) \le 1$$

[1] This equation is a popular model for the 'real' NLS equation, where instead of the convolution term $V * u$ we have the potential term Vu. Considering this model we remove some technical difficulties, which are not related to the main ones.

for each $a \in \mathcal{A}$. The $(\#\mathcal{A})$-dimensional torus

$$\frac{1}{2}(\xi_a^2 + \eta_a^2) = r_a(0), \quad a \in \mathcal{A},$$

is invariant for the Hamiltonian flow when $\varepsilon = 0$. In a neighborhood of this torus we introduce action-angle variables (φ_a, r_a)

$$\xi_a = \sqrt{2(r_a(0) + r_a)} \cos(\varphi_a), \ \eta_a = \sqrt{2(r_a(0) + r_a))} \sin(\varphi_a).$$

The Hamiltonian now becomes

$$H = \sum_{a \in \mathcal{A}} \omega_a r_a + \frac{1}{2} \sum_{a \in \mathcal{L}} \Omega_a(\xi_a^2 + \eta_a^2) + \varepsilon h(\varphi, r, \xi, \eta),$$

where

$$\omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{A},$$

are the *basic frequencies*, and

$$\Omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A},$$

are the *normal frequencies*.

We define (for $m^* \in \mathbb{N}$) the complex domain

$$O^\gamma(\rho, \sigma) = \begin{cases} |\Im\varphi| < \rho, \ |r| < \sigma^2 \\ \|\xi\|_\gamma = \sqrt{\sum_{a \in \mathcal{L}} |\xi_a|^2 |a|^{2m^*} e^{2\gamma|a|}} < \sigma, \ \|\eta\|_\gamma < \sigma \end{cases}$$

H is analytic on $O^\gamma(\rho, \sigma)$ and its (local) hamiltonian flow is well-defined on this domain.

The basic frequencies $\omega = \{\omega_a : a \in \mathcal{A}\}$ will be our free parameters belonging to a set

$$U \subset \{\omega \in \mathbb{R}^{\mathcal{A}} : |\omega| \le C_3\}.$$

The normal frequencies should verify

$$\begin{array}{ll} |\Omega_a + \Omega_b|, \ |\Omega_a| \ge C_4 & \forall\, a, b \in \mathcal{L}, \\ |\Omega_a - \Omega_b| \ge C_4 & \forall\, a, b \in \mathcal{L}, |a| \ne |b|. \end{array}$$

Theorem 1. *Under the above assumptions, for ε sufficiently small there exist a Borel subset $U' \subset U$,*

$$\mathrm{Leb}\,(U \setminus U') \le \text{cte. } \varepsilon^{exp_1},$$

and, for each $\omega \in U'$, a real analytic symplectomorphism

$$\Phi_\omega : \Omega^{\gamma/2}(\rho/2, \sigma/2) \to O^{\gamma/2}(\rho, \sigma),$$

such that $|\Phi_\omega - \mathrm{id}| \le \text{cte. } \varepsilon^{\exp_2}$ *and*

$$\begin{aligned} H \circ \Phi_\omega(\varphi, r, \xi, \eta) \quad &= l\omega', r + \\ &+ l\xi, \Omega_1'\xi + l\eta, \Omega_1'\eta + 2l\xi, \Omega_2'\eta + \varepsilon h(\varphi, r, \xi, \eta), \end{aligned}$$

where $\Omega' = \Omega_1' + i\Omega_2'$ is Hermitian and block-diagonal with finite-dimensional blocks, and where

$$h' \in \mathcal{O}^2(r, \xi, \eta) \cup \mathcal{O}^3(\xi, \eta).$$

The constant cte. *only depends on the dimensions d and n and on $C_1, \dots, C_4$. The exponents* $\exp_1$ *and* $\exp_2$ *only depend on the dimensions d and n.*

A first consequence of this statement is the *persistence* of quasi-periodic solutions: The torus $\Phi_\omega(\mathbb{T}^n \times \{0\} \times \{0\})$ is invariant for the Hamiltonian flow and the flow on it is conjugate to the linear flow

$$t \to \varphi + t\omega' .$$

A second consequence is the *reducibility* of the linearized system on this torus (ie. the variational equations of the quasi-periodic solution) to the constant coefficient system determined by

$$l\omega', r + l\xi, \Omega_1'\xi + l\eta, \Omega_1'\eta + 2l\xi, \Omega_2'\eta.$$

Due to its form it follows that all Lyapunov exponents of the solutions vanish.

Some references. For finite dimensional Hamiltonian systems the first proof of persistence and reducibility of stable (i.e. vanishing of all Lyapunov exponents) isotropic tori was obtained by Eliasson [1, 2]. This has been improved in many works and the situation in finite dimension is pretty well understood. Not so, however, in infinite dimension.

For space–one–dimensional ($d = 1$) equations, if the space-variable x belongs to a finite segment and the equation is supplemented by the Dirichlet or Neumann boundary conditions, the same result was obtained by Kuksin in [3]. The case periodic boundary conditions was treated later by Bourgain in [4], using another multi–scale scheme, suggested by Fröhlich–Spencer in their work on the Anderson localisation, and later exploited by Craig–Wayne to construct time–periodic solutions of nonlinear PDEs. Due to these and other publications, the perturbation theory for quasiperiodic solutions of 1d Hamiltonian PDE is now sufficiently well developed.

For space–multi–dimensional ($d \geq 2$) equations the situation is much less understood. Developing further the scheme, suggested by Fröhlich–Spencer, Bourgain managed to prove persistence for the 2d case [5]. Finally, he has recently announced that the new techniques allow to establish persistence of quasi-periodic solutions for any d. (A detailed proof has not been given yet but the ideas are explained in his book [6].) It should be mentionned that the multi–scale-scheme developped by these authors does not (at least not immediately) give vanishing of the Lyapunov exponents nor reducibility of the linearized equation.

Main ideas. Very briefly, our main idea is to put under strict control the linear parts of the transformations, forming the KAM–procedure, defined by the homological equation.The solution, with estimates, of this equation requires control of the "small divisors" which imposes conditions on $\omega \in U$. These conditions are relatively easy to fulfill when $\mathcal{L}$ is a finite set in $\mathbb{Z}^d$ or when $\mathcal{L} \subset \mathbb{Z}^1$ because then the equation imposes on finitely many conditions on ω on every scale. In the case when $\mathcal{L}$ is an infinite subset of $\mathbb{Z}^d$, $d \geq 2$, the equation imposes infinitely many conditions on ω on every scale.

To verify that these conditions can be fulfilled in the n-parameter family $\omega \in U$, we make use of a special property of infinite–dimensional matrices — the Töplitz-Lipschitz property. This property has two nice features. These matrices form an algebra: one can multiply them and solve linear differential equations [7]. They permit a "compactification of the dimensions": If the Hessian (with respect to (ξ, η)) of the Hamiltonian is Töplitz-Lipschitz, then the infinitely many small divisor conditions needed to solve the homological equation reduce to finitely many conditions [8].

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Sard, Whitney, Assouad and Mather

Albert Fathi

Suppose L is a Tonelli Lagrangian on the compact manifold M, i.e.:

1) $L\colon TM \to \mathbb{R}$ is C^r, $r \geq 2$.
2) $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite for all $(x,v) \in TM$.
3) $\frac{L(x,v)}{\|v\|_x} \to \infty$ as $\|v\|_x \to \infty$.

For $x, y \in M$, $t > 0$, we introduce

$$h_t(x,y) = \inf \left\{ \int_0^t L(\gamma(s), \dot\gamma(s))ds \mid \gamma\colon [0,t] \to M,\ \gamma(0) = x,\ \gamma(t) = y \right\}$$

John Mather has shown that there exists a unique value $c[0]$ such that

$$h(x,y) = \liminf_{t\to\infty} h_t(x,y) + c[0]t$$

is finite everywhere. Moreover, the function $h\colon M \times M \to \mathbb{R}$ is Lipschitz, and satisfies $h(x,x) \geq 0$ and $h(x,z) \leq h(x,y) + h(y,z)$, so that $\delta_M(x,y) := h(x,y) + h(y,x)$ is a semi-metric on $\mathcal{A} = \{x \mid h(x,x) = 0\}$, which is the projected Aubry set. We introduce the equivalence relation $x \sim y$ if $\delta_M(x,y) = 0$; on $\widetilde{\mathcal{A}}_M = \mathcal{A}/\sim$, the function δ_M becomes a genuine metric $\widetilde{\delta}_M$. This metric space is called the Mather quotient of L.

John Mather has shown in [1] that for every $r \geq 2$ there exists a C^r Tonelli Lagrangian on some high-dimensional torus T^N such that the Mather quotient $(\widetilde{\mathcal{A}}_M, \widetilde{\delta}_M)$ is isometric to $([0,1], |x-y|)$.

We show how to partially extend this result: For every doubling compact metric space (X, δ) we find a C^r Lagrangian L on some high-dimensional torus T^N such that $(\widetilde{\mathcal{A}}_M, \widetilde{\delta}_M)$ is Lipschitz equivalent to (X, δ).

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About M. Herman's proof of 'Arnold's theorem' in celestial mechanics

Jacques Féjoz

Consider $1+n$ point bodies with masses $m_0, \epsilon m_1, \dots, \epsilon m_n > 0$ ($\epsilon > 0$) and position vectors $x_0, x_1, \dots, x_n \in \mathbb{R}^3$. According to Newton's equations we have

$$\ddot{x}_j = m_0 \frac{x_0 - x_j}{||x_0 - x_j||^3} + \epsilon \sum_{k \neq j} m_k \frac{x_k - x_j}{||x_k - x_j||^3} \quad (j = 1, \dots, n).$$

These equations have a limit when $\epsilon \to 0$, for which each planet (masses ϵm_j) undergoes the only attraction of the sun (mass m_0). If the energies of the planets are negative, planets describe Keplerian ellipses with some given semi major axes and excentricities. As a whole, the system is quasiperiodic with n frequencies. In 1963, V. Arnold [A] published the following remarkable result.

Theorem 1. *For every $m_0, m_1, \dots, m_n > 0$ and for every $a_1 > \cdots > a_n > 0$ there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, in the phase space in the neighborhood of circular and coplanar Keplerian motions with semi major axes $a_1, \dots, a_n$, there is a subset of positive Lebesgue measure of initial conditions leading to quasiperiodic motions with $3n-1$ frequencies.*

The proof of this theorem is rendered difficult by the multitudinous degeneracies of the planetary problem. Arnold's initial proof does not fully describe these degeneracies and actually misses one of them. Hence it is wrong in the case of $n \geq 3$ planets in space. In 1998, in a series of lectures M. Herman sketched a complete and more conceptual proof of this theorem [F]. I will now review a couple of ideas which make this proof so powerful and, I believe, elegant. These ideas mainly pertain to some normal forms of Hamiltonians, which might not surprise the specialists but which epitomize the structure of KAM theory as understood by M. Herman.

Let $X = \mathbb{T}^p \times \bar{\mathbb{B}}^p$, $\mathbb{T}^p = \mathbb{R}^p/\mathbb{Z}^p$ and $\bar{\mathbb{B}}^p$ the closed p-dimensional unit Euclidean ball. Endow X with the natural coordinates (θ, r) and the standard symplectic form $\omega = \sum_{j=1}^p d\theta_j \wedge dr_j$. If $H \in C^\infty(X)$ is a smooth Hamiltonian, its Hamiltonian vector field is $\dot{\theta} = \partial_r H, \quad \dot{r} = -\partial_\theta H$. Denote by R_α, $\alpha \in \mathbb{R}$, the Hamiltonian defined by $R_\alpha = \alpha \cdot r$. Let $\mathcal{N}_\alpha = \{R_\alpha + O(r^2)\}$ be the space of Hamiltonians for whose flow the torus $\mathbb{T}^p_0 = \mathbb{T}^p \times \{0\}$ is invariant and quasiperiodic with frequency vector α. Let also $\mathcal{G}$ be some space of Hamiltonian diffeomorphisms, which we will not fully describe here, but which is diffeomorphic to a neighborhood of $(0, id)$ in the product $\mathbb{B}^\infty_1(\mathbb{T}^p) \times \mathrm{Diff}^\infty_o(\mathbb{T}^p)$, where $\mathbb{B}^\infty_1(\mathbb{T}^p)$ is the space (acting by translation in the r direction) of closed one-forms on $\mathbb{T}^p$ and $\mathrm{Diff}^\infty_o(\mathbb{T}^p)$ is the space (acting contragrediently) of diffeomorphisms of the torus which fix the origin. Let ϕ_α be the map

$$\begin{array}{rlcl} \phi_\alpha : & \mathcal{N}_\alpha \times \mathcal{G} \times \mathbb{R}^p & \to & C^\infty_+(X) \\ & (N, G, \hat{\alpha}) & \mapsto & H = N \circ G + R_{\hat{\alpha}}, \end{array}$$

where $C^\infty_+(X)$ is the quotient of the space of Hamiltonians by the real constants. The Hamiltonian $N \circ G$ is symplectically conjugate to N by G; hence for the flow of $N \circ G$ the torus $G^{-1}(\mathbb{T}^p_0)$ is invariant and α-quasiperiodic. The term $R_{\hat{\alpha}}$, which tunes the frequency, unfortunately breaks down the dynamical conjugacy; hence I call $(N, G, \hat{\alpha})$ a *twisted conjugacy* of H, and in general H does not have an invariant torus. Eventually, define

$$HD_{\gamma,\tau} = \{\alpha \in \mathbb{R}^p : |k \cdot \alpha| \geq \gamma ||k||^{-\tau} \ \ \forall\, k \in \mathbb{Z}^p \setminus 0\} \quad (\gamma, \tau > 0).$$

Theorem 2 (Twisted conjugacy, M. Herman)**.** *For every $\alpha \in HD_{\gamma,\tau}$ and for every $N^o \in \mathcal{N}_\alpha$, the map ϕ_α is a local (tame in the sense of Hamilton) C^∞-diffeomorphism in a neighborhood of $(N^o, id, 0) \mapsto N^o$; in particular, the $\mathcal{G}$-orbit of $\mathcal{N}_\alpha$ defines a germ of submanifold of codimension p of $C^\infty_+(X)$. Moreover, the germ of map $(H, \alpha) \mapsto \phi_\alpha^{-1}(H)$ is C^∞-smooth in the sense of Whitney.*

Sketch of proof. We want to solve the equation $\phi_\alpha(N, G, \hat{\alpha}) = H$ for H close enough to N^o in the C^∞-topology. In this setting, small denominators manifest themselves in the loss of differentiability of ϕ_α, which prevents from choosing Banach norms at the source and target spaces of ϕ_α for which this operator is both bounded and coercive. A way out is to use scaled Fréchet structures and the Nash-Moser inverse function theorem. For the sake of simplicity, the version due to Sergeraert and Hamilton in the C^∞-category can be used. Then the problem boils down to inverting the linear operator $d\phi_\alpha(N, G, \hat{\alpha})$ for $(N, G, \hat{\alpha})$ close, but not necessarily equal, to $(N^o, id, 0)$. This inversion is equivalent to one step in the induction of Kolmogorov's original proof of the invariant torus theorem.

In order to get rid of the twist of the conjugacy, a natural idea could be to tune the frequency before conjugating by G, i.e., to consider $\psi_\alpha : (N, G, \hat{\alpha}) \mapsto (N + R_{\hat{\alpha}}) \circ G$ instead of ϕ_α. But ψ_α is glaringly not a local diffeomorphism – if it were, the property of having an invariant torus would be open in the space of Hamiltonians! We will use this idea in a more sophisticated manner. Let

$$\mathcal{N} = \cup_{\alpha \in \mathbb{R}^p} \mathcal{N}_\alpha = \{\alpha \cdot r + O(r^2)\}_{\alpha \in \mathbb{R}^p}.$$

Corollary 1 (Hypothetical conjugacy)**.** *For every $N^o \in \mathcal{N}$ there is a (non unique) germ of C^∞-diffeomorphism*

$$\begin{array}{rlcl} \Theta : & C^\infty_+(X) & \to & \mathcal{N} \times \mathcal{G} \\ & H & \mapsto & (N_H, G_H), \quad N_H = \alpha_H \cdot r + O(r^2), \end{array}$$

at $N^o \mapsto (N^o, id)$ *such that for every* H *the following implication holds :*

$$\alpha_H \in HD_{\gamma,\tau} \Longrightarrow H = N_H \circ G_H.$$

I call (N_H, G_H) a *hypothetical conjugacy* of H because the property $H = N_H \circ G_H$ depends on arithmetical conditions involving the unknown frequency α_H.

Proof. According to Theorem 2, the equality $\tilde{\Theta}(H, \alpha) = {\phi_\alpha}^{-1}(H)$ defines a germ of map

$$\tilde{\Theta} : C^\infty_+(X) \times HD_{\gamma,\tau} \to \mathcal{N} \times \mathcal{G} \times \mathbb{R}^p$$

at N^o which is Whitney-smooth. According to Whitney's extension theorem, this germ extends to a smooth germ

$$\tilde{\Theta} : C^\infty_+(X) \times \mathbb{R}^p \to \mathcal{N} \times \mathcal{G} \times \mathbb{R}^p.$$

Now, the equality $N^o = (N^o + R_{\alpha - \alpha^o}) \circ id + R_{\alpha^o - \alpha}$ shows that

$$\left.\frac{\partial \hat{\alpha}}{\partial \alpha}\right|_{\{G = id\}} = -id_{\mathbb{R}^p}.$$

Hence, the usual implicit function theorem entails that there is a unique germ of function $\alpha = \bar{\alpha}(H)$ such that $\hat{\alpha}(\bar{\alpha}) = 0$. There only remains to set $\Theta(H) = \tilde{\Theta}(H, \bar{\alpha}(H))$.

Now assume that the perturbed Hamiltonian H depends on some parameter $s \in \mathbb{B}^t$; if H is close to some completely integrable Hamiltonian, s may be the action coordinate and, in the case of Arnold's theorem, s represents the semi-major axes, excentricities and inclinations. By composition with Θ, H determines a frequency map $s \mapsto \alpha_s$, which is C^∞-close to the frequency map $s \mapsto \alpha^o_s$ of the unperturbed Hamiltonian N^o.

Theorem 3 (Arnold, Margulis, Pyartli)**.** *If some real-analytic map* $s \in \mathbb{B}^t \mapsto \alpha^o_s \in \mathbb{R}^p$ *is non-planar in the sense that its image is nowhere locally contained in some proper vector space of* $\mathbb{R}^p$*, the Lebesgue measure of* $\{s \in \mathbb{B}^t,\ \alpha^o_s \in HD_{\gamma,\tau}\}$ *is positive provided that* γ *is small enough and* τ *large enough.*

There exists a similar statement in the smooth setting, involving finitely many derivatives of the frequency map. By combining the two latter statements and using the fact that being non planar is an open property in the C^∞-topology, we get an invariant tori theorem. Unfortunately, the following holds.

Theorem 4 (M. Herman)**.** *The frequency map* α^o *of the first order secular system – that is, the Birkhoff normal form of the planetary problem along circular and coplanar Keplerian* n*-tori –, as a function of the semi major axes, has its image lying entirely in a plane* P *of codimension* 2*. Moreover, its image lies in no plane of higher codimension.*

The theorem can be proved by induction on the number of planets and by complexifying the semi-major axes. The first resonance comes from Galilean symmetry and disappears when fixing the direction of the angular momentum. The second resonance is mysterious and seems not to have been noticed before. According to numerical evidence, it vanishes when the secular system is fully reduced by rotations, or for the second order secular system; but one precisely wants to avoid these computations in the general case (quoting M. Herman, 'BLC' for 'Bonjour Les Calculs'!).

M. Herman's resonance can actually be taken care of by looking to the planetary problem in a well-chosen rotating frame of reference, or, equivalentely, by adding to the Hamiltonian a term proportional to the vertical angular momentum. Abstracly, this is tantamount to applying Corollary 1 not merely to H but to $H + R_\beta$ with β varying in the set V of frequencies spanned by the first integrals of H. Let $\alpha = \alpha_{H,\beta}$ be the frequency of $H + R_\beta$, as defined by Θ. Since $\partial\alpha/\partial\beta|_{\{G=id\}} = id_{\mathbb{R}^p}$, the deformation $\alpha^1 : (s,\beta) \in \mathbb{B}^t \times V \mapsto \alpha_{N^o,\beta}$ is non planar in $P + V$. If H has enough 'transversal' first integrals in respect of the resonances of α^o, i.e., if $P + V = \mathbb{R}^p$ (which is the case for the planetary system), the map α^1 is non-planar and Theorem 3 applies. Due to Fubini's theorem, there is a fixed β for which the inverse image of $HD_{\gamma,\tau}$ by the partial map $s \mapsto \alpha^1_{s,\beta}$ has a positive measure. Hence $H + R_\beta$ has a set of invariant tori of positive measure. The so obtained invariant tori are Lagrangian, hence invariant for H itself.

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(See further references therein.)

Zygmund foliations and rigidity

Boris Hasselblatt

(joint work with Patrick Foulon)

For Anosov systems, both diffeomorphisms and flows, interesting phenomena of smooth and geometric rigidity have been observed in connection with the degree of (transverse) regularity of the (weak) stable and unstable subbundles of these systems. The seminal result was the study of volume-preserving Anosov flows on 3-manifolds by Hurder and Katok [4], which showed that the weak-stable and weak-unstable foliations are $C^{1+\text{Zygmund}}$ and that there is an obstruction to higher regularity whose vanishing implies smoothness of these foliations. This, in turn, happens only if the Anosov flow is smoothly conjugate to an algebraic one. The cocycle obstruction described by Katok and Hurder was first observed by Anosov and is the first nonlinear coefficient in the Moser normal form. Therefore one might call it the KAM-cocycle.

The present work, presented here for the first time, is aimed at showing some analogous rigidity features associated with the *longitudinal* direction, i.e. associated with various degrees of regularity of the sum of the *strong* stable and unstable subbundles. In [2] we showed that for a volume-preserving Anosov flow on a 3-manifold the strong stable and unstable foliations are Zygmund-regular [6, Section II.3, (3·1)], and we exhibited an obstruction to higher regularity, which admits a direct geometric interpretation. Vanishing of this obstruction implies high smoothness of the joint strong subbundle and that the flow is either a suspension or a contact flow. The work in progress presented here is aimed at a similar understanding of higher-dimensional systems.

Definition 1 ([5]). *Let M be a manifold, $\varphi\colon \mathbb{R}\times M \to M$ a smooth flow. Then φ is said to be an Anosov flow if the tangent bundle TM splits as $TM = E^{\varphi}\oplus E^{u}\oplus E^{s}$, where $E^{\varphi}(x) = \mathbb{R}\dot{\varphi}(x) \neq \{0\}$ for all $x \in M$, in such a way that there are constants $C > 0 < \lambda < 1 < \eta$ such that for $t > 0$ we have*

$$\|D\varphi^{-t} \restriction E^{u}\| \leq C\eta^{-t} \text{ and } \|Df^{t} \restriction E^{s}\| \leq C\lambda^{t}.$$

The subbundles are then invariant and (Hölder-) continuous and have smooth integral manifolds W^u and W^s that are coherent in that $q \in W^u(p) \implies W^u(q) = W^u(p)$. W^u and W^s define laminations (continuous foliations with smooth leaves).

Definition 2. *A function f between metric spaces is said to be* Hölder *continuous if there is an $H > 0$, called the Hölder exponent, such that $d(f(x), f(y)) \leq const.d(x, y)^H$ whenever $d(x, y)$ is sufficiently small. We specify the constant by saying that a function is H-Hölder. A continuous function $f\colon U \to \mathbb{R}$ on an open set $U \subset \mathbb{R}$ is said to be Zygmund-regular if there is $Z > 0$ such that $|f(x+h)+f(x-h)-2f(x)| \leq Z|h|$ for all $x \in U$ and sufficiently small h. To specify a value of Z we may refer to a function as being Z-Zygmund. The function is said to be "little Zygmund" (or "zygmund") if $|f(x+h)+f(x-h)-2f(x)| = o(|h|)$. For maps between manifolds these definitions are applied component-wise in smooth local coordinates.*

Zygmund regularity implies modulus of continuity $O(|x \log |x||)$ and hence H-Hölder continuity for all $H < 1$ [6, Theorem (3·4)]. It follows from Lipschitz continuity and hence from differentiability. Being "little Zygmund" implies having modulus of continuity $o(|x \log |x||)$.

The regularity of the unstable subbundle E^u is usually substantially lower than that of the weak-unstable subbundle $E^u \oplus E^{\varphi}$. The exception are geodesic flows, where the strong unstable subbundle is obtained from the weak-unstable subbundle by intersecting with the kernel of the invariant contact form. This has the effect that the strong-unstable and weak-unstable subbundles have the same regularity. However, time changes affect the regularity of the strong-unstable subbundle, and this is what typically keeps its regularity below C^1. In [2] we presented a *longitudinal KAM-cocycle* that is the obstruction to differentiability, and we derived higher regularity from its vanishing.

Theorem 1 ([2, Theorem 3])**.** *Let M be a 3-manifold, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent:*

(1) *$E^u \oplus E^s$ is "little Zygmund" (see Definition 2).*
(2) *The longitudinal KAM-cocycle is a coboundary.*
(3) *$E^u \oplus E^s$ is Lipschitz.*
(4) *$E^u \oplus E^s \in C^{k-1}$.*
(5) *φ is a suspension or contact flow.*

In (5) no stronger rigidity should be expected because $E^u \oplus E^s$ is smooth for all suspensions and contact flows.

The work by Hurder and Katok in [4] inspired developments of substantial extensions to higher dimensions, see, for example, [3]. The present work tries to make some analogous progress for higher-dimensional systems in this "longitudinal" context. We assume uniform quasiconformality, that is, boundedness of

$$K_i(x,t) := \frac{\max\{\|d\varphi^t(u)\| \mid u \in E^i(x),\ \|u\| = 1\}}{\min\{\|d\varphi^t(u)\| \mid u \in E^i(x),\ \|u\| = 1\}}$$

on $M \times \mathbb{R}$ for $i = u, s$.

Theorem 2. *Let M be a compact Riemannian manifold, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a uniformly quasiconformal volume-preserving C^k Anosov flow with $\dim E^u = \dim E^s$. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that defines the cohomology class of the* longitudinal KAM-cocycle*, and the following are equivalent:*

(1) *$E^u \oplus E^s$ is "little Zygmund" (see Definition 2).*
(2) *The longitudinal KAM-cocycle is a coboundary.*
(3) *$E^u \oplus E^s$ is Lipschitz.*

Unlike in dimension 3 we do not know whether these in turn imply that $E^u \oplus E^s \in C^{k-1}$. On the other hand, going far beyond our earlier assertion that smoothness of $E^u \oplus E^s$ implies that φ is a suspension or contact flow, there is a rigidity theorem by Fang:

Theorem 3 ([1, Theorem 1])**.** *Let M be a compact Riemannian manifold, $\varphi\colon \mathbb{R} \times M \to M$ a uniformly quasiconformal volume-preserving C^∞ Anosov flow with $\dim E^u, \dim E^s \geq 2$ and $E^u \oplus E^s \in C^\infty$. Then up to a constant time change and finite covers, φ^t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical perturbation of the geodesic flow of a hyperbolic manifold.*

Here, a *canonical perturbation* of a flow of a vector field X is that of the vector field $X/(1 + \alpha(X))$ for a closed C^∞ 1-form α such that $1 + \alpha(X) > 0$.

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Normally hyperbolic spheres in the coupling of oscillators

Mathilde Kammerer-Colin de Verdière

(joint work with Marc Chaperon)

Our main result is the existence of invariant, repelling or attracting, spheres in the neighboorhood of an elliptic fixed point for generic families of vector fields with a large space of parameters.

Theorem 1. *Let $(u,x) \mapsto X_u(x) \in \mathbb{R}^{2n}$ be a generic sufficiently smooth family of vector fields with parameter $u \in \mathbb{R}^{n^2+n}$, defined in a neighbourhood of $0 \in \mathbb{R}^{2n} \times \mathbb{R}^{n^2+n}$, such that 0 is an elliptic fixed point of X_0 without resonances of order lower than 4. Assume X_0 is formally linearizable at order 3 at $0 \in \mathbb{R}^{2n}$. Under these hypotheses, there exists an open neighbourhood U of $0 \in \mathbb{R}^{n^2+n}$ such that for every $u \in U$, the vector field X_u has a repelling (or attracting) invariant manifold W_u diffeomorphic to S^{2n-1}. The submanifold W_u depends continuously on the parameter $u \in U$ and tends to $\{0\}$ when $u \to 0$.*

Sktech of proof in the case $n = 2$. The standard normal form theory provides a local change of coordinates $\mathbb{R}^4 = \mathbb{C}^2$; the normal form N_u induces a planar vector field in the modulus plane : $\dot{r_j^2} = r_j^2(\lambda_j + a_j r_1^2 + b_j r_2^2)$ for $j = 1, 2$ with $u = (\lambda_j, a_j, b_j)_{j=1,2}$ by genericity [2]. Choose $\varepsilon \in \mathbb{R}_+^*$. In the first quadrant, the sphere $S = S(0, \varepsilon)$ is represented by a line segment; for very particular values of the parameter u_0 this segment is repelling and made of fixed points of N_{u_0}. This means that the sphere $S \subset \mathbb{R}^4$ is foliated by invariant tori for N_{u_0}. The standard normal hyperbolicity theory implies: for parameter values not too far from u_0 and for ε small enough the vector field X_u admits an invariant repelling manifold close to S. More precisely, we have used the very simple result in [1] which is stated for maps and admits an analogous formulation for vector fields [3].

Comments.

- The open set U_n is made of neighboorhoods of very particular values of the parameters for which the normal form induces no dynamic on the segment lines representing the sphere; it is obviously far from being as large as possible.
- It is very easy to see the invariant sphere in the normal form N_u because of the big dimension of our parameter space;
- this result holds for diffeomorphisms with the same assumptions of genericity, elliptic fixed point and non resonance: $h_u\colon (\mathbb{R}^{2n},0) \to (\mathbb{R}^{2n},0)$ with $dX_0(0)$ having n pairs of eigenvalues $(e^{\pm i\alpha_j})_{1\leqslant j\leqslant n}$ or $h_u\colon (\mathbb{R}^{2n-1},0) \to (\mathbb{R}^{2n-1},0)$ with $dX_0(0)$ having $(n-1)$ pairs of eigenvalues $(e^{\pm i\alpha_j})_{1\leqslant j\leqslant n-1}$ and the eigenvalue -1.

More details can be found in [3].

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Two new approaches to local rigidity of abelian actions: KAM and algebraic K-theory

Anatole Katok

We consider two classes of algebraic partially hyperbolic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$: actions by automorphisms of a torus and restrictions of the Weyl Chamber flow to an intermediate subgroup of rank ≥ 2.

For the first class we prove local differential rigidity by a KAM-type method. Vanishing of the obstructions for the linearized equation and tame estimation for the non-linear conjugacy equation involve "the higher-rank" trick and an estimate of the solution of the second cohomology equation.

For the second class the method is very different and complementary. It uses the construction of solutions for the cohomolgy equation for the perturbed action. The key element is a representation of generators and relations in $\mathrm{SL}(n;\mathbb{R})$ following classical work of R. Steinberg and J. Milnor.

Variation on a Theorem of C. Conley

Janko Latschev

For a smooth flow on a smooth compact manifold, a fundamental theorem of Conley asserts the existence of a smooth Lyapunov function, that is a function whose critical set coincides with the chain-recurrent set R of the flow and whose derivative in the flow direction is negative outside R. One may interpret this result as giving (via Lusternik-Schnirelman theory) a lower bound on the complexity of the chain recurrent set in terms of the category of M.

In my talk, I discussed the problem of existence of Lyapunov 1-forms, as described by the following definition:

Definition 1. [3] *Let φ_t be a smooth flow on the smooth closed manifold M, and let $Z \subset M$ be a closed subset invariant under the flow of φ_t. A* Lyapunov 1-form for *(φ_t, Z) is a smooth closed 1-form λ such that*

(L1) *λ vanishes pointwise on Z and is exact in some neighborhood of Z, and*
(L2) *$\lambda(V) < 0$ on $M \setminus Z$, where V is the generating vector field for the flow φ_t.*

There is also a parallel notion of a Lyapunov 1-form for continuous flows on compact metric spaces, which was introduced in [4]. Part of the results described below have analogues in that context as well.

The existence question is motivated in part by recent results about flows that admit Lyapunov 1-forms. For example, Farber [2] introduced a category-type invariant $cat(M, \xi)$ associated to a cohomology class $\xi \in H^1(M; \mathbb{R})$. When $\xi = 0$, it agrees with the usual Lusternik–Schnirelman category, and for $\xi \neq 0$ and M connected it takes values between 0 and $cat(M) - 1$. Farber showed that his invariant can be effectively bounded from below in terms of homotopy theory. In [2], he proved a first version of the following result, which later appeared as stated in [5].

Theorem 1 ([2, 5])**.** *Let φ_t be a smooth flow on a smooth closed manifold M, and let Z be an isolated invariant set with finitely many components $Z_1, \ldots, Z_k$. Suppose there exists a Lyapunov 1-form for (φ_t, Z) representing $\xi \in H^1(M; \mathbb{R})$.*

Then either

$$cat(M, \xi) \leq \sum_{i=1}^{k} cat_M(Z_i),$$

or there exist points $x_1, \ldots, x_r, x_{r+1} = x_1 \in M \setminus Z$ with $r \leq k$ such that for each $i = 1, \ldots, r$ the forward limit set of x_i and the backward limit set of x_{i+1} are contained in the same connected component of Z.

Other interest in this question comes from the general philosophy that there should be a theory generalizing Novikov theory for 1-forms with Morse singularities in the same way that Conley's theory generalizes ordinary Morse theory. First results in this direction have been obtained by Fan and Jost [1].

There is a classical theorem by Schwartzman [7] that can be rephrased as an existence result for Lyapunov 1-forms with $Z = \varnothing$. In fact, given a finite invariant measure μ for the flow φ_t on M generated by the vector field V, the *asymptotic cycle of* μ is defined as the homomorphism $A_\mu : H^1(M; \mathbb{R}) \to \mathbb{R}$ given by mapping the cohomology class $[\alpha]$ of a closed form α to

$$A_\mu([\alpha]) := \int_M \alpha(V)\, d\mu.$$

Theorem 2 (Schwartzman [7]). *Let φ_t be a smooth flow on a smooth closed manifold M. Then there exists a smooth Lyapunov 1-form for $(\varphi_t, \varnothing)$ representing the class $\xi \in H^1(M;\mathbb{R})$ if and only if*

$$A_\mu(\xi) < 0$$

for every finite positive invariant measure μ.

In the joint work [3] we proved a first existence result for Lyapunov 1-forms with nonempty zero set, which is also formulated in terms of asymptotic cycles of finite invariant measures. Unfortunately, it requires the zero set of the form to be isolated in the chain recurrent set, an assumption that is rather restrictive and also nearly impossible to check in practice.

It turns out that to get necessary and sufficient conditions in a more general situation, one has to appropriately enlarge the class of measures one considers. In fact, given the flow φ_t and the closed invariant subset $Z \subset M$, one is lead to consider locally finite invariant measures μ on $M \setminus Z$. These still define relative asymptotic cycles $A_\mu : H^1(M, Z; \mathbb{R}) \to \mathbb{R}$, where the cohomology group $H^1(M, Z; \mathbb{R})$ is computed using forms compactly supported in $M \setminus Z$. Since we assume that the Lyapunov 1-form is exact near Z, it seems reasonable to restrict our search to classes $\xi \in H^1(M;\mathbb{R})$ that vanish in some neighborhood of Z. In terms of the long exact sequence in cohomology

$$\cdots \to H^0(Z;\mathbb{R}) \to H^1(M, Z;\mathbb{R}) \xrightarrow{j^*} H^1(M;\mathbb{R}) \xrightarrow{i^*} H^1(Z;\mathbb{R}) \to \dots,$$

where $H^*(Z;\mathbb{R})$ is computed from germs of differential forms near Z, we are thus interested in classes $\xi \in H_Z := \ker i^* = \operatorname{im} j^*$. It turns out that the measures carrying the relevant homological information are those whose relative asymptotic cycle descends as a well-defined homomorphism to H_Z. We thus arrive at the following definition.

Definition 2. *A locally finite invariant measure μ on $M \setminus Z$ is called* coherent relative to Z *if $A_\mu : H^1(M, Z;\mathbb{R}) \to \mathbb{R}$ vanishes on $\ker j^*$ and so descends to a homomorphism $\widetilde{A_\mu} : H_Z \to \mathbb{R}$. More explicitly, this means that*

$$\int_{M\setminus Z} dg(V)\, d\mu = 0$$

for all smooth functions $g : M \to \mathbb{R}$ whose differential vanishes in some neighborhood of Z.

The easiest examples of coherent measures are given by finite invariant measures μ that restrict non-trivially to $M \setminus Z$. Next there are examples arising from homoclinic orbits, i.e. orbits in $M \setminus Z$ whose forward and backward limit set are contained in the same component of Z, or chains similar to those described in the first theorem above. The main result of [6] now reads as follows.

Theorem 3 ([6]). *Let φ_t be a smooth flow on a smooth closed manifold M, and let $Z \subset M$ be an isolated invariant set for φ_t. Then there exists a smooth Lyapunov*

1-form for (φ_t, Z) *representing* $\xi \in H_Z$ *if and only if*

$$\widetilde{A_\mu}(\xi) < 0$$

for every positive measure μ *coherent relative to* Z.

In the talk, I also mentioned two results from [6] that hold in the special case where we consider an integral cohomology class $\xi \in H^1(M;\mathbb{Z})$. The first one characterizes the possible closed invariant subsets $Z \subset M$ for which there can be a Lyapunov 1-form representing ξ as countable intersections of isolated invariant sets admitting such forms. The second characterizes the smallest set Z for which there exists a Lyapunov 1-form for (φ_t, Z) representing $\xi \in H^1(M;\mathbb{Z})$ as a certain subset of the chain recurrent set naturally associated to ξ, under the assumption that there is at least one Lyapunov 1-form representing ξ.

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Multiple closed geodesics on Finsler 2-spheres

Yiming Long†

(joint work with Victor Bangert)

The study of closed geodesics on spheres is a classical and important problem in both dynamical systems and differential geometry. The results of V. Bangert in 1993 and J. Franks in 1992 prove that for every Riemannian metric on S^2 there exist infinitely many geometrically distinct closed geodesics. In 1973, A. Katok constructed a remarkable irreversible Finsler metric on S^2 which possesses precisely two distinct prime closed geodesics. Based on this result, D. V. Anosov in his ICM report of 1974 proposed the following question: "*For the* n*-dimensional sphere* S^n*, Katok's example gives an irreversible Finsler metric, arbitrarily near to the 'standard' metric (to the metric of constant curvature) which has* $2[n/2]$ *closed geodesics. This number coincides with the lower bound which one naturally expects for irreversible Finsler metrics on* S^n *and which can be proved for metrics sufficiently near the 'standard' metric*". Here we denote by $[a] = \max\{k \in \mathbf{Z} \,|\, k \le a\}$ for any $a \in \mathbf{R}$. Note that the existence of one closed geodesic on any Finsler

2-sphere follows from the proof of the classical theorem of Lyusternik-Fet in 1951. We are only aware of a few results on the existence of at least 2 closed geodesics on Finsler 2-spheres under certain non-degeneracy conditions.

In a recent paper we proved the following theorem which confirmed Anosov's conjecture for all Finsler 2-spheres.

Theorem 1 (V. Bangert and Y. Long). *For every Finsler metric F on the 2-sphere S^2, there exist at least two distinct prime closed geodesics.*

Our proof depends on the following four main ingredients: (1) the precise index iteration formulae of Y. Long established in 2000, (2) Morse inequality, (3) two theorems of N. Hingston proved in 1993 and 1997, and (4) a new exact sequence method.

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Log-Riemann Surfaces

Ricardo Perez-Marco

We exposed our joint project with Kingshook Biswas on building an algebraic theory for transalgebraic curves. Log-Riemann surfaces correspond to Riemann's point of view of Riemann surfaces, that is with canonical charts. These canonical charts allow to write formulas (for uniformization, transabelian integrals, etc.) and allow to make the link of the geometry with the world of special functions.

We defined Caratheodory convergence and the associated Caratheodory theorem. As a corollary we proved Euler's formula

$$e^z = \lim_{n\to+\infty}\left(1+\frac{z}{n}\right)^n = \left(1+\frac{z}{\infty}\right)^{+\infty}$$

(apparently nobody understood why that was a geometric proof of the formula ... but this was due to the esotheric exposition of the author).

Fiberwise volume growth via Lagrangian intersections

Felix Schlenk

(joint work with Urs Frauenfelder and Leonardo Macarini)

1. Topological entropy and volume growth. The topological entropy $h_{\rm top}(\varphi)$ of a compactly supported C^1-diffeomorphism φ of a smooth manifold X is a basic numerical invariant measuring the orbit structure complexity of φ. There are various ways of defining $h_{\rm top}(\varphi)$, see [5]. If φ is C^∞-smooth, a geometric way was found by Yomdin and Newhouse in their seminal works [11] and [7]: Fix a Riemannian metric g on X. For $j \in \{1,\dots,\dim X\}$ denote by Σ_j the set of smooth compact (not necessarily closed) j-dimensional submanifolds of X, and by $\mu_g(\sigma)$ the volume of $\sigma\in\Sigma_j$ computed with respect to the measure on σ induced by g. The *j'th volume growth* of φ is defined as

$$v_j(\varphi) \;=\; \sup_{\sigma\in\Sigma_j}\liminf_{m\to\infty}\frac{1}{m}\log\mu_g\left(\varphi^m(\sigma)\right),$$

and the *volume growth* of φ is defined as $v(\varphi)=\max_{1\le j\le \dim X} v_j(\varphi)$. Newhouse proved in [7] that $h_{\rm top}(\varphi)\le v(\varphi)$, and Yomdin proved in [11] that $h_{\rm top}(\varphi)\ge v(\varphi)$ provided that φ is C^∞-smooth, so that

$$h_{\rm top}(\varphi) \;=\; v(\varphi)\quad \text{if } \varphi \text{ is } C^\infty\text{-smooth.} \tag{1}$$

The topological entropy measures the *exponential* growth rate of the orbit complexity of a diffeomorphism. It therefore vanishes for many interesting dynamical systems. Following [6, 4] we thus also consider the *j'th slow volume growth*

$$s_j(\varphi) \;=\; \sup_{\sigma\in\Sigma_j}\liminf_{m\to\infty}\frac{1}{\log m}\log\mu_g\left(\varphi^m(\sigma)\right)$$

and the *slow volume growth* $s(\varphi)=\max_{1\le j\le\dim X}s_j(\varphi)$. It measures the *polynomial* volume growth of the iterates of the most distorted smooth j-dimensional family of initial data. Note that $v_j(\varphi)$, $v(\varphi)$, $s_j(\varphi)$, $s(\varphi)$ do not depend on the choice of g, and that $v_{\dim X}(\varphi) = s_{\dim X}(\varphi) = 0$.

The aim of our work is to give uniform lower estimates of localized versions of $v(\varphi)$ and $s(\varphi)$ for certain symplectomorphisms of cotangent bundles. We consider a smooth closed d-dimensional Riemannian manifold (M, g) and the cotangent bundle T^*M over M endowed with the induced Riemannian metric g^* and the standard symplectic form $\omega = \sum_{j=1}^{d} dp_j \wedge dq_j$. We abbreviate

$$D(r) \;=\; \{(q,p) \in T^*M \mid |p| \le r\} \qquad \text{and} \qquad D_q(r) \;=\; T_q^*M \cap D(r).$$

Let φ be a C^1-smooth symplectomorphism of (T^*M, ω) which preserves $D(r)$. If φ is C^∞-smooth, (1) says that the maximal orbit complexity of $\varphi|_{D(r)}$ is already contained in the orbit of a single submanifold of $D(r)$. Usually, lower estimates of the topological entropy do not give any information on the dimension or the location of such a submanifold. We consider for φ as above the *uniform fiberwise volume growth*

$$v_{\text{fibre}}(\varphi; r) \;=\; \inf_{q \in M} \liminf_{m \to \infty} \frac{1}{m} \log \mu_{g^*}\left(\varphi^m\left(D_q(r)\right)\right)$$

and the *uniform slow fiberwise volume growth*

$$s_{\text{fibre}}(\varphi; r) \;=\; \inf_{q \in M} \liminf_{m \to \infty} \frac{1}{\log m} \log \mu_{g^*}\left(\varphi^m\left(D_q(r)\right)\right).$$

Writing $v(\varphi; r) = v\left(\varphi|_{D(r)}\right)$ and so on, we clearly have

$$v(\varphi; r) \;\ge\; v_d(\varphi; r) \;\ge\; v_{\text{fibre}}(\varphi; r), \tag{2}$$

$$s(\varphi; r) \;\ge\; s_d(\varphi; r) \;\ge\; s_{\text{fibre}}(\varphi; r). \tag{3}$$

We obtain lower estimates of $v_{\text{fibre}}(\varphi; r)$ and $s_{\text{fibre}}(\varphi; r)$ in terms of the growth of certain homotopy-type invariants of M. For the sake of brevity we assume from now on that $\pi_1(M)$ is finite.

2. Rationally elliptic and hyperbolic manifolds. A closed connected manifold M with finite fundamental group $\pi_1(M)$ is said to be *rationally elliptic* if the total rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is finite dimensional, and M is said to be *rationally hyperbolic* if the integers

$$\sum_{j=0}^{m} \dim \pi_j(M) \otimes \mathbb{Q}$$

grow exponentially. It is shown in [2, 3] that every closed manifold M with $\pi_1(M)$ finite is either rationally elliptic or rationally hyperbolic. "Most" manifolds with finite fundmamental group are rationally hyperbolic. We refer to [2, 3, 8] for more information on rationally elliptic and hyperbolic manifolds.

3. Main result. Consider a closed Riemannian manifold (M, g) with $\pi_1(M)$ finite, and let $H \colon [0,1] \times T^*M \to \mathbb{R}$ be a C^2-smooth Hamiltonian function meeting the following assumption: There exists $r_H > 0$ and a function $f\colon [0,\infty) \to \mathbb{R}$ with $f'(r_H) \neq 0$ such that

$$H(t,q,p) \;=\; f\left(|p|\right) \qquad \text{for } \; |p| \geq r_H. \tag{4}$$

The flow φ_H^t of the time-dependent vector field X_H given by $\omega\left(X_{H_t},\cdot\right) = -dH_t\left(\cdot\right)$ is defined for all $t \in [0,1]$. We abbreviate $\varphi_H = \varphi_H^1$.

Theorem 1. *Let (M,g) and H be as above. If M is rationally elliptic, then*

$$s_{\text{fibre}}(\varphi_H; r_H) \;\geq\; 1. \tag{5}$$

If M is rationally hyperbolic, then

$$v_{\text{fibre}}(\varphi_H; r_H) \;\geq\; f'(r_H)\, r_H\, C_1 \tag{6}$$

for some positive constant C_1 depending only on (M,g).

Discussion 1. (i) There are rationally elliptic manifolds for which all the numbers in (3) are 1, see Discussion 2 (i) below. The estimate (5) is thus sharp.

(ii) If H is C^∞-smooth, then $h_{\text{top}}(\varphi_H; r_H) \geq v_{\text{fibre}}(\varphi_H; r_H)$ by Yomdin's theorem and (2), so that (6) yields a positive lower bound for $h_{\text{top}}(\varphi_H; r_H)$. This bound implies and is implied by the estimate

$$h_{\text{top}}(g) \;\geq\; C_1(M,g)$$

for the topological entropy of the geodesic flow on the unit sphere bundle $\partial D(1)$ of a C^∞-smooth Riemannian metric g on a rationally hyperbolic manifold, which is due to Gromov and Paternain (see [8, Corollary 5.21]).

(iii) Theorem 1 extends well-known results from the study of geodesic flows, see [8, Corollary 3.9 and Chapter 5]: These results imply Theorem 1 if there exists an $\epsilon > 0$ such that $H = \frac{1}{2}|p|^2$ on $D(r_H) \setminus D(r_H - \epsilon)$.

(iv) As the identity mapping illustrates, the assumption $f'(r) \neq 0$ in (4) is essential.

(v) Assume that all geodesics of (M,g) are closed. This is so, e.g., for the canonical Riemannian structures on compact rank one symmetric spaces. In this situation one can define a compactly supported twist-like symplectomorphism ϑ on T^*M, see [9, 10]. For this map one computes $s\big(\vartheta\big) = s_{\text{fibre}}\big(\vartheta\big) = 1$, so that Corollary 1 is sharp. The estimate (5) implies that $s_{\text{fibre}}(\varphi; r) \geq 1$ for each symplectomorphism in the symplectic isotopy class of ϑ and r large enough. This complets the main result of [4]. $\Diamond$

The proof of Theorem 1 is along the following lines. Using an idea from [4] we first show that fiberwise volume growth is a consequence of the growth of the dimension of certain Floer homology groups. Applying the isotopy invariance of Floer homology and a recent result of Abbondandolo and Schwarz [1], these homology groups are seen to be isomorphic to the homology of the space of based loops in M not exceeding a certain length. Their dimension can be estimated from below by results of Gromov and Serre.

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Tire track geometry and flotation problems

Serge Tabachnikov

There are three motivation for this work.

First, a bicycle leaves two tracks on the ground, the front wheel track Γ and the rear one γ (smooth plane curves). The relation between them is that the positive tangent segment of fixed length L (the length of the bicycle frame) at every point of γ has its end-point on Γ; see [5]. Generically, given two tire tracks, one can determine the direction of motion. The problem is to describe pairs of closed convex curves Γ and γ such that the direction of motion cannot be determined. An example is a pair of concentric circles of radii R and r with $R^2 = r^2 + L^2$. Are there other examples?

The second motivation comes from flotation theory. S. Ulam [6] asked whether the round ball is the only solid of uniform density that floats in equilibrium in all positions. This is an open problem, and one may ask about its 2-dimensional analog (a uniform cylindrical log). Surprisingly, this problem is equivalent to the previous one. This flotation problem has attracted substantial attention; see [1, 2, 3, 8].

The third motivation comes from the theory of billiards. The dual (or outer) billiard is a discrete-time dynamical system outside a smooth closed convex curve γ: two points, x and y, are related by the dual billiard map if xy is tangent to γ and the tangency point bisects the segment xy. Can it be that the dual billiard map has an invariant curve different from the circle such that all the segments xy have the same length, say, $2L$? This problem is equivalent to the previous two. For inner billiards, an analogous problem was studied by Gutkin [4].

A closed convex plane curve Γ is called a *bicycle curve* if two points x and y can move around Γ so that the lengths of the arc xy and of the chord xy remain constant. The ratio ρ of the arc length xy to the perimeter length of Γ is called the *rotation number*. The problem is whether circles are the only bicycle curves. This talk is based on [7].

Let me describe some of the results. There is a functional space of bicycle curves with $\rho = 1/2$. Consider a closed piece-wise smooth curve γ having a total rotation of π and without inflection points; γ has an odd number of cusps. Place a segment of length $2L$ so that it is tangent to γ in its mid-point and move it around γ. The end-points of the segment will trace a closed curve, and for L sufficiently large, this is a bicycle curve with rotation number $1/2$.

There is a number of necessary conditions a bicycle curve must satisfy. Assume that the total perimeter length is 2π. Then every arc of length $2\pi\rho$ must contain a curvature extremum (for ρ arbitrarily small this implies that Γ is a circle). The total number of curvature extrema is not less than 6. If $\rho = 1/3$ or $1/4$ then Γ is a circle.

A bicycle curve Γ is uniquely characterized by the angle $\alpha(x)$ made by the segments xy with Γ (the two angles are equal; x is an arc-length parameter on Γ). This function satisfies the equation

$$\sin\alpha(x+\pi\rho) - \sin\alpha(x-\pi\rho) = L(\alpha'(x+\pi\rho) + \alpha'(x-\pi\rho)),$$

and the study of this equation is an interesting problem on its own right.

Consider infinitesimal deformations of a circle as a bicycle curve with rotation number ρ. One has a mode-locking phenomenon: such a (non-trivial) deformation exists if and only if $n\tan(\pi\rho) = \tan(n\pi\rho)$ for some integer $n \geq 2$. This equation has no solutions for rational ρ.

A discrete version of a bicycle curve is called a *bicycle* (n,k)*-gon*: this is a convex equilateral n-gon whose k-diagonals have equal lengths. The problem again is whether regular n-gons are the only examples.

In some situations, this is indeed the case: bicycle $(n,2), (2n+1,3), (2n+1,n)$ and $(3n,n)$-gons are regular. On the other hand, for even n and odd k, there exist 1-parameter families of (non-regular) bicycle (n,k)-gons.

The problem of infinitesimal deformations of regular n-gons as bicycle (n,k)-gons has the following solution: such a non-trivial deformation exists if and only if

$$\tan\left(kr\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) = \tan\left(k\frac{\pi}{n}\right)\tan\left(r\frac{\pi}{n}\right)$$

for some $2 \leq r \leq n-2$.

This equation is interesting to study on its own right. If n is even and k is odd, one may set $r = n/2$, and this corresponds to the deformations mentioned above. There are other solutions. For example, if $n = 2(k+r)$ and n divides

$(k+1)(r+1)$, then (n, k, r) is a solution (discovered by B. Csikos and communicated by R. Connelly). Do these solutions correspond to actual deformations?

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Normal forms and the Nonlinear Schrödinger approximation to water waves

C. Eugene Wayne
(joint work with G. Schneider)

The study of waves on the surface of an inviscid fluid undergoing irrotational motion has a very long history. In particular, the study of nonlinear effects on periodic wave trains dates back to the work of Stokes in the 1840's [1]. However, the partial differential equations describing this system are so complicated that their solutions are understood only in very special circumstances. The complication of the underlying equations, coupled with the importance of wave phenomena in applications lead to the development of a host of approximate equations that model the motion of such waves. Probably the most famous of the equations is the Kortewg-de Vries (KdV) equation which models the motion of small amplitude, long-wavelength waves. These model equations were typically derived by a formal asymptotic analysis and despite the ubiquity and importance of these equations for a great while there were few attempts to mathematically analyze how good an approximation they actually provided. A major step in their justification was taken by W. Craig [2] who showed that solutions of the water wave problem for certain classes of initial data could be well approximated by the Korteweg-de Vries equation or the Boussinesq equation.

In this talk I described a part of an on-going research program with Guido Schneider to give a mathematical justification for the approximating equations of water waves in a variety of different physical settings. In an earlier work [3] we showed that general long wavelength, small amplitude initial data for the water wave problem can be described by the solutions of a pair of uncoupled KdV equations, one describing the left moving part of the solution and the other describing the right moving part.

In the present work [4] we are investigating the approximation of slowly varying periodic wave trains. In 1968, Zakharov argued that if one considered the modulation of a small amplitude wave train with wave number k_0 and frequency $\omega(k_0)$, the surface elevation, η, should be approximated by

$$\eta(x,t) = \epsilon A(\epsilon[x + c_g t], \epsilon^2 t) e^{i[k_0 x + \omega(k_0) t]} + \text{complex conjugate}$$

where the amplitude function $A(X,T)$ is a solution of the nonlinear Schrödinger equation (NLS)

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2 \ .$$

The goal of our present research is to show that the Zakharov approximation does give an accurate approximation to the water wave problem over the very long time scale of $O(\epsilon^{-2})$ which the formal calculation indicates is appropriate.

In this talk I focused on one particular aspect of this approximation problem, the construction of a normal form for the partial differential equation that describes how the difference between the true solution and the NLS approximation evolves.

Denotes the true solution of the water wave problem by $v(x,t)$ and the NLS approximation by $\epsilon\psi_{NLS}$. We formally write the water wave problem as

$$\partial_t v = \Lambda v + B(v,v) + \ldots$$

In our representation of the water wave problem v is a vector valued function with three components, Λ is a diagonal operator whose eigenvalues are given in the Fourier transform representation by 0, $-\omega(k)$ and $\omega(k)$, where

$$\omega(k) = i \ \mathrm{sgn}(k)\sqrt{k \tanh(k)} \ .$$

If we now write the true solution as the sum of approximation $\epsilon\psi_{NLS}$ given by the NLS equation, plus a correction $\epsilon^\beta R$, i.e., if we write $v = \epsilon\psi_{NLS} + \epsilon^\beta R$, with $\beta \geq 2$, then the "correction" term R satisfies the equation

$$\partial_t R = \Lambda R + \epsilon(B(\psi_{NLS}, R)) + \epsilon^\beta B(R,R) + \epsilon^{-\beta} Res(\epsilon_{NLS}) \ .$$

(For simplicity we have assumed that the bilinear term B is symmetric in its arguments.) The "residue" $\epsilon^{-\beta} Res(\epsilon_{NLS})$ describes the amount by which the formal approximation fails to satisfy the original equation at any given instant of time, and by modifying the original approximation by terms of $O(\epsilon^2)$ or higher, one can make this term arbitrarily small without changing the fact that the leading order approximation is given by the solution of the NLS equation. The nonlinear term can be controlled by the use of Gronwall's inequality. From the explicit form of the operator Λ that appears in the water wave problem we know that it generates a uniformly bounded semi-group. Thus, if it were not for the presence of the terms $\epsilon B(\psi_{NLS}, R)$ in the equation for R it would be straightforward to show that solutions of this equation remained of order one for the long times ($O(\epsilon^{-2})$) of interest in this problem and thus that the true solution v is given by the NLS approximation plus higher order corrections.

Our goal is to remove these terms by means of a normal form transformation. In the water wave problem the Fourier transform of the bilinear terms can all be written as integrals of the form

$$\hat{B}(\psi_{NLS}, R)(k) = \int \hat{b}(k, k-\ell, \ell)\hat{\psi}_{NLS}(k-\ell)R(\ell)d\ell \ ,$$

and thus we look for a normal form transformation of the form $R = w + \epsilon M(\psi_{NLS} w)$, where

$$\hat{M}(\psi_{NLS}, R)(k) = \int \hat{m}(k, k-\ell, \ell)\hat{\psi}_{NLS}(k-\ell)R(\ell)d\ell \ .$$

If we now use the equation for $\partial_t R$ to derive the evolution equation for w, we find that all the terms of $O(\epsilon)$ vanish if we choose the kernel function $\hat{m}$ to satisfy

$$\hat{m}(k, k-\ell, \ell) = \frac{2b(k, k-\ell, \ell)}{i(\omega(k) - \omega(k-\ell) - \omega(\ell))} \ .$$

From an analytic point of view the difficulty now arises in determining whether or not the denominator of this expression vanishes, given the form of $\omega(k)$ that occurs in the water wave problem. At first sight it appears that since k and ℓ can range over the entire real line there will inevitably be zeros in the denominator, however, by taking advantage of special features of the problem, such as the exact formulas for the numerator $\hat{b}$ (which can cancel zeros in the denominator if it happens to vanish for the same values of k and ℓ) and by using the fact that $\hat{\psi}_{NLS}$ is strongly localized in Fourier space, which further restricts the values of k and ℓ that one must consider, we have shown that one generates a well defined and bounded normal form transformation on certain spaces of analytic functions.

One must now show that one can solve the resulting partial differential equations for the transformed variables w in these function spaces over the long time scales of relevance for this problem but we believe that can also be done and will yield a proof of the validity of Zakharov's approximation of such modulated, periodic wave trains by the solution of the nonlinear Schrodinger equation. This in turn implies that all the phenomena known to occur in the NLS equation can also be seen, at least approximately, in the dynamics of water waves.

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Noncontractible periodic orbits in cotangent bundles

Joa Weber

We report on results obtained in [6]. Let M be a closed connected smooth manifold of finite dimension.

Its cotangent bundle T^*M carries the canonical symplectic form $\Omega = -d\theta$, where θ denotes the Liouville form. A function $H \in C^\infty(S^1 \times T^*M, \mathbb{R})$ is called a *Hamiltonian.* Throughout S^1 is identified with $\mathbb{R}/\mathbb{Z}$ and we think of H as a smooth function on $\mathbb{R} \times T^*M$, $(t,q,p) \mapsto H(t,q,p) =: H_t(q,p)$, satisfying $H_{t+1} = H_t$. The *Hamiltonian vector field* X_{H_t} is determined by the identity $dH_t(\cdot) = \Omega(X_{H_t}, \cdot)$. Our main object of interest is the set of 1-periodic orbits

$$\mathcal{P}(H) := \{z \in C^\infty(S^1, T^*M) \mid \dot{z}(t) = X_{H_t}(z(t)), \forall t \in S^1\}.$$

It coincides with the set of critical points of the *symplectic action functional* $\mathcal{A}_H(x,y) = \int_0^1 \langle y(t), \dot{x}(t)\rangle - H(t,x(t),y(t))\, dt$. Here x is a smooth loop in M and $t \mapsto y(t) \in T^*_{x(t)}M$ depends smoothly on $t \in S^1$.

Let us choose, in addition, a Riemannian metric on M and denote by DT^*M the open unit disk cotangent bundle. Given a homotopy class α of free loops in M, let ℓ_α be the smallest length of periodic geodesics representing α.

Theorem 1. *Every compactly supported $H \in C^\infty(S^1 \times DT^*M, \mathbb{R})$ which satisfies*

$$\sup_{S^1 \times M} H =: -c \leq -\ell_\alpha$$

admits a 1-periodic orbit z whose projection to M represents α and $\mathcal{A}_H(z) \geq c$.

In comparison to the contractible case the search for noncontractible periodic orbits has a short history. First steps have been taken by Gatien-Lalonde [3] and by Biran-Polterovich-Salamon [2] which both impose rather strong conditions on the Riemannian manifold. For instance, Theorem 1 is proved in [2] in case that M is either the euclidean torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ or of negative sectional curvature.

Theorem 1 holds with S^1 replaced by the interval $[0,1]$. In the noncontractible case the inequality in Theorem 1 is sharp. The zero section $\mathcal{O}_M = M$ in the inequality cannot be replaced by an arbitrary smooth section. (See [6].)

Idea of proof. It suffices to show nontriviality of action filtered Floer homology $\mathrm{HF}_*^{(c,\infty)}(H;\alpha)$, because its chain groups are generated by $\mathcal{P}^{(c,\infty)}(H;\alpha)$, the set of 1-periodic orbits of action strictly larger than c whose projections to M represent α. Here we need to assume that c is not element of the *action spectrum* $\mathcal{S}(H;\alpha) := \mathcal{A}_H(\mathcal{P}^{(-\infty,\infty)}(H;\alpha))$. A limit argument takes care of the other case. Throughout all homologies will be with $\mathbb{Z}_2$-coefficients.

A standard method to prove nontriviality of Floer homology is to *prepare a Hamiltonian sandwich*: construct functions $f \leq H \leq h$ whose Floer homology is known to be nontrivial and such that the continuation homomorphism

$$\sigma^{hf} : \mathrm{HF}_*^{(c,\infty)}(f;\alpha) \to \mathrm{HF}_*^{(c,\infty)}(h;\alpha),$$

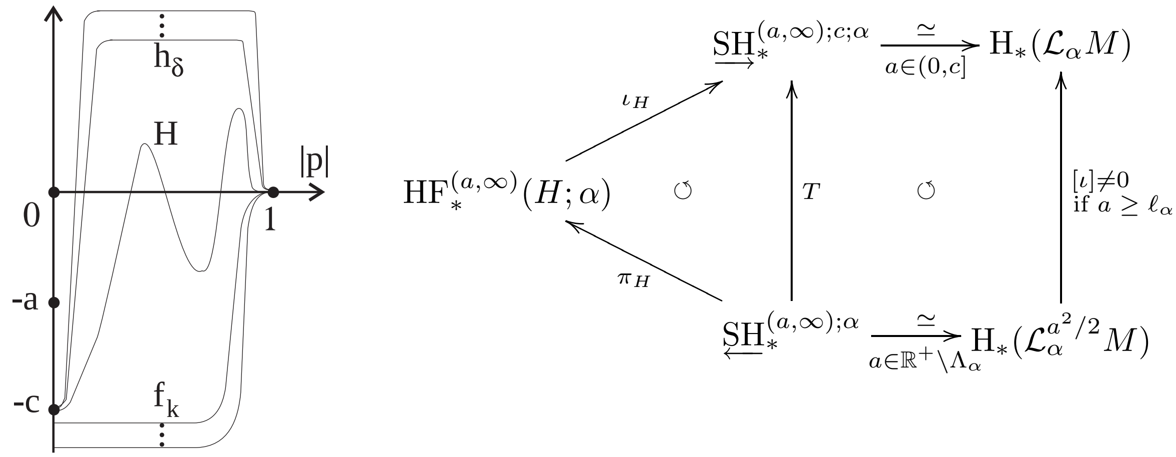

Figure 1. Idea of proof of Theorem 1

which is associated to a homotopy F_s from f to h satisfying the *monotonicity condition* $\partial_s F_s \geq 0$, is nonzero. These so called *monotone homomorphisms* preserve action windows, are independent of the choice of the monotone homotopy and satisfy $\sigma^{hf} = \sigma^{hH}\sigma^{Hf}$. This implies nontriviality. For later use let us call a homotopy F_s *action-regular* if the boundary c of the action window is – throughout the homotopy – a regular value of $\mathcal{A}_{F_s}$ (restricted to the appropriate component of the free loop space).

Since f and h cannot take care of all possible Hamiltonians H at once, we construct sequences f_k and h_δ as indicated in Figure 1. Calculation of their Floer homologies relies on the fact that continuation homomorphisms associated to action-regular homotopies are isomorphisms. The main point is then to construct action-regular homotopies from f_k and h_δ, respectively, to *convex radial* Hamiltonians (smooth symmetric convex functions of $|p|$). The reason is that we can extend the main result of [4] (see also [5] and [1]), namely

$$\mathrm{HF}_*^{(-\infty,a)}(\tfrac{1}{2}|p|^2;\alpha) \simeq \mathrm{H}_*(\mathcal{L}_\alpha^a M),$$

to general convex radial Hamiltonians. Here the right hand side denotes singular homology of a sublevel set of the free loop space component $\mathcal{L}_\alpha M$ with respect to the classical action $\mathcal{S}_0(x) = \frac{1}{2}\int_0^1 |\dot{x}(t)|^2\, dt$.

Convenient tools for book keeping all these Floer homologies simultaneously are symplectic homology $\underleftarrow{\mathrm{SH}}$ of DT^*M and relative symplectic homology $\underrightarrow{\mathrm{SH}}$ of (DT^*M, M), respectively. The result of our calculation is presented by the commutative diagram in Figure 1. Here ι denotes the natural inclusion of a sublevel set and Λ_α is the set of lengths of all periodic geodesics representing α. The monotone homomorphisms descend to a homomorphism T and $a \in (\ell_\alpha, c] \setminus \Lambda_\alpha$ implies $T \neq 0$. Hence $\mathrm{HF}_*^{(a,\infty)}(H;\alpha) \neq 0$ for every element a of the open and dense subset $(\ell_\alpha, c] \setminus (\Lambda_\alpha \cup \mathcal{S}(H;\alpha))$ of $(\ell_\alpha, c]$. $\square$

A consequence of Theorem 1 is that the *relative Biran-Polterovich-Salamon capacity* $c_{BPS}(DT^*M, M;\alpha)$ equals ℓ_α. In the terminology of [2] this means that *every* $\alpha \neq 0$ is symplectically essential. Therefore we obtain the following two *multiplicity results* (both are proved in [2] for symplectically essential α).

Theorem 2. *Let $H : T^*M \to \mathbb{R}$ be a smooth Hamiltonian which is proper and bounded from below. Suppose that the sublevel set $\{H < c\}$ contains M. Then for every $\alpha \neq 0$ there exists a dense subset $S_\alpha \subset (c, \infty)$ such that the following is true. For every $s \in S_\alpha$, the level set $\{H = s\}$ contains a periodic orbit $z = (x, y)$ of H with $[x] = \alpha$ and $\int_0^1 \langle y(t), \dot{x}(t)\rangle dt > 0$.*

Note that the period of the orbit in the previous theorem is not specified. Moreover, the theorem is not true in case $\alpha = 0$ as the example of the euclidean torus and the Hamiltonian $H(x, y) = \frac{1}{2}|y|^2$ shows.

The following corollary of Theorem 2 is related to *Weinstein's conjecture*: given a symplectic manifold (N, ω) and any compact hypersurface $Q \subset N$ of contact type, then the characteristic foliation of Q has a closed leaf.

Recall that Q is of *contact type* if there exists a smooth vector field Z with $\mathcal{L}_Z\omega = \omega$, defined on a neighbourhood U of Q and pointing outward along Q. The characteristic line bundle is given by $\mathcal{L}_Q := \ker\,(\omega_{can}\ |_{TQ})$. The associated foliation is called *characteristic foliation* and its leaves *characteristics*. The Reeb vector field is a nonvanishing section of $\mathcal{L}_Q$ inducing its *canonical orientation*, thereby orienting the characteristics.

Theorem 3. *Let $M \subset W \subset T^*M$ be an open set with compact closure and smooth boundary $Q = \partial\overline{W}$ of contact type. Let the characteristic line bundle $\mathcal{L}_Q$ be equipped with its canonical orientation. Then for every $\alpha \neq 0$ the characteristic foliation of Q has a closed leaf $z \subset Q$ with $j_\#[z] = \alpha$, where $j : Q \hookrightarrow T^*M \to M$ is the composition of inclusion and projection.*

We refer to [6] for references concerning the history of Theorem 3.

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Holomorphic foliations and 3-dimensional Reeb dynamics

Chris Wendl

Many interesting results in the dynamics of Reeb vector fields on contact 3-manifolds arise from the following construction: Consider, for some 3-manifold M with contact form λ, a 2-dimensional foliation of $\mathbb{R} \times M$ such that every leaf is the image of some embedded pseudoholomorphic curve of finite energy, and the foliation is invariant under the natural $\mathbb{R}$-action on $\mathbb{R} \times M$. Such "finite energy foliations" have been constructed for generic contact forms on the tight 3-sphere by Hofer, Wysocki and Zehnder [2]. Due to the intimate connection between holomorphic curves and Reeb orbits, it follows for example that generic Reeb vector fields in this setting admit exactly either two or infinitely many periodic orbits. Holomorphic foliations derived from Giroux's open book decompositions of contact 3-manifolds have also been used by Abbas, Cieliebak and Hofer [1] to prove the Weinstein conjecture for planar contact structures.

We discuss a program for extending these existence results to a general theory in which embedded holomorphic curves of low Fredholm index in 3-manifolds are analyzed via the foliations that they generate. These constructions exhibit a wealth of powerful stability and compactness phenomena, which can be summarized by the motto, "if holomorphic curves are everywhere, it's hard to kill them." Such phenomena have led Hofer to suggest using existing foliations to create new ones via a homotopy and stretching argument, which leads to the notion of a *concordance*: a holomorphic foliation of a cylindrical symplectic cobordism that interpolates between $\mathbb{R}$-invariant foliations for two distinct contact forms. In principle, this can be used to define an equivalence relation for holomorphic foliations, and it is then interesting to ask: given M and a contact structure ξ, what is the set of all concordance classes of foliations on (M, ξ)? One can regard this problem as a distinctly 3-dimensional version of Symplectic Field Theory, with Floer-type algebras used to distinguish concordance classes of foliations; a rough outline of this theory is given in [3], and the analytical foundations are currently being developed [4]. As an application, we conjecture that there is only one concordance class of foliations on the tight 3-sphere, but more than one for a certain overtwisted contact structure on $S^1 \times S^2$. The answers to such questions seem to be related to the topology of the underlying manifold, and may also yield insights into Reeb dynamics.

As a step toward the development of this program, we prove the existence of finite energy foliations for every overtwisted contact structure. This is another example of the motto mentioned above: in this case, one starts with a foliation on S^3, cuts out certain pieces to form a foliation with totally real boundary conditions, then performs surgery on the region that has been removed. The foliation with boundary converges to a new foliation under a twisting process, and compactness in this setting arises from a uniquely 3-dimensional argument, using the topological constraints imposed by the existing foliation and its associated Reeb dynamics. This result is also one step in a program suggested by Hofer for proving the Weinstein conjecture in dimension three. Details of the argument and an outline of the wider program may be found in [3].

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Nonequilibrium steady-states for Hamiltonian and stochastic systems

Lai-Sang Young

(joint work with Jean-Pierre Eckmann)

The material of this talk is based on [1].

As models of generalized conduction or transport in one dimension, we consider chains of dynamical systems in which transfer of energy is mediated by tracer particles. Coupling the two ends of the chain to unequal tracer-heat reservoirs and allowing the system to settle down to a nonequilibrium steady state, we seek to derive macroscopic conduction laws (such as the distribution of energy, heat flux, and tracer flux) from the microscopic dynamics.

We introduce a class of models that can be seen as an abstraction of certain types of mechanical models. These models are simple enough to be amenable to analysis, and complex enough to have fairly rich dynamics. They have in common the following basic set of characteristics: Each model is made up of an array of identical cells that are linearly ordered. Energy is carried by two types of agents: storage receptacles (called "tanks") that are fixed in place, and tracer particles that move about. Direct energy exchange is permitted only between tracers and tanks. The two ends of the chain are coupled to infinite reservoirs that emit tracer particles at characteristic rates and characteristic temperatures; they also absorb those tracers that reach them. To allow for a broad range of examples, we do not specify the rules of interaction between tracers and tanks. All the rules considered in this paper have a Hamiltonian character, involving the kinetic energy of tracers. Formally they may be stochastic or purely dynamical, resulting in what we will refer to as *stochastic* and *Hamiltonian models*.

Via the models in this class, we seek to clarify the relation among several aspects of conduction, including the role of conservation laws, their relation to the dynamics within individual cells, and the notion of "local temperature". We propose a simple recipe for deducing various macroscopic profiles from local rules. Our recipe is generic; it does not depend on specific characteristics of the system. When the local rules are sufficiently simple, it produces explicit formulas that depend on exactly 4 parameters: the temperatures and rates of tracer injection at the left and right ends of the chain.

For demonstration purposes, we carry out this proposed program for two examples, one stochastic and the other Hamiltonian. Our main stochastic example, dubbed the "random-halves model", is particularly simple: A clock rings with rate proportional to $\sqrt{x}$ where x is the (kinetic) energy of the tracer; at the clock, energy exchange between tracer and tank takes place; and the rule of exchange consists simply of pooling the two energies together and randomly dividing – in an unbiased way – the total energy into two parts. Our main Hamiltonian example is a variant of the model studied in [2]. Here the role of the "tank" is played by a rotating disk nailed down at its center, and stored energy is ω^2 where ω is the angular velocity of the disk. Explicit formulas for the profiles in question are correctly predicted in all examples.

In terms of methodology, this paper has a theory part and a simulations part. The theory part is rigorous in the sense that all points that are not proven are isolated and stated explicitly as "assumptions" (see the next paragraph). It also serves to elucidate the relation between various concepts regardless of the extent to which the assumed properties hold. Simulations are used to verify these properties for the models considered.

Our main assumption is in the direction of local thermodynamic equilibrium, a phenomenon widely accepted in physics. For our stochastic models, a proof of this property seems within reach though technically involved; no known techniques are available for Hamiltonian systems. Two extra assumptions are needed to make the Hamiltonian study viable. The first is ergodicity; it is easy to "improve ergodicity" via model design, harder to mathematically eliminate the possibility of all (small) invariant regions. The second is the near perfect mixing within cells. Our prediction of energy profiles etc. are for the idealized limit when such mixing is perfect, i.e. before a tracer exits a site the loss of memory is complete.

Under these assumptions, explicit formulas for stored energy and tracer density profiles along the chain are predicted, as are energy and tracers (mass) transported per unit time. Agreement between results of simulations and predictions is excellent.

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