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Differentialgeometrie im Großen

Organised by
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Abstract. The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. While global Riemannian geometry with its connections to geometric analysis and topology remained an important focus of the conference, this time special emphasis was given to complex and symplectic geometry.

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Introduction by the Organisers

The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. While global Riemannian geometry with its connections to geometric analysis and topology remained an important focus of the conference, this time special emphasis was given to complex and symplectic geometry.

There were 50 participants from 8 countries, more specifically, 22 participants from Germany, 9 from other European countries, 1 from China, 19 from North-America. 10% of the participants were women. More than half of the participants (about 29) were young researchers (less than 10 years after diploma or B.A.), both on doctoral and postdoctoral level.

The official scientific program consisted of 22 one hour talks leaving plenty of time for informal discussions.

The organizers would like to thank the institute staff for their great hospitality and support before and during the conference.

Workshop: Differentialgeometrie im Großen

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Abstracts

A flow for real hypersurfaces in unimodular complex surfaces

Robert L. Bryant

A *unimodular complex surface* (X, Υ) is a complex 2-manifold X endowed with a holomorphic volume form Υ. A local coordinate chart $(w, z) : U(\subseteq X) \to \mathbb{C}^2$ is *unimodular* if $U^*\Upsilon = \mathrm{d}w{\wedge}\mathrm{d}z$.

Let $M \subset X$ be a real hypersurface that is CR-nondegenerate (i.e., if $D = TM \cap \mathrm{i}(TM) = \mathrm{i}(D)$, then D is a contact 2-plane field on M). For any point $p \in M$, there exist local unimodular coordinates $(w, z) : U \to \mathbb{C}^2$ centered on p such that, in U, the hypersurface M is defined by a relation of the form $\mathrm{Im}(w) = F\big(z, \mathrm{Re}(w)\big)$ where F is a real function vanishing to order at least 2 at $\big(z, \mathrm{Re}(w)\big) = (0, 0)$ and satisfying $F_{z\bar{z}}(0, 0) > 0$.

I first show that there is a unique pair of 1-forms $\theta = \bar{\theta}$ and η on M that satisfy $M^*\Upsilon = \theta{\wedge}\eta$ and $\mathrm{d}\theta = \mathrm{i}\,\eta{\wedge}\bar{\eta}$. This coframing (θ, η) depends on 3 derivatives of a local defining function F. I also show that there exist functions $a = \bar{a}$ and b on M such that $\mathrm{d}\eta = 2\mathrm{i}\,\theta{\wedge}(a\,\eta + b\bar{\eta})$.

These functions a and b are the lowest order scalar invariants of M under the unimodular biholomorphism group and depend on 4 derivatives of the defining function F. In fact, given $p \in M$, one can always find local unimodular coordinates $(w, z) : U \to \mathbb{C}^2$ centered on p such that, in U, the hypersurface M is defined by a relation of the form

$$\mathrm{Im}(w) = \tfrac{1}{2} z\bar{z}\left(1 + b(p)\, z^2 + \tfrac{3}{2} a(p)\, z\bar{z} + \overline{b(p)}\, \bar{z}^2\right) + R_5\big(z, \mathrm{Re}(w)\big)$$

where R_5 is a real function vanishing to order at least 5 at $\big(z, \mathrm{Re}(w)\big) = (0, 0)$. Finally, I remark that the quantities (θ, η, a, b) give a complete set (in Cartan's sense) of invariants of M under unimodular biholomorphisms. All other finite-order smooth pointwise invariants are expressed in terms of a and b and their derivatives with respect to the coframing (θ, η). For example, the condition for CR-flatness of M is the vanishing of the 6th order expression $s = a_{\bar{\eta}\bar{\eta}} + 2\mathrm{i}\, b_\theta + 6ab$.

I give various interpretations of the invariants a and b. For example, the equation $a = 0$ is the Euler-Lagrange equation for variations of the integral of the volume form $\theta{\wedge}\mathrm{d}\theta$.

I discuss the locally homogeneous examples, i.e., the cases in which a and b are constant, as well as the cohomogeneity 1 examples.

I then define a geometrically natural normal vector to M. Let R be the Reeb vector field of θ, i.e., $\theta(R) = 1$ and $\eta(R) = 0$. Then $\mathrm{i}R$ is a vector field along M that is nowhere tangent to M. This canonical normal (which is not invariant under the full biholomorphism pseudo-group) then allows one to define a third-order flow for CR-nondegenerate hypersurfaces M.

I illustrate various properties of this flow, showing that it can collapse a hypersurface to a point or a surface in finite time by considering the homogeneous examples, and showing that it reduces to known flows for curves in the cohomogeneity one case. I also show that, after reparametrizing the flow in a noncanonical way, one can replace the flow by a second order, weakly parabolic flow.

I point out that short time existence for the flow is not known, nor is it understood what sorts of singularities can develop in finite time in general. Much work remains to be done.

For details and references, see [1].

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Tropical curves, surfaces and beyond

Grigory Mikhalkin

The talk was devoted to the geometric structures on certain piecewise-linear polyhedral complexes that appear in tropical geometry. This structure can be thought of as the extension of the so-called integer affine structure, however it can be defined not only for smooth manifolds, but also for polyhedral complexes.

Recall that *the integer affine structure* on a smooth k-manifold X is the atlas $\{U_j, \phi_j : U_j \to \mathbb{R}^k\}$, such that $\{U_j\}$ is an open covering of X, ϕ_j is an embedding and each overlapping map $\phi_j \circ \phi_l^{-1}$ is an *integer affine map*, i.e. is a restriction of the composition of a linear map over $\mathbb{Z}$ and a translation in $\mathbb{R}^k$.

A similar structure exists also in the case when X is a polyhedral complex of pure dimension k. In this case the maps ϕ_j embed neighborhoods U_j to $\mathbb{R}^{n_j}$, with $n_j \geq k$ so that the image $\phi_j(U_j)$ is contained in the so-called balanced piecewise-linear complex of pure dimension k in $\mathbb{R}^{n_j}$ (see [3], [4], [5], [6] for a discussion of balanced (or zero-tension) piecewise-linear polyhedral complexes).

In the case $k = 1$ (i.e. if X is *a tropical curve*) this structure can be reformulated in much more familiar terms – it amounts to be the same as the inner metric on a finite graph X provided that X has no 1-valent vertices.[1] It is a much more delicate structure in the case of higher dimension, see e.g. [1] or [2] for a discussion of tropical K3-surfaces.[2]

One way to approach higher-dimensional tropical varieties is via the tropical branched coverings of $\mathbb{R}^k$, see Figure 1 for the case of quadric and cubic surfaces presented as the double branched coverings of $\mathbb{R}^2$. The solid lines depict the branching curves. The thin edges are of weight 1 and separate 1-sheeted and 2-sheeted parts of the covering. The thick edges are of weight 2 and are projections of two distinct edges on the surfaces (note that they have the same projection

[1] For simplicity in this abstract we do not consider the general "log-tropical" case when X is allowed to have 1-valent vertices at "the boundary" and the polyhedral complexes have to be considered in $\mathbb{T}^{n_j} = (\mathbb{R} \cup \{-\infty\})^{n_j}$ instead.

[2] Note that these papers consider the relevant *singular* tropical structure which allows X to be still a manifold topologically.

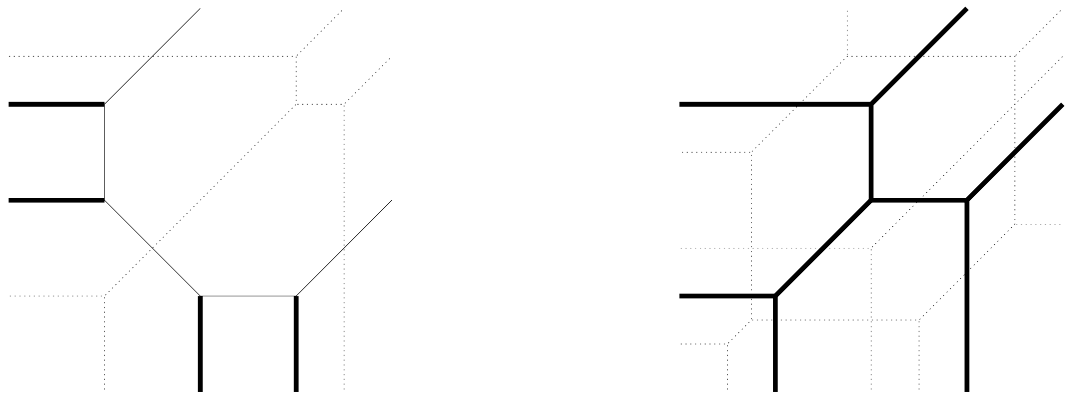

FIGURE 1. Tropical quadric and cubic surfaces as double branched coverings of the plane.

image in $\mathbb{R}^2$). The covering is 2-sheeted both sides of the thick edges. The inverse image of a point on a dotted edge contains a half-infinite ray. The right-hand side of Figure 1 can be used to verify that a smooth tropical cubic surface has to contain 27 distinct tropical lines.

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Contractible surfaces and Stein exotica

IVAN SMITH

(joint work with Paul Seidel)

Introduction: This talk concerned the symplectic geometry of Stein manifolds, a subject initiated by Eliashberg and Gromov in [3] and [4]. A finite type complete Stein manifold is also an exact symplectic manifold, with the symplectic structure being canonically defined up to symplectomorphism; the natural deformation equivalences for such Stein manifolds are symplectomorphisms; and moreover amongst non-compact symplectic manifolds this class is particularly well-behaved. They are also ubiquitous; on the one hand Eliashberg has given an essentially topological characterisation of which open manifolds are Stein, and on the other the complements of ample divisors in closed Kähler manifolds give rise to Stein manifolds (which are finite type, and can be canonically completed). In particular, there is an interesting circle of questions concerning the symplectic geometry of affine algebraic varieties.

Our work is intended to illustrate this (largely unexplored) area by studying from a symplectic viewpoint the first ever example of a contractible affine variety which is not complex isomorphic to Euclidean space. The example, due to Ramanujam [7], is surprisingly simple. Inside $\mathbb{CP}^2$ take a singular curve given by the union of a smooth conic C, and a cuspidal cubic C', meeting at two points (both smooth on the cubic), one a transverse intersection and the other a 5-fold tangency. Blow up the transverse intersection point and remove from the resulting surface $\mathbb{F}_1$ the union $\tilde{D} = \tilde{C} \cup \tilde{C}'$ of the proper transforms of the conic and cubic; Ramanujam showed that the complement $M = \mathbb{F}_1 \backslash \tilde{D}$ is contractible. Since the surface has log general type, it cannot be algebraically isomorphic to $\mathbb{C}^2$; indeed, it is not simply connected at infinity. The product M^m, with $m > 1$, is diffeomorphic to $\mathbb{R}^{4m}$ but still of log general type, hence still as an affine algebraic variety is distinct from Euclidean space. The main result of [8] is then:

Theorem: *With its canonical symplectic structure, and for any $m > 0$, M^m cannot be symplectically embedded in any subcritical Stein manifold.*

A (complex) n-dimensional Stein manifold is subcritical if it admits a plurisubharmonic exhausting function with all critical points of index at most $n-1$. In particular, M^m does not embed in $\mathbb{R}^{4m}$, and hence we construct exotic symplectic structures on Euclidean space (the first known examples which are conical at infinity). A consequence, answering a question of Eliashberg from [4], is that there are Stein manifolds diffeomorphic to the ball but with no *plurisubharmonic* Morse function having a single critical point.

Remarks on the proof: It is not hard to give a sketch of the proof of the theorem. The basic idea goes back to work of Biran and Cieliebak [1] together with deep results of Chekanov [2]; the latter shows that if a Lagrangian submanifold bounds no holomorphic discs, then it can't be displaced from itself by any Hamiltonian isotopy, whilst the former showed that in a subcritical Stein manifold all Lagrangian submanifolds can be displaced. It would therefore be sufficient to exhibit a Lagrangian submanifold in M which bounds no holomorphic discs. In fact, we prove something slightly weaker, and strengthen the results of Biran-Cieliebak accordingly; we show that there is a Lagrangian submanifold L in $M = M_0$ which, for any real $E > 0$, can be deformed through an exact Lagrangian isotopy $L_t \subset M_t$ inside a Stein deformation of M, so that at time $t = 1$ all non-constant holomorphic discs lying on L_1 have energy at least E. Such a submanifold we call "Stein-essential". A series of technical lemmas shows that the presence of a Stein-essential Lagrangian submanifold is indeed enough to show that M cannot be embedded into a subcritical Stein manifold.

The key object is, therefore, a certain Lagrangian torus $L \subset M$, which we take to lie in a neighbourhood of the cusp point of the curve $\tilde{C}'$ at infinity. Recall that a cusp singularity is the cone on a trefoil knot κ, and L is given by taking a tubular neighbourhood of this trefoil (so is made up of a meridian of the knot, together with an arbitrary choice of longitude). This naturally lies inside M, and is in fact Lagrangian. For later use, recall that Dehn's Lemma asserts that the inclusion of this torus into the knot complement $S^3 \backslash \kappa$ induces an injection on fundamental groups. There is an obvious isotopy $\{L_t\}$ of this torus which shrinks it back into the cusp point itself (collapsing the torus to a point, so at time $t = 1$ the isotopy becomes singular). Suppose that at all times $t < 1$ in this isotopy, we have some non-trivial holomorphic disc $u_t : D \to M$ with boundary on L_t, with the discs having energy bounded by E. Then by Gromov compactness we would expect that there is a limiting object (stable disc) $\overline{u}$ at time $t = 1$. Any closed component of this $\overline{u}$ would be a curve in $\mathbb{F}_1$ lying in the complement of the *ample* curve $\tilde{C}$, which is absurd, whilst if the limiting map is constant, then for t very close to 1 the disc $u_t(D)$ lies inside a neighbourhood of the cusp, which given Dehn's Lemma contradicts Stokes' theorem (we would have a holomorphic disc with boundary a contractible loop on L). With this contradiction in hand, we hope to conclude that in fact L is Stein-essential, since no such family of discs could have existed.

The technical problem here is that since the total space of the isotopy $\cup_{t\in[0,1]} L_t \subset \mathbb{F}_1 \times \mathbb{C}$ is singular, Gromov compactness does not obviously apply. This issue is avoided by exchanging the complication of the family of Lagrangians inside the trivial degeneration $\mathbb{F}_1 \times \mathbb{C}$ by a Lagrangian which is obviously smooth but which lives inside a more complicated degeneration. By blowing up the cusp and the 5-fold tangency point repeatedly, we obtain a different compactification $\hat{\mathbb{F}}_1$ of M, with a normal crossing divisor $\Delta = \hat{\mathbb{F}}_1 \backslash M$ at infinity; the torus L now lies near a crossing point p of Δ. We study the topologically non-trivial degeneration $\mathcal{X} \to \mathbb{C}$ with generic fibre $\hat{\mathbb{F}}_1$ but with special fibre at zero having an additional component coming from blowing up $\hat{\mathbb{F}}_1 \times \mathbb{C}$ at $(p, 0)$. For suitable Kähler forms, this degeneration contains a Lagrangian $L \times [0, 1]$ which in the zero fibre meets the exceptional component $\mathbb{P}^2$ in a Clifford torus (so on blowing everything back down, we recover the naive situation studied above). Gromov compactness applies to a family of holomorphic discs with boundary in this new Lagrangian, and with suitable changes to take account of the new framework (for instance the ample divisor $\tilde{C}$ is replaced by an appropriate nef divisor in $\hat{\mathbb{F}}_1$), the previous argument can still be applied to achieve the required contradiction.

Speculation: There are other directions in which one can develop even this example, some of which are addressed in [9] (in progress). For instance, existence of a Stein-essential Lagrangian submanifold already implies a nonvanishing result for Viterbo's symplectic cohomology as defined in [10], that is $SH^*(M^n) \neq 0$ (cf. [6]). The full computation of SH^* seems much harder, and indeed unpublished work of Seidel indicates that such computations cannot in general be made algorithmic. That $SH^*(M^m) \neq 0$ can in turn be used to show that the naturally induced (homotopically standard) contact structure on $S^7 = \partial_\infty(M \times M)$ is exotic; such examples were first constructed, by quite different means, by Geiges [5]. It is tempting to believe that the boundary connect sums of M^n with itself are pairwise distinct, and that there are countably many finite type exotic Stein structures on higher-dimensional balls (but this is still work in progress and beliefs may change!). More generally, one might envisage in the symplectic study of Stein manifolds a role for the contractible affine varieties somewhat analogous to the role played by exotic spheres in the classical differential topology of closed manifolds. Whether such an analogy is really helpful, of course, remains to be seen.

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Ricci curvature for metric-measure spaces

John Lott
(joint work with Cédric Villani)

We describe results obtained with Cédric Villani [1, 2]. Related results were obtained independently by Karl-Theodor Sturm [3].

We use optimal transport and displacement convexity in order to define a notion of a measured length space (X, d, ν) having Ricci curvature bounded below. For simplicity, we focus on the case of nonnegative Ricci curvature and assume that the relevant length space X is compact. If $N \in [1, \infty]$ is a parameter (playing the role of a dimension) then we will define a notion of (X, d, ν) having nonnegative N-Ricci curvature.

Let $P(X)$ denote the space of Borel probability measures on X, equipped with the 2-Wasserstein metric. Then $P(X)$ is also a length space.

Let $U : [0, \infty) \to \mathbf{R}$ be a continuous convex function with $U(0) = 0$. Given a reference probability measure $\nu \in P(X)$, define the function $U_\nu : P(X) \to \mathbf{R} \cup \{\infty\}$

by

$$U_\nu(\mu) = \int_X U(\rho(x))\, d\nu(x) + U'(\infty)\, \mu_s(X),$$

where

$$\mu = \rho\nu + \mu_s$$

is the Lebesgue decomposition of μ with respect to ν into an absolutely continuous part $\rho\nu$ and a singular part μ_s.

If $N \in [1, \infty)$ then we define DC_N to be the set of such functions U so that the function

$$\psi(\lambda) = \lambda^N U(\lambda^{-N})$$

is convex on $(0, \infty)$. We further define DC_∞ to be the set of such functions U so that the function

$$\psi(\lambda) = e^\lambda U(e^{-\lambda})$$

is convex on $(-\infty, \infty)$. A relevant example of an element of DC_N is given by

$$U_N(r) \;=\; \begin{cases} Nr(1 - r^{-1/N}) & \text{if } N < \infty, \\ r \log r & \text{if } N = \infty. \end{cases}$$

Definition 1. *Given $N \in [1, \infty]$, we say that a compact measured length space (X, d, ν) has nonnegative N-Ricci curvature if for all $\mu_0, \mu_1 \in P(X)$ with $supp(\mu_0) \subset supp(\nu)$ and $supp(\mu_1) \subset supp(\nu)$, there is* some *Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1 so that for all $U \in DC_N$ and all $t \in [0, 1]$,*

$$U_\nu(\mu_t) \;\le\; t\, U_\nu(\mu_1) \;+\; (1 - t)\, U_\nu(\mu_0).$$

Note that the inequality is only assumed to hold along *some* Wasserstein geodesic from μ_0 to μ_1, and not necessarily along all such geodesics. This is what we call *weak displacement convexity.*

Naturally, one wants to know that in the case of a Riemannian manifold, our definition is equivalent to the classical one. Let M be a smooth compact connected n-dimensional manifold with Riemannian metric g. We let (M, g) denote the corresponding metric space. Given $\Psi \in C^\infty(M)$ with $\int_M e^{-\Psi}\, dvol_M \;=\; 1$, put $d\nu \;=\; e^{-\Psi}\, dvol_M$.

Definition 2. *For $N \in [1, \infty]$, let the N-Ricci tensor Ric_N of (M, g, ν) be defined by*

$$Ric_N \;=\; \begin{cases} Ric \;+\; Hess(\Psi) & \text{if } N = \infty, \\ Ric \;+\; Hess(\Psi) \;-\; \frac{1}{N-n}\, d\Psi \otimes d\Psi & \text{if } n \;<\; N \;<\; \infty, \\ Ric \;+\; Hess(\Psi) \;-\; \infty\, (d\Psi \otimes d\Psi) & \text{if } N = n, \\ -\infty & \text{if } N < n, \end{cases}$$

where by convention $\infty \cdot 0 \;=\; 0$.

Theorem 3. *For $N \in [1, \infty]$, the measured length space (M, g, ν) has nonnegative N-Ricci curvature if and only if $Ric_N \;\ge\; 0$.*

In the special case when Ψ is constant, and so $\nu = \frac{dvol_M}{vol(M)}$, the theorem shows that we recover the usual notion of a Ricci curvature bound from our length space definition, as soon as $N \geq n$.

The next theorem says that our notion of N-Ricci curvature has good behavior under measured Gromov-Hausdorff limits.

Theorem 4. *Let $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ be a sequence of compact measured length spaces with $\lim_{i\to\infty}(X_i, d_i, \nu_i) = (X, d, \nu)$ in the measured Gromov-Hausdorff topology. For any $N \in [1, \infty)$, if each (X_i, d_i, ν_i) has nonnegative N-Ricci curvature then (X, d, ν) has nonnegative N-Ricci curvature.*

The above two theorems imply that measured Gromov-Hausdorff limits (X, d, ν) of smooth manifolds $\left(M, g, \frac{dvol_M}{vol(M)}\right)$ with nonnegative Ricci curvature fall under our considerations. Additionally, we obtain the following new characterization of such limits (X, d, ν) which happen to be *smooth*, meaning that (X, d) is a smooth n-dimensional Riemannian manifold (B, g_B) and $d\nu = e^{-\Psi}\, dvol_B$ for some $\Psi \in C^\infty(B)$.

Corollary 5. *If (B, g_B, ν) is a measured Gromov-Hausdorff limit of Riemannian manifolds with nonnegative Ricci curvature and dimension at most N then $Ric_N(B) \geq 0$.*

There is also a partial converse to the corollary 5.

Finally, if a measured length space has lower Ricci curvature bounds then there are analytic consequences, such as local and global Poincaré inequalities. We define the *gradient norm* of a Lipschitz function f on X by the formula

$$|\nabla f|(x) = \limsup_{y\to x} \frac{|f(y) - f(x)|}{d(x, y)}.$$

Theorem 6. *Suppose that a compact measured length space (X, d, ν) has nonnegative N-Ricci curvature. Suppose in addition that for almost all $(x_0, x_1) \in X \times X$, there is a unique minimizing geodesic from x_0 to x_1. Then for all balls $B_r(x)$ and all $f \in Lip(X)$, we have*

$$\frac{1}{\nu(B_r(x))} \int_{B_r(x)} |f - \overline{f}|\, d\nu \leq 2^{2N+1}\, r\, \frac{1}{\nu(B_{2r}(x))} \int_{B_{2r}(x)} |\nabla f|\, d\nu.$$

Theorem 7. *Suppose that a compact measured length space (X, d, ν) has N-Ricci curvature bounded below by $K > 0$. Then for all $f \in Lip(X)$ with $\int_X f\, d\nu = 0$, we have*

$$\int_X f^2\, d\nu \leq \frac{N-1}{NK} \int_X |\nabla f|^2\, d\nu.$$

In the case of Riemannian manifolds, one recovers the Lichnerowicz inequality for the smallest positive eigenvalue of the Laplacian.

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Classification of negatively pinched manifolds with amenable fundamental groups

Vitali Kapovitch
(joint work with Igor Belegradek)

We study manifolds of the form X/Γ, where X is a simply-connected complete Riemannian manifold with sectional curvatures pinched between two negative constants, and Γ is a discrete torsion free subgroup of the isometry group of X. According to [BS87], if Γ is amenable, then either Γ stabilizes a biinfinite geodesic, or else Γ fixes a unique point z at infinity. In the former case the normal exponential map to the Γ-invariant geodesic is a Γ-equivariant diffeomorphism, hence X/Γ is a vector bundle over S^1; there are only two such bundles each admitting a complete hyperbolic metric.

If Γ fixes a unique point z at infinity (such groups are called *parabolic*), then Γ stabilizes horospheres centered at z and permutes geodesics asymptotic to z, so that given a horosphere H, the manifold X/Γ is diffeomorphic to the product of H/Γ with $\mathbb{R}$. We refer to H/Γ as a *horosphere quotient*. In this case a delicate result of B. Bowditch [Bow93] shows that Γ must be finitely generated, which by Margulis lemma [BGS85] implies that Γ is virtually nilpotent.

Our main result [BKa] is a diffeomorphism classification of horosphere quotients.

Theorem 1. *For a smooth manifold N the following are equivalent:*
(1) *N is a horosphere quotient;*
(2) *N is diffeomorphic to an infranilmanifold;*
(3) *N is the total space of a flat Euclidean vector bundle over a compact infranilmanifold.*

By an *infranilmanifold* we mean the quotient of a simply-connected nilpotent Lie group G by the action of a torsion free discrete subgroup Γ of the semidirect product of G with a compact subgroup of $\mathrm{Aut}(G)$.

The implication (3) $\Rightarrow$ (2) is straightforward, (2) $\Rightarrow$ (1) is proved by constructing an explicit warped product metric of pinched negative curvature. The proof of (1) $\Rightarrow$ (3) is the hard part of Theorem 1, and depends on the collapsing theory of J. Cheeger, K. Fukaya, and M. Gromov [CFG92].

If N is compact (in which case the conditions (2), (3) are identical), the implication (1) $\Rightarrow$ (2) follows from Gromov's classification of almost flat manifolds, as improved by E. Ruh, while the implication (2) $\Rightarrow$ (1) is new.

A direct algebraic proof of (2) $\Rightarrow$ (3) was given in [Wil00b, Theorem 6], but the case when N is a nilmanifold was already treated in [Mal49], where it is shown that any nilmanifold is diffeomorphic to the product of a compact nilmanifold and a Euclidean space.

By [Wil00a] any flat Euclidean bundle with virtually abelian holonomy is isomorphic to a bundle with finite structure group, so the vector bundle in (3) becomes trivial in a finite cover. Thus any horosphere quotient is finitely covered by the product of a compact nilmanifold and a Euclidean space.

Corollary 2. *A smooth manifold M with amenable fundamental group admits a complete metric of pinched negative curvature if and only if it is diffeomorphic to the Möbius band, or to the product of a line and the total space a flat Euclidean vector bundle over a compact infranilmanifold.*

The pinched negative curvature assumption in Corollary 2 cannot be relaxed to $-1 \leq \sec \leq 0$ or $\sec \leq -1$, e.g. because these assumptions do not force the fundamental group to be virtually nilpotent [Bow93, Section 6]. More delicate examples come from the work of M. Anderson [And87] that the total space E of any vector bundle over a torus admits a metric with $-1 \leq \sec \leq 0$.

Also $-1 \leq \sec(M) \leq 0$ can be turned into $\sec(M \times \mathbb{R}) \leq -1$ for the warped product metric on $M \times \mathbb{R}$ with warping function e^t [BO69], hence Anderson's examples carry metrics with $\sec \leq -1$ after taking product with $\mathbb{R}$. On the other hand, by Corollary 2, if E has a nontrivial Pontrjagin class, then $E \times \mathbb{R}^l$ carries no metric of pinched negative curvature. Anderson also showed that every vector bundle over a closed negatively curved manifold admits a complete Riemannian metric of pinched negative curvature, hence amenability of the fundamental group is indispensable.

In [Bow95] Bowditch developed several equivalent definitions of geometrical finiteness for pinched negatively curved manifolds, and conjectured the following result.

Corollary 3. *Any geometrically finite pinched negatively curved manifold X/Γ is diffeomorphic to the interior of a compact manifold with boundary.*

Given a smooth manifold M, we define $\text{pinch}^{\text{diff}}(M)$ to be the infimum of $a^2 \geq 1$ such that M admits a complete Riemannian metric of $-a^2 \leq \sec(M) \leq -1$. If M admits no complete metric of pinched negative curvature, it is convenient to let $\text{pinch}^{\text{diff}}(M) = +\infty$. We then define $\text{pinch}^{\text{top}}(M)$ to be the infimum of all $\text{pinch}^{\text{diff}}(N)$ where N is homeomorphic to M, and define $\text{pinch}^{\text{hom}}(M)$ to be the infimum of $\text{pinch}^{\text{diff}}(N)$'s where N is manifold with $\dim(N) = \dim(M)$ that is homotopy equivalent to M. Of course, $\text{pinch}^{\text{diff}}(N) \geq \text{pinch}^{\text{top}}(M) \geq \text{pinch}^{\text{hom}}(N) \geq 1$.

In general, the pinching invariants are hard to estimate and even harder to compute (see [Gro91] and [Bel01, Section 5] for surveys).

If $\Gamma = \pi_1(M)$ is nilpotent parabolic acting cocompactly on horospheres then it follows from the proof of Gromov's theorem of almost flat manifolds (see [BK81, Corollary 1.5.2]) that $\text{pinch}^{\text{hom}}(M) \geq k^2$ where k is the nilpotency length of Γ.

More recently, Gromov sketched in [Gro91, p.309] a proof of the more general estimate

$$a \geq \frac{k}{r+1} \quad \text{for} \quad r = \left[\frac{\dim(M) - 1 - \mathrm{cd}(\Gamma)}{2}\right],$$

where $[x]$ denotes the largest integer satisfying $\leq x$. If $k \leq r+1$, the estimate gives no information, so Gromov asked [Gro91, p.309] whether it can be improved to an estimate that is nontrivial for all $\mathrm{cd}(\Gamma) < \dim(M)$.

In [BKb] we prove

Theorem 4. *If M be a pinched negatively curved manifold such that $\pi_1(M)$ has a k-step nilpotent subgroup of finite index, then $pinch^{\mathrm{diff}}(M) = pinch^{\mathrm{top}}(M) = pinch^{\mathrm{hom}}(M) = k^2$.*

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Rigidity of cone-3-manifolds

Hartmut Weiss

A cone-manifold of curvature $\kappa \in \{-1, 0, 1\}$ is a metric space X, which is homeomorphic to a manifold and whose local geometry is modelled on the κ-cone over a cone-manifold of curvature $+1$ and one dimension lower. The set of points whose link (i.e. the cross-section of the model cone) is isometric to the standard round sphere is called the smooth part of the cone-manifold, its complement is called the singular locus and is usually denoted by Σ. In the following we will be concerned with 2- and 3-dimensional cone-manifolds only.

In two dimensions the singular locus consists of isolated points. The link of each cone point is isometric to the circle of a certain length, which we will refer to as the cone angle associated to that point. In three dimensions the singular locus is an embedded geodesic graph. The cone-angle associated to an edge will be the cone-angle of a transverse disk.

If cone-angles are $\leq 2\pi$, these spaces satisfy a lower curvature bound in the triangle comparison sense. If cone-angles are $\leq \pi$, the geometry is even more restricted, for example the Dirichlet-polyhedron will be convex and the valency of a vertex of the singular locus (in the 3-dimensional case) will be at most 3.

The concept of cone-3-manifold is a natural generalization of the concept of geometric 3-orbifold, where the cone-angles are of the form $2\pi/n$, $n \in \mathbb{Z}, n \geq 2$. Cone-3-manifolds play a significant role in the proof of the Orbifold Theorem, which has recently been accomplished by M. Boileau, B. Leeb and J. Porti, cf. [1]. The Orbifold Theorem states that a similar geometric decomposition as conjectured for 3-manifolds holds true for 3-orbifolds with non-empty singular locus. It was announced by W. Thurston around 1982.

In this talk I discussed the following global rigidity results for hyperbolic and spherical cone-3-manifolds, cf. [7]:

Theorem 1. *Let X, X' be hyperbolic cone-3-manifolds with cone-angles $\leq \pi$. If there exists a homeomorphism of pairs $(X, \Sigma) \cong (X', \Sigma')$ such that the cone-angles around corresponding edges coincide, then X and X' are isometric.*

Theorem 1 generalizes the global rigidity result of S. Kojima, cf. [4], which states the same rigidity property under the additional assumption that the singular locus is a link, i.e. a union of embedded circles. In the statement of Theorem 2, following [5], we call a cone-3-manifold X *Seifert fibered*, if X is Seifert fibered in the usual sense and if in addition Σ is a union of fibers.

Theorem 2. *Let X, X' be spherical cone-3-manifolds with cone-angles $\leq \pi$, which are not Seifert fibered. If there exists a homeomorphism of pairs $(X, \Sigma) \cong (X', \Sigma')$ such that the cone-angles around corresponding edges coincide, then X and X' are isometric.*

I indicated the proof of Theorem 1: We follow the same strategy as Kojima in [4], namely we construct a continuous path of hyperbolic cone-manifold structures on X with singular locus Σ and decreasing cone angles, which starts at the given structure and terminates at a complete hyperbolic structure of finite volume, possibly with totally geodesic boundary consisting of thrice-punctured spheres if vertices are present. This reflects the fact that the geometry of links of vertices has to change from spherical through horospherical to hyperspherical as we decrease cone angles. Here the link of a vertex is called horospherical if it is the horospherical cross-section of a singular cusp (i.e. a Euclidean turnover) and hyperspherical if it is a totally geodesic boundary component (i.e. a hyperbolic turnover).

To be able to carry out this strategy, we need to know that we can always deform cone-angles in an essentially unique way, furthermore we need to rule out degenerations as we decrease cone-angles. The first issue is handled by a local rigidity theorem for hyperbolic cone-manifold structures with singular cusps and totally geodesic boundary as above. This generalizes the local rigidity theorem for compact hyperbolic cone-3-manifolds in [6] and replaces the use of the local rigidity theorem of C.D. Hodgson and S.P. Kerckhoff, cf. [3], in Kojima's proof. The proof uses analytic techniques developed by J. Brüning and R. Seeley in [2]. As in [6], the assumption that cone angles are $\leq \pi$ is essential.

To rule out degenerations, we use the geometric results of [1], which enter the proof of the Orbifold Theorem. The description of thin parts of hyperbolic cone-3-manifolds with lower diameter bound and cone-angles $\leq \alpha < \pi$, which is achieved in [1], allows us to conclude that at most very mild degenerations can occur, namely tubes around short closed (smooth or singular) geodesics opening into rank-2 cusps. These in turn can be ruled out by a hyperbolic Dehn-surgery theorem in the setting of cone-manifolds.

If we are now given hyperbolic cone-manifolds X and X' as in Theorem 1, we can consider the corresponding paths of cone-manifold structures terminating at the complete structures. By Mostow-Prasad rigidity, these will be isometric. Using local rigidity along the paths, we conclude that X and X' are isometric.

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Theorem of de Rham for metric spaces

Alexander Lytchak
(joint work with Thomas Foertsch)

Theorem of de Rham ([deRham52]) states that if the holonomy group of a simply connected complete Riemannian manifold acts in a reducible way on the tangent space of a point, then the manifold must be a direct product of two other manifolds. As a consequence this implies that each simply connected complete Riemannian manifold M has a unique decomposition as a product $M = M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Euclidean space and M_i for $i > 0$ are irreducible Riemannian manifolds, that are not equal to the real line.

The statement is that M_i are determined not only up to an abstract isometry, but that the M_i-factor through a given point x is well defined (up to permutation of isometric factors). Observe that the Euclidean space plays a special role, since it has many different decompositions as a product of real lines.

In [EH98] the statement about the uniqueness of the decomposition $M = M_0 \times \cdots \times M_k$ was generalized to non-simply connected Riemannian manifolds, by studying the action of the fundamental group of M on the decomposition of the universal covering.

Our main result is the following theorem, that shows that the uniqueness of the decomposition holds true in a much greater generality

Theorem 1. *Let X be a geodesic metric space of finite topological dimension. Then X has a unique decomposition as $X = X_0 \times X_1 \times \cdots \times X_k$, where X_0 is a Euclidean space $\mathbb{R}^l$ and the other factors X_i are irreducible spaces that are not equal to the real line.*

The uniqueness should be understood in the same way as in the classical theorem of de Rham. As a direct application we obtain:

Corollary 2. *Let X, Y be geodesic metric spaces of finite topological dimension. If X and Y have no isometric factors, then for the isometry groups we get $Iso(X \times Y) = Iso(X) \times Iso(Y)$.*

The most striking part of our theorem is the following rigidity statement. Let X be a geodesic metric space of finite dimension with decompositions $X = Y \times \bar{Y}$ and $X = Z \times \bar{Z}$. Assume that at some point $x \in X$ the fibers $Y_x, \bar{Y}_x, Z_x, \bar{Z}_x$ intersect pairwise only in x. Then X is isometric to a Euclidean space.

The proof of the main theorem goes along the following lines. We say that a metric space is affine if it is isometric to a linear convex subset of a normed vector space. The main technical observation is that if a subset C of a product $Y \times \bar{Y}$ is affine then its projections onto the factors are affine too. As an immediate consequence one obtains that if C is a maximal affine subset of a metric space X then for each decomposition $X = Y \times \bar{Y}$ the subset C is the product of its projection to Y resp. $\bar{Y}$.

This allows us to find sufficiently many affine subsets of each geodesic metric space X and to prove that if the result was wrong for X it must be wrong for one of its affine subsets.

Finally we show that the theorem is true for finite dimensional affine spaces. This finishes the proof.

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Geometry and dynamics of discrete subgroups of higher rank Lie groups

Gabriele Link

Let X be a globally symmetric space of noncompact type, $o \in X$ and $G = \mathrm{Isom}^o(X)$ the connected component of the identity. We will denote by ∂X the geometric boundary of X endowed with the cone topology. In this talk we intend to give more insight into the dynamics of certain individual isometries of X and describe geometrically the structure of the limit set $L_\Gamma := \overline{\Gamma \cdot o} \cap \partial X$ of discrete isometry groups $\Gamma \subset G$. The main difficulties we face in the higher rank case compared to the situation in manifolds with pinched negative curvature are due to the more complicated structure of the geometric boundary. In fact, to each point $\xi \in \partial X$ we can associate a unique "direction" in a fixed Weyl chamber of X. If the direction of ξ is in the interior of the Weyl chamber, we say that ξ belongs to the regular boundary $\partial X^{reg} \subseteq \partial X$. Every point in the G-orbit of ξ possesses the same direction, in particular G does not act transitively on the geometric boundary if the rank of X is greater than one.

Concerning the dynamics of parabolic isometries, there are only partial results for example by P. Eberlein ([E, chapter 4.1]), A. Parreau ([P, chapter I.2]) and the author ([L, chapter 4.4]). For axial (sometimes also called loxodromic) isometries, however, we are able to describe precisely the action on the geometric boundary. Namely, if $\gamma^+ \in \partial X^{reg}$ denotes the attractive fixed point of an axial isometry $\gamma \in G$, then for any point $\xi \in \partial X^{reg}$ which can be joined to γ^+ by a geodesic we have

$$\lim_{j\to\infty} \gamma^j \xi = \gamma^+ .$$

Similarly, a dense and open subset of the Furstenberg boundary is moved by γ towards the asymptotic Weyl chamber containing γ^+.

As an application of this result we describe a new geometric construction of Schottky groups in higher rank symmetric spaces.

We then generalize appropriately the notion of "nonelementary" groups well-known in the context of manifolds of pinched negative curvature to higher rank symmetric spaces. Our notion is weaker than Zariski density, which has been used for discrete subgroups of real reductive linear Lie groups by Y. Benoist ([B]) and Conze-Guivarc'h ([CG]). Also, the definition is more natural and easily understandable from a geometrical point of view.

Unfortunately, the incomplete picture we have about the dynamics of parabolic isometries makes it difficult to describe the structure of the limit set of discrete isometry groups which, in general, always contain parabolics. For the large class of nonelementary groups, however, we can use a so-called "approximation argument" in order to reduce the problem to understanding the action of sequences of axial isometries. A direct consequence of this approximation argument is the fact that for nonelementary groups Γ, the set of attractive fixed points of axial isometries is dense in the geometric limit set L_Γ. Together with a proposition concerning the dynamics of sequences of axial isometries, it is one of the key ingredients in the proof of our main

Theorem 1. *Let $\Gamma \subset G$ be a nonelementary discrete group. Then*

(1) *the limit set K_Γ, considered as a subset of the Furstenberg boundary, is a minimal closed set under the action of Γ,*
(2) *the regular limit set L_Γ^{reg} splits as a product $K_\Gamma \times P_\Gamma^{reg}$, where P_Γ denotes the set of directions of limit points, and P_Γ^{reg} the set of directions of regular limit points.*

To each axial isometry of a higher rank symmetric space we can associate a so-called "translation vector", a notion introduced by A. Parreau ([P]) which generalizes the translation length in rank one spaces. Let ℓ_Γ denote the set of translation vectors of axial isometries in Γ. As a further result we have

Theorem 2. *If Γ is a nonelementary group, then P_Γ is equal to the closure of ℓ_Γ.*

Although these results are already known in the context of Zariski dense subgroups of real reductive linear groups (see i.e. [B], [CG]), the advantage of our approach is its purely geometric nature which allows to easily adapt the methods to products of manifolds of pinched negative curvature (compare [DK]). Precise statements and the proofs of the above-mentioned results can be found in [Li].

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Singular Minimal Hypersurfaces and Scalar Curvature

Joachim Lohkamp

(joint work with Ulrich Christ)

We present joint work with U. Christ on the resolving of hypersurface singularities in the context of preserving positive scalar curvature. The main result is that doubling the hypersurface after deleting a suitably chosen neighborhood of the singular set leads to a manifold which admits positive scalar curvature. This involves a detailed study of how the surface is approximated by tangent cones and how the analysis of the conformal Laplacian transfers from the hypersurface to such cones.

Positive scalar curvature with symmetry

Bernhard Hanke

We develop equivariant analogues of the construction techniques introduced by Gromov-Lawson and Schoen-Yau for positive scalar curvature metrics. Part of the nonequivariant discussion can be translated more or less directly to the equivariant context. As shown in [1], this applies in particular to the surgery principle (cf. [2]) which states that the class of smooth manifolds admitting metrics of positive scalar curvature is closed under surgery of codimension at least 3. However, the following equivariant bordism principle requires a refined argument because it is based on handle cancelation techniques that in general do not carry over to the equivariant world (which is illustrated by the failure of equivariant analogous of the h- and s-cobordism theorems):

Theorem A. *Let G be a compact Lie group and let Z be a compact connected oriented G-bordism (with an orientation preserving G-action) between the closed G-manifolds X and Y. Assume the following:*

- (i) *The cohomogeneity of Z is at least 6,*
- (ii) *the inclusion of maximal orbits $Y_{max} \hookrightarrow Z_{max}$ is a (nonequivariant) 2-equivalence (i.e. a bijection on π_0, an isomorphism on π_1 and a surjection on π_2),*
- (iii) *each singular stratum of codimension 2 in Z meets Y.*

Then, if X admits a G-invariant metric of positive scalar curvature, the same is true for Y.

We remark that by a classical result of Lawson-Yau [3], closed connected effective G-manifolds admit G-invariant metrics of positive scalar curvature if the identity component of G is non-abelian. Hence, Theorem A is useful mainly for finite or for toral G.

Our Theorem A is almost a direct analogue of the corresponding nonequivariant result (see [6], Theorem 3.3). In particular, the dimension restriction i. and the connectivity restriction for the inclusion $Y_{max} \hookrightarrow Z_{max}$ stated in point ii. translate to analogous requirements in the nonequivariant setting if $G = \{1\}$. However, if G is not trivial, we need an additional assumption on codimension-2 singular strata.

Theorem A is useful for constructing equivariant metrics of positive scalar curvature only if it can be combined with powerful structure results for geometric equivariant bordism groups implying that the manifold X in Theorem A can be assumed to admit an equivariant positive scalar curvature metric under general assumptions on the manifold Y. Two main difficulties occur at this point. Firstly, explicit geometric generators of equivariant bordism groups are known only in a very limited number of cases. Secondly, whereas conditions i. and ii. in Theorem A can be achieved under fairly general assumptions on the manifold Y (by performing appropriate surgeries on Z_{max}), it is a priori not clear under what circumstances condition iii. holds.

We present a solution to the last mentioned problem if $G = S^1$ and the G-action on Z is fixed point free. The idea we use is to alter a given bordism Z by cutting out equivariant tubes connecting Y with each of the codimension-2 singular strata in Z that are disjoint from Y. This replaces the bordism Z and the manifold Y by other manifolds Z' and Y' so that each codimension-2 singular stratum in Z' meets Y'. In particular, Theorem A can be applied to Z' (after some more manipulations of Z', but we omit these details here). We must now understand how Y can be recovered from Y'. A closer inspection of the situation shows that Y' is obtained from Y by adding certain codimension-2 singular strata with finite isotropies. Conversely, Y can be reconstructed from Y' by a kind of codimension-2 surgery process that removes these additional singular strata and puts back free ones instead. We show by a somewhat involved geometric argument that this surgery step preserves the existence of S^1-invariant positive scalar curvature metrics under fairly general assumptions. Roughly speaking, we replace the "bending outwards" process in the surgery step due to Gromov-Lawson and Schoen-Yau by a "bending inwards" process. We remark that this kind of positive scalar curvature preserving codimension-2 surgery only works under the additional S^1-symmetry on Z.

With the help of this surgery method, we conclude that the original manifold Y admits an invariant metric of positive scalar curvature if the manipulated one Y' admits such a metric. Arguing in this rather roundabout manner, assumption iii. of Theorem A is no longer a true obstacle against the construction of equivariant positive scalar curvature metrics on fixed point free S^1-manifolds. This insight is now combined with a classical theorem of Ossa [4], which states that fixed point free S^1-manifolds are S^1-boundaries, to complete the proof of the following result, an equivariant generalization of the well known existence result of positive scalar curvature metrics on closed simply connected non-spin manifolds of dimension at least 5 due to Gromov and Lawson [2]:

Theorem B. *Let M be a connected closed oriented fixed point free S^1-manifold so that all normal bundles around H-fixed components ($H \subset S^1$ being a closed subgroup) in M are complex S^1-bundles. If the dimension of M is at least 6, the union of maximal orbits of M is simply connected and does not admit a spin structure, then M admits an S^1-invariant metric of positive scalar curvature.*

We remark that no additional assumption on codimension-2 singular strata in M is needed. It is not clear at present to what extent Ossa's theorem can be generalized to the spin case so that we leave the discussion of a corresponding S^1-equivariant analogue of Stolz' theorem [5] for later consideration.

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Foliation by holomorphic discs and applications to Kähler geometry

Xiuxiong Chen

(joint work with Gang Tian)

We study partial regularity of WZW Harmonic map from Σ (unit disc in $\mathbb{C}$) to $\mathcal{H}$ (the space of Kähler potentials in a compact manifold $(M, [\omega])$). The equation can be reduced to the following homogeneous complex MA equation

$$\begin{cases} \det \begin{pmatrix} \left(g_{ij} + \frac{\partial^2 \varphi}{\partial z_i \partial y_{\bar{j}}}\right)_{n\times n} & \frac{\partial^2 \varphi}{\partial \bar{z}_i z_j} \\ \frac{\partial^2 \varphi}{\partial z_i \partial z_{\bar{j}}} & \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \end{pmatrix} = 0 \\ \varphi|_{\partial\Sigma} = \psi|_{\partial\Sigma} \quad : \quad \partial\Sigma \to \mathcal{H} \end{cases}$$

The existence of $C^{1,1}$ solutions was established by the first named author and we (joint with Tian) prove essentially that the solution is almost everywhere smooth (except at most a codimension 2 set) for generic boundary data. The main idea is to transform this into a problem of smoothness of certain moduli space of holomorphic discs whose boundary lies in a totally submanifold

$$f : (\Sigma, \partial\Sigma) \to (\mathcal{W}, \bar{\Lambda}_\psi).$$

Let me explain here the notation appears in the above formula. First. $\mathcal{W}$ is essentially T^*M with a non-traditional gluing of sections of T^*M. Suppose $U_\alpha, \alpha \in \mathcal{I}$ is a finite covering of M where $\mathcal{I}$ is an index set. For any $\alpha \in \mathcal{I}$, set ρ_α be the local Kähler potential for ω_g in the coordinate chart U_α. For any pair of index $\alpha, \beta \in \mathcal{I}$ such that $U_\alpha \bigcap U_\beta \neq \emptyset$, the sections of T^*U_α and T^*U_β can be identified by

$$\forall\, (z,\xi) \in T^*U_\alpha, \;\; (w,\eta) \in T^*U_\beta,$$

we identify these two element

$$(z,\xi) \equiv (w,\eta),$$

if and only if

$$z = w, \text{and } \xi + \rho_\alpha = \eta + \rho_\beta.$$

Under this gluing map, $\Theta = d\,z \wedge d\,\xi = d\,w \wedge d\,\eta$ is a global holomorphic 2-forms in $\mathcal{W}$ at the expense of metric structure. For any Kähler potential $\varphi \in \mathcal{H}$, denote Λ_φ as the graph of $\partial\varphi$ in $\mathcal{W}$. It is straightforward to see that Λ_φ is a totally real submanifold of $\mathcal{W}$ and $\bar{\Lambda}_\psi = \bigcup_{s \in S^1} \Lambda_{\psi(s)}$ is a totally real submanifold in $\Sigma \times \mathcal{W}$. The defomation problem (which includes the existence and regularity problem) of holomphic discs in $(\Sigma \times \mathcal{W}, \bar{\Lambda}_\psi)$ is well known as Riemann Hilbert problem and there is a wealth of literatures on that subject. If, we have a smooth solution to the geodesic equation above, then $\Sigma \times M$ admits a smooth foliation by holomorphic discs. Moreover, this smooth foliation of holomorphic can be lifted to the moduli space holomophic discs in $(\mathcal{W}, \bar{\Lambda}_\psi)$ and the image of such a lifting is full and the moduli space is regular in the sense of Fredholm. In fact, according to Donaldson, the image is a **super regular** foliation in the sense it is lifted from a nowhere degenerated foliation in $\Sigma \times M$. The amazing story here is that the converse is also true: Given a super regular moduli space of holomorphic discs in $(\mathcal{W}, \bar{\Lambda}_\psi)$, there is a smooth solution to geodesic equation with boundary map ψ. To explain this part of story, we need to Go back to this global holomorphic 2-forms Θ. It turns out that Λ_φ is a Lagrange-Sympletic submanifold in $\mathcal{W}$. There is an important correspondence between a Kähler metric in $[\omega_g]$ and a Hamiltonian class of a given Lagrange-Sympletic submanifold (with respect to Θ). According to S. Semmes, further clarified by Donaldson later, a super regular moduli space of holomorphic discs in $(\mathcal{W}, \bar{\Lambda}_\psi)$ corresponds to a smooth disc family of Lagrange-Sympltic submanifolds in $\mathcal{W}$. From here, one recover a smooth smooth solution to the HCMA equation above with prescribed boundary map $\psi : \partial\Sigma \to \mathcal{H}$. In this correspondence, it is important that solution is smooth and the moduli is super regular. Donaldson use this correspondence, to establish a fact that any smooth solution to geodesic equation can be perturbed by boundary data. The story looks nice for setting up a continuous defomation theorem. Unfortunately, there is certain boundary map where the corresponding moduli space can not be super regular. In other words, the above geodesic equation can be solved smoothly for all boundary datas.

In this work, we first devised a notion of almost smooth solution of geodesic equation and almost regular moduli space of holomorphic discs and we established an equivalence relation between the two. There are serious difficulty arised when one moves from the equivalence observed by S. Semmes or S. K. Donaldson to this one. More importantly, we proved that almost regular moduli space is open and close under generic peturbation of the boundary maps. We then established partial regularity for geodesic equation for generic boundary data.

For application in Kähler geometry, the crucial observation is that the K-energy functional (whose critical point is c.s.c. metric) is subharmonic when restricted to this disc family of potentials. And this fact is used to prove the uniqueness of extremel metric in all Kähler classes. As a consequence, c.s.c implies that K-energy has a lower bound, which implies K-semistability.

Energy functionals and canonical Kähler metrics

Ben Weinkove
(joint work with Jian Song)

The problem of finding necessary and sufficient conditions for the existence of extremal metrics, which include Kähler-Einstein metrics, on a compact Kähler manifold M has been the subject of intense study over the last few decades and is still largely open. If M has zero or negative first Chern class then it is known by the work of Yau [Ya1] and Yau, Aubin [Ya1], [Au] that M has a Kähler-Einstein metric. When $c_1(M) > 0$, so that M is Fano, there is a well-known conjecture of Yau [Ya2] that the manifold admits a Kähler-Einstein metric if and only if it is stable in the sense of geometric invariant theory.

There are now several different notions of stability for manifolds. It is conjectured by Tian [Ti2] that the existence of a Kähler-Einstein metric should be equivalent to his 'K-stability'. This stability is defined in terms of the Futaki invariant [Fu], [DiTi] of the central fiber of degenerations of the manifold.

The behavior of Mabuchi's [Ma] energy functional is central to this problem. It was shown by Bando and Mabuchi [BaMa], [Ba] that if a Fano manifold admits a Kähler-Einstein metric then the Mabuchi energy is bounded below. Recently, it has been shown by Chen and Tian [ChTi3] that if M admits an extremal metric in a given class then the (modified) Mabuchi energy is bounded below in that class. Moreover, if a lower bound on the Mabuchi energy is given then the class is K-semistable [Ti1], [Ti2], [PaTi]. Conversely, Donaldson [Do1] showed that, for toric surfaces, K-stability implies the lower boundedness of the Mabuchi energy.

In addition, the existence of a Kähler-Einstein metric on a Fano manifold has been shown to be equivalent to the 'properness' of the Mabuchi energy [Ti2]. Tian conjectured [Ti3] that the existence of a constant scalar curvature Kähler metric be equivalent to this condition on the Mabuchi energy. This holds when the first Chern class is a multiple of the Kähler class.

In this talk, we discuss a family of functionals E_k, for $k = 0, \ldots, n$, which were introduced by Chen and Tian [ChTi1]. They are generalizations of the Mabuchi energy, with E_0 being precisely Mabuchi's functional.

A critical metric ω of E_k is a solution of the equation

$$\sigma_{k+1}(\omega) - \triangle(\sigma_k(\omega)) = \text{constant},$$

where $\sigma_k(\omega)$ is the kth elementary symmetric polynomial in the eigenvalues of the Ricci tensor of the metric ω. Notice that the critical metrics for E_0 are precisely the constant scalar curvature metrics. Kähler-Einstein metrics are solutions to the above equation for all k.

The functionals E_k were used by Chen and Tian [ChTi1, ChTi2] to obtain convergence of the normalized Kähler-Ricci flow on Kähler-Einstein manifolds with positive bisectional curvature. The Mabuchi energy is decreasing along the flow. The functional E_1 is also decreasing, as long as the sum of the Ricci curvature and the metric is nonnegative.

In a recent preprint, Chen [Ch1] has proved a stability result for E_1 for Fano manifolds in the sense of the Kähler-Ricci flow (shortly after this paper was first posted we learned that, in an unpublished work [Ch2], Chen has proved the following: if there exists a Kähler-Einstein metric then E_1 is bounded below along the Kähler-Ricci flow.). In [Ch1], Chen asked whether E_1 is bounded below or proper on the full space of potentials (not just along the flow) if there exists a Kähler-Einstein metric.

We now state our results. For ϕ in $P(M,\omega)$, let ϕ_t be a path in $P(M,\omega)$ with $\phi_0 = 0$ and $\phi_1 = \phi$. The functional $E_{k,\omega}$ for $k = 0, \ldots, n$ is defined by

$$\begin{aligned} E_{k,\omega}(\phi) &= \frac{k+1}{V}\int_0^1\int_M (\triangle_{\phi_t}\dot{\phi}_t)\,\mathrm{Rc}(\omega_{\phi_t})^k \wedge \omega_{\phi_t}^{n-k} dt \\ &\quad - \frac{n-k}{V}\int_0^1\int_M \dot{\phi}_t\,(\mathrm{Rc}(\omega_{\phi_t})^{k+1} - \mu_k\,\omega_{\phi_t}^{k+1}) \wedge \omega_{\phi_t}^{n-k-1} dt, \end{aligned} \tag{1}$$

where μ_k is a constant depending only on the classes $[\omega]$ and $c_1(M)$. We will often write $E_k(\omega, \omega_\phi)$ instead of $E_{k,\omega}(\phi)$.

Suppose now that M has positive first Chern class and denote by $\mathcal{K}$ the space of Kähler metrics in $2\pi c_1(M)$.

Theorem 1. *Let (M, ω_{KE}) be a Kähler-Einstein manifold with $c_1(M) > 0$. Then, for $k = 0, \ldots, n$, and for all $\tilde{\omega} \in \mathcal{K}$ with $Rc(\tilde{\omega}) \geq 0$,*

$$E_k(\omega_{KE}, \tilde{\omega}) \geq 0,$$

and equality is attained if and only if $\tilde{\omega}$ is a Kähler-Einstein metric.

In the case of E_1 we have:

Theorem 2. *Let (M, ω_{KE}) be a Kähler-Einstein manifold with $c_1(M) > 0$ or $c_1(M) = 0$. Then for all Kähler metrics ω' in the class $[\omega_{KE}]$,*

$$E_1(\omega_{KE}, \omega') \geq 0,$$

and equality is attained if and only if ω' is a Kähler-Einstein metric.

We have the following result on the properness of E_1:

Theorem 3. *Let* (M, ω_{KE}) *be a compact Kähler-Einstein manifold with* $c_1(M) > 0$*. Then there exists* δ *depending only on* n *such that the following hold:*

(i) *If* M *admits no nontrivial holomorphic vector fields then there exist positive constants* C *and* C' *depending only on* ω_{KE} *such that for all* θ *in* $P(M, \omega_{KE})$,

$$E_{1,\omega_{KE}}(\theta) \geq CJ_{\omega_{KE}}(\theta)^{\delta} - C'.$$

(ii) *In general, let* Λ_1 *be the space of eigenfunctions of the Laplacian for* ω_{KE} *with eigenvalue 1. Then for all* θ *in* $P(M, \omega_{KE})$ *satisfying*

$$\int_M \theta\, \psi\, \omega_{KE}^n = 0, \quad \text{for all } \psi \in \Lambda_1,$$

there exist constants C *and* C' *depending only on* ω_{KE} *such that*

$$E_{1,\omega_{KE}}(\theta) \geq CJ_{\omega_{KE}}(\theta)^{\delta} - C'.$$

We also have a converse to Theorem 3.

Theorem 4. *Let* (M, ω) *be a compact Kähler manifold with* $c_1(M) > 0$*. Suppose that* $\omega \in 2\pi c_1(M)$*. Then the following hold:*

(i) *Suppose that* (M, ω) *admits no nontrivial holomorphic vector fields. Then* M *admits a Kähler-Einstein metric if and only if* E_1 *is proper on* $P(M, \omega)$*.*

(ii) *In general, let* G *be a maximal compact subgroup in* $Aut(M)$*. Suppose that* ω *is a* G*-invariant Kähler metric. Then* M *admits a* G*-invariant Kähler-Einstein metric if and only if* E_1 *is proper on* $P_G(M, \omega)$*, the space of* G*-invariant potentials for* ω*.*

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Toric anti-self-dual four manifolds via complex geometry

Joel Fine

(joint work with Simon K. Donaldson)

Review of twistor geometry. Twistor geometry was invented by Penrose in the context of general relativity. For a description from this point of view see [1]. A thorough description in Riemannian signature is given in [2].

On an oriented Riemannian four-manifold M, the two-forms split $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ as the ± 1-eigenspaces of the Hodge star operator. This induces an additional decomposition of the curvature tensor not present in other dimensions: considered as homomorphism $\Lambda^2 \to \Lambda^2$, the curvature has a block decomposition

$$\begin{pmatrix} s + W^+ & \mathrm{Rc}_0 \\ \mathrm{Rc}_0^* & s + W^- \end{pmatrix}$$

where s is (up to a constant factor) the scalar curvature, Rc_0 is the trace-free Ricci curvature and $W^\pm$ are the self-dual and anti-self-dual Weyl curvatures. A metric is called *anti-self-dual* if $W^+ = 0$. Both the whole Weyl curvature $W = W^+ + W^-$ and the splitting of Λ^2 are conformally invariant, hence so is anti-self-duality.

The Weyl curvature vanishes if and only if conformally Euclidean coordinates exist. It is natural then to ask for the integrability interpretation of anti-self-duality. This is provided by Penrose's famous twistor construction. The *twistor space* of M is a six-manifold Z which fibres over M, the fibre over a point x being the set of all almost complex structures on T_xM which are compatible with both the metric and the orientation. The fibres are symmetric spaces $\mathrm{SO}(4)/\mathrm{U}(2)$, in other words metric two-spheres. The Levi–Civita connection induces a connection on Z and using this there is natural almost complex structure on Z: it is the natural complex structure on S^2 in the vertical directions and is tautological in the horizontal directions. The metric on M is anti-self-dual if and only if the almost complex structure on Z is integrable.

Assuming this to be the case, the fibres of Z — called *twistor lines* — are rational curves whose normal bundles can be seen to be $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Kodaira's theorem says such a curve moves in a four complex dimensional family N. The antipodal map on S^2 induces a fixed-point-free antiholomorphic involution on Z

and hence an involution on N; the original four-manifold $M \subset N$ is recovered as the fixed points.

Finally, the conformal structure can be recovered from Z. A tangent vector in $T_L N$ is said to be null if the corresponding section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ has a zero. This defines the null cone in $T_L N$ and hence a complex conformal structure on N. Its restriction to M recovers the original Riemannian conformal structure. This sets up a natural correspondence between anti-self-dual four-manifolds with a fixed base point and complex three-folds with a fixed-point-free real involution containing a real, rational curve with normal bundle $\mathcal{O}(1) \oplus O(1)$.

Twistor spaces of toric anti-self-dual metrics. Special cases of toric anti-self-dual manifolds have been studied before. See, for example, [3, 4]

We consider the case where the anti-self-dual metric on M admits two linearly independent commuting conformally Killing fields (the germ of a conformal $\mathbb{R}^2$-action). The fields lift to holomorphic fields on Z where they define the germ of $\mathbb{C}^2$-action. Let Σ denote the subset where this action fails to be free.

The case $\dim \Sigma = 2$ can be shown to be equivalent to M being hypercomplex. At least in the hyperkähler situation, such metrics are well understood via the Gibbons–Hawking ansatz. This partially justifies the restriction from now on to the case $\dim \Sigma < 2$. When this holds, Σ does not meet the generic twistor line. In other words, above the generic point of M, the induced $\mathbb{C}^2$-action on Z is free.

This free action can be used to define two holomorphic invariants which completely characterise the local geometry of Z. Let L denote a twistor line not meeting Σ. Since its normal bundle is $\mathcal{O}(1) \oplus \mathcal{O}(1)$, there are two distinguished points on L, 0 and ∞ say, at which the $\mathbb{C}^2$-orbits are tangential to order one; elsewhere on L the orbits are transverse. Since the orbit through 0 is tangential to order one, an orbit through a nearby point z meets L in a second nearby point $\tau(z)$. This defines a holomorphic involution τ near 0. Moreover, since z and $\tau(z)$ lie on the same orbit, there is a complex number $\phi(z)$ such that $\phi(z) \cdot z = \tau(z)$. This defines a $\mathbb{C}^2$-valued holomorphic function near 0 which is odd with respect to τ. There is a similar picture near ∞, but this is related to the picture near 0 by the real structure and so carries no more information.

The data (τ, ϕ) characterises Z. This is shown by explicitly reconstructing the twistor space from (τ, ϕ). For example, the geometry near a point of tangency is modeled on the quotient of $D \times \mathbb{C}^2$ where (z, v) is identified with $(\tau(z), v + \phi(z))$.

In rebuilding Z, a point 0 is fixed in L and a choice of antipodal map is made. This determines the coordinate on L up to rotations about the axis through 0. In other words, there is a natural S^1-action on the pairs (τ, ϕ) and two twistor spaces are biholomorphic if and only if their pairs lie in the same S^1-orbit. The twistor correspondence now says that germs of toric anti-self-dual four-manifolds taken at a generic point are in natural one-to-one correspondence with pairs (τ, ϕ) modulo the S^1-action.

To make use of this classification, it is useful to be able to describe the metric explicitly. By the twistor correspondence, this amounts to describing lines near to the central line L. If L' is a nearby line, it has two tangential orbits just as L does.

These meet L in four points, $a, \tau(a)$ near 0 and $b, \tau(b)$ near ∞. Well away from 0 and ∞, the $\mathbb{C}^2$-action identifies L and L'; that is L' is the graph of a $\mathbb{C}^2$-valued function f defined on the middle portion of L. To make f well defined near 0 and ∞ it is necessary to cut L from a to $\tau(a)$ and from b to $\tau(b)$. Then f extends to the cut plane and jumps

$$f(z) \mapsto f(\tau(z)) + \phi(z)$$

over the cuts. Explicitly describing the lines in Z near L amounts to describing such functions with prescribed jumps.

It is possible to express this problem in terms of prescribing the Čech coboundary of a 0-cochain on a certain elliptic curve. This is a classical problem and can be solved via contour integrals. In favourable cases the integrals can be computed explicitly yielding concrete metric formulae.

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Hamiltonian quantum product in equivariant cohomology

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(joint work with Gang Tian)

1. Introduction

The Gromov–Witten (GW) invariants of a symplectic manifold can be used to define a deformation of the usual ring structure of the singular cohomology of the manifold tensored by Novikov's ring. This deformed ring is associative, essentially as a consequence of a gluing property of GW invariants.

We describe an analogue of this deformed ring structure in the setting of compact symplectic manifolds endowed with a Hamiltonian circle action. GW invariants are replaced by the so-called Hamiltonian Gromov–Witten (HGW) invariants and the new ring structure is defined on the equivariant cohomology of the symplectic manifold.

2. GW invariants and quantum cohomology

In this section we review the main ideas in the definition of GW invariants and quantum cohomology (see for example [3] for details). Let (X, ω) be a a compact symplectic manifold and pick an almost complex structure $I \in \operatorname{End} TX$ such that $\omega(\cdot, I\cdot)$ is a Riemannian metric. Define for any $A \in H_2(X;\mathbb{Q})$

$$\mathcal{M}_A = \{\phi : \mathbf{C}P^1 \to X \mid \phi_*[\mathbf{C}P^1] = A,\ \overline{\partial}_I \phi = 0\},$$

where $\overline{\partial}_I \phi = d\phi \circ J - I \circ d\phi$ and $J \in \operatorname{End} T\mathbf{C}P^1$ is the standard complex structure on the projective line. Take three different points $x_0, x_1, x_2 \in \mathbf{C}P^1$ and define $e_j : \mathcal{M}_A \to X$ as $e_j(\phi) = \phi(x_j)$. Consider on $\mathcal{M}_A$ the compact-open topology. One of the main results in GW theory is the fact that $\mathcal{M}_A$ admits a compactification $\overline{\mathcal{M}}_A$ which (roughly speaking) carries a canonical homology class $[\overline{\mathcal{M}}_A] \in H_{2(\langle c_1(TX),A\rangle+n)}(\overline{\mathcal{M}}_A)$ and such that the maps e_0, e_1, e_2 can be extended to maps from $\overline{\mathcal{M}}_A$ to X.

Pick cohomology classes $\alpha, \beta, \gamma \in H^*(X)$. The GW invariant is defined as $\Phi_A(\alpha,\beta,\gamma) = \langle e_0^*\alpha \cup e_1^*\beta \cup e_2^*\gamma, [\overline{\mathcal{M}}_A]\rangle$. Write the Poincaré dual of the diagonal in $X \times X$ using Künneth as $PD[\Delta_X] = \sum u_i \otimes v_i \in H^*(X) \otimes H^*(X)$. The quantum product of two cohomology classes $\alpha, \beta \in H^*(X)$ is defined as $\alpha * \beta = \sum_A \sum_i \Phi_A(\alpha,\beta,u_i) v_i q^A$. This takes values in $QH^*(X) := H^*(X)[\{q^{A_i}\}]$, where $A_1, \dots, A_r$ is a basis of $H_2(X;\mathbb{Q})$. Extending the product $*$ so that it is q^A linear, we get the ring structure in quantum cohomology $* : QH^*(X) \otimes QH^*(X) \to QH^*(X)$. Associativity is equivalent to

$$\sum_{A_1+A_2=A} \sum_i \Phi_{A_1}(\alpha,\beta,u_i)\Phi_{A_2}(v_i,\gamma,\delta) = \sum_{A_1'+A_2'=A} \sum_i \Phi_{A_1'}(\gamma,\beta,u_i)\Phi_{A_2'}(v_i,\alpha,\delta)$$

holding for any $\alpha,\beta,\gamma,\delta \in H^*(X)$. This equality is proved using a cobordism argument which uses crucially Gromov's compactness theorem for holomorphic maps. One of the important features of this theorem is the following: suppose that $\{\Sigma_k\}_{k\in(n)}$ is a sequence of stable curves which converges as $k \to \infty$ to a nodal curve Σ; if for each k there is a holomorphic map $\phi_k : \Sigma_k \to X$ representing a homology class independent of k, then passing to a subsequence there is a limit map $\phi_{k_i} \to \phi$, where $\phi : \Sigma \to X$ is continuous and is holomorphic when pulled back to the normalization of Σ. Continuity of the limit map is a consequence of the following remarkable fact: there are constants K, ϵ such that if $\phi : [0,N] \times S^1 \to X$ is a holomorphic map, where N *is arbitrary* and the energy of ϕ is less than ϵ, then the diameter of the image of ϕ is less than K times the energy of ϕ.

3. Hamiltonian Gromov Witten invariants

Suppose that X carries a Hamiltonian action of the circle with moment map $\mu : X \to \mathbf{i}\mathbb{R}$, and assume that I is invariant under the action. If $P \to \mathbf{C}P^1$ is a circle bundle and ϕ is a continuous section of $P \times_{S^1} X$, then combining the classifying map of P with ϕ one gets a map $\rho(P,\phi)$ from $\mathbf{C}P^1$ to the Borel construction $X_{S^1} = ES^1 \times_{S^1} X$. Given an equivariant homology class $B \in H_2^{S^1}(X;\mathbb{Q})$ and $c \in \mathbf{i}\mathbb{R}$ consider the moduli space

$$\mathcal{M}_{B,c} = \{(P,A,\phi) \mid \rho(P,\phi)_*[\mathbf{C}P^1] = B,\ \overline{\partial}_A \phi = 0,\ *F_A + \mu(\phi) = c\}/\sim,$$

where $\overline{\partial}_A\phi = d_A \circ J - I \circ d_A$, d_A is the covariant derivative, $*$ is the Hodge star on $\mathbf{C}P^1$ (we assume a volume form has been chosen), F_A is the curvature of A and $\sim$ denotes gauge equivalence. The objects parameterized by $\mathcal{M}_{B,c}$ are called twisted holomorphic maps. If c does not belong to a certain finite critical set $\mathcal{C} \subset \mathbf{i}\mathbb{R}$, then one can define a Poincaré bundle $\mathcal{P} \to \mathcal{M}_{B,c} \times \mathbf{C}P^1$ and a universal section Φ of $\mathcal{P} \times_{S^1} X$. Restricting both objects to $\mathcal{M}_{B,c} \times \{x_j\}$ one gets a map $e_j : \mathcal{M}_{B,c} \to X_{S^1}$. The following is proved in [7]

Theorem 1. *(-,Tian) If $c \notin \mathcal{C}$ then one can define a compactification $\overline{\mathcal{M}}_{B,c}$ of $\mathcal{M}_{B,c}$ in such a way the maps e_j extend to $\overline{\mathcal{M}}_{B,c}$. Furthermore, there is a canonical homology class $[\overline{\mathcal{M}}_{B,c}] \in H_{2(\langle c_1^{S^1}(TX),B\rangle+n-1)}(\overline{\mathcal{M}}_{B,c};\mathbb{Q})$.*

Given equivariant cohomology classes $\alpha, \beta, \gamma \in H^*_{S^1}(X)$ one defines the HGW invariant (see [1, 2, 4]) as $\Psi_{B,c}(\alpha,\beta,\gamma) = \langle e_0^*\alpha \cup e_1^*\beta \cup e_2^*\gamma, [\overline{\mathcal{M}}_{B,c}]\rangle$. To encode the HGW invariants in an associative ring structure one cannot use the diagonal class as in GW theory. This can be motivated geometrically by the following (see [6]): if $\Sigma_k \to \Sigma$ is a sequence of smooth curves converging to a nodal curve, and (P_k, A_k, ϕ_k) is a twisted holomorphic map on Σ_k such that $\rho(P_k,\phi_k)_*[\Sigma_k]$ is independent of k, then passing to a subsequence one obtains a limit triple (P, A, ϕ) defined on Σ, which becomes a twisted holomorphic map when pullbacked to the normalization of Σ, but in which ϕ is not a continuous section when defined on Σ. Instead, the images of ϕ on the two preimages of any node are in general connected by a gradient line of $-\mathbf{i}\mu$. This happens because the uniform bound on the diameter of holomorphic cylinders quoted above does not hold for twisted holomorphic maps. Here is an example. Let $f : \mathbb{R} \to X$ be a nonconstant gradient line of $-\mathbf{i}\mu$ and define for any $l > 0$ the renormalized map $f_l(t) = f(lt)$. The energy $\|f_l'\|_{L^2}$ converges to 0 as $l \to 0$. Now take on the trivial circle bundle over $\mathbb{R} \times S^1$ the connection A such that $d_A = d - \mathbf{i}l d\theta$ and the section ϕ represented by the map $\phi(t,\theta) = f_l(t)$. We then have $\overline{\partial}_A\phi = 0$, the energy $\|d_A\phi\|^2_{L^2} = 2\|f_l'\|^2_{L^2} \to 0$ as $l \to 0$, but the diameter of ϕ is always the same nonzero number. (We ignore here the vortex equation $*F_A + \mu(\phi) = c$, since it never poses a problem on long cylinders taken as conformal models for neighborhoods of vanishing cycles, because the volume form becomes nearly zero away from the boundary of the cylinder.)

4. The thick diagonal

Let $\xi_t : X \to X$ denote the gradient flow at time $t \in \mathbb{R}$ of the function $-\mathbf{i}\mu$. Define the set $\Delta^*_{\mathcal{C}} = \{(x,y) \in X \times X \mid \exists t \text{ such that } \xi_t(x) \in S^1 \cdot y\}$. This is an open submanifold of $X \times X$, but its closure is not in general a submanifold.

Theorem 2. *(-) There exists a cohomology class $[\Delta_{\mathcal{C}}] \in H^*_{S^1\times S^1}(X \times X)$ "representing the Poincaré dual of the closure of $\Delta^*_{\mathcal{C}}$".*

(See [5].) Since $(X \times X)_{S^1\times S^1} \sim X_{S^1} \times X_{S^1}$, we can write $[\Delta_{\mathcal{C}}] = \sum r_i \otimes s_i \in H^*_{S^1}(X) \otimes H^*_{S^1}(X)$. If c is a regular value of the moment map, $X_c = \mu^{-1}(c)/S^1$ is the symplectic quotient, and $\kappa_c : H^*_{S^1}(X) \to H^*(X_c)$ is the Kirwan map, then $\sum_i \kappa(r_i) \otimes \kappa(s_i)$ represents the diagonal in X_c. This implies that the map l_c :

$H^*(\mu^{-1}(c)/S^1) \to H^*_{S^1}(X)$ defined as $l_c(a) = \sum_i(\int_{X_c} a \cup r_i)s_i$ is a right inverse of the Kirwan map. Define a new product in $H^*_{S^1}(X)$ as follows: for any $\alpha, \beta \in H^*_{S^1}(X)$, $\alpha \cup_c \beta = l_c(\kappa_c(\alpha) \cup \kappa_c(\beta))$. Since $\kappa_c \circ l_c = \mathrm{id}$, this product is associative.

5. Hamiltonian quantum product

Given classes $\alpha, \beta \in H^*_{S^1}(X)$, define

$$\alpha *_c \beta = \sum_B \sum_i \Psi_{B,c}(\alpha, \beta, r_i) s_i q^B \in QH^*_{S^1}(X) = H^*_{S^1}(X)[\{q^{B_i}\}],$$

where $\{B_i\}$ is a basis of $H_2^{S^1}(X)$, and extend the product q^B-linearly to get $*_c : QH^*_{S^1}(X) \otimes QH^*_{S^1}(X) \to QH^*_{S^1}(X)$. We have:

Theorem 3. *(-,Tian) The product $*_c$ is associative. Restricting to $q^B = 0$ one recovers the product $\cup_c$ defined above.*

This is proven in [7] using the same cobordism argument as in GW theory and the compactness theorem proven in [6].

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Link homology theories from symplectic geometry

Ciprian Manolescu

Khovanov and Rozansky [3] associated to every link $\kappa \subset S^3$ a series of bigraded cohomology theories $KR^{i,j}_{(n)}(\kappa)$ for $n > 0$ and showed that they are link invariants. Their theories can be interpreted as categorifications of the $sl(n)$ quantum link polynomial $P_{(n)}$, in the sense that

$$P_{(n)}(\kappa) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}} KR^{i,j}_{(n)}(\kappa).$$

When $n = 2$, they recover the older categorification of the Jones polynomial due to Khovanov [2].

Khovanov-Rozansky homology is particularly interesting because it is conjectured to be related to the knot Floer homology of Ozsváth-Szabó and Rasmussen

[5], [6]. Knot Floer homology is an invariant defined used Lagrangian Floer homology, and an important question is to find a way to compute it algorithmically. Its graded Euler characteristic is the Alexander polynomial $P_{(0)}$. On the other hand, Khovanov-Rozansky homology is algorithmically computable by definition, and the hope is to be able to extract the case $n = 0$ from the $n > 0$ theories. A precise conjecture in this direction was made by Dunfield, Gukov and Rasmusen in [1], and a potentially useful triply graded categorification of the HOMFLY polynomial was constructed by Khovanov and Rozansky in the sequel [4].

In this talk we present the construction of a sequence of link invariants $KR^*_{(n)\,\mathrm{symp}}$ using Lagrangian Floer theory. This was done by Seidel and Smith in the case $n = 2$ [7], and our work is a generalization of theirs. We conjecture that our invariants are related to Khovanov-Rozansky homology of the mirror link $\kappa^!$ in the following way:

$$KR^k_{(n)\,\mathrm{symp}}(\kappa) \otimes \mathbb{Q} = \bigoplus_{i+j=k} KR^{i,j}_{(n)}(\kappa^!).$$

Our construction is inspired from that of Seidel and Smith. We start by presenting the link κ as the closure of an m-stranded braid $\beta \in Br_m$. The rough idea is to find a symplectic manifold (M, ω) with an action of the braid group by symplectomorphisms $\phi : Br_m \to Symp(M, \omega)$, to take a specific Lagrangian $L \subset M$ and to consider the Floer cohomology of L and $\phi(\beta)L$ in M. Our example of a manifold M comes from Lie theory. Consider the adjoint action of the Lie group $SL(mn)$ on its Lie algebra. We define the bipartite configuration space

$$BConf_m = \Big\{ \big(\{\lambda_1, \dots, \lambda_m\}, \{\mu_1, \dots, \mu_m\}\big) \mid \lambda_i, \mu_j \in \mathbb{C} \text{ are all distinct} \Big\}.$$

The elements in each of the two sets $\lambda = \{\lambda_1, \dots, \lambda_m\}$ and $\mu = \{\mu_1, \dots, \mu_m\}$ are not ordered, but the pair $\tau = (\lambda, \mu)$ is ordered. The fundamental group of $BConf_m$ is the colored braid group on two colors $Br_{m,m}$. This has a (noncanonical) subgroup isomorphic to Br_m, which corresponds to keeping the μ's fixed.

Consider the diagonal matrix D_τ with entries $\lambda_1, \dots, \lambda_m$ (each with multiplicity 1) and $\mu_1, \dots, \mu_m$ (each with multiplicity $n-1$), chosen so that the trace of D_τ is zero. Let O_τ be the orbit of D_τ under the adjoint action. We define M to be the intersection of O_τ with a transverse affine slice to the nilpotent made of n Jordan blocks of size m. As we vary $\lambda_1, \dots, \lambda_m$ keeping their sum constant, the respective manifolds M form a symplectic fibration, and that induces the desired action ϕ by symplectomorphisms. Using an inductive procedure in m, we construct a Lagrangian $L \subset M$ diffeomorphic to the product of m copies of $\mathbb{CP}^{n-1}$.

Our main result is that the shifted Floer cohomology groups

$$KR^*_{(n)\,\mathrm{symp}}(\kappa) = HF^{*+(n-1)(m+w)}(L, \phi(\beta)L)$$

are link invariants. The proof of this involves checking invariance under the Markov moves I and II which relate braids with the same closure.

We managed to compute the groups $KR_{(n)\,\mathrm{symp}}$ in a few cases. For the unknot we have $KR^*_{(n)\,\mathrm{symp}}(\mathrm{unknot}) = H^{*+n-1}(\mathbb{CP}^{n-1})$, while for the unlink of p components we get the tensor product of p copies of the same group. The first nontrivial computation is for the trefoil T, for which we have the following result, consistent with the formula in [1, Proposition 6.6]:

$$KR^*_{(n)\,\mathrm{symp}}(T) = H^{*-n+1}(\mathbb{CP}^{n-1}) \oplus H^{*-n-1}(UT\mathbb{CP}^{n-1}),$$

where $UT\mathbb{CP}^{n-1}$ is the unit tangent bundle to $\mathbb{CP}^{n-1}$.

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Broken pencils and four-manifold invariants

Tim Perutz

In this talk I outlined the construction of an invariant, the 'Lagrangian matching invariant', attached to a pair (X, π) comprising a smooth, closed, oriented four-manifold X and a *broken fibration* π—a smooth map onto a surface with special requirements on its critical points. For any X with $b_2^+ > 0$, the blow-up $X \# n\overline{\mathbb{C}P^2}$ admits broken fibrations for $n \gg 0$. The invariant is constructed using standard symplectic techniques—holomorphic curves with Lagrangian boundary conditions—-and generalises the Donaldson-Smith invariant of Lefschetz fibrations [2]. It can be arranged as a map

$$\mathrm{Spin}^c(X) \supset \mathcal{S} \xrightarrow{L_{X,\pi}} \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H^1(X),$$

where $\mathcal{S}$ is a certain subset of the set of Spin^c-structures.[1] The right-hand side is a graded ring with $\deg(U) = 2$; with respect to this grading,

$$\deg(L_{X,\pi}(\mathfrak{s})) = \frac{1}{4}\big(c_1(\mathfrak{s})^2 - 2e(X) - 3\sigma(X)\big).$$

[1] In principle it should be possible to set $\mathcal{S} = \mathrm{Spin}^c(X)$, but at present there are technical obstacles to doing so.

The Seiberg-Witten invariant of X can be formulated as a map of the same kind (defined on the whole of $\mathrm{Spin}^c(X)$), and I conjecture that

$$L_{X,\pi}(\mathfrak{s}) = \pm \mathrm{SW}_X(\mathfrak{s}), \quad \mathfrak{s} \in \mathcal{S}.$$

This has been verified in some cases, including ones where the Seiberg-Witten invariant is highly non-trivial. The Lagrangian matching invariant arises from a field theory for fibrations over 1-manifolds and singular fibrations over surfaces with boundary, and this resembles a version of Seiberg-Witten monopole Floer homology. Details are given in the author's Ph.D. thesis [3].

Context. One of the strands of current 4-manifold research is the attempt to understand non-symplectic four-manifolds from a symplectic point of view. The hope is to extend the techniques which have had such success in elucidating symplectic 4-manifolds—Taubes' interpretation of the Seiberg-Witten invariant as a Gromov invariant counting holomorphic curves, and Donaldson's theorem on the existence of Lefschetz pencils—to smooth but non-symplectic manifolds. This is sensible because any closed, oriented, smooth four-manifold X with $b^+ > 0$ admits a *near-symplectic form* in the sense of [1], that is a closed two-form ω such that at any point $x \in X$, either $\omega_x^2 > 0$, or else $\omega_x = 0$, and in the latter case satisfying a transversality condition: the image of the (intrinsic) gradient $\nabla_x \omega \colon T_x X \to \Lambda^2 T_x^* X$, which is a positive-definite subspace for the wedge-square form, should be as as large as possible, i.e. 3-dimensional. A near symplectic-form is symplectic off its zero-set $Z = \omega^{-1}(0)$ (a 1-submanifold).

Two major results in near-symplectic geometry motivate our constructions:

1. Theorem of Taubes [4]**.** Let (X, ω) be closed and near-symplectic, with ω self-dual for a metric g, so that on $X \setminus Z$ one has a compatible almost complex structure J_g. Then any sequence (A_n, ψ_n) of solutions to the Seiberg-Witten equations on (X, g), with a perturbation term $n\omega$, has a subsequence which concentrates along a finite-area J_g-holomorphic curve in $X \setminus Z$. This carries a fundamental class in $H_2(X, Z)$ which maps to $[Z] \in H_1(Z)$—thus, in a homological sense at least, the curve bounds Z. Taubes has a programme to recover the Seiberg-Witten invariant by 'counting' such curves using methods from symplectic field theory. The technical difficulties involved in establishing such a count are, however, formidable.

2. Theorem of Auroux, Donaldson and Katzarkov [1]. This states that any closed near-symplectic 4-manifold (X, ω) admits a 'singular Lefschetz pencil' $\pi \colon X \setminus \{b_1, \dots, b_n\} \to S^2$ with critical points along its zero-set Z. I call these objects *broken pencils* instead. Here are the defining conditions.

Near a 'basepoint' b_i, the map is modelled, in local complex coordinates, on the projectivisation map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$. The fibres of π have compact closures in X. The set of critical points of π is the union of a discrete set D and the 1-manifold Z. Each point $p \in D$ has a neighbourhood U such that the map $U \to \pi(U)$ is smoothly equivalent to $(z_1, z_2) \mapsto z_1 z_2$. Each point $p \in Z$ has a neighbourhood U such that $U \to \pi(U)$ is equivalent to one of the two maps $\mathbb{R}^4 \to \mathbb{R}^2$, $(t, x_1, x_2, x_3) \mapsto (t, \pm(x_1^2 - x_2^2 - x_3^2))$. Furthermore, each component of Z is mapped diffeomorphically to its image.

In fact, a near-symplectic manifold (X,ω) admits a broken pencil *such that* $\pi(Z)$ *consists of just one circle in* S^2.

When $Z=\emptyset$, π is, by definition, a topological Lefschetz pencil. A *broken fibration* is a broken pencil for which the set of basepoints b_i is empty. One converts a broken pencil over S^2 into a broken fibration by blowing up the basepoints.

Three-dimensional Morse theory gives another source of broken fibrations: let $f\colon Y^3\to S^1$ be a circle-valued Morse function without extrema; then

$$f\times \mathrm{id}_{S^1}\colon Y\times S^1\to S^1\times S^1$$

is a broken fibration, with one circle of critical points for each critical point of f. There are Lagrangian matching invariants in this situation (though not at present for all Spin^c-structures) and these coincide with the Seiberg-Witten invariants, which are Turaev torsions of Y.

Idea of the construction. Consider, for simplicity, a broken fibration over S^2 with just one circle Z of critical points, mapping to an 'equator' $\pi(Z)$ lying in an 'equatorial' annulus $A=S^1\times[-1,1]$. We suppose that none of the discrete critical values lies in A. Thus $\overline{S^2\setminus A}$ is the union of two discs $D^\pm$, and over them we have Lefschetz fibrations $X^\pm\to D^\pm$. The topology of the smooth fibres changes as one crosses the equator—on side, over D^+ say, they are connected of genus g, while on the other side, over D^-, they are either connected of genus $g-1$ or else disconnected with components of genera h and $g-h$.

The restriction of π to ∂X^+ is a bundle over its image S^1. We fix $r\in\mathbb{N}$ and consider its *relative symmetric product* $\mathrm{Sym}^r_{S^1}(\partial X^+)\to S^1$ (the quotient of the r-fold fibre product by the symmetric group). Similarly we have a bundle $\partial X^-\to S^1$, and we consider its symmetric product $\mathrm{Sym}^{r-1}_{S^1}(\partial X^-)\to S^1$. The construction of $L_{X,\pi}$ hinges on the existence of a sub-fibre bundle

$$\mathcal{L}\subset \mathrm{Sym}^r_{S^1}(\partial X^+)\times_{S^1}\mathrm{Sym}^{r-1}_{S^1}(\partial X^-),$$

canonical up to isotopy, and fibrewise middle-dimensional.

Now, $\mathrm{Sym}^r_{S^1}(\partial X^+)$ is the boundary of $\mathrm{Sym}^r_{D^+}(D^+)$. The latter is singular (because of the nodal fibres of $X^+\to D^+$) but it has a natural resolution $\mathrm{Hilb}^r_{D^+}(X^+)\to \mathrm{Sym}^r_{D^+}(X^+)$ relative to D^+, the *relative Hilbert scheme of r points*, which does not affect the non-singular fibres. Thus $\mathrm{Sym}^r_{S^1}(\partial X^+)$ is also the boundary of $\mathrm{Hilb}^r_{D^+}(X^+)$. Likewise, $\mathrm{Sym}^{r-1}_{S^1}(\partial X^-)$ is the boundary of $\mathrm{Hilb}^{r-1}_{D^-}(X^-)$. These Hilbert schemes admit symplectic forms, $\Omega^\pm$, canonical up to deformation, such that $\mathcal{L}$ is *isotropic* with respect to $-\Omega^+\oplus\Omega^-$. After choosing suitable almost complex structures on the two Hilbert schemes, making the projections to $D^\pm$ holomorphic, one has a Fredholm moduli space of pairs (u^+,u^-) of holomorphic sections of the two Hilbert schemes, taking boundary values in $\mathcal{L}$. The invariant is derived from this moduli space. It has 'topological sectors' which correspond to a certain subset of $\mathrm{Spin}^c(X)$; one cuts down positive-dimensional sectors to zero dimensions via certain cycles in fibres, and this procedure gives rise to the group $\mathbb{Z}[U]\otimes\Lambda^*H^1(X)$. Compactness can be established for many values of r, though it is problematic in certain cases (assuming π has connected fibres, the difficulties arise when $r\in(g/2,g-1)$).

What is $\mathcal{L}$? Each of its fibres is a 'vanishing cycle' for the relative Hilbert scheme of r points of an elementary Lefschetz fibration over a disc.

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Rigidity of Asymptotically Hyperbolic Manifolds

Yuguang Shi

(joint work with Gang Tian, MIT)

During these years, conformally compact manifolds are one of favorite manifolds in mathematic physics.

Definition 1. *A complete Riemannian (M, g) is conformally compact if:*

- *M is diffeomorphic to the interior of a compact manifold $\bar{M}$ with non empty boundary;*
- *There is a smooth function f on $\bar{M}$ which is positive in interior of $\bar{M}$ and is zero on $\partial\bar{M}$ and $|df|_{\partial\bar{M}} \neq 0$ such that $\bar{g} = f^2 g$ can be extended smoothly on $\bar{M}$.*

Example 2. *Hyperbolic space $\mathbb{H}^{n+1} = (\mathbb{B}^{n+1}, \frac{4dS^2}{(1-\|x\|^2)^2})$ is conformally compact manifold, here $\mathbb{B}^{n+1}$ is the unit ball in $\mathbb{R}^{n+1}$, $\|.\|$ is the standard Euclidean normal, dS^2 is the standard Euclidean metric.*

In the above example, $M = \mathbb{B}^{n+1}$, $\bar{M} = \bar{\mathbb{B}}^{n+1}$, $g = \frac{4dS^2}{(1-\|x\|^2)^2}$ and $f = \frac{1-\|x\|^2}{2}$

In a conformally compact manifold, it is interesting to see that the conformal structure at infinity of the manifold is completely determined by the conformal structure of the manifold itself.

Conformally compact manifolds have deep relations with a kind of manifolds so called asymptotically hyperbolic manifolds.

Definition 3. *A complete Riemannian manifold (M, g) is called asymptotically hyperbolic if:*

- *$M \setminus K$ is diffeomorphic to $\mathbb{H}^{n+1} \setminus K'$, here K and K' are compact sets of M and $\mathbb{H}^{n+1}$ respectively, in particular, there are global coordinates at the infinity of M;*
- *In the coordinates at the infinity of M, the metric matrix (g_{ij}) becomes closer and closer to that of the standard hyperbolic metric at infinity of M in a certain sense.*

In fact, many conformally compact manifolds are asymptotically hyperbolic, for instance, one can show that conformally compact Einstein manifolds are asymptotically hyperbolic.

There are many results on rigidity of asymptotically hyperbolic manifolds. For instance,

Theorem 4. *(M. Min-Oo, 1989) Suppose* (X^{n+1}, g) *is a spin manifold and it is asymptotically hyperbolic in stronger sense. If* $R \geq -n(n+1)$*, then* (X^{n+1}, g) *is isometric to* $\mathbb{H}^{n+1}$*.*

and

Theorem 5. *(Qing, 2003) Suppose* (X^{n+1}, g) *is a conformally compact Einstein manifold and* $3 \leq n \leq 6$*. If the conformal structure at infinity of the manifold is equivalent to that of the standard sphere, then* (X^{n+1}, g) *is isometric to* $\mathbb{H}^{n+1}$*.*

Remark: By the assumption of the theorem, one can prove that the manifold is actually asymptotically hyperbolic.

In all the results above, one needs to assume that there are nice coordinates at infinity and in such coordinates, the metrics tensor behaves well. In view of geometry, it would be natural to ask:

Problem 6. *Can we define conformally compact manifolds and asymptotically hyperbolic manifolds in an intrinsic way? Is a rigidity theorem still true in such a category?*

One years ago, Tian and I gave a partial answer to this question. In order to state the result, we have to introduce some notation first.

Let $(X^{n+1},\ g)$ be a complete noncompact Riemannian manifold, we call it an asymptotically locally hyperbolic manifold, which we abbreviate as ALH in the following, of order α if $|K(x) + 1| = O(e^{-\alpha\rho(x)})$, where $K(x)$ is the sectional curvature of g at the point x in any direction and $\rho(x) = dist_g(x, o)$.

Recall that a Riemannian manifold X has a pole o if the exponential map $exp_o : T_oX \to X$ is a diffeomorphism. Without loss of generality, in our case, we may assume that the sectional curvature is negative outside a unit ball of (X, g). We have:

Theorem 7. *Suppose that* (X^{n+1}, g) $n \geq 2$ *and* $n \neq 3$ *is an ALH manifold of order* α *with a pole and there is a* $\rho > 1$ *such that the geodesic sphere with radius* ρ *and center at the pole is convex. If we further have* $\alpha > 2$ *and* $Ric(g) \geq -ng$*, then* (X^{n+1}, g) *is isometric to* $\mathbb{H}^{n+1}$*.*

As a corollary, we have:

Corollary 8. *Suppose that* (X^{n+1}, g) $n \geq 2$ *and* $n \neq 3$ *is a simply connected ALH manifold of order* α *(*$\alpha > 2$*),* $K \leq 0$ *and* $Ric(g) \geq -ng$*, then* (X^{n+1}, g) *is isometric to* $\mathbb{H}^{n+1}$*.*

Let Rm^0 denotes the traceless part of the curvature tensor[1], $\|Rm^0\|$ denote the norm of the tensor for (X, g), then for $n = 3$, we have:

Theorem 9. *Suppose that (X^4, g) is an ALH manifold of order $\alpha > 2$ with a pole and there is a $\rho > 1$ such that the geodesic sphere with radius ρ and center at the pole is convex. If we further have $\|Rm^0\| \in L^1(X)$ and $Ric(g) \geq -3g$, then (X^4, g) is isometric to $\mathbb{H}^4$.*

Remark: The assumption $\alpha > 2$ should be optimal, since there are many asymptotically hyperbolic Einstein metrics on $\mathbb{B}^4$ with $\alpha = 2$. In the case of $n = 3$, in order to show locally conformal flatness of the boundary (we need this fact in the proof), one has to check that certain linear combinations of covariant derivatives of the Schouten tensor vanish. For the time being, we do not know how to deduce this from the assumption $\alpha > 2$; this is the reason why we need the extra assumption $\|Rm^0\|_g \in L^1(X)$, we doubt its necessity. We also think that the assumption on existence of a pole is unnecessary.

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Foliations of asymptotically flat 3-manifolds by 2-surfaces prescribed mean curvature

Jan Metzger

Surfaces with prescribed mean curvature play an important role for example in the field of general relativity. In our paper [2], which is presented here, we consider

[1] The metric g is of constant sectional curvature iff Rm^0 vanishes. This property determines Rm^0 uniquely.

the subsequent slicing of a three dimensional spacelike slice by two dimensional spheres with prescribed mean curvature in the three geometry.

To be more precise, let (M, g, K) be a set of initial data. That is, (M, g) is a three dimensional Riemannian manifold and K is a symmetric bilinear form on M. This can be interpreted as the extrinsic curvature of M in the surrounding four dimensional space time. We consider 2-surfaces Σ satisfying the quasilinear degenerate elliptic equations $H + P = \text{const}$ where H is the mean curvature of Σ in (M, g) and $P = \operatorname{tr}^\Sigma K$ is the two dimensional trace of K.

In the case where $K \equiv 0$ this equation particularizes to $H = \text{const}$, which is the Euler-Lagrange equation of the isoperimetric problem. This means that surfaces satisfying $H = \text{const}$ are stationary points of the area functional with respect to volume preserving variations. Yau suggested to use such surfaces to describe physical information in terms of geometrically defined objects. Indeed Huisken and Yau [1] have shown that the asymptotic end of an asymptotically flat manifold, with appropriate decay conditions on the metric, is uniquely foliated by such surfaces which are stable with respect to the isoperimetric problem. The Hawking mass

$$m_{\mathrm{H}}(\Sigma) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_\Sigma H^2 d\mu \right)$$

of such a surface Σ is monotone on this foliation and converges to the ADM-mass. This foliation can also be used to define the center of mass of an isolated system since for growing radius, the surfaces approach Euclidean spheres with a converging center. Therefore the static physics of an isolated system considered as point mass is contained in the geometry of the $H = \text{const}$ foliation. However, these surfaces are defined independently of K, such that no dynamical physics can be found in their geometry. A different proof of the existence of CMC surfaces is due to Ye [4].

Our result generalizes the CMC foliations to include the dynamical information into the definition of the foliation. The equation $H + P = \text{const}$ was chosen since apparent horizons satisfying $H = 0$ in the case $K \equiv 0$ generalize to surfaces satisfying $H + P = 0$ when K does not necessarily vanish.

We consider asymptotically flat data describing isolated gravitating systems. For constants $m > 0$, $\sigma \geq 0$, and $\eta \geq 0$ data (M, g, K) will be called (m, σ, η)-asymptotically flat if there exists a compact set $B \subset M$ and a diffeomorphism $x : M \setminus B \to \mathbf{R}^3 \setminus B_\sigma(0)$ such that in these coordinates g is asymptotic to the conformally flat spatial Schwarzschild metric g^S representing a static black hole of mass m. Here, $g^S = \phi^4 g^e$, where $\phi = 1 + \frac{m}{2r}$, g^e is the Euclidean metric, and r is the Euclidean radius. The asymptotics we require for g and K are

$$\sup_{\mathbf{R}^3 \setminus B_\sigma(0)} \left(r|g - g^S| + r^2 |\nabla - \nabla^S| + r^3 |\mathrm{Ric} - \mathrm{Ric}^S| \right) < \eta m \,,$$

$$\sup_{\mathbf{R}^3 \setminus B_\sigma(0)} \left(r^2 |K| + r^3 |\nabla^g K| \right) < \eta m \,.$$

Here ∇ and ∇^S denote the Levi-Civita connections of g and g^S on TM, such that $\nabla - \nabla^S$ is a $(1, 2)$-tensor. Furthermore Ric and Ric^S denote the respective Ricci tensors of g and g^S. That is, we consider data arising from a perturbation of the Schwarzschild data $(g^S, 0)$.

The main theorem will be proved for small $\eta > 0$. These conditions are optimal in the sense that we only impose conditions on geometric quantities, not on partial derivatives. They include far more general data than similar results. Huisken and Yau [1] for example demand that $g - g^S$ decays like r^{-2} with corresponding conditions on the decay of the derivatives up to fourth order, while we only need derivatives up to second order. In particular we allow data with nonzero ADM-momentum. For such data we can prove the following:

Theorem 1. *There exists $\eta_0 > 0$, such that if the data (M, g, K) are (m, σ, η_0)-asymptotically flat for some $\sigma > 0$, there is $h_0 = h_0(m, \sigma)$ and a differentiable map*

$$F : (0, h_0) \times S^2 \to M : (h, p) \mapsto F(h, p)$$

satisfying the following statements.

(i) *The map $F(h, \cdot) : S^2 \to M$ is an embedding. The surface $\Sigma_h = F(h, S^2)$ satisfies $H + P = h$ with respect to (g, K). Each Σ_h is convex, that is it satisfies $|A|^2 \leq C(m, \sigma) \det A$. Here A denotes the second fundamental form of Σ_h.*
(ii) *There is a compact set $\bar{B} \subset M$, such that $F\big((0, h_0), S^2\big) = M \setminus \bar{B}$.*
(iii) *The surfaces $F(h, S^2)$ form a regular foliation.*
(iv) *Every convex surface Σ with $H + P = h$ and $|A|^2 \leq C' \det A$, contained in $\mathbf{R}^3 \setminus B_{(ch)^{-1}}(0)$ equals Σ_h, provided $0 < h < h_1 = h_1(\sigma, C')$. Hence the foliation is unique in the class of convex foliations.*

Remarks. (i) This theorem does not need that (M, g, K) satisfy the constraint equations. It can be generalized to give the existence of a foliation satisfying $H + P_0(\nu) = \text{const}$, where $P_0 : SM \to \mathbf{R}^3$ is a function on the sphere bundle of M with the same decay as K.
(ii) By methods of Quing and Tian [3], the uniqueness part can be improved to hold for $\Sigma_h \subset \mathbf{R}^3 \setminus B_{h^{-\gamma}}(0)$, for any $\gamma > 0$, provided the fall-off of g and K are better by a factor $r^{-\delta}$. Then the h_1 in the statement also depends on γ.
(iii) Our result includes the existence results from Huisken and Yau [1] for CMC foliations. In this setting convexity can be replaced by stability.

The relation to linear momentum in general relativity is given by the next theorem. To state it we introduce the tensor

$$K^Y_{ij} = \tfrac{3}{2r^2}\big(P_i\rho_j + \rho_i P_j + P_k\rho^k(\delta_{ij} - \rho_i\rho_j)\big)\,,$$

where $\rho_i = x_i/r$ is the radial direction and $P_i \in \mathbf{R}^3$ is a fixed vector representing the ADM-momentum of K_{ij}. This tensor was proposed by York [5] as a model for the highest order term in K, representing linear momentum. We then consider data (g, K) with the asymptotics

$$\sup_{\mathbf{R}^3 \setminus \bar{B}_\sigma(0)} \left\{ r^{1+\delta}|g - g^S| + r^{2+\delta}|\nabla - \nabla^S| + r^{3+\delta}|\mathrm{Ric} - \mathrm{Ric}^S| \right\} < \infty\,,$$

$$\sup_{\mathbf{R}^3 \setminus \bar{B}_\sigma(0)} \left\{ r^{2+\delta}|K - K^Y| + r^{3+\delta}|\nabla(K - K^Y)| \right\} < \infty\,,$$

for some $\delta > 0$. Then we can prove the following position estimate. We denote by $d\mu^e$ the volume form of the Euclidean background and by $|\Sigma|_e$ the area of Σ with respect to the Euclidean metric.

Theorem 2. *Let Σ_h be the surface with $H + P = h$ from the existence theorem. Let*

$$R(h) = \sqrt{\frac{|\Sigma_h|_e}{4\pi}} \quad \text{and} \quad a(h) = \frac{1}{4\pi R^2} \int_{\Sigma_h} \mathrm{id}_{\Sigma_h}\, d\mu^e\,.$$

Then the center $a(h)$ satisfies the estimate

$$\left| \frac{a(h)}{R(h)} - \tau\left(\frac{|P|}{m}\right)\bar{P} \right| \le \frac{C}{R(h)^\delta} \quad \text{with} \quad \tau(s) = \frac{1 - \sqrt{1 - s^2}}{s} \quad \text{and} \quad \bar{P} = \frac{P}{|P|}\,.$$

The momentum therefore governs the translation of $a(h)$ as $R(h) \to \infty$.

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Singular semi-flat Calabi-Yau metrics on S^n

John Loftin

(joint work with S.T. Yau, Eric Zaslow)

The famous conjecture of Strominger-Yau-Zaslow [7] predicts the geometry of Calabi-Yau manifolds near or at a large complex structure limit point at the boundary of the moduli space. In particular, such a Calabi-Yau space M should be the total space of a fibration over a base $\bar{B}$ of real half-dimension, and the generic fiber should be a special Lagrangian torus. Over a singular locus $S \subset \bar{B}$ of codimension two, the fibers degenerate. $\bar{B}$ is expected to be a topological sphere.

The Calabi-Yau structure on M (at least over the nonsingular locus $B = \bar{B} \setminus S$) is determined by the data of an integral affine structure and a semi-flat

Calabi-Yau metric on B. An affine structure on a manifold is a maximal atlas of affine coordinate charts with gluing maps consisting of locally constant maps in $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$. An affine structure is integral if the maps are further restricted to lie in $SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$. An affine Kähler structure is a Riemannian metric locally given by the real Hessian of a potential function ϕ with respect to local affine coordinates. Note that the restricted gluing maps ensure that the affine Kähler metric $\phi_{ij}dx^i dx^j$ is a tensor. As first discussed by Cheng-Yau [2], the structure of an affine Kähler manifold is the natural notion of a real slice of a Kähler metric. In particular, the tangent bundle TB of an affine Kähler manifold B naturally carries a Kähler metric $\phi_{i\bar{j}}dz^i\overline{dz^j}$ for $z^i = x^i + \sqrt{-1}y^i$, the y variables represent the fiber tangent spaces, and ϕ is locally extended to the tangent bundle by making it constant in y. The metric on TB is Calabi-Yau if ϕ satisfies the real Monge-Ampère equation

$$\det \frac{\partial^2\phi}{\partial x^i \partial x^j} = 1.$$

In this case, the affine Kähler metric on B is called semi-flat Calabi-Yau (since the Kähler metric on TB is flat along the fibers). Moreover, if the affine structure is integral, then it is possible to define a lattice bundle $\Lambda \subset TB$ which is locally constant. The fiberwise quotient TB/Λ is then a Calabi-Yau manifold which is the total space of a fibration by special Lagrangian tori.

We are concerned with constructing semi-flat Calabi-Yau metrics on appropriate base spaces B. In dimension $n = 2$, the degeneration question has been satisfactorially answered by Gross-Wilson [3], who study limits of elliptic K3 surfaces. The smooth bases of the limits constructed are S^2 minus 24 points. In [4], we produce many more singular semi-flat Calabi-Yau metrics on S^2, though the vast majority of these will not come from limits of K3 surfaces.

Much less in known in dimension 3. The singular set should be a graph in S^3, which generically should have only vertices of valence 3. One of the main sticking points has been the lack of a good model for the semi-flat Calabi-Yau structure on a 3-ball minus such a “Y” vertex of a graph. In [5], we construct such an example by indirect means.

The techniques we use come from affine differential geometry. The graph of a solution ϕ to the real Monge-Ampère equation is a parabolic affine sphere in $\mathbb{R}^{n+1}$. Such hypersurfaces have been well studied, going back to the work of Blaschke in the 1920s. In particular, there are structure equations for such hypersurfaces. When $n = 2$, Simon-Wang can integrate these equations by using the affine metric to find a complex structure on the surface [6] as long as a certain integrability condition is satisfied. The integrability condition is a semilinear elliptic PDE in the conformal factor e^ψ of the metric:

$$\psi_{z\bar{z}} + |U|^2 e^{-2\psi} = 0,$$

where U is a holomorphic cubic differential. We solve this equation for any U on $\mathbb{CP}^1$ with poles of order 1 allowed [4]. This provides a semi-flat Calabi-Yau structure on $\mathbb{CP}^1$ minus the pole set.

The case $n = 3$ proceeds in much the same way [5]. By a result of Baues-Cortés [1], a convex solution of the real Monge-Ampère equation ϕ which is homogeneous of order 2 has level sets which are elliptic affine spheres of dimension 2. In particular, we use Simon-Wang's developing map and a similar semilinear PDE to construct an elliptic affine sphere structure on $\mathbb{CP}^1$ minus 3 points. A homothetic expansion then gives a solution on $\mathbb{R}^3$ minus the "Y" vertex of a graph.

Many questions remain about the examples constructed. In both cases, we construct an affine structure, but whether these structures are integral is hard to determine. This problem is an artifact of our method of proof: In passing from affine coordinates to the conformal coordinates determined by the metric, we forget the affine structure. The good thing about this approach is that we produce an affine structure along with the metric from purely holomorphic data. On the other hand, at least away from the singular points, we lose control of the affine holonomy. Not all information is lost, as the affine structure is determined by integrating the structure equations. To do this, however, depends on the solution to the semilinear PDE, which we only hope to control near the singular set. A possible way around this problem is to show that for all affine structures in a given large class which include the integral ones, that we may find a cubic differential U whose semi-flat Calabi-Yau structure has any given structure. (We may be able to say much more in the $n = 2$ case, as the equations involved are completely integrable.)

In $n = 3$, the Baues-Cortés elliptic affine sphere model may be too restrictive. In particular, the use of homogeneous solutions to the Monge-Ampère equation implies that the singular local at the "Y" vertex consists of straight lines with respect to the affine flat coordinates. The singular locus may consist of curved lines, as suggested in Zharkov [8]. New models should be found to address this case. Then solutions near the "Y" vertex must be patched together on all of the base $B = S^3$ minus a graph, and the metric should be "fattened" to one on a nearby Calabi-Yau manifold.

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A fully nonlinear scalar curvature

Guofang Wang

In this talk, we address some geometric and analytic aspects of k-scalar curvature, which was recently introduced by Viaclovsky in [18]. Since then there have been a lot of work related to this k-scalar curvature. See for instance [2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20]. In this detailed abstract, I will only mention my work (joint with, Pengfei Guan, J. Viaclovsky, C. -S. Lin and Yuxin Ge).

Let (M, g_0) be a compact, oriented Riemannian manifold with metric g_0 and $[g_0]$ the conformal class of g_0. Let Ric_g and R_g be the Ricci tensor and scalar curvature of a metric g respectively. And let S_g be the Schouten tensor of the metric g defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

Define $\sigma_k(g)$ be the *σ_k-scalar curvatures* or *k-scalar curvature* by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$ and σ_k is the kth elementary symmetric function. Here for an $n \times n$ symmetric matrix A we define $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \cdots, \lambda_n)$ is the set of eigenvalues of A. It is clear that $\sigma_1(g)$ is a constant multiple of the scalar curvature R_g. The k-scalar curvature is a quite natural generalization of the scalar curvature.

Let

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \leq k\}$$

be Garding's cone. A metric g is said to be *k-positive* or simply $g \in \Gamma_k^+$ if $g^{-1} \cdot S_g \in \Gamma_k^+$ for every point $x \in M$. We call u is *k-admissible* if $e^{-2u} g_0 \in \Gamma_k^+$. A metric is 1-positive is equivalent to that the metric has positive scalar curvature.

To this new scalar curvature function, one may ask the following questions:

1. Which manifolds admit a metric of k-positive?
2. Are there topological obstructions for the existence of such a metric?
3. Is there a metric in a given conformal class with constant σ_k-scalar curvature?
4. Any interesting applications?

We try to answer such questions.

Theorem 1 ([9]). *Let $2 \leq k < n/2$, and let M_1^n and M_2^n be two compact manifolds (not necessary locally conformally flat) of positive Γ_k-curvature. Then the connected sum $M_1 \# M_2$ also admits a metric of positive Γ_k-curvature. If in addition, M_1 and M_2 are locally conformally flat, then $M_1 \# M_2$ admits a locally conformally flat structure with positive Γ_k-curvature.*

This implies that on

$$(\mathbb{S}^{n-1}\times\mathbb{S})\#\mathbb{S}^{n-1}\times\mathbb{S})\#\cdots\#(\mathbb{S}^{n-1}\times\mathbb{S})$$

admits a metric of k-positive for $k < n/2$. Hence, any free product of finitely many copies of $\mathbb{Z}$ with finite many copies of the fundamental group of spherical space forms is the fundamental group of a manifold of positive Γ_k-curvature, for $k < n/2$. When $k = 1$, this is a result of Gromov-Lawson [8] and Schoen-Yau [16].

Theorem 2 ([9])**.** *Let (M^n, g) be a compact, locally conformally flat manifold with $\sigma_1(g) > 0$.*

(i). *If $g \in \overline{\Gamma}_k^+$ for some $2 \le k < n/2$, then the q-th Betti number $b_q = 0$ for*

$$\left[\frac{n+1}{2}\right]+1-k \le q \le n - \left(\left[\frac{n+1}{2}\right]+1-k\right).$$

(ii). *Suppose $g \in \Gamma_2^+$, then $b_q = 0$ for $\left[\frac{n-\sqrt{n}}{2}\right] \le q \le \left[\frac{n+\sqrt{n}}{2}\right]$. If $g \in \overline{\Gamma}_2^+$, $p = \frac{n-\sqrt{n}}{2}$ and $b_p \neq 0$, then (M, g) is a quotient of $\mathbb{S}^{n-p}\times H^p$.*

(**iii**). *If $k \ge \frac{n-\sqrt{n}}{2}$ and $g \in \Gamma_k^+$, then $b_q = 0$ for any $2 \le q \le n-2$. If $k = \frac{n-\sqrt{n}}{2}$, $g \in \overline{\Gamma}_k^+$, and $b_2 \neq 0$, then (M, g) is a quotient of $\mathbb{S}^{n-2}\times H^2$.*

Here $\mathbb{S}^{n-p}$ is the standard sphere of sectional curvature 1 and H^p is a hyperbolic plane of sectional curvature -1.

When $k = 1$, it was obtained by Bourguignon in [1].

Question 3 is the so-called the σ_k Yamabe problem. The corresponding equation is

$$\sigma_k^{1/k}\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = e^{-2u}. \tag{1}$$

When $g_0 \in \Gamma_k^+$, equation (1) is elliptic. For this equation, we first have a local estimate for solutions.

Theorem 3 ([10])**.** *Let $u \in C^4$ be an admissible solution of (1) in B_r, the geodesic ball of radius r in a Riemannian manifold (M, g_0). Then, there exists a constant $c > 0$ depending only on r, $\|g_0\|_{C^4(B_r)}$ and $\|f\|_{C^2(B_r)}$ (independent of $\inf f$), such that*

$$|\nabla u|^2 + |\nabla^2| \le c(1 + e^{-2\inf_{B_r} u}).$$

This gives a first example of a fully nonlinear equation, which has local estimates and becomes an analytic foundation for the σ_k-Yamabe problem. Now we mention all existence results for the σ_k-Yamabe problem.

Theorem 4. *Let $g_0 \in \Gamma_k^+$. The σ_k-Yamabe is solvable in the following cases:*

(i) *$k = n$, there is a sufficient condition given in* [20].
(ii) *$n = 4 = 2k$ (*[3]*)*
(iii) *(M, g) is locally conformally flat (*[11], [15]*)*
(iv) *$k > n/2$ (*[12], [14]*)*
(v) *$k = 2$ (*[6] *for $n > 8$ and* [17] *for $n > 4$)*

The problem remains open for $2 < k \leq n/2$ and M is not locally conformally flat.

The analysis developed for equation (1) has a nice application in [4]. Very recently, I have an application in estimating eigenvalue of the Dirac operator ([21]).

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Fundamental groups of manifolds with Ricci curvature bounded below

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(joint work with Vitali Kapovitch)

In general the class of compact manifolds with given lower sectional bound is much more rigid than the class of compact manifolds with a lower Ricci curvature bound. However there is a general belief that the same structure theorems for fundamental groups should hold. We present several results which confirm this philosophy.

First we give an analogue of Gromov's bound on the number of generators of the fundamental group. Recall that Gromov showed that for a compact manifold (M, g) with sectional curvature $K \geq -1$ and diameter $\mathrm{diam}(M) \leq D$ there is a constant such that the fundamental group is generated by at most C elements, where C only depends on D and $\dim(M)$. Gromov actually was able to give effective estimates for C.

Our first result asserts that the same conclusions holds if one replaces $K \geq -1$ by the corresponding Ricci curvature condition $\mathrm{Ric} \geq -(n-1)$. However our estimate for C is highly ineffective. There is another difference: Gromov was able to bound the length of a short generator system at every point in M. We can only show that for same points in M there is an an a-priori bound on the number of short generators. As a corollary we mention.

Corollary 1. *For each n there is a constant C such that the following holds. Let (M, g) be a n-dimensional complete manifold with nonnegative Ricci curvature. If $\pi_1(M)$ is finitely generated, then it is generated by at most C elements.*

However, the Milnor conjecture whether the fundamental group of an open Riemannian manifold of nonnegative Ricci curvature is finitely generated remains open. We also show the following

Theorem 2. *In each dimension n there are positive constants $C(n)$ and $\varepsilon(n)$ such that each compact n-manifold (M, g) with $\mathrm{diam}(M, g)^2 \cdot \mathrm{Ric} \geq -\varepsilon(n)$ satisfies the following. There is a nilpotent normal subgroup $N \subset \pi_1(M)$ of index at most C Furthermore there is a chain of subgroups $1 = N_1 \subset \cdots \subset N_h = N$ with cyclic factor groups and with $h \leq n$.*

We should mention that in the case of sectional curvature the same result was only established recently by Kapovitch, Petrunin and Tuschmann improving the fundamental work of Fukaya and Yamaguchi. The prove of the main theorem relies on the foundational work of Cheeger and Colding on limit spaces of manifolds with Ricci curvature bounded below.

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Rozansky–Witten Invariants for Quaternionic Kähler Manifolds

Gregor Weingart

In a sense the theory of characteristic classes culminating in the construction of the Chern–Weil homomorphism tells us that a rather small subring of the ring of invariant polynomials in the curvature tensor of Riemannian manifolds is preferred by topology. The preferred polynomials lead to characteristic numbers, all other polynomials only lead to topologically uninteresting numerical invariants. This monopoly was broken a couple of years ago for compact hyperkähler manifolds, when Rozansky–Witten showed that the Chern–Weil homomorphism can be extended to an apparently larger ring of polynomials, which still enjoy many properties of the polynomials resulting in characteristic numbers.

In his thesis J. Sawon studied the Rozansky–Witten invariants of hyperkähler manifolds in more detail and calculated many of them for the known low-dimensional, compact examples. In particular he noticed a numerical relation in these examples between the value of a specific Rozansky–Witten invariant given by a suitable power of the L^2-norm of the curvature tensor R^{hyper} and a well-known characteristic number:

Theorem (Hitchin–Sawon [1]). *On an irreducible, compact hyperkähler manifold M of quaternionic dimension n the L^2-norm of the curvature tensor R^{hyper} raised to the power $2n$ is proportional to the characteristic number*

$$\langle\, \widehat{A}(\, E^{\text{hyper}}M\,),\, [M]\,\rangle \quad = \quad \frac{1}{\text{vol}(M)^{n-1}} \left(\frac{\|R^{\text{hyper}}\|^2_{L^2}}{192\,\pi^2 n} \right)^n$$

where $E^{\text{hyper}}M \cong T^{1,0}M$ is by definition the holomorphic tangent bundle of M.

The proof given by Hitchin–Sawon involves three key ingredients:

- A graphical calculus for invariant polynomials on representations of the symplectic group specifically on $\text{Sym}^3 E$. In the original work of Rozansky–Witten [2] this graphical calculus was interpreted as the set of Feynman rules associated to a suitable quantum field theory.
- A powerful relation in this graphical calculus called the IHX-relation allowing us to modify graphs without changing the resulting invariant of compact hyperkähler manifolds. The IHX-relation links the Rozansky–Witten invariants to the theory of Vasiliev invariants of knots.

- A particular instance of the IHX-relation, which relates the sequence of graphs with associated Rozansky–Witten invariant $\langle \widehat{A}(E^{\text{hyper}}M), [M]\rangle$ to the powers of the graph describing $\|R^{\text{hyper}}\|^2_{L^2}$. Hitchin–Sawon show that this IHX-relation is a special case of the so-called wheeling theorem.

Two of these three ingredients are more or less combinatorial in nature independent of the specific hyperkähler geometry. Hence ever since I first heard about this result I had been asking myself, whether it is possible to fix the crucial IHX-relation, which is no longer valid for quaternionic Kähler manifolds, in order to prove similar relations between characteristic numbers and the hyperkähler part R^{hyper} of the curvature tensor.

It is definitely tempting to pursue this idea, because quaternionic Kähler manifolds of scalar curvature $\kappa \neq 0$ are always irreducible and moreover possess a specific characteristic class u, the first Pontrjagin class of HM, which allows us to use graphs of arbitrary even degree in the definition of Rozansky–Witten type numerical invariants instead of graphs of degree $2n$ only. On the other hand it is easily seen that the IHX-relation is broken on quaternionic Kähler manifolds by error terms proportional to the scalar curvature κ and associated to graphs with an odd number of vertices, which can impossibly result in characteristic numbers as associated Rozansky–Witten invariants.

There is at least one interesting graph with three vertices however, which governs the L^2-norm of the covariant derivative ∇R^{hyper} (sic!) of the hyperkähler part of the curvature tensor in the sense that its associated Rozansky–Witten invariant reads $(\,R^{\text{hyper}} \star R^{\text{hyper}},\, R^{\text{hyper}}\,)_{L^2}$ with some bilinear multiplication $\star$ appearing prominently in the Weitzenböck formula:

$$-\,\nabla^*\nabla R^{\text{hyper}} \;=\; R^{\text{hyper}} \star R^{\text{hyper}} \;+\; \frac{\kappa}{2n}\, R^{\text{hyper}}$$

It turns out that this specific graph with three vertices is exactly the lowest order summand of the error term breaking the IHX-relation responsible for the Hitchin–Sawon identity on hyperkähler manifolds. Working out the combinatorial details of this interesting twist of nature proves the following theorem:

Theorem. *The L^2-Norm of the covariant derivative ∇R^{hyper} of the hyperkähler part of the curvature tensor R of a quaternionic Kähler manifold M of quaternionic dimension n and positive scalar curvature $\kappa > 0$ is given by*

$$\frac{1}{4\pi^3}\left(\frac{\kappa}{16\pi n(n+2)}\right)^{2n-3}\|\nabla R^{\text{hyper}}\|^2_{L^2} \;-\; \frac{2}{3}\,\frac{n+2}{(2n-1)!}\,\langle\, u_1 u^{n-1},\,[M]\,\rangle$$
$$=\;\frac{1}{(2n-2)!}\,\langle\,(7u_1^2 - 4u_2)u^{n-2},\,[M]\,\rangle \;-\; 5\,\frac{2n+1}{(2n-1)!}\,\frac{\langle\, u_1 u^{n-1},\,[M]\,\rangle^2}{\langle\, u^n,\,[M]\,\rangle}$$

where u is the first Pontrjagin class of HM and u_1 and u_2 are the first and second Pontrjagin class of $E^{\text{hyper}}M := HM \oplus EM - \mathbb{C}^{2n+2}M$ respectively.

The left hand side of this formula is proportional to $(\,R^{\text{hyper}} \star R^{\text{hyper}},\, R^{\text{hyper}}\,)_{L^2}$ and thus to the error term breaking the IHX-relation leading to the Hitchin–Sawon identity. Answering a question of I. Smith from the talk I can add that the right hand side vanishes on compact hyperkähler manifolds, because the IHX-relation is strictly valid in this case, provided u is interpreted as an arbitrary non-zero multiple of the Kraines form.

Actually I have to admit that the formula I wrote down in the talk in Oberwolfach differed somewhat in the numerical constants, evidently I had some trouble fixing the combinatorial problems. In the meantime however I have checked the corrected version of this formula for a couple of cases, say the values of the relevant characteristic numbers on the exceptional Wolf spaces are

	n	$\langle u^n, [M] \rangle$	$\langle u^{n-1}u_1, [M] \rangle$	$\langle u^{n-2}u_2, [M] \rangle$	$\langle u^{n-2}u_1^2, [M] \rangle$
$\mathbf{G}_2$	2	$\frac{9}{16}$	$-\frac{21}{16}$	$-\frac{21}{16}$	$\frac{49}{16}$
$\mathbf{F}_4$	7	$\frac{39}{256}$	$-\frac{156}{256}$	$\frac{174}{256}$	$\frac{624}{256}$
$\mathbf{E}_6$	10	$\frac{37791}{524288}$	$-\frac{188955}{524288}$	$\frac{328185}{524288}$	$\frac{944775}{524288}$
$\mathbf{E}_7$	16	$\frac{17678835}{1073741824}$	$-\frac{123751845}{1073741824}$	$\frac{351295560}{1073741824}$	$\frac{866262915}{1073741824}$
$\mathbf{E}_8$	28	$\frac{7933594805325}{9007199254740992}$	$-\frac{87269542858575}{9007199254740992}$	$\frac{428414119487550}{9007199254740992}$	$\frac{959964971444325}{9007199254740992}$

and all lead to $\|\nabla R^{\text{hyper}}\|^2_{L^2} = 0$, after all the Wolf spaces are symmetric spaces.

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Existence of Engel structures

Thomas Vogel

Engel structures form a class of non-integrable distributions on manifolds which is closely related to contact structures. By definition, an Engel structure is a smooth distribution $\mathcal{D}$ of rank 2 on a manifold M of dimension 4 which satisfies the non-integrability conditions

$$\text{rank}[\mathcal{D},\mathcal{D}] = 3 \qquad \text{rank}[\mathcal{D},[\mathcal{D},\mathcal{D}]] = 4 \ .$$

Similarly to contact structures and symplectic structures, all Engel structures are locally isomorphic: Every point of an Engel manifold has a neighborhood with local coordinates x, y, z, w such that the Engel structure is the intersection of the kernels of the 1-forms

$$\alpha = dz - x\,dy \qquad \beta = dx - w\,dy \ .$$

This normal form was obtained first by F. Engel in [2]. Together with the fact that a C^2-small perturbation of an Engel structure is again an Engel structure, this implies that Engel structures are stable in the sense of singularity theory. In

[5] R. Montgomery classified all distribution types with this stability property as follows

- foliations of rank one on manifolds of arbitrary dimension
- contact structures on manifolds of odd dimension
- even contact structures on manifolds of even dimension
- Engel structures on manifolds of dimension 4.

Thus Engel structures are special among general distributions and even among the stable distribution types they seem to be exceptional since they are a peculiarity of dimension 4.

Important examples can be obtained from contact structures on 3-manifolds.

Example 1. *Let $\mathcal{C}$ be a contact structure on the 3-manifold N. On the total space of the fibration $\pi : \mathbb{P}\mathcal{C} \longrightarrow N$ (where $\mathbb{P}\mathcal{C}$ is the space of Legendrian lines in $\mathcal{C}$) the distribution*

$$\mathcal{D}_{\mathcal{C}} = \{v \in T_\lambda \mathbb{P}\mathcal{C} \mid \pi_* v \in \lambda\}$$

is an Engel structure.

The existence of an Engel structure leads to restrictions on the topology of the underlying manifold: The existence of a non-singular plane field on a 4-manifold already imposes topological conditions, cf. [3].

By its definition, an Engel structure $\mathcal{D}$ induces a distribution of hyperplanes $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ which is an even contact structure, i.e. $[\mathcal{E}, \mathcal{E}] = TM$. An even contact structure $\mathcal{E}$ contains a distinguished line field $\mathcal{W}$ characterized by the property $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. It is easy to show that if $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is associated to an Engel structure, then $\mathcal{W} \subset \mathcal{D}$. Thus an Engel structure $\mathcal{D}$ on M induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} = [\mathcal{D}, \mathcal{D}] \subset TM \tag{1}$$

such that each distribution has corank one in the next one. From the properties of $\mathcal{W}$ and $\mathcal{E}$ it follows that hypersurfaces transverse to $\mathcal{W}$ carry a contact structure together with a Legendrian line field. We will refer to this line field as the intersection line field.

Using relations between orientations of the distributions appearing in (1) one obtains the following result.

Proposition 2. *If an orientable 4-manifold admits an orientable Engel structure, then it has trivial tangent bundle.*

A proof can be found in [4] where it is attributed to V. Gershkovich. We develop a construction of Engel manifolds allowing us to prove the converse of Proposition 2.

Theorem 3. *Every parallelizable 4-manifold admits an orientable Engel structure.*

Proof. We give only a brief outline. A detailed exposition can be found in [6], see also [7]. The construction is based on decompositions of manifolds into round handles and we first recall the relevant definitions and theorems.

A round handle of dimension n and index $k \in \{0, \ldots, n-1\}$ is defined to be $R_k = D^k \times D^{n-k-1} \times S^1$. Round handles are attached to manifolds with boundary using embeddings of $\partial_- R_k = \partial D^k \times D^{n-k-1} \times S^1$ into the boundary. We say that M admits a round handle decomposition if M can be obtained from the disjoint union of several round handles of index 0 by successively attaching round handles.

Theorem 4 (Asimov [1]). *A closed connected manifold of dimension $n \neq 3$ admits a decomposition into round handles if and only if its Euler characteristic vanishes.*

One can assume that the round handles are ordered according to their index.

We fix a set of model Engel structures on round handles such that the characteristic foliation is transverse to $\partial_- R_l$ and $\partial_+ R_l = \partial R_l \setminus \partial_- R_l$. These model Engel structures are constructed using a perturbation of the Engel structures from Example 1. The contact structure on each $\partial_- R_l$ depends essentially only on l while the intersection line field and the orientation of the contact structure are different for different model Engel structures on R_l.

When we attach a round handle R_l with a model Engel structure to an Engel manifold M' with transverse boundary we have to ensure that the attaching map preserves the oriented contact structures and the intersection line field if we want to extend the Engel structure from M' to $M' \cup R_l$ by the model Engel structure. In order to satisfy these conditions we isotope the attaching map and we choose a model Engel structure suitably. For this we use several constructions of contact topology. It turns out that it is convenient to ensure that the contact structures on boundaries transverse to the characteristic foliation are overtwisted throughout the construction.

Since $M' \cup R_l$ is again an Engel manifold with transverse boundary we can iterate this construction. For the construction of an Engel structure on a manifold M with trivial tangent bundle we fix a round handle decomposition of M and we use the model Engel structures to extend the Engel structure when the round handles are attached successively. The condition that M has trivial tangent bundle is be used to show that there is a model Engel structure with the desired properties in our collection of model Engel structures on R_l for $l = 1, 2, 3$. □

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