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C^*-Algebren

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Introduction by the Organisers

The aim of the workshop *C^*-algebras* was to bring together researchers from basically all areas related to operator algebra theory. This gives a unique opportunity to obtain a broader view of the subject and to create new interactions between researchers with different background. The organizers, Claire Anantharaman-Delaroche, Siegfried Echterhoff, Uffe Haagerup, and Dan Voiculescu took special care to invite a good number of young researchers, some of them already being leading experts in their fields. As a result, several contributions in this report are from researchers who, at the time of the workshop, were less then 30 years old.

There were 29 lectures presented at this workshop with topics from Ergodic Theory, L^2-(co-)homology, classification of C^*-algebras, Operator Theory, von Neumann algebras, KK-theory and the Baum-Connes conjecture, quantum spaces and quantum groups, mathematical physics, non-commutative probability theory, and the theory of operator spaces. To name some special highlights we can mention the reports on recent developements in the study of "boundary actions" of quantum groups due to S. Vaes and R. Vergnioux, the new results on classification theory of amenable C^*-algebras in terms of studying algebras which are stable under tensoring with the Jiang-Su algebra $\mathcal{Z}$ (see the lectures of A. Toms and M. Rørdam), or the report by S. Popa on recent progress in the study of strong rigidity for II_1 factors associated to equivalence relations. But this is only a very small selection of the interesting lectures on new results presented at this workshop.

It is a pleasure for the organizers of the conference to use this opportunity to thank all participants of the workshop for their contributions—either in lectures held at the workshop or in stimulating discussions following the lectures. We also thank the Mathematisches Forschungsinstitut Oberwolfach for providing a great environment and strong support for organizing this conference. Special thanks go to the very competent and helpful staff of the institute and to the chef de cuisine.

Workshop: C*-Algebren

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Abstracts

Boundaries and exactness for discrete quantum groups

Stefaan Vaes
(joint work with Roland Vergnioux)

Since Murray and von Neumann introduced *von Neumann algebras*, the ones associated with discrete groups played a prominent role. The main aim of this talk is to show how concrete examples of discrete quantum groups give rise to interesting C*- and von Neumann algebras. We show that the *universal discrete quantum groups* [1, 5] admit a boundary, with an amenable boundary action. This allows to prove that the reduced C*-algebras are *exact* and that the *Akemann-Ostrand property* holds. We conclude that the associated von Neumann algebras are *full prime factors*. In this way, we obtain new examples of *solid type* II_1 *factors* in the sense of Ozawa. We obtain as well the *simplicity* of the reduced C*-algebra. Finally, the boundary that we construct, can be identified with a *Martin or Poisson boundary* of a quantum random walk. We mainly report on [4].

Discrete quantum groups have essentially been introduced by Woronowicz as the dual of a compact quantum group.

Definition 1 (Woronowicz [6])**.** A *compact quantum group* $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$, where

- $C(\mathbb{G})$ is a unital C*-algebra;
- $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital *-homomorphism satisfying the *co-associativity* relation
$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \; ;$$
- $\mathbb{G}$ satisfies the *left and right cancellation property* expressed by
$$\Delta(C(\mathbb{G}))(1 \otimes \mathbb{G}) \quad \text{and} \quad \Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1) \quad \text{are total in } \; C(\mathbb{G}) \otimes C(\mathbb{G}) \, .$$

The two major aspects of the general theory of compact quantum groups are the existence of a *unique invariant state* h on $C(\mathbb{G})$ and the theory of *unitary representations* parallelling Peter-Weyl theory for compact groups. The GNS construction for h allows to define a *reduced C*-algebra* $C(\mathbb{G})_{\mathrm{red}}$ and a *von Neumann algebra* $C(\mathbb{G})''_{\mathrm{red}}$.

For the purposes of this note, the following definition is sufficiently general.

Definition 2. A *unitary representation* of the compact quantum group $\mathbb{G}$ on the Hilbert space $\mathbb{C}^n$ is a unitary matrix (U_{ij}) with entries in $C(\mathbb{G})$ satisfying $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

It is straightforward to define the tensor product of unitary representations and to introduce notions as irreducible representations, direct sums, etc.

The previous definition allows to define the universal orthogonal quantum group $A_o(F)$ [1, 5]. Take $F \in \mathrm{GL}(n, \mathbb{C})$ satisfying $F\overline{F} = \pm 1$. The compact quantum group $\mathbb{G} = A_o(F)$ is defined by taking as $C(\mathbb{G})$ the universal C*-algebra generated by the entries of an n by n unitary matrix (U_{ij}) with $(U_{ij}) = F(U_{ij}^*)F^{-1}$. The comultiplication Δ is (uniquely) defined such that U is a unitary representation of $\mathbb{G}$. *For the rest of this note, we fix* $\mathbb{G} = A_o(F)$.

In [1], Banica determined the representation theory of $\mathbb{G} = A_o(F)$ and showed that $\mathbb{G}$ has the same fusion rules as the compact group SU(2). So, for every $n \in \mathbb{N}$, we can take an irreducible unitary representation U^n on the Hilbert space H_n such that

$$U^n \otimes U^m \cong U^{|n-m|} \oplus U^{|n-m|+2} \oplus \cdots \oplus U^{n+m} .$$

The (discrete) dual of $\mathbb{G}$ is defined as follows:

$$\ell^\infty(\widehat{\mathbb{G}}) := \prod_{n\in\mathbb{N}} \mathrm{B}(H_n) .$$

The quantum group structure is expressed by a comultiplication $\widehat{\Delta} : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}})$. We write moreover $c_0(\widehat{\mathbb{G}}) = \bigoplus_{n\in\mathbb{N}} \mathrm{B}(H_n)$.

A suitable *compactification* for $\widehat{\mathbb{G}}$ should be a unital C*-algebra $\mathcal{B}$ satisfying $c_0(\widehat{\mathbb{G}}) \subset \mathcal{B} \subset \ell^\infty(\widehat{\mathbb{G}})$, the *boundary* $\mathcal{B}_\infty$ being $\mathcal{B}/c_0(\widehat{\mathbb{G}})$. The fusion rules of $\mathbb{G}$ yield isometries $V(x \otimes y, x+y) : H_{x+y} \to H_x \otimes H_y$ that intertwine U^{x+y} and $U^x \otimes U^y$. Such an isometry is determined up to a number of modulus one, which makes the following maps canonically defined

$$\psi_{x+y,x} : \mathrm{B}(H_x) \to \mathrm{B}(H_{x+y}) : \psi_{x+y,x}(A) = V(x \otimes y, x+y)^*(A \otimes 1)V(x \otimes y, x+y) ,$$

for all $A \in \mathrm{B}(H_x)$.

Definition 3. We define the linear subspace $\mathcal{B}_0 \subset \ell^\infty(\widehat{\mathbb{G}})$ consisting of elements (A_n) such that there exists n with $A_m = \psi_{m,n}(A_n)$ for all $m \geq n$. We define $\mathcal{B}$ as the norm closure of $\mathcal{B}_0$.

Intuitively, $\mathcal{B}_\infty$ is a direct limit along the inductive system of maps $\psi_{m,n}$. But, since these completely positive maps are by no means multiplicative, the following theorem needs a careful analysis.

Proposition 4. *The closed subspace $\mathcal{B}$ actually is a unital C*-subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$.*

In a next step, we study the action of $\widehat{\mathbb{G}}$ on the boundary $\mathcal{B}_\infty$ by left translations. We show that this is an *amenable* action. This allows to prove the following result.

Theorem 5. *The reduced C*-algebra $C(\mathbb{G})_{\mathrm{red}}$ is exact. Moreover, $\mathbb{G}$ satisfies the* Akemann-Ostrand property*, which means that the map*

$$C(\mathbb{G})_{\mathrm{red}} \otimes_{\mathrm{alg}} C(\mathbb{G})_{\mathrm{red}} \to \frac{\mathrm{B}(L^2(\mathbb{G}))}{\mathcal{K}(L^2(\mathbb{G}))} : a \otimes b \mapsto \pi(\lambda(a)\rho(b))$$

is continuous with respect to the minimal *tensor product norm.*

Note that $L^2(\mathbb{G})$ denotes the GNS Hilbert space of the Haar state h and that λ and ρ are the left and the right regular representations of $C(\mathbb{G})_{\mathrm{red}}$ on $L^2(\mathbb{G})$.

We also study factoriality of the von Neumann algebra $C(\mathbb{G})''_{\mathrm{red}}$ and simplicity of the C*-algebra $C(\mathbb{G})_{\mathrm{red}}$. The tool is a study of an analogue of the operator

$\frac{1}{\#S}\sum_{g\in S}\lambda_g\rho_g$ on $\ell^2(\Gamma)$ associated with a finite set of generators S of a discrete group Γ. Under the right assumptions, such an operator has a spectral gap at 1 and one derives factoriality and fullness. Combining such a spectral gap with the Property of Rapid Decay, established by Vergnioux, one obtains simplicity of the reduced C*-algebra.

Theorem 6. *Let $n \geq 3$ and $F \in \mathrm{GL}(n,\mathbb{C})$ with $F\overline{F} = \pm 1$.*

- $C(\mathbb{G})''_{\mathrm{red}}$ *is a full prime factor when* $\dfrac{\|F\|^2}{\mathrm{Tr}(F^*F)} \leq \dfrac{1}{\sqrt{5}}$.
- $C(\mathbb{G})_{\mathrm{red}}$ *is a simple C*-algebra when* $\dfrac{\|F\|^8}{\mathrm{Tr}(F^*F)} \leq \dfrac{3}{8}$.

The previous theorem applies in particular to $F = I_n$, the n by n identity matrix with $n \geq 3$. In that case, $C(\mathbb{G})''_{\mathrm{red}}$ is a *solid* II_1 *factor* and $C(\mathbb{G})_{\mathrm{red}}$ is a *simple C*-algebra with unique tracial state.*

Remark 7. Of course, the conditions on $\|F\|$ in the theorem above are ad hoc. It is our belief that the result is true without these conditions, although a different technique of proof would be needed.

Finally, we observe that the boundary $\mathcal{B}_\infty$ for the dual of $A_o(F)$ can be identified with a Martin or a Poisson boundary. The notion of a Poisson boundary for discrete quantum groups is due to Izumi [2], who identified the Poisson boundary of the dual of $\mathrm{SU}_q(2)$ with (von Neumann algebraic) Podles' sphere. Martin boundaries for discrete quantum groups were introduced in [3]: the Martin boundary of the dual of $\mathrm{SU}_q(2)$ is isomorphic with the C*-version of Podles' sphere.

Without going into details, our result roughly goes as follows. The boundary $\mathcal{B}_\infty$ admits a natural *harmonic state* ω_∞. Taking the GNS construction, the generated von Neumann algebra $(\mathcal{B}_\infty, \omega_\infty)''$ is isomorphic with the Poisson boundary of $\widehat{\mathbb{G}}$ with respect to a generating state. The boundary $\mathcal{B}_\infty$ itself is on the nose the Martin boundary of $\widehat{\mathbb{G}}$ with respect to a generating state with finite first moment.

References

[1] T. Banica, Théorie des représentations du groupe quantique compact libre $O(n)$. *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), 241–244.

[2] M. Izumi, Non-commutative Poisson boundaries and compact quantum group actions. *Adv. Math.* **169** (2002), 1–57.

[3] S. Neshveyev & L. Tuset, The Martin boundary of a discrete quantum group. *J. reine und angewandte Math.* **568** (2004), 23–70.

[4] S. Vaes & R. Vergnioux, The boundary of universal discrete quantum groups, exactness and factoriality. *Preprint.*

[5] A. Van Daele & S. Wang, Universal quantum groups. *Internat. J. Math.* **7** (1996), 255–263.

[6] S.L. Woronowicz, Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, North-Holland, Amsterdam, 1998, pp. 845–884.

The Property of Rapid Decay for Free Quantum Groups

Roland Vergnioux

The Property of Rapid Decay (RD) was first considered by Haagerup in a famous article about the convolution algebras of the free groups [2]. A general theory was then developped and studied by Jolissaint, with applications to K-theory. We report here on the extension of this theory to discrete quantum groups : definition, examples, applications. Proofs and details can be found in [5].

Let us denote by (S, δ) the Hopf C*-algebra of c_0-functions on a discrete quantum group [4], and by h_L, h_R its Haar weights. As a C*-algebra, S is a direct sum $\bigoplus_{\alpha\in\mathcal{I}} B(H_\alpha)$ of matrix algebras, and we denote by $p_\alpha \in S$ the minimal central projection associated to α. We moreover identify the index set $\mathcal{I}$ with the set of classes of irreducible representations of S. It is equipped with a tensor product, a conjugation and a unit object, which are respectively induced by the coproduct δ, the antipode κ and the co-unit ε.

On the other hand we denote by $(\hat{S}_{\text{red}}, \hat{\delta})$ the reduced dual Hopf C*-algebra of (S, δ), and by $\hat{h}$ its Haar state. The unital C*-algebra $\hat{S}_{\text{red}}$ is the object of interest, and Property RD is a tool to study it. We will make use of a densely defined Fourier transform $\mathcal{F} : S \supset \mathcal{S} \to \hat{S}_{\text{red}}$ which induces an identification between the GNS spaces relative to h_R and $\hat{h}$.

1. Extension of the Definition

A length on (S, δ) is an unbounded multiplier $L \in S^\eta$ such that

$$L \geq 0,\ \varepsilon(L) = 0,\ \kappa(L) = L \text{ and } \delta(L) \leq L \otimes 1 + 1 \otimes L.$$

We denote by p_n the spectral projection of L associated with $[n, n+1[$. Interesting examples of lengths are the word lengths: assume that (S, δ) is finitely generated — ie $\mathcal{I}$ is generated by a finite subset $\mathcal{D} = \bar{\mathcal{D}}$ —, we put then $L = \sum l(\alpha) p_\alpha$ with

$$l(\alpha) = \min\{k \mid \exists\ \beta_1, \ldots, \beta_k \in \mathcal{D}\ \alpha \subset \beta_1 \otimes \cdots \otimes \beta_k\}.$$

We define the Sobolev norms of an element $a \in \mathcal{S}$ by the standard formulas $||a||_2 := \hat{h}_R(a^*a)$ and $||a||_{2,s} = ||(1+L)^s a||_2$. We denote by $H \supset H_s$ the respectively associated completions, and we put $H_\infty = \bigcap_{s\geq 0} H_s$. Since H identifies via the Fourier transform with the GNS space of $\hat{h}$, the space $\hat{S}_{\text{red}}$ can also be considered a subspace of H.

We are now ready to give the following Definition which, like in the classical case, is about controlling the norm of $\hat{S}_{\text{red}}$ by the Sobolev norms.

Definition 1. Let L be a central length on (S, δ). We say that (S, δ, L) has Property RD if the following equivalent conditions are satisfied:

(1) $\exists C, s \in \mathbb{R}_+\ \forall a \in \mathcal{S}\ ||\mathcal{F}(a)|| \leq C||a||_{2,s}$,
(2) $H_\infty \subset \hat{S}_{\text{red}}$ inside H,
(3) $\exists P \in \mathbb{R}[X]\ \forall k, l, n\ \forall a \in p_n\mathcal{S}\ ||p_l \mathcal{F}(a) p_k|| \leq P(n)||a||_2$.

The centrality assumption about the length may seem too restrictive in the quantum case, and one could actually give a definition for arbitrary lengths, using for instance the first condition. However in the finitely generated case all lengths are dominated by word lengths, which are central, so that it is enough to consider central lengths in the study of Property RD.

In the case of a discrete group Γ, one recovers the classical notion of Property RD. In the (quantum) ammenable case, one can show that Property RD is still equivalent to polynomial growth, and in particular duals of connected compact Lie groups G always have Property RD. In this case we have $\hat{S}_{\text{red}} = C(G)$ by definition and the embedding $H_\infty \subset \hat{S}_{\text{red}}$ corresponds to the inclusion $C^\infty(G) \subset C(G)$.

2. The Free Quantum Groups

Apart from discrete groups and duals of compact groups, the first test examples for a quantum Property RD should be the free quantum groups introduced by Wang [6], which are quantum analogues of the free groups.

We start by presenting a necessary condition which proves to be usefull in that context. Replacing the "projections onto the spheres" in condition (3) by "projections onto the points of the spheres" and restricting to multiplicity-free cases we obtain the following "local version" of Property RD: there exists a polynomial $P \in \mathbb{R}[X]$ such that, for any multiplicity-free inclusion $\gamma \subset \beta \otimes \alpha$ of elements of $\mathcal{I}$

$$\forall a \in p_\alpha S \; ||p_\gamma \mathcal{F}(a) p_\beta|| \leq P(|\alpha|) ||a||_2,$$

where $|\alpha|$ is the positive number such that $Lp_\alpha = |\alpha| p_\alpha$.

The interesting point about this condition is that it can be reformulated in a way that makes no reference anymore to the norm of $\hat{S}_{\text{red}}$:

$$\forall a \in p_\alpha S,\; b \in p_\beta S \; ||\delta(p_\gamma)(b \otimes a)\delta(p_\gamma)||_2 \leq \sqrt{\frac{m_\gamma}{m_\beta m_\alpha}} P(|\alpha|) ||b \otimes a||_2.$$

This inequality of Hilbert-Schmidt norms of matrices over $H_\beta \otimes H_\alpha$ is in fact an assertion about the relative positions in $H_\beta \otimes H_\alpha$ of the subspace equivalent to H_γ and of the cone of decomposable tensor products. Note that it is trivially verified in the case of discrete groups, since all spaces H_α are then 1-dimensional.

Using this necessary condition with the inclusions $\varepsilon \subset \bar{\alpha} \otimes \alpha$, we see that our theory is a unimodular one, although this is not apparent in the definition:

Proposition 2. *Non-unimodular discr. quantum groups can't have Property RD.*

On the other hand, our necessary condition happens to be sufficient in the case of free quantum groups. In the orthogonal case this is trivial since the spheres in $\mathcal{I}$ are singletons, whereas in the unitary case this is an adaptation of the proof of Haagerup for free groups, using the freeness properties of $(\mathcal{I}, \otimes)$. By a finer study of the geometry of the fusion rules of free quantum groups, one can investigate this condition and we have finally the following quantum analogue of Haagerup's founding result:

Theorem 3. *The orthogonal and unitary free quantum groups have Property RD* **iff** *they are unimodular.*

3. Applications

The applications to K-theory are the first ones that come to mind to check wether the quantum theory is usable. They go back to Jolissaint [3] and rely, in the quantum case too, on a technical description of H_∞. More precisely, let L be a word length on a finitely generated discrete quantum group and denote by D the closed inner derivation by L on $B(H)$.

Proposition 4. *We have* $\hat{S}_{red} \cap \mathrm{Dom}\, D^k \subset H_k$ *and, if* (S, δ, L) *has Property RD with exponent* s*,* $H_{k+s} \subset \hat{S}_{red} \cap \mathrm{Dom}\, D^k$ *hence* $H_\infty = \bigcap_k \mathrm{Dom}\, D^k \cap \hat{S}_{red}$.

Standard general results about domains of closed derivations imply then that H_∞ is a full subalgebra of $\hat{S}_{\mathrm{red}}$, and in particular they have the same K-theory. Using the same techniques one can also generalize the result of V. Lafforgue stating that H_s is already a full subalgebra of $\hat{S}_{\mathrm{red}}$ for s big enough.

Finally, let us mention another application, which is part of a joint work with S. Vaes. Let $U \in M_N(\mathbb{C}) \otimes \hat{S}_{\mathrm{red}}$ be the fundamental corepresentation of a unimodular orthogonal free quantum group and consider the operator of "conjugation by the generators" $\Psi : \hat{S}_{\mathrm{red}} \to \hat{S}_{\mathrm{red}}$, $x \mapsto (\mathrm{Tr} \otimes \mathrm{id})(U^*(1 \otimes x)U)/N$.

Proposition 5. *If* $N \geq 3$*, there exists* $\lambda < 1$ *such that*

$$\forall x \in \hat{S}_{red}\ \hat{h}(x) = 0 \ \Rightarrow\ ||\Psi(x)||_2 \leq \lambda ||x||_2.$$

This technical result clearly implies that $\hat{S}''_{\mathrm{red}}$ is a full factor. In fact, combining the Proposition with Property RD one can transfer this "hilbertian simplicity" to the C*-algebraic level and prove that $\hat{S}_{\mathrm{red}}$ is simple with a unique trace. The corresponding result in the unitary case was proved by Banica [1] using the freeness in $\mathcal{I}$, a method that cannot apply in the orthogonal case.

References

[1] T. Banica, *Le groupe quantique compact libre* $U(n)$, Comm. Math. Phys. **190** (1997), 143–172.

[2] U. Haagerup, *An example of a nonnuclear* C^**-algebra, which has the metric approximation property*, Invent. Math. **50** (1978/79), 279–293.

[3] P. Jolissaint, K*-theory of reduced* C^**-algebras and rapidly decreasing functions on groups*, K-Theory **2** (1989), 723–735.

[4] P. Podleś and S.L. Woronowicz, *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431.

[5] R. Vergnioux, *The Property of Rapid Decay for Free Quantum Groups*, J. Operator Theory, to appear.

[6] S. Wang, *Free products of compact quantum groups*, Comm. Math. Phys. **167** (1995), 671–692.

Spectral measures of free quantum groups

Teodor Banica

This is a report on joint work with Julien Bichon, Dietmar Bisch, Benoit Collins and Sergiu Moroianu, part of which is in preparation [1, 2, 3, 4].

A compact quantum group is an abstract object, dual to a Hopf $\mathbb{C}^*$-algebra. Such a Hopf algebra is by definition a pair (A, Δ), where A is a $\mathbb{C}^*$-algebra with unit, and Δ is a morphism of $\mathbb{C}^*$-algebras

$$\Delta : A \to A \otimes A$$

subject to certain axioms, discovered by Woronowicz in the late 80's.

The very first example is $A = \mathbb{C}(G)$, where G is a compact group. Here the comultiplication map Δ comes from the multiplication map $m : G \times G \to G$.

The other basic example is $A = \mathbb{C}^*(\Gamma)$, where Γ is a discrete group. Here the comultiplication is defined on generators by $\Delta(g) = g \otimes g$.

In general, a Hopf $\mathbb{C}^*$-algebra A can be thought of as being of the form

$$A = \mathbb{C}(G) = \mathbb{C}^*(\Gamma)$$

where G is a compact quantum group, and Γ is a discrete quantum group.

The free analogues of $\mathbb{C}(U(n))$, $\mathbb{C}(O(n))$, $\mathbb{C}(S_n)$ are the universal Hopf $\mathbb{C}^*$-algebras $A_u(n)$, $A_o(n)$, $A_s(n)$, constructed by Wang in the 90's:

$$\begin{aligned} A_u(n) &= \mathbb{C}^* \left(u_{ij} \mid u = \text{ unitary}, \ \bar{u} = \text{ unitary} \right) \\ A_o(n) &= \mathbb{C}^* \left(u_{ij} \mid u = \bar{u} = \text{ unitary} \right) \\ A_s(n) &= \mathbb{C}^* \left(u_{ij} \mid u = \text{ magic biunitary} \right) \end{aligned}$$

Here $u = u_{ij}$ is a $n \times n$ matrix, and $\bar{u} = u_{ij}^*$. The magic biunitarity condition says that all entries u_{ij} are projections, and on each row and each column of u these projections are orthogonal, and sum up to 1. See [1] for details.

The fundamental question is: who are these algebras?

In other words, we would like to have models for $A_u(n)$, $A_o(n)$, $A_s(n)$, where generators u_{ij} correspond to explicit operators, say in some known $\mathbb{C}^*$-algebra.

Theorem 1. *We have the following models for Wang's algebras:*
– *an embedding* $A_u(n) \subset \mathbb{C}^*(\mathbb{Z}) * A_o(n)$.
– *an isomorphism* $A_o(2) = \mathbb{C}(SU(2))_{-1}$.
– *an isomorphism* $A_s(n) = \mathbb{C}(S_n)$ *for* $n = 1, 2, 3$.
– *an inner faithful representation* $A_s(4) \to \mathbb{C}(SU(2), M_4(\mathbb{C}))$.

Here the middle assertions are easy, and provide models for $A_o(2)$, $A_s(2)$, $A_s(3)$. The first assertion is proved in my thesis, and reduces study of $A_u(n)$ to that of $A_o(n)$ (in particular, we get a model for $A_u(2)$). As for the last assertion, this is based on a realisation of the universal 4×4 magic biunitary matrix, obtained with S. Moroianu by using the magics of Pauli matrices [4].

All proofs are based on the following key lemma.

Lemma 2. *A morphism* $(A, u) \to (B, v)$ *is faithful if and only if the spectral measures of* $\chi(u)$ *and of* $\chi(v)$ *are the same.*

Here $\chi(w) = w_{11} + w_{22} + \ldots + w_{nn}$ is the character of $w = w_{ij}$. In the self-adjoint case the spectral measure of $\chi(w)$ is the real probability measure coming from Haar integration; in the general case, it is the $*$-distribution.

The key lemma tells us that in order to find models for $A_u(n)$, $A_o(n)$, $A_s(n)$, the very first thing to be done is to compute the spectral measure of $\chi(u)$. This was done by myself in the 90's, with the following conclusion.

Theorem 3. *We have the following spectral measures:*
– *for $A_u(n)$ the variable $\chi(u)$ is circular.*
– *for $A_o(n)$ the variable $\chi(u)$ is semicircular.*
– *for $A_s(n)$ the variable $\chi(u)$ is free Poisson.*

This gives some indication about where to look for models (and that is how theorem 1 was found!), but in general, the fundamental problem is still there.

The recent work [1, 2, 3] focuses on three related problems.

A first natural question is to find analogues of theorem 2, for other classes of universal Hopf algebras. One would like of course to investigate "simplest" such Hopf algebras, and according to general theory of Jones and Bisch-Jones (the "2-box" philosophy), these are algebras $A(X)$, with X finite graph.

If X is a finite graph having n vertices, the algebra $A(X)$ is by definition the quotient of $A_s(n)$ by the commutation relation $[u, d] = 0$, where d is the Laplacian of X. This algebra $A(X)$ corresponds to a so-called quantum permutation group.

In joint work with J. Bichon [1] we investigate several formulae of type $\mu(X \times Y) = \mu(X) \times \mu(Y)$, where $\mu(Z)$ is the spectral measure of the character of $A(Z)$. In particular we obtain evidence for the following conjecture.

Conjecture 4. *We have an equality of spectral measures*

$$\mu(X * Y) = \mu(X) \boxtimes \mu(Y)$$

where X, Y are colored graphs, $X * Y$ is obtained by "putting a copy of X at each vertex of Y", and $\boxtimes$ is Voiculescu's free multiplicative convolution.

We prove this statement in two simple situations: one using work of Bisch-Jones and Landau, the other one using work of Nica-Speicher and Voiculescu. In the general case we have no proof, but we suspect that our free product operation $*$ is a a graph-theoretic version of the free product operation for planar algebras discovered by Bisch and Jones. See the report of Bisch in these Proceedings.

A second natural question is to find finer versions of theorem 2, with $\chi(u)$ replaced by arbitrary coefficients of u. This would no doubt give more indication about what models for $A_u(n)$, $A_o(n)$, $A_s(n)$ should look like.

We are currently investigating this problem, in joint work with B. Collins [3]. The idea is to use an old idea of Weingarten, recently studied in much detail by Collins and Sniady, for classical groups. So far, we have several results for $A_o(n)$, including a general integration formula, and a formula for moments of diagonal coefficients of the form $o_{sn} = u_{11} + \ldots + u_{ss}$.

Theorem 5. *The odd moments of o_{sn} are all 0, and the even ones are given by*

$$\int o_{sn}^{2k} = Tr(A_{kn}^{-1}A_{ks})$$

where A_{kn} is the Gram matrix of Temperley-Lieb diagrams in $TL(k,n)$.

As a corollary, the normalised variable $(n/s)^{1/2}o_{sn}$ with $n\to\infty$ is asymptotically semicircular. We are trying now to get more information about o_{sn}, along with some similar results for variables u_{sn}, s_{sn}, corresponding to $A_u(n)$, $A_s(n)$.

A third natural question is whether similar problems can be asked about subfactors. A much studied invariant of subfactors (Jones, Bisch-Jones) is a series with integer coefficients, called Poincaré series. In case the subfactor comes from a Hopf $\mathbb{C}^*$-algebra (A,v), the Poincaré series is nothing but the Stieltjes transform of the spectral measure of $\chi(v)$. In other words, the question is to compute the measure-theoretic version of the Poincaré series, for various subfactors.

We investigate this problem for subfactors of index ≤ 4, in joint work with D. Bisch [2]. These are known to be classified by ADE graphs. We use Jones' change of variables $z = q/(1+q)^2$: at level of measures, this leads to consideration of a certain probability measure ε supported by the unit circle, that we call spectral measure of the graph. Our key remark is that ε is given by a nice formula.

Theorem 6. *The spectral measures of AD graphs are given by*

$$\begin{array}{rcl} A_{n-1} & \to & \alpha\, d_n \\ D_{n+1} & \to & \alpha\, d_n' \\ A_\infty & \to & \alpha\, d \\ A_{2n}^{(1)} & \to & d_n \\ A_{-\infty,\infty} & \to & d \\ D_{n+2}^{(1)} & \to & d_1'/2 + d_n/2 \\ D_\infty & \to & d_1'/2 + d/2 \end{array}$$

where d, d_n, d_n' are the uniform measures on the unit circle, on $2n$-th roots of unity, and on $4n$-th roots of unity of odd order, and $\alpha(u) = 2Im(u)^2$.

This is closely related to work of Reznikoff; she computes moments of the spectral measures of ADE graphs by counting planar modules, via a theorem of Jones.

We are trying now to find a nice formula for E graphs, plus of course to formulate some kind of relevant question regarding graphs of small index > 4.

References

[1] T. Banica and J. Bichon, *Free product formulae for quantum permutation groups*, submitted for publication, arxiv:math.QA/0503461.

[2] T. Banica and D. Bisch, *Spectral measures of small index graphs*, in preparation.

[3] T. Banica and B. Collins, *Integration over compact quantum groups*, in preparation.

[4] T. Banica and S. Moroianu, *On the structure of quantum permutation groups*, Proc. Amer. Math. Soc., to appear, arxiv:math.QA/0411576.

Deformation of infinite projections

Etienne Blanchard

A classification programme of nuclear C*-algebras through K-theoretic invariants has been launched by G. Elliott. If $\mathcal{O}_\infty$ is the unital C*-algebra generated by an infinite sequence of isometries s_j, $j \in \mathbb{N}$, then $K_*(A) \cong K_*(A \otimes \mathcal{O}_\infty)$ for all C*-algebra A. Hence any classification programme can only be performed for C*-algebras A which absorb $\mathcal{O}_\infty$, id est (i.e.) such that $A \cong A \otimes \mathcal{O}_\infty$.

In order to study these C*-algebras, recall that two projections p, q in a C*-algebra A are said to be Murray-von Neumann *equivalent* (resp. p *dominates* q) if there exists a partial isometry $v \in A$ with $v^*v = p$ and $vv^* = q$ (resp. $v^*v \leq p$ and $vv^* = q$). For short we write $p \sim q$ (resp. $q \precsim p$). The non-zero projection p is said to be *infinite* (resp. *properly infinite*) if p is equivalent to a proper subprojection $q < p$ (resp. p is equivalent to two mutually orthogonal projections p_1, p_2 with $p_1 + p_2 \leq p$) and p is *finite* otherwise.

Definition 1 (Cuntz [5])**.** A simple $A \neq \mathbb{C}$ is said to be purely infinite (p.i.) if and only if (iff) every nonzero hereditary C*-subalgebra in A contains an infinite projection.

Then one has the following characterization of the pure infiniteness in the simple nuclear case.

Proposition 2 (Kirchberg, Phillips)**.** *If A is a simple nuclear C*-algebra, then the following are equivalent:*

i) *A is p.i.,*
ii) *For all $a, b \in A_+ \setminus \{0\}$ there exists an element $d \in A$ with $\|b - d^*ad\| < 1$,*
iii) *$A \cong A \otimes O_\infty$.*

Possible generalisations to the non-simple case are the following:

Definition 3. ([7]) A C*-algebra A is said to be purely infinite (p.i.) iff

i) $\forall\, a, b \in A_+$ with $b \in \overline{AaA}$, there exists $d \in A$ such that $\|b - d^*a\| < 1$ and
ii) There is no character on the C*-algebra A.

Definition 4. ([4]) A C*-algebra A is said to be locally purely infinite (l.p.i.) iff for all element $b \in A$ and all two sided closed ideal $J \triangleleft A$ such that $b \notin J$, there exists a stable C*-subalgebra D_J of the hereditary C*-subalgebra $\overline{b^*Ab}$ which is not totally contained in J.

M. Rørdam has proved in [8] that any p.i. C*-algebra A is always l.p.i. The converse implication holds if the primitive ideal space $Prim(A)$ of the C*-algebra A is Hausdorff and of finite topological dimension ([3]). In order to study the infinite dimensional case, we fix a compact Hausdorff space X and we look more generally at a unital $C(X)$-algebras A which admits a faithful $C(X)$-representation π on a Hilbert $C(X)$-module $E = (E_x)_{x \in X}$ ([2]) such that the projection $\pi_x(1_A)$ is properly infinite in $\pi_x(A)$ (resp. in $L(E_x)$) for all $x \in X$.

Then the unit $p \otimes 1_n$ of $M_n(A)$ is properly infinite in $M_n(A)$ for large enough n as soon as π is a continuous field of faithful $*$-representations ([1], [3]). This is also the case in $\mathcal{L}_{C(X)}(E)$ if the topological dimension of X is finite and each Hilbert space E_x is infinite dimensional ([6]). But the C*-algebra $\mathcal{L}(E)$ can have a finite (and even stably finite) unit in general.

References

[1] E. Blanchard, *Déformations de C*-algèbres de Hopf.* Bull. Soc. Math. France **24** (1996), 141–215.

[2] E. Blanchard, *A few remarks on C(X)-algebras*, Rev. Roumaine Math. Pures Appl. **45** (2001), 565–576.

[3] E. Blanchard, E. Kirchberg, *Non-simple purely infinite C*-algebras: the Hausdorff case*, J. Funct. Anal. **207** (2004), 461–513.

[4] E. Blanchard, E. Kirchberg, *Global Glimm halving for C*-bundles*, J. Op. Th. **52** (2004), 385–420.

[5] J. Cuntz, *K-theory for certain C*-algebras*, Ann. of Math. **113** (1981), 181–197.

[6] J. Dixmier, A. Douady, *Champs continus d'espaces hilbertiens et de C*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[7] E. Kirchberg, M. Rørdam, *Non-simple purely infinite C*-algebras*, Amer. J. Math. **122** (2000), 637–666.

[8] M. Rørdam, *Stable C*-algebras*, Advanced Studies in Pure Mathematics **38** "Operator Algebras and Applications" (2004), 177-200.

Continuous Fields of C*-algebras

Marius Dadarlat

We report on results from two recent papers [2] and [3], the first of which is joint work with George Elliott.

In [2] it is proved that all unital separable continuous fields of C*-algebras over $[0, 1]$ with fibers isomorphic to the Cuntz algebra $\mathcal{O}(n)$ $(2 \leq n \leq \infty)$ are trivial. More generally, we show that if A is a separable unital continuous field over $[0, 1]$ of Kirchberg C*-algebras satisfying the UCT and having finitely generated K-theory groups, then A is isomorphic to a trivial field if and only if the associated K-theory presheaf is isomorphic to the presheaf of a trivial field. We also show that, under the additional assumption that the fibers have torsion free K_0-group and trivial K_1-group, the K_0-(pre)sheaf is a complete invariant for separable unital continuous fields of Kirchberg algebras.

In [3] the approach of [1] and [2] is extended to higher dimensional spaces. Let X be a finite dimensional compact metrizable space. We prove that all separable unital continuous fields of C*-algebras over X with fibers isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ are trivial. In a more general context, assuming that X is locally contractible, we show that if A is a separable unital continuous field over X with fibers Kirchberg C*-algebras satisfying the universal coefficient theorem in KK-theory (UCT) and having finitely generated K-theory groups, then A is isomorphic to a locally trivial field if and only if A satisfies a natural Fell-type condition in K-theory (which is automatically satisfied if the K-theory presheaf associated to A is locally trivial). As a corollary we obtain that any separable unital continuous field of C*-algebras over a finite dimensional locally contractible compact metrizable space with fibers isomorphic to $\mathcal{O}_n$ is locally trivial. Using certain approximation and deformation techniques, we show that the C*-algebra of sections associated to a separable continuous field of C*-algebras over X satisfies UCT, provided that each fiber is nuclear and satisfies the UCT.

References

[1] M. Dadarlat and C. Pasnicu, *Continuous Fields of Kirchberg C*-algebras*, J. Funct. Anal., **226** (2), (2005), 257–506.

[2] M. Dadarlat and G. Elliott, *One-parameter Continuous Fields of Kirchberg algebras*, Preprint 2005.

[3] M. Dadarlat, *Continuous fields of C*-algebras over finite dimensional spaces*, Preprint 2005.

Pseudo-multiplicative unitaries on C*-modules, Hopf C*-families and duality

Thomas Timmermann

The aim of this talk is to generalise Pontrjagin duality and Takesaki-Takai duality to locally compact quantum groupoids. We do not have a definition of this category, yet we obtain partial results on a duality of objects and a satisfactory duality theory of C*-dynamical systems. The results presented in this talk were obtained in my PhD thesis which was supervised by Joachim Cuntz.

1. Duality of objects

1.1. **Background on generalised Pontrjagin duality.** Given a locally compact abelian group G, the set $\hat{G}$ of characters forms a locally compact abelian group, and the natural map $G \to \hat{\hat{G}}$ is an isomorphism. A satisfactory generalisation to non-abelian groups was achieved by Stefaan Vaes and Johan Kustermans [5] within the theory of locally compact quantum groups. Here, the construction of the dual object proceeds in two steps: the quantum group, which is a Hopf C*-algebra with a Haar weight, gives rise to a multiplicative unitary. From this unitary, one constructs a pair of Hopf C*-algebras [1, 10, 5] which underly the initial quantum group and its dual, respectively. The theory simultaneously covers measurable quantum groups – they correspond bijectively with locally compact ones.

For groupoids, the correspondence between the locally compact and the measurable category is lost. Whereas Pontrjagin duality has been generalised to measurable quantum groupoids [6] by Franck Lesieur, a parallel theory of locally compact quantum groupoids is missing. In Lesieur's theory, pseudo-multiplicative unitaries on Hilbert spaces take the rôle multiplicative unitaries have for quantum groups.

1.2. **Pseudo-multiplicative unitaries on C*-modules.** Although the von Neumann algebraic theory of measurable quantum groupoids can not easily be translated into a C*-algebraic theory of locally compact quantum groupoids, one can do so with the notion of a pseudo-multiplicative unitary: Given a C*-module E over a C*-algebra A and two commuting representations π_s, π_r of A on E, a unitary $V\colon E\ {}_{\pi_s}\!\otimes E \to E \otimes_{\pi_r} E$ is called *pseudo-multiplicative* if it satisfies the pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$ and some intertwining conditions with respect to π_s and π_r. Here, the domain and range of V are internal tensor products formed with respect to π_s acting on the first and π_r acting on the second factor, respectively. If E is a Hilbert space and π_s, π_r are given by scalar multiplication, one obtains an ordinary multiplicative unitary. For a more interesting example, let G be a locally compact groupoid with a left Haar system λ. Put $E = L^2(G,\lambda)$, $A = C_0(G^0)$ and denote by π_s and π_r the representations induced by the source and range map of G, respectively. Then the formula $(Vf)(x,y) := f(x,x^{-1}y)$ defines a pseudo-multiplicative unitary [7, 9]. Further examples arise e.g. from inclusions of C*-algebras [8] and deformations of multiplicative unitaries [2].

1.3. **Twisted C*-bimodule theory and Hopf C*-families.** How does a multiplicative unitary give rise to a pair of Hopf C*-algebras? For a finite group G, the associated unitary V simultaneously encodes $C_0(G)$ and $C^*_r(G)$ in the sense that $V = \sum_{x\in G} \delta_x \otimes \lambda_x$, where δ_x and λ_x denote the multiplication and left convolution operators associated to an element $x \in G$. The general case behaves similarly: the operators obtained by slicing V with maps of the form $1 \otimes \omega$ or $\omega \otimes 1$, where ω varies over a certain space of functionals, generate two Hopf C*-algebras [1, 10].

To general pseudo-multiplicative unitaries on C*-modules, these constructions do not carry over since they do not respect the module structures and are not compatible with the internal tensor product. We solve these problems by means of *twisted C*-bimodule theory* if the underlying C*-bimodules satisfy a *decomposability* condition which is inspired by r-discrete groupoids. The general idea is to allow module structures to be preserved only up to twists by partial automorphisms which are kept track of by additional book-keeping. As an example, a C*-bimodule E over a C*-algebra A is *decomposable* if it is the closed linear span of its *homogeneous* elements, and an element $\xi \in E$ is *homogeneous* of degree α, where α is a partial automorphism of A, if $\xi \in E\mathrm{Dom}(\alpha)$ and $\xi a = \alpha(a)\xi$ for all $a \in \mathrm{Dom}(\alpha)$. Likewise, we introduce *homogeneous operators* on C*-bimodules and C*-families which roughly are graded C*-algebras of such operators. The category of C*-families has an internal tensor product which we use to define Hopf C*-families. Now, the construction outlined above carries over – to each decomposable pseudo-multiplicative unitary we asscoiate a dual pair of Hopf C*-families.

2. Duality of C*-dynamical systems

2.1. **Background on generalised Takesaki-Takai duality.** Given an action α of a locally compact group G on a C*-algebra, one can form the (reduced) crossed product $A \rtimes_{(r)} G$ which encodes the C*-dynamical system (A, G, α). If G

is abelian, the crossed product carries a dual action of $\hat{G}$, and the iterated crossed product $A \rtimes_r G \rtimes_r \hat{G}$ is equivariantly Morita equivalent to A. Baaj and Skandalis generalised this result to a large class of quantum groups [1], replacing the dual pair of groups by a dual pair of Hopf C*-algebras arising from a Kac system, and actions of the groups by suitable coactions of the Hopf C*-algebras. Here, a Kac system consists of a multiplicative unitary and a unitary antipode.

2.2. **Pseudo-Kac systems.** To further generalise the duality to (quantum) groupoids, we need to define coactions of Hopf C*-families on C*-algebras and to reconsider the notion of a Kac system. Clearly, the multiplicative unitary should be replaced by a pseudo-multiplicative unitary on C*-modules, but already for locally compact groupoids, the unitary antipode can not be defined. In this example, one can prove a duality if one simultaneously considers the C*-modules associated to a left and to a right Haar system, the isometry between both induced by the involution of the groupoid, and various unitary maps between internal tensor products of these C*-modules all given by the same formula as the pseudo-multiplicative unitary in subsection 1.2. The definition of a *pseudo-Kac system* puts this into an axiomatic framework. If the system is *decomposable*, it gives rise to a dual pair of Hopf C*-families. For coactions of these Hopf C*-families on C*-algebras, we carry over the definition of reduced crossed products and the duality theorem.

In particular, we obtain a satisfying duality theorem for r-discrete groupoids, and more general for groupoids which are extensions of r-discrete groupoids by group bundles.

3. Non-Hausdorff groupoids

The theory developed so far applies primarily to Hausdorff groupoids. For a locally compact, non-Hausdorff groupoid G with a left Haar system λ, already the definition of the natural C*-module $L^2(G,\lambda)$ poses a problem. A replacement $L^2(G,\lambda)_{KS}$ was proposed by Mahmoud Khoshkam and Georges Skandalis [4]. Building on a Hausdorff compactification introduced by James Fell [3], we give a nice geometric description of this C*-module. In fact, the construction of Fell, applied to a locally compact non-Hausdorff groupoid G, yields a locally compact groupoid $\mathfrak{H}G$, and if G is r-discrete, then $L^2(G,\lambda)_{KS} = L^2(\mathfrak{H}G,\lambda')$. Here, λ and λ' denote the families counting measures. This result suggests that for the approach to locally compact quantum groupoids taken above, G can be replaced by $\mathfrak{H}G$.

References

[1] Saad Baaj and Georges Skandalis. *Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres.* Ann. scient. Ec. Norm. Sup., **26**:425–488, 1993.

[2] Étienne Blanchard. *Déformations de C*-algèbres de Hopf.* Bull. Soc. Math. France, **124(1)**:141–215, 1996.

[3] James Fell. *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space.* Proc. Amer. Math. Soc. **13**:472-476, 1962.

[4] Mahmood Khoshkam and Georges Skandalis. *Regular representation of groupoid C*-algebras and applications to inverse semigroups.* J. reine angew. Math., **546**:47–72, 2002.

[5] Johan Kustermans and Stefaan Vaes. *Locally compact quantum groups.* Ann. Sci. École Norm. Sup. **4**:837-834, 2000.
[6] Franck Lesieur. *Groupoïdes quantiques mesurés: axiomatique, étude, dualité, exemples.* PhD thesis, Universite d'Orléans, 2003.
[7] Moto O'uchi. *Pseudo-multiplicative unitaries on Hilbert C^*-modules.* Far East J. Math. Sci. (FJMS), (Special Volume, Part II):229–249, 2001.
[8] Moto O'uchi. *Pseudo-multiplicative unitaries associated with inclusions of finite-dimensional C^*-algebras.* Linear Algebra Appl. **341**:201-218, 2002.
[9] Thomas Timmermann. *Pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2005.
[10] Stanislaw L. Woronowicz. *From multiplicative unitaries to quantum groups.* Internat. J. Math. (7) **1**:127-149, 1996.

The structure of crossed products by minimal homeomorphisms

N. Christopher Phillips

This is joint work with Huaxin Lin, except for Example 4, which is joint work with Benjamín Itzá-Ortíz.

We consider the classification of crossed products by minimal homeomorphisms, in the real rank zero case. For any unital C*-algebra B, following the usual notation, let $\rho = \rho_B \colon K_0(B) \to \mathrm{Aff}(T(B))$ be the standard map from $K_0(B)$ to the affine functions on the tracial state space of B. If B is simple and has real rank zero, then ρ has dense range.

Theorem 1 ([10]). *Let X be an infinite compact metric space with finite covering dimension, and let $h \colon X \to X$ be a minimal homeomorphism. Set $A = C^*(\mathbf{Z}, X, h)$. Suppose that $\rho(K_0(A))$ is dense in $\mathrm{Aff}(T(A))$. Then A has tracial rank zero in the sense of* [8] *(is tracially AF in the sense of* [7]*).*

Since $C^*(\mathbf{Z}, X, h)$ is automatically simple and unital, and satisfies the Universal Coefficient Theorem, H. Lin's classification theorem [9] applies, and the isomorphism class of this algebra is determined, within a large class, by its Elliott invariant. It is often not difficult to verify the hypotheses of the theorem and compute the Elliott invariant directly from X and h, without knowing $C^*(\mathbf{Z}, X, h)$. For example, $T(C^*(\mathbf{Z}, X, h))$ is just the space of h-invariant Borel probability measures on X, the groups $K_*(C^*(\mathbf{Z}, X, h))$ can be calculated using the Pimsner-Voiculescu exact sequence [13], and, when X is connected, $\rho_{C^*(\mathbf{Z},X,h)}$ can be calculated using ideas systematized in [2].

Theorem 1, together with classification [9], implies known structure results, for example that the irrational rotation algebras are AT algebras (first proved by Elliott and Evans in [1]) and that the transformation group C*-algebras of minimal homeomorphisms of the Cantor set are AT algebras (essentially due to Putnam [16]). It also gives new isomorphism results for transformation group C*-algebras. Here is a selection.

Example 2. For each minimal homeomorphism $h_0 \colon X_0 \to X_0$ of the Cantor set, Gjerde and Johansen [4] construct a minimal homeomorphism $h \colon X \to X$ of a highly disconnected one dimensional space, such that $C^*(\mathbf{Z}, X, h)$ has the same Elliott invariant as $C^*(\mathbf{Z}, X_0, h_0)$. We show that $C^*(\mathbf{Z}, X, h) \cong C^*(\mathbf{Z}, X_0, h_0)$.

Example 3. Consider two Furstenberg transformations on the torus $S^1 \times S^1$, say

$$h_j(\zeta_1, \zeta_2) = \big(\exp(2\pi i\theta)\zeta_1,\ (\exp(2\pi i f_j(\zeta_1))\zeta_1^d\zeta_2\big)$$

for $j = 1, 2$, with $\theta \in \mathbf{R} \setminus \mathbf{Q}$ and $d \in \mathbf{Z} \setminus \{0\}$ fixed (the same for both), and with $f_1, f_2 \colon S^1 \to S^1$ continuous. Suppose h_1 and h_2 are uniquely ergodic. (In particular, this happens whenever f_1 and f_2 are Lipschitz. See [3].) Then $C^*(\mathbf{Z},\ S^1 \times S^1,\ h_1) \cong C^*(\mathbf{Z},\ S^1 \times S^1,\ h_2)$. (The case in which f_1 and f_2 are smooth is covered by [12].) An analogous statement is true on $(S^1)^d$ for $d \geq 3$.

Example 4. For most choices of a nondegenerate real skew symmetric $d \times d$ matrix θ with $d \geq 3$, the simple higher dimensional noncommutative torus A_θ is isomorphic to the transformation group C*-algebra of a minimal homeomorphism $h \colon X \to X$ of a one dimensional space X, obtained in the following way. Choose a suitable minimal homeomorphism $h_0 \colon X_0 \to X_0$ of the Cantor set, obtained as the restriction to its unique minimal set of a suitable Denjoy homeomorphism of the circle. Consider the suspension flow $t \mapsto H_t$ of h_0 on the space $X = (X_0 \times [0,1])/\sim$, where $(x, 1) \sim (h(x), 0)$. The flow is given by moving points up at unit speed, and following the equivalence relation when one hits the top. Then, for a suitable (irrational) choice of t, the homeomorphism $h = H_t$ is the one desired. See [5].

The proof of Theorem 1 has two main ingredients. First, we give a convenient condition for a simple unital C*-algebra to have tracial rank zero.

Lemma 5. *Let A be a simple unital C*-algebra. Suppose that for every finite subset $S \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a simple unital subalgebra $B \subset pAp$ with tracial rank zero such that:*

(1) $\| [a, p] \| < \varepsilon$ *for all* $a \in S$.
(2) $\operatorname{dist}(pap, B) < \varepsilon$ *for all* $a \in S$.
(3) $1 - p$ *is Murray-von Neumann equivalent to a projection in* $\overline{cAc}$.

Then A has tracial rank zero.

Now fix $h \colon X \to X$ satisfying the hypotheses. Set $A = C^*(\mathbf{Z}, X, h)$. Let $u \in A$ be the standard unitary, satisfying $ufu^* = f \circ h^{-1}$ for $f \in C(X)$. For $Y \subset X$ closed, let A_Y be the C*-subalgebra of $C^*(\mathbf{Z}, X, h)$ generated by $C(X)$ and all uf for $f \in C(X)$ such that $f = 0$ on Y. This algebra is the C*-algebra of a subgroupoid of the transformation group groupoid. If $\operatorname{int}(Y) \neq \varnothing$, then A_Y is a recursive subhomogeneous C*-algebra in the sense of [14]. The important subalgebra is $A_{\{y\}}$, the one obtained from a one point set. It is a simple direct limit, with no dimension growth, of subalgebras A_{Y_n} with $\operatorname{int}(Y) \neq \varnothing$. Moreover, $T(A) \to T(A_{\{y\}})$ is an isomorphism [11], and $K_0(A_{\{y\}}) \to K_0(A)$ is an isomorphism [17]. If $T(A)$ has only countably many extreme points, it follows from general theory [15] that $A_{\{y\}}$ has tracial rank zero. In the general case, using [6], this can be proved directly, at least for y in a dense G_δ-set in X. This is the first main ingredient.

The second main ingredient is an adaptation of Putnam's version of Berg's technique, from [16]. We want to verify the hypotheses of Lemma 5. In this outline, we omit discussion of Condition (2) there. The finite subset S can be taken to have the form $S = S_0 \cup \{u\}$ for some finite subset $S_0 \subset C(X)$. Moreover, we may assume that the functions in S_0 are constant on any predetermined finite collection of disjoint compact subsets of X whose diameters are all sufficiently small.

Choose and fix $y \in X$ such that $A_{\{y\}}$ has tracial rank zero. Choose a large number n, and use minimality to choose $N \geq n$ such that $h^{N-k}(y)$ is very close to $h^{-k}(y)$ for $0 \leq k \leq n$. Choose a neighborhood U of y such that, with $Z = \overline{U}$, the sets

$$h^{-n}(Z),\, h^{-n+1}(Z),\, \ldots, h^{N}(Z)$$

are all disjoint, and furthermore the sets

$$h^{-n}(Z) \cup h^{N-n}(Z), \quad h^{-n+1}(Z) \cup h^{N-n+1}(Z), \quad \ldots, \quad Z \cup h^{N}(Z)$$

and

$$h(Z),\, h^{2}(Z),\, \ldots, h^{N-n-1}(Z)$$

all have very small diameter. We will take the functions in S_0 to be constant on each of the sets in the second and third lists above.

Since $A_{\{y\}}$ has real rank zero, it is possible to find functions $g_0, g_1 \in C(X)$ and a projection $e \in A_{\{y\}}$ such that $g_0 = 1$ on a neighborhood of y, $eg_0 = g_0$, $g_1 e = e$, and $\mathrm{supp}(g_1) \subset U$. One can show that the projections $e_j = u^j e u^{-j}$ are all in $A_{\{y\}}$, and they are all unitarily equivalent in $A_{\{y\}}$ because $K_0(A_{\{y\}}) \to K_0(A)$ is an isomorphism and $A_{\{y\}}$ has tracial rank zero. Since each $f \in S_0$ is constant on $h^j(Z)$, it follows that f commutes with e_j.

The projection $\sum_{j=0}^{N-1} e_j$ thus commutes with every $f \in S_0$. However, it does not even approximately commute with u because $e_N = u^n e_0 u^{-n}$ is not close to e_0. This can be fixed as follows. There is a subalgebra $D \subset A_{\{y\}}$ isomorphic to M_2, with e_{N-n} and e_{-n} corresponding to the diagonal rank one projections. There is a discrete path of projections in D from e_{N-n} to e_{-n}, say $e_{N-n} = r_0, r_1, \ldots, r_{n-1}, r_n = e_{-n}$, with $\|r_j - r_{j-1}\| \leq \pi/n$. Note that $u^j r_j u^{-j}$ commutes with every $f \in S_0$, because f is constant on $h^{-n+j}(Z) \cup h^{N-n+j}(Z)$. Instead of using $e_0, e_1, \ldots, e_{N-1}$, we add up the projections

$$e_0,\ e_1,\ \ldots,\ e_{N-n} = r_0,\ u r_1 u^*,\ \ldots,\ u^{n-1} r_{n-1} u^{-(n-1)},$$

noting that $u^n r_n u^{-n} = u^n e_{-n} u^{-n} = e_0$. Call the result q. Then $\|uq - qu\| \leq \pi/n$. If moreover U is small enough, then q will be small in the tracial sense. The required projection p is then $p = 1 - q$. As noted above, we do not describe how to show that $pap \in A_{\{y\}}$ for $a \in S$, but it is not particularly hard; the key property is that $eg_0 = g_0$ with $g_0 = 1$ on a neighborhood of y.

References

[1] G. A. Elliott and D. E. Evans, *The structure of the irrational rotation algebra*, Ann. of Math. (2) **138** (1993), 477–501.

[2] R. Exel, *Rotation numbers for automorphisms of C*-algebras*, Pacific J. Math. **127** (1987), 31–89.
[3] H. Furstenberg, *Strict ergodicity and transformation of the torus*, Amer. J. Math. **83** (1961), 573–601.
[4] R. Gjerde and Ø. Johansen, *C*-algebras associated to non-homogeneous minimal systems and their K-theory*, Math. Scand. **85** (1999), 87–104.
[5] B. Itzá-Ortiz and N. C. Phillips, in preparation.
[6] J. Kulesza, *Zero-dimensional covers of finite dimensional dynamical systems*, Ergod. Th. Dynam. Sys. **15** (1995), 939–950.
[7] H. Lin, *Tracially AF C*-algebras*, Trans. Amer. Math. Soc. **353** (2001), 693–722.
[8] H. Lin, *The tracial topological rank of C*-algebras*, Proc. London Math. Soc. **83** (2001), 199–234.
[9] H. Lin, *Classification of simple C*-algebras with tracial topological rank zero*, Duke Math. J. **125** (2005), 91–119.
[10] H. Lin and N. C. Phillips, *Crossed products by minimal homeomorphisms*, preprint (arXiv: math.OA/0408291).
[11] Q. Lin, *Analytic structure of the transformation group C*-algebra associated with minimal dynamical systems*, preprint.
[12] Q. Lin and N. C. Phillips, *Direct limit decomposition for C*-algebras of minimal diffeomorphisms*, pages 107–133 in: *Operator Algebras and Applications,* Advanced Studies in Pure Mathematics vol. 38, Mathematical Society of Japan, 2004.
[13] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C*-algebras*, J. Operator Theory **4** (1980), 93–118.
[14] N. C. Phillips, *Recursive subhomogeneous algebras*, Trans. Amer. Math. Soc., to appear.
[15] N. C. Phillips, *Real rank and property (SP) for direct limits of recursive subhomogeneous algebras*, preprint (arXiv: math.OA/0405265).
[16] I. F. Putnam, *On the topological stable rank of certain transformation group C*-algebras*, Ergod. Th. Dynam. Sys. **10** (1990), 197–207.
[17] I. F. Putnam, *On the K-theory of C*-algebras of principal groupoids*, Rocky Mtn. J. Math. **28** (1998), 1483–1518.

C^*–algebras of real rank zero

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Dedicated to the memory of Gert Kjærgaard Pedersen

Larry Brown and Gert Kjærgaard Pedersen introduced in the paper, [1], the notion of real rank for C^*–algebras. The lowest possible real rank is zero; and a commutative C^*–algebra is of real rank zero if and only if its spectrum is zero-dimensional (i.e., is totally disconnected). In general a (possibly non-commutative) C^*–algebra A is of real rank zero if the set of self-adjoint elements in A with finite spectrum is a norm-dense subset of the set of all self-adjoint elements in A. (This is one of several equivalent conditions for real rank zero listed in the paper by Brown and Pedersen.)

The present talk focuses on recent developments and open problems concerning C^*–algebras of real rank zero, in particular in regards to two set of problems. Describe the structure of C^*–algebras of real rank zero (having in mind the intuive assumption that these are "zero-dimensional" non-commutative spaces); and characterize which C^*–algebras are of real rank zero. In particular we report on the results of our recent paper, [4]. We also mention the papers [3] and [2] on embedding properties for unital C*–algebras of real rank zero. Our study of C*–algebras of real rank zero is motivated by the following open problems for simple C*–algebras.

Question 1. Is any simple C*–algebra of real rank zero either stably finite or purely infinite?

Question 2. Is the ordered group $K_0(A)$ of a simple real rank zero C*–algebra A necessarily weakly unperforated?

Question 3 (Elliott's Conjecture)**.** Is (K_0, K_0^+, K_1) a complete invariant for the class of simple, separable, nuclear, stable C*–algebras of real rank zero?

One characterization of when a C*–algebra is of real rank zero applies to C*–algebras A where the so-called *Cuntz semigroup* $W(A)$ is weakly unperforated (i.e., if $nx \le my$ and $n > m$ for some $x, y \in W(A)$ and some $n, m \in \mathbb{N}$, then $x \le y$). The semigroup $W(A)$ is defined to be $M_\infty(A)$ modulo the equivalence arising from Cuntz comparison. Addition is given by direct sum, and the ordering by the Cuntz ordering (see [4] for details).

Theorem 4. *Let A be a simple unital exact C*–algebra with* $\mathrm{sr}(A) = 1$ *and with weakly unperforated Cuntz semigroup $W(A)$. Then* $\mathrm{RR}(A) = 0$ *if and only if $K_0(A)$ is uniformly dense in* $\mathrm{Aff}(T(A))$.

We show in [4] that if A is a finite simple $\mathcal{Z}$-absoring C*–algebra, then $\mathrm{sr}(A) = 1$ and $W(A)$ is weakly unperforated. We therefore obtain the following:

Corollary 5. *Let A be a unital simple exact $\mathcal{Z}$-absorbing C*–algebra. Then* $\mathrm{RR}(A) = 0$ *if and only if $K_0(A)$ is uniformly dense in* $\mathrm{Aff}(T(A))$.

The literature contains several results along the line of the corollary above for other classes of C*–algebras.

We now turn to the non-simple case, where the situation is less settled. Let us first mention the following result from [1].

Theorem 6 (Brown–Pedersen)**.** *Given a short exact sequence of C*–algebras* $0 \to A \to E \to B \to 0$. *Then* $\mathrm{RR}(E) = 0$ *if and only if* $\mathrm{RR}(A) = \mathrm{RR}(B) = 0$ *and the index map* $\delta\colon K_0(B) \to K_1(A)$ *is zero.*

The latter condition gives rise to the following defintion:

Definition 7. A (non-simple) C*–algebra A is said to be K_0*-liftable* if for every pair of closed two-sided ideals I, J in A such that $I \subseteq J$ the index map $K_0(J/I) \to K_1(I)$, arising from the short exact sequence $0 \to I \to J \to J/I \to 0$, is zero.

Every C*–algebra of real rank zero is K_0-liftable. It follows from Brown and Pedersen's extension theorem that a purely infinite C*–algebra with finite ideal lattice is of real rank zero if and only if it is K_0-liftable.

If we look at more general non-simple C*–algebras (possibly with infinitely many ideals), then we must also look at its spectrum, which necessarily is totally disconnected when the C*–algebra is of real rank zero. In the converse direction we have the following:

Theorem 8 (Kirchberg)**.** *Let A be a separable, nuclear C*–algebra such that $A \cong A \otimes \mathcal{O}_2$. It follows that* $\mathrm{RR}(A) = 0$ *if and only if the spectrum of A is totally disconnected.*

It would be interesting to know when non-simple purely infinite (and non-simple $\mathcal{Z}$-absorbing) C*–algebras are of real rank zero. It seems plausible that the two necessary conditions mentioned above are also sufficient in the purely infinite case.

References

[1] L. G. Brown and G. K. Pedersen, *C*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

[2] G. A. Elliott and M. Rørdam, *Perturbation of Hausdorff moment sequences, and an application to the theory of C*-algebras of real rank zero*, preprint, 2005.

[3] F. Perera and M. Rørdam, *AF-embeddings into C*-algebras of real rank zero*, J. Funct. Anal. **217** (2004), no. 1, 142–170.

[4] M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras*, International J. Math. **15** (2004), no. 10, 1065–1084.

Dimension growth for C*-algebras

Andrew Toms

A recurring theme in theorems confirming the Elliott conjecture is that of minimal rank. There are various notions of rank for C*-algebras — the real rank, the stable rank, the tracial topological rank, and the decomposition rank — see [1], [7], [6] and [4], respectively, for definitions and basic properties — which attempt to capture a non-commutative version of dimension. A natural and successful approach to proving classification theorems for separable and nuclear C*-algebras has been to assume that one or more of these ranks is minimal (see [2] and [5], for instance). But there are examples which show these minimal rank conditions to be variously too strong or too weak to characterise those algebras for which the Elliott conjecture will be confirmed. There are many reasons to assume $\mathcal{Z}$-stability — the condition that a C*-algebra absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially — instead (see [8] for a discussion), but a fair objection has been that this assumption seems unnatural. We overcome this objection by situating $\mathcal{Z}$-stability as the minimal instance of a well-behaved rank for C*-algebras.

Definition 1. Let A be a C*-algebra. The *growth rank* $\mathrm{gr}(A)$ is the least natural number n such that

$$A^{\otimes n} \stackrel{\text{def}}{=} \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}$$

is $\mathcal{Z}$-stable, assuming the minimal tensor product. If no such integer exists, then say $\mathrm{gr}(A) = \infty$.

The growth rank inherits excellent behaviour with respect to common operations from the robustness of $\mathcal{Z}$-stability. Its name is motivated by the fact that it may be viewed as giving a theory of dimension growth for AH algebras, and, more generally, locally type-I C*-algebras. Of course, it is of little consequence if its range is not exhausted.

Theorem 2. *For every $n \in \mathbb{N} \cup \{\infty\}$ there exists a simple, separable and nuclear C*-algebra of growth rank n.*

The algebras constructed in the proof of this theorem are new and rather exotic: for all but two of them, the other ranks for C*-algebras above are simultaneously infinite. We use these algebras to obtain an unexpected creature.

Corollary 3. *There is a simple, nuclear, and non-$\mathcal{Z}$-stable C*-algebra which is not tensorially prime.*

Formally, the growth rank suggests that infinite tensor products of unital and nuclear C*-algebras should be $\mathcal{Z}$-stable (although one must exclude algebras with one-dimensional representations). If $A^{\otimes \mathrm{gr}(A)} \cong A^{\otimes \mathrm{gr}(A)} \otimes \mathcal{Z}$ when $\mathrm{gr}(A)$ is finite, then why not $A^{\otimes\infty} \cong A^{\otimes\infty} \otimes \mathcal{Z}$ whenever this makes sense? This leads us to consider:

Universal Property 4. *Let $\mathcal{C}$ be a class of unital and nuclear C*-algebras. If A in $\mathcal{C}$ is such that*

(i) $A^{\otimes\infty} \cong A$, *and*
(ii) $B^{\otimes\infty} \otimes A \cong B^{\otimes\infty}$ *for every B in $\mathcal{C}$,*

then A is unique up to $*$-isomorphism.

Proof. Suppose that A, B in $\mathcal{C}$ satisfy (i) and (ii) above. Then,

$$A \overset{\text{(i)}}{\cong} A^{\otimes\infty} \overset{\text{(ii)}}{\cong} A^{\otimes\infty} \otimes B \overset{\text{(i)}}{\cong} A \otimes B^{\otimes\infty} \overset{\text{(ii)}}{\cong} B^{\otimes\infty} \overset{\text{(i)}}{\cong} B.$$

□

Condition (i) is known to hold for $\mathcal{Z}$. With $A = \mathcal{Z}$, condition (ii) asks for infinite tensor products to be $\mathcal{Z}$-stable, as suggested formally by the growth rank. This suggestion turns out to be prophetic. With $A = \mathcal{Z}$, we can verify condition (ii) inside a large — read "beyond the scope of classification results" — class of separable, unital, nuclear, and locally subhomogeneous C*-algebras which, significantly, contains projectionless algebras. This represents the first uniqueness theorem for $\mathcal{Z}$ among projectionless algebras which does not require the said algebras to be classified via the Elliott invariant. The greatest possible generalisation of this theorem would come from a positive answer to:

Question 5. Let A be a separable, unital, and nuclear C*-algebra having no one-dimensional representations. Is $A^{\otimes\infty}$ $\mathcal{Z}$-stable?

References

[1] Brown, L. G. and Pedersen, G. K.: *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131-149
[2] Elliott, G. A., and Gong, G.:On the classification of C^*-algebras of real rank zero. II. Ann. of Math. (2) **144** (1996), 497-610
[3] Jiang, X. and Su, H.: *On a simple unital projectionless C^*-algebra*, Amer. J. Math. **121** (1999), 359-413
[4] Kirchberg, E. and Winter, W.: *Covering dimension and quasidiagonality*, Intern. J. Math. **15** (2004), 63-85
[5] Lin, H.: *Classification of simple tracially AF C^*-algebras*, Canad. J. Math. **53** (2001), 161-194
[6] Lin, H.: *The tracial topological rank of C^*-algebras*, Proc. London Math. Soc. (3) **83** (2001), 199-234
[7] Rieffel, M. A.: *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. (3) **46** (1983), 301-333
[8] A. S. Toms: *Dimension growth for C^*-algebras*, arXiv preprint math.OA/0509159 (2005)

Axiomatics as a technique of proof—the classification theorem of Niu

George A. Elliott

The following result was obtained recently by Zhuang Niu (in his University of Toronto Ph.D. thesis):

Theorem 1. *Let A and B be simple TAS-algebras (see below) with the same naive K-theory invariant (i.e., the ordered group K_0, with the class of the unit, the pairing of K_0 with the simplex of tracial states, and the group K_1; see* [3], [4]*). Suppose that A and B are separable, amenable, and satisfy the Universal Coefficient Theorem (see* [10]*). It follows that A and B are isomorphic.*

The class of TAS-algebras is a considerable extension of the class of TAI-algebras, introduced by Lin in [14]. The analogue of Theorem 1 for TAI-algebras (a special case of Theorem 1) was proved by Lin in [14]—and also by Niu in [16]—both authors using the substantial earlier papers [11] and [15] of Lin. Very briefly, the difference between TAI-algebras and TAS-algebras is that the ordered group K_0 is constrained to have the Riesz decomposition property in the case of a TAI-algebra but not in the case of a TAS-algebra.

Let there be given a class $\mathscr{C}$ of separable unital C*-algebras. Let us say that a given separable unital C*-algebra A belongs to the inductive limit closure of $\mathscr{C}$—and write $A \in \mathrm{A}\mathscr{C}$—if A is the inductive limit of a sequence of C*-algebras in the class $\mathscr{C}$.

If $\mathscr{F}$ denotes the class of finite-dimensional C*-algebras, then the class $\mathrm{A}\mathscr{F}$ is the class of AF-algebras (approximately finite-dimensional C*-algebras) considered (and classified) by Bratteli in [1] and by the present author in [2].

If $\mathscr{H}$ denotes the class of finite direct sums of matrix algebras over separable unital commutative C*-algebras (together, more precisely, with the unital hereditary sub-C*-algebras of these), then $\mathrm{A}\mathscr{H}$ is the class of AH-algebras (approximately homogeneous C*-algebras) considered by numerous authors and classified by Gong, Li, and the present author in [6] in the simple case, under the important restriction that the spectra of the commutative C*-algebras appearing in the decomposition are of bounded dimension. (This restriction was shown by Villadsen in [18] not to be redundant.)

If $\mathscr{S}$ denotes the class of what were called splitting interval C*-algebras by Jiang and Su in [9]—containing the C*-algebra $C[0,1]$, matrix algebras over this, and also the subalgebras of these consisting of matrix-valued functions with specified block-diagonal form at either end of the interval—then the class of simple C*-algebras in $A\mathscr{S}$ was classified in [9].

Let us say that a given separable unital C*-algebra A belongs approximately to the class $\mathscr{C}$ (of separable unital C*-algebras) in the sense of approximation in trace, or tracial approximation—briefly, that A is tracially approximately in $\mathscr{C}$—and write $A \in \mathrm{TA}\mathscr{C}$—if for every finite subset of A there exists a non-zero projection p in A commuting approximately with the elements of this subset, such that the elements pap with a in this subset can be approximated by a unital sub-C*-algebra of pAp belonging to the class $\mathscr{C}$—both of these approximations being arbitrarily close—, and such that the projection $1-p$ is Murray-von Neumann equivalent to a projection belonging to an arbitrarily given non-zero hereditary sub-C*-algebra of A. (Note that even if the projection p may always be taken to be the unit for a particular A, it is not clear that A belongs to $\mathrm{A}\mathscr{C}$, although this is known for instance in the case $\mathscr{C} = \mathscr{F}$. One might say if this is possible that A belongs approximately to $\mathscr{C}$ in the sense of approximation in norm, or norm approximation.)

The class $\mathrm{TA}\mathscr{F}$, with $\mathscr{F}$ as above the class of finite-dimensional C*-algebras, is the class of TAF-algebras (tracially approximately finite-dimensional C*-algebras) introduced by Lin in [12] (following a proposal of Popa in [17] in which an approximation as above was required but with the projection p assumed only to be non-zero, not necessarily close to 1 in the sense described above). (Lin also referred to TAF-algebras as C*-algebras of tracial rank zero.) In [12], [13], and [15], Lin established Theorem 1 as stated above in the special case of TAF-algebras. In fact, Lin proved this by showing that the class of TAF-algebras referred to is contained in the class of AH-algebras classified by the present author and Gong in [5] (namely, simple, of real rank zero—and with the pertinent spectra of bounded dimension). In fact, it was shown in [5] that the algebras considered had the TAF property, and so the classes considered in [15] and [5] are coextensive; one has an axiomatization of the class considered in [5].

Similarly, it was shown in [8] (see also [6]) that the C*-algebras classified in [6] were TAI. Therefore, the result of [14] and [16] mentioned above—Theorem 1 in the case of TAI-algebras—is an axiomatization of the class of AH-algebras considered in [6] (namely, simple, with the pertinent spectra of bounded dimension).

Theorem 1 is proved in the general case using very much the same techniques as in the special cases of TAF-algebras and TAI-algebras mentioned above. The difference is that a peculiarity of the logical structure of the proof in the earlier two cases, irrelevant in those cases of the theorem as the associated class of AH-algebras had already been classified, now leaps to the fore: the proof of Theorem 1 consists first in showing that there exists an algebra in the class $\mathrm{A}(\mathscr{S} \cup \mathscr{H})$ which is TAS and simple and, moreover, has the same invariant as a given TAS-algebra as in the statement of Theorem 1—this is joint work with the author of the present abstract—see [7]—and then in showing that any such algebra is isomorphic to A. (The second step is of course just Theorem 1 in the case $B \in \mathrm{A}(\mathscr{S} \cup \mathscr{H})$.)

In applying these two steps also to a second algebra B with the same invariant as A, one need only take the precaution of choosing the same inductive limit comparison algebras—B and A are then isomorphic to the same comparison algebras and therefore to each other! (In the TAF or TAI case, the comparison algebras are automatically the same, by the inductive limit isomorphism theorem already known in that case.)

The following result is a consequence of Theorem 1:

Corollary 2. *Let A and B be separable simple unital C^*-algebras which are inductive limits of sequences of C^*-algebras each of which is a finite direct sum of algebras in either the class $\mathscr{H}$ or the class $\mathscr{S}$ (briefly, A and B belong to $A(\mathscr{H} \cup \mathscr{S})$). Suppose that the center of the C^*-algebras appearing in the inductive limit decompositions of A and B have spectra of bounded dimension. If A and B have isomorphic naive K-theory invariants (see e.g.* [3]*), then A and B are isomorphic.*

Proof. It follows immediately from the fact (proved in [8]) that the algebras of [6] are TAI that A and B are TAS. □

References

[1] Ola Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[2] George A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

[3] George A. Elliott, *The classification problem for amenable C*-algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 922–932, Birkhäuser, Basel, 1995.

[4] George A. Elliott, *An invariant for simple C*-algebras*, Canadian Mathematical Society, 1945–1995, Vol. **3**, 61–90, Canad. Math. Soc., Ottawa, 1996.

[5] George A. Elliott and Guihua Gong, *On the classification of C*-algebras of real rank zero. II*, Ann. of Math. (2) **144** (1996), 497–610.

[6] George A. Elliott, Guihua Gong, and Liangqing Li, *On the classification of simple inductive limit C*-algebras. II. The isomorphism theorem*, preprint, 1998.

[7] George A. Elliott and Zhuang Niu, *On the invariants of TAS-algebras*, in preparation.

[8] Guihua Gong, *On the classification of simple inductive limit C*-algebras. I. The reduction theorem*, Doc. Math. **7** (2002), 255–461.

[9] Xinhui Jiang and Hongbing Su, *A classification of simple limits of splitting interval algebras*, J. Funct. Anal. **151** (1997), 50–76.

[10] Huaxin Lin, *An introduction to the classification of amenable C*-algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xii+320 pp. ISBN: 981-02-4680-3.

[11] Huaxin Lin, *A classification theorem for nuclear simple C*-algebras of stable rank one. I and II*, preprint, 2000.

[12] Huaxin Lin, *Tracially AF C*-algebras*, Trans. Amer. Math. Soc. **353** (2001), 693–722.
[13] Huaxin Lin, *Classification of simple tracially AF C*-algebras*, Canad. J. Math. **53** (2001), 161–194.
[14] Huaxin Lin, *Simple nuclear C*-algebras of tracial topological rank one*, preprint, 2003, 2004.
[15] Huaxin Lin, *Classification of simple C*-algebras of tracial topological rank zero*, Duke Math. J. **125** (2004), 91–119.
[16] Zhuang Niu, *On the classification of TAI algebras*, C. R. Math. Acad. Sci. Soc. R. Canada **26** (2004), 18–24.
[17] Sorin Popa, *On local finite-dimensional approximation of C*-algebras*, Pacific J. Math. **181** (1997), 141–158.
[18] Jesper Villadsen, *Simple C*-algebras with perforation*, J. Funct. Anal. **154** (1998), 110–116.

Uniform embeddings via group actions

Erik Guentner
(joint work with Marius Dadarlat)

Gromov introduced the notion of uniform embedding and suggested it may be relevant for the Novikov conjecture [6]. Subsequently Yu proved that a uniformly embeddable discrete group satisfies the Novikov conjecture [13, 14]. Motivated in part by these results, the class of uniformly embeddable groups has attracted much attention [2, 3, 4, 7, 8].

We recall that a function $f : X \to Y$ between metric spaces is a *uniform embedding* if there exist proper and nondecreasing functions $\rho_\pm : [0,\infty) \to [0,\infty)$ such that for all x, $y \in X$

$$\rho_-(d_X(x,y)) \leq d_Y(f(x),f(y)) \leq \rho_+(d_X(x,y)).$$

A metric space X is *uniformly embeddable* (*in Hilbert space* $\mathcal{H}$) if there exists a uniform embedding $f : X \to \mathcal{H}$. An easy consequence of the definitions is that if Y is uniformly embeddable and there exists a uniform embedding $X \to Y$ then X is uniformly embeddable.

Let Γ be a countable discrete group. In order to apply the concept of uniform embeddability we view Γ as a metric space. We equip Γ with a (left invariant) metric associated to a *proper* length function. Any two such metrics are *coarsely equivalent*; indeed, the identity $\Gamma \to \Gamma$ is a uniform embedding. As a consequence, the property of Γ being uniformly embeddable is independent of the choice of metric.

Theorem (Fibering Theorem). Let the discrete group Γ act on the metric spaces X and Y by isometries, and let $p : X \to Y$ be a Lipschitz, Γ-equivariant map. Assume that Y is an exact space (see below) and that the action of Γ on Y is cobounded. If there exists $y_0 \in Y$ such that for all $n \in \mathbb{N}$ the set $p^{-1}(B(y_0,n))$ is uniformly embeddable then X is uniformly embeddable.

A metric space Y is *exact* if it satisfies a certain partition of unity condition. Rather than giving a precise definition we place exactness in context with the remark that a countable discrete group Γ is exact as a metric space if and only if it is C^*-exact. (See [4] for the definition and additional details.)

An important special case of the theorem occurs when $X = \Gamma$ and the map $p : \Gamma \to Y$ is the orbit map $p(g) = g \cdot y_0$. In this case the theorem states that if Y is an exact space and if, for every $n \in \mathbb{N}$, the *coarse stabilizer*

$$\{\, g \in \Gamma : d(g \cdot y_0, y_0) \leq n \,\} \tag{1}$$

is uniformly embeddable then Γ itself is uniformly embeddable.[1] A necessary condition for Γ to be uniformly embeddable is that the stabilizers of the action on Y are uniformly embeddable; the hypothesis concerns how copies of these stabilizers fit together to form coarse stabilizers.

The Fibering Theorem, and its variants, have several applications:

- (i) extensions
- (ii) free products (with amalgam)
- (iii) relatively hyperbolic groups

The applications to extensions and relativly hyperbolic groups were described in the talk.

Theorem (Extensions). Let $1 \to H \to \Gamma \to G \to 1$ be an extension of discrete groups. If H is uniformly embeddable and G is C^*-exact then Γ is uniformly embeddable. □

For several equivalent formulations of relative hyperbolicity we refer to [1, 5, 11]. In its outline, the proof of the following theorem follows the proof of the analogous result of Osin concerning finite asymptotic dimension [10].

Theorem (Relatively hyperbolic groups). Let Γ be a finitely generated discrete group which is relatively hyperbolic with respect to a subgroup H. Then Γ is uniformly embeddable if and only if H is uniformly embeddable. □

The Fibering Theorem has a parallel version for exact spaces; simply replace 'uniformly embeddable' by 'exact' in the statement. The version for exact spaces yields parallel applications to C^*-exact groups. We recover the result of Kirchberg and Wassermann on extensions of C^*-exact groups [9]. An alternate approach to C^*-exactness of relatively hyperbolic groups is given by Ozawa [12].

The results described here are based on [4], to which we refer for details.

References

[1] B. H. Bowditch. Relatively hyperbolic groups. Preprint, 1999.

[2] X. Chen, M. Dadarlat, E. Guentner, and G. Yu. Uniform embeddability and exactness of free products. *Jour. of Funct. Anal.*, 25:168–179, 2003.

[3] M. Dadarlat and E. Guentner. Constructions preserving Hilbert space uniform embeddability of discrete groups. *Trans. Amer. Math. Soc.*, 355:3253–3275, 2003.

[4] M. Dadarlat and E. Guentner. Uniform embeddability of relatively hyperbolic groups. Preprint, 2005.

[1] Replacing Y by the orbit $\Gamma \cdot y_0$ we see that the assumption that the action on Y be cobounded is superfluous.

[5] B. Farb. Relatively hyperbolic groups. *Geom. Funct. Anal.*, 8(5):810–840, 1998.
[6] M. Gromov. Asymptotic invariants of infinite groups. In A. Niblo and M. Roller, editors, *Geometric Group Theory*, number 182 in London Mathematical Society Lecture Notes, pages 1–295. Cambridge University Press, 1993.
[7] E. Guentner. Exactness of one relator groups. *Proc. of the Amer. Math. Soc.*, 130:1087–1093, 2002.
[8] E. Guentner, N. Higson, and S. Weinberger. The Novikov conjecture for linear groups. Preprint, 2003.
[9] E. Kirchberg and S. Wassermann. Permanence properties of C^*-exact groups. *Documenta Mathematica*, 4:513–558, 1999.
[10] D. Osin. Asymptotic dimension of relatively hyperbolic groups. Preprint GR/0411585, 2004.
[11] D. Osin. Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems. Preprint GR/0404040, 2004.
[12] N. Ozawa. Boundary amenability of relatively hyperbolic groups. Preprint GR/0501555, 2005.
[13] G. Skandalis, J. L. Tu, and G. Yu. Coarse Baum-Connes conjecture and groupoids. *Topology*, 41:807–834, 2002.
[14] G. Yu. The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Inventiones Math.*, 139:201–240, 2000.

Chern character for twisted K-theory of orbifolds

Jean-Louis Tu
(joint work with Ping Xu)

Let M be a manifold (compact for simplicity). The Chern character establishes an isomorphism $\mathrm{ch}\colon K^i(M) \otimes \mathbb{C} \to H^{i+2\mathbb{Z}}(M, \mathbb{C})$. Moreover, the map ch factors through the Connes-Karoubi non-commutative Chern character $K^i(M) \otimes \mathbb{C} \to HP_*(C^\infty(M))$ and the Hochschild-Kostant-Rosenberg map $a_0 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} a_0 da_1 \cdots da_k$. This result has been generalized in many directions by different authors, including Burghelea, Feigin, Tsygan, Nistor, Brylinsky, Baum, Connes, Block, Getzler and Crainic.

In this talk, we will focus on orbifolds. The definition of orbifold we will use is Moerdijk's one, namely, an orbifold is a Morita equivalence class of étale proper groupoids. Thus, a crossed-product of a manifold M by a discrete group G acting properly by diffeomorphisms is an orbifold groupoid; conversely, any orbifold is locally of the form $M \times G$.

A result, essentially due to Baum and Connes [1], says that for any orbifold groupoid Γ, the K-theory group $K_*(C^*(\Gamma)) \otimes \mathbb{C}$ is isomorphic to the de Rham cohomology with compact supports of the inertia groupoid $H^*_{dR,c}(\Lambda\Gamma, \mathbb{C})$. More precisely, let $S\Gamma = \{\gamma \in \Gamma |\ s(\gamma) = t(\gamma)\}$ be the space of closed loops of Γ. Then $S\Gamma$ is a manifold endowed with the action of Γ by conjugation, and the inertia groupoid $\Lambda\Gamma$, which is again an orbifold, is by definition the crossed-product $S\Gamma \times \Gamma$. The de Rham cohomology $H^*_{dR,c}(\Lambda\Gamma, \mathbb{C})$ is the cohomology of the complex $\Omega_c(S\Gamma, \mathbb{C})^\Gamma$ of invariant differential forms endowed with the de Rham differential.

Let us now come to twisted K-theory. Given a compact manifold M and a cohomology class $\alpha \in H^3(M,\mathbb{Z})$, one can associate a continuous field of C*-algebras with fiber $\mathcal{K}$ and satisfying Fell's condition; α is called the Dixmier-Douady invariant. Then, the twisted K-theory group $K^i_\alpha(M)$ is by definition the K-theory group of A_α. A recent theorem by Mathai and Stevenson [2] says that $K^i_\alpha(M)\otimes\mathbb{C}$ is isomorphic to the twisted cohomology group $H^{i+2\mathbb{Z}}_\alpha(M,\mathbb{C})$, defined as the cohomology of the complex $(\Omega^*(M)((u)), d+u\Omega\wedge\cdot)$, where $\Omega\in\Omega^3(M)$ is a 3-form whose cohomology class $[\Omega]\in H^3(M,\mathbb{R})$ is the image of α by the canonical map $H^3(M,\mathbb{Z})\to H^3(M,\mathbb{R})$.

Suppose now that $\mathcal{G}$ is an orbifold groupoid and $\alpha\in H^3(\mathcal{G},\mathbb{Z})$. Then α can be represented by a central extension

$$S^1\to\tilde{\Gamma}\to\Gamma.$$

Denote again by A_α the C*-algebra of the central extension $A_\alpha = C^*(\tilde{\Gamma})^{S^1}$. Let $L=\tilde{\Gamma}\times_{S^1}\mathbb{C}$ be the associated line bundle, and denote by $\pi\colon L\to\Gamma$ the projection and by $L_g=\pi^{-1}(g)$ the fibers. Then the groupoid law on $\tilde{\Gamma}$ induces an associative product

$$L_g\otimes L_h\to L_{gh}. \tag{1}$$

Moreover, there exists a connection ∇ for the line bundle $\pi\colon L\to\Gamma$ which satisfies Leibniz' rule with respect to the product (1). The connection ∇ is not unique, but its restriction ∇' to $S\Gamma\subset\Gamma$ is unique and is flat.

We are now ready to define twisted cohomology: consider $\Omega^*(S\Gamma,L)^\Gamma((u))$, the complex of formal Laurent series (u being a formal variable of degree -2), with coefficients in the invariant differential forms on $S\Gamma$ with values in L. This complex is endowed with the differential $\nabla'+u\Omega\wedge\cdot$, where $\Omega\in\Omega^3(\Gamma^{(0)})^\Gamma$ is the "curving" of ∇. Our main result is that $K_*(A_\alpha)\otimes\mathbb{C}$ is isomorphic to the cohomology of the above complex.

As a final remark, let us note that our methods do not directly apply to crossed-products of manifolds by Lie groups, because the de Rham complex for a groupoid is not endowed with a super-commutative cup-product, and we used the fact that $\Omega\wedge\Omega=0$ in a crucial way. However, it should be possible to use the Cartan model instead. Since we work on groupoids up to Morita equivalence, it seems necessary to generalize Cartan's model to a wider class of groupoids than those of the form $M\times G$, namely to pseudo-étale groupoids in the sense of X. Tang [3]

References

[1] P. Baum and A. Connes, *Chern character for discrete groups*, A fête of topology, 163–232, Academic Press, Boston, MA, 1988.
[2] V. Mathai and D. Stevenson, *On a generalized Connes-Hochschild-Kostant-Rosenberg theorem*, To appear in Advances in Mathematics.
[3] X. Tang, *Deformation Quantization of Pseudo Symplectic(Poisson) Groupoids*, ArXiv math.QA/0405378.

Almost connected groups, asymptotic dimension and an exactness question

Agata Hanna Przybyszewska
(joint work with Uffe Haagerup)

In my talk I have demonstrated that every almost connected locally compact, second countable group has finite asymptotic dimension.

Definition 1. Following [1, p. 10], a metric space is called *proper* if all closed bounded sets are compact. When G is a group, this reduces by the left invariance of the group metric to the requirement, that for every $M > 0$ all the closed balls $D(e,M) = \{h \in G : \ d(e,h) \leq M\}$ are compact.

A metric d on a group which is proper, left invariant, and generates the topology of the group is called a *plig*.

First, we show that a proper left invariant metric that generates the group topology is a coarse invariant for locally compact, second countable groups:

Theorem 2. *Let G be a locally compact, second countable group. Assume that the metrics d_1, d_2 on G are both pligs. Then the map*

$$Id\colon (G,d_1) \to (G,d_2)$$

is a coarse equivalence.

Then we exhibit how to construct a metric on a compactly generated locally compact second countable group

Theorem 3. *Let G be a compactly generated, locally compact, second countable group. There exists a plig d on G, such that $B_d(e,n) \subset B_d(e,1)^{2n+1}$.*

Such a metric can be used to construct a proper isometric action on the reflexive, strictly convex Banach space $\oplus_{n=1}^{\infty} L^{2n}(G,\mu)$.

We use theorem 3, to construct a plig on a given locally compact second countable group:

Theorem 4. *Let G be a locally compact, second countable group, then there exists a plig d on G, such that*

$$\mu(B_d(e,n)) \leq C \cdot \mu(B_d(e,1))^{2n+1},$$

where μ is denoting the Haar measure.

Having constructed a plig, we go on to prove a key lemma:

Lemma 5. *Let G be a group, and let d be a plig on G, and let H be a closed, normal subgroup.*

Define the quotient metric *d_q on the left coset space G/H by*

$$d_q(aH,bH) = \inf\{d(x,y) : x \in aH, y \in bH\} \tag{1}$$

Then the quotient metric d_q is a plig on G/H, and

$$\forall_{x \in G, yH \in G/H} \quad \exists_{h_1 \in H} \quad d(x, yh_1) = d_q(xH, yH). \tag{2}$$

Lemma 2, combined with a theorem of Laffourge and Higson, [2], gives us the following important observation:

Lemma 6. *Let G be a locally compact, second countable group, and let H be a normal subgroup of G. Assume that* $\operatorname{as.dim}(H) = d$ *and* $\operatorname{as.dim}(G/H) = d$. *Then*

$$\operatorname{as.dim}(G) \leq dk + d + k.$$

Every connected Lie group has a Levi decomposition:

Theorem 7. *Let G be a connected Lie group. Then G is decomposed as follows*

$$G = (Rad(G))L,$$

where the subgroup L is a connected, semisimple Lie group, and the radical is the maximal solvable Lie subgroup of G.

We use the Levi decomposition together with lemma 6 to show, that both the radical, and a connected semisimple Liegroup has finite asymptotic dimension, and we use the Levy Decomposition to prove, that every connect Lie group has finite asymptotic dimension. Finally, we can conlcude with the use of group structure theory, that:

Theorem 8. *Every almost connected, locally compact, second countable group has finite asymptotic dimension.*

We introduce the notion of O-exactness:

Definition 9. A locally compact group G is called *O-exact*, or we say that it has an *O-kernel* if:

$$\forall_{\epsilon>0} \forall_{R>0} \exists u : G \times G \to \mathbb{C}$$

such that:

- u is a continuous, positive definite kernel
- $\exists_{S>0} \quad u(x,y) \neq 0 \Rightarrow d(x,y) \leq S$
- $|1 - u(x,y)| < \epsilon$ when $d(x,y) \leq R$.

We conclude by showing that

Theorem 10. *Let G be a locally compact, second countable group. If G has finite asymptotic dimension, then G is O-exact.*

Having an O-kernel is in the case of a discrete, finitely generated group equivalent to exactness of the group, [3], and thus we have a new proof of the fact that a closed, discrete subgroups of an almost connected locally compact group is exact, which was originally shown using different methods in [4].

It is a question for future research what the relation between O-exactness and C^*-exactness is.

References

[1] J. Roe, *Index Theory, Coarse Geometry, and Topology of Manifolds*, American Mathematical Society **32** (1995).

[2] N. Higson, V. Lafforgue, *Espaces a geometrie grossiere de dimension finie.*, Preprint **32** (1999).

[3] C. Anantharaman-Delaroche, *Amenability and exactness for dynamical systems and their C^*-algebras*, Preprint **32** (1990), 120–140.

[4] E. Kirchberg, S. Wasserman, *Permanence Properties of C^*-Exact Groups*, Doc. Math. J. **32** (1999).

Strong rigidity for II_1 factors and equivalence relations

Sorin Popa

I first presented[1] a history of rigidity results in the theory of von Neumann algebras (II_1 factors) and orbit equivalence relations, explaining how these two subjects evolved in parallel, since the group-measure space construction of Murray-von Neumann in 1936.

Then I presented some recent rigidity results I have obtained by myself or with collaborators (2001-2005). A sample such result shows that any isomorphism from a factor coming from an action of an icc property (T) group onto a factor coming from a Bernoulli action of an icc group essentially comes from a conjugacy of the actions. In particular, the groups are necessarily isomorphic. Many other applications were given including a superrigidity result for orbit equivalence relations.

L^2-cohomology and derivations

Andreas Berthold Thom

This is a report about work in progress. In [1], Alain Connes and Dimitri Shlyakhtenko have defined a notion of L^2-Betti numbers for $*$-sub-algebras of finite von Neumann algebras. Let A be a $*$-sub-algebra of a finite von Neumann algebra, equipped with a fixed faithful trace τ. They define

$$\beta_k^{(2)}(A,\tau) = \dim_{M\overline{\otimes}M^o} \operatorname{Tor}_k^{A\otimes A^o}(A, M\overline{\otimes}M^o).$$

Here, dim denote the dimension function for arbitrary modules over a finite von Neumann algebra, which was introduced by Wolfgang Lück, see [2].

Let Γ be a discrete group. An easy computation, relying on work of Wolfgang Lück (see [2]), shows that $\beta_k^{(2)}(\mathbf{C}\Gamma,\tau)$ equals $\beta_k^{(2)}(\Gamma)$, the k-th L^2-Betti number of the group Γ.

Definition 1. The L^2-Betti number of a finite von Neumann algebra M, equipped with a fixed trace τ, is $\beta_k^{(2)}(M,\tau)$.

[1] This abstract, typed by the reporter, is based on the handwritten abstract by S. Popa in the "Book of Talks" of the MFO.

Our work aims at a computation of these L^2-Betti numbers in certain cases. In particular we are interested in the computation of the first L^2-Betti number. In [1], there was given some evidence for the possibility of the following conjecture, which we better formulate as a question. (The underlying conjecture being, that the answer is affirmative in a lot of cases.)

Question 2. Let Γ be a discrete group and let $L\Gamma$ be the associated finite von Neumann algebra with its canonical trace. Is it true that $\beta_k^{(2)}(L\Gamma, \tau) = \beta_k^{(2)}(\Gamma)$?

In [1], it is shown that the answer is affirmative if Γ is abelian. However, even for $k = 1$ in the case of amenable i.c.c. groups, it remained open. It is clear, that an affirmative answer for a wide range of finite von Neumann algebras would have important consequences. In particular, it would resolve the Non-isomorphism Conjecture for free group factors.

In our talk, we present the following structural results about the L^2-Betti numbers of von Neumann algebras.

Theorem 3. *Let M and N be finite von Neumann algebras, each equipped with a fixed trace. Let ${}_M L_N$ be a $M \otimes N^o$-module, which is finitely generated projective as M and as N^o-module. Then the following coupling formula holds for all natural numbers k:*

$$\frac{\beta_k^{(2)}(M)}{\dim_M {}_M L} = \frac{\beta_k^{(2)}(N)}{\dim_{N^o} L_N}.$$

The proof is an application of homological algebra. It has to be seen in anology of Roman Sauers proof of Gabouriaus coupling formula for L^2-Betti numbers for groups, see [3]. Note, that the assuptions of the theorem are met in the case of a sub-factor of finite index. The computations in this case are compatible with the classical Nielsen-Schreier formula and its extensions to the world of interpolated free group factors by Radulescu.

Theorem 4. *Let M be a finite von Neumann algebra. If M has diffuse centre, then, for all natural numbers k,*

$$\beta_k^{(2)}(M) = 0.$$

Again, the proof is in anology to the diffuse abelian case, extracting the necessary homological conditions.

Our second group of results is related to a new approach to L^2-cohomology. First of all, several ring theoretic properties of the ring of operators affiliated with a finite von Neumann algebra are observed. These results are well-known to the experts.

Theorem 5. *Let M be a finite von Neumann algebra. Denote by $U(M)$ its ring of affiliated operators.*

- *$U(M)$ is von Neumann regular, i.e. all modules are flat,*
- *$U(M)$ is self-injective,*
- *$U(M)$ is flat as a M module, and*

- *$U(M)$ is a complete separable metrizable complex algebra in the topology of convergence in measure.*

The results above imply that certain Tor and Ext-terms are dual to each other, so that the ring-theoretic dual of L^2-homology with coefficients on the ring of affiliated operators of $M\overline{\otimes}M^o$ is identified with L^2-cohomology with coefficients in the ring of affiliated operators of $M\overline{\otimes}M^o$. Furthermore, it is shown that duality preserves the dimension of a module. (Note that this is far from being true in the case of modules over M itself.)

These results together, using a standard description of the first Hochschild cohomology using derivations, imply the following theorem concerning the first L^2-Betti number.

Theorem 6. *Let A be a $*$-sub-algebra of a finite von Neumann algebra (M, τ).*

$$\dim_{U(M\overline{\otimes}M^o)} Der(A, U(M\overline{\otimes}M^o)) = \beta_1^{(2)}(A,\tau) - \beta_0^{(2)}(A,\tau) + 1.$$

Here, $Der(A, K)$ denotes the space of derivations of A with values in the bimodule K. Note that $U(M\overline{\otimes}M^o)$ is a $M \otimes M^o$-module via left multiplication. The dimension is taken with respect to the second commuting $U(M\overline{\otimes}M^o)$-module structure, given by right multiplication.

The theorem above has several implications. In particular, the following conclusions can be drawn. Let A be a $*$-sub-algebra of (M, τ). Denote by M^A the smallest subalgebra of M, which is closed under taking weak closures of abelian $*$-sub-algebras.

Theorem 7. *Let A be a $*$-sub-algebra of M. Then,*

$$\beta_1^{(2)}(M^A, \tau) \leq \beta_1^{(2)}(A, \tau).$$

The proof relies on an interpretation of the vanishing of the first L^2-Betti number in the case of abelian von Neumann algebras, proved in [1]. Using the duality above, it leads to the observation that all derivation as above restricted to abelian von Neumann algebras are inner. In particular, they are continuous in the various topologies; i.e. extensions are unique.

We end this short summary by some obvious questions related to the results above.

Question 8. Let A be a weakly dense $*$-sub-algebra of (M, τ). Under what circumstances is it true that $M^A = M$?

Question 9. Let $\Delta : M \to U(M\overline{\otimes}M^o)$ be a derivation. Is it automatically continuous from the topology of bounded convergence in measure to the topology of convergence in measure? (Same question for other reasonable topology on both sides.)

References

[1] A.Connes and D.Shlyakhtenko, *L^2-homology for von Neumann algebras*, math.OA/0309343 (2003).

[2] W.Lück, *L^2-invariants: Theory and Applications to Geometry and K-theory*, Springer Verlag, Berlin (2002).
[3] R.Sauer, *L^2-invariants of discrete measured groupoids*, thesis

On Hilbert 17th Problem and aspects of Connes Embedding Conjecture

Florin Rădulescu

The Hilbert 17th Problem has been solved by Artin. It asserts that every positive polynomial is a sum of squares (of fractions). Moreover there exist positive polynomials that are not a sum of squares of polynomials (Motzkin, Schmudgen). Recently Helton proved a non-commutative version of Artin's theorem in which he shows that a non-commutative polynomial which is positive definite when evaluated on matrices is a sum of squares of polynomials.

We prove a non-commutative version of the Hilbert's 17th problem, giving a characterization of the class of non-commutative polynomials in n-undeterminates that have positive trace when evaluated in n-selfadjoint elements in an arbitrary II_1 von Neumann algebra. These polynomials are limits of a sum of squares modulo terms of zero trace.

As a corollary it follows that the Connes's embedding conjecture is equivalent to a statement that can be formulated entirely in the context of finite matrices.

More precisely the Connes's embedding conjecture is equivalent to proving that every order 4, non-commutative polynomial that has positive trace when evaluated in selfadjoint matrices of arbitrary size is a sum of squares. One can further reduce this to polynomials of a very specific shape

$$p(x_1, ...x_n) = \sum x_i^4 + \sum a_{ij} x_i x_j + \sum b_i x_i + c.$$

References

[1] A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$*, Ann. of Math. (2) **104** (1976), no. 1, 73–115.
[2] J. W. Helton, *"Positive" noncommutative polynomials are sums of squares*, Ann. of Math. **56** (2002), no. 2, 675–694.
[3] E. Kirchberg, *Discrete groups with Kazhdan's property* T *and factorization property are residually finite*, Math. Ann. **299** (1994), no. 3, 551–563.
[4] F. Rădulescu, *Convex sets associated with von Neumann algebras and Connes' approximate embedding problem*, Math. Res. Lett. **6** (1999), no. 2, 229–236.
[5] F. Rădulescu, *A non-commutative, analytic version of Hilbert's 17-th problem in type II_1 von Neumann algebras*, arXiv:math.OA/0404458.

Classification of non-simple purely infinte C*-algebras

Eberhard Kirchberg

(partially joint work with E. Blanchard, H. Harnisch and M. Rørdam)

We reported[1] on four results related to the classification of non-simple purely infinite C*-algebras:

(1) Permanence properties of spi-algebras, in particular on the new result that $A \otimes^{\min} B$ *is spi if* A *or* B *is exact and the other is spi* (and related results).

(2) The relations between
- operator-convex (sub-)cones $\mathcal{C} \subset \mathrm{CP}(A, B)$ (= cone of completely positive maps from A into B),
- lower semi-continuous lattice maps

$$\psi \colon \mathcal{I}(B \otimes^{\max} C^*(F_2)) \to \mathcal{I}(A \otimes^{\max} C^*(F_2)),$$

  and
- some (special type of) Hilbert A-B-modules.

This can be used to construct e.g. a Hilbert $\mathcal{C}_0(P, \mathcal{K})$-$\mathcal{C}_0(P, \mathcal{K})$-module $\mathcal{H}$ that is defined by a sup-inf closed sublattice $\mathcal{O}(X)$ of $\mathcal{O}(P) \cong \mathcal{I}(\mathcal{C}_0(P))$ (here: $\mathcal{I}(\mathcal{C}_0(P))$ is the lattice of open ideals of $\mathcal{C}_0(P)$, P is a locally compact metric space that is second countable, i.e. P is a locally compact Polish space; $\mathcal{O}(P)$ means the lattice of open subsets of P. A "point complete" (= "sober" = "spectral") T_0-space X appears here naturally as a completion of the quotient T_0-space of P defined by the topology on P given by the sublattice of $\mathcal{O}(P)$.)

(3) In recent work (with H. Harnisch) it is shown that for this bimodule $\mathcal{H}$ the Toeplitz-algebra $\mathcal{T}(\mathcal{H})$ and the Cuntz-Pimsner-algebra $\mathcal{O}(\mathcal{H})$ are the same, and $\mathcal{O}(\mathcal{H})$ is a strongly purely infinite crossed product $E \rtimes \mathbb{Z}$ with E= an inductive limit of type I C*-algebras, that has $\mathcal{O}(X)$ as lattice of open subsets of $\mathrm{Prim}(\mathcal{O}(\mathcal{H}))$, i.e. $X \cong \mathrm{Prim}(\mathcal{O}(\mathcal{H}))$.

    In conjunction with joint work with M. Rørdam this result gives a characterization of the primitive ideal spaces X of separable nuclear C*-algebras A: $X \cong \mathrm{Prim}(A)$ *for some separable nuclear* A *if and only if* $\mathcal{O}(X)$ *is a sup-inf-closed sublattice of* $\mathcal{O}(P)$ *for some locally compact Polish space* P.

(4) A closer look on the results of Pimsner on the KK-theory of Toeplitz-algebras $\mathcal{T}(\mathcal{H})$ shows that the inclusion map $\mathcal{C}_0(P) \hookrightarrow \mathcal{T}(\mathcal{H}) = \mathcal{O}(\mathcal{H})$ defines an element of $\mathrm{KK}(X; \cdot, \cdot)$-equivalence. Interestingly there are many possible choices for P in general, that have even different KK-theory (in the ordinary sense). Thus the $(\mathcal{C}_0(P), \mathcal{O}(X) \subset \mathcal{O}(P))$ are candidates for a generalization of the UCT-class for the $\mathrm{KK}(X; \cdot, \cdot)$-theory.

[1]This abstract, typed by the reporter, is based on the handwritten abstract by E. Kirchberg in the "Book of Talks" of the MFO.

The Kasparov groups $\mathrm{KK}(X;A,B)$ can be defined more generally as $\mathrm{KK}(\mathcal{C};A,B)$ where $\mathcal{C}$ is a suitable operator convex cone (given by the duality to obtain maps from $\mathcal{I}(A)$ to $\mathcal{I}(B)$ as outlined in part (2)). One simply takes the Grothendieck group of the semigroup of those Kasparov modules $(\mathcal{H}_B,\ d\colon A\to\mathcal{L}(\mathcal{H}_B),\ F)$ such that the c.p.-maps $a\in A\mapsto\langle d(a)x,x\rangle\in B$ are in $\mathcal{C}$ for all $x\in\mathcal{H}_B$.

Estimates on free entropy dimension

Dimitri Shlyakhtenko

Based in part on joint work with A. Connes and on joint work with I. Mineyev.

General setting: $X_1,\dots,X_n\in M$ self-adjoint generators, $\tau: M\to\mathbb{C}$ trace.

$$\delta^*(X_1,\dots,X_n)=n-\limsup_{t\to 0}\frac{\chi^*(X_1^t,\dots,X_n^t)}{\log t^{1/2}}\in[0,n],$$

where $X_j^t=X_j+\sqrt{t}S_j$, $(S_1,\dots,S_n)$ free semicircular family, free from M. This quantity is called Voiculescu's *non-microstates free entropy dimension* [Voi94, Voi98].

The big question: is δ^* an invariant of M?

This is true if $n=1$. If μ is the spectral measure of $X=X_1$,

$$\delta^*(X)=1-\sum_{t\in\mathbb{R}}\mu(\{t\})^2.$$

Notation 1. $HS=HS(L^2(M))=L^2(M)\bar{\otimes}L^2(M^o)$, $X_j^t=X_j+\sqrt{t}S_j$.

Fix $T=(T_1,\dots,T_n)\in HS^n$ and define a derivation $\partial_T^t: Alg(X_1^t,\dots,X_n^t)\to L^2(X_1,\dots,X_n,S_1,\dots,S_n)^{\otimes 2}$ by

$$\partial_T^t(X_j^t)=T_j\#S_j.$$
$$(a\otimes b)\#S=aSb.$$

Then ∂_T^t is closable and

$$J_T^k(t)=(\partial_T^t)^*(S_k)\qquad\text{exists}\in L^2(X_1^t,\dots,X_n^t).$$

The main estimate. Consider the spaces:

$$\begin{aligned}
H^0 &= cl\{(T_1,\dots,T_n)\in HS:\ \exists D\in B(L^2(M)),\ [D,X_j]=T_j\}\\
&= cl\{(T_1,\dots,T_n)\in HS:\ \exists D \text{ ess. s.a. } 1\in\mathrm{dom}D,\ [D,X_j]=T_j\}\\
H^1 &= cl\{T=(T_1,\dots,T_n)\in HS:\ L^2\text{-}\lim_{t\to 0}(J_T^k(t)) \text{ exists}\}\\
&= cl\{(T_1,\dots,T_n)\in HS:\ \exists D \text{ closable } 1\in\mathrm{dom}D,\ [D,X_j]=T_j\}\\
H^2 &= cl\{T=(T_1,\dots,T_n)\in HS:\ \lim_{t\to 0}t\sum_k\|J_T^k(t)\|_2^2=0\}\\
H^3 &= \{(Q_1,\dots,Q_n)\in FR:\ \sum[Q_j,X_j]=0\}^{\perp}
\end{aligned}$$

Theorem 2. [Shl04, CS, Shl05, MS05] *(a)* H^0, H^1, H^2, H^3 *are modules over* $M_2 = M\bar{\otimes}M^o$ *acting on* HS *by* $(m \otimes n) \cdot T = Jm^*J\ T\ Jn^*J$.
(b) The following inclusions hold:

$$H^0 \subset H^1 \subset H^2 \subset H^3$$

(c) One has

$$\begin{aligned}\dim_{M_2} H^0 &\leq \dim_{M_2} H^1 \leq \dim_{M_2} H^2 \\ &\leq \delta^*(X_1, \dots, X)_n) \leq \dim_{M_2} H^3.\end{aligned}$$

Idea of proof.
H^0 vs H^1 and H^2: there is a formula for $J_T^k(t)$ if such D exists.
H^2 vs H^3: somewhat technical, but elaboration of H^0 vs H^3:

$$\begin{aligned}(Q_1, \dots, Q_n) &\in FR : \sum [Q_j, X_j] = 0, \quad T_j = [D, X_j] \in HS \\ \sum Tr(Q_j T_j) &= \sum Tr(Q_j [D, X_j]) \\ &= \sum Tr([X_j, Q_j]D) = 0.\end{aligned}$$

Application: lower estimates for δ^*. (1) Take $D \in HS$. Then the range of

$$HS \ni D \overset{\phi}{\mapsto} ([D, X_1], \dots, [D, X_n])$$

is contained in H^0. $\implies$ $\dim_{M_2}(H^0) \geq 1 - \dim_{M_2} \ker \phi$

(a) $n = 1$, in which case $\ker \phi$ measures atoms of the law of X (compare $\delta^* = 1 - \sum \mu(\{t\})^2$)

(b) M is diffuse, $\ker \phi = 0$ ($\implies$ $\delta^* \geq 1$).

(2) Semicirular systems $S_j = \ell_j + \ell_j^*$. Can take $D_k = r_k$, $[D_k, S_j] = -\delta_{kj} P_1$ ($\implies$ $\delta^* = n$)

(3) q-Semicircular systems $S_j = \ell_j^q + \ell_j^{q*}$. Can take $D_k = r_k$; $[D_k, S_j] \in HS$ for small q. ($\implies$ $\delta^* > 1$)

(3) Free groups: $F =$ projection onto words that end with a fixed letter. Then $[F, \lambda(\gamma)] \in FR$. ($\implies$ $\delta^* = n$).

Aside on solid von Neumann algebras. Sometimes the same D is useful in verifying Ozawa's condition AO ($\implies$ M is solid: $N' \cap M$ hyperfinite for all $N \subset M$ diffuse).

Example. Free group factors; q-semicircular systems for q small.

Let

$$A = C^*(S_1, \dots, S_n), \quad B = C^*(r_j + r_j^*) \subset C = C^*(r_j).$$

Then $A = JBJ$, $M = W^*(A)$, $[A, C] \subset K$ and C is nuclear. Hence the map

$$a \otimes b \mapsto ab \mod K$$

extends to $A \otimes (C/K)$ and hence to $A \otimes_{\min} B$.

This has type III consequences as well, e.g.:

Theorem 3. *The following hold:*
** free Araki-Woods factors M with almost-periodic weights are prime*
** and "solid":*

$$N \subset M, \qquad E : M \to N \text{ cond. expectation}$$
$$\implies N' \cap M \text{ hyperfinite.)}$$

** There is an example of a non-hyperfinite type III_1 factor M, so that M^ϕ is hyperfinite for all ϕ.*

Upper and lower estimates for groups. $X_1, \ldots, X_n \in \mathbb{C}\Gamma$ generators of group ring.

Theorem 4. [MS05] *(a)* $\dim_{M_2}(H^0) \geq \beta^1_{(2)}(\Gamma) - \beta^0_{(2)}(\Gamma) + 1$ *(ℓ^2 cohomology)*
(b) $\dim_{M_2} H^3 \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$ *(ℓ^2 homology).*
(c) $\dim_{M_2} H^0 = \dim_{M_2} H^1 = \cdots = \dim_{M_2} H^3 = \delta^* = \beta_1 - \beta_0 + 1$.

Thus the estimates for δ^* are in this case optimal.

The estimate $\dim_{M_2}(H^0) \geq \beta^1_{(2)}(\Gamma) - \beta^0_{(2)}(\Gamma) + 1$. Given an ℓ^2-cocycle $c : \Gamma \to \ell^2$, write $c = df$, for f an unbounded function on Γ. Then $c(\gamma) = f - \lambda_\gamma f \in \ell^2$, for all $\gamma \in \Gamma$.

Let $D = m_f$. Then

$$[\lambda(\gamma), m_f] = \lambda(\gamma)c(\gamma) \in HS$$

and hence

$$(c(\gamma_1), \ldots, c(\gamma_n)) \mapsto (\lambda(\gamma_1)c(\gamma_1), \ldots, \lambda(\gamma_n)c(\gamma_n)) \in H^1$$

$\implies$ get a lower estimate on H^1.

The estimate $\dim_{M_2} H^3 \leq \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$. If c is an exact ℓ^2-cycle, it can be approximated by df with f of compact support; hence one can get $f_1, \ldots, f_n \in C_c(\Gamma)$ so that

$$\sum \lambda_{\gamma_j} f_j - f_j = 0.$$

From this one can manufacture $(Q_1, \ldots, Q_n) \in FR^n$ so that $\sum [Q_j, X_j] = 0 \implies (Q_1, \ldots, Q_n) \in H^{3\perp} \implies$ upper estimate.

Some more applications. M finite dimensional $\implies$ δ^* is independent of generators and equals $\beta_1^{(2)}(M) - \beta_0^{(2)}(M) + 1$.

Semicontinuity question (Voiculescu). $X_p^j \to X_p$ strongly with bounded norm. Is it true that

$$\liminf \delta(X_1^j, \ldots, X_n^j) \geq \delta(X_1, \ldots, X_n)?$$

The answer is NO: Counterexample comes from considering semicontinuity of H^j (see [Shl05]).

References

[CS] A. Connes and D. Shlyakhtenko, *L^2-homology for von Neumann algebras*, Preprint math.OA/0309343, to appear in J. Reine Angew. Math.

[MS05] I. Mineyev and D. Shlyakhtenko, *Non-microstates free entropy dimension for groups*, Geom. Func. Anal. **15** (2005), 476–490.

[Shl04] D. Shlyakhtenko, *Some estimates for non-microstates free entropy dimension, with applications to q-semicircular families*, IMRN **51** (2004), 2757–2772.

[Shl05] ———, *Remarks on free entropy dimension*, Preprint, math.OA/0504062, 2005.

[Voi94] D.-V. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory II*, Invent. Math. **118** (1994), 411–440.

[Voi98] ———, *The analogues of entropy and of Fisher's information measure in free probability, V*, Invent. Math. **132** (1998), 189–227.

Spectral Subspaces of Operators in a II_1-factor

Hanne Schultz
(joint work with Uffe Haagerup)

It is shown that to every operator T in a general von Neumann factor $\mathcal{M}$ of type II_1 and to every Borel set B in the complex plane $\mathbb{C}$, one can associate a maximal, closed, T-invariant subspace, $\mathcal{K} = \mathcal{K}_T(B)$, affiliated with $\mathcal{M}$, such that the Brown measure of $T|_{\mathcal{K}}$ is concentrated on B. Moreover, $\mathcal{K}$ is T-hyperinvariant, and the Brown measure of $P_{\mathcal{K}^\perp}T|_{\mathcal{K}^\perp}$ is concentrated on $\mathbb{C} \setminus B$. In particular, if $T \in \mathcal{M}$ has a Brown measure which is not concentrated on a singleton, then there exists a non-trivial, closed, T-hyperinvariant subspace.

References

[1] U. Haagerup, H. Schultz, *Invariant Subspaces of Operators in a General II_1-factor*, preprint 2005. Preliminary version available at http://www.imada.sdu.dk/~schultz/publications.html.

Bounded generation and amenability of C*-algebras

Gilles Pisier

We prove that a discrete group G is amenable iff it is strongly unitarizable in the following sense: every unitarizable representation π on G can be unitarized by an invertible chosen in the von Neumann algebra generated by the range of π. Analogously a C*-algebra A is nuclear iff any bounded homomorphism $u : A \to B(H)$ is strongly similar to a $*$-homomorphism in the sense that there is an invertible operator ξ in the von Neumann algebra generated by the range of u such that $a \to \xi u(a)\xi^{-1}$ is a $*$-homomorphism. An analogous characterization holds in terms of derivations. We apply this to answer several questions left open in our previous work concerning the length $\ell(A, B)$ of the maximal tensor product $A \otimes_{\max} B$ of two unital C*-algebras, when we consider its generation by the subalgebras $A \otimes 1$ and $1 \otimes B$. We show that if $\ell(A, B) < \infty$ either for $B = B(\ell_2)$ or when B is the C*-algebra (either full or reduced) of a non Abelian free group, then A must be nuclear. We also show that $\ell(A,B) \leq d$ iff the canonical quotient map from the unital free product $A * B$ onto $A \otimes_{\max} B$ remains a complete quotient map when restricted to the closed span of the words of length $\leq d$.

In 1950, J. Dixmier and M. Day proved that any amenable group G is unitarizable, i.e. any uniformly bounded representation $\pi\colon\ G \to B(H)$ is similar to a unitary representation. More precisely there is an invertible operator $\xi\colon\ H \to H$ such that $\xi\pi(\cdot)\xi^{-1}$ is a unitary representation of G. The proof uses a simple averaging argument from which it can be seen that ξ can be chosen with the additional property that ξ commutes with any unitary U commuting with the range of π. Equivalently, ξ can be chosen in the von Neumann algebra generated by $\pi(G)$. (See [5] for more on this). For convenience, let us say that π (resp. G) is strongly unitarizable if it has this additional property (resp. if every uniformly bounded π on G is strongly unitarizable).

It is still an open problem whether "unitarizable" implies "amenable" (see [8]). However, we will show that G is amenable iff it is strongly unitarizable. Moreover, we will show an analogous result for C*-algebras, as follows.

Theorem 1. *The following properties of a C*-algebra A are equivalent.*

- (i) *A is nuclear.*
- (ii) *For any c.b. homomorphism $u\colon\ A \to B(H)$ there is an invertible operator ξ on H belonging to the von Neumann algebra generated by $u(A)$ such that $a \to \xi u(a)\xi^{-1}$ is a $*$-homomorphism.*
- (iii) *For any C*-algebra B, the pair (A,B) has the following simultaneous similarity property: for any pair $u\colon\ A \to B(H)$, $v\colon\ B \to B(H)$ of c.b. homomorphisms with commuting ranges there is an invertible ξ on H such that both $\xi u(\cdot)\xi^{-1}$ and $\xi v(\cdot)\xi^{-1}$ are $*$-homomorphisms.*
- (iv) *Same as (iii) but with v assumed to be itself a $*$-homomorphism.*
- (v) *For any embedding $A \subset B(H)$, there is a constant C such that any inner derivation $\delta\colon\ A \to B(H)$ can be written as $\delta(a) = aT - Ta$ for an operator T in the von Neumann algebra generated by A and $\delta(A)$ with $\|T\| \leq C\|\delta\|_{cb}$.*

Remark 2. It is possible that (iii) or (iv) for a fixed given B implies that $A\otimes_{\min}B = A \otimes_{\max} B$ but this is not clear (at the time of this writing).

Corollary 3. *If a discrete group G is strongly unitarizable then G is amenable.*

Proof. Let $A = C^*(G)$. Any bounded homomorphism $u\colon\ A \to B(H)$ restricts to a uniformly bounded representation π on G. Note that $\pi(G)$ and $u(A)$ generate the same von Neumann algebra M. Thus if G is strongly amenable, A satisfies (ii) in Theorem 1, hence is nuclear and, as is well known, this implies G amenable in the discrete case (see [3]). □

Actually, we obtain a stronger statement:

Corollary 4. *If every* unitarizable *representation π on a discrete group G is strongly unitarizable then G is amenable.*

Proof. Indeed, in Theorem 1, u is assumed c.b. on $A = C^*(G)$, so the corresponding π is unitarizable. □

Remark 5. Assume G amenable with invariant mean ϕ. Consider a uniformly bounded representation π on G. Then, in the proof of the Day-Dixmier theorem, the invertible ξ that unitarizes π can be described (non rigorously) by the weakly convergent integral

$$\xi = (\int \pi(g)^*\pi(g)\phi(dg))^{1/2}.$$

This formula makes it clear that ξ is in the von Neumann algebra generated by the range of π.

Remark 6. Note that, by [1], $C^*(G)$ is nuclear for any separable, connected locally compact group G, hence every continuous unitarizable representation on G is strongly unitarizable; therefore we definitely must restrict the preceding Corollary 4 to the discrete case.

References

[1] A. Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. Ann. of Math. 104 (1976) 73–115.

[2] U. Haagerup. All nuclear C^*-algebras are amenable. Invent. Math. 74 (1983) 305–319.

[3] C. Lance. On nuclear C^*-algebras. J. Functional Analysis 12 (1973) 157–176.

[4] C. Le Merdy A strong similarity property of nuclear C^*-algebras. Rocky Mountain J. Math. 30 (2000), no. 1, 279–292.

[5] M. Nagisa and S. Wada. Simultaneous unitarizability and similarity problem. Sci. Math. 2 (1999) 255-261 (electronic).

[6] G. Pisier. Joint similarity problems and the generation of operator algebras with bounded length. Integral Equations Operator Theory 31 (1998), no. 3, 353–370.

[7] G. Pisier. A similarity degree characterization of nuclear C^*-algebras. Publl. RIMS Kyoto. To appear.

[8] G. Pisier. Are unitarizable groups amenable ? Proceedings (Gaeta). To appear.

Some examples of masas in II_1–factors

Ken Dykema

We study maximal abelian ∗–subalgebras of II_1–factors, which are known as masas. In particular, we consider masas in the hyperfinite II_1–factor R and in the free group factor $L(\mathbf{F}_2)$. Following Dixmier [1], a masa $A \subset M$ is said to be Cartan if its normalizer

$$N(A) = \{u \in \mathcal{U}(M) \mid u^*Au = A\}$$

generates M and is singular if $N(A) \subset A$. Our interest in masas is partially inspired by Sorin Popa's conjecture that of $A \subset L(\mathbf{F}_2)$ is a masa that is maximal hyperfinite, then A is freely complemented in $L(\mathbf{F}_2)$.

The Pukánszky invariant $\mathrm{Puk}(A)$ or $\mathrm{Puk}_M(A)$ of a masa $A \subset M$ was introduced in [9]. If J denotes the canonical involution on $L^2(M)$, then the abelian von Neumann algebra $\mathcal{A}$ generated by A and JAJ has a type I commutant $\mathcal{A}'$. The projection e_A onto $L^2(A)$ lies in $\mathcal{A}$ and $\mathcal{A}'(1-e_A)$ decomposes into a direct sum of type I_{n_i} algebras, where $1 \le n_i \le \infty$. Those n_i's appearing in this direct sum form $\mathrm{Puk}(A)$, a subset of $\mathbb{N} \cup \{\infty\}$. This quantity is invariant under the action of any automorphism of M, and so serves as an aid to distinguishing pairs of masas. In [9, 7, 11], various values of the invariant were found for masas, primarily in the hyperfinite type II_1 factor R. In particular, Neshveyev and Størmer [7] showed that for every subet S of $\mathbb{N} \cup \{\infty\}$ with $1 \in S$, there is a masa in the hyperfinite II_1–factor having Pukánsky invariant S; in [11], Sinclair and Smith showed that for certain abelian subgroups H of discrete i.c.c. groups G, the inclusion $L(H) \subset L(G)$ is a masa whose Pukánsky invariant can be computed in terms of the double coset structure of H in G. They give examples of such subgroups yielding for any $a, b, c \in \mathbb{N}$ a masa in the hyperfinite II_1–factor having Pukánsky invariant $\{a, b, abc\}$.

We now consider some particular examples.

Example 1. ([11]). In the mutliplicative group $\mathbb{Q}^*$ of nonzero rational numbers, consider the infinite index subgroup

$$P_\infty = \{\frac{p}{q} \mid p, q \in \mathbb{Z}^*,\, p, q \text{ odd}\}$$

and, for $n \in \mathbb{N}$, let

$$P_n = \{\frac{p}{q} 2^{kn} \mid p, q \in \mathbb{Z}^*,\, p, q \text{ odd},\, k \in \mathbb{Z}\},$$

which has index n in $\mathbb{Q}^*$. For $n \in \mathbb{N} \cup \{\infty\}$, consider the matrix group

$$G_n = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \Big| f \in P_n,\, x \in \mathbb{Q} \right\}$$

and let $H_n \subset G_n$ be the set of diagonal matrices in G_n. Then $L(H_n) \subset L(G_n)$ is a singular masa in the hyperfinite II_1–factor R whose Pukánsky invariant is $\{n\}$.

Using Example 1 and the Theorem 2.1 of [11], we immediatly see the following.

Example 2. Let $n \in \mathbb{N}$ and let $A_n = L(H_n \times H_\infty) \subset L(G_n \times G_\infty)$. Then A_n is a singular masa in the hyperfinite II_1–factor R whose Pukánsky invariant is $\{n, \infty\}$.

In [5] we observe the following examples of masas, using techniques from [11].

Example 3. Let $S \subseteq \mathbb{N} \cup \{\infty\}$ be such that $\infty \in S$. Then there is a discrete, i.c.c. amenable group G_S and abelian subgroup H_S such that $A_S = L(H_S) \subset L(G_S) = R$ is a singular masa in the hyperfinite II_1–factor R whose Pukánsky invariant is S. To illustrate, if $S = \{n_1, n_2, \infty\}$ with n_1 and n_2 distinct, then we take

$$G_S = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & f_1 2^{n_1 k} & 0 \\ 0 & 0 & f_2 2^{n_2 k} \end{pmatrix} \Bigg| f_1, f_2 \in P_\infty,\, x_1, x_2 \in \mathbb{Q},\, k \in \mathbb{Z} \right\}$$

and H_S to be the set of diagonal matrices belonging to G_S. The general case is done analogously.

We now turn to some examples of masas in the free group factor $L(\mathbf{F}_2)$ and their Pukánsky invariants. To begin with, we have the classical cases of a freely complemented masa which has Pukánsky invariant $\{\infty\}$ and the radial masa, which was shown by Rădulescu [10] also to have Pukánsky invariant $\{\infty\}$.

By [4], (generalizing work of Ge [6]), $L(\mathbf{F}_2)$ has no masas of finite multiplicity. From this, Sinclair and Smith [11] deduced the corollary that the Pukánsky invariant of any masa in $L(\mathbf{F}_2)$ must either contain ∞ or be an infinite set. It is an open question whether all masas in $L(\mathbf{F}_2)$ must have ∞ in their Pukánsky invariants.

The next two results, which are from [5], show that all sets containing ∞ do arise as the Pukánsky invariants of masas in $L(\mathbf{F}_2)$.

Example 4. With $A_S \subset R$ the masa from Example 3, take $A_S \subset R * L(\mathbb{Z})$, where the free product is taken with respect to the canonical traces. Then $R * L(\mathbb{Z}) \cong L(\mathbf{F}_2)$, (by [2]). Moreover, A_S is a singular masa of $L(\mathbf{F}_S)$ having Pukansky invariant equal to S.

Example 5. Let S_0 be a nonempty subset of $\mathbb{N}$ and let

$$A = \bigoplus_{n\in S_0} A_n \subset \bigoplus_{n\in S_0} R \subset \Big(\bigoplus_{n\in S_0} R\Big) * L(\mathbb{Z}) \cong L(\mathbf{F}_2),$$

where we take the free product with respect to a normal faithful tracial state on $\bigoplus R$ and the canonical trace on $L(\mathbb{Z})$, and where the isomorphism $\cong L(\mathbf{F}_2)$ follows from [3]. Then A is a singular masa of $L(\mathbf{F}_2)$, whose Pukánsky invariant is equal to $S_0 \cup \{\infty\}$.

The masas in Examples 4 and 5 have the same Pukánsky invariant, but they can be distinguished using a well known invariant which we now describe.

Notation 6. Let $B = C(X)$ be a separable, unital, abelian C*–algebra and let $\pi : B \to \mathcal{B}(\mathcal{H})$ be a unital $*$–representation. Then we can write $\mathcal{H}$ as a direct integral

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x \, d\mu(x)$$

for a Borel measure μ on X such that $\forall f \in C(X)$, $\pi(f)$ is decomposable and

$$\pi(f)_x = f(x)\mathrm{id}_{\mathcal{H}_x}.$$

The measure μ is not unique, but its class $[\mu]$ is unique. We call $[\mu]$ the *measure class* of π, and we call $m(x) = \dim \mathcal{H}_x$ the *multiplicity function* of π, which is unique up to redifinition on sets of μ–measure zero.

Let M be a II_1 factor with separable predual and with tracial state τ. Let $A \subset M$ be a masa. The invariant for $A \subset M$ that we use is defined as follows. Let $1 \in C(Y) \subset A$ be a separable unital C*–subalgebra that is weakly dense in A. Let ν be the Borel probability measure on Y such that

$$\tau(f) = \int_Y f \, d\nu \qquad (f \in C(Y)).$$

Let $\pi : C(Y) \otimes C(Y) \to \mathcal{B}(L^2(M))$ be the "left–right representation" of the C*–algebra tensor product given by $\pi(a \otimes b) = aJbJ$. We identity $C(Y) \otimes C(Y)$ with $C(Y \times Y)$ and we let $[\eta]$ be the measure class and m the multiplicity function of π; thus η is a measure on $Y \times Y$. Taking η to be a finite measure without loss of generality, the two coordinate projections of η send the class $[\eta]$ to the class $[\nu]$. The invariant for $A \subset M$ that we consider is the equivalence class of $(Y, [\eta], m)$, where we define equivalence by

$$(Y, [\eta], m) \sim (Y', [\eta'], m')$$

if and only if there is an a.e.–defined, a.e.–bijective, measure–preserving tranformation $F : Y \to Y'$ such that $[(F \times F)_*\eta] = [\eta']$ and $m' \circ (F \times F) = m$ a.e.

This invariant has been considered by several authors. Neshveyev and Størmer [7] observed that it is a complete invariant for the pair (A, J) acting on $L^2(M)$.

We use this invariant to distinguish the masas in Examples 4 and 5. In order to compute this invariant, we apply the following proposition.

As above, let $A \subset M$ be a masa in a II_1–factor with separable predual, let $1 \in C(Y) \subset A$ be a weakly dense, separable C*–subalgebra, and let ν denote the measure on Y given by the trace on M. Let Q be any von Neumann algebra not equal to $\mathbf{C}$ with a specified normal faithful tracial state and let $N = M * Q$ be the free product taken with respect to the trace on M and this specified trace on Q. We regard A as a embedded in N. By [8], A is known to be a masa in N. Let π_1 and, respectively, π, denote the left–right representation of $C(Y) \otimes C(Y)$ on $L^2(M)$ and, respectively, $L^2(N)$.

Proposition 7. ([5]). *The representation π is unitarily conjugate to the representation*

$$\pi_1 \oplus (\lambda \otimes \mathrm{id}_{\ell^2(\mathbb{N})} \otimes \rho)$$

on $L^2(M) \oplus (L^2(M) \otimes \ell^2(\mathbb{N}) \otimes L^2(M))$, where here λ denotes the restriction to $C(Y)$ of the usual left action of M on $L^2(M)$ and ρ denotes the restriction to $C(Y)$ of the right representation $a \mapsto J\lambda(a)J$ of A on $L^2(M)$. Moreover, the measure class of $\lambda \otimes \mathrm{id}_{\ell^2(\mathbb{N})} \otimes \rho$ is $[\nu \otimes \nu]$ and the multiplicity function is constant ∞.

References

[1] J. Dixmier, Sous–anneaux abéliens maximaux dans les facteurs de type fini, *Ann. of Math. (2)* **59** (1954), 279-286.

[2] K.J. Dykema, On certain free product factors via an extended matrix model, *J. Funct. Anal.* **112** (1993), 31-60.

[3] K.J. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, *Duke Math. J.* **69** (1993), 97-119.

[4] K.J. Dykema, Two applications of free entropy, *Math. Ann.* **308** (1997), 547–558.

[5] K.J. Dykema, A.M. Sinclair and R.R. Smith, in preparation.

[6] L. Ge, Applications of free entropy to finite von Neumann algebras, *Amer. J. Math.* **119** (1997), 467-485.

[7] S. Neshveyev and E. Størmer, Ergodic theory and maximal abelian subalgebras of the hyperfinite factor, *J. Funct. Anal.* **195** (2002), 239-261.

[8] S. Popa, Orthogonal pairs of ∗-subalgebras in finite von Neumann algebras, *J. Operator Theory* **9** (1983), 253–268.

[9] L. Pukánszky, On maximal abelian subrings of factors of type II_1, *Canad. J. Math.* **12** (1960), 289-296.

[10] F. Rădulescu, Singularity of the radial subalgebra of $L(F_N)$ and the Pukánszky invariant, *Pacific J. Math.* **151** (1991), 297-306.

[11] A.M. Sinclair and R.R. Smith, The Pukánszky invariant for group von Neumann algebras, *Illinois J. Math.*, to appear.

A notion of *free product* for planar algebras

Dietmar Bisch
(joint work with Vaughan Jones)

Let $N \subset M$ be an inclusion of II_1 factor with finite Jones index ([9]). The *standard invariant* of $N \subset M$, given by the system of higher relative commutants (see for instance [8])

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} = M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \cdots \end{array}$$

can be axiomatized as a (subfactor) *planar algebra* ([10], [12]). It is a complete invariant for *amenable* subfactors ([11]). Popa's abstract characterization of standard invariants in [12] allows one to study subfactors by working purely on the level of the standard invariants. The planar algebra formalism of Jones is a tool which facilitates computations with standard invariants. It gives a new approach to the analsyis of subfactors which leads in a natural way to a generators and relations approach to investigating the structure of subfactors. From this point of view the "simplest" subfactors are those whose standard invariants are generated (as planar algebras) by the fewest elements satisfying the simplest relations (see [5], [6]). Furthermore, it becomes natural to try to use planar algebra techniques to construct new standard invariants, and hence new subfactors by [12], from given ones.

It turns out the the *Fuss-Catalan algebras* $(FC_k(a,b))_k$ of [3] are examples of this type. They can be constructed as certain "free products" of two Temperley-Lieb planar algebras $(TL_n(a))_n$ and $(TL_n(b))_n$, where a and b correspond to the indices of the intermediate subfactors in [3], which we assume to be generic for simplicity. A basis diagram of a Fuss-Catalan algebra $FC_k(a,b)$ ([3]) can be thought of as being obtained through a *planar concatenation* of two Temperley-Lieb basis diagrams, one from the algebra $TL_k(a)$ and the other from $TL_k(b)$, in an obvious way. This procedure realizes the algebra $FC_k(a,b)$ as a natural (proper) subalgebra of the tensor product algebra $TL_k(a) \otimes TL_k(b)$. It is this idea of "planar concatenation" which leads to the notion of *free product* of planar algebras ([7]): Suppose $\mathcal{P} = (P_k)_k$ and $\mathcal{Q} = (Q_k)_k$ are two subfactor planar algebras with parameters δ_1 resp. δ_2. We define a new planar algebra $\mathcal{P} * \mathcal{Q} = ((\mathcal{P} * \mathcal{Q})_k)_k$ with parameter $\delta_1 \cdot \delta_2$ by letting $(\mathcal{P} * \mathcal{Q})_k$ be the natural subalgebra of $P_k \otimes Q_k$ spanned by those diagrams from P_k resp. Q_k which stay planar when submitted to the same planar concatenation as above. More precisely, we label the diagrams representing operators from P_k by "a" and those from Q_k by "b" and concatenate these a-diagrams and b-diagrams using the FC pattern "abbabbabba..." to obtain new diagrams. If this process yields a *planar* diagram it is by definition in $(\mathcal{P} * \mathcal{Q})_k$.

The first step is to show that we obtain indeed a planar algebra in the sense of [10]. The next step is then to determine the structure of this new planar algebra, in particular its dimension. Recall that if $\mathcal{P} = (P_k)_k$ is a planar algebra, then we call the formal power series $\sum_{k=0}^{\infty} (\dim P_k) z^k$ the *dimension* (or the *dimension generating function*) of $\mathcal{P}$. For instance, the dimension of the Temperley-Lieb planar algebra is given by $G_{TL}(z) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^n = \frac{1-\sqrt{1-4z}}{2z}$. If one computes the free multiplicative convolution $G_{TL} \boxtimes G_{TL}$ in the sense of Voiculescu [14] then one finds the dimension generating function $G_{FC}(z) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{3n}{n} z^n$ of the Fuss-Catalan planar algebra (to be more precise one computes the free multiplicative convolution of the associated distributions). In [7] we prove

Theorem 1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be planar algebras arising as the standard invariants of (extremal) subfactors and let $G_{\mathcal{P}}$ resp. $G_{\mathcal{Q}}$ be their dimension generating functions. Then the dimension generating function $G_{\mathcal{P}*\mathcal{Q}}$ of the free product planar algebra $\mathcal{P} * \mathcal{Q}$ can be computed from $G_{\mathcal{P}}$ and $G_{\mathcal{Q}}$ using Voiculescu's free multiplicative convolution, namely $G_{\mathcal{P}*\mathcal{Q}} = G_{\mathcal{P}} \boxtimes G_{\mathcal{Q}}$.*

The proof of this theorem uses the following idea. If $N \subset P$ and $P \subset M$ are subfactors then the N-P bimodule ${}_N L^2(P)_P$ and the P-M bimodule ${}_P L^2(M)_M$ generate fusion algebras of P-P-bimodules, denoted by A resp. B. As usual a basis of A resp. B is given by the irreducible P-P bimodules appearing in the decomposition of bimodule tensor powers of these bimodules (see for instance [1] for details on fusion algebras associated to subfactors). If the fusion algebra generated by A and B is the free product $A * B$ of the two fusion algebras, then every "word" in letters alternating from A and B is again an *irreducible* P-P bimodule. We obtain in this way a noncommutative probability space $A * B$ in which (nontrivial) P-P bimodules from A resp. B are free (the state is given by the multiplicity of the trivial P-P bimodule). Hence the techniques of [13] apply. The difficulty consists then in identifying the planar algebra picture with the bimodule picture appropriately. Note that the fusion algebras of the Fuss-Catalan subfactors of [3] are worked out in [4].

It is perhaps worth mentioning that for the group-like subfactors $N = P^H \subset P \subset M = P \rtimes K$ ([2]), where H and K are finite groups with an outer action on the II_1 factor P, the irreducible P-P bimodules are simply labelled by the group G generated by H and K in the outer automorphism group of P. If P is the hyperfinite II_1 factor then any group G which is the free product of two finite groups H and K has an outer action on P and hence we get a class of explicit examples of hyperfinite subfactors for which the above theorem applies. In general, since we work on the level of the planar algebras, the associated subfactors obtained via Popa's reconstruction theorem [12] will no longer be hyperfinite.

References

[1] D. Bisch, *Bimodules, higher relative commutants and the fusion algebra associated to a subfactor*, The Fields Institute for Research in Mathematical Sciences Communications Series **13**, 13–63, 1997, AMS, Providence, Rhode Island.

[2] D. Bisch, U. Haagerup, *Composition of subfactors: new examples of infinite depth subfactors*, Ann. Scient. Éc. Norm. Sup. **29** (1996), 329–383.

[3] D. Bisch, V.F.R. Jones, *Algebras associated to intermediate subfactors*, Invent. Math. **128** (1997), 89–157.

[4] D. Bisch, V.F.R. Jones, *A note on free composition of subfactors*, in "Geometry and Physics", Marcel Dekker, Lecture Notes in Pure and Applied Mathematics, **184** 339–361, 1997.

[5] D. Bisch, V.F.R. Jones, *Singly generated planar algebras of small dimension*, Duke Math. Journal **101** (2000), no. 1, 41–75.

[6] D. Bisch, V.F.R. Jones, *Singly generated planar algebras of small dimension, Part II*, Adv. Math. **175** (2003), no. 2, 297–318.

[7] D. Bisch, V.F.R. Jones, *A notion of* free product *for planar algebras*, in preparation.

[8] F. Goodman, P. de la Harpe, V.F.R. Jones, *Coxeter graphs and towers of algebras*, Springer Verlag, MSRI publications, 1989.

[9] V.F.R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[10] V.F.R. Jones, *Planar algebras I*, preprint, available at `http://www.math.berkeley.edu/~ vfr/`.

[11] S. Popa, *Classification of amenable subfactors of type II*, Acta Math. **172** (1994), 352–445.

[12] S. Popa S., *An axiomatizaton of the lattice of higher relative commutants*, Invent. Math. **120** (1995), 427–445.

[13] D. Voiculescu D., *Symmetries of some reduced free product C^*-algebras*, in Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, Springer Verlag **1132** (1985), 556–588.

[14] D. Voiculescu, *Multiplication of certain noncommuting random variables*, J. Operator Theory **18** (1987), no. 2, 223–235.

Subfactors from Braided C* Tensor Categories

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(joint work with Juliana Erlijman)

It was noted by Vaughan Jones that his examples of subfactors gave rise to unitary braid representations. By this we mean representations of the infinite braid group B_∞ defined by infinitely many generators σ_1, σ_2, ... which satisfy the familiar braid representations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$. Subsequently, unitary braid representations were used by Ocneanu and by Wenzl to construct new examples of subfactors; here the subfactor is generated by the subgroup $B_{2,\infty}$ generated by σ_2, σ_3, ... This construction was denoted as the one-sided subfactor construction by Erlijman, as opposed to her multi-sided subfactors, constructed as follows: For given integer $s > 1$, the s-sided subfactor is obtained as a suitable inductive limit of the embeddings of the braid groups $B_n^s = B_n \times \ldots \times B_n$ (s times) into B_{ns} for $n \to \infty$. She also computed the indices of these subfactors and their relative commutants.

The main motivation for this paper was to calculate the higher relative commutants of Erlijman's subfactors. To do this it is convenient to generalize the above mentioned constructions to the setting of a braided C* tensor category $\mathcal{C}$ with only finitely many simple objects up to isomorphism. By definition of such a category, we obtain a unitary representation of B_n in $\mathrm{End}(X^{\otimes n})$ for any object X in $\mathcal{C}$. The constructions in our paper in the category setting follow closely the above-mentioned braid constructions, and reduce to them in case that $\mathrm{End}(X^{\otimes n})$ is generated by B_n for all $n \in \mathbb{N}$.

The main results of our paper are as follows. We show that the first principal graph is given by the fusion graph of $(\mathcal{C}')^s$, where $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ generated by the objects which appear in the same tensor powers of X as the trivial representation appears. The fusion graph describes the decomposition of the tensor product of s simple objects of $\mathcal{C}'$ into irreducibles. Hence the even vertices are labeled by s-tuples of the labeling set Λ' of simple objects of $\mathcal{C}'$, and the odd ones by the elements of Λ'; the number of edges connecting ν with and s-tuple $(\lambda_1,\ ...,\ \lambda_s)$ is equal to the multiplicity of X_ν in $X_{\lambda_1} \otimes\ ...\ \otimes X_{\lambda_s}$.

The situation is more complicated for the dual (or second) principal graph. If a certain matrix depending on the braiding structure, and called the S-matrix for the category $\mathcal{C}'$, is invertible, the dual principal graph coincides with the principal graph. We do not have a general complete result in the case of a noninvertible S-matrix. It is known that in this case there is a canonical subcategory $\mathcal{T}$ of $\mathcal{C}'$ which is equivalent to the representation category of a finite group G. If G is abelian, we obtain an action of G on the set of irreducible objects of $\mathcal{C}$, which is given by a labeling set Λ. The dual principal graph can now be fairly precisely characterized in terms of the orbits of the action of the group $G_1^s = \{(g_1,\ ...\ g_s) \in G^s,\ g_1 g_2\ ...\ g_s = 1\}$ on Λ^s. This is often referred to as an orbifold.

The basic idea of our paper is that we explicitly construct a number of $\mathcal{A}-\mathcal{B}$ bimodules, with $\{\mathcal{A},\mathcal{B}\} \subset \{\mathcal{N},\mathcal{M}\}$ and with $\mathcal{N} \subset \mathcal{M}$ our s-sided inclusion. We show that these examples of bimodules are closed under induction and restriction. One deduces from this that the induction-restriction graph for these bimodules must coincide with the principal or dual principal graph under some mild additional assumptions.

Our findings are related to a number of results by different authors. If $s = 2$, our subfactors correspond to the subfactors obtained from the asymptotic inclusion of certain one-sided subfactors. In this case, the orbifold phenomenon for the dual principal graph has first been observed by Ocneanu for the example of the asymptotic inclusion of certain Jones subfactors. Further results have been obtained in papers by Evans and Kawahigashi and by Izumi for Hecke type subfactors. In particular, some of our proofs have been inspired by these results.

References

[E1] Erlijman, J., Multi-sided braid type subfactors, Canad. J. Math. 53 (2001), 546-564.
[E2] Erlijman, J., Multi-sided braid type subfactors. II. Canad. Math. Bull. 46 (2003), no. 1, 80-94.

[EK1] Evans, David E.; Kawahigashi, Yasuyuki Quantum symmetries on operator algebras. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.
[EK2] Evans, David E.; Kawahigashi, Yasuyuki Orbifold subfactors from Hecke algebras. II. Quantum doubles and braiding. Comm. Math. Phys. 196 (1998), no. 2, 331–361.
[Iz] Izumi, Masaki The structure of sectors associated with Longo-Rehren inclusions. I. General theory. Comm. Math. Phys. 213 (2000), no. 1, 127–179.
[Jo] Jones, V.F.R., Index for subfactors, Invent. Math 72 (1983) 1–25.
[W] Wenzl, H., Hecke algebras of type A_n and subfactors, Invent. Math. 92 (1988) 349-383.

Generalised Hecke algebras

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(joint work with Magnus B. Landstad)

A Hecke pair (G, H) consists of a group G together with a subgroup H such that every double coset with respect to H contains finitely many left cosets. Equivalently, the index $L(x) = [H : H \cap xHx^{-1}]$ must be finite for all $x \in G$ (and, equivalently, left cosets can be replaced by right cosets).

As a vector space, the *Hecke algebra* $\mathcal{H}(G, H)$ consists of functions $f : G \to \mathbb{C}$ which are H-biinvariant, i.e. $f(hxk) = f(x)$ for all $x \in G$, $h, k \in H$, and which have finite support when regarded on the double coset space $H \setminus G/H$. We let $\Delta(x) = L(x)/L(x^{-1})$ for $x \in G$. The convolution and involution on $\mathcal{H}(G, H)$ are defined by

$$f * g(x) = \sum_{y \in G/H} f(y)g(y^{-1}x) \tag{1}$$

and

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}, \tag{2}$$

where $x \in G$ and the notation "$y \in G/H$" means that the sum taken is over a set of representatives for the left coset space.

A powerful method to study $\mathcal{H}(G, H)$ is by constructing a *Schlichting completion* $(\overline{G}, \overline{H})$ of (G, H). This is an essentially unique pair consisting of a locally compact group $\overline{G}$ together with a compact, open subgroup $\overline{H}$, see [5, 3, 4]. Since then the Hecke algebra $\mathcal{H}(G, H)$ is isomorphic to the corner of the $*$-algebra $C_c(\overline{G})$ (endowed with its usual operations) determined by the projection $\chi_{\overline{H}}$, one can study the representation theory and the possible enveloping C*-algebra of $\mathcal{H}(G, H)$ by means of familiar techniques.

The interest in Hecke algebras and their C*-completions is rooted in the work of Bost and Connes [1] on phase transitions with symmetry breaking arising in number-theoretic context.

In [2], Curtis considers a Hecke algebra associated to a pair (G, H) and a unitary representation of H. By generalising to triples (G, H, σ) the Schlichting completion as defined by Tzanev in [5], she studies a von Neumann algebra arising from a regular representation of the Hecke algebra in the space of the induced representation of σ from H to G.

Our interest lies in studying the Hecke algebra associated to a Hecke pair (G, H) and a finite character σ of H. One reason for restricting our attention to one-dimensional representations of H is that the construction and the properties of the Schlichting completion become clearer, and therefore easier to employ when studying examples.

We make the following definition. The *generalised Hecke algebra* associated to a Hecke pair (G, H) and a character σ of H is the vector space $\mathcal{H}_\sigma(G, H)$ of functions $f : G \to \mathbb{C}$ such that

$$f(hxk) = \sigma(h)f(x)\sigma(k), \forall x \in G, \forall h, k \in H, \tag{3}$$

and the support of f, when regarded in $H\backslash G/H$, is finite. Then $\mathcal{H}_\sigma(G, H)$ becomes a $*$-algebra with the operations defined in (1) and (2).

In [4], a Hecke pair (G, H) is endowed with the *Hecke topology* defined by declaring a subbase for the neighbourhoods at e to consist of all the conjugates xHx^{-1} for $x \in G$. For this topology to be Hausdorff it is necessary and sufficient that the pair (G, H) be *reduced*, i.e. $\bigcap_{x\in G} xHx^{-1} = \{e\}$. Assuming this to be the case, the Schlichting completion of (G, H) is simply the locally compact closure $\overline{G}$ together with the compact, open subgroup $\overline{H}$, where both closures are taken in the Hecke topology from (G, H).

Given a reduced Hecke pair (G, H) and a finite character σ on H, we define the Schlichting completion to be the pair $(\overline{G}_\sigma, \overline{H}_\sigma)$ consisting of the closures of G and H with respect to the Hecke topology from the reduced Hecke pair $(G, \ker\sigma)$. The character σ has an extension $\overline{\sigma}$ to $\overline{H}_\sigma$, and the triple $(\overline{G}_\sigma, \overline{H}_\sigma, \overline{\sigma})$ is unique up to isomorphism.

One crucial observation is that $\mathcal{H}_\sigma(G, H)$ and $\mathcal{H}_\sigma(\overline{G}_\sigma, \overline{H}_\sigma)$ are isomorphic $*$-algebras. Another important fact is that the formula

$$p_\sigma(x) := \overline{\sigma}(x)\chi_{\overline{H}_\sigma}(x), x \in \overline{G}_\sigma,$$

defines a projection in $C_c(\overline{G}_\sigma)$, and then $\mathcal{H}_\sigma(G, H)$ will be equal to $p_\sigma C_c(\overline{G}_\sigma)p_\sigma$.

Two immediate interesting questions arise, and we give some answers to them. One is whether $p_\sigma C^*(\overline{G}_\sigma)p_\sigma$ is the largest C*-completion of $\mathcal{H}_\sigma(G, H)$, the other is to what extend the ideal $\overline{C^*(\overline{G}_\sigma)p_\sigma C^*(\overline{G}_\sigma)}$, to which $p_\sigma C^*(\overline{G}_\sigma)p_\sigma$ is Morita-Rieffel equivalent, can be characterised. The first question has an affirmative answer when, as in the case of Hecke pairs [4], there is a normal subgroup N of G such that H is normal in N, if moreover the identity

$$\sigma(nhn^{-1}) = \sigma(h)$$

is satisfied for all $n \in N$ and $h \in H$. Under the condition of normality we can also answer the second question, and describe the ideal as a crossed product involving an action of G/N. We can illustrate applications of these results to several examples. In the case of the rational Heisenberg group, slightly modified so as to produce a reduced Hecke pair with its subgroup consisting of integer entries, the universal C*-completion of the generalised Hecke algebra is the same as for $\mathcal{H}(G,H)$, for all finite characters. However, this does not happen for the rational "$ax+b$"-group or variants of it. As known from [5] or [4], the Hecke algebra of the infinite dihedral group does not possess a largest C*-norm, and we find that a similar behaviour is valid for its generalised counterpart. The examples suggest possible new methods for further investigation.

References

[1] J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) **1** (1995), 411–457.

[2] R. Curtis, *Hecke algebras associated with induced representations*, C. R. Acad. Sci. Paris, Ser. I **334** (2002), 31–35.

[3] H. Glöckner and G. A. Willis, *Topologization of Hecke pairs and Hecke C*-algebras*, Topology Proceedings **26** (2001/2002), 565–591.

[4] S. Kaliszewski, Magnus B. Landstad and J. Quigg, *Hecke C*-algebras, Schlichting completions, and Morita equivalence*, in preparation.

[5] K. Tzanev, *Hecke C*-algebras and amenability*, J. Operator Theory **50** (2003), 169–178.

Exact C*-bundles

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(joint work with Etienne Blanchard)

Given continuous C*-bundles $\mathcal{A}=\{A,X,\pi_x^A : A\to A_x\}$ and $\mathcal{B}=\{B,X,\pi_x^B : B\to B_x\}$ on a compact Hausdorff space X, the amalgamated minimal tensor product $\mathcal{A}\otimes_{C(X)}\mathcal{B}$ is the C*-bundle $\{A\otimes_{C(X)}^{min}B, X, \pi_x^A\otimes\pi_x^B : A\otimes_{C(X)}B\to A_x\otimes_{min}B_x\}$ on X, where the bundle C*-algebra $A\otimes_{C(X)}^{min}B$ is the image of $A\otimes_{min}B$ under the *-homomorphism $\oplus_{x\in X}(\pi_x^A\otimes\pi_x^B)$. When $\mathcal{A}$ is the trivial bundle on X with constant fibre A, $\mathcal{A}\otimes_{C(X)}\mathcal{B}$ is denoted by $A\otimes_{min}\mathcal{B}$. This C*-bundle is always lower semicontinuous, but not necessarily continuous. Kirchberg and Wassermann [2] considered the continuity of such C*-bundles and obtained the following continuity criteria.

1. A C*-algebra A is exact if and only if the tensor product bundle $A\otimes_{min}\mathcal{B}$ is continuous for any continuous C*-bundle $\mathcal{B}$ on $\hat{\mathbb{N}}$.

2. If $\mathcal{A}$ is an *exact* continuous C*-bundle, that is, $\mathcal{A}$ has exact bundle algebra, then $\mathcal{A}\otimes_{C_0(X)}\mathcal{B}$ is continuous for all continuous $\mathcal{B}$, and, if $\mathcal{A}$ is an arbitrary continuous C*-bundle with exact fibres, then the converse holds.

We have extended these results by showing that if X has no isolated points, then for arbitrary $\mathcal{A}$, the continuity of $\mathcal{A}\otimes_{C(X)}\mathcal{B}$ for all continuous C*-bundles $\mathcal{B}$ implies the exactness of all the fibre algebras of $\mathcal{A}$, yielding the following characterisation of exact C*-bundles on X.

Theorem 1 *If X is a compact Hausdorff space with no isolated points, then a continuous C*-bundle $\mathcal{A}$ on X is exact if and only if for any continuous C*-bundle $\mathcal{B}$ on X, $\mathcal{A} \otimes_{C(X)} \mathcal{B}$ is continuous.*

This result is proved in stages. The first step is to prove

Proposition 2 *If a continuous C*-bundle $\mathcal{A}$ satisfies the conditions of Theorem 1 with $X = \hat{\mathbb{N}}$, the one-point compactification of $\mathbb{N}$, then A_∞, the fibre of $\mathcal{A}$ at infinity, is exact.*

This is proved by repeated application of criteria 1 and 2 above. The next stage is to prove

Proposition 3 *The assertion of Theorem 1 holds if X is a compact metric space with no isolated points.*

This is proved using the following extension result for continuous C*-bundles.

Theorem 4 *Let X be a compact metric space and let Y be a closed subset of X. If $\mathcal{A}$ is a continuous C*-bundle on Y then there exists a continuous C*-bundle $\bar{\mathcal{A}}$ on X such that*

$$\bar{\mathcal{A}}|_Y = \text{ cone}(\mathcal{A}) \ (= C((0,1]) \otimes \mathcal{A}).$$

This result together with Proposition 2 implies that if $\mathcal{A}$ satisfies the hypotheses of Proposition 3, then the fibre algebra A_x of $\mathcal{A}$ at any point $x \in X$ is exact. Proposition 3 is then a consequence of criterion 2 above. The proof of Theorem 1 for general X follows by representing the C*-bundles concerned as inductive limits of separable C*-subbundles and the space X as a projective limit of associated compact metric spaces. Proposition 3 can then be applied to show that if $\mathcal{A}$ satisfies the hypotheses of Theorem 1, then for $x \in X$ any separable C*-subalgebra of the fibre algebra A_x is exact, which implies the exactness of A_x itself. Theorem 1 is then a consequence of criterion 2.

The extension result (Theorem 4) has a number of other consequences. For example it can be used to show that in criterion 2, the space $\hat{\mathbb{N}}$ can be replaced by any given infinite compact Hausdorff space, a result first obtained in [1, Corollary 4].

References

[1] S. CATTERALL, S. WASSERMANN, *Continuous bundles of C*-algebras with discontinuous tensor products*, Bull. London Math. Soc., *to appear.*

[2] E. KIRCHBERG, S. WASSERMANN, *Operations on continuous bundles of C*-algebras*, Math. Ann. **303** (1995), 677–697.

Types of von Neumann algebras arising from boundary actions of hyperbolic groups

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(joint work with Masaki Izumi and Rui Okayasu)

Let Γ be a non-elementary hyperbolic group, μ a non-degenerate finitely supported probability measure on Γ. It is known that both the Martin and the Poisson boundaries of the random walk on Γ defined by μ can be identified with the Gromov boundary $\partial\Gamma$, see [1, 4]. Let ν be the corresponding harmonic measure on $\partial\Gamma$, and K the Martin kernel. The measure ν is quasi-invariant with respect to Γ. We thus get an amenable ergodic type III (by [3]) orbit equivalence relation $\mathcal{R}$ on $(\partial\Gamma, \nu)$. We want to compute the ratio set of $\mathcal{R}$. To formulate the main result, let g be an infinite order element. Then the sequence $\{g^n\}_{n=1}^{\infty}$ converges to an element g^+ on the boundary. Set

$$r(g) = K(g^{-1}, g^+).$$

Theorem 1. *The number $r(g)$ lies in the interval $(0,1)$ and belongs to the ratio set of $\mathcal{R}$. In particular, $\mathcal{R}$ has type III_λ with $0 < \lambda \le 1$.*

The proof is inspired by Bowen's computation of the ratio set of the Gibbs measure in [2]. On the way we prove that the Martin kernel is Hölder continuous, which extends the result of Ledrappier for free groups [5].

References

[1] A. Ancona, *Positive harmonic functions and hyperbolicity*, in: Potential theory – surveys and problems (Prague, 1987), 1–23, Lecture Notes in Math., **1344**, Springer, Berlin, 1988.
[2] R. Bowen, *Anosov foliations are hyperfinite*, Ann. of Math. (2) **106** (1977), no. 3, 549–565.
[3] V. Kaimanovich, *The Poisson boundary of covering Markov operators*, Israel J. Math. **89** (1995), no. 1-3, 77–134.
[4] V. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*, Ann. of Math. (2) **152** (2000), no. 3, 659–692.
[5] F. Ledrappier, *Some asymptotic properties of random walks on free groups*, in: Topics in probability and Lie groups: boundary theory, 117–152, CRM Proc. Lecture Notes, **28**, Amer. Math. Soc., Providence, RI, 2001.

Semi-groups of partial homeomorphisms

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Since the introduction of crossed products in the theory of C*-algebras nearly fifty years ago, many variations and extensions have been designed. The subject of this talk is a particular crossed product construction, first introduced by the author in [13] to describe Cuntz algebras as groupoid algebras and generalized to arbitrary surjective local homeomorphisms by V. Deaconu in [5]. One associates to a local homeomorphism T of a compact space X an étale groupoid $\mathcal{O}_T$ (a kind of semi-direct product), hence a C*-algebra O_T. When T is a homeomorphism, O_T is the usual crossed product $C(X)\rtimes_T \mathbf{Z}$. When T is not invertible, the new feature is an equivalence relation $\mathcal{F}_T \subset \mathcal{O}_T$ and its associated C*-algebra $F_T \subset O_T$. This construction is related to the theory of Smale spaces, initiated by D. Ruelle in [17] and developed by I. Putnam in [12]: F_T is essentially the stable algebra and O_T the Ruelle algebra of this theory. More generally, the same construction applies when N is a subsemi-group of a discrete abelian group Z and T is a homomorphism of N into the semi-group $\mathrm{End}(X)$ of local homeomorphisms of X. We first review the groupoid construction of this crossed product, give some properties of the C*-algebras so obtained and finally give some algebraic construction of O_T. This last part is based on discussions with R. Exel. These constructions also appear in [6].

A groupoid construction. Let N be a subset of a discrete countable abelian group Z (noted additively) containing 0 and stable under addition. We define an action of N on a compact metric space X as a semi-group homomorphism $T : N \to \mathrm{End}(X)$. When $N = \mathbf{N}$, we do not distinguish the action T and its generator T_1. The semi-direct groupoid of the action T of N on X is

$$\mathcal{O}_T = \{(x, m-n, y) : \quad x, y \in X \quad m, n \in N \quad T_m(x) = T_n(y)\}.$$

It is a subgroupoid of $X \times Z \times X$ endowed with its natural groupoid structure. Its topology is defined by the basic open subsets obtained by fixing $m, n \in N$ and taking x, y in open subsets U, V of X such that the restrictions $T_{m|U}$ and $T_{n|V}$ are homeomorphisms. This turns $\mathcal{O}_T$ into an étale second countable locally compact groupoid. The C*-algebra $O_T = C^*(\mathcal{O}_T)$ is our crossed product. Just as for group actions, the homomorphism $c : (x, k, y) \in \mathcal{O}_T \mapsto k \in Z$ is an essential feature of this construction. The novelty is that the subgroupoid

$$\mathcal{F}_T = c^{-1}(0) = \{(x, y) \in X \times X : \quad \exists n \in N : \quad T_n(x) = T_n(y)\}$$

may not be reduced to the diagonal. It is an *approximately proper equivalence relation* in the sense of [16]: it is an increasing union of open proper subgroupoids. Its C*-algebra $F_T = C^*(\mathcal{F}_T)$ is the fixed point subalgebra of O_T under the gauge automorphism group defined by c. One obtains immediately (cf.[2, 15]) that the groupoid $\mathcal{O}_T$ is amenable, its full and reduced C*-algebras coincide and O_T is a nuclear C*-algebra. One says that the action T is *essentially free* if for all $n \neq m$ in N, there exists no non-empty open subset U on which T_n and T_m agree, *exact* if for all non-empty open subset U and all $m \in N$, there exists $n \in N$ such that $T_{m+n}(U) = X$ and *expansive* if there exists $\epsilon > 0$ such that for all $x \neq y$ in X, there exists $n \in N$ such that $d(T_n(x), T_n(y)) > \epsilon$.

Theorem 1. *Assume that N is finitely generated and that the action T of N on X is essentially free, exact and expansive. Then*

(1) *O_T is a Kirchberg algebra;*
(2) *F_T is simple, stably finite and has a unique tracial state.*

The simplicity of O_T and F_T results from [14, Corollary 4.6]. The pure infiniteness of O_T results from [1, Section 2.4]. The unique ergodicity of F_T is obtained in [16, Theorem 6.1] in the case of a single endomorphism. The main tool is a *dimension group*, analogous to Elliott's dimension group of [9] for AF algebras and Krieger's dimension group of [11]. In the case of a single endomorphism, this is the inductive limit of the stationary system $C(X,\mathbf{Z}) \xrightarrow{\mathcal{L}} C(X,\mathbf{Z}) \xrightarrow{\mathcal{L}} \dots$, where $\mathcal{L}f(x) = \sum_{T(y)=x} f(y)$. There is a similar definition in the semi-group case.

The classical example of this construction is the one-sided shift T on the infinite product space $X = \prod_1^\infty \{1,\dots,d\}$. Then O_T is the Cuntz algebra O_d and $F_T = F_d$ is the UHF algebra of type d^∞. More generally, the one-sided subshift of finite type T_A, where $A \in M_d\{0,1\}$ yields the Cuntz-Krieger algebra O_A. The assumptions of the theorem are satisfied when the matrix A is primitive. Graph algebras also fit within this construction. More recently, in [18, 19] (see also [8]), F. Shultz has applied this construction and used the dimension group to study piecewise monotonic interval maps. Since these maps are not local homeomorphisms, he first constructs a space X over the interval $[0,1]$ and a local homeomorphism $T \in \mathrm{End}(X)$ above the given interval map which retains most dynamical properties of that map. A good example of an action of $\mathbf{N}^2$ is provided by the maps $S(z) = z^p$ and $T(z) = z^q$ on the circle, where p, q are relatively prime integers. Let $\Delta_{\mathbf{Z}}$ be the diagonal subgroup of $\mathbf{N}^2$. Then, $c^{-1}(\Delta_{\mathbf{Z}})$ and its C*-algebra describe the polymorphism (S,T) (see [20] and [3, Section 5]). The following example appears in the work [4] of J.-B. Bost and A. Connes (see also [3, Section 6]): the semi-group is the multiplicative semi-group $\mathbf{N}^*$ of positive integers (as a subset of the multiplicative group $\mathbf{Q}_+^*$); it acts on the infinite product $X = \prod \mathbf{Z}_p\}$ over the set of prime numbers, where $\mathbf{Z}_p$ is the ring of p-adic integers by the diagonal action. In this example, the local homeomorphisms T_n are injective and F_T is simply $C(X)$.

An algebraic construction. From an algebraic viewpoint, the initial data consist of a C*-algebra with unit, namely $A = C(X)$, an abelian semi-group N and a semi-group homomorphism $\alpha : N \to \mathrm{End}(A)$, where $\alpha_n(f) = f \circ T_n$. In the case of $\mathbf{N}$, it is shown in [10] that O_T is the *Exel crossed product* $A \rtimes_{\alpha,\mathcal{L}} \mathbf{N}$, where $\mathcal{L}f(x) = \sum_{T(y)=x} \rho(y) f(y)$ and ρ is an arbitrary positive continuous function on X such that $\sum_{T(y)=x} \rho(y) = 1$ for all x. Equivalently, O_T can be obtained as an augmented Pimsner-Cuntz algebra (see [7]). A simple-minded extension of this result to an arbitrary abelian semi-group N requires finding positive continuous functions ρ_n on X such that $\rho_{m+n}(x) = \rho_m(x)\rho_n(T_n x)$ for all $m, n \in N, x \in X$ and $\sum_{T_n(y)=x} \rho_n(y) = 1$ for all $n \in N, x \in X$, which we can only do in a few cases.

References

[1] C. Anantharaman-Delaroche: Purely infinite C^*-algebras arising from dynamical systems, *Bull. Soc. Math. France*, **125** (1997), 199–225.

[2] C. Anantharaman-Delaroche and J. Renault: *Amenable groupoids*, Monographie de l'Enseignement Mathématique, **36**, Genève, 2000.

[3] V. Arzumanian and J. Renault: Examples of pseudogroups and their C^*-algebras, *Operator Algebras and Quantum Field Theory*, S. Doplicher, R. Longo, J. E. Roberts and L. Zsido, editors, International Press 1997, 93-104.

[4] J.-B. Bost and A. Connes: Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math.* (New Series) **1** (1995), 411-457.
[5] V. Deaconu: Groupoids associated with endomorphisms, *Trans. Amer. Math. Soc.*, **347** (1995), 1779–1786.
[6] V. Deaconu: C*-algebras of commuting endomorphisms, *arXiv:math.*OA/0406624.
[7] V. Deaconu and P. Muhly: C*-algebras associated with branched coverings, *Proc. Amer. Math. Soc.*, **129** (2001), no.4, 1077–1086.
[8] V. Deaconu and F. Shultz: C*-algebras associated with interval maps, *arXiv:math.*OA/0405469.
[9] G. Elliott: On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. Algebra*, **38** (1976), 29–44.
[10] R. Exel and A. Vershik: C*-algebras of irreversible dynamical systems, *Can. J. Math.*, to appear, arXiv:math.OA/0203185.
[11] W. Krieger: On a dimension for a class of homeomorphism groups, *Math. Ann.*, **252** (1980), 87–95.
[12] I. Putnam: C^*-algebras from Smale spaces, *Can. J. Math.*, **48** (1996), 175-195.
[13] J. Renault: *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, Vol. **793** Springer-Verlag Berlin, Heidelberg, New York (1980).
[14] J. Renault: The ideal structure of groupoid crossed product C^*-algebras, *J. Operator Theory* **25** (1991), 3-36.
[15] J. Renault: Cuntzlike-algebras, *Proceedings of the 17th International Conference on Operator Theory* (Timisoara 98), The Theta Fondation (2000).
[16] J. Renault: The Radon-Nikodym problem for approximately proper equivalence relations, *Ergodic Theory Dynam. Systems*, to appear, arXiv:math.OA/0211369.
[17] D. Ruelle: Non-commutative algebras for hyperbolic diffeomorphisms, *Invent. Math.* **93**(1988), 1-13.
[18] F. Shultz: Dimension groups for interval maps, *arXiv:math.*DS/0405466.
[19] F. Shultz: Dimension groups for interval maps II: the transitive case, *arXiv:math.*DS/0405467.
[20] A. Vershik: Many-valued measure-preserving mappings (polymorphisms) and Markovian operators, translated from *Zapiski Nauchnnyk Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR*,**72**, 26-61, (1977)). 0090-4104 /83 /2303-2243

Spectral triples and the Cantor set

Erik Christensen
(joint work with Cristina Antonescu)

Alain Connes has extended the notion of a compact metric space to the non commutative setting of C*-algebras and unbounded operators on Hilbert spaces, [Co1, Co2]. For a compact, spin, Riemannian manifold $\mathcal{M}$, Connes has shown that the geodesic distance can be expressed in terms of an unbounded Fredholm module over the C*-algebra $\mathrm{C}(\mathcal{M})$, such that the distance between two points p, q in $\mathcal{M}$ is obtained via the Dirac operator D by the formula

$$d(p,q) = \sup\{|a(p) - a(q)\mid \ : \ a \in \mathrm{C}(\mathcal{M}), \|[D,a]\| \leq 1\}.$$

If one replaces $\mathrm{C}(\mathcal{A})$ by a C*-algebra and thinks of the points p and q as states, one can see how the concept of a compact metric space can be transformed into a non commutative setting. Connes has formalized this in the notion called a *spectral triple*

Definition 1. Let $\mathcal{A}$ be a unital C*-algebra. A *spectral triple* $(\mathcal{A}, H, D)$ is a C*-algebra $\mathcal{A}$, a Hilbert space H and an unbounded self-adjoint operator D on H such that:

(i) The Hilbert space H is a left $\mathcal{A}$-module, i. e. there is a *-representation π of $\mathcal{A}$ on H.

(ii) The set given as

$$\{a \in \mathcal{A} : [D, \pi(a)] \text{ is densely defined and extends to a bounded operator on } H\}$$

is norm dense in $\mathcal{A}$.

(iii) The operator $(I + D^2)^{-1}$ is compact.

Condition (iii) is quite often strengthened in the way that the module is said to be finitely summable or *p-summable*, [Co1, Co2], if for some $p > 0$

$$\text{trace}\left(\left(I + D^2\right)^{-p/2}\right) < \infty$$

Given a spectral triple $(\mathcal{A}, H, D)$, one can then introduce a pseudo-metric on the state space $\mathcal{A}(\mathcal{A})$ of $\mathcal{A}$ by the formula

$$\forall \phi, \psi \in \mathcal{A}(\mathcal{A}) : \ d(\phi, \psi) := \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{A}, \|[D, a]\| \leq 1\}.$$

We use the term *pseudo-metric* because it is not clear that $d(\varphi, \psi) < \infty$ for all pairs, but the other axioms of a metric are fulfilled. Marc A. Rieffel has studied several aspects of this extension of the concept of a compact metric space to the framework of C*-algebras, and he has obtained a lot of results [Ri1, Ri2, Ri3, Ri4]. Among the questions he has dealt with, we have been most attracted by the one which asks whether a spectral triple will induce a metric for the weak*-topology on the state space. In the article [OR] by Ozawa and Rieffel the situation for the discrete, word hyperbolic groups of Gromov were studied and a complete result telling that the metric has the right properties were obtained. One of the ingredients in Rieffel's and Ozawa's proof is their use of the filtration which the length function induce on the group C*-algebra. The construction of spectral triples based on filtrations were also studied by Connes [Co1, Co2] and Voiculsecu [Vo] and we were inspired to investigate the possibilities of constructing spectral triples for AF C*-algebras. An AF C*-algebra $\mathcal{A}$ has a natural filtration since , by definition, $\mathcal{A}$ is the norm closure of an increasing sequence of finite dimensional C*-algebras $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$.

Based on the GNS-representation coming from a faithful state and the given increasing sequence of subalgebras, we get immediately an increasing sequence of finite dimensional subspaces of the Hilbert space of this cyclic representation. Following the canonical ideas of Connes' in [Co1] the eigen spaces of the Dirac operator shall be chosen as the sequence of difference spaces induced by this increasing sequence. Based on the previous examples by Connes and Rieffel we first thought that the eigen values for the Dirac operator ought to be the non negative integers, but it turns out, *to our big surprise,* that in the case of AF C*-algebras one can choose the eigen values rather arbitrarily and still get a spectral triple.

Theorem 2. *Let $\mathcal{A}$ be an infinite dimensional unital AF C*-algebra acting on a Hilbert space H and let ξ be a separating and cyclic, unit, vector for $\mathcal{A}$ and let $(\mathcal{A}_n)$ be an increasing sequence of finite dimensional C*-algebras whose union is dense in $\mathcal{A}$. Then there exists a sequence of pair wise orthogonal finite dimensional projections (Q_n) and a sequence of real numbers (α_n) such that $\alpha_0 = 0$ and $|\alpha_n| \to \infty$ and such that for any other sequence (β_n) of real numbers satisfying $|\beta_n| \geq |\alpha_n|$ the selfa-djoint operator given by $D_\beta = \sum_{n=1}^{\infty} \beta_n Q_n$ on H has the properties:*

(i) *For any $a \in \cup_{\mathbb{N}} \mathcal{A}_n \quad \| \, [D_\beta, a] \, \| < \infty$.*
(ii) *The set $(\mathcal{A}, H, D_\beta)$ is a spectral triple.*
(iii) *The metric induced by D_β on the state space generates the w*-topology.*
(iv) *For any $p > 0$ the sequence (β_n) may be chosen such that the unbounded Fredholm module is p-summable.*

This result should be seen in connection with Connes' fundamental result from [Co1] that for a non amenable discrete group the reduced group C*-algebra can never have a finitely summable spectral triple !
In the article [Vo] Voiculescu also investigates the possibilities of constructing spectral triples based on filtrations and it turns out that it is difficult to determine if a general filtration will induce a spectral triple. This indicates that AF C*-algebras are special in the way that they allow an abundance of in equivalent spectral triples. This point of view is stressed in the following theorem, where we have obtained a partial inverse to the theorem above.

Theorem 3. *Let $\mathcal{A}$ be a C*-algebra acting on a Hilbert space H, ξ be a cyclic and separating unit vector for $\mathcal{A}$ and $(Q_n)_{n \in \mathbb{N}_0}$ a sequence of pair wise orthogonal finite-dimensional projections with sum I such that $Q_0 \xi = \xi$. For any sequence of real numbers (λ_n) such that $|\lambda_n| \to \infty$ the symbol D_λ shall denote the closed self adjoint operator which formally can be written as $\sum \lambda_n Q_n$. The common domain of definition, given as* span $(\cup Q_n H)$, *for all the operators D_λ is denoted $\mathcal{A}_0$.*
If $\mathcal{A}$ contains a dense subset $\mathcal{A}$ such that for any s in $\mathcal{A}$ and any D_λ the commutator $[D_\lambda, s]$ is defined and bounded on $\mathcal{A}_0$ then $\mathcal{A}$ is an AF C-algebra.*

The continuous functions on the Cantor set is a unital AF C*-algebra and we have tried to see what our results mean for this C*-algebra. If one looks at the Cantor set as a countable infinite product of the two point set $\mathbb{Z}_2$ then one may think of the Cantor set as the compact group $\prod_{\mathbb{N}} \mathbb{Z}_2$, and the continuous functions on this group is nothing but the reduced C*-algebra for the dual group, which is $\oplus_{\mathbb{N}} \mathbb{Z}_2$. In [Co1] Connes constructs a spectral triple for a the group C*-algebra of a discrete group with a proper length function, and it turns out that an application of his construction to the group $\oplus_{\mathbb{N}} \mathbb{Z}_2$ gives the same spectral triple as we get in the AF C*-algebra setting. Except, of course, for the actual choice of the eigen values. We studied then a special case where the eigen values are inverse powers of a real α, such that $0 < \alpha < 1$, and such that the sequence of eigen values, say λ_n, becomes

α^{-n}. Let us call the Dirac operator induced by this sequence of eigen values D_α, then that Dirac operator induces a metric for the topology on the Cantor set. Any two such metrics are in equivalent and the Hausdorff dimension of the metric space for a given α agrees with the Minkowski dimension and can be computed as $\frac{\log 2}{-\log \alpha}$. Further it turns out, *still for a given* α, that the metric induced by D_α is bi-Lipschitz equivalent to the one called δ_α defined on $\prod_{\mathbb{N}} \mathbb{Z}_2$ by

$$\forall x,\, y \in \prod_{\mathbb{N}} \mathbb{Z}_2 : \quad \delta_\alpha(x\,, y) := \sum_{n=1}^{\infty} |x_n - y_n| \alpha^{(n-1)}(1-\alpha).$$

References

[AC] C. Antonescu, E. Christensen, *Metrics on group C*-algebras and a non-commutative Arzelà-Ascoli theorem*, J. Funct. Anal. 214 **214** (2004), 247–259.

[Co1] A. Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic. Theor. and Dynam. Systems, **9** (1989), 207–220.

[Co2] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[Fa] K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, Wiley 1990.

[OR] N. Ozawa, M. A. Rieffel, *Hyperbolic group C*-algebras and free product C*-algebras as compact quantum metric spaces*, Canad. J. Math., to appear, arXiv:math. OA/0302310 v1, 2003.

[Ri1] M.A. Rieffel *Metrics on states from actions of compact groups*, Doc. Math., **3** (1998), 215–229.

[Ri2] M.A. Rieffel *Metrics on state spaces*, Doc. Math. **4** (1999), 559–600.

[Ri3] M.A. Rieffel *Group C*-algebras as compact quantum metric spaces*, Doc. Math. **7** (2002), 605–651.

[Ri4] M.A. Rieffel *Compact Quantum Metric Spaces*, arXiv: math. OA/0308207 v1, 2003.

[Vo] D. V. Voiculescu *On the existence of quasicentral approximate units relative to normed ideals. Part I*, J. Funct. Anal. **91** (1990), 1–36.

Von Neumann algebras in elliptic cohomology?

Antony Wassermann

I briefly reported[1] on the possible applications of von Neumann algebras to the Teichner-Stolz programme for making precise Graeme Segal's proposal for elliptic cohomology.

One success has been the use of homomorphisms into the outer automorphism group of a factor to describe the spin structure on a loop space, I described two constructions: the type III_1 construction using loop groups; and the type II_1 construction based on the group-measure space construction and Singer's 1-cocycle perturbations.

After having described fermionic modular functor of Graeme Segal, I explained how the open string version of this theory might be formulated in terms of fusion of bimodules over type III_1 factors. On the one hand, the combinatorial computations of Feng Xu suggest that such geometrical interpretation of fusion should be

[1]This abstract, typed by the reporter, is based on the handwritten abstract by A. Wassermann in the "Book of Talks" of the MFO.

possible; on the other hand, the theory would require glueing Riemann surfaces with corners – even for domains in $\mathbb{C}$, it is at present not clear how to handle certain technical difficulties (well-known from the Toeplitz story).

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