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Cohomology of Finite Groups: Interactions and Applications

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Abstract. This is a report on a meeting on interactions and applications of the cohomology of finite groups. Besides several talks on the cohomology of finite groups there were talks on related subjects, in particular on the cohomology of infinite groups, on the theory of transformation groups and p-compact groups, on modular representation theory and commutative algebra.
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Introduction by the Organisers

The aim of this meeting was to bring together people from different fields in which the cohomology of finite groups is an important tool. Several such meetings have taken place in the past (for example at Oberwolfach in 2000) and they have contributed substantially to the development of interactions between such fields, in particular between commutative algebra, homological algebra, homotopy theory, modular representation theory and transformation groups.

The meeting was attended by 54 participants from about a dozen countries. There were 24 talks of various length. Several of them were directly concerned with the cohomology of finite groups, but beyond that there were talks in which the relation of the cohomology of finite groups with other topics was emphasized: among them talks on the cohomology of infinite groups, of quantum groups and of local rings, on functor cohomology, on transformation groups and p-compact groups, on homological algebra and triangulated categories, on endotrivial modules in modular representation theory and on the Alperin conjecture as well as on

K-theory, on invariant theory, on commutative algebra and relations to stable homotopy theory.

The schedule allowed for enough time for extensive and lively interactions between the participants. Besides the traditional and popular Wednesday afternoon hike (during a week with an exceptionally nice weather) an excellent concert was organized on Thursday evening by some of the participants of the conference. As always the very pleasant and stimulating atmosphere at the institute contributed to make this a very successful meeting.

Workshop: Cohomology of Finite Groups: Interactions and Applications

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Abstracts

The classification of 2-compact groups

Kasper K. S. Andersen
(joint work with Jesper Grodal)

A p-compact group, as introduced by Dwyer-Wilkerson [7], is a homotopy theoretic version of a compact Lie group, but with all its structure concentrated at a single prime p. In this talk I will announce a proof of the classification of 2-compact groups, joint with J. Grodal, hence completing the classification of p-compact groups at all primes p. Our work will appear in the papers [2] and [3]; a longer and more detailed summary than this can be found in [4].

Recall that a p-compact group is a triple $(X, BX, e : X \xrightarrow{\simeq} \Omega BX)$ where BX is a pointed, connected, p-complete space of the homotopy type of a CW-complex, X satisfies that $H^*(X; \mathbb{F}_p)$ is finite over $\mathbb{F}_p$, and e is a homotopy equivalence. A p-compact group is said to be *connected* if X is a connected space. Our main theorem is the following

Theorem 1 ([3]). *Let $(X, BX, e : X \xrightarrow{\simeq} \Omega BX)$ be a connected 2-compact group. Then*

$$BX \simeq (BG)^{\wedge}_2 \times B\,\mathrm{DI}(4)^s$$

where $(BG)^{\wedge}_2$ is the 2-completion of a connected compact Lie group G, and $B\,\mathrm{DI}(4)$ is the classifying space of the exotic 2-compact group $\mathrm{DI}(4)$ constructed by Dwyer-Wilkerson in [6], $s \geq 0$.

A corresponding statement for odd primes was proved by the authors together with Møller and Viruel in [5]. Partial results for $p = 2$ have been obtained by Dwyer-Miller-Wilkerson, Notbohm, Viruel, and Vavpetič-Viruel. Independently Møller has also announced a proof of the classification relying on computer algebra. Our proof is a self-contained induction.

There is a better, more precise, formulation of our theorem which both makes it clear why it is the correct 2-local version of the classification of compact Lie groups and suggests our strategy of proof. It is based on the notion of root data for p-compact groups as introduced by [9] and further developed in [2].

For a principal ideal domain R, an *R-root datum* is defined to be a triple $(W, L, \{Rb_\sigma\})$, where L is a free R-module of finite rank, $W \subseteq \mathrm{Aut}_R(L)$ is a finite subgroup generated by reflections (i.e., elements σ such that $1 - \sigma \in \mathrm{End}_R(L)$ has rank one), and $\{Rb_\sigma\}$ is a collection of rank one submodules of L, indexed by the reflections σ in W, satisfying

$$\mathrm{im}(1-\sigma) \subseteq Rb_\sigma \text{ and } w(Rb_\sigma) = Rb_{w\sigma w^{-1}} \text{ for all } w \in W$$

The element $b_\sigma \in L$, called the *coroot* corresponding to σ, is determined up to a unit in R. Together with σ it determines a *root* $\beta_\sigma : L \to R$ via the formula $\sigma(x) = x + \beta_\sigma(x)b_\sigma$.

For $R = \mathbb{Z}$ there is a 1-1-correspondence between $\mathbb{Z}$-root data and classically defined root data, by to $(W, L, \{\mathbb{Z}b_\sigma\})$ associating $(L, \{\pm b_\sigma\}, L^*, \{\pm\beta_\sigma\})$; see [9, Prop. 2.16]. Two R-root data $\mathbf{D} = (W, L, \{Rb_\sigma\})$ and $\mathbf{D}' = (W', L', \{Rb'_\sigma\})$ are said to be *isomorphic* if there exists an isomorphism $\varphi : L \to L'$ such that $\varphi W \varphi^{-1} = W'$ and $\varphi(Rb_\sigma) = Rb'_{\varphi\sigma\varphi^{-1}}$. In particular the *automorphism group* is given by $\mathrm{Aut}(\mathbf{D}) = \{\varphi \in N_{\mathrm{Aut}_R(L)}(W) | \varphi(Rb_\sigma) = Rb_{\varphi(\sigma)}\}$ and we define the *outer automorphism group* as $\mathrm{Out}(\mathbf{D}) = \mathrm{Aut}(\mathbf{D})/W$.

We now quickly recall Dwyer-Wilkersons construction [9] of $\mathbb{Z}_p$-root data for p-compact groups. Dwyer and Wilkerson [7] showed that any p-compact group (X, BX, e) has a maximal torus, that is a map $i : BT = (B(S^1)^r)\hat{_p} \to BX$ whose fiber has finite $\mathbb{F}_p$-cohomology and non-trivial Euler characteristic. Replacing i by an equivalent fibration, we define the *Weyl space* $\mathcal{W}_X$ as the topological monoid of self-maps $BT \to BT$ over i and the *Weyl group* as $W_X = \pi_0(\mathcal{W}_X)$. By definition W_X acts on $L = \pi_2(BT)$, and Dwyer-Wilkerson [7] proved that for X connected, this gives a faithful representation of W_X in $\mathrm{Aut}_{\mathbb{Z}_p}(L)$ as a finite $\mathbb{Z}_p$-reflection group. Next let $\mathcal{N}_X$ be the *maximal torus normalizer*, given by $\mathcal{N}_X = \Omega B\mathcal{N}_X$ where $B\mathcal{N}_X = BT_{h\mathcal{W}_X}$. Finally one defines the coroots b_σ in terms of $\mathcal{N}_X$, for details see [9] or [2]. This gives a root datum $\mathbf{D}_X = (W_X, L, \{b_\sigma\})$ over the p-adic integers $\mathbb{Z}_p$ for any connected p-compact group X. (For p odd, a $\mathbb{Z}_p$-root datum is in fact completely determined by the finite $\mathbb{Z}_p$-reflection group (W, L), which explains the formulation of our classification theorem with Møller and Viruel in [5].) We are now ready to state the precise version of our main theorem.

Theorem 2 ([3])**.** *The assignment which to a connected 2-compact group X associates its $\mathbb{Z}_2$-root datum $\mathbf{D}_X$ root gives a one-to-one correspondence between connected 2-compact groups and $\mathbb{Z}_2$-root data. Furthermore there is an isomorphism $\pi_0(\mathrm{Aut}(BX)) \xrightarrow{\cong} \mathrm{Out}(\mathbf{D}_X)$, and $B\,\mathrm{Aut}(BX)$ is the unique total space of a split fibration*

$$B^2\mathcal{Z}(\mathbf{D}_X) \to B\,\mathrm{Aut}(BX) \to B\,\mathrm{Out}(\mathbf{D}_X)$$

Here $\mathcal{Z}(\mathbf{D})$ is the center of the root datum $\mathbf{D}$, defined so as to agree with the formula for the center of a p-compact group given in [8], and $B\,\mathrm{Aut}(BX)$ denotes the classifying space of the topological monoid of self-homotopy equivalences of BX.

The main theorem has a number of corollaries. The most important is perhaps that it gives a proof of the maximal torus conjecture, giving a purely homotopy theoretic characterization of compact Lie groups amongst finite loop spaces.

Theorem 3 (Maximal torus conjecture [3])**.** *The classifying space functor, which to a compact Lie group G associates the finite loop space $(G, BG, e : G \xrightarrow{\simeq} \Omega BG)$ gives a one-to-one correspondence between compact Lie groups and finite loop spaces with a maximal torus. (Moreover, if G is connected we have a split fibration $B^2\mathcal{Z}(\mathbf{D}_G) \to B\,\mathrm{Aut}(BG) \to B\,\mathrm{Out}(\mathbf{D}_G)$.)*

The fact that the functor "B" is faithful was already known by work of Notbohm, Møller, and Osse and the statement about the space $B\,\mathrm{Aut}(BG)$ follows easily from earlier work of Jackowski-McClure-Oliver and Dwyer-Wilkerson. The new, and a priori quite surprising, result here is the statement that if a finite loop space has a maximal torus, then it has to come from a compact Lie group.

Another application of the classification of 2-compact groups is to give an answer to the so-called Steenrod problem for $p = 2$ (see [13] and [12]), which asks which graded polynomial algebras can occur as the mod 2 cohomology ring of a space? Steenrod's problem was solved for p "large enough" by Adams-Wilkerson [1] and for all odd primes by Notbohm [11] using a partial classification of p-compact groups, p odd.

Theorem 4 (Steenrod's problem for $p = 2$ [3])**.** *Suppose that P^* is a graded polynomial algebra over $\mathbb{F}_2$ in finitely many variables. If P^* occurs as $H^*(Y;\mathbb{F}_2)$ for some space Y, then P^* is isomorphic, as a graded algebra, to*

$$H^*(BG;\mathbb{F}_2) \otimes H^*(B\,\mathrm{DI}(4);\mathbb{F}_2)^{\otimes s} \otimes Q^*$$

where G is a connected semi-simple Lie group and Q^* is a polynomial ring with generators in degrees one and two.

In particular if the generators of P^* have degree ≥ 3, then P^* is a tensor product of the following graded algebras:

$\mathbb{F}_2[x_4, x_6, \ldots, x_{2n}]$ (SU(n)), $\mathbb{F}_2[x_4, x_8, \ldots, x_{4n}]$ (Sp(n)),
$\mathbb{F}_2[x_4, x_6, x_7, x_8]$ (Spin(7)), $\mathbb{F}_2[x_4, x_6, x_7, x_8, y_8]$ (Spin(8)),
$\mathbb{F}_2[x_4, x_6, x_7, x_8, x_{16}]$ (Spin(9)), $\mathbb{F}_2[x_4, x_6, x_7]$ (G_2),
$\mathbb{F}_2[x_4, x_6, x_7, x_{16}, x_{24}]$ (F_4), $\mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}]$ (DI(4))

It seems reasonable that one can in fact list all polynomial rings which occur as $H^*(BG;\mathbb{F}_2)$ for G semi-simple, although we have not been able to locate such a list in the literature; for G simple a list can be found in [10].

We finally point out that many classical theorems from Lie theory by Borel, Bott, Demazure, Steinberg, and others also carry over to 2-compact groups via the classification. Applications of this type were already pointed out in [5], to which we refer.

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Weight conjectures and fusion

Geoffrey R. Robinson

Since Alperin's Weight Conjecture (AWC) was formulated, a number of conjectures have attempted to make precise how p-local structure of a finite group determines various invariants of its characteristic p-representation theoretic. One of these is the Ordinary Weight Conjecture (OWC). The key distinction of this conjecture, which is locally equivalent to Dade's Projective Conjecture (DPC), is that for a given block B, only chains of B-subpairs beginning with subpairs (U, b_U) such that $Z(U)$ is a defect group of b_U and $O_p(\frac{N_G(U,b_U)}{UC_G(U)}) = 1$ contribute to the alternating sums appearing in the conjectural formulae. In various contexts, such subpairs play an important role (e.g. control of cohomology, control of fusion), and we call such subpairs "Alperin-Goldschmidt" subpairs.

The relationship between control of fusion and control of representation theoretic invariants is a subtle one. For example, if x is an element of order p such that $C_G(x)$ controls strong p-fusion in G, then there is a bijection between p-blocks of G whose defect group contains a conjugate of x and p-blocks of $C_G(x)$, such that corresponding blocks have equivalent module categories (for $p = 2$ this follows from Glaubeman's Z^*-theorem – for odd p it follows from CFSG, through a direct proof would be desirable). Such a statement would fail in general if $N_G(\langle x\rangle)$ controls fusion.

A key role is played by the Külshammer-Puig extension L_U, an extension of $\frac{N_G(U,b_U)}{UC_G(U)}$ by U such that the fusion of p-subgroups in L_U containing U corresponds to fusion of subpairs in $N_G(U, b_U)$ containing (U, b_U), with various other properties. However, it is important to work with a p-central extension, $\tilde{L}_U$ of L_U, by the Külshammer-Puig 2-cocycle which arises from the action of $\frac{N_G(U,b_U)}{UC_G(U)}$ on the unique simple module in b_U (or its endomorphism ring). For representation-theoretic invariants, these 2-cocycles generally must be confronted.

In alternating sums of OWC, it is possible to calculate the contribution from chains beginning with (U, b_U) by working entirely in $\tilde{L}_U$. The same $\tilde{L}_U$ can occur for many blocks B and many groups G.

In our talk, we calculate the contributions in the case that U is metacyclic. If U is not a defect group, the only cases in which a non-zero contribution is obtained is when $U \cong Q_8$, or $U \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ for $n > 1$, $p \leq 3$, and we calculate the contributions explicitly in that case. In this situation, the Külshammer-Puig 2-cocycle is trivial.

In another direction, we show that if a block B satisfies OWC and contains irreducible characters of very large height (small defect), then for some Alperin-Goldschmidt B-subpair (U, b_U), the Külshammer-Puig extension L_U involves $Qd(p)$ (the semi-direct product of $\mathbb{Z}/p \times \mathbb{Z}/p$ by $SL(2,p)$ with its natural action.

Cohomology primitives associated to central extensions

Nicholas J. Kuhn

1. Introduction

Let p be a fixed prime, and let $H^*(G)$ mean $H^*(G;\mathbb{Z}/p)$. In the mid 1990's, H.-W. Henn, J. Lannes, and L. Schwartz [HLS] studied mod p group cohomology using technology developed to study the categories of unstable modules and algebras over the mod p Steenrod algebra A. Among their discoveries was that the low dimensional cohomology of the centralizers of the elementary abelian p–subgroups of a finite group G determines $H^*(G)$. As an aspect of this, they prove the existence of a 'detection number' $d_0(G)$ defined below.

Thinking about possible extensions of their work has led me to a number of formulae involving cohomology primitives associated to central group extensions. As an application, I am able to calculate $d_0(G)$ when G is 'p-central': a group in which every element of order p is central.

2. The detection number $d_0(G)$

Recall that Quillen considered the category $\mathcal{A}_p(G)$ with objects the elementary p–subgroups of G, and with morphisms the group homomorphisms between objects generated by subgroup inclusions and conjugation by elements in G.

Note that the inclusion $V \to G$ extends to a group homomorphism

$$V \times C_G(V) \to G.$$

For a fixed $d \geq 0$, these induce a map

$$H^*(G) \longrightarrow \prod_{V \in \mathcal{A}_p(G)} H^*(V) \otimes H^{\leq d}(C_G(V)), \tag{1}$$

where $M^{\leq d}$ denotes a graded vector space M^* modulo all elements of degree greater than d. Then $d_0(G)$ is defined as the smallest d such that map (1) is a monomorphism. (That the map is monic for *some* d is shown in [HLS].)

Note that $d_0(G) = 0$ exactly when the $V \in \mathcal{A}_p(G)$ detect all of $H^*(G)$.

A general property is that $d_0(G \times H) = d_0(G) + d_0(H)$. As examples, one has $d_0(\Sigma_n) = 0$ for all n (and all p), $d_0(\mathbb{Z}/p^k) = 1$ if $k > 1$, and $d_0(Q_8) = 3$. As any finite group embeds in a symmetric group, it follows from the first of these that $d_0(G)$ is not, in general, well behaved with respect to subgroup inclusion.

[HLS] use a theorem of J.Duflot to prove rough upper bounds like the following: if G admits a faithful complex representation of (complex) dimension n, then $d_0(G) \leq n^2$. However, the precise calculation of $d_0(G)$, especially in cases where $H^*(G)$ has not itself been completely calculated, has been elusive for most groups.

3. Primitives

Let $C \simeq (\mathbb{Z}/p)^c < G$ be central. The multiplication map $m : C \times G \to G$ induces a map of unstable A–algebras $m^* : H^*(G) \to H^*(G) \otimes H^*(C)$ making $H^*(G)$ into a $H^*(C)$–comodule. ($H^*(C)$ is a Hopf algebra.)

Definition 1. In this situation, we let

$$\begin{aligned} P_C H^*(G) &= \{x \in H^*(G) \mid m^*(x) = x \otimes 1\} \\ &= \text{ equalizer } \{m^*, \pi^* : H^*(G) \to H^*(C \times G)\}. \end{aligned}$$

What sort of beast is $P_C H^*(G)$? We begin by observing that $P_C H^*(G)$ is an unstable algebra, and also that it becomes a $H^*(G/C)$–module via the inflation map $H^*(G/C) \to H^*(G)$.

Recall that the Krull dimension of $H^*(G)$ equals the p–rank of G.

Theorem 2. *$P_C H^*(G)$ is a finitely generated $H^*(G/C)$–module, and thus is a Noetherian ring. It has dimension equal to (the p–rank of G) - (the rank of C).*

4. General formulae involving the primitives

If G is a finite group, we let $C_p(G)$ denote the elementary abelian p–part of its center, and then let $\mathcal{A}_p^C(G)$ denote the full subcategory of $\mathcal{A}_p(G)$ with objects $V < G$ such that $C_p(G) \subseteq V$. The definition of $d_0(G)$ remains the same if we replace $\mathcal{A}_p(G)$ by the smaller category $\mathcal{A}_p^C(G)$ in (1).

The associated twisted arrow category $\mathcal{A}_p^C(G)^{\#}$ has objects $V_1 \xrightarrow{\alpha} V_2$, and morphisms $\alpha \rightsquigarrow \beta$ consisting of commutative squares in $\mathcal{A}_p^C(G)$

$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_2 \\ \downarrow & & \uparrow \\ W_1 & \xrightarrow{\beta} & W_2 \end{array}$$

For M is an unstable A–module, let

$$M_{loc.fin.} = \{x \mid Ax \subset M \text{ is finite dimensional}\}.$$

Proposition 3. $H^*(G)_{loc.fin.} = \lim\limits_{V_1 \xrightarrow{\alpha} V_2 \in \mathcal{A}_p^C(G)^{\#}} P_{\alpha(V_1)} H^*(C_{\alpha(V_1)}(V_2)).$

An unstable module M has a canonical 'nilpotent' filtration

$$\cdots \subseteq nil_2 M \subseteq nil_1 M \subseteq nil_0 M = M.$$

In general, $nil_d M/nil_{d+1} M = \Sigma^d R_d(M)$, where $R_d(M)$ is reduced, i.e, has no nonzero nilpotent elements. We let $\bar{R}_d(M)$ denote the nilclosure of $R_d(M)$.

The module $nil_d H^*(G)$ identifies with the kernel of the map (1). Thus $d_0(G)$ is the length of the filtration of $H^*(G)$, and so also is the biggest d such that $\bar{R}_d(H^*(G)) \neq 0$.

Proposition 4. $\bar{R}_d(H^*(G)) = \lim_{V_1 \xrightarrow{\alpha} V_2 \in \mathcal{A}_p^C(G)^{\#}} H^*(V_1) \otimes P_{\alpha(V_1)} H^d(C_{\alpha(V_1)}(V_2))$.

5. p–central groups

Now suppose that G is p–central, and let $C = C_p(G)$ denote its maximal elementary abelian p–subgroup. The category $\mathcal{A}_p^C(G)^{\#}$ has only one object and morphism, and the propositions of the last section simplify as follows.

Corollary 5. *$H^*(G)_{loc.fin.} = P_C H^*(G)$, and $\bar{R}_d(H^*(G)) = H^*(C) \otimes P_C H^d(G)$. Thus $d_0(G)$ is the degree of the top nonzero primitive class.*

Now we describe how to compute this top degree. For simplicity, assume that G, of p–rank r, cannot be written as $H \times \mathbb{Z}/p$. Then the image of restriction, $Res_G^C : H^*(G) \to H^*(C)$, will be a Hopf algebra of the form $\mathbb{Z}/p[y_1^{p^{e_1}}, \ldots, y_r^{p^{e_r}}]$, where each y_i has degree 2.

Theorem 6. *In this situation, $d_0(G) = \sum_{i=1}^r (2p^{e_i} - 1)$.*

Corollary 7. *If G is p–central, and $H < G$, then $d_0(H) \leq d_0(G)$.*

Examples include the following. If $W(k,p)$ is the universal p–central group with quotient $(\mathbb{Z}/p)^k$ (as studied in papers of Adem, Pakianathan, and Karagueuzian), then $d_0(W(k,p)) =$ the p–rank $= k(k+1)/2$. We note that $H^*(W(k,p))$ has not been completely calculated in all cases. If P is the 2–Sylow subgroup of $SU_3(\mathbb{F}_4)$, a group of order 64 and rank 2, then $d_0(P) = 14$. David Green has recently shown that this last group is the smallest group having nonzero products in its essential cohomology, and the fact that its d_0 is so large seems related.

6. Ideas behind the proofs of the theorems

One studies the Serre spectral sequence for the extension $C \to G \to G/C$. Note that $E_\infty^{0,*} = Im(Res_G^C)$. For all r, $E_r^{*,*}$ is simultaneously an $E_\infty^{0,*} \otimes H^*(G/C)$–module and an $H^*(C)$–comodule, such that

$$E_\infty^{0,*} \otimes E_r^{*,*} \to E_r^{*,*}$$

is a map of $H^*(C)$–comodules.

One can then deduce that, for all r, (a) the composite $P_C E_r^{*,*} \to E_r^{*,*} \to Q_{E_\infty^{0,*}} E_r^{*,*}$ is monic, and (b) $E_r^{*,*}$ is a free $E_\infty^{0,*}$–module.

For Theorem 2, by induction on r, one shows that $E_r^{*,*}$ is a finitely generated $E_\infty^{0,*} \otimes H^*(G/C)$–module. Thus $Q_{E_\infty^{0,*}} E_\infty^{*,*}$ is a finitely generated $H^*(G/C)$–module. By (a), so is $P_C E_\infty^{*,*}$, and Theorem 2 easily follows.

For Theorem 6, by a theorem of Benson and Carlson [BC], $Q_{E_\infty^{0,*}} E_\infty^{*,*}$ is the associated graded of a finite dimensional Poincare duality algebra of degree equal to the formula given in Theorem 6. By (a), we are done, after noting that the top indecomposable class must be represented by a primitive.

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The classification of endo-permutation modules

Serge Bouc

The Dade group : • Let k be a field of characteristic $p > 0$, and let P be a finite p-group. A finitely generated kP-module M is called an *endo-permutation module* if $\mathrm{End}_k(M)$ is a permutation kP-module, i.e. admits a P-invariant k-basis. Such modules are ubiquitous in the modular representation theory of finite groups. In particular, they appear as sources of simple modules for p-solvable groups (Dade, Puig), or in the local study of Morita, stable or derived equivalences between blocks.

• If M is an endo-permutation kP-module, then M is called *capped* if M admits an indecomposable summand with vertex P. In this case, such a summand is unique up to isomorphism, and it is called the *cap* of M. Two capped endo-permutation kP-modules are said to be *equivalent* if their caps are isomorphic. The set of equivalence classes for this relation is denoted by $D(P) = D_k(P)$.

• The *tensor product* $M \otimes_k N$ of two capped endo-permutation kP-modules M and N is again a capped endo-permutation kP-module, and this endows $D(P)$ with an (abelian) group structure. This group was introduced by E.C. Dade (1978), and it is now called the *Dade group* of P (over k). Dade also determined the structure of $D(P)$ when P is abelian.

• In a series of fundamental papers, J. Carlson and J. Thévenaz recently determined the structure of the subgroup $T(P)$ of $D(P)$, consisting of the images of *endo-trivial* modules. This is an essential step in understanding the structure of $D(P)$.

Operations on the Dade group : • Let Q be a subgroup of P. *Restriction* of modules from P to Q induces a group homomorphism $\mathrm{Res}^P_Q : D(P) \to D(Q)$. In the same situation, *tensor induction* of modules from Q to P gives a group homomorphism $\mathrm{Ten}^P_Q : D(Q) \to D(P)$. Now if N is a normal subgroup of P. *Inflation* of modules from P/N to P induces group homomorphism $\mathrm{Inf}^P_{P/N} : D(P/N) \to D(P)$. In the same situation, there is a group homomorphism in the other direction, introduced by Dade as the *slash construction*, that we now call *deflation*, and denote by $\mathrm{Def}^P_{P/N} : D(P) \to D(P/N)$. Finally, if $\varphi : P \to P'$ is a *group isomorphism*, there is a corresponding group isomorphism $\mathrm{Iso}(\varphi) : D(P) \to D(P')$.

• All these operations are subject to various composition conditions, e.g. transitivity of restrictions and tensor inductions, Mackey formula, etc... They can be unified in a single formalism, using *bisets* : if P and Q are finite p-groups, if U is a finite (Q,P)-biset, then one can define a group homomorphism $D(U) : D(P) \to D(Q)$ associated to U, and one recovers all the previous operations for some specific biset in each case.

• There one more operation on the Dade group, which appears when composing the above operations. It is a map $\gamma_a : D(P) \to D(P)$ associated to any endomorphism a of the ground field k, called *Galois torsion*. This map has to be taken into account for theoretical reasons, though the final classification theorem shows that it is almost always the identity map.

Biset functors : • A biset functor F (over p-groups) consists of the following data : for each finite p-group P, an abelian group $F(P)$, and for each finite (Q,P)-biset U (where Q and P are finite p-groups), a group homomorphism $F(U) : F(P) \to F(Q)$. These data are subject to the following four conditions :

(1) If U and U' are isomorphic (Q,P)-bisets, then $F(U) = F(U')$.
(2) If $U = A \sqcup B$ is the disjoint union of two (Q,P)-bisets A and B, then $F(U) = F(A) + F(B)$.
(3) If U is the (P,P)-biset P for left and right multiplication, then $F(U)$ is the identity map.
(4) If P, Q, and R are finite p-groups, if U is a (Q,P)-biset and V is an (R,Q)-biset, then $F(V) \circ F(U) = F(V \times_Q U)$.

• There is an obvious notion of *natural transformation* or *morphism* of biset functors. So biset functors over p-groups form a category $\mathcal{F}_p$. This category is an *abelian category.*

Examples of biset functors : K, B, $R_{\mathbb{Q}}$: • If P is a finite p-group, denote by $B(P)$ the Burnside group of P. If Q is a finite p-group, and if U is a finite (Q,P)-biset, denote by $B(U) : B(P) \to B(Q)$ the map induced by the correspondence sending a finite P set X to the finite Q-set $U \times_P X$. Then B is a biset functor, called the *Burnside functor*.

• In the same situation, denote by $R_{\mathbb{Q}}(P)$ the group of rational representations of P. Denote by $R_{\mathbb{Q}}(U)$ the map induced by the correspondence sending a finite dimensional $\mathbb{Q}P$-module M to the finite dimensional $\mathbb{Q}Q$-module $\mathbb{Q}U \otimes_{\mathbb{Q}P} M$. Then $R_{\mathbb{Q}}$ is a biset functor, called *the functor of rational representations*.

• The natural morphism $\chi_P : B(P) \to R_{\mathbb{Q}}(P)$ sending the isomorphism class of the finite P-set X to the class of the permutation module $\mathbb{Q}X$ is surjective, by the Ritter-Segal theorem, since P is a p-group. This gives a surjective morphism $\chi : B \to R_{\mathbb{Q}}$ in the category $\mathcal{F}_p$.

• Set $K = \operatorname{Ker}\chi$, i.e. $K(P) = \operatorname{Ker}\chi_P$, for any P. This gives the following short exact sequence in $\mathcal{F}_p$:

$$0 \to K \to B \to R_{\mathbb{Q}} \to 0$$

• **An *almost* example :** The correspondence $P \mapsto D(P)$ is *not* a biset functor in general, because of Galois torsions. Fortunately, there is a big enough subobject of D which *is* a biset functor.

From K to D : • Let X be a finite P-set. Denote by ω_X the linear form on $B(P)$, with values in $\mathbb{Z}$, defined by $\omega_X(P/Q) = 1$ if $X^Q \neq \emptyset$, and $\omega_X(P/Q) = 0$ otherwise. In the same situation, denote by Ω_X the kernel of the augmentation map $kX \to k$. Then Ω_X is an endo-permutation module (Alperin), called the *syzygy* of the trivial module relative to X. Denote by $D^\Omega(P)$ the subgroup of $D(P)$ generated by the images in $D(P)$ of all these Ω_X. Then the correspondence $P \mapsto D^\Omega(P)$ is a biset functor, called *functor of relative syzygies* in the Dade group.

• The link between K and the Dade group is a kind of duality, given by the following theorem :

• **Theorem :**

* *Let B^* denote $\mathbb{Z}$-dual of B. Then $B^* \in \mathcal{F}_p$, and there is a surjective natural transformation $\Theta : B^* \to D^\Omega$, such that $\Theta_P(\omega_X) = \Omega_X$, for any p-group P and any finite P-set X.*

* *This natural transformation gives rise to a short exact exact sequence in $\mathcal{F}_p$:*

$$0 \to R_\mathbb{Q}^* \to B^* \to D^\Omega/D^\Omega_{tors} \to 0 \ ,$$

where $R_\mathbb{Q}^*$ is the $\mathbb{Z}$-dual of $R_\mathbb{Q}$, and D^Ω_{tors} is the torsion part of D^Ω_{tors}.

Generating K : • **Notation :**

(1) If $p \neq 2$, denote by X_{p^3} an extraspecial group of order p^3 and exponent p, and by Z its center. Choose two non conjugate non central subgroups I and J of order p in X_{p^3}. Let δ be the element of $B(X_{p^3})$ defined by $\delta = (X_{p^3}/I - X_{p^3}/IZ) - (X_{p^3}/J - X_{p^3}/JZ)$.

(2) If $p = 2$, and if $n \geq 3$ is an integer, denote by D_{2^n} a dihedral group of order 2^n, and by Z its center. Choose two non conjugate non central subgroups I_n and J_n of order 2 in D_{2^n}. Let δ_n be the element of $B(D_{2^n})$ defined by $\delta_n = (D_{2^n}/I_n - D_{2^n}/I_nZ) - (D_{2^n}/J_n - D_{2^n}/J_nZ)$.

• **Theorem :** *If $p \neq 2$ (resp. if $p = 2$), then the functor K is generated by δ (resp. by all the δ_n's, for $n \geq 3$).*

• **Corollary :** *$D = D^\Omega + D_{tors}$. If moreover $p \neq 2$, then $D = D^\Omega$.*

Genetic subgroups : • When studying biset functors for p-groups, some special subgroups of a p-group P, called *genetic subgroups*, appear as an important tool. These are defined as follows : if Q is a subgroup of P, let $Z_P(Q)$ denote the subgroup of $N_P(Q)$ defined by $Z_P(Q)/Q = Z\Big(N_P(Q)/Q\Big)$. The subgroup Q is called *genetic* if the following two conditions are satisfied : firstly, the group $N_P(Q)/Q$ has normal p-rank 1. And secondly, if $x \in P$, then $Q^x \cap Z_P(Q) \subseteq Q$ if and only if $Q^x = Q$. The isomorphism class of the group $N_P(Q)/Q$ is called the *type* of Q.

• Two genetic subgroups Q and R are said to be *linked modulo* P (notation $Q \,\textbf{—}_P\, R$), if there exist elements x and y in P such that $Q^x \cap Z_P(R) \subseteq R$ and $R^y \cap Z_P(Q) \subseteq Q$. In this case in particular Q and R have the same type. The relation $\textbf{—}_P$ is an equivalence relation on the set of genetic subgroups of P. A *genetic basis* of P is a set of representatives of these equivalences classes.

The structure of $D(P)$: • Let $[s_P]$ denote a set of representatives of conjugacy classes of subgroups of P, and let $\mathcal{G}$ be a genetic basis of P. Using results of Carlson-Thévenaz and B.-Mazza, one can show the following :

• **Theorem :** *The natural map* $\underset{S\in\mathcal{G}}{\oplus} T_{tors}(N_P(S)/S) \to D_{tors}(P)$ *obtained by inflation and tensor induction, is a group isomorphism.*

• Let $(\delta_{P/Q})_{Q\in[s_P]}$ denote the basis of $B^*(P)$, dual to the canonical basis of $B(P)$. Denote by $\Delta_{P/Q}$ the image of $\delta_{P/Q}$ in $D^\Omega(P)$ by the map Θ_P. If $p = 2$, let $\mathcal{Q}$ be the subset of $\mathcal{G}$ consisting of subgroups Q of generalized quaternion type (of order at least 16 if k has no non trivial cubic roots of unity). To each $Q \in \mathcal{Q}$, one can associate an element Λ_Q in $D(P)$, called *exotic*, defined by $\Lambda_Q = \mathrm{Ten}_{N_P(Q)}^P \mathrm{Inf}_{T_Q}^{N_P(Q)} \eta_{T_Q}$, where $T_Q = N_P(Q)/Q$, and η_{T_Q} is one of the two elements of order 2 in $D(T_Q) - D^\Omega(T_Q)$ (Carlson-Thévenaz).

• **Theorem :** *The group $D(P)$ is generated by the elements $\Delta_{P/Q}$, for $Q \in [s_P]$, and by the elements Λ_Q, for $Q \in \mathcal{Q}$ (when $p = 2$), subject to the following relations :*

* *one relation between the $\Delta_{P/Q}$'s, for each $S \in \mathcal{G}$, namely*

$$\tau_S \sum_{Q\in[s_P]} (a_S i_P(Q,S) + j_P(Q,S))\Delta_{P/Q} = 0 \ ,$$

where, setting $T_S = N_P(S)/S$, the integer τ_S is equal to 1 if T_S is of order at most 2 or dihedral, to 2 if T_S is cyclic of order at least 3 or semi-dihedral, and to 4 if T_S is generalized quaternion, and the integer a_S is equal to 1 if T_S is cyclic or generalized quaternion, and to 2 if T_S is dihedral or semi-dihedral. Moreover, the integer $i_P(Q,S)$ is the number of $x \in Q\backslash P/N_P(S)$ such that $Q^x \cap N_P(S) \subseteq S$, whereas $j_P(Q,S)$ is the number of such x such that the group $(Q^x \cap N_P(S))S/S$ has order p and is not contained in the center of $N_P(S)/S$.

* $2\Lambda_Q = 0$*, for each $Q \in \mathcal{Q}$.*

Moreover this is a presentation of $D(P)$ as an abelian group.

• **Corollary :** *There is a group isomorphism*

$$D(P) \cong \mathbb{Z}^{nc_P} \oplus (\mathbb{Z}/4\mathbb{Z})^{a_P} \oplus (\mathbb{Z}/2\mathbb{Z})^{b_P} \ ,$$

where nc_P is the number of conjugacy classes of non cyclic subgroups of P, where a_P is the number of elements of $\mathcal{G}$ whose type is generalized quaternion, and b_P is the number of elements of $\mathcal{G}$ whose type is cyclic of order at least 3, semidihedral, or generalized quaternion (of order at least 16 if k has only one cubic root of unity).

• **Corollary :** *"Endo-permutation kP-modules can be lifted to endo-permutation $\mathcal{O}P$-lattices."*

• The following completes the proof of the two conjectures I proposed in 2003 in Oberwolfach meeting "Darstellungen endlicher Gruppen" :

• **Corollary :** *When $p = 2$, the quotient D/D^Ω is a biset functor, isomorphic to the subfunctor H_Q of $\mathbb{F}_2 R_\mathbb{Q}$, where $Q = Q_8$ if k has all cubic roots of unity, and $Q = Q_{16}$ otherwise.*

Approaching the Alperin weight conjecture

Jeff Smith

The weight conjecture relates the modular representation theory of a finite group to the representation theory of its local subgroups. Let G be a finite group. A weight is a conjugacy class of pairs (D, P) where D is a p-group in G and P is a projective simple module over $\bar{\mathbb{F}}_p ND/D$. The weight conjecture is that :

$$\# \text{ of simple } \bar{\mathbb{F}}_p G\text{-modules} = \# \text{ of weights.}$$

This approach attempts to prove the weight conjecture when the subgroup complex of G is simply connected.

Let $X_p G$ be the nerve of the partially ordered set of non-trivial p-subgroups of G. Let $L_G = \mathrm{Hom}_G(\mathbb{F}_p X_p G, \mathrm{Hom}(\bar{\mathbb{F}}_p G, \mathbb{F}_p G))$. L_G is a cosimplicial $\bar{\mathbb{F}}_p$-algebra. The weight conjecture would follow from two assertions in the situation that $X_p G$ is simply connected :

(a) $rank K_0 L_G = \#$ of weights with non-trivial defect
(b) L_G is Rickard equivalent to $B =$ sum block of $\mathbb{F}_p G$ that have non-trivial defect.

Cohomology for quantum groups

Daniel K. Nakano

(joint work with Christopher P. Bendel, Brian J. Parshall and Cornelius Pillen)

Given a semisimple algebraic group G over an algebraically closed field k of characteristic $p > 0$, one of the central problems is to find a character formula for the finite-dimensional simple G-modules. For $p \geq h$, a character formula is given by a conjectural formula due to Lusztig.

In the analogous world of quantum groups over the complex numbers, this formula is verified for $l \geq h$ where ζ is a primitive lth root of unity. The first proof used an equivalence of categories due to Kazhdan and Lusztig between quantum groups and affine Lie algebras. Kashiwara and Tanisaki subsequently verified the character formula for affine Lie algebras. A second proof was found recently by Arkhipov, Bezrukavnikov and Ginzburg [ABG]. One of the key components of their proof involved employing the computation of the cohomology of quantum groups for $l > h$ due to Ginzberg and Kumar [GK] in 1993.

The main purpose of our work is to demonstrate how to compute cohomology for quantum groups when $l \leq h$. This computation entails many beautiful results:

1) Realization of the "restricted nullcone" due to Carlson, Lin, Nakano and Parshall [CLNP].

2) Combinatorics involving the decomposition of the exterior algebra via the Steinberg representation. Our decomposition results makes use of MAGMA computations on root systems for exceptional Lie algebras.

3) Powerful vanishing results on line bundle cohomology proved via complex algebraic geometry (i.e. Grauert-Riemenschnieder theorem).

4) Normality results on the closures of nilpotent orbits due to Kraft-Procesi [KP1, KP2], Sommers [So1, So2], Broer [Br], Kumar-Lauritzen-Thomsen [KLT].

Our results show that the cohomology ring is finitely generated. This allows us to define support varieties and compute the support varieties for quantum Weyl modules in the case when $(l, p) = 1$ where p is any bad prime for the underlying root system.

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Rank varieties for a class of finite-dimensional local algebras

Karin Erdmann

(joint work with D. Benson and M. Holloway)

Assume k is an algebraically closed field. The theory of the rank variety associated to a finite-dimensional module over a group algeba of an elementary abelian p-group when k has characteristic $p > 0$ was introduced by Jon Carlson [3]. Since then the theory has proven to be extremely sueful, and analogous structures have been discovered in several other settings, especially p-restricted Lie algebras, and finite group schemes.

We develop a rank variety for a class of finite-dimensional k-algebras, with k not necessarily of prime chacteristic, which can be viewed as quantum complete intersections. Included in this class are the truncated polynomial algebras $\Lambda_m^n := k[X_1, \ldots, X_n]/(X_i^n)$. This generalizes [4] where the case $n = 2$ was considered.

Let A be a finite-dimensional k-algebra, we would like to define for any A-module M, a 'rank variety' $V_A^r(M)$, which can be explicitly computed, and which has as many of the properties that hold in the group algebra case as make sense. Apart from additivity, and the usual relationship between terms of short exact sequences, we require that 'Dade's lemma' should hold. That is, is $V_A^r(M)$ is trivial if and only if M is projective.

Let $\underline{q} = [q_{ij}] \in \mathrm{Mat}_m(k)$ be a commutation matrix, that is $q_{ii} = 1$ for all i and $q_{ij}q_{ji} = 1$ for $i \neq j$. The quantum symmetric algebra, as introduced by Manin, is the algebra $k_{\underline{q}}[X] := k\langle X_1, \ldots, X_m\rangle/(X_iX_j - q_{ij}X_jX_i)$. We assume that all q_{ij} are roots of unity, and let $r \geq 1$ such that $q_{ij}^r = 1$ for all i, j. For $n \geq 2$, let

$$A_{\underline{q}}^n := k_{\underline{q}}[X]/(X_i^n)$$

a 'quantum complete intersection'.

We start with the special case for which rank varieties can be defined most naturally. Let $A := A_\zeta$ be the quantum complete intersection with $q_{ij} = \zeta$ for $i < j$ where ζ is a primitive n'-th root of unity, for n' the p'-part of n.

For $0 \neq \underline{\lambda} := (\lambda_1, \ldots, \lambda_m) \in k^m$, define $u_{\underline{\lambda}} := \sum_{i=1}^m \lambda_i X_i \in A$; it is crucial that $u_{\underline{\lambda}}^n = 0$ for any such $\underline{\lambda}$. Let $k[u_{\underline{\lambda}}]$ be the subalgebra of A generated by $u_{\underline{\lambda}}$. For an A-module M we define

$$V^r(M) := \{0\} \cup \{\underline{\lambda} \neq 0 :\in k^m : M \text{ as a module for } k[u_{\underline{\lambda}}] \text{ is not projective } \}.$$

This is a homogeneous affine variety and when $n = p$ we recover the original definition by Jon Carlson. By adapting the proof of Dade's Lemma from [1] we show:

Theorem *$V^r(M) = \{0\}$ if and only if M is projective.*

We extend the classes of algebras for which rank varieties are defined, using the following setup.

Theorem *Let A and A' be two finite-dimensional self-injective k-algebras and suppose the following conditions hold.*

(1) There exists a rank variety theory for A-modules, V_A^r.

(2) There exists a finite-dimensional k-algebra B and a Morita equivalence G : A'-mod $\rightarrow$ $B-$mod .

(3) A is a subalgebra of B such that B is projective as an A-module, and moreover every B-module is relative A-projective.

Then A'-mod has rank varieties, defined by $V_{A'}^r(M) := V_A^r(GM\downarrow_A)$. If V_A^r satisfies Dade's Lemma then so does $V_{A'}^r$.

We apply this with $A' = A_{\underline{q}}[X]/(X_i^n)$, and $A = A_\zeta$. We construct a simple algebra C, which can be viewed as a 'generalized Clifford algebra' such that the tensor product $B := C \otimes_k A'$ has a subalgebra isomorphic to A_ζ, satisfying the hypotheses of this second theorem. Hence the above quantum complete intersections have rank varieties, satisfying Dade's Lemma.

As a corollary we deduce that 'Webb's theorem' holds for these quantum complete intersections. This describes the possible tree classes of the stable Auslander Reiten quiver. Furthermore, the class of these quantum complete intersections contain the algebras defined by Benson and Green in [2]. They define rank varieties by identifying their algebra with a block of a group algebra, and then using Carlson's definition. Our rank variety is isomorphic to theirs.

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Levels in triangulated categories and finite group actions

Srikanth B. Iyengar

(joint work with Luchezar Avramov, Ragnar-Olaf Buchweitz, Claudia Miller)

The results reported in this talk are motivated in part by the following theorem proved by Carlsson [5], when $p = 2$, and by Allday and Puppe [1], when p is odd.

Theorem 1. Let k a field of positive characteristic p, and let G be an elementary abelian p-group of rank c. If G acts freely on a finite CW-complex X, then

$$\sum_{n\in\mathbb{Z}} \ell\ell_{k[G]} \operatorname{H}_n(X; k) \geq c + 1$$

In the statement, $\operatorname{H}_*(X; k)$ is the singular homology of X with coefficients in k. For any $k[G]$-module H, the number

$$\ell\ell_{k[G]}(H) = \inf\{i \geq 0 \mid \mathfrak{m}^i H = 0\}$$

where $\mathfrak{m}$ is the augmentation ideal of $k[G]$, is the *Loewy length* of H.

Now, for G as in the theorem, the group algebra $k[G]$ is isomorphic to the commutative local ring $R = Q/I$, where $Q = k[x_1, \dots, x_c]$ and $I = (x_1^p, \dots, x_c^p)$. There is a natural R-module structure on I/I^2, and it is not hard to check that, for this structure, $I/I^2 \cong R^c$. Moreover, the hypotheses on X implies that, as a complex of R-modules, the chain complex that computes singular cohomology is equivalent to a finite free complex. Thus, Theorem 1 is a special case of

Theorem 2. Let R be a local ring and let $R = Q/I$ where $(Q, \mathfrak{q}, k)$ is a local ring with $I \subseteq \mathfrak{q}^2$. If F is a finite free complex of R-modules with $\mathrm{H}(F) \neq 0$, then

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \,\mathrm{H}_n(F) \geq \text{f-rank}_R(I/I^2) + 1$$

where $\text{f-rank}_R(I/I^2)$ denotes the maximal rank of R-free summands of I/I^2.

This result reveals an unexpected relationship between (free ranks of) conormal modules and the homology of finite free complexes. In [2], it is used to derive new lower bounds on Loewy lengths of modules of finite projective dimension.

In my talk I gave an overview of the proof of Theorem 2. A crucial new ingredient in it is the consideration of numerical invariants of objects in arbitrary triangulated categories, called *levels*. Their introduction is motivated in part by work of Dwyer, Greenlees, and Iyengar [6], where a notion of *building* objects (modules, complexes, etc.) from a given one was transported into commutative algebra from algebraic topology; levels provide a way to *quantify* the complexity of the "building process".

Levels can be defined with respect to an arbitrary class of objects: given a non-empty class $\mathcal{C}$ in a triangulated category $\mathcal{T}$, an object $T \in \mathcal{T}$ has $\mathrm{level}^{\mathcal{C}}_{\mathcal{T}}(T) \leq n$ if it is isomorphic to a direct summand of an n-fold extension of finite direct sums of shifts of objects in $\mathcal{C}$. The utility of this notion was suggested to us by the spectacular recent work of Rouquier [10, 11], see also Bondal and Van den Bergh [4], on the representation dimension, in the sense of Auslander, of exterior algebras.

Two levels in the derived category $\mathcal{D}R$ of R-modules play a special role in this work: one, with respect to the class of simple modules, tracks Loewy length; the other, with respect to the class of projective modules, tracks projective dimension. It is remarkable that homological invariants, such as projective dimension, as well as ring theoretic invariants, such as Loewy length, are captured by the same formalism. This attests to the flexibility afforded by the notion of levels.

To elaborate on this point, we consider levels with respect to k, the residue field of R. If C is a complex of R-modules, then

$$(\dagger) \qquad \sum_{n\in\mathbb{Z}} \ell\ell_R \,\mathrm{H}_n(C) \geq \mathrm{level}^k_{\mathcal{D}R}(C) + 1 \geq \max_{n\in\mathbb{Z}}\{\ell\ell_R \,\mathrm{H}_n(C)\}$$

In particular, for an R-module M one has

$$\mathrm{level}^k_{\mathcal{D}R}(M) = \ell\ell_R M - 1$$

On the other hand, levels with respect to R have the property that every finite free complex $F : 0 \to F_l \to \cdots \to F_0 \to 0$ with $\mathrm{H}(F) \neq 0$ satisfies an inequality:

$$(\ddagger) \qquad l \geq \mathrm{level}^R_{\mathcal{D}R}(F)$$

The inequality is typically strict; however, when F is the free resolution of an R-module M, a result of Krause and Kussin [9] implies that

$$\mathrm{level}^R_{\mathcal{D}R}(M) = \mathrm{level}^R_{\mathcal{D}R}(F) = l = \mathrm{proj\,dim}_R\, M$$

Now, if A is a DG algebra, the function $\mathrm{level}^{A}_{\mathcal{D}A}(-)$, defined on the derived category $\mathcal{D}A$ of DG modules over A, provides an analogue of projective dimension of modules. Indeed, many of the formal properties of projective dimension for modules over rings extend to this setting.

An outline of the proof of Theorem 2. The argument involves a passage from the derived category $\mathcal{R}$ of R-modules, to the derived category $\mathcal{S}$ of DG modules over a polynomial ring $S = k[x_1, \cdots, x_c]$, where k is the residue field of R, the degree of x_i is -2 for each i, and $c = \text{f-rank}_R(I/I^2)$. The transition from $\mathcal{R}$ to $\mathcal{S}$ is via a chain of exact functors of triangulated categories:

$$\mathcal{R} \xrightarrow{\mathsf{K}=K\otimes_R -} \mathcal{K} \xrightarrow[\sim]{\mathsf{B}} \mathcal{B} \xrightarrow{\mathsf{L}} \mathcal{L} \xrightarrow[\sim]{\mathsf{S}} \mathcal{S}$$

In this diagram $\mathcal{K}$ is the derived category of DG modules over the Koszul complex K on a minimal generating set for the maximal ideal of R. The presence of a free summand of rank c in I/I^2 entails that K is quasi-isomorphic to a DG algebra B of the form $C \otimes_k \Lambda$, where Λ is an exterior algebra on the k-vectorspace k^c in degree 1; this was proved in [8]. This quasi-isomorphism induces an equivalence B between $\mathcal{K}$ and $\mathcal{B}$, the derived category of DG modules over B. The inclusion of DG algebras $\Lambda \to C \otimes_k \Lambda$ yields the functor L from $\mathcal{B}$ to $\mathcal{L}$, the derived category of DG modules over Λ. The equivalence S is a DG version of the Bernstein-Gelfand-Gelfand [3] correspondence.

Let N be the image of the complex F, in the statement of Theorem 2, under the composite functor $\mathcal{R} \to \mathcal{S}$. We prove:

(a) F finite free implies that the graded S-module $\mathrm{H}(N)$ has finite length;

(b) $\mathrm{level}^{k}_{\mathcal{D}R}(F) \geq \mathrm{level}^{S}_{\mathcal{D}S}(N) + 1$.

The proof is based on the fact that, in contrast to many homological invariants, levels behave predictably under changes of categories. To complete the proof of Theorem 2 it remains to recall (†), and apply the next result:

Theorem 3. Let S be a graded noetherian commutative ring with S_0 a local ring containing a field. If N is a DG S-module with $\mathrm{length}_S \mathrm{H}(N)$ finite and non-zero, then $\mathrm{level}^{S}_{\mathcal{D}S}(N) \geq \dim S$.

This result improves upon the New Intersection Theorem [7] for local rings containing a field: if S is concentrated in degree 0 and $0 \to P_l \to \cdots \to P_0 \to 0$ is a finite free complex with homology non-zero and of finite length, inequality (‡) and Theorem 3 yield $l \geq \dim S$.

The proof of Theorem 3 builds on an idea of Carlsson [5], by using big Cohen-Macaulay modules constructed by Hochster [7]; this remark explains the hypothesis that S contains a field.

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Group actions on homogically finite dimensional spaces

William Browder

Much of the technology of the homological study of finite group actions necessitates geometrical assumptions such as finite dimensionality or stronger assumptions. On might ask whether there is some body of results which can be proved simply assuming that the homology of the space is finite dimensionality. We study an even weaker content : let G be a finite p-group and C a projective $\mathbb{Z}G$ chain complex with $C_i = 0$ for $i < 0$ and $H_i(C) = 0$ for $i > n$.

Definition 1. A 'fixed point' for G on C is a splitting of G-chain complexes $C \rightleftarrows P[n]$ where P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and $[n]$ indicates the shift by n of the grading (we say this is a 'fixed point' of degree n).

The existence of a fixed point implies that C_G is not homologically finite dimensional. Note that :

Theorem 2. If C/G is homologically finite dimensional, then $\dim C/G = \dim C$ (i.e. the dimension of the highest non-zero homology group).

(This is proved by considering $C \otimes D$ where $D = C(W)$ where W is a closed manifold, n connected for large n (>> hom dim of C/G) and comparing the two spectral sequences, filtering on $\dim C$ or $\dim D$.)

Theorem 3. If $n = \text{hom}\dim C$ and $H_n(C) \cong \mathbb{Z}$ and if $j^* : H^n(C/G) \to H^n(C)$ is surjective then G has a degree n fixed point in C.

¿From this one can derived :

Theorem 4. If $n = \text{hom}\dim C$, $H_n(C)$ and if C/G is hom. finite dimensional then degree $j : C \to C/G$ is $|G|$.

In particular if $j^*(H^n(C/G))$ is $qH^n(C)$, $q < |G|$ then some $\mathbb{Z}/p \subset G$ has a fixed point on C.

Conjecture 5. If $G = (\mathbb{Z}/p)^r$ and $j^*H^n(C/G) = p^k H^n(C)$ then some subgroup isomorphic to $(\mathbb{Z}/p)^{r-k}$ has a fixed point on C.

I proved such a theorem about 1990 with a strong geometrical hypothesis, of G acting on a connected oriented compact near manifold (i.e. a space which is a manifold off a subset of codimension 2).

Bockstein Closed Central Extensions of Elementary Abelian 2-Groups; Binding Operators

Ergün Yalçin

(joint work with Jonathan Pakianathan)

We consider central extensions of the form

$$E : 0 \to V \to G \to W \to 0$$

where V and W are elementary abelian 2-groups. Let n and m denote the ranks of V and W respectively, and let $q \in H^2(W, V)$ be the extension class for E. Choosing a basis for V, we can write $q = (q_1, \dots, q_n)$ with $q_i \in H^2(W, \mathbb{Z}/2)$. The elements $q_1, \dots, q_n$ are called the components of q. Note that if $\{x_1, \dots, x_m\}$ is a basis for the dual space of W, the components of q can be considered as homogeneous quadratic polynomials in $x_1, \dots, x_m$.

Definition 1. Let $I(q) \subseteq H^*(W, \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_m]$ denote the ideal generated by the components of q. We say E is Bockstein closed if the ideal $I(q)$ is closed under the Bockstein operator $\beta : H^*(W, \mathbb{Z}/2) \to H^{*+1}(W, \mathbb{Z}/2)$ (or equivalently if it is Steenrod closed).

It is easy to see that E is a Bockstein closed extension if and only if there is a one dimensional class $L \in H^1(W, \mathrm{End}(V))$ such that $\beta(q) = Lq$. Here, the multiplication Lq is given by the composite

$$H^1(W, \mathrm{End}(V)) \otimes H^2(W, V) \xrightarrow{\cup} H^3(W, \mathrm{End}(V) \otimes V) \xrightarrow{(ev)^*} H^3(W, V)$$

where the second map is induced by the evaluation map $ev : \mathrm{End}(V) \otimes V \to V$. If L is a class satisfying the equation $\beta(q) = Lq$, we say L is a binding operator for E. Note that it is possible to have more then one binding operator for a given Bockstein closed extension.

A rich source of Bockstein closed extensions comes from matrix groups. Let G be the kernel of the mod 2 reduction map $GL_n(\mathbb{Z}/8) \to GL_n(\mathbb{Z}/2)$. It is easy to show that G fits into a central extension of the form

$$E : 0 \to \mathfrak{gl}_n(\mathbb{F}_2) \to G \to \mathfrak{gl}_n(\mathbb{F}_2) \to 0$$

where $\mathfrak{gl}_n(\mathbb{F}_2)$ is the vector space of $n \times n$-matrices over $\mathbb{F}_2$. In this case the binding operator L corresponds to the homomorphism $L : W \to \mathrm{End}(V)$ defined by $L(A)(B) = AB + BA$. So, E is Bockstein closed (we actually verify this using a more general theorem we prove for Bockstein closed quadratic maps $Q : W \to V$).

Note that restrictions of E to suitable subspaces of $\mathfrak{gl}_n(\mathbb{F}_2)$, such as $\mathfrak{sl}_n(\mathbb{F}_2)$ or $\mathfrak{u}_n(\mathbb{F}_2)$, also give us Bockstein closed extensions.

The cohomology of the extension group has been studied for various Bockstein closed extensions of elementary abelian groups. Under the assumption that the extension is 2-power exact (Frattini, effective, and $\dim V = \dim W$), the mod p cohomology calculations have been completed by Browder and Pakianathan [1] for $p > 2$, and by Minh and Symonds [2] for $p = 2$. In this research, we concentrate on the connections between the binding operators and the uniform liftings of Bockstein extensions.

It is well known that Bockstein closed extensions are precisely the extensions which uniformly lift to extensions of the form

$$\widetilde{E} : 0 \to M \to \widetilde{G} \to W \to 0,$$

where M is a $\mathbb{Z}/4$-free $\mathbb{Z}/4[W]$-module ($\mathbb{Z}/4[W]$-lattice). By uniform lifting, we mean that the extension $0 \to M/2M \to \widetilde{G}/2M \to W \to 0$ is equivalent to E. In this case M is called a lifting module for E. Note that a lifting module M has the property that $2M \cong M/2M \cong V$ is a trivial $\mathbb{Z}/2[W]$-module. We are interested with the following question: Given a $\mathbb{Z}/4[W]$-lattice M having the property $2M \cong M/2M \cong V$, is there a direct way to tell if it is a lifting module for E? We give the following nice characterization:

Theorem 2. *Let $E : 0 \to V \to G \to W \to 0$ be a central extension with V and W are elementary abelian 2-groups. Suppose M is a $\mathbb{Z}/4[W]$-lattice with $2M \cong M/2M \cong V$. Associated to M there is a cohomology class $L_M \in H^1(W, \mathrm{End}(V))$, defined in a specific way, such that M is a lifting module for E if and only if L_M is a binding operator for E.*

We now describe how the class L_M is defined. Let $\rho_M \in \mathrm{Hom}(W, \mathrm{Aut}(M))$ denote the representation associated to M. Since both $2M$ and $M/2M$ are trivial $\mathbb{Z}/2[W]$-modules we can write

$$\rho_M = I_M + 2 \log \rho_M$$

with $\log \rho_M$ is defined only modulo 2. We consider $\log \rho_M$ as an homomorphism from W to $\mathrm{End}(V)$, and define the class L_M in $H^1(W, \mathrm{End}(V))$ as the image of $\log \rho_M$ under the isomorphism

$$\mathrm{Hom}(W, \mathrm{End}(V)) \xrightarrow{\cong} H^1(W, \mathrm{End}(V)).$$

Theorem 2 can be used in many ways to study Bockstein closed extensions. In particular, we use it to study diagonalizable and triangulable extensions. An extension E is called diagonalizable if there exists a basis for V such that the components of q are individually Bockstein closed. There is a well known result that a homogeneous quadratic polynomial is Bockstein closed if and only if it is product of two linear polynomials. So, the extension class of a diagonalizable extension has a very special form. We find the following equivalent statements to diagonalizability of extensions:

Theorem 3. *Let $E : 0 \to V \to G \to W \to 0$ be a central extension with V and W are elementary abelian 2-groups. The following are equivalent:*
(i) E is diagonalizable.
(ii) There is a choice of basis of V, such that the components q_i of the extension class q all decompose as $q_i = u_i v_i$ where u_i, v_i are linear polynomials.
(iii) There exists a diagonalizable binding operator $L \in \mathrm{Hom}(W, \mathrm{End}(V))$ such that $\beta(q) = Lq$. This is characterized exactly by the equation $\beta(L) + L^2 = 0$.
(iv) E lifts uniformly to an extension $0 \to M \to \widetilde{G} \to W \to 0$ where M is a $\mathbb{Z}/4[W]$-lattice which is a direct sum of one dimensional $\mathbb{Z}/4[W]$-lattices.
(v) E lifts uniformly to an extension $0 \to M \to \widetilde{G} \to W \to 0$ where M is a $\mathbb{Z}/8[W]$-lattice.
(vi) E lifts uniformly to an extension $0 \to M \to \widetilde{G} \to W \to 0$ where M is a $\mathbb{Z}[W]$-lattice.

Some of the implications in Theorem 3 are well known to specialist and they are proved using spectral sequence arguments. We provide easier arguments which only require simple extension theory. The most interesting implication above is $(vi) \Rightarrow (i)$ which says the Bockstein closed extensions coming from integral extensions are always diagonalizable. We use Theorem 2 to prove this, and we do not know any other way to prove this statement.

Another striking consequence of Theorem 3 is that using the characterization given in (iii), one can easily construct Bockstein closed extensions which are not diagonalizable. For this it is enough to find a 2-power exact Bockstein closed extension where the binding operator L does not satisfy the equation $\beta(L)+L^2 = 0$. Since in this case the binding operator L is unique, the extension will not be diagonalizable. We give an explicit example satisfying these properties, hence provide an example of a Bockstein closed extension which is not diagonalizable. This example is a counterexample to a result in the literature by Dave Rusin (Lemma 20 on page 11 of [3].)

Next, we consider triangulable extensions. We say E is triangulable if there is a basis for V such that the components $q_1, \dots, q_n$ of q has the property that for each $i = 1, \dots, n$, the ideal $(q_i, q_{i+1}, \dots, q_n)$ is a Bockstein closed ideal. We show that the triangulability of E is equivalent to E having a lifting lattice M with the property that M has a filtration $0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_m = M$ such that each factor M_i/M_{i-1} is a one dimensional $\mathbb{Z}/4[W]$-lattice. We would like to have more equivalent conditions for the triangulability of extensions, like in the case of diagonalizable extension, but so far we were not successful in our attempts. Also, we do not know any Bockstein closed 2-power exact extensions which are not triangulable.

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Generic Jordan type of modular representations

Julia Pevtsova
(joint work with E. Friedlander and A. Suslin)

Elementary abelian p-subgroups of a finite group G capture significant aspects of the cohomology and representation theory of G. For example, if k is a field of characteristic $p > 0$, then a theorem of D. Quillen [9] asserts that the Krull dimension of the cohomology algebra $H^\bullet(G,k)$ is equal to the maximum of the ranks of elementary abelian p-subgroups of G and a theorem of L. Chouinard [3] asserts that a kG-module is projective if and only if its restrictions to all elementary abelian p-subgroups of G are projective. Quillen's geometric description of $\operatorname{Spec} H^\bullet(G,k)$ [9] provides the basis for interesting invariants of kG-modules, most notably the cohomological support variety $|G|_M$ of a kG-module M.

The geometric methods developed by Quillen were further applied to study representations of other algebraic structures, such as restricted Lie algebras. E. Friedlander and B. Parshall developed a theory of support varieties for finite dimensional p-restricted Lie algebras g over a field k of characteristic $p > 0$ (e.g., [4]). For restricted Lie algebras, the role of the group algebra kG of the finite group G is played by the restricted enveloping algebra $u(g)$. Indeed, restricted Lie algebras lead one to more interesting geometrical structure than do finite groups, and seemingly lead to stronger results. For example, the theorem of G. Avrunin and L. Scott [1] identifying the cohomological support variety $|E|_M$ of a finite dimensional kE module M for an elementary abelian p-group E with the rank variety of J. Carlson [2] admits a formulation in the case of a restricted Lie algebras g in terms of closed subvarieties of the p-nilpotent cone of g (cf. [4], [8], [11]).

A uniform approach to the study of the cohomology and related representation theory of all finite group schemes was presented in [5], [6]. This approach involves the use of π-points of G, which are finite flat maps of K-algebras $K[t]/t^p \to KG$ for field extensions K/k; these play the role of "cyclic shifted subgroups" in the case that G is an elementary abelian p-group and the role of 1-parameter subgroups in the case that G is an infinitesimal group scheme. In [6], the space $\Pi(G)$ of equivalence classes $[\alpha_K]$ of π-points $\alpha_K : K[t]/t^p \to KG$ of G is given a scheme structure without reference to cohomology such that $\Pi(G)$ is isomorphic as a scheme to $\operatorname{Proj} H^\bullet(G,k)$. In particular, there is a natural bijection between such equivalence classes of π-points and homogeneous prime ideals of $H^\bullet(G,k)$.

The purpose of the talk, though, is to demonstrate that one can go further in the search of information about modules encoded geometrically.

For a finite group scheme G over a field k of characteristic $p > 0$, we associate new invariants to a finite dimensional kG-module M. Namely, for each generic point of the projectivized cohomological variety $\operatorname{Proj} H^\bullet(G,k)$ we exhibit a "generic Jordan type" of M. We prove the following theorem which both describes the invariant and demonstrates that it is well-defined.

Theorem 1. *Let G be a finite group scheme, let M be a finite dimensional G-module and let $\alpha_K : K\mathbb{Z}/p \to KG$ be a π-point of G which represents a generic point $[\alpha_K] \in \Pi(G)$. Then the Jordan type of $\alpha_K(t)$ viewed as a nilpotent operator on M_K depends only upon $[\alpha_K]$ and not the choice of α_K representing $[\alpha_K]$.*

The Jordan type of such $\alpha_K(t)$ is called the *generic Jordan type* of M. In a special case when $G = E$ is an elementary abelian p-group, the theorem specializes to the non-trivial observation that the Jordan type obtained by restricting M via a generic cyclic shifted subgroup does not depend upon a choice of generators for E.

We verify that sending a module M to its generic Jordan type $[\alpha_K]^*(M_K)$ for generic $[\alpha_K] \in \operatorname{Proj} H^\bullet(G,k)$ determines a well-defined tensor triangulated functor on stable module categories

$$[\alpha_K]^* : stmod(kG) \to stmod(K[t]/t^p).$$

The second invariant we present is the *non-maximal support variety* of a finite-dimensional kG-module M, $\Gamma(G)_M \subset \operatorname{Proj} H^\bullet(G,k)$. The non-maximal support variety coincides with the "classical" support when the module is generically projective but gives a new non-tautological invariant in the case when the "classical" support variety of M is the entire cohomological spectrum: for example, when the dimension of M is not divisible by p.

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Brave new commutative algebra and modular representation theory

John Greenlees
(joint work with D.J.Benson, building on work with W.C.Dwyer and S.B.Iyengar)

The underlying philosophy is that for a finite group G and a field k one can move between representation theory and modules over the group cohomology ring using a morita equivalence of derived categories

$$Ho(C^*(BG;k)-mod) \underset{\mathrm{Hom}_{C^*Bg}(k,-)}{\overset{\otimes_{kG}k}{\leftrightarrows}} Ho(Mod-kG)$$

Here $C^*(BG;k) := \mathrm{map}(BG, Hk)$ is a strictly commutative ring spectrum. Hence one may prove Benson's conjecture about the Benson-Carlson-Rickard module $\mathcal{K}_\mathcal{P}$ picking out the subquotient of $D(kG)$ corresponding to the prime ideal $\mathcal{P}$ of H^*BG.

This is done by working in the category of modules over $C^*(BG)$ using methods of commutative algebra. In this context everything is a formal consequence of the fact that kG is a Frobenius algebra.

Hilbert series		Modules		Derived categories over $R = C^*BG$		Derived category over kG
				R is hGorenstein	$\Leftrightarrow$	kG Frobenius
				$\Downarrow$		
Benson-Carlson duality	$\Leftarrow$	Local cohomology spectral sequence	$\Leftarrow$	hGorenstein duality for R		
		$\Downarrow$		$\Downarrow$		
		Dual localized local cohomology spectral sequence (with Lynbeznik)	$\Leftarrow$	hGorenstein duality for $R_\mathcal{P}$	$\Rightarrow$	Benson's conjecture on $\mathcal{K}_\mathcal{P}$

From finite groups to infinite groups: Baum-Connes Conjecture, equivariant stable cohomotopy, Segal Conjecture

Wolfgang Lück

The purpose of this talk is to explain how methods, constructions and definitions for finite groups can be transferred and applied to infinite groups. We will focus on questions about the algebraic K- and L-theory of group rings, topological K-theory of group C^*-algebras, Burnside rings and equivariant stable cohomotopy. Some of the results and notions can be found in [1] and [3].

When we will consider actions of an infinite group, we will only consider proper actions. Recall that a G-CW-complex is proper if and only if all its isotropy groups are finite. The role which the one-point-space plays for finite groups is now taken over by the following notion (see for instance [2]).

Definition 1. A *classifying space for proper G-actions* is a G-CW-complex $\underline{E}G$ such that $\underline{E}G^H$ is contractible for $H \subseteq G$ with $|H| < \infty$ and $\underline{E}G^H$ is empty for $H \subseteq G$ with $|H| = \infty$.

If X is a proper G-CW-complex, then there is up to G-homotopy precisely one G-map $X \to \underline{E}G$. In particular two models for $\underline{E}G$ are G-homotopy equivalent. We have $\underline{E}G = \{*\}$ if and only if G is finite. We get $\underline{E}G = EG$ if and only if G is torsionfree. If L is a connected Lie group and $K \subseteq L$ its maximal compact subgroup, then for any discrete subgroup $G \subseteq L$ the space L/K is a model for $\underline{E}G$. For instance Rips complexes for word-hyperbolic groups and Teichmüller spaces for mapping class groups are models for $\underline{E}G$.

Next we deal with the question how the notion of the representation ring $R_K(G)$ and of the Burnside ring $A(G)$ for finite groups carry over to infinite groups. There are several possible definitions. Each represents a different aspect of the original notions. All of them coincide with the original notion in the case of a finite group. The table 1 briefly lists the various definitions.
Next we consider

Conjecture 2 (Baum-Connes)**.** The assembly map

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Here the target is the group of interest. The assembly map is essential given by taking an equivariant index. The source is in comparison easier to compute since it is given by an equivariant homology theory and standard tools from algebraic topology apply. The Baum-Connes Conjecture is known to be true for many groups and has a lot of applications. For a survey on the Baum-Connes Conjecture we refer for instance to [4].

A rational computation of its source can be given in terms of Chern characters. For $H \subseteq G$ let N_GH be its normalizer, C_GH be its centralizer and $W_GH = N_GH/H \cdot C_GH$ be its Weyl group. The latter is finite if H is finite. Let $\mathbb{Z} \subseteq \Lambda^G \subseteq \mathbb{Q}$ be the ring obtained from the integers by inverting the orders of the finite

Table 1. Definitions

$R_F(G)$	$A(G)$	key words
$K_0(FG)$	$\underline{A}(G)$	universal additive invariant, equivariant Euler characteristic
$\mathrm{Sw}^f(G;F)$	$\overline{A}(G)$	induction theory, Green functors
$R_{\mathrm{cov},F}(G) := \mathrm{colim}_{H\subseteq G,\|H\|<\infty} R_F(H)$	$A_{\mathrm{cov}}(G) := \mathrm{colim}_{H\subseteq G,\|H\|<\infty} A(H)$	collecting all values for finite subgroups with respect to induction
$R_{\mathrm{inv},F}(G) := \mathrm{invlim}_{H\subseteq G,\|H\|<\infty} R_F(H)$	$A_{\mathrm{inv}}(G) := \mathrm{invlim}_{H\subseteq G,\|H\|<\infty} A(H)$	collecting all values for finite subgroups with respect to restriction
$K_G^0(\underline{E}G)$	$A_{\mathrm{ho}}(G) := \pi_G^0(\underline{E}G)$	completion theorems, equivariant vector bundles
$K_0^G(\underline{E}G)$	$\pi_0^G(\underline{E}G)$	representation theory, Baum-Connes Conjecture, equivariant homotopy theory

subgroups of G. For a finite cyclic subgroup $C \subseteq G$ let $\theta_C \in \Lambda^G \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(C)$ be the idempotent whose character evaluated at a generator of C is 1 and is zero otherwise. It acts on $R_{\mathbb{C}}(C)$.

Theorem 3. Let X be a proper G-CW-complex. Then there is for $n \in \mathbb{Z}$ a natural isomorphism called *equivariant Chern character*

$$\bigoplus_{(C)} \Lambda \otimes_{\mathbb{Z}} K_n(C_GC\backslash X^C) \otimes_{\Lambda^G[W_GC]} \left(\theta_C \cdot \Lambda^G \otimes_{\mathbb{Z}} R_{\mathbb{C}}(C)\right) \xrightarrow{\cong} \Lambda^G \otimes_{\mathbb{Z}} K_n^G(X),$$

where (C) runs through the conjugacy classes of finite cyclic subgroups of G.

This implies the following corollary which illustrates how a K-group associated to an infinite group can be rationally computed by group homology and representation theory of finite cyclic groups.

Corollary 4. If the Baum-Connes Conjecture holds for G, we get an isomorphism

$$\bigoplus_{(C)} \bigoplus_{k\in\mathbb{Z}} H_p(BC_GC,\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} (\theta_C \cdot \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{C}}(C)) \xrightarrow{\cong} \mathbb{Q} \otimes_{\mathbb{Z}} K_p(C_r^*(G)).$$

Finally we explain how one can extend the definition of equivariant stable cohomotopy for finite groups. The main idea is to use instead of representations equivariant vector bundles and their associated sphere bundles for the stabilization. We show

Theorem 5. Equivariant stable cohomotopy $\pi_?^*$ defines an equivariant cohomology theory with multiplicative structure for cocompact proper equivariant CW-complexes. For every finite subgroup H of a group G the abelian groups $\pi_G^n(G/H)$ and π_H^n are isomorphic for every $n \in \mathbb{Z}$ and the rings $\pi_G^0(G/H)$ and $\pi_H^0 = A(H)$ are isomorphic.

A (hard) test case for the question whether this theory is worthwhile studying is to prove the following version of the Segal Conjecture. It boils down for finite groups to the classical Segal Conjecture which has been proved by Carlsson.

Conjecture 6 (Segal Conjecture for infinite groups)**.** Let G be a group with a cocompact model for $\underline{E}G$. Then there is an isomorphism

$$\pi^0(BG) \cong A_{\text{ho}}(G)\widehat{_I},$$

where I is kernel of the augmentation homomorphism $A_{\text{ho}}(G) \to \mathbb{Z}$.

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Cohomology of Bifunctors

Eric M. Friedlander

(joint work with Vincent Franjou)

The following paragraphs are a slight modification of the introduction of the preprint [3] on which Friedlander's talk was based.

We fix a prime p, a base field k of characteristic p, and consider the category $\mathcal{V}$ of finite dimensional k-vector spaces and k-linear maps. The study of the cohomology of categories of functors from $\mathcal{V}$ to k-vector spaces has had numerous applications, including insight into the structure of modules for the Steenrod algebra [2] and proof of finite generation of the cohomology of finite group schemes [4]. The computational power of functor cohomology arises as follows: the abelian category of strict polynomial functors $\mathcal{P}$ of bounded degree enjoys many pleasing properties which lead to various cohomological computations (cf. [5]); this cohomology for the category $\mathcal{P}$ is closely related to the cohomology for the abelian category of all functors $\mathcal{F}$ provided that our base field k is finite; for k finite, the cohomology of finite functors $F \in \mathcal{F}$ is equal to the stabilized cohomology of general linear groups with coefficients determined by F.

On the other hand, many natural coefficients modules for the general linear group are not given by functors but by bifunctors (contravariant in one variable, covariant in the other variable). Initially motivated by the quest to determine the group cohomology $H^*(GL(n, \mathbb{Z}/p^2), k)$, efforts have been made to compute the cohomology of $GL(n, k)$ with coefficients in symmetric and exterior powers of the adjoint representation $g\ell_n$ (cf. [1]). These coefficients are not given by functors but by bifunctors. The purpose of this paper is to provide computational tools and first computations towards the determination of the stable (with respect to n) values of $H^*(GL(n, k), S^d(g\ell))$ and $H^*(GL(n, k), \Lambda^d(g\ell))$.

Our first task is to formulate in terms of Ext groups in the category $\mathcal{P}^{op} \times \mathcal{P}$ of strict polynomial bifunctors the stable version of rational cohomology of the algebraic group GL with coefficients determined by the given bifunctor. We show that

$$\mathrm{Ext}^*_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^d(g\ell), T) \ \cong H^*_{rat}(GL_n, T(k^n, k^n))$$

where T is a strict polynomial bifunctor of homogeneous bidegree (d, d) with $n \geq d$. In the special case that T is of the form $A(g\ell)$ (for example, $S^d(g\ell)$), we write this as

$$H^*_{\mathcal{P}}(GL, A) \ \cong H^*_{rat}(GL_n, T(k^n, k^n)).$$

As for rational cohomology, the most relevant coefficients are given by beginning with a strict polynomial bifunctors T and applying the Frobenius twist operation (i.e., $I^{(1)} \circ (-)$) sufficiently often until the Ext-group of interest stabilizes. This "generic" strict polynomial bifunctor cohomology is our main target of computations.

In [4], the fundamental computation of $\mathrm{Ext}^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is achieved, modeled on the computation of $\mathrm{Ext}^*_{\mathcal{F}}(I, I)$ in [2]. For bifunctor cohomology, the computation of

$$H^*_{\mathcal{P}}(GL, \otimes^{n(r)}) \equiv \mathrm{Ext}_{\mathcal{P}^{op}\times\mathcal{P}}(\Gamma^{np^r}, \mathcal{H}om(I^{(r)}, I^{(r)})^{\otimes n})$$

which we present plays an analogous role.

We prove various useful formal results concerning bifunctor cohomology. For example, we relate the strict polynomial bifunctor cohomology with coefficients in a functor of separable type (i.e., of the from $\mathcal{H}om(A, B)$ where A, B are strict polynomial functors) to *Ext* computations in the category $\mathcal{P}$. We establish a base change result (one of the important advantages of strict polynomial functors/bifunctors in contrast to "ususal" functors/bifunctors) and a twist stability theorem; both results follow from analogus results proved in [5] for Ext-groups in the category $\mathcal{P}$.

We consider bifunctor cohomology such as $H^*_{\mathcal{P}}(GL, S^d(g\ell^{(r)}))$ where d is less than p. This is in principle completely computable. However, computations for

$p \leq d$ would appear to be much more difficult. We work out the case $p = 2 = d$, a computation which is quite involved. The applicability of this computation is extended to other examples.

Finally, we relate our computations of strict polynomial bifunctor cohomology to the cohomology of the finite groups $GL(n,k)$ where k is a finite field of characteristic p. We develop sufficient formalism for bifunctor cohomology to enable comparison of this bifunctor cohomology with both the cohomology of strict polynomial bifunctors and with group cohomology. Many explicit computations of group cohomology are obtained; in particular, we extend the low degree, unstable computations of [1] to stable (i.e., for the infinite general linear group) computations for all degrees.

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Endotrivial modules for finite groups of Lie type

Nadia Mazza

(joint work with Jon F. Carlson and Daniel K. Nakano)

Let G be a finite group and k be a field of characteristic $p > 0$. An endotrivial kG-module is a finitely generated kG-module M whose k-endomorphism ring is isomorphic to a trivial module in the stable module category. That is, M is an endotrivial module provided $\mathrm{Hom}_k(M,M) \cong k \oplus P$ where P is a projective kG-module. Now recall that as kG-modules, $\mathrm{Hom}_k(M,M) \cong M^* \otimes_k M$ where $M^* = \mathrm{Hom}_k(M,k)$ is the k-dual of M. Hence, the functor "$\otimes_k M$" induces an equivalence on the stable module category and the collection of all endotrivial modules makes up a part of the Picard group of all stable equivalences of kG-modules. In particular, equivalence classes of endotrivial modules modulo projective summands form a group that is an essential part of the group of stable self-equivalences.

Endotrivial modules were first defined by Dade in [Da]. He demonstrated that for p-groups, the endotrivial modules formed the building blocks of the endo-permutation modules which he proved are the sources for the irreducible modules in finite p-nilpotent groups. Dade also showed that if G is an abelian p-group, then any endotrivial kG-module is the direct sum of a syzygy of the trivial module ($\Omega^n(k)$ for some integer n) and a projective module. More recently, the first author and Thévenaz have given complete classification of the endotrivial modules for p-groups (cf. [CaTh3]). The group $T(G)$ of endotrivial modules is torsion-free except in the cases that the group G is cyclic, quaternion or semi-dihedral (cf. [CaTh1]). The torsion-free rank of $T(G)$ was determined by Alperin in [Al]. This rank depends on the number of conjugacy classes of maximal elementary abelian p-subgroups of p-rank 2. A complete set of generators for the group of endotrivial modules can also be constructed (cf. [Ca]).

The purpose of this joint work is to determine the group of endotrivial modules in the defining characteristic for all finite groups of Lie type, including those of twisted type. It is well understood that if G is an arbitrary finite group and M is an endotrivial kG-module, then both the Green correspondent and the source of M are endotrivial modules. For this reason we first consider the endotrivial modules for a Sylow p-subgroup U and its normalizer B, a Borel subgroup, of a given finite group G of Lie type. For the unipotent and Borel subgroups we present a complete classification of the endotrivial modules. For the finite groups G of Lie type, $T(G)$ has torsion-free rank one and is generated by the class of $\Omega(k)$ except in cases where the Lie rank is small and the field of the group is close to the prime field. In these exceptional cases, we find the torsion-free rank of $T(G)$. It would seem that finding a complete set of generators for the group of endotrivial modules would require a more detailed knowledge of the cohomology ring $H^*(G,k)$ than is currently available.

In the process of classifying the endotrivial modules for finite groups of Lie type, many of the results for p-groups are extended to arbitrary finite groups. We first introduce the group $T(G)$ of endotrivial kG-modules and show that it is a finitely generated abelian group. Hence it is the direct sum of its torsion subgroup $TT(G)$ and a torsion-free subgroup $TF(G)$ which we identify with the image of the product of the restriction maps onto the groups of endotrivial modules of elementary abelian p-subgroups of G of p-rank at least 2. Then, we prove that Alperin's theorem on the rank of $T(G)$ holds also for all finite groups, not just p-groups. Thereafter, we focus on the finite groups of Lie type, starting with the case where the Sylow p-subgroups are trivial intersection subgroups. Then we handle the larger groups, where it turns out that $T(G)$ is cyclic.

The following statements summarize the results of our investigations.

Theorem A:

(a) If G is not of type $A_1(p)$ $(p>2)$, ${}^2A_2(p)$, or ${}^2B_2(2^{\frac{1}{2}})$, then the torsion subgroup $TT(U)$ of $T(U)$ is trivial, by the classification of endotrival modules over p-groups.
(b) The torsion subgroup $TT(B)$ of $T(B)$ is isomorphic to the direct sum of $TT(U)$ and the character group of the torus $T \cong B/U$.
(c) If G is not of type $A_1(p)$ $(p>2)$, ${}^2A_2(p)$, or ${}^2B_2(2^{\frac{1}{2}})$, then the torsion subgroup $TT(G)$ of $T(G)$ is trivial.

The torsion-free group $TF(G)$ is described as follows.

Theorem B: The ranks of $TF(U)$, $TF(B)$ and $TF(G)$ are determined entirely by the number of conjugacy classes of maximal elementary abelian p-subgroups of p-rank 2 in the groups U, B and respectively G.

(a) If G has type $A_1(p)$ $(p > 2)$, or ${}^2B_2(2^{\frac{1}{2}})$, then $TF(U)$, $TF(B)$ and $TF(G)$ are all trivial.
(b) If G is one of the of the following, then the rank of $TF(G)$ is explored in detail:
  (i) G has type $A_2(p)$,
  (ii) G has type $B_2(p)$,
  (iii) G has type $G_2(p)$,
  (iv) G has type ${}^2A_2(p)$,
  (v) G has type ${}^2B_2(2^{a+\frac{1}{2}})$ (for $a \geq 1$),
  (vi) G has type ${}^2G_2(3^{a+\frac{1}{2}})$ (for $a \geq 0$).
(c) In all the other cases, the ranks of $TF(U)$, $TF(B)$ and $TF(G)$ are one.
(d) A complete set of generators for $TF(U)$ and $TF(B)$ can be specified.

It is worth stating that in the process of proving part (b) of Theorem B we enumerate the conjugacy classes of maximal elementary abelian p-subgroups of G. These conjugacy classes are in one-to-one correspondence with the components of the maximal ideal spectrum $V_G(k)$ of the cohomology ring $H^*(G, k)$. Hence the results are of some interest, independent of the structure of endotrivial modules.

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Invariance of generalized Reynolds ideals under derived equivalence

Alexander Zimmermann

Let k be an algebraically closed field of characteristic $p > 0$ and let A be a symmetric finite dimensional k-algebra with symmetrising bilinear form $(\ , \ )$. Let KA be the k-vector space generated by the subset $\{xy - yx | \ x, y \in A\}$ of A. This space was defined and used by R. Brauer in [1].

Külshammer defined in [3] and [4] for any integer n the spaces

$$T_n(A) := \{x \in A \mid x^{p^n} \in KA\}$$

and $T_n(A)^\perp$ the orthogonal space to $T_n(A)$ with respect to the symmetrising form $(\ , \ )$. Then, $T_n(A)^\perp$ is an ideal of the centre $Z(A)$ of A.

$$Z(A) = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \cdots \supseteq \bigcap_{n \in \mathbb{N}} T_n(A)^\perp = soc(A) \cap Z(A)$$

Héthelyi et al. show in [2] that $Z_0A \subseteq (T_1A^\perp)^2 \subseteq HA$, where HA is the Higman ideal, that is the image of the trace map of A, and where Z_0A is the sum of the centres of those blocks of A which are simple algebras. They show that for odd p the left inclusion is an equality, whereas for $p = 2$ one gets $Z_0A = (T_1A^\perp)^3 = (T_1A^\perp) \cdot (T_2A^\perp)$. Many more interesting properties of these ideals are given there.

The authors of [2] show that the ideals are invariant under Morita equivalence in the obvious sense and they ask if for derived equivalent symmetric algebras A and B there is an isomorphism $\varphi : Z(A) \longrightarrow Z(B)$ with $\varphi(T_n(A)^\perp) = T_n(B)^\perp$ for all $n \in \mathbb{N}$.

Let B be a k-algebra. By Rickard's theory [5] given an equivalence $D^b(A) \simeq D^b(B)$ as triangulated categories there is a complex X in $D^b(B \otimes_k A^{op})$ so that

$$X \otimes^{\mathbb{L}}_A - : D^b(A) \longrightarrow D^b(B)$$

is an equivalence, called "of standard type". Now, for a symmetric algebra A, given such an equivalence the algebra B is symmetric as well again by [5], or in a more general context by [6]. Then, replacing X by a suitable isomorphic copy consisting of a complex formed of left and right projective A-modules,

$$X \otimes_A - \otimes_A Hom_k(X, k) : D^b(A \otimes_k A^{op}) \longrightarrow D^b(B \otimes_k B^{op})$$

is an equivalence. Moreover, this equivalence maps the bimodule A to B and therefore induces an isomorphism

$$HH^n(A) = Hom_{D^b(A \otimes_k A^{op})}(A, A[n]) \simeq Hom_{D^b(B \otimes_k B^{op})}(B, B[n]) = HH^n(B)$$

between the degree n Hochschild cohomology of A and B. Now, observe that $HH^0(A) = Z(A)$ and $HH^0(B) = Z(B)$.

Theorem. [7] Let k be an algebraically closed field of characteristic $p > 0$ and let A and B be finite dimensional k-algebras. Then, the isomorphism $\varphi : ZA \longrightarrow ZB$ between the centres ZA of A and ZB of B induced by an equivalence $D^b(A) \simeq D^b(B)$ of standard type has the property $\varphi(T_nA^\perp) = T_nB^\perp$ for all positive integers $n \in \mathbb{N}$.

Remark: As an application of the theorem it is possible to distinguish the derived categories of certain algebras arising as blocks of group rings of tame representation type, which could not be distinguished otherwise. This will be subsequent joint work with Thorsten Holm.

The proof of the theorem uses first that the ideals $T_n(A)^\perp$ are images of mappings ζ_n defined by the property

$$(\zeta_n(x), y)^{p^n} = (x, y^{p^n}) \ \forall x \in Z(A) \forall y \in A/KA\ .$$

Then, we study in detail the mapping ζ_n as a composition of mappings

$$\begin{array}{rcl} Hom_{A\otimes_k A^{op}}(A,A) & \longrightarrow & Hom_{A\otimes_k A^{op}}(A, Hom_k(A,k)) \\ & \xrightarrow{\psi} & Hom_{A\otimes_k A^{op}}(A, Hom_k(A,k)) \\ & \longrightarrow & Hom_{A\otimes_k A^{op}}(A,A). \end{array}$$

Here ψ is the composition of the n fold p-power mapping and the inverse of the Frobenius mapping. It is then possible to study the behaviour under a derived equivalence and this discussion gives the statement.

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Stable cohomology algebra of local rings

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(joint work with Oana Veliche)

Stable cohomology theory assigns to each pair (M,N) of modules over an associative ring R groups $\widehat{\mathrm{Ext}}^n_R(M,N)$ defined for every $n\in\mathbb{Z}$, which vanish if either M or N has finite projective dimension. The prototype is Tate cohomology ($\widehat{\mathrm{Ext}}^n_{\mathbb{Z}G}(\mathbb{Z},N)=\widehat{\mathrm{H}}(G,N)$ when G is a finite group), based on complete resolutions. Buchweitz [4] extended this approach to produce a stable theory when R is noetherian with finite self injective dimension on each side. Vogel, see [6], Benson and Carlson [3], and Mislin [9] have given (equivalent) general constructions; see [7] for background and details, [2] for interpretations and further generalizations.

For applications to commutative algebra we fix a commutative noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field $k=R/\mathfrak{m}$. Let d denote the Krull dimension of R and e the minimal number of generators of $\mathfrak{m}$; one has $e\geq d$.

We characterize three important classes of local rings in terms of the stable cohomology of k. The results reflect a basic hierarchy: regular $\subseteq$ complete intersection $\subseteq$ Gorenstein. In each case, the difficult part is the first implication.

Regular local rings, defined by $e=d$, are precisely the local rings of finite global dimension, and can be described by the condition $\mathrm{Ext}^{e+1}_R(k,k)=0$. We prove:

1. $\widehat{\mathrm{Ext}}^n_R(k,k)=0$ *for some* $n\in\mathbb{Z}\implies R$ *regular* $\implies\widehat{\mathrm{Ext}}^n_R(k,k)=0$ *for all* $n\in\mathbb{Z}$.

Complete intersections, or c.i., rings are (essentially) quotients of regular rings by regular sequences; they are characterized by $\mathrm{rank}_k\,\mathrm{Ext}^2_R(k,k)=\binom{e+1}{2}-d$.

2. *For every singular (that is, not regular) ring R and each $n\in\mathbb{Z}$ one has*

$$\mathrm{rank}_k\,\widehat{\mathrm{Ext}}^n_R(k,k)\geq\sum_{i=0}^{d}\binom{d}{i}\left(\binom{e-d+n-i-1}{e-d-1}+\binom{e-n-i-2}{e-d-1}\right).$$

Equality holds for some $n\notin[d-2,1]\implies R$ c.i. $\implies$ equalities hold for all $n\in\mathbb{Z}$.

Gorenstein local rings are the local rings of finite self-injective dimension. No characterization of these rings is known in terms of the absolute cohomology groups $\mathrm{Ext}^n_R(k,k)$. For that reason, the first implication below is surprising.

3. $\mathrm{rank}_k\,\widehat{\mathrm{Ext}}^n_R(k,k)$ *is finite for some* $n\in\mathbb{Z}\implies R$ *Gorenstein* $\implies$
$\mathrm{rank}_k\,\widehat{\mathrm{Ext}}^n_R(k,k)$ *is finite for all* $n\in\mathbb{Z}$.

The statements above concern k-vector spaces, but their proofs use the graded k-algebras $\mathcal{E}=\mathrm{Ext}^*_R(k,k)$ and $\mathcal{S}=\widehat{\mathrm{Ext}}^*_R(k,k)$ and a canonical homomorphism $\iota\colon\mathcal{E}\to\mathcal{S}$. Martsinkovsky [8] proved that ι is injective, unless R is regular.

Following Félix *et al* [5], using the left $\mathcal{E}$-module $\mathcal{E}$ and $k=\mathcal{E}/\mathcal{E}^{\geqslant 1}$ we set

$$\operatorname{depth}\mathcal{E}=\inf\{n\in\mathbb{Z}\mid\mathrm{Ext}^n_{\mathcal{E}}(k,\mathcal{E})\neq 0\},$$

An inequality $\operatorname{depth}\mathcal{E}\geq 1$ holds if and only if R is singular. We assume this is the case, as otherwise there is not much to say, see Theorem 1.

Let $\mathcal{J}$ denote $\operatorname{Hom}_k(\mathcal{E}, k)$ with the canonical graded left $\mathcal{E}$-module structure. Set $\mathcal{T} = \{\sigma \in \mathcal{S} \mid \mathcal{E}^{\geqslant n} \cdot \sigma = 0 \text{ for some } i \geq 0\}$; this is a $\mathcal{E}$-subbimodule of $\mathcal{S}$.

The next result contains most of what we know about $\mathcal{S}$ in general.

4. *For a singular local ring R the following hold.*

(1) *There is an exact sequence of left $\mathcal{E}$-modules*

$$(*) \qquad 0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\eth} \coprod_{i=\operatorname{depth} R}^{\infty} (\Sigma^{1-i}\mathcal{J})^{\mu^i} \longrightarrow 0$$

where $g = \operatorname{depth} R$ and $\mu^i = \operatorname{rank}_k \operatorname{Ext}^i_R(k, R)$, and there are equalities

$$\mathcal{S} = \iota(\mathcal{E}) + \mathcal{E} \cdot \mathcal{S}^{\leqslant 0} \qquad \textit{and} \qquad \iota(\mathcal{E}) \cap \mathcal{T} = 0\,.$$

(2) *If $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}'$ for some graded left $\mathcal{E}$-submodule $\mathcal{T}' \subseteq \mathcal{S}$, then $\mathcal{T}' = \mathcal{T}$.*

(3) *If* $\operatorname{depth} \mathcal{E} \geq 2$, *then $\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T}$ as graded $\mathcal{E}$-bimodules.*

(4) *If $R = Q/(f)$, where $(Q, \mathfrak{n})$ is a singular local ring, f is a non-zero-divisor, and f is in $\mathfrak{n}^2$, then* $\operatorname{depth} \mathcal{E} \geq 2$ *and $\mathcal{T}$ is a two-sided ideal of $\mathcal{S}$ such that*

$$\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T} \qquad \textit{and} \qquad \mathcal{T} \cdot \mathcal{T} = 0\,.$$

As one already has $\operatorname{depth} \mathcal{E} \geq 1$, the hypothesis $\operatorname{depth} \mathcal{E} \geq 2$ covers, in some sense, the ‘generic, case. The assertion in (2) shows that if the exact sequence $(*)$ admits any left $\mathcal{E}$-linear splitting, then such a splitting is unique and is automatically right $\mathcal{E}$-linear as well. The sequence shows that, as a left module, $\operatorname{Coker}(\iota)$ is a direct sum of shifts of $\mathcal{J}$, so the left action of $\mathcal{E}$ on $\operatorname{Coker}(\iota)$ is locally nilpotent.

The theorem provides no information about the right action of $\mathcal{E}$. Furthermore, it says nothing about rings with $\operatorname{depth} \mathcal{E} = 1$. In the example below $\mathcal{E}$ is the tensor algebra of the vector space $\mathcal{E}^1$, isomorphic to k^e; as tensor algebras have global dimension 1, on gets $\operatorname{depth} \mathcal{E} = 1$. (We show that all Golod rings have this property.) The simple nature of R and $\mathcal{E}$ notwithstanding, the structure of $\mathcal{S}$ turns out to be intrinsically more complicated than in the cases covered by Theorem 4.

5. *If $\mathfrak{m}^2 = 0$ and the ideal $\mathfrak{m}$ is not principal, then $(*)$ does not split as a sequence of left $\mathcal{E}$-modules, and the right action of $\mathcal{E}$ on* $\operatorname{Coker}(\iota)$ *is not locally nilpotent.*

The behavior of Bass numbers displays a striking cohomological dichotomy: If R is not Gorenstein, then $\mu^i > 0$ for all $i \geq \operatorname{depth} R$; else, $\mu^{\operatorname{depth} R} = 1$ and $\mu^i = 0$ for $i \neq \operatorname{depth} R$. In view of the exact sequence $(*)$, this property accounts for Theorem 3. It also shows that when R is Gorenstein the sequence $(*)$ simplifies to

$$(**) \qquad 0 \longrightarrow \mathcal{E} \xrightarrow{\iota} \mathcal{S} \xrightarrow{\eth} \Sigma^{1-d}\mathcal{J} \longrightarrow 0$$

We prove that splitting of this sequence largely determines the algebra structure.

6. *Let R be a Gorenstein local ring with $e - d \geq 2$. If $(**)$ splits as a sequence of left $\mathcal{E}$-modules, then $\mathcal{T}$ is a two-sided ideal of $\mathcal{S}$ and satisfies*

$$\mathcal{S} = \iota(\mathcal{E}) \oplus \mathcal{T} \qquad \textit{and} \qquad \mathcal{T} \cdot \mathcal{T} = 0\,.$$

As a graded left $\mathcal{E}$-module $\mathcal{T}$ is generated by $\mathcal{S}^{<0}$ and is isomorphic to $\Sigma^{1-d}\mathcal{J}$.

The theorem applies to Gorenstein rings described by any of the conditions: $e-d=2$; $e-d=3$; multiplicity equal to $e-d+2$; localizations of graded Koszul algebras at the maximal homogeneous ideal; tensor products of singular algebras over a field; isomorphic to $Q/(f)$ with $(Q,\mathfrak{n})$ singular and a non-zero-divisor $f\in\mathfrak{n}^2$.

The latter class includes all complete intersections with $d-e\geq 2$. Rings of the form $Q/(f)$ with Q regular and $f\in\mathfrak{n}^2$ are called hypersurface rings; they essentially coincide with Gorenstein rings with $e-d=1$. Their stable cohomology, computed by Buchweitz's [4], should be contrasted with the result of Theorem 4(4):

7. *If R is a hypersurface ring, then $\mathcal{S}=\mathcal{E}[\vartheta^{-1}]$, where $\vartheta\in\mathcal{E}^2$ is a central non-zero-divisor and $\mathcal{E}/(\vartheta)$ is an exterior algebra on e generators of degree* 1*.*

It is easy to see that hypersurface rings have depth $\mathcal{E}=1$ and that for them the sequence (∗∗) does not split. Non-hypersurface Gorenstein rings with depth $\mathcal{E}=1$ do exist, but in all known examples the defining exact sequence split.

Our results for Gorenstein rings offer striking parallels to theorems of Benson and Carlson [3], relating the structure of the Tate cohomology algebra $\widehat{\mathrm{H}}^*(G,k)$of a finite group G with that of the absolute cohomology algebra $\mathrm{H}^*(G,k)$. The parallelism might have been anticipated additively, at the level of the underlying vector spaces, where it can be traced to the self-injectivity of the group ring kG and the finite self-injective dimension of the Gorenstein local ring R.

On the other hand, the similarities of the graded algebra structures are completely unexpected. Indeed, both algebras $\mathrm{H}^*(G,k)$ and $\widehat{\mathrm{H}}^*(G,k)$ are graded-commutative; moreover, the first one is finitely generated over k, and hence noetherian. In stark contrast, $\mathcal{E}$ may not be finitely generated; it is noetherian if and only if R is complete intersection; it is commutative if and only if, it is c.i. and the length of the R-module $R/\mathfrak{m}^3$ is equal to $\binom{e+2}{2}$ (which is the largest possible value).

Proofs of the results above and further developments can be found in [1].

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Homotopy rank and small rank groups

Michael A. Jackson

Recall that the p-rank of a finite group G, $\mathrm{rk}_p(G)$, is the largest rank of an elementary abelian p-subgroup of G and that the rank of a finite group G, $\mathrm{rk}(G)$, is the maximum of $\mathrm{rk}_p(G)$ taken over all primes p. We define the homotopy rank of a finite group G, $\mathrm{h}(G)$, to be the minimal integer k such that G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$. Benson and Carlson [3] have conjectured that for any finite group G, $\mathrm{rk}(G) = \mathrm{h}(G)$. The case of their conjecture, when G is a rank one group, is a direct result of Swan's theorem [9]. Benson and Carlson's conjecture has also been verified by Adem and Smith [1] for rank two p-groups as well as for all rank two finite simple groups except $PSL_3(\mathbb{F}_p)$ for p an odd prime.

Recall that $\mathrm{Qd}(p)$ is the semidirect product of $(\mathbb{Z}/p\mathbb{Z})^2$ by $SL_2(\mathbb{F}_p)$ with the natural action. We say that a group G does not involve $\mathrm{Qd}(p)$ if no subquotient of G is isomorphic to $\mathrm{Qd}(p)$. In addition, a group that does not involve $\mathrm{Qd}(p)$ is called $\mathrm{Qd}(p)$-free. Here we will verify Benson and Carlson's conjecture for $\mathrm{Qd}(p)$-free finite groups of rank two. A result of Heller [5] states that if $\mathrm{h}(G) = 2$, then $\mathrm{rk}(G) = 2$; therefore, the conjecture holds for a given rank two group G, if G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$. We will find such actions using a recent result of Adem and Smith, but we need to include two definitions before stating their result.

Definition 1. Let $\varphi : BG \to BU(n)$ be a map and let $\alpha \in H^{2n}(BU(n), \mathbb{Z})$ be the Euler class (top Chern class) of $U(n)$. The Euler class in $H^{2n}(BG, \mathbb{Z})$ associated to φ is $\varphi^*(\alpha)$.

Definition 2. A cohomology class $\alpha \in H^*(BG, \mathbb{Z})$ is called effective if for each elementary abelian subgroup $E \subseteq G$ with $\mathrm{rk}(E) = \mathrm{rk}(G)$, $\mathrm{res}^G_E(\alpha) \neq 0$.

Theorem 3 (Adem and Smith [1])**.** Let G be a finite group with $\mathrm{rk}(G) = 2$. If the Euler class associated to some map $\varphi : BG \to BU(n)$ is effective, then $\mathrm{h}(G) = 2$.

In light of Theorem 3, verifying Benson and Carlson's conjecture for a rank two group G can be reduced to finding a particular map $\varphi : BG \to BU(n)$ with an effective Euler class. Two properties of maps from BG to $BU(n)$ will prove useful: homotopic maps have the same Euler class; and if G is a p-group for some prime p, then maps from BG to $BU(n)$ are in one-to-one correspondence with complex characters of degree n. The second property uses a result of Dwyer and Zabrodsky [4].

In light of the second property above, we will be relating maps from BG to $BU(n)$ to characters. To do so, we must introduce the following notation: G_p will denote a Sylow p-subgroup of G; $\mathrm{Char}_n(G_p)$ will denote the set of degree n complex characters of G_p; and $\mathrm{Char}_n^G(G_p)$ will denote the subset of $\mathrm{Char}_n(G_p)$ consisting of those degree n complex characters of G_p that are the restrictions of class functions on G, meaning that they respect fusion in G.

We now define a map $\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$, by using the following composition:

$$[BG, BU(n)] \stackrel{\cong}{\to} \prod_{p||G|} [BG, BU(n)_p^\wedge] \stackrel{\text{res}}{\to} \prod_{p||G|} [BG_p, BU(n)_p^\wedge] \stackrel{\cong}{\to} \prod_{p||G|} \mathrm{Char}_n(G_p).$$

Notice that spaces in the center and at the right of the top row contain $BU(n)_p^\wedge$, which is the p-completion of the space $BU(n)$. The left bijection is a result by Jackowski, McClure, and Oliver [6] while the right bijection follows is the previously mentioned property. The restriction map *res* is induced by the inclusion of the Sylow p-subgroups G_p into G. We notice that the image of the composition above lies in the subset $\prod_{p||G|} \mathrm{Char}_n^G(G_p)$; therefore, we let ψ_G be the composition above with the range restricted to $\prod_{p||G|} \mathrm{Char}_n^G(G_p)$. We get the following result concerning the map ψ_G:

Theorem 4 (Jackson [7, Theorem 1.3])**.** If G is a finite group of rank two, then the natural mapping $\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$ is a surjection.

Using Theorem 4, we see that a map from BG to $BU(n)$ with an effective Euler class can be demonstrated by giving appropriate characters in $\mathrm{Char}_n^G(G_p)$ for each prime p dividing the order of G, which leads to Definition 5 and Theorem 6

Definition 5. Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. A character χ of G_p is called a *p-effective character of G* if $\chi \in \mathrm{Char}_n^G(G_p)$ and if for each elementary abelian subgroup $E \subseteq G_p$ with $\mathrm{rk}(E) = \mathrm{rk}(G)$, the trivial character of E is not an irreducible summand of the character $\chi|_E$.

Theorem 6 (Jackson[8])**.** Let G be a finite group. If for each prime p dividing $|G|$ there exists a p-effective character of G, then there is a map $\varphi : BG \to BU(n)$ whose associated Euler class is effective. If in addition $\mathrm{rk}(G) = 2$, then $\mathrm{h}(G) = 2$

Theorem 6 has reduced the process of showing that a rank two group has homotopy rank two to finding p-effective characters for each prime p dividing the order of the group. A definition from group theory is necessary in demonstrating the existence of p-effective characters.

Definition 7. Let G be a finite group, and let H and K be subgroups such that $H \subset K$. We say that H is *strongly closed in K with respect to G* if for each $g \in G$, $H^g \cap K \subseteq H$.

We are now able to show a sufficient condition for the existence of a p-effective character.

Proposition 8. Let G be a finite group, p a prime divisor of $|G|$, and G_p a Sylow p-subgroup of G. If $H \subseteq Z(G_p)$ exists such that H is non-trivial and strongly closed in G_p with respect to G, then G has a p-effective character.

Recall that $\Omega_1(P)$, for a p-group P, is the subgroup of P generated by the order p elements of P. Notice that if P is abelian, then $\Omega_1(P)$ is elementary abelian. The next theorem shows that in many cases the sufficient condition may be applied.

Theorem 9 (Jackson [8])**.** Let G be a finite group, p a prime with $\mathrm{rk}_p(G) = \mathrm{rk}(G) = 2$, and $G_p \in \mathrm{Syl}_p(G)$. If $\Omega_1(Z(G_p))$ is not strongly closed in G_p with respect to G, then either p is odd and $\mathrm{Qd}(p)$ is involved in G, or $p = 2$ and G_2 is dihedral, semi-dihedral, or wreathed.

The prime 2 portion of Theorem 9 is a result of Alperin, Brauer, and Gorenstein [2, Proposition 7.1]. As a result of Theorem 9, a rank two finite group has a 2-effective character if its Sylow 2-subgroups are not dihedral, semi-dihedral or wreathed. The cases of dihedral, semi-dihedral and wreathed Sylow 2-subgroups are shown to have 2-effective characters in Theorem 9.

Theorem 10 (Jackson [8])**.** If G is a finite group with a dihedral, semi-dihedral, or wreathed Sylow 2-subgroup such that $\mathrm{rk}(G) = 2$, then G has a 2-effective character.

Theorem 10 is shown by explicitly demonstrating the 2-effective character in each case.

Theorem 11 (Jackson)**.** Let G be a finite group such that $\mathrm{rk}(G) = 2$. G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ unless for some odd prime p, G involves $\mathrm{Qd}(p)$. In particular, if G is a rank two group that is $\mathrm{Qd}(p)$-free for each odd prime p, then $\mathrm{h}(G) = 2$.

We end by pointing out that for an odd prime p, $\mathrm{Qd}(p)$ does not have a p-effective character.

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Sturm sequences and H_2 of the hyperbolic homomorphism

Jean Lannes
(joint work with Jean Barge)

Short abstract: Let R be an arbitrary ring (let's say commutative). Let $\mathrm{GL}(R)$ and $\mathrm{Sp}(R)$ be the "infinite" general linear group and symplectic group; let $\mathrm{H} : \mathrm{GL}(R) \to \mathrm{Sp}(R)$ be the hyperbolic homomorphism. Let $\mathrm{ESp}(R)$ be the "elementary subgroup" of $\mathrm{Sp}(R)$, $\mathrm{ESp}(R) \cdot \mathrm{GL}(R)$ the subgroup of $\mathrm{Sp}(R)$ generated by $\mathrm{ESp}(R)$ and $\mathrm{GL}(R)$, and ${}^{\mathrm{r}}\mathrm{H} : \mathrm{GL}(R) \to \mathrm{ESp}(R)\cdot\mathrm{GL}(R)$ the group homomorphism induced by H. Finally let $\mathrm{I}(R)$ be the fundamental ideal of the Witt ring of R (the one of the theory of non-degenerate symmetric bilinear forms, defined here in terms of free R-modules). Our aim is to show that the homology group $\mathrm{H}_2({}^{\mathrm{r}}\mathrm{H})$, group which takes place naturally in a five term exact sequence

$$\mathrm{K}_2(R) \longrightarrow \mathrm{KSp}_2(R) \longrightarrow \mathrm{H}_2({}^{\mathrm{r}}\mathrm{H}) \longrightarrow \mathrm{K}_1(R) \longrightarrow \mathrm{KSp}_1(R) \quad ,$$

is the fibre product of $\mathrm{I}(R)$ and $\mathrm{K}_1(R)$ over $\mathrm{K}_1(R)/(1+\tau)$, τ denoting the involution of $\mathrm{K}_1(R)$ defined by matrix transposition.

This result is a variant of results of R. W. Sharpe [On the structure of the unitary Steinberg group, *Ann. of Math.*, **96** (1972), 444-479]. Our method of proof is distantly related to the classical theory of Sturm sequences, hence our title.

1. Variations on the Hopf formula

Let $\rho : G \to G'$ be a group homomorphism. One may choose for definition of the homology of ρ, the reduced homology of the cone of the application

$$\mathrm{B}\rho : \mathrm{B}G \to \mathrm{B}G' \qquad ;$$

so one has a long exact sequence

$$\ldots \to \mathrm{H}_n G \to \mathrm{H}_n G' \to \mathrm{H}_n \rho \to \mathrm{H}_{n-1} G \to \mathrm{H}_{n-1} G' \to \ldots \qquad .$$

Proposition (Hopf formula) **.** *Let $\rho : G \to G'$ be a group epimorphism. Then one has a canonical isomorphism*

$$\mathrm{H}_2\,\rho \;\cong\; \ker\rho\,/\,[G, \ker\rho] \qquad .$$

In other words, "centralization" of the exact sequence

$$1 \longrightarrow \ker\rho \longrightarrow G \xrightarrow{\ \rho\ } G' \longrightarrow 1$$

gives a central extension of the form

$$1 \longrightarrow \mathrm{H}_2\,\rho \longrightarrow G\,/\,[G, \ker\rho] \xrightarrow{\ \rho\ } G' \longrightarrow 1 \qquad .$$

2. Symplectic groups

Let R be a commutative ring and L a free R-module of finite dimension.

One denotes by $\mathrm{H}(L)$ the R-module $L \oplus L^*$ equipped with the alternating bilinear form

$$\big((x,\xi)\,,\,(y,\eta)\big) \mapsto \langle x,\eta\rangle - \langle y,\xi\rangle$$

(one says that $\mathrm{H}(L)$ is the *symplectic hyperbolic space* associated to L and that the bilinear form above is its *symplectic form*). One denotes by Sp_L the group made of the automorphisms of the R-module $\mathrm{H}(L)$ which preserve the symplectic form.

One denotes by GL_L the group of automorphisms of the R-module L. The map from GL_L to Sp_L,

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{bmatrix} \quad ,$$

is a group homomorphism which is denoted by H.

Let $q : L \to L^*$ (resp. $q : L^* \to L$) an R-module homomorphism. The automorphism

$$\begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix} \quad (\text{resp. } \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix})$$

of the R-module $\mathrm{H}(L)$ belongs to Sp_L if and only if one has $q = q^*$. One will observe that such a q identifies with a symmetric bilinear form on L (resp. L^*) ; we will denote by $\mathcal{S}_L$ (resp. $\mathcal{S}_{L^*}$) the submodule of $\mathrm{Hom}_R(L, L^*)$ (resp. $\mathrm{Hom}_R(L^*, L)$) made of the homomorphisms q satisfying $q = q^*$. The symplectic automorphisms of $\mathrm{H}(L)$ of the preceeding type are called *elementary*; we will denote by ESp_L the subgroup of Sp_L they generate.

One denotes by GL_n, Sp_n, ESp_n,..., the functors $R \mapsto \mathrm{GL}_{R^n}$, $R \mapsto \mathrm{Sp}_{R^n}$, $R \mapsto \mathrm{ESp}_{R^n}$,.... One denotes by GL, Sp, ESp,...the functors colimits of these functors.

The three group homomorphisms

$$\mathrm{GL}_L \to \mathrm{Sp}_L \ , \ a \mapsto \begin{bmatrix} a & 0 \\ 0 & a^{*-1} \end{bmatrix} \ ; \ \mathcal{S}_L \to \mathrm{Sp}_L \ , \ q \mapsto \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix} \ ; \ \mathcal{S}_{L^*} \to \mathrm{Sp}_L \ , \ q \mapsto \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}$$

induce a homomorphism from the semi-direct product $(\mathcal{S}_L * \mathcal{S}_{L^*}) \rtimes \mathrm{GL}_L$ to Sp_L (corresponding to the natural left action of GL_L on $\mathcal{S}_L$ and $\mathcal{S}_{L^*}$). We set $\Theta_L = (\mathcal{S}_L * \mathcal{S}_{L^*}) \rtimes \mathrm{GL}_L$ and denote by $\rho_L : \Theta_L \to \mathrm{Sp}_L$ the canonical homomorphism just evoked. From the very definition, the image of ρ_L is the subgroup $\mathrm{ESp}_L{\cdot}\mathrm{GL}_L$ of Sp_L generated by the subgroups ESp_L and GL_L.

Now one centralizes the exact sequence

$$1 \longrightarrow \ker\rho_L \longrightarrow \Theta_L \xrightarrow{\rho_L} \mathrm{ESp}_L{\cdot}\mathrm{GL}_L \longrightarrow 1 \qquad .$$

Let $[\Theta_L, \ker\rho_L]$ be the (normal) subgroup of Θ_L generated by commutators of elements of Θ_L and of $\ker\rho_L$; we denote respectively by Γ_L et A_L the quotient groups $\Theta_L/[\Theta_L, \ker\rho_L]$ and $\ker\rho_L/[\Theta_L, \ker\rho_L]$, we denote by $\pi : \Gamma_L \to \mathrm{Sp}_L$ the homomorphism induced by ρ_L. By the very contruction the exact sequence

$$1 \longrightarrow \mathrm{A}_L \longrightarrow \Gamma_L \xrightarrow{\pi} \mathrm{ESp}_L{\cdot}\mathrm{GL}_L \longrightarrow 1$$

is a central extension.

Our aim is to study the central extension

$$1 \longrightarrow \mathrm{A}(R) \longrightarrow \Gamma(R) \xrightarrow{\pi} \mathrm{ESp}(R)\cdot\mathrm{GL}(R) \longrightarrow 1$$

colimit of the central extensions

$$1 \longrightarrow \mathrm{A}_n(R) \longrightarrow \Gamma_n(R) \xrightarrow{\pi} \mathrm{ESp}_n(R)\cdot\mathrm{GL}_n(R) \longrightarrow 1 \quad .$$

Proposition. *The group* $\mathrm{A}(R)$ *is canonically isomorphic to the* H_2 *of the hyperbolic homomorphism* $\mathrm{GL}(R) \to \mathrm{ESp}(R)\cdot\mathrm{GL}(R)$.

Corollary. *The group* $\mathrm{A}(R)$ *takes place in a five term exact sequence:*

$$\mathrm{K}_2(R) \longrightarrow \mathrm{KSp}_2(R) \longrightarrow \mathrm{A}(R) \longrightarrow \mathrm{K}_1(R) \longrightarrow \mathrm{KSp}_1(R) \quad .$$

3. Identification of the group $\mathrm{A}(R)$ in terms of symmetric bilinear forms

3.1. The group $\mathrm{V}(R)$

One considers the isomorphism classes $[L; q_0, q_1]$ of triples $(L; q_0, q_1)$ of the following type:

- L is a free R-module of finite dimension;
- q_0 and q_1 are symmetric bilinear forms on L which are assumed to be non-degenerate.

One denotes by $\mathrm{V}(R)$ the quotient of the abelian Grothendieck group generated by these $[L; q_0, q_1]$, with the orthogonal sum for group law, by the subgroup generated by the elements of the form

$$[L; q_0, q_1] + [L; q_1, q_2] - [L; q_0, q_2] \quad .$$

Proposition. *One has a canonical cartesian diagram of abelian groups*

$$\begin{array}{ccc} \mathrm{V}(R) & \longrightarrow & \mathrm{I}(R) \\ \downarrow & & \downarrow \\ \mathrm{K}_1(R) & \longrightarrow & \mathrm{K}_1(R)/(1+\tau) \end{array} \quad .$$

3.2. The homomorphism $\mu : \mathrm{A}(R) \to \mathrm{V}(R)$

Sturm forms

Let L be a finite dimensional free R-module.

Let k be an integer, we set

$$L_k = \begin{cases} L & \text{for } k \text{ even,} \\ L^* & \text{for } k \text{ odd.} \end{cases}$$

Let $\underline{q} = (q_m, q_{m+1}, \dots, q_n)$ be a sequence with $q_k \in \mathcal{S}_{L_k}$ for $m \leq k \leq n$ (we call this type of sequence a *Sturm sequence*).

We set $L_{m,n} = \bigoplus_{k=m}^{n} L_k$ and we denote by $\mathrm{S}(\underline{q})$ the symmetric bilinear form on the finite dimensional free R-module $L_{m,n}$ whose matrix is the following one:

$$
\begin{array}{c} \\ L_m^* \\ L_{m+1}^* \\ \dots \\ \dots \\ \dots \\ \dots \\ L_n^* \end{array}
\begin{array}{c}
\begin{array}{ccccccc} L_m & L_{m+1} & \dots & \dots & \dots & \dots & L_n \end{array} \\
\left[\begin{array}{ccccccc}
(-1)^m q_m & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & (-1)^{m+1} q_{m+1} & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & \dots & 1 & 0 & 0 & 0 \\
\dots & 0 & 1 & \dots & \dots & \dots & \dots \\
\dots & \dots & 0 & \dots & \dots & 1 & \dots \\
\dots & \dots & \dots & \dots & 1 & \dots & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & (-1)^n q_n
\end{array}\right]
\end{array}
$$

(it is this symmetric bilinear form that we call *Sturm form*).

Let us say that an element θ of Θ_L is represented by a finite sequence of elements of $\mathcal{S}_L \coprod \mathcal{S}_{L^*} \coprod \mathrm{GL}_L$, if θ is the product of the elements of this sequence.

Proposition-definition. *Let L be a finite dimensional free R-module. There exists a unique group homomorphism*

$$\mu_L : \ker \rho_L \to \mathrm{V}(R)$$

such that the image by μ_L of an element of $\ker \rho_L$ represented by a sequence

$$(q_0, q_1, \dots, q_{2m-1}; a) \quad ,$$

with $m \geq 1$, $q_k \in \mathcal{S}_{L_k}$ for $0 \leq k \leq 2m-1$ and $a \in \mathrm{GL}_L$, is the element of $\mathrm{V}(R)$ represented by the triple

$$(L_{0,2m-1}; \mathrm{S}(0, 0, \dots, 0), \mathrm{S}(q_0, q_1, \dots, q_{2m-1}))$$

This homomorphism is trivial on $\ker \rho_L / [\Theta_L, \ker \rho_L]$ and so induces a homomorphism still denoted by $\mu_L : \mathrm{A}_L \to \mathrm{V}(R)$. One denotes by

$$\mu : \mathrm{A}(R) \to \mathrm{V}(R)$$

the homomorphism induced by these μ_L.

Here is the main result of the work I am reporting on:

Theorem. *The homomorphism $\mu : \mathrm{A}(R) \to \mathrm{V}(R)$ is an isomorphism.*

Cyclic group actions on polynomial rings

Peter Symonds

Consider a cyclic group of order p^n acting on a polynomial ring $S = k[x_1, \dots, x_r]$, where k is a field of characteristic p; this is equivalent to the symmetric algebra

$S^*(V)$ on the module V generated by $x_1, \ldots, x_r$. We would like to know the decomposition of S into indecomposables.

This was calculated by Almkvist and Fossum in [1] in the case $n = 1$. They reduced the problem to the calculation of the exterior powers of V, and then gave a formula for these.

In this note we accomplish the first part for general n, that is to say the reduction of the calculation of the symmetric algebra to that of the exterior algebra. Many of the results extend to a group with normal cyclic Sylow p-subgroup, in particular to any finite cyclic group. for more details see [4].

We wish to thank Dikran Karagueuzian for providing the computer calculations using Magma that motivated this work.

Let C_{p^n} denote the cyclic group of order p^n and consider kC_{p^n}-modules, where k is a field of characteristic p.

Recall that $kC_{p^n} \cong k[X]/X^{p^n}$, where X corresponds to $g-1$ for some generator $g \in C_{p^n}$. The indecomposable representations of kC_{p^n} are $V_t = k[X]/X^t$, up to isomorphism, where $1 \leq t \leq p^n$.

First we use the main theorem of [3] to prove a conjecture of Hughes and Kemper in [2].

Theorem 1. Suppose that V is an indecomposable kC_{p^n}-module. Then $S^*(V) \cong k[a] \otimes B$ modulo modules that are induced from proper subgroups, where a is an eigenvector for C_{p^n} of degree p^n and B is a sum of homogeneous submodules of degree strictly less than $\deg a$.

Next we show that it is sufficient to work modulo induced modules.

Lemma 2. If V is a kC_{p^n} module that is known up to induced summands both over kC_{p^n} and on restriction to subgroups then V is determined up to isomorphism.

Our main results depend on analyzing Koszul complexes with a group action. They are as follows, where Ω denotes the Heller translate.

Theorem 3. For $r < p^n$ and $p^{n-1} \leq t \leq p^n$ we have $S^r(V_t) \cong \Omega^{-r}\Lambda^r(V_{p^n-t})$ modulo induced modules. In particular, if $r + t > p^n$ then $S^r(V_t)$ is induced.

We adopt the convention that $S^r = 0$ for $r < 0$.

Corollary 4. With the same conditions as above, $S^r(V_t) \cong \Omega^{p^n-t}S^{p^n-t-r}(V_t)$ modulo induced modules.

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Equivariant Gysin maps and pulling back fixed points

Bernhard Hanke
(joint work with Volker Puppe)

In [1] W. Browder proved the following result:

Let G be a finite abelian p-group, let M be an oriented smooth G-manifold and let N be an oriented PL G-manifold. Assume that M and N are without boundary and have the same dimension. Let

$$f : M \to N$$

be a continuous proper G-equivariant map. Additionally, assume that if $p = 2$ and $H < G$, then the normal bundle of M^H in M has the structure of a complex linear G-bundle. Then, if the degree of f is not divisible by p, the induced map of fixed point sets

$$f^G : M^G \to N^G$$

is surjective (put differently, any point in N^G can be pulled back under f to a point in M^G).

This result was shown by an involved argument relying heavily on the differentiability of the action on M. As Browder pointed out at the end of his paper, it would be desirable to prove versions of his theorem in more general contexts, above all to weaken the differentiability assumption on M.

In our work [2], we provide such generalizations and in particular, if G is a finite cyclic p-group (p odd), we remove the assumption of the differentiability of the G-action on M. Our approach is different from Browder's: Based on a combination of the Atiyah-Segal-tom Dieck localization theorem with equivariant Gysin maps for generalized equivariant cohomology theories, we prove a general pulling back fixed points theorem whose precise form depends on the particular cohomology theory chosen. In this respect, our discussion provides a link of Browder's result to methods building on classical Smith theory.

Starting from this general fact, ordinary cohomology with $\mathbb{F}_p$-coefficients (as in classical Smith theory) leads to a pulling back fixed points theorem for topological actions of elementary abelian p-groups, Browder's original theorem can be derived (for smooth actions) with the use of p-local unitary bordism and the above mentioned generalization to topological actions of cyclic p-groups relies on p-local K-theory together with Sullivan's K-theoretic orientation of topological bordism. A generalization of Browder's result to topological actions of finite abelian p-groups would be possible, if the following question has an affirmative answer: Let G be a finite abelian p-group (p odd) and let V be a complex G-representation without trivial direct summand. Is the Euler class of V in topological bordism theory different from zero?

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A universal construction of support varieties

Henning Krause
(joint work with Aslak B. Buan, Øyvind Solberg)

Given a category of representations (abelian or triangulated) endowed with a tensor product, we discuss a universal construction which assigns to each object its support. In many cases, all thick tensor ideals can be classified in terms of this support. The idea of this construction is based on work of Balmer [1] and has been extended more recently in joint work with Aslak Buan and Øyvind Solberg [3]. One obtains an elegant and conceptual explanation of various existing classifications. This includes the classification of thick tensor ideals for the category of perfect complexes on a scheme by Hopkins, Neeman, and Thomason [5] and a similar classification for the stable category of representations of a finite group by Benson, Carlson, and Rickard [2].

The first basic observation is the following. We assume that we work in an abelian or triangulated tensor category $\mathcal{C}$. Then two objects of $\mathcal{C}$ have the same support if and only if they generate the same thick tensor ideal of $\mathcal{C}$. Therefore only the lattice of thick tensor ideals of $\mathcal{C}$ is relevant if we want to study the support of objects of $\mathcal{C}$. Note that the tensor product on $\mathcal{C}$ induces a multiplication on this lattice of ideals. This observation motivates the following general set-up.

We define an *ideal lattice* to be a partially ordered set $L = (L, \leq)$, together with an associative multiplication $L \times L \to L$, such that the following holds.

(L1) The poset L is a *complete lattice*, that is,

$$\bigvee_{a \in A} a = \sup A \quad \text{and} \quad \bigwedge_{a \in A} a = \inf A$$

exist in L for every subset $A \subseteq L$.

(L2) The lattice L is *compactly generated*, that is, every element in L is the supremum of compact elements. (An element $a \in L$ is *compact*, if for all $A \subseteq L$ with $a \leq \sup A$ there exists some finite $A' \subseteq A$ with $a \leq \sup A'$.)

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The element $1 = \sup L$ is compact, and $1a = a = a1$ for all $a \in L$.

(L5) The product of two compact elements is again compact.

For example, the thick tensor ideals of a small triangulated tensor category form such an ideal lattice. The compact elements are precisely the finitely generated ideals.

Call $1 \neq p \in L$ *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in L$. Let $\operatorname{Spec} L$ denote the set of all primes in L and consider the usual Zariski topology. Thus the Zariski closed subsets are those of the form

$$V(a) = \{p \in \operatorname{Spec} L \mid a \leq p\}$$

for some $a \in L$. This space is *spectral* in the sense of Hochster [4] and this observation justifies the use of another topology on $\operatorname{Spec} L$, where basic Zariski open sets are turned into basic closed sets. We denote this new space by $\operatorname{Spec}^* L$ and define the support of each $a \in L$ to be

$$\operatorname{supp}(a) = \{p \in \operatorname{Spec}^* L \mid a \not\leq p\}.$$

This set is closed whenever a is compact.

A *support datum* on an ideal lattice L is a pair (X, σ) consisting of a topological space X and a map σ which assigns to each compact $a \in L$ a closed subset $\sigma(a)$ such that for all $a, b \in L$

(1) $\sigma(a \vee b) = \sigma(a) \cup \sigma(b)$,
(2) $\sigma(ab) = \sigma(a) \cap \sigma(b)$,
(3) $\sigma(1) = X$.

A *morphism* $f\colon (X, \sigma) \to (X', \sigma')$ of support data is a continuous map $f\colon X \to X'$ such that $\sigma(a) = f^{-1}(\sigma'(a))$ for all compact $a \in L$.

The following theorem formulates the universal property of the support $\operatorname{supp}(a)$.

Theorem 1. *Let L be an ideal lattice. Then the pair $(\operatorname{Spec}^* L, \operatorname{supp})$ is a support datum on L. For every support datum (X, σ) on L, there exists a unique continuous map $f\colon X \to \operatorname{Spec}^* L$ such that $\sigma(a) = f^{-1}(\operatorname{supp}(a))$ for every compact $a \in L$. The map f is defined by*

$$f(x) = \bigvee_{\substack{x \notin \sigma(c) \\ c \text{ compact}}} c \quad \text{for} \quad x \in X.$$

Let us return to our applications. We fix an abelian or triangulated tensor category $\mathcal{C}$ and let $L(\mathcal{C})$ be the lattice of thick tensor ideals. We define the spectrum of $\mathcal{C}$

$$\operatorname{Spec} \mathcal{C} = \operatorname{Spec}^* L(\mathcal{C})$$

and the support of an object $C \in \mathcal{C}$ is by definition the closed subset $\operatorname{supp}(\langle C \rangle)$ where $\langle C \rangle$ denotes the smallest thick tensor ideal of $\mathcal{C}$ containing C.

It is interesting to compute $\operatorname{Spec} \mathcal{C}$ in some specific examples and to see that this abstract notion of support coincides (up to a homeomorphism) with the more familiar definitions.

Example 1. Let $(X, \mathcal{O}_X)$ be a noetherian scheme and let $\mathcal{C}$ be the abelian category of coherent $\mathcal{O}_X$-modules. Given an object $C \in \mathcal{C}$, denote by $\operatorname{supp}_X(C)$ the usual support of X (which is a closed subset of the underlying space of X). Then $(X, \operatorname{supp}_X)$ is a support datum and the induced map $\operatorname{Spec} \mathcal{C} \to X$ is a homeomorphism.

Example 2. Let $(X, \mathcal{O}_X)$ be a quasi-compact and quasi-separated scheme and let $\mathcal{C}$ be the triangulated category of perfect complexes on X. Given an object $C \in \mathcal{C}$, denote by $\mathrm{supp}_X(C)$ the usual support of X (which is a closed subset of the space X). Then (X, supp_X) is a support datum and the induced map $\mathrm{Spec}\,\mathcal{C} \to X$ is a homeomorphism.

Example 3. Let G be a finite group and k be a field. Let $\mathcal{C}$ the stable category of finite dimensional k-linear representations of G. Given an object $C \in \mathcal{C}$, denote by $\mathrm{supp}_G(C)$ the usual cohomological support of X (which is a closed subset of the projective variety $X = \mathrm{Proj}\, H^*(G, k)$). Then (X, supp_G) is a support datum and the induced map $\mathrm{Spec}\,\mathcal{C} \to X$ is a homeomorphism.

One can define a sheaf of rings on $\mathrm{Spec}\,\mathcal{C}$, and it is interesting to note that in all three examples the homeomorphism $\mathrm{Spec}\,\mathcal{C} \to X$ can be extended to an isomorphism of ringed spaces. We refer to [1] and [3] for details.

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On spaces of homomorphisms

Frederick R. Cohen[1]

(joint work with Alejandro Adem[2])

1. Introduction

Let Γ denote a finitely generated discrete group and G a finite dimensional Lie group. The purpose of this lecture is to give properties of $Hom(\Gamma, G)$ the space of all group homomorphisms from Γ to G topologized with the compact open topology. Many beautiful properties of these spaces have been developed by Goldman, Akbulut-McCarthy, Dwyer-Wilkerson, Lannes and others . One interesting feature of these spaces is that in case Γ is finite, Jean Lannes proved that these spaces are manifolds (possibly not connected).

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2. Homologically toroidal groups

The results in this section are joint work with A. Adem and D. Cohen.

Definition 1. (1) The groups $KU^0_{\mathrm{rep}}(B\Gamma)$ and $KO^0_{\mathrm{rep}}(B\Gamma)$ are defined to be the subgroups of $KU^0(B\Gamma)$ and $KO^0(B\Gamma)$ generated by the images, for all $n \geq 1$, of the maps

$$Hom(\Gamma, U(n)) \to [B\Gamma, BU] \qquad \text{and} \qquad Hom(\Gamma, O(n)) \to [B\Gamma, BO].$$

(2) A discrete group Γ is said to be *homologically toroidal* if there is a homomorphism $\mathcal{F} \to \Gamma$ inducing a split epimorphism in integral homology, where $\mathcal{F}$ is a finite free product of free abelian groups of finite rank.

Proposition 2. *Let Γ be the fundamental group of the complement of a $K(\Gamma,1)$ arrangement. Then Γ is homologically toroidal.*

Theorem 3. *Let Γ be homologically toroidal and let ζ_1 and ζ_2 be arbitrary classes in $H^1(\Gamma;\mathbb{Z}/2\mathbb{Z})$ and $H^2(\Gamma;\mathbb{Z}/2\mathbb{Z})$. Then there is a finite dimensional orthogonal representation of Γ which factors through the abelianization of Γ with first and second Stiefel-Whitney classes given by ζ_1 and ζ_2 respectively. Moreover for these groups the Stiefel-Whitney classes induce an isomorphism*

$$KO^0_{\mathrm{rep}}(B\Gamma) \cong H^1(\Gamma, \mathbb{Z}/2) \oplus H^2(\Gamma, \mathbb{Z}/2).$$

Theorem 4. *Let Γ be homologically toroidal.*

(1) The spaces $Hom(\Gamma, U(n))$ are path-connected.

(2) For n sufficiently large, the number of path components of $Hom(\Gamma, O(n))$, $\#\pi_0(Hom(\Gamma, O(n))$, is bounded below by

$$\#\pi_0(Hom(\Gamma, O(n)) \geq |H^1(\Gamma, \mathbb{F}_2)||H^2(\Gamma, \mathbb{F}_2)|.$$

(3) Let $\mathcal{M}$ denote the complement of an arrangement of hyperplanes which happens to be aspherical; if π denotes its fundamental group and n is sufficiently large, then $\#\pi_0(Hom(\pi, O(n)) \geq |H^1(\pi, \mathbb{F}_2)||H^2(\pi, \mathbb{F}_2)|$.

3. Commuting n-tuples

A special case of a homologically toroidal group is $\Gamma = \mathbb{Z}^n$. Natural subspaces of $Hom(\mathbb{Z}^n, G)$ arise from the so-called fat wedge filtration of the product G^n where the base-point of G is 1_G. Thus F_jG^n is the subspace of G^n with at least j coordinates equal to 1_G. Define subspaces of $Hom(\mathbb{Z}^n, G)$ by the formula

$$S_n(j, G) = Hom(\mathbb{Z}^n, G) \cap F_jG^n.$$

Notice that $S_n(n-1, G) = \vee_n G$. Consider the natural inclusions $S_n(n-1, G) \to Hom(\mathbb{Z}^n, G)$ with mapping cone denoted $A_n(G)$. This section is joint work with A. Adem.

Proposition 5. *If G is a Lie group (not necessarily compact), then there are homotopy equivalences $\Sigma(\vee_n G) \vee \Sigma(A_n(G)) \to \Sigma(Hom(\mathbb{Z}^n, G))$.*

A more precise result applies in special cases.

Definition 6. A Lie group G is said to be good if the natural inclusion $I_j : S_n(j,G) \to S_n(j-1,G)$ is a cofibration for all n and j for which both spaces are non-empty.

Theorem 7. *If G is good, there are homotopy equivalences*

$$\vee_{1\leq k\leq n}\Sigma(\vee_{\binom{n}{k}} Hom(\mathbb{Z}^k,G)/S_k(1,G)) \to \Sigma(Hom(\mathbb{Z}^n,G)).$$

Remark: This theorem is a special case of properties for certain choices of simplicial spaces.

Theorem 8. *If G is either $SU(2)$ or $G = GL(n,\mathbb{F})$ for $\mathbb{F}$ given by $\mathbb{C}$, or $\mathbb{R}$, then G is good.*

Corollary 9. *If $G = SU(2)$, there are homotopy equivalences*

$$\Sigma(G\vee G)\vee\Sigma(S^6 - SO(3)) \to \Sigma(Hom(\mathbb{Z}^2,G)),$$

and

$$\vee_3\Sigma^3(S^6 - SO(3)) \to Hom(\mathbb{Z}^3,G)/S_3(G)).$$

Remark. Notice that $\Sigma(Hom(\mathbb{Z}^n,SU(2)))$ has $\vee_{q=1,2,3}(\vee_{\binom{n}{q}} Hom(\mathbb{Z}^q,G)/S_q(1,G))$ as a retract and that these summands have been identified in 9 .

Examples concerning the singular homology for $Hom(\mathbb{Z}^n,SU(2))$ are given next.

Theorem 10. *(1) The integral cohomology of the space of (ordered) commuting pairs in $SU(2)$ is given by*

$$H^i(Hom(\mathbb{Z}\oplus\mathbb{Z},SU(2)),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i=0 \\ 0 & \text{if } i=1 \\ \mathbb{Z} & \text{if } i=2 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } i=3 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i=4 \\ 0 & \text{if } i\geq 5 \end{cases}$$

(2) The integral cohomology of the space of (ordered) commuting triples in $SU(2)$ is given by

$$H^i(Hom(\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z},SU(2)),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i=0 \\ 0 & \text{if } i=1 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } i=2 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } i=3 \\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \text{if } i=4 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } i=5 \\ 0 & \text{if } i=6 \\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & \text{if } i=7 \\ 0 & \text{if } i\geq 8 \end{cases}$$

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