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## Mini-Workshop: Thick Subcategories - Classifications and Applications

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ABSTRACT. Thick subcategories of triangulated categories arise in various mathematical areas, for instance in algebraic geometry, in representation theory of groups and algebras, or in stable homotopy theory. The aim of this workshop has been to bring together experts from these fields and to stimulate interaction and exchange of ideas.

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### Introduction by the Organisers

Thick subcategories of triangulated categories have been the main topic of this workshop. Triangulated categories arise in many areas of modern mathematics, for instance in algebraic geometry, in representation theory of groups and algebras, or in stable homotopy theory. We give three typical examples of such triangulated categories:

- the category of perfect complexes of  $\mathcal{O}_X$ -modules over a scheme  $X$ ,
- the stable category of finite dimensional representations of a finite group,
- the stable homotopy category of finite spectra.

In each case, there is a classification of thick subcategories under some appropriate conditions. Recall that a subcategory of a triangulated category is *thick*, if it is a triangulated subcategory and closed under taking direct factors. Historically, the first classification has been established by Hopkins and Smith for the stable homotopy category, using the nilpotence theorem. A similar idea was then applied by Hopkins and Neeman to categories of perfect complexes over commutative noetherian rings. Later, Thomason extended this classification to schemes. For stable

categories of finite group representations, the classification of thick subcategories is due to Benson, Carlson, and Rickard.

The format of the workshop has been a combination of introductory survey lectures and more specialized talks on recent progress and open problems. The mix of participants from different mathematical areas and the relatively small size of the workshop provided an ideal atmosphere for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

**Workshop: Mini-Workshop: Thick Subcategories - Classifications and Applications****Table of Contents**

Luchozar L. Avramov	
<i>Support varieties over complete intersections</i> . . . . .	465
Paul Balmer	
<i>The spectrum of prime ideals in tensor triangulated categories</i> . . . . .	468
Apostolos Beligiannis	
<i>A Note on Virtually Gorenstein Algebras</i> . . . . .	469
Andrei Căldăraru	
<i>Derived categories of sheaves on varieties</i> . . . . .	471
Sunil K. Chebolu	
<i>Thick subcategories in stable homotopy theory (work of Devinatz, Hopkins, and Smith)</i> . . . . .	472
J.P.C. Greenlees	
<i>Triangulated categories of rational equivariant cohomology theories</i> . . . . .	480
Lutz Hille (joint with Markus Perling)	
<i>Strongly exceptional sequences of line bundles on toric varieties</i> . . . . .	489
Osamu Iyama (joint with Idun Reiten)	
<i>Tilting modules over Calabi-Yau algebras</i> . . . . .	492
Srikanth Iyengar	
<i>Thick subcategories of perfect complexes over a commutative ring</i> . . . . .	495
Bernhard Keller (joint with Marco Porta)	
<i>On well generated triangulated categories</i> . . . . .	499
Wendy Lowen	
<i>On the Gabriel-Popescu theorem</i> . . . . .	501
Amnon Neeman	
<i>A question arising from a theorem of Rosicky</i> . . . . .	504



## Abstracts

### Support varieties over complete intersections

LUCHEZAR L. AVRAMOV

The classification of thick subcategories of the stable derived category of modules over a finite group  $G$ , see [5], makes heavy use of cohomological support varieties of  $G$ -modules. We discuss constructions of such varieties for modules over commutative rings and applications in commutative algebra.

Let  $(R, \mathfrak{m}, k)$  be a local ring, that is, a commutative ring with unique maximal ideal  $\mathfrak{m}$  and residue field  $k$ . Set  $\mathcal{E} = \text{Ext}_R^*(k, k)$  and  $e = \text{rank}_k \mathcal{E}^1$ . One has

$$e \geq d \quad \text{and} \quad \text{rank}_k \mathcal{E}^2 \geq \binom{e}{2} + e - d$$

where  $d$  is the Krull dimension of  $R$ . When equality holds in the first, respectively, the second, formula  $R$  is said to be *regular*, respectively *complete intersection*.

Extremal conditions on numerical invariants of local rings often imply strong structural properties. For instance,  $R$  is regular if and only if it has finite global dimension. It is *complete intersection* if and only if its  $\mathfrak{m}$ -adic completion  $\widehat{R}$  is isomorphic to  $Q/(\mathbf{f})$  where  $Q$  is a regular local ring and  $\mathbf{f} = f_1, \dots, f_c$  is a  $Q$ -regular sequence; i.e.,  $f_i$  is a non-zero-divisor on  $Q/(f_1, \dots, f_{i-1})$  for  $i = 1, \dots, c$ . One may assume  $\mathbf{f} \subseteq \mathfrak{q}^2$ , where  $\mathfrak{q}$  is the maximal ideal of  $Q$ . For simplicity, we assume  $R$  is complete and  $k$  is algebraically closed, and fix  $Q$  and  $\mathbf{f}$  as above.

- (1) Let  $\mathcal{S} = R[\chi_1, \dots, \chi_c]$  be a polynomial ring where each indeterminate  $\chi_i$  has cohomological degree 2. Let  $M, N$  be finitely generated  $R$ -modules. There exist natural homomorphisms of graded rings

$$\text{Ext}_R^*(M, M) \xleftarrow{\xi_M} \mathcal{S} \xrightarrow{\xi_N} \text{Ext}_R^*(N, N)$$

(Ext's are equipped with Yoneda products) and each  $\gamma \in \text{Ext}_R^*(M, N)$  satisfies

$$\xi_N(\chi_j) \cdot \gamma = \gamma \cdot \xi_M(\chi_j) \quad \text{for} \quad j = 1, \dots, c.$$

Thus,  $\xi_M(\mathcal{S})$  and  $\xi_N(\mathcal{S})$  are central subalgebras and both maps induce the same structure of  $\mathcal{S}$ -module on  $\text{Ext}_R^*(M, N)$ ; it is finitely generated, see [9].

Set  $\mathcal{R} = \mathcal{S}/\mathfrak{m}\mathcal{S}$ . The map  $\xi_k$  induces an isomorphism  $\mathcal{R} \cong \xi_k(\mathcal{S})$ . This polynomial ring is used in [1] to introduce a *cohomological support variety*  $V_R^*(M) \subseteq k^c$ .

- (2) As  $\mathfrak{m}$  annihilates  $\mathcal{M} = \text{Ext}_R^*(M, k)$ , this graded  $R$ -module is naturally a graded  $\mathcal{R}$ -module. Let  $V_R^*(M)$  be the zero-set in  $k^c$  of the ideal  $\text{ann}_{\mathcal{R}} \mathcal{M} \subseteq \mathcal{R}$ . This is an algebraic cone, with  $V_R^*(M) = \{0\}$  if and only if  $\text{proj dim}_R M < \infty$ .

In fact,  $V_R^*(M)$  does not depend on  $Q$ , and admits a different description:

- (3) Identify  $k^c$  with  $k \otimes_Q(\mathbf{f})$  by sending the  $i$ th element of the standard basis to  $1 \otimes f_i$ , and let  $\bar{a}$  denote the image in  $k$  of  $a \in R$ . One then has

$$V_R^*(M) \cong \{(\bar{a}_1, \dots, \bar{a}_c) \in k^c \mid \text{proj dim}_{Q/(a_1 f_1 + \dots + a_c f_c)} M = \infty\}$$

As  $Q$  is regular,  $\text{proj dim}_Q M < \infty$  always holds. Thus,  $V_R^*(M)$  is a set of unstable directions in the ‘space of relations’ of  $R$ .

For an elementary abelian  $p$ -group  $G = (\mathbb{Z}/p\mathbb{Z})^c$  over a field of characteristic  $p > 0$ , the group ring  $kG$  is isomorphic to  $S = k[x_1, \dots, x_c]/(x_1^p, \dots, x_c^p)$ , a complete intersection, and  $V_G(M) \cong V_S^*(M)$ . There is a catch:  $V_G(M)$  is a subset of  $kx_1 + \dots + kx_c$ , while  $V_S^*(M)$  lives in the  $kx_1^p + \dots + kx_c^p$ .

Using both descriptions above, it was shown in [1] that varieties over complete intersections satisfy analogs of most properties of varieties over group rings, see [4], but the following question was left unanswered at the time:

- (4) Is *every* algebraic cone in  $k^c$  equal to  $V_R^*(M)$  for some  $R$ -module  $M$ ?

The proof for groups hinges on an equality,  $V_G(M) \cap V_G(N) = V_G(M \otimes_k N)$ , that has no analog over  $R$  where there is no substitute for the diagonal action of  $G$  on a tensor product. An interpretation of the intersection was found in [2]:

- (5) Let  $V_R^*(M, N)$  denote the zero-set in  $k^c$  of the ideal  $\text{ann}_{\mathcal{R}}(\text{Ext}_R^*(M, N) \otimes_R k)$ . By (1) it is an algebraic cone. One has  $V_R^*(M, k) = V_R^*(M) = V_R^*(k, N)$ , and  $V_R^*(M, N) = \{0\}$  if and only if  $\text{Ext}_R^n(M, N) = 0$  for all  $n \gg 0$ . In general,

$$V_R^*(M, N) = V_R^*(M) \cap V_R^*(N)$$

The last equality uncovers a striking property of complete intersections: **(ee)** A pair  $(M, N)$  of  $R$ -modules satisfies  $\text{Ext}_R^n(M, N) = 0$  for all  $n \gg 0$  if and only if  $\text{Ext}_R^n(N, M) = 0$  for all  $n \gg 0$ . On the other hand, if  $R$  satisfies **(ee)**, then by varying  $N$  in the pair  $(R, N)$  one gets  $\text{inj dim}_R R < \infty$  is finite; that is,  $R$  is Gorenstein. Two other conditions interpolate between complete intersection and Gorenstein: **(gap)** There is an integer  $g(R)$ , such that the vanishing of  $\text{Ext}_R^n(M, N)$  for  $g(R)$  consecutive values of  $n$  implies its vanishing for all  $n \gg 0$ , and **(ab)**  $R$  is Gorenstein, and there is an integer  $h(R)$  such that  $\text{Ext}_R^n(M, N) = 0$  for all  $n \gg 0$  implies  $\text{Ext}_R^n(M, N) = 0$  for all  $n > h(R)$ . From [2], [8], one gets implications

$$\text{Complete intersection} \implies \mathbf{(gap)} \implies \mathbf{(ab)} \implies \mathbf{(ee)} \implies \text{Gorenstein}$$

Relatively simple examples in [8], [12], show that the arrow on the left is not invertible. Subtle examples in [11] prove that neither is the one on the right.

These results raise interesting questions:

- (6) Do the intermediate conditions above define distinct classes of rings? What are the ring-theoretic and homological properties of rings in these classes?

Constructions of support varieties have been proposed for local rings that are not assumed to be complete intersection. Even when  $\mathbf{f}$  is not a regular sequence, the equality in (3) defines an algebraic set, see [10]. On the other hand, when  $R$  contains a field  $K$  the Hochschild cohomology of  $R$  over  $K$  is a graded-commutative

algebra and  $\text{Ext}_R^*(M, N)$  is a graded module over it; this structure is used to introduce a notion of cohomological variety in [13].

Examples show that, in general, neither construction reflects faithfully the homological properties of  $M$ . This should not come as a surprise, as cohomological constructions over commutative rings are not well suited to produce classical geometric objects. For instance, the algebra  $\mathcal{E} = \text{Ext}_R^*(k, k)$  is noetherian precisely when  $R$  is complete intersection, see [6]. Else, it is highly non-commutative, in the precise sense that it is the universal enveloping algebra of an infinite dimensional Lie algebra that has a finite dimensional solvable radical. In several cases  $\mathcal{E}$  is known to contain a free non-commutative graded subalgebra, and it is conjectured that it always does, unless  $R$  is complete intersection.

On the other hand, when  $R$  is complete intersection both ‘general’ constructions above yield a variety isomorphic to  $V_R^*(M, N)$ , see (5). The approach through Hochschild cohomology is used in [7] to answer question (4) in the affirmative when  $R$  contains a field. A positive answer over arbitrary complete intersections is obtained in [3] as part of a more general program. Such results suggest that cohomological support varieties over a complete intersection local ring  $R$  might provide tools for classifying thick subcategories of the stable derived category of  $R$ -modules, even though it does not have internal tensor products.

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## The spectrum of prime ideals in tensor triangulated categories

PAUL BALMER

We define the *spectrum* of a tensor triangulated category  $\mathcal{K}$  as the set of so-called *prime ideals*, endowed with a suitable topology, called the Zariski topology. In this very generality, the spectrum is the universal space in which one can define *supports* for objects of  $\mathcal{K}$  in a reasonable way. This construction is functorial with respect to all tensor triangulated functors. Several elementary properties of schemes hold for such spaces, *e.g.* the existence of generic points for irreducible closed subsets, as well as quasi-compactness. It is in fact a spectral space in the sense of Hochster.

We establish in complete generality a classification of (radical) thick  $\otimes$ -ideal subcategories in terms of arbitrary unions of closed subsets of the spectrum having quasi-compact complements (Thomason's theorem for schemes, *mutatis mutandis*). We also equip this spectrum with a sheaf of rings, turning it into a locally ringed space. We show that our spectrum unifies the schemes of algebraic geometry and the projective support varieties of modular representation theory. This relies upon a sort of converse to the classification theorem, which asserts that a spectral space which classifies (radical) thick  $\otimes$ -ideal subcategories is necessarily isomorphic to the spectrum. This result was first obtained by myself assuming the candidate space be topologically noetherian, and was generalized as stated above by Buan, Krause and Solberg.

The computation of the spectrum in examples uses this key result and the various classifications, by Hopkins (homotopy theory), Hopkins, Neeman, Thomason (algebraic geometry), Benson, Carlson, Rickard (modular representation theory of finite groups), Friedlander and Pevtsova (finite group schemes).

In the sequel, we consider strongly closed tensor triangulated categories meaning that we assume the symmetric monoidal structure  $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$  is closed, *i.e.* admits an internal Hom functor which is bi-exact, and that all objects are strongly dualizable, *i.e.* that the formula  $\underline{Hom}(a, b) = \underline{Hom}(a, 1) \otimes b$  holds for all  $a, b \in \mathcal{K}$ .

Our first main theorem in this subject (that we can call Tensor Triangular Geometry) is the following:

**Theorem 1.** *Let  $\mathcal{K}$  be a strongly closed tensor triangulated category. Assume that  $\mathcal{K}$  is idempotent complete. Then, if the support of an object  $a \in \mathcal{K}$  can be decomposed as  $\text{supp}(a) = Y_1 \cup Y_2$  for disjoint closed subsets  $Y_1, Y_2 \subset \text{Spec}(\mathcal{K})$ , with each open complement  $\text{Spec}(\mathcal{K}) \setminus Y_i$  quasi-compact, then the object itself can be decomposed as a direct sum  $a \simeq a_1 \oplus a_2$  with  $\text{supp}(a_i) = Y_i$  for  $i = 1, 2$ .*

For  $G$  a finite group and  $k$  a field of positive characteristic (dividing the order of the group), and for  $\mathcal{K} = kG\text{-stab}$ , the stable category of  $kG$ -modules modulo projective-injective ones, the above result is a celebrated theorem of Carlson.

When  $\text{Spec}(\mathcal{K})$  is noetherian, we can use the above result to describe the subquotients of the filtration of  $\mathcal{K}$  by (any) dimension of the support of objects:  $\mathcal{K} \supset \cdots \mathcal{K}_{(p)} \supset \mathcal{K}_{(p-1)} \supset \cdots$  as follows. The idempotent completion of  $\mathcal{K}_{(p)}/\mathcal{K}_{(p-1)}$



is equivalent to the idempotent completion of the coproduct over all points of  $\text{Spec}(\mathcal{K})$  of dimension exactly  $p$  of the categories of finite length objects in the corresponding localizations.

This result seems new both in representation theory and in algebraic geometry and provides new spectral sequences in  $K$ -theory or in Witt theory, for instance.

## A Note on Virtually Gorenstein Algebras

APOSTOLOS BELIGIANNIS

Throughout  $\Lambda$  denotes an Artin algebra. We denote by  $\text{Mod}(\Lambda)$  the category of right  $\Lambda$ -modules and by  $\text{mod}(\Lambda)$  the full subcategory of finitely generated  $\Lambda$ -modules. We also let  $\mathcal{P}_\Lambda$ , resp.  $\mathcal{I}_\Lambda$ , be the category of finitely generated projective, resp. injective, right  $\Lambda$ -modules. A full subcategory  $\mathcal{U}$  of  $\text{Mod}(\Lambda)$  or of  $\text{mod}(\Lambda)$  is called **thick** if  $\mathcal{U}$  is closed under direct summands, extensions, kernels of epimorphisms and cokernels of monomorphisms. Interesting examples of thick subcategories include the full subcategories  $\mathcal{P}_\Lambda^{<\infty}$  and  $\mathcal{I}_\Lambda^{<\infty}$  of  $\text{mod}(\Lambda)$  consisting of all modules with finite projective dimension and finite injective dimension respectively. For a full subcategory  $\mathcal{U}$  of  $\text{mod}(\Lambda)$ , we denote by  $\text{Thick}(\mathcal{U})$  the thick subcategory of  $\text{mod}(\Lambda)$  generated by  $\mathcal{U}$ , e.g. the smallest thick subcategory of  $\text{mod}(\Lambda)$  which contains  $\mathcal{U}$ . For example  $\text{Thick}(\mathcal{P}_\Lambda) = \mathcal{P}_\Lambda^{<\infty}$  and  $\text{Thick}(\mathcal{I}_\Lambda) = \mathcal{I}_\Lambda^{<\infty}$ .

**Problem.** Let  $\mathcal{U}$  be a full subcategory of  $\text{mod}(\Lambda)$ .

- (1) Describe the thick subcategory  $\text{Thick}(\mathcal{U})$ . For instance, when  $\text{Thick}(\mathcal{U})$  is contravariantly finite?
- (2) In particular when  $\text{Thick}(\mathcal{P}_\Lambda \cup \mathcal{I}_\Lambda)$  is contravariantly finite?

Note that by a result of Krause and Solberg [4] contravariant finiteness of  $\text{Thick}(\mathcal{U})$  is equivalent to covariant finiteness. The second question in the above problem is related to virtually Gorenstein Algebras, introduced in [3] and studied further in [1] (see [2] for a short account). This class of algebras form a natural generalization of Gorenstein algebras and algebras of finite representation type. To recall the definition of virtually Gorenstein algebras and to explain the connection with thick subcategories, we need some definitions and notation. For a subcategory  $\mathcal{A}$  of  $\text{Mod}(\Lambda)$  or  $\text{mod}(\Lambda)$ ,  ${}^\perp\mathcal{A}$  consists of all modules  $X$  in  $\text{Mod}(\Lambda)$  or  $\text{mod}(\Lambda)$  respectively such that  $\text{Ext}_\Lambda^n(X, A) = 0$ ,  $\forall A \in \mathcal{A}$ ,  $\forall n \geq 1$ . The subcategory  $\mathcal{A}^\perp$  is defined dually. We denote by  $\text{CM}(\mathbf{P}_\Lambda)$  the full subcategory of  $\text{Mod}(\Lambda)$  consisting of the *Cohen-Macaulay* modules, i.e. modules  $A$  for which there exists an exact sequence  $0 \rightarrow A \rightarrow P^0 \rightarrow P^1 \rightarrow \dots$  where the  $P^i$  are projective and  $\text{Ker}(P^i \rightarrow P^{i+1}) \in {}^\perp\Lambda$ ,  $\forall i \geq 0$ . We set  $\mathfrak{P}_\Lambda^{<\infty} = \text{CM}(\mathbf{P}_\Lambda)^\perp$  and  $\mathfrak{I}_\Lambda^{<\infty} = {}^\perp\text{CoCM}(\mathbf{I}_\Lambda)$ , and  $\text{CM}(\mathcal{P}_\Lambda) = \text{CM}(\mathbf{P}_\Lambda) \cap \text{mod}(\Lambda)$  and  $\text{CoCM}(\mathcal{I}_\Lambda) = \text{CoCM}(\mathbf{I}_\Lambda) \cap \text{mod}(\Lambda)$ . Note that  $\text{CM}(\mathcal{P}_\Lambda)$  is the full subcategory of  $\text{mod}(\Lambda)$  of modules with  $G$ -dimension zero in the sense of Auslander.

**Definition.** The algebra  $\Lambda$  is called **virtually Gorenstein** if  $\mathfrak{P}_\Lambda^{<\infty} = \mathfrak{I}_\Lambda^{<\infty}$ .

It is shown in [1] that for any algebra it holds  $\mathfrak{P}_\Lambda^{\prec\infty} \cap \text{mod}(\Lambda) = \mathfrak{I}_\Lambda^{\prec\infty} \cap \text{mod}(\Lambda)$ , which is a thick subcategory of  $\text{mod}(\Lambda)$ , and moreover  $\text{CM}(\mathbf{P}_\Lambda)$  is always contravariantly finite in  $\text{Mod}(\Lambda)$ . Moreover  $\Lambda$  is virtually Gorenstein iff  $\mathfrak{P}_\Lambda^{\prec\infty} \cap \text{mod}(\Lambda)$  is contravariantly finite (or equivalently covariantly finite) in  $\text{mod}(\Lambda)$ . Henning Krause observed that we always have an equality:  $\text{Thick}(\mathcal{P}_\Lambda \cup \mathcal{I}_\Lambda) = \mathfrak{P}_\Lambda^{\prec\infty} \cap \text{mod}(\Lambda)$ . In other words virtual Gorensteinness, originally defined using infinitely generated modules, may be defined using thick subcategories of finitely generated modules as follows:

**Lemma.**  $\Lambda$  is virtually Gorenstein iff  $\text{Thick}(\mathcal{P}_\Lambda \cup \mathcal{I}_\Lambda)$  is contravariantly finite (or equivalently covariantly finite) in  $\text{mod}(\Lambda)$ . Moreover if  $\Lambda$  is virtually Gorenstein, then  $\text{CM}(\mathcal{P}_\Lambda)$  is contravariantly finite in  $\text{mod}(\Lambda)$ .

If  $\Lambda$  is in particular Gorenstein, then it is not difficult to see that we have an equality:  $\text{Thick}(\mathcal{P}_\Lambda \cup \mathcal{I}_\Lambda) = \mathcal{P}_\Lambda^{\prec\infty} = \mathcal{I}_\Lambda^{\prec\infty}$ . A natural question then arises, see [1]:

**Question.** Are all Artin algebras virtually Gorenstein?

This question gains its interest from the fact that virtually Gorenstein algebras, besides that they provide a natural enlargement of Gorenstein algebras and algebras of finite representation type, form a class of algebras which is very well behaved from many different aspects, see [1] for more information. In particular a virtually Gorenstein algebra satisfies the Gorenstein Symmetry Conjecture [1, 3].

Last years there was a common feeling shared by many people that there should exist an example of an algebra which is not virtually Gorenstein. By the above Lemma, to find such an example it suffices to find an example of an algebra  $\Lambda$  such that  $\text{CM}(\mathcal{P}_\Lambda)$  fails to be contravariantly finite in  $\text{mod}(\Lambda)$ . During the Workshop, Osamu Iyama pointed out that such an example was recently constructed by Yuji Yoshino [6], see also [5]. (Srikanth Iyengar also pointed out that such examples should possibly constructed using recent results of David Jorgensen and Liana Segal).

**The Example.** (Yoshino [6]) Let  $\mathbb{K}$  be a field and consider the ideal  $I := \langle xz - y^2, yx - z^2, zy - x^2 \rangle$  in the polynomial ring  $\mathbb{K}[x, y, z]$ . Set  $\Gamma := \mathbb{K}[x, y, z]/I$ . This is a one-dimensional Cohen-Macaulay non-Gorenstein homogeneous ring over the field  $\mathbb{K}$ . Let  $\Lambda := \Gamma/x^2\Gamma$  which is a 6-dimensional  $\mathbb{K}$ -algebra with radical cubed zero which is non-Gorenstein. More explicitly it is easy to see that  $\Lambda$  has the following presentation  $\Lambda = \mathbb{K}[x, y, z]/J$  where  $J := \langle x^2, yz, y^2 - xz, z^2 - yx \rangle$ . It is shown in [6] (in a more general setting) that the trivial  $\Lambda$ -module  $\mathbb{K}$  has no right  $\text{CM}(\mathcal{P}_\Lambda)$ -approximation, hence  $\text{CM}(\mathcal{P}_\Lambda)$  fails to be contravariantly finite in  $\text{mod}(\Lambda)$ . The proof uses the graded structure of  $\Lambda$  and its Hilbert series. Consequently, by the above Lemma,  $\Lambda$  is an example of a finite-dimensional  $\mathbb{K}$ -algebra which is not virtually Gorenstein.

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## Derived categories of sheaves on varieties

ANDREI CĂLDĂRARU

One of the most active areas of study of triangulated categories is in algebraic geometry, primarily motivated by Kontsevich’s homological mirror symmetry conjecture [8]. My talk consisted of a survey of the most important results in the field, with the emphasis being placed on the new notion of stability condition on a triangulated category due to Bridgeland [4].

The first half of the talk was devoted to presenting by-now classic results. Mukai’s equivalence [9]  $\mathbf{D}_{\text{coh}}^b(A) \cong \mathbf{D}_{\text{coh}}^b(\hat{A})$  between the derived category of an abelian variety  $A$  and its dual  $\hat{A}$  was presented alongside Bondal and Orlov’s rigidity result [1], which can be stated as saying that varieties with ample or anti-ample canonical class have no non-trivial derived equivalent partners. We stated Orlov’s result [10] that any derived equivalence between smooth varieties in characteristic zero is induced by a unique kernel, and we used a technique similar to those of Happel [6] and Keller [7] to argue that this implies the derived equivalence of Hochschild homology and cohomology groups. We also presented Bridgeland’s theorem [2] on the invariance of derived categories of threefolds under flops, as well as the Bridgeland-King-Reid [3] version of the McKay correspondence: under certain assumptions, if  $G$  is a finite group acting on a smooth manifold  $X$ , the derived category  $\mathbf{D}_{\text{coh}}^b([X/G])$  of  $G$ -equivariant sheaves on  $X$  is equivalent to the derived category  $\mathbf{D}_{\text{coh}}^b(Y)$  of a crepant resolution  $Y$  of the scheme quotient  $X/G$ .

During the second half of the talk we reviewed Bridgeland’s definition of a stability condition on a triangulated category [4], following ideas of Douglas [5].

As an application, we discussed stability conditions on the derived category of sheaves on a smooth curve  $X$ , and in particular we argued that while rotating the standard stability structure on  $X$ , one reaches a point where the structure sheaf of a point  $P$ ,  $\mathcal{O}_P$ , stops being an element of the heart of the t-structure of the corresponding stability condition. At that point the usual triangle

$$\mathcal{J}_P \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_P \rightarrow \mathcal{J}_P[1]$$

which arises from the short exact sequence

$$0 \rightarrow \mathcal{J}_P \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_P \rightarrow 0$$

in the standard t-structure, becomes the short exact sequence

$$0 \rightarrow \mathcal{O}_P[-1] \rightarrow \mathcal{J}_P \rightarrow \mathcal{O}_X \rightarrow 0$$

in the new t-structure.

A discussion relating this to tilting a t-structure with respect to a torsion pair and interpretations from physics ended the talk.

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### **Thick subcategories in stable homotopy theory (work of Devinatz, Hopkins, and Smith).**

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In this series of lectures we give an exposition of the seminal work of Devinatz, Hopkins, and Smith which is surrounding the classification of the thick subcategories of finite spectra in stable homotopy theory. The lectures are expository and are aimed primarily at non-homotopy theorists. We begin with an introduction to the stable homotopy category of spectra, and then talk about the celebrated thick subcategory theorem and discuss a few applications to the structure of the Bousfield lattice. Most of the results that we discuss below were conjectured by Ravenel [Rav84] and were proved by Devinatz, Hopkins, and Smith [DHS88, HS98].

## 1. THE STABLE HOMOTOPY CATEGORY OF SPECTRA

Recall that in homotopy theory one is interested in studying the homotopy classes of maps between CW complexes (spaces that are built in a systematic way by attaching cells): If  $f$  and  $g$  are maps (continuous) between CW complexes  $X$  and  $Y$ , we say that they are *homotopic* if there is a map from the cylinder  $X \times [0, 1]$  to  $Y$  whose restriction to the two ends (top and bottom) of the cylinder gives  $f$  and  $g$  respectively. The homotopy classes of maps between  $X$  and  $Y$  is denoted by  $[X, Y]$ . In stable homotopy theory one studies a weaker notion of homotopy called stable homotopy – maps  $f$  and  $g$  as above are said to be *stably homotopic* if  $\Sigma^n f$  and  $\Sigma^n g$  are homotopic for some  $n$ . ( $\Sigma$  denotes the reduced suspension functor on the homotopy category of pointed CW complexes.) The notion of stable homotopy is much weaker than homotopy. For example, the obvious quotient map from the torus to the two sphere is not null homotopic but stably null homotopic. The importance of stable homotopy classes of maps comes from an old result due to Freudenthal which implies that if  $X$  and  $Y$  are finite CW complexes, then the sequence

$$[X, Y] \rightarrow [\Sigma X, \Sigma Y] \rightarrow [\Sigma^2 X, \Sigma^2 Y] \rightarrow \dots$$

eventually stabilises. The stable homotopy classes of maps from  $X$  to  $Y$  is precisely the above colimit. In particular when  $X$  is the  $n$ -sphere  $S^n$ , we get the  $n$ -th stable homotopy group of  $Y$ , denoted  $\pi_n^s(Y)$ . Computing stable homotopy groups is, in general, a more manageable problem than that of homotopy groups. However, it became abundantly clear to homotopy theorists by 1960s that in order to do serious stable calculations efficiently it is absolutely essential to have a nice category in which the objects are stabilised analogue of spaces each of which represent a cohomology theory. The finite objects of such a category can be easily described. This is called the (finite) Spanier-Whitehead category which captures finite stable phenomena, and is defined as follows. The objects are ordered pairs  $(X, n)$  where  $X$  is a finite CW complex and  $n$  is an integer, and morphisms between objects  $(X, n)$  and  $(Y, m)$  are given by

$$\{(X, n), (Y, m)\} := \operatorname{colim}_k [\Sigma^{n+k} X, \Sigma^{m+k} Y].$$

This category has a formal suspension  $\Sigma(X, n) := (X, n + 1)$  which agrees with the geometric suspension, i.e.,  $(X, n + 1) \cong (\Sigma X, n)$ . While there is no geometric desuspension, there is a formal desuspension  $\Sigma^{-1}(X, n) = (X, n - 1)$ . Thus by passing to the Spanier-Whitehead category we have inverted the suspension functor on CW complexes! Moreover, this category has a tensor triangulated structure: exact triangles are induced by mapping sequences and the product comes from the smash product of CW complexes. Although this category is the right stabilisation of finite CW complexes, it has its limitations. The key point here is that one needs infinite dimensional CW complexes to understand finite CW complexes. For example, the singular cohomology theories on finite CW complexes are represented by Eilenberg-Mac Lane spaces which are infinite dimensional. So naturally one has to enlarge the finite Spanier-Whitehead category so that it has all the desired properties; one at least demands arbitrary coproducts and the Brown representability

theorem. Building the “stable category” with all the desired properties is quite challenging. Several stable categories have been proposed; the first satisfactory category was constructed by Mike Boardman in his 1964 Warwick thesis [Boa64], and then by Frank Adams [Ada74], followed by several others. All these models share a set of properties which can be taken to be the defining properties of the stable homotopy category. Following Margolis [Mar83], we take this axiomatic approach.

**Theorem 1.** *There is a category  $\mathcal{S}$  called the stable homotopy category (whose objects are called spectra) which has the following properties.*

- (1)  $\mathcal{S}$  is a triangulated category which admits arbitrary set indexed coproducts.
- (2)  $\mathcal{S}$  has a unital, commutative and associative smash product which is compatible with the triangulation.
- (3) The sphere spectrum is a graded weak generator:  $\pi_*(X) = 0$  implies  $X = 0$ .
- (4) The full subcategory of compact objects of  $\mathcal{S}$  is equivalent to the Spanier-Whitehead category of finite CW complexes.

Note that there are a lot of categories in the literature which satisfy the first three properties. It is the property (4) that makes the theorem very unique, important and non-trivial. It is also worth pointing out that the study of spectra is equivalent to that of generalised homology theories on CW complexes (theories which satisfy all the Eilenberg-Steenrod axioms except the dimension axiom.) Some standard examples of such theories are the singular homology, complex K-theory and Complex bordism which are represented by the Eilenberg-Mac Lane spectrum, the K-theory spectrum  $BU$  and the Thom spectrum  $MU$  respectively. The study of these two subjects is in turn essentially equivalent to the study of infinite loop spaces. To get a better picture of the strong connections between spectra, generalised homology theories and infinite loop spaces, we refer the reader to Adams excellent account [Ada78].

The stable homotopy category is very rich in its structural complexity, and one of the goals of the subject is to understand the global structure of this category. Doug Ravenel in the late 70s suspected some deep and interesting structure in this category (which was inspired by his algebraic calculations) and has formulated seven conjectures [Rav84] on the structure of  $\mathcal{S}$ . All but one of them have been solved by 1986, due largely to the seminal work of Devinatz, Hopkins, and Smith [DHS88, HS98]. We discuss some of these conjectures which are surrounding the thick subcategory theorem.

To start, let  $f : X \rightarrow Y$  be a map between spectra. Then we can ask several questions, the first one is when is  $f$  null-homotopic? Detecting null homotopy of maps is an extremely difficult problem. A long standing conjecture of Peter Freyd [Fre66] called Generating Hypothesis says that if  $X$  and  $Y$  are finite spectra, then  $f$  is null homotopic if  $\pi_*(f)$  is zero. Some partial results are known due to Devinatz [Dev90], the conjecture remains open; see [Fre66] for some very interesting consequences of this conjecture. The second question is when is  $f$  nilpotent under composition. The nilpotence theorem which was conjectured by Ravenel gives an

answer to this question when the spectra in question are finite. This theorem is very deep and its proof involves some hard homotopy theory. It generalises a well-known theorem of Nishida which tells that every positive degree self map of the sphere spectrum is nilpotent.

**Theorem 2.** [DHS88] (Nilpotence theorem) *There is a generalised homology theory known as  $MU_*(-)$  (complex bordism) such that a map  $f : X \rightarrow Y$  between finite spectra is nilpotent if and only if  $MU_*(f)$  is nilpotent.*

A much sharper view of the stable homotopy theory is obtained when one localises at the prime  $p$  and studies the  $p$ -local stable category whose objects are spectra whose homotopy groups are  $p$ -local, i.e.,  $\pi_*(X) \cong \pi_*(X) \otimes \mathbb{Z}_{(p)}$ . It is a very standard practise in stable homotopy theory to localise at a prime  $p$ . When this is done, there are distinguished field objects known as Morava K-theories  $K(n)$  (with the prime  $p$  suppressed) which play a key role in the  $p$ -local stable category. We now begin discussing these objects which also play an important role in the thick subcategory theorem.

## 2. MORAVA K-THEORIES AND THE THICK SUBCATEGORY THEOREM.

To set the stage, let  $\mathcal{F}$  denote the category of compact objects in the  $p$ -local stable homotopy category  $\mathcal{S}$ . There are many naturally arising properties of spectra called *generic properties* which are properties that are preserved under cofibrations, retractions and suspensions. Recall that a subcategory is *thick* precisely when it is closed under these operations. Thus one is naturally led to the study the lattice of thick subcategories of  $\mathcal{F}$ .

The lattice of thick subcategories of  $\mathcal{F}$  is determined by the Morava K-theories. For each  $n \geq 1$  there is a spectrum  $K(n)$  called the  $n$ -th Morava K-theory whose coefficient ring  $K(n)_*$  is isomorphic to  $\mathbb{F}_p[v_n, v_n^{-1}]$  with  $|v_n| = 2(p^n - 1)$ . We also set  $K(0)$  to be the rational Eilenberg-Mac Lane spectrum and  $K(\infty)$  the mod- $p$  Eilenberg-Mac Lane spectrum. These theories have the following pleasant properties.

- (1) For every spectrum  $X$ ,  $K(n) \wedge X$  has the homotopy type of a wedge of suspensions of  $K(n)$ .
- (2) Künneth isomorphism:  $K(n)_*(X \wedge Y) \cong K(n)_*X \otimes_{K(n)_*} K(n)_*Y$ . In particular  $K(n)_*(X \wedge Y) = 0$  if and only if either  $K(n)_*X = 0$  or  $K(n)_*Y = 0$ .
- (3) If  $X \neq 0$  and finite, then for all  $n \gg 0$ ,  $K(n)_*X \neq 0$ .
- (4) For each  $n$ ,  $K(n+1)_*X = 0$  implies  $K(n)_*X = 0$
- (5) (Nilpotence theorem) Morava K-theories detect ring spectra: If  $R$  is a non-trivial ring spectrum, then there exists an  $n$  ( $0 \leq n \leq \infty$ ) such that  $K(n)_*R \neq 0$

The first three properties can be easily derived from the fact that every graded module over  $K(n)_*$  is a direct sum of suspensions of  $K(n)$ . The third property is proved in [Rav84], and the last property can be derived from the  $MU$ -version of the nilpotence theorem stated above; see [HS98] for a proof of this implication. In view of the above properties, one prefers to work with  $K(n)$  as opposed to  $MU$

because it is easier to do computations with  $K(n)$ . Set  $\mathcal{C}_0 = \mathcal{F}$ , and for  $n \geq 1$ , let  $\mathcal{C}_n := \{X \in \mathcal{F} : K(n-1)_*X = 0\}$ , and finally let  $\mathcal{C}_\infty$  denote the subcategory of contractible spectra. We can now state the celebrated thick subcategory theorem.

**Theorem 3.** [HS98] (Thick subcategory theorem) *A subcategory  $\mathcal{C}$  of  $\mathcal{F}$  is thick if and only if  $\mathcal{C} = \mathcal{C}_n$  for some  $n$ . Further these subcategories form a nested decreasing filtration of  $\mathcal{F}$ :*

$$\mathcal{C}_\infty \subsetneq \cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0$$

We say that a spectrum  $X$  is of type- $n$  if it belongs to  $\mathcal{C}_n - \mathcal{C}_{n+1}$ , and we write  $\text{type}(X) = n$ . For example the sphere spectrum is of type 0 and the mod- $p$  Moore spectrum is of type-1. The existence of type- $n$  spectra was first proved by Mitchell [Mit85].

It is not hard to prove the above theorem using the nilpotence theorem. The proof we sketch below is following Rickard. The only other tool that we need is finite localisation.

**Theorem 4.** [Mil92] (Finite Localisation) *Let  $\mathcal{C}$  be a thick subcategory of  $\mathcal{F}$ , and let  $\mathcal{D}$  denote the localising subcategory generated by  $\mathcal{C}$ . Then there is a localisation functor  $\mathcal{L}_\mathcal{C}^f : \mathcal{S} \rightarrow \mathcal{S}$  called "finite localisation away from  $\mathcal{C}$ " which has the following properties.*

- (1) *For  $X$  finite,  $\mathcal{L}_\mathcal{C}^f X = 0$  if and only if  $X$  belongs to  $\mathcal{F}$ .*
- (2) *For  $X$  arbitrary,  $\mathcal{L}_\mathcal{C}^f X = 0$  if and only if  $X$  belongs to  $\mathcal{D}$ .*
- (3)  *$\mathcal{L}_\mathcal{C}^f$  is a smashing localisation functor, i.e.,  $\mathcal{L}_\mathcal{C}^f X \cong \mathcal{L}_\mathcal{C}^f S^0 \wedge X$ , where  $S^0$  is the  $p$ -local sphere spectrum.*

The idea involved in the construction of such a localisation functor is well-known to homotopy theorists. For instance, it shows up in the proof of the Brown representability theorem. A very good treatment of this construction is also given by Rickard [Ric97] where he constructs idempotent modules in the stable module category using finite localisation.

We should mention at this point that a version of Ravenel's telescope conjecture states that every smashing localisation functor (a localisation functor that satisfies property (3) above) on  $\mathcal{S}$  is isomorphic to  $\mathcal{L}_{\mathcal{C}_n}^f$  for some integer  $n$ . This is the only conjecture of Ravenel that is still open; some experts believe that it is false, see [Rav92].

Now we give a proof (due to Rickard) of the thick subcategory theorem. Let  $\mathcal{C}$  be a non-zero thick subcategory of  $\mathcal{F}$ . Then define

$$n := \max \{l : \mathcal{C} \subseteq \mathcal{C}_l\}.$$

From property (3) of the Morava  $K$ -theories we infer that  $n$  is a well-defined non-negative integer. We claim that  $\mathcal{C} = \mathcal{C}_n$ . Note that we only have to show that  $\mathcal{C}_n \subseteq \mathcal{C}$ . In showing this inclusion we use the finite localisation functors  $\mathcal{L}_\mathcal{C}^f$ . So let  $X$  in  $\mathcal{C}_n$ . Now to show that  $X$  is in  $\mathcal{C}$ , it is enough to show that  $\mathcal{L}_\mathcal{C}^f X = 0$ . But since these functors are smashing we have to show  $X \wedge \mathcal{L}_\mathcal{C}^f S^0 = 0$ . Since every finite spectrum ( $X$ ) is Bousfield equivalent to a ring spectrum ( $X \wedge DX$ ),



we can assume without loss of generality that  $X$  is a ring spectrum. Then by property (5) it suffices to show that for all  $0 \leq l \leq \infty$ ,  $K(l)_*(X \wedge \mathcal{L}_C^f S^0) = 0$ . Further by property (2) we have to show that for each  $l$ , either  $K(l)_* X = 0$  or  $K(l)_*(\mathcal{L}_C^f S^0) = 0$ . Since  $X$  is in  $\mathcal{C}_n$ , the former holds for all  $0 \leq l < n$  by property (4). So we have to show that the latter holds for  $n \leq l \leq \infty$ . Now by the definition of  $n$ , we have for each  $n \leq l \leq \infty$ , a spectrum  $X_l$  in  $\mathcal{C}$  such that  $K(l)_*(X_l) \neq 0$ . Since  $X_l$  is in  $\mathcal{C}$ , we have  $\mathcal{L}_C^f X_l = X_l \wedge \mathcal{L}_C^f S^0 = 0$ . So clearly  $K(l)_* \mathcal{L}_C^f X_l = 0$ , but since  $K(l)_*(X_l) \neq 0$ , we must have for all  $n \leq l \leq \infty$ , that  $K(l)_* \mathcal{L}_C^f S^0 = 0$  as desired.

Note that this proof highlights the key properties of Morava K-theories which are used in proving the thick subcategory theorem, and therefore it can be adapted easily to the other algebraic settings such as derived categories of rings and stable module categories of group algebras. The role played by the Morava K-theories in the former are the residue fields and in the latter are the kappa modules; see [HPS97] for a thick subcategory theorem in an axiomatic stable homotopy category.

We now illustrate how one can use the thick subcategory theorem. Suppose  $P$  is some generic property of spectra and we want to identify the subcategory of finite spectra which satisfy  $P$ . If we can find a type- $k$  spectrum which satisfies  $P$  and a type- $(k-1)$  spectrum which does not satisfy  $P$ , that forces the subcategory in question to be  $\mathcal{C}_k$ . For example, consider the generalised homology theory  $BP\langle n \rangle$  whose coefficient ring is given by  $\mathbb{Z}_p[v_1, v_2, \dots, v_n]$  with  $|v_i| = 2(p^i - 1)$ . Using the above strategy one can easily show that the full subcategory of finite spectra which have bounded  $BP\langle n \rangle$  homology (spectra  $X$  such that  $BP\langle n \rangle_i X = 0$  for  $i \gg 0$ ) is precisely  $\mathcal{C}_{n+1}$ .

### 3. BOUSFIELD CLASSES OF FINITE SPECTRA

There are several interesting applications of the thick subcategory theorem. We focus on its applications to the Bousfield lattice – an important lattice which encapsulates the gross structure of stable homotopy theory. This was introduced by Bousfield in [Bou79a, Bou79b]. Given a spectrum  $E$ , define its *Bousfield Class*  $\langle E \rangle$  to be the collection of all spectra which are invisible to the  $E$ -homology theory, i.e., spectra  $X$  such that  $E_*(X) = 0$  or equivalently  $E \wedge X = 0$ . Then we say that spectra  $E$  and  $F$  are Bousfield equivalent if  $\langle E \rangle = \langle F \rangle$ . It is a result by Ohkawa that there is only a set of Bousfield classes. With the partial order given by reverse inclusion, the set of Bousfield classes form a lattice which is called the Bousfield lattice and will be denoted by  $\mathbf{B}$ . One can perform various operations on  $\mathbf{B}$ . The two important ones being the wedge ( $\vee$ ):  $\langle X \rangle \vee \langle Y \rangle = \langle X \vee Y \rangle$  and the smash ( $\wedge$ ):  $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$ . In this lattice, the Bousfield class of the sphere spectrum is the largest element and that of the trivial spectrum is the smallest. This lattice plays an important role in the study of modern stable homotopy theory. While much of the current knowledge about the  $\mathbf{B}$  is only conjectural, the thick subcategory theorem completely determines the Bousfield classes of finite spectra. We describe them in the next theorem which was conjectured by Ravenel.

**Theorem 5.** [HS98] (Class-invariance theorem) *Let  $X$  and  $Y$  be finite  $p$ -local spectra, then  $\langle X \rangle \leq \langle Y \rangle$  if and only if  $\text{type}(X) \geq \text{type}(Y)$ .*

Although this theorem follows as an immediate corollary to the thick subcategory theorem, it is a very non-trivial statement about finite spectra. It says that the Bousfield class of a finite spectrum is completely determined by its type.

We now discuss the Boolean algebra conjecture of Ravenel which identifies the Boolean subalgebra generated by the Bousfield classes of finite  $p$ -local spectra. But first we have to introduce some important non-nilpotent maps of finite spectra called  $v_n$ -self maps. A self map  $f : \Sigma^? X \rightarrow X$  is a  $v_n$ -self map ( $n \geq 1$ ) if  $K(n)_*(f)$  is an isomorphism and  $K(m)_*(f)$  is zero for  $m \neq n$ . For example, the degree  $p$  map on the sphere spectrum is a  $v_0$ -self map, and the Adams map [Ada66] on the Moore spectrum:  $\Sigma^? M(p) \rightarrow M(p)$  which induces isomorphism in complex  $K$ -theory is a  $v_1$ -self map. These  $v_n$ -self maps are important because give rise to periodic families in the stable homotopy groups of spheres. For example, one can iterate Adams maps and get a periodic family in  $\pi_*(S^0)$  called the  $\alpha$ -family. Showing the existence of such maps is highly non-trivial. A deep result of Hopkins and Smith called the periodicity theorem produces a wealth of such maps. More precisely:

**Theorem 6.** [HS98] (Periodicity Theorem)

- (1) *Every type- $n$  spectrum admits an asymptotically unique  $v_n$ -self map  $\phi_X : \Sigma^? X \rightarrow X$*
- (2) *If  $h : X \rightarrow Y$  is a map between type- $n$  spectra, then there exists integers  $i$  and  $j$  such that the follow diagram commutes:*

$$\begin{array}{ccc} \Sigma^? X & \xrightarrow{\Sigma^? h} & \Sigma^? Y \\ \phi_X^i \downarrow & & \downarrow \phi_Y^j \\ X & \xrightarrow{h} & Y \end{array}$$

Using this periodicity theorem it is not hard to show that the full subcategory of finite  $p$ -local spectra admitting  $v_n$  self map is precisely  $\mathcal{C}_n$ . So this theorem gives another characterisation of the thick subcategories of  $\mathcal{F}$ .

For every positive integer  $n$ , let  $F(n)$  denote some spectrum of type- $n$ . Note that the Bousfield class of  $F(n)$  is well-defined by the class-invariance theorem. Now the periodicity theorem says that  $F(n)$  admits an essentially unique  $v_n$ -self map. So let  $T(n)$  denote the mapping telescope of this  $v_n$ -self map. It follows that the Bousfield classes of  $T(n)$  is also well-defined.

A Bousfield class  $\langle E \rangle$  is said to be complemented if there exists another class  $\langle F \rangle$  such that  $\langle E \rangle \wedge \langle F \rangle = \langle 0 \rangle$  and  $\langle E \rangle \vee \langle F \rangle = \langle S^0 \rangle$ . The collection of all complemented Bousfield classes forms a Boolean algebra with respect to the smash and wedge operations and will be denoted by **BA**. Bousfield [Bou79b] showed that every possibly infinite wedge of finite spectra belongs to **BA**. A pleasant consequence of the thick subcategory theorem is the classification of the Boolean subalgebra

generated by the finite  $p$ -local spectra and their complements in the  $p$ -local sphere spectrum.

**Theorem 7.** [Rav84] (Boolean Algebra Theorem) *Let  $\mathbf{FBA}$  denote the Boolean subalgebra generated by the Bousfield classes of the finite  $p$ -local spectra and their complements in  $\langle S^0 \rangle$ . Then  $\mathbf{FBA}$  is the free (under complements, finite unions and finite intersections) Boolean algebra generated by the Bousfield classes of the telescopes  $\langle T(n) \rangle$  for  $n \geq 0$ .*

So by this theorem one can identify  $\mathbf{FBA}$  with the Boolean algebra of finite and cofinite subsets of non-negative integers: the Bousfield class  $\langle T(n) \rangle$  corresponds to the subset  $\{n\}$ , and  $\langle F(n) \rangle$  corresponds to the subset  $\{n, n+1, n+2, \dots\}$ . By the way, the Boolean algebra conjecture of Ravenel uses  $K(n)$  instead of  $T(n)$  in the above theorem; according to telescope conjecture (which is still open) these two spectra are Bousfield equivalent.

There are several other interesting sublattices of  $\mathbf{B}$  which have been studied. For example there is a distributive lattice  $\mathbf{DL}$  which consists of the Bousfield classes  $\langle X \rangle$  such that  $\langle X \rangle \wedge \langle X \rangle = \langle X \rangle$  which has some nice properties. A good discussion on the structure of the Bousfield lattice can be found in [HP99]. These authors use lattice theoretic methods to explore the structure of the Bousfield lattice. They also pose a number of interesting conjectures and study their implications.

We end by mentioning briefly one other application of the thick subcategory theorem. Thomason has given a brilliant K-theory recipe [Tho97] which refines the thick subcategory theorem and gives a classification of the triangulated subcategories of finite spectra. This recipe amounts to computing the Grothendieck groups of the thick subcategories of the finite  $p$ -local spectra. We refer the reader to [Che06] where we use this recipe to study the lattice of triangulated subcategories of finite spectra.

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## Triangulated categories of rational equivariant cohomology theories

J.P.C. GREENLEES

### 1. INTRODUCTION.

This article is designed to provide an introduction to some examples of triangulated categories that arise in the study of  $G$ -equivariant cohomology theories for a compact Lie group  $G$ . We focus on cohomology theories whose values are rational vector spaces since one may often give explicit algebraic constructions of the triangulated category in that case.

As general references for equivariant cohomology theories see [3, 13, 14].

### 2. EXAMPLES OF EQUIVARIANT COHOMOLOGY THEORIES

Here are some examples of reduced equivariant cohomology theories on a based  $G$ -space  $X$ .

- **Borel cohomology theories:**  $F^*(EG_+ \wedge_G X)$  for any non-equivariant cohomology theory  $F^*(\cdot)$ . [Here  $EG$  is the universal free  $G$  space, and  $EG_+$  is the same space with a  $G$ -fixed basepoint adjoined].
- **Equivariant K-theory**  $K_G^*(X)$ : The theory is defined for unbased compact  $G$ -spaces  $Y$  by taking  $K_G(Y)$  to be the Grothendieck group of equivariant vector bundles on  $Y$ . This defines  $K_G^0(X) = \ker(K_G(X) \longrightarrow$

$K_G(*)$ ) in the usual way, and this is extended to all degrees by Bott periodicity. Note that  $K_G^0 = K_G(*) = R(G)$ , the complex representation ring, and  $K_G^1 = 0$ .

- **Equivariant Bordism  $MU_G^*(X)$ :** The stabilized form of bordism of  $G$ -manifolds with a complex structure on their stable normal bundle defined by tom Dieck. [15]

### 3. DEFINITION OF GENUINE COHOMOLOGY THEORIES

A naïve equivariant cohomology theory is a contravariant exact functor

$$F_G^* : \text{Based } G\text{-spaces} \longrightarrow \text{Graded abelian groups.}$$

If in addition  $F_G^*(\cdot)$  is equipped with an extension to an  $RO(G)$ -graded theory in such a way that

$$F_G^{k+V}(S^V \wedge X) \cong F_G^k(X)$$

for any representation  $V$  (where  $S^V$  is the one point compactification of  $V$ ), we say that  $F_G^*(\cdot)$  is a ‘genuine’ equivariant cohomology theory. The Examples in Section 2 all have the stronger property that they are *complex stable* in the sense that

$$F_G^{k+|V|}(S^V \wedge X) \cong F_G^k(X)$$

for every complex representation  $V$ . This is clear by the Serre spectral sequence for the Borel theories, it follows from Bott periodicity for equivariant K theory, and it is built into the definition for stabilized bordism.

One reason (beyond the existence of interesting examples) for considering genuine cohomology theories is that if  $H \subseteq K$  there is an induction map  $\text{ind}_H^K : F_H^*(X) \longrightarrow F_K^*(X)$  in addition to the restriction map  $\text{res}_H^K : F_K^*(X) \longrightarrow F_H^*(X)$  that exists for any naïve theory and is induced by the projection  $G/H_+ \longrightarrow G/K_+$ . Henceforth we drop the adjective ‘genuine’ since all cohomology theories will be genuine.

It is convenient to work in a context where equivariant cohomology theories are represented. Indeed, one may form the model category  $G$ -spectra [3], which can be thought of as a category of stable based  $G$ -spaces. Thus every based  $G$ -space gives rise to a suspension spectrum  $\Sigma^\infty X$ , and for every  $G$ -equivariant cohomology theory  $F_G^*(\cdot)$  there is a  $G$ -spectrum  $F$  so that

$$F_G^*(X) = [\Sigma^\infty X, F]_G^*,$$

where  $[A, B]^G$  means maps in the homotopy category of  $G$ -spectra. Henceforth we omit the notation  $\Sigma^\infty$  for the suspension spectrum functor.

As in the case of non-equivariant spectra, one may attempt to classify thick subcategories of finite  $G$ -spectra, but there are some additional complications. For instance, if  $X$  is a finite  $p$ -local  $G$ -spectrum the geometric fixed point spectrum  $X^H$  has a chromatic type  $n_X(H)$ . N.P.Strickland [18] has studied the functions  $n_X$  that can occur. For example, chromatic Smith theory shows that  $n_X(H) \geq n_X(K) - 1$  if  $K$  is normal and of index  $p$  in  $H$ .

## 4. ORDINARY EQUIVARIANT COHOMOLOGY AND MACKEY FUNCTORS

The basic building blocks for  $G$ -spaces are the cells  $(G/H \times D^n, G/H \times S^{n-1})$  for closed subgroups  $H$  and  $n \geq 0$ . Thus the relevant 0-spheres are  $G/H_+$ . Accordingly a cohomology theory  $F_G^*(\cdot)$  satisfies the dimension axiom if  $F_G^i(G/H_+) = 0$  for  $i \neq 0$  and all closed subgroups  $H$ . A cohomology theory satisfying the dimension axiom is called an *ordinary* cohomology theory. Note also that

$$F_G^i(G/H_+) = [G/H_+, F]_G^i = [S^0, F]_H^i = \pi_{-i}^H(F)$$

so  $F$  represents an ordinary cohomology theory if and only if all its equivariant homotopy groups are concentrated in degree 0.

However the groups  $F_G^i(G/H_+)$  for various subgroups  $H$  are related. First, define the stable orbit category  $\mathcal{SO}$  to be the full subcategory of the homotopy category of  $G$ -spectra with objects  $G/H_+$ , it has morphisms  $\mathcal{SO}(G/H_+, G/K_+) = \lim_{\rightarrow V} (S^V \wedge G/H_+, S^V \wedge G/K_+)^G$ , where  $(A, B)^G$  denotes homotopy classes of  $G$ -maps. We may then define an additive functor

$$\underline{\pi}_i^G(F) : \mathcal{SO} \longrightarrow \text{Ab}$$

by  $\underline{\pi}_i^G(F)(G/H_+) = \pi_i^H(F)$ . Quite generally, any additive functor  $M : \mathcal{SO} \longrightarrow \text{Ab}$  is called a *Mackey functor*, and if we rewrite it by taking  $M'(H) := M(G/H_+)$  then the way to think of a Mackey functor is that if  $K \subseteq H$  then there is a restriction map  $\text{res}_K^H : M'(H) \longrightarrow M'(K)$  (induced by the projection  $\pi : G/K \longrightarrow G/H$ ), a conjugation map  $c_g : M'(H) \longrightarrow M'(H^g)$  (induced by right multiplication by  $g^{-1}$  as a map  $G/H^g \longrightarrow G/H$ ), and an induction map  $\text{ind}_K^H : M'(K) \longrightarrow M'(H)$  (induced by a certain *stable* map  $G/H \longrightarrow G/K$  (the dual of  $\pi$  if  $G$  is finite)). These satisfy the Mackey induction restriction formula (or Feshbach's generalization if  $G$  is of positive dimension [2]). If  $G$  is finite there is a purely algebraic definition [1], which can be shown to be equivalent to this definition via topology.

**Lemma 1.** *Ordinary cohomology theories correspond bijectively to Mackey functors.*

**Proof:** We have seen that the zeroth homotopy group of an ordinary cohomology theory defines a Mackey functor, and conversely, given a Mackey functor  $M$  we may construct a cohomology theory  $H_G^*(\cdot; M)$  by using cellular chain complexes, or alternatively construct the representing Eilenberg-MacLane  $G$ -spectrum  $HM$  directly by realising a resolution of  $M$  by free Mackey functors.  $\square$

## 5. ALL COHOMOLOGY IS ORDINARY FOR FINITE GROUPS.

It is an immediate consequence of Serre's calculation of the rational homotopy of spheres that every non-equivariant rational cohomology theory is ordinary. Here is a generalization to any finite group; a precursor for equivariant K-theory was the early result of Slominska [17]

**Theorem 1.** [12] *If  $G$  is a finite group then every rational cohomology theory  $F_G^*(\cdot)$  is ordinary:*

$$F_G^k(X) \cong \prod_n H_G^{k+n}(X; \underline{\pi}_n^G(F)).$$

**Proof:** For the proof we define a related cohomology theory. Given any injective rational Mackey functor  $I$  we may define a cohomology theory  $hI_G^*(\cdot)$  by

$$hI_G^n(X) = \text{Hom}(\underline{\pi}_n^G(X), I),$$

There are two special facts about finite groups that let us proceed.

**Lemma 2.** *If  $G$  is finite every rational Mackey functor is injective.*

**Proof:** This is due to the fact that the rational Burnside ring splits as a product of copies of  $\mathbb{Q}$  and Maschke's theorem.  $\square$

For each  $n$  we may therefore choose the map  $F \rightarrow \Sigma^n h\underline{\pi}_n^G(F)$  corresponding to the identity map of  $\underline{\pi}_n^G(F)$ , and we may assemble these to give a map

$$F \xrightarrow{\simeq} \prod_n \Sigma^n h\underline{\pi}_n^G(F).$$

**Lemma 3.** *The spectrum  $hI$  is an Eilenberg-MacLane spectrum:  $hI = HI$ .*

**Proof:** Since  $\underline{\pi}_0^G(G/H_+) = [\cdot, G/H_+]^G$  is the free functor, it is clear that  $hI$  has the correct homotopy groups in degree 0. We must calculate  $hI_G^n(G/H_+)$  for each subgroup  $H$ , and show that it is zero if  $n \neq 0$ .

For this we need to examine the functor  $\underline{\pi}_n^G(G/H_+)$ , which is made up from the groups  $\pi_n^K(G/H_+)$ . The tom Dieck splitting theorem for the  $G$ -space  $X$  states

$$\pi_n^K(X) = \bigoplus_{(L)} \pi_n(EW_K(L)_+ \wedge_{W_K(L)} X^L)$$

where the sum is over  $K$ -conjugacy classes of subgroups  $L$ . Since we are working rationally, the homotopy may be replaced by homology, and since the groups concerned are finite, there is no higher homology; since  $X = G/H_+$  is zero dimensional the result follows.  $\square$

It follows that the map  $\nu$  induces an isomorphism of  $\pi_n^H$  for all  $n$  and  $H$  and is therefore an equivalence by the Whitehead theorem: the  $G$ -spectrum  $F$  splits as a product of Eilenberg-MacLane spectra

$$F \xrightarrow{\simeq} \prod_n \Sigma^n H\underline{\pi}_n^G(F).$$

The statement about cohomology theories follows.  $\square$

## 6. ALGEBRAIC MODELS FOR CATEGORIES OF RATIONAL COHOMOLOGY THEORIES.

The idea is that for any compact Lie group  $G$  there is an abelian category  $\mathcal{A}(G)$  modelling rational  $G$ -equivariant cohomology theories. On a practical level, we want to be able to *calculate* in this homotopy category, but if we understand the category completely we can also *construct* interesting new cohomology theories [8]. The idea is that objects of  $\mathcal{A}(G)$  should be a rather small, and based on information easily accessible from the cohomology theories they represent.

**Conjecture 2.** For a compact Lie group  $G$  there is an abelian category  $\mathcal{A}(G)$  and a Quillen equivalence

$$G\text{-spectra}/\mathbb{Q} \simeq dg\mathcal{A}(G)$$

such that

- (1)  $\mathcal{A}(G)$  is abelian
- (2)  $\text{InjDim}(\mathcal{A}(G)) = \text{rank}(G)$
- (3) The category consists of sheaves of modules over a space of closed subgroups of  $G$ ; the object corresponding to a cohomology theory  $E_G^*(\cdot)$  has fibre over  $H$  built from the Borel theory  $E_{TW_G(H)}^*(ETW_G(H)_+ \wedge X^H)$ . The additional structure is built from these Borel theories using their relation under localization and inflation.
- (4) The model of  $E_G^*(\cdot)$  is built from its values on spheres and a little extra structure.

**6.1. Consequences of the conjecture.** Note immediately that if the conjecture holds we have an equivalence of homotopy categories

$$\text{Ho}(G\text{-spectra}/\mathbb{Q}) \simeq D(\mathcal{A}(G))$$

as triangulated categories. This reduces to algebra the problem of classifying rational equivariant cohomology theories and the process of calculation with them. Furthermore, it provides a universal homology theory

$$\pi_*^{\mathcal{A}(G)} : G\text{-spectra} \longrightarrow \mathcal{A}(G)$$

and an Adams spectral sequence

$$\text{Ext}_{\mathcal{A}(G)}^{*,*}(\pi_*^{\mathcal{A}(G)}(X), \pi_*^{\mathcal{A}(G)}(Y)) \Rightarrow [X, Y]_*^G$$

for calculation. Finally, because of the injective dimension of  $\mathcal{A}(G)$ , the Adams spectral sequence is only non-zero on  $s$ -line for  $0 \leq s \leq r$ , so the calculation is very accessible.

**6.2. Status of the conjecture.**

- **$G$  finite.** The conjecture is true. From the result of Section 5 it is not hard to see

$$\mathcal{A}(G) = \prod_{(H)} \mathbb{Q}W_G(H)\text{-mod.}$$



- **The circle group  $G = T$ .** Again the theorem is true. Indeed, [5] constructs  $\mathcal{A}(G)$  and shows that there is a triangulated equivalence of homotopy categories. Shipley [16] upgraded this to a Quillen equivalence. We describe the models for free  $T$ -spectra in Section 7 and the model for semifree spectra in Section 8.
- **The groups  $G = O(2), SO(3)$  and their double covers.** In this case the equivalence of homotopy categories is proved in [6, 7].
- **The tori  $G = T^g$ .** The Adams spectral sequence exists [9], and in [10, 11] we show that the Quillen equivalence holds.

## 7. FREE $T$ -SPACES.

We spend the rest of the article on the circle group  $G = T$ , and to simplify the discussion we consider actions with restricted isotropy. In this section we give a complete classification of rational cohomology theories on free  $T$ -spaces.

First note that an arbitrary space  $X$  is equivalent to a free  $T$ -space if and only if the map  $ET_+ \wedge X \rightarrow S^0 \wedge X = X$  is an equivalence. For homotopical work it is convenient to adopt this as the definition of a free space or spectrum.

**Lemma 4.** *Cohomology theories on free  $T$ -spaces are represented by free spectra.*

**Proof:** If  $X$  is free then  $[X, F]_T^* \leftarrow [X, F \wedge ET_+]_T^*$  is an isomorphism since maps from a free space into a non-equivariantly contractible space are null-homotopic. Hence  $F_T^*(\cdot)$  is represented by the free  $T$ -spectrum  $ET_+ \wedge F$ .  $\square$

We may thus concentrate on classifying free  $T$ -spectra.

**Lemma 5.** *For any free  $X$ , the homotopy groups  $\pi_*^T(X)$  are naturally a module over  $\mathbb{Q}[c]$ , where  $c$  is of degree  $-2$ . Furthermore,  $\pi_*^T(X)$  is torsion in the sense that every element is annihilated by a power of  $c$ .*

**Proof:** Note that  $[ET_+, ET_+]_*^T$  acts on  $\pi_*^T(X) = \pi_*^T(X \wedge ET_+)$ . Now calculate

$$[ET_+, ET_+]_*^T = [ET_+, S^0]_*^T = [BT_+, S^0]_*^T = H^*(BT_+) = \mathbb{Q}[c].$$

For the torsion statement, note that any element  $x \in \pi_*^T(X)$  is supported on a finite subcomplex  $K$ , and  $\pi_*^T(K)$  is bounded below. Since  $c$  is in degree  $-2$ , the statement follows.  $\square$

We are now equipped to state the classification.

**Theorem 3.** *Associating the module  $\pi_*^T(F \wedge ET_+)$  to the cohomology theory  $F_T^*(\cdot)$  gives a bijective correspondence*

$$\text{Cohomology theories on free } T\text{-spaces} \leftrightarrow \text{Torsion } \mathbb{Q}[c]\text{-modules}$$

Furthermore, for any free  $T$ -spectra  $X$  and  $F$ , there is a short exact sequence

$$0 \longrightarrow \text{Ext}_{\mathbb{Q}[c]}(\pi_*^T(\Sigma X), \pi_*^T(F)) \longrightarrow [X, F]_*^T \longrightarrow \text{Hom}_{\mathbb{Q}[c]}(\pi_*^T(X), \pi_*^T(F)) \longrightarrow 0.$$

We first need the Whitehead theorem.

**Lemma 6.** *If  $X$  and  $Y$  are free and  $f : X \rightarrow Y$  is a map inducing an isomorphism of  $\pi_*^T$ , then  $f$  is an equivalence.*

**Proof:** Change of groups and the Gysin sequence. □

**Proof:** The short exact sequence is an Adams spectral sequence. The method of proof is therefore standard. We need only note that there are realizable injectives  $F$  for which the short exact sequence exists, and that any free  $T$ -spectrum can be resolved by these.

To realize an injective, note that

$$\pi_*^T(ET_+) = \pi_*(\Sigma BT_+) = H_*(\Sigma BT_+) = \Sigma^{-1}\mathbb{Q}[c, c^{-1}]/\mathbb{Q}[c]$$

is  $c$ -divisible and hence injective. It is easy to show

$$[X, ET_+]_*^T \cong \text{Hom}_{\mathbb{Q}[c]}(\pi_*^T(X), \pi_*^T(ET_+))$$

(both sides are cohomology theories in  $X$  so only need to check on  $X = T_+$ ). This establishes the injective case.

Let us now show there are enough injectives of this form, and that they and the resulting resolutions are realizable. First, there are algebraically enough injectives of the form  $\pi_*^T(ET_+)$ . For simplicity assume that  $F$  is bounded below and of finite type. Hence we may construct an embedding  $\pi_*^T(F) \rightarrow \pi_*^T(\bigvee_i \Sigma^{n_i} ET_+)$  where the wedge is locally finite and hence equivalent to the product. We may then lift it to a map  $F \rightarrow \bigvee_i \Sigma^{n_i} ET_+ =: I$ . Since  $\mathbb{Q}[c]$  is of injective dimension 1, the mapping cone  $J$  also has injective homotopy, and, as a matter of algebra, this is necessarily isomorphic to  $\pi_*^T(\bigvee_j \Sigma^{n_j} ET_+)$  for suitable integers  $n_j$ . By the injective case of the short exact sequence we can construct a map from the cofibre to this wedge, and by 6 it is an equivalence. This gives a cofibre sequence  $F \rightarrow I \rightarrow J$ , realizing the injective resolution of  $\pi_*^T(F)$ , and where  $I$  and  $J$  are both wedges of suspensions of  $ET_+$  (for which the theorem is known). Now apply  $[X, \cdot]_*^T$  and obtain the exact sequence.

The classification of free  $T$ -spectra now follows easily. To construct enough spectra we realize a resolution of a torsion  $\mathbb{Q}[c]$ -module. To show that if  $\pi_*^T(X) \cong \pi_*^T(Y)$  then  $X \simeq Y$ , we just lift the algebraic isomorphism to a map  $X \rightarrow Y$  and then apply 6 to deduce it is an equivalence. □

**Corollary 1.** There is an equivalence of triangulated categories

$$\text{Free } T\text{-spectra} \simeq D(\text{Isomorphism classes of torsion } \mathbb{Q}[c]\text{-modules})$$

where the derived category on the right is obtained from dg torsion  $\mathbb{Q}[c]$ -modules by inverting homology isomorphisms.

**Proof:** Use the Adams short exact sequence and the fact that  $c$  is in degree 2, together with a Toda bracket argument. □

8. SEMI-FREE  $T$ -SPACES.

In this section we give a complete classification of rational cohomology theories on semi-free  $T$ -spaces (those whose isotropy groups are either 1 or  $T$ ). The pattern of the argument is very similar to that in Section 7, so we will omit most of the proofs.

First note that an arbitrary space  $X$  is equivalent to a semi-free  $T$ -space where  $E\mathcal{F}$  is the universal  $\mathcal{F}$ -space, where  $\mathcal{F}$  is the set of finite subgroups. For homotopical work it is convenient to adopt this as the definition of a semi-free space or spectrum.

**Lemma 7.** *Cohomology theories on semi-free  $T$ -spaces are represented by semi-free spectra.*  $\square$

Thus we turn to the study of semifree  $T$ -spectra.

Now, for any semi-free  $X$  we have a cofibre sequence

$$ET_+ \wedge X \longrightarrow X \longrightarrow \tilde{E}\mathcal{F} \wedge X,$$

where  $\tilde{E}\mathcal{F} = \bigcup_{V^T=0} S^V$  is  $H$ -contractible for all finite  $H$ . We described the spectra  $ET_+ \wedge X$  in Section 7, and it is easy to see that  $\tilde{E}\mathcal{F} \wedge X \simeq \tilde{E}\mathcal{F} \wedge X^T$ , so that  $\tilde{E}\mathcal{F} \wedge X$  is determined by the graded rational vector space  $\pi_*(X^T)$ . It thus remains to describe how to splice these two pieces of information. For this we take the cue from the classical Localization theorem which states that if  $X$  is finite and semifree then

$$\begin{aligned} H^*(ET_+ \wedge_T X)[1/c] &\cong H^*(ET_+ \wedge_T X^T)[1/c] \cong H^*(BT_+)[1/c] \otimes H^*(X^T) \\ &= \mathbb{Q}[c, c^{-1}] \otimes H^*(X^T). \end{aligned}$$

Thus the Borel cohomology of  $X$  very nearly determines the cohomology of the fixed point space. Inspired by this we may define an appropriate category.

**Definition 1.** The localization category  $\mathcal{A}$  has objects  $\beta : N \longrightarrow \mathbb{Q}[c, c^{-1}] \otimes V$ , where  $N$  is a  $\mathbb{Q}[c]$ -module, and  $\beta$  is a  $\mathbb{Q}[c]$ -map which becomes an isomorphism when  $c$  is inverted. We call  $N$  the *nub*,  $V$  the *vertex* and  $\beta$  the *basing map*. The morphisms in  $\mathcal{A}$  are given by commutative squares in which the map is the identity on  $\mathbb{Q}[c, c^{-1}]$ .

The following lemma is an elementary exercise.

**Lemma 8.** *The category  $\mathcal{A}$  is abelian and of injective dimension 1. In fact the objects  $(I \longrightarrow 0)$  with  $I$  an injective torsion  $\mathbb{Q}[c]$ -module, and  $(\mathbb{Q}[c, c^{-1}] \otimes V \xrightarrow{1} \mathbb{Q}[c, c^{-1}] \otimes V)$  together give enough injectives.*  $\square$

The final three results are direct counterparts of results in Section 7.

**Lemma 9.** *For any semi-free  $X$ , the object*

$$\pi_*^{\mathcal{A}}(X) = \left( \pi_*^T(X \wedge DET_+) \longrightarrow \pi_*^T(X \wedge DET_+ \wedge \tilde{E}\mathcal{F}) \cong \mathbb{Q}[c, c^{-1}] \otimes \pi_*(X^T) \right)$$

*is an object of  $\mathcal{A}$ .*

**Proof:** The cofibre of the map  $X \wedge DET_+ \longrightarrow X \wedge DET_+ \wedge \tilde{E}\mathcal{F}$  is  $X \wedge DET_+ \wedge ET_+$ ; this is free and hence its homotopy is annihilated when  $c$  is inverted.  $\square$

We are now equipped to state the classification.

**Theorem 4.** *Associating the module  $\pi_*^{\mathcal{A}}(F)$  to the cohomology theory  $F_T^*(\cdot)$  gives a bijective correspondence*

*Cohomology theories on semifree  $T$ -spaces  $\leftrightarrow$  Isomorphism classes of objects of  $\mathcal{A}$*

*Furthermore, for any semifree  $T$ -spectra  $X$  and  $F$ , there is a short exact sequence*

$$0 \longrightarrow \text{Ext}_{\mathcal{A}}(\pi_*^T(\Sigma X), \pi_*^T(F)) \longrightarrow [X, F]_*^T \longrightarrow \text{Hom}_{\mathcal{A}}(\pi_*^T(X), \pi_*^T(F)) \longrightarrow 0. \quad \square$$

**Corollary 2.** There is an equivalence of triangulated categories

$$\text{Semifree } T\text{-spectra} \simeq D(\mathcal{A})$$

where the derived category on the right is obtained from dg objects of  $\mathcal{A}$  by inverting homology isomorphisms.  $\square$

## 9. SOME APPLICATIONS.

Here are some consequences which do not require much explanation to state.

- The Atiyah-Hirzebruch spectral sequence for  $F_T^*(X)$  with  $X$  free collapses if and only if  $\pi_*^T(F \wedge ET_+)$  is injective over  $\mathbb{Q}[c]$ .
- (McClure) The Atiyah-Hirzebruch spectral sequence for  $K_T^*(X)$  with  $X$  free always collapses.
- For an arbitrary semifree space  $X$ , the K-theory  $K_T^*(X)$  is determined by the map

$$H_*(ET_+ \wedge_T X^T) \longrightarrow H_*(ET_+ \wedge_T X).$$

- There are infinitely many non-isomorphic finite indecomposable semi-free spectra with  $\pi_*^T(ET_+ \wedge X) \cong \pi_*^T(ET_+ \wedge (S^0 \vee S^2 \vee S^4))$  and  $\pi_*(X^T) \cong \pi_*(S^0 \vee S^2 \vee S^4)$

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**Strongly exceptional sequences of line bundles on toric varieties**

LUTZ HILLE

(joint work with Markus Perling)

Let  $k$  be an algebraically closed field. All varieties and algebras are over the fixed ground field  $k$ . A classical result of Beilinson states that the bounded derived category of coherent sheaves  $\mathcal{D}^b(\mathbb{P}^n)$  on the projective space is equivalent to the bounded derived category  $\mathcal{D}^b(\text{mod-}A)$  of finitely generated right modules over the

finite dimensional algebra  $A := \text{End}(\oplus_{i=0}^n \mathcal{O}(i))$ . More precisely, the functor

$$\mathbb{R}\text{Hom}(\oplus_{i=0}^n \mathcal{O}(i), -) : \mathcal{D}^b(\mathbb{P}^n) \longrightarrow \mathcal{D}^b(\text{mod-}A)$$

is an equivalence of triangulated categories. It is an open question whether a similar equivalence exists for any smooth projective toric variety  $X$ . The aim of this talk is to discuss new results in this direction. In the first part we present some motivation for the problem. In the second part we consider surfaces and threefolds.

## 1. MOTIVATION

We shortly mention some motivation for considering derived equivalences for toric varieties.

1) *Homological mirror symmetry.*

The famous conjecture of Kontsevich states that the bounded derived category of coherent sheaves  $\mathcal{D}^b(Y)$  for a Calabi-Yau variety  $Y$  is equivalent to the Fukaya category on the mirror dual of  $Y$  ([Ko]). So it is of interest to understand the category  $\mathcal{D}^b(Y)$ .

2) *Batyrev's construction of Calabi-Yau varieties.*

There is a construction of Batyrev for Calabi-Yau varieties as complete intersections in toric varieties  $X$ . So it is useful to understand first  $\mathcal{D}^b(X)$  (for  $X$  toric) and then the subcategory  $\mathcal{D}^b(Y)$  for  $Y$  a Calabi-Yau subvariety.

3) *The endomorphism algebra of a direct sum of line bundles on toric varieties.*

The endomorphism algebra of a direct sum of line bundles on a toric variety is well understood. In particular, in case we can find a full strongly exceptional sequence (see the following definition) of line bundles on a toric variety  $X$  we obtain a derived equivalence as above.

## 2. THE CONJECTURE

For an understanding of the derived equivalences the following two notions are crucial. For we work in an abelian or bounded triangulated category  $\mathcal{D}$ , so that  $\oplus_l \text{Ext}^l(M, N)$  is finite dimensional for all objects  $M$  and  $N$ . The examples we have in mind are the category of coherent sheaves on a smooth projective variety, the category of finitely generated modules over a finite dimensional algebra of finite global dimension and the corresponding derived categories.

**DEFINITION.** A sequence  $\varepsilon = (L_1, \dots, L_t)$  of objects in  $\mathcal{D}$  is called *exceptional* if  $\text{Ext}^l(L_i, L_i) = 0$  for all  $i = 1, \dots, t$ ;  $l \neq 0$ ,  $\text{End}(L_i) = k$  (we also say  $L_i$  is exceptional) and  $\text{Ext}^l(L_j, L_i) = 0$  for all  $j > i$  and all  $l$ . Such a sequence is called *strongly exceptional* if in addition  $\text{Ext}^l(L_i, L_j) = 0$  for all  $l \neq 0$ . Finally, it is called *full* if  $t$  is the rank of the Grothendieck group  $K_0(\mathcal{D})$  (this condition is slightly weaker than the usual one).

It was conjectured that on any smooth toric variety a full strongly exceptional sequence of line bundles exists (see e. g. [AKO, CM, Hi, Ka] and the references therein for some recent related results). We present a counter example to this

conjecture ([HP]). However, for toric Fano varieties of dimension less or equal to three we can prove the conjecture. We discuss the problem for surfaces and threefolds in more detail.

### 3. TORIC SURFACES

Let  $X$  be a toric surface (always smooth and projective). We denote by  $D_i$  for  $i = 1, \dots, t$  the  $T$ -invariant toric prime divisors numbered so that  $D_i D_{i+1} = 1$  for  $i \in \mathbb{Z}/t$ . Note that the rank of the Grothendieck group is  $t$ . We consider the following sequence of line bundles

$$\varepsilon := (\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \mathcal{O}(D_1 + D_2 + D_3), \dots, \mathcal{O}(D_1 + D_2 + \dots + D_{t-2} + D_{t-1})).$$

Note that we can not proceed, since  $\text{Ext}^2(\mathcal{O}(D_1 + D_2 + \dots + D_{t-1} + D_t), \mathcal{O}) \neq 0$ . Moreover, the only choice we have to form  $\varepsilon$  is to chose the starting divisor  $D_1$  and then we have two choices for the next one (corresponding to the two possible orientations).

**Theorem 1.** *a) The sequence  $\varepsilon$  is full exceptional.*

*b) The sequence  $\varepsilon$  is strongly exceptional precisely when the self intersection numbers satisfy  $D_i^2 \geq -1$  for all  $i = 1, \dots, t - 1$ .*

Note that the condition  $D_i^2 \geq -1$  for all  $i$  corresponds to  $X$  is Fano (there exist precisely 5 non-isomorphic toric Fano surfaces). Moreover, note that we do not need any assumption on  $D_i^2$  in part b). In particular, this construction yields a full strongly exceptional sequence of line bundles on each Hirzebruch surface  $\mathbb{F}_m$ . Looking closer to the construction, one can show that a possible counter example can occur only for  $t \geq 7$ . We now present such a counter example:

We start with the second Hirzebruch surface and its  $T$ -invariant prime divisors with self intersection numbers  $D_1^2 = D_3^2 = 0$ ,  $D_2^2 = -2$ , and  $D_4^2 = 2$ . Now we blow up iteratively as follows:  $X_1$  is the blow up in  $D_1 \cap D_4$  with exceptional divisor  $E_1$ ,  $X_2$  is the blow up in  $D_1 \cap E_1$  with exceptional divisor  $E_2$  and finally  $\overline{X}$  is the blow up of  $D_1 \cap E_2$  with exceptional divisor  $E_3$ . Thus  $\overline{X}$  has seven  $T$ -invariant prime divisors  $\overline{D}_i$  (ordered as above), where  $\overline{D}_i = E_i$ ,  $i = 1, 2, 3$  and  $\overline{D}_{i+3} = \pi^{-1}D_i$ ,  $i = 1, 2, 3, 4$  (here  $\pi : \overline{X} \rightarrow X$  is the projection). For the self intersection numbers we get  $\overline{D}_4^2 = -3$ ,  $\overline{D}_i^2 = -2$  for  $i = 1, 2, 5$ ,  $\overline{D}_3^2 = -1$ ,  $\overline{D}_6^2 = 0$ , and  $\overline{D}_7^2 = 1$ .

**Theorem 2.** [HP] *On the variety  $\overline{X}$  constructed above a full strongly exceptional sequence of line bundles cannot exist.*

### 4. TORIC THREEFOLDS

Now let  $X$  be a toric threefold. We keep the notation from above:  $D_i$  for  $i = 1, \dots, t$  are the  $T$ -invariant toric prime divisors. Note that there does not exist a natural ‘‘cyclic order’’ on the divisors. So we start with an arbitrary order and consider again the sequence  $\varepsilon$  of line bundles on  $X$  as in the previous part. To formulate the result we need to intersect the fan of  $X$  with the two-sphere and obtain a graph  $\Gamma$  (the vertices correspond to the divisors  $D_i$  and an edge between

$D_i$  and  $D_j$  corresponds to a non-empty intersection between  $D_i$  and  $D_j$ ). Note that the construction also makes sense in higher dimension. Recall that a cycle in a graph is called *hamiltonian* if it meets each vertex of the graph precisely once.

**Theorem 3.** (we can assume here that  $X$  is of arbitrary dimension)

- a) If  $\varepsilon$  is exceptional then  $D_1, D_2, \dots, D_t, D_1$  is an Hamiltonian cycle in  $\Gamma$ .  
 b) Assume  $\dim X \leq 3$  and  $D_1, D_2, \dots, D_t, D_1$  is an Hamiltonian cycle in  $\Gamma$  then  $\varepsilon$  is exceptional.

Note first that the theorem generalizes the result in the previous section. Moreover, if  $\dim X \geq 3$  then the sequence is only full if  $X$  is the projective space. So we need to construct further bundles. Finally, the condition is only a condition on the topology of the fan of  $X$ , whereas it turns out (as already can be seen in dimension 2) that strongly exceptional depends also on the convex geometric properties of the fan. Finally, we can prove the following result based on the classification of toric Fano threefolds (see [O], fig. 2.6).

**Theorem 4.** Let  $X$  be a toric Fano threefold. Then the sequence  $\varepsilon$  is strongly exceptional and can be completed, using various hamiltonian cycles in  $\Gamma$ , to a full strongly exceptional sequence of line bundles.

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### Tilting modules over Calabi-Yau algebras

OSAMU IYAMA

(joint work with Idun Reiten)

We review some results in [IR]. Throughout let  $R$  be a  $d$ -dimensional normal complete Gorenstein local ring and  $\Lambda$  a module-finite  $R$ -algebra. We denote by  $\text{mod } \Lambda$  the category of finitely generated  $\Lambda$ -modules, and by  $\text{fl } \Lambda$  the category of  $\Lambda$ -modules of finite length.



**1. Calabi-Yau algebras and symmetric orders**

We denote by  $D$  the Matlis duality on  $\text{fl } R$ . For an integer  $n$ , we call  $\Lambda$   $n$ -Calabi-Yau ( $n$ -CY for short) if there exists a functorial isomorphism

$$\text{Hom}_{\mathcal{D}^b(\text{mod } \Lambda)}(X, Y[n]) \simeq D \text{Hom}_{\mathcal{D}^b(\text{mod } \Lambda)}(Y, X) \quad (*)$$

for any  $X, Y \in \mathcal{D}^b(\text{fl } \Lambda)$ . Similarly, we call  $\Lambda$   $n$ -Calabi-Yau<sup>-</sup> ( $n$ -CY<sup>-</sup> for short) if there exists a functorial isomorphism  $(*)$  for any  $X \in \mathcal{D}^b(\text{fl } \Lambda)$  and  $Y \in \mathcal{D}^{\text{perf}}(\text{mod } \Lambda)$ . By a theorem of Rickard, any  $n$ -CY algebra is  $n$ -CY<sup>-</sup>. On the other hand, we call a module-finite  $R$ -algebra  $\Lambda$  *symmetric* if  $\text{Hom}_R(\Lambda, R)$  is isomorphic to  $\Lambda$  as a  $(\Lambda, \Lambda)$ -module. We call  $\Lambda$  an  $R$ -order if  $\Lambda$  is a (maximal) Cohen-Macaulay  $R$ -module. Our first result is a characterization of  $n$ -CY and  $n$ -CY<sup>-</sup> algebras.

**Theorem 1.1** *Assume that the structure morphism  $R \rightarrow \Lambda$  is injective.*

- (1) *If  $\Lambda$  is  $n$ -CY or  $n$ -CY<sup>-</sup>, then  $n = d (= \dim R)$ .*
- (2)  *$\Lambda$  is  $d$ -CY<sup>-</sup> if and only if  $\Lambda$  is a symmetric  $R$ -order.*
- (3)  *$\Lambda$  is  $d$ -CY if and only if  $\Lambda$  is a symmetric  $R$ -order with  $\text{gl.dim } \Lambda = d$ .*

**Examples 1.2** (1) Assume that  $\Lambda$  is commutative. Then  $\Lambda$  is  $d$ -CY<sup>-</sup> if and only if it is Gorenstein, and  $d$ -CY if and only if it is regular.

(2) A finite dimensional algebra over a field is 0-CY if and only if it is semisimple.

(3) Let  $R$  be a complete discrete valuation ring and  $\Lambda$  a module-finite  $R$ -algebra. If  $\Lambda$  is 1-CY, then it is a maximal  $R$ -order. The converse is not true in general.

(4) Let  $k$  be a field of characteristic zero and  $G$  a finite subgroup of  $\text{SL}_d(k)$  acting on  $V := k^d$  naturally. The action of  $G$  naturally extends to  $S := k[[x_1, \dots, x_d]]$ . We denote by  $S^G$  the invariant subring, and by  $S * G$  the skew group ring, i.e. a free  $S$ -module with a basis  $G$ , where the multiplication is given by  $(s_1 g_1) \cdot (s_2 g_2) = (s_1 g_1(s_2))(g_1 g_2)$  for  $s_i \in S$  and  $g_i \in G$ . Then  $S^G$  is  $d$ -CY<sup>-</sup> and  $S * G$  is  $d$ -CY.

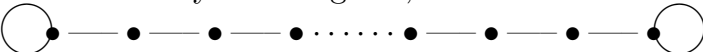
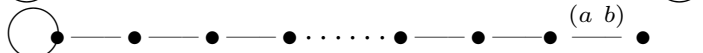
In the rest we study tilting modules over  $d$ -CY algebras  $\Lambda$ . If  $d = 0$  or  $1$ , then any tilting  $\Lambda$ -module is projective since  $\Lambda$  is Morita equivalent to a finite product of local rings by (2) and (3) above. Let us consider the case  $d = 2$ .

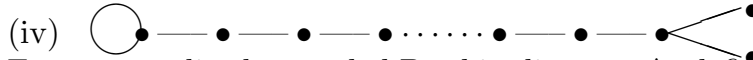
**2. Tilting modules over 2-Calabi-Yau algebras**

Let  $\Lambda$  be a basic ring-indecomposable 2-CY algebra and  $\text{tilt}_1 \Lambda$  the set of isoclasses of basic tilting  $\Lambda$ -modules of projective dimension at most one. Let  $e_1, \dots, e_n$  be a complete set of orthogonal primitive idempotents of  $\Lambda$ . Put  $I_i := \Lambda(1 - e_i)\Lambda$ . The set of 2-sided ideals of  $\Lambda$  forms a monoid by multiplication of ideals. We denote by  $\mathcal{I}(\Lambda)$  the submonoid generated by the ideals  $I_1, \dots, I_n$ .

**Theorem 2.1**  $\mathcal{I}(\Lambda) = \text{tilt}_1 \Lambda$  and  $\mathcal{I}(\Lambda) = \text{tilt}_1 \Lambda^{\text{op}}$  hold.

We give an explicit description of  $\mathcal{I}(\Lambda)$ . Following Happel-Preiser-Ringel [HPR], we call the valued graphs below *generalized extended Dynkin diagrams*.

- (i) An extended Dynkin diagram,
- (ii) 
- (iii)   $(a, b) = (2, 1)$  or  $(1, 2)$



For a generalized extended Dynkin diagram  $\Delta$ , define a valued quiver called the *double* of  $\Delta$  as follows: We replace a valued edge  $\bullet \xrightarrow{(a \ b)} \bullet$  by two valued arrows  $\bullet \xrightarrow{(a \ b)} \bullet$  and  $\bullet \xleftarrow{(b \ a)} \bullet$  of opposite direction. We replace a loop by a loped arrow.

**Theorem 2.2** *The valued quiver of  $\Lambda$  is a double of a generalized extended Dynkin diagram. The corresponding affine Weyl group acts transitively and freely on  $\mathcal{I}(\Lambda)$ .*

**3. Tilting modules over 3-Calabi-Yau algebras**

We call  $X \in \text{mod } \Lambda$  *reflexive* if the natural map  $X \rightarrow \text{Hom}_R(\text{Hom}_R(X, R), R)$  is an isomorphism. We denote by  $\text{ref } \Lambda$  the category of reflexive  $\Lambda$ -modules. Following Van den Bergh [V], we say that  $M$  gives a *non-commutative crepant resolution* (NCCR for short)  $\Gamma$  of  $\Lambda$  if

- (1)  $M \in \text{ref } \Lambda$ , and  $M_\varphi$  is a generator of  $\Lambda_\varphi$  for any  $\varphi \in \text{Spec } R$  with  $\text{ht } \varphi = 1$ ,
- (2)  $\Gamma = \text{End}_\Lambda(M)$  is an  $R$ -order with  $\text{gl.dim } \Gamma = d (= \dim R)$ .

In [V], Van den Bergh gave a non-commutative analogue of a conjecture of Bondal-Orlov: *All NCCR of a normal Gorenstein domain  $\Lambda$  are derived equivalent.* The following theorem shows that his conjecture is true for  $d = 3$ .

**Theorem 3.1** *Let  $\Lambda$  be a module-finite algebra with  $d = 3$ . If  $M_i \in \text{ref } \Lambda$  ( $i = 1, 2$ ) gives a NCCR  $\Gamma_i := \text{End}_\Lambda(M_i)$  of  $\Lambda$ , then  $U := \text{Hom}_\Lambda(M_1, M_2)$  is a reflexive tilting  $\Gamma_1$ -module with  $\text{End}_{\Gamma_1}(U) = \Gamma_2$ . Consequently, all NCCR of  $\Lambda$  are derived equivalent.*

For 3-CY<sup>-</sup> algebras we have the following stronger assertion.

**Theorem 3.2** *Let  $\Lambda$  be 3-CY<sup>-</sup> and  $M$  a  $\Lambda$ -module giving a NCCR  $\Gamma := \text{End}_\Lambda(M)$ . Then the equivalence  $\text{Hom}_\Lambda(M, -) : \text{ref } \Lambda \rightarrow \text{ref } \Gamma$  gives a one-one correspondence between  $\Lambda$ -modules giving NCCR and reflexive tilting  $\Gamma$ -modules.*

Choosing  $M := \Lambda$  in 3.2 for a 3-CY algebra, we have the relationship below between tilting modules and NCCR. Note that any  $M \in \text{ref } \Lambda$  over a 3-CY algebra  $\Lambda$  has projective dimension at most one.

**Corollary 3.3** *Let  $\Lambda$  be 3-CY. Then  $\Lambda$ -modules giving NCCR are exactly reflexive tilting  $\Lambda$ -modules.*

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## Thick subcategories of perfect complexes over a commutative ring

SRIKANTH IYENGAR

Let  $R$  a commutative noetherian ring and  $\mathcal{D}$  the derived category of  $R$ -modules. A *perfect complex* of  $R$ -modules is one of the form

$$0 \rightarrow P_s \rightarrow \cdots \rightarrow P_i \rightarrow 0$$

where each  $P_i$  is a finitely generated projective  $R$ -module. Let  $\mathcal{P}$  the full subcategory of  $\mathcal{D}$  consisting of complexes isomorphic to perfect complexes. These are precisely the compact objects, also called small objects, in  $\mathcal{D}$ .

These notes are an abstract of two lectures I gave at the workshop. The main goal of the lectures was to present various proofs of a theorem of Hopkins [7] and Neeman [8], Theorem 1 below, that classifies the thick subcategories of  $\mathcal{P}$ , and to discuss results from [5], which is inspired by this circle of ideas.

As usual  $\text{Spec } R$  denotes the set of prime ideals in  $R$  with the Zariski topology; thus, the closed subsets are precisely the subsets  $\text{Var}(I) = \{\mathfrak{p} \supseteq I \mid \mathfrak{p} \in \text{Spec } R\}$ , where  $I$  is an ideal in  $R$ . A subset  $V$  of  $\text{Spec } R$  is *specialization closed* if it is a (possibly infinite) union of closed subsets; in other words, if  $\mathfrak{p}$  and  $\mathfrak{q}$  are prime ideals such that  $\mathfrak{p}$  is in  $V$  and  $\mathfrak{q} \supseteq \mathfrak{p}$ , then  $\mathfrak{q}$  is in  $V$ .

For a prime ideal  $\mathfrak{p}$ , we write  $k(\mathfrak{p})$  for  $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ , the residue field of  $R$  at  $\mathfrak{p}$ . The *support* of a complex of  $R$ -modules  $M$  is the set of prime ideals

$$\text{Supp}_R M = \{\mathfrak{p} \in \text{Spec } R \mid k(\mathfrak{p}) \otimes_R^{\mathbf{L}} M \neq 0\}$$

In the literature, this is sometimes referred to as the *homological support*, while the word ‘support’ refers to the set of primes  $\mathfrak{p}$  such that  $M_{\mathfrak{p}} \neq 0$ ; this latter set contains  $\text{Supp}_R M$ , but is typically larger. They coincide when the  $R$ -module  $H(M)$  is finitely generated, in which case  $\text{Supp}_R M$  is a closed subset of  $\text{Spec } R$ .

With this notation, the theorem of Hopkins and Neeman is as follows.

**Theorem 1.** *There is a bijection of sets*

$$\left\{ \begin{array}{c} \text{Thick subcategories} \\ \text{of } \mathcal{P} \end{array} \right\} \begin{array}{c} \xrightarrow{\text{S}} \\ \xleftarrow{\text{T}} \end{array} \left\{ \begin{array}{c} \text{Specialization closed} \\ \text{subsets of } \text{Spec } R \end{array} \right\}$$

where the maps in question are

$$\text{S}(\mathcal{T}) = \bigcup_{M \in \mathcal{T}} \text{Supp}_R M \quad \text{and} \quad \text{T}(V) = \{M \mid \text{Supp}_R M \subseteq V\}$$

*Proof.* Note that both  $\text{S}$  and  $\text{T}$  are inclusion reversing.

It is easy to prove  $\text{ST}(V) = V$  when  $V \subseteq \text{Spec } R$  is specialization closed.

Indeed, it is clear from definitions that  $\text{ST}(V) \subseteq V$ . Conversely, given  $\mathfrak{p}$  in  $V$ , pick a set  $\{x_1, \dots, x_n\}$  which generates the ideal  $\mathfrak{p}$ , and let  $K$  be the Koszul complex on  $\mathbf{x}$ . It is readily verified that  $\text{Supp}_R K = \text{Var}(\mathbf{x}) = \text{Var}(\mathfrak{p}) \subseteq V$ , so  $K$  is in  $\text{T}(V)$ , and hence  $\mathfrak{p} \in \text{Var}(\mathfrak{p}) = \text{Supp}_R K \subseteq \text{ST}(V)$ . Therefore,  $V \subseteq \text{ST}(V)$ .

Let  $\mathcal{T}$  be a thick subcategory of  $\mathcal{P}$ . Evidently  $\mathcal{T} \subseteq \text{TS}(\mathcal{T})$ , so it remains to verify the reverse inclusion.

Suppose  $M$  is in  $\mathbf{TS}(\mathcal{T})$ , so that  $\mathrm{Supp}_R M \subseteq \mathbf{S}(\mathcal{T})$ . Since the  $R$ -module  $H(M)$  is finitely generated,  $\mathrm{Supp}_R M$  is a closed subset of  $\mathrm{Spec} R$ , and hence it has finitely many minimal primes. Therefore, there exist complex  $N_1, \dots, N_s$  in  $\mathcal{T}$  such that

$$\mathrm{Supp}_R M \subseteq \bigcup_{i=1}^s \mathrm{Supp}_R N_i = \mathrm{Supp}_R \left( \bigoplus_{i=1}^s N_i \right).$$

It remains to invoke Theorem 2 below.  $\square$

In what follows, given complexes  $M$  and  $N$  in (some full subcategory) of  $\mathcal{D}$ , we say that  $N$  *builds*  $M$ , and write  $N \implies M$  if  $M$  is in the thick subcategory generated by  $N$ ; when  $R$  needs to be specified, we write  $N \xRightarrow[R]{} M$ .

Note that if  $N \implies M$ , then  $\mathrm{Supp}_R M \supseteq \mathrm{Supp}_R N$ .

**Theorem 2.** *If  $M$  and  $N$  in  $\mathcal{P}$  are such that  $\mathrm{Supp}_R M \subseteq \mathrm{Supp}_R N$ , then  $N \xRightarrow[R]{} M$ .*

There are (at least) three proofs of this theorem.

*First proof of Theorem 2.* This is due to Neeman. The basic idea is to classify the localizing subcategories of  $\mathcal{D}$ . These turn out to be in bijection with arbitrary subsets of  $\mathrm{Spec} R$ , see [8]. Hence, if  $\mathrm{Supp}_R M \subseteq \mathrm{Supp}_R N$ , then  $M$  is in the localizing subcategory of  $\mathcal{D}$  generated by  $N$ . Since  $M$  and  $N$  are both in  $\mathcal{P}$ , they are compact objects in  $\mathcal{D}$ , so another result of Neeman's [9, (2.2)], implies that  $M$  is in fact in the thick subcategory generated by  $N$ .  $\square$

*Second proof of Theorem 2.* This is inspired by work of Dwyer and Greenlees [3]. Consider the DG algebra  $\mathcal{E} = \mathbf{RHom}_R(N, N)$ , the right  $\mathcal{E}$ -module  $\mathbf{RHom}_R(N, M)$ , and the following natural morphism in  $\mathcal{D}$

$$\theta : \mathbf{RHom}_R(N, M) \otimes_{\mathcal{E}}^{\mathbf{L}} N \longrightarrow M.$$

The point is that one knows *a posteriori* that  $\theta$  represents the natural morphism  $\mathbf{R}\Gamma_I(M) \rightarrow (M)$ , where  $\mathbf{R}\Gamma_I(M)$  is local cohomology with respect to the ideal  $I$ , with  $\mathrm{Supp}_R N = \mathrm{Var}(I)$ . Thus, since  $\mathrm{Supp}_R M$  is contained in  $\mathrm{Var}(I)$ , it must be that  $\theta$  is an isomorphism. This can be proved directly, as follows:

An elementary calculation shows that the support of  $\mathrm{cone}(\theta)$  is a subset of  $\mathrm{Supp}_R M \cup \mathrm{Supp}_R N$ , and hence of  $\mathrm{Supp}_R N$ , by hypothesis. On the other hand,  $\mathbf{RHom}_R(N, \mathrm{cone}(\theta)) \simeq 0$ , since  $\mathbf{RHom}_R(N, \theta)$  is isomorphism, as can be easily verified, keeping in mind that  $N$  is compact. Given this, it is not difficult to prove that  $\mathrm{cone}(\theta) \simeq 0$ , so  $\theta$  is an isomorphism.

Now, the  $R$ -algebra  $H(\mathcal{E})$  is noetherian, and  $H(\mathbf{RHom}_R(N, M))$  is finitely generated over  $H(\mathcal{E})$ , so in the derived category of right  $\mathcal{E}$ -modules, one has that

$$\mathbf{RHom}_R(N, M) \simeq \mathrm{hocolim}_n X^n$$

where  $X^n$  is in the thick subcategory generated by  $\mathcal{E}$ , that is to say,  $\mathcal{E} \xRightarrow[\mathcal{E}]{} X^n$ .

Therefore, in  $\mathcal{D}$ , one has isomorphisms

$$M \simeq \mathbf{RHom}_R(N, M) \otimes_{\mathcal{E}}^{\mathbf{L}} N \simeq \mathrm{hocolim}_n (X^n \otimes_{\mathcal{E}}^{\mathbf{L}} N),$$

where the first isomorphism is  $\theta$ , and the second one is obtained by base change. Since  $M$  is compact, a standard argument yields that  $M$  is a retract of  $X^n \otimes_{\mathcal{E}}^{\mathbf{L}} N$ , for some  $N$ . It remains to note that by base change

$$\mathcal{E} \xrightarrow{\theta} X^n \quad \text{implies} \quad N \xrightarrow{R} X^n \otimes_{\mathcal{E}}^{\mathbf{L}} N.$$

Therefore,  $M$  is in the thick subcategory generated by  $N$ , as desired.  $\square$

*Third proof of Theorem 2.* This proof is the original one, due to Hopkins [7], also see [8], especially the discussion on the first page, and Thomason's article [10]. The main step in it is the proof of the following 'tensor-nilpotence' theorem.

**Theorem 3.** *Let  $\alpha : X \rightarrow Y$  be a morphism of perfect complexes. If for each  $\mathfrak{p}$  in  $\text{Spec } R$ , one has  $H(k(\mathfrak{p}) \otimes_R \alpha) = 0$ , then there exists an integer  $n \geq 0$  such that*

$$\alpha^n = 0 : X^{\otimes n} \longrightarrow Y^{\otimes n}.$$

In my lectures, I discussed a proof of this result, and also Hopkins' argument for Theorem 2; see [7], [8], and [10].  $\square$

Theorem 2 gives a new approach to some problems concerning descent of properties along a *local homomorphism*

$$\varphi : (Q, \mathfrak{q}, h) \longrightarrow (R, \mathfrak{m}, k).$$

This notation means that  $Q$  and  $R$  are (commutative noetherian) local rings, with maximal ideals  $\mathfrak{q}$  and  $\mathfrak{m}$ , residue fields  $h$  and  $k$ , and  $\varphi$  is a homomorphism of rings with  $\varphi(\mathfrak{q}) \subseteq \mathfrak{m}$ . This is the context for the rest of this write-up.

It is a classical result, found already in Cartan and Eilenberg [2], that for any  $R$ -module  $M$ , if  $\text{flat dim}_Q R$  and  $\text{flat dim}_R M$  are both finite, then so is  $\text{flat dim}_Q M$ . A complex over a ring is said to have finite flat dimension if it is isomorphic, in the derived category of the ring, to a bounded complex of flat modules.

In [6], Foxby and I proved the following converse.

**Theorem 4.** *Let  $M$  be a perfect complex of  $R$ -modules, with  $H(M) \neq 0$ .*

*If  $\text{flat dim}_Q M$  is finite, then  $\text{flat dim}_Q R$  is finite as well.*

In [5], we use Theorem 2 to give a totally different proof of this result. The argument requires the following basic facts.

- (1)  $\text{flat dim}_Q R < \infty$  if and only if  $\sup(h \otimes_Q^{\mathbf{L}} R) = \sup\{n \mid H_n(h \otimes_Q^{\mathbf{L}} R) \neq 0\} < \infty$ .
- (2) The subcategory  $\{X \in \mathcal{D}(R) \mid \sup(h \otimes_Q^{\mathbf{L}} X) < \infty\}$  of  $\mathcal{D}(R)$  is thick.
- (3) Let  $K$  be the Koszul complex on a finite set of elements in  $R$  and  $Y$  be a complex of  $R$ -modules such that the  $R$ -module  $H_n(Y)$  is finitely generated for each  $n$ . If  $\sup(Y \otimes_Q K)$  is finite, then  $\sup(Y)$  is finite.

Indeed, when  $\text{flat dim}_Q R$  is finite, it is clear that  $\sup(h \otimes_Q^{\mathbf{L}} R)$  is finite. The converse is a result of André, and is proved by a standard argument: since  $h$  is the only simple  $Q$ -module, induction on length yields that  $\sup(L \otimes_Q^{\mathbf{L}} R)$  is finite for any finite length  $Q$ -module  $L$ . This is the basis of an induction on the Krull dimension of  $L$  that proves that  $\sup(L \otimes_Q^{\mathbf{L}} R)$  is finite for any finitely generated  $Q$ -module  $L$ , and hence that  $\text{flat dim}_Q R$  is finite. This justifies the first claim.

The second claim is a straightforward verification. As to (3), the Koszul complex on an element  $x$  of  $R$  is the mapping cone of the morphism  $R \xrightarrow{x} R$ , so one obtains an exact sequence of complexes

$$0 \longrightarrow Y \longrightarrow Y \otimes_Q K \longrightarrow \Sigma Y \longrightarrow 0.$$

The homology long exact sequence and Nakayama's lemma imply that when  $\text{sup}(Y \otimes_Q K)$  is finite, so is  $\text{sup}(Y)$ , as desired. The general case is settled by an induction on the number of elements, for the corresponding Koszul complex can be realized as an iterated mapping cone.

*Proof of Theorem 4.* Let  $K$  be the Koszul complex on a finite set of generators for  $\mathfrak{m}$ . Since  $\mathfrak{m}$  is the unique closed point of  $\text{Spec } R$ , and  $\text{Supp}_R M$  is a closed subset of  $\text{Spec } R$ , Theorem 2 implies that  $M \implies K$ . In view of (2) above, this explains the third implication in the chain below:

$$\begin{aligned} \text{flat dim}_Q N < \infty &\implies \text{sup}(h \otimes_Q^{\mathbf{L}} N) < \infty \\ &\implies \text{sup}((h \otimes_Q^{\mathbf{L}} R) \otimes_R^{\mathbf{L}} N) < \infty \\ &\implies \text{sup}((h \otimes_Q^{\mathbf{L}} R) \otimes_R^{\mathbf{L}} K) < \infty \\ &\implies \text{sup}(h \otimes_Q^{\mathbf{L}} R) < \infty \\ &\implies \text{flat dim}_Q R < \infty. \end{aligned}$$

The first implication is clear, the second one is by the associativity of tensor products, the fourth follows from (3) above applied with  $Y = h \otimes_Q^{\mathbf{L}} R$ , while the fifth is by (1). This completes the proof of Theorem 4.  $\square$

The paradigm of the preceding proof, see [5, (5.2)], is applicable to other homological invariants as well, and yields new results in commutative algebra, some of which are not, as yet, accessible by more 'traditional' methods.

Note that the argument above allows for a stronger conclusion: all one needs is that the thick subcategory generated by  $N$  contains a, homologically non-zero, small (i.e., compact) object; in other words,  $N$  is *virtually small*, in the terminology of [5]. This notion was suggested to us by the work in [4].

Evidently, any small object is virtually small; in [5], we identify various other classes of virtually small objects. One noteworthy result in this direction is:

**Theorem 5.** *Let  $R$  be a complete intersection local ring. Any complex  $M$  of  $R$ -modules with  $H(M)$  finitely generated, is virtually small.*

Compare this to the result that when  $R$  is a regular local ring, any complex  $M$  of  $R$ -modules with  $H(M)$  finitely generated is small. This is one direction of a theorem of Auslander, Buchsbaum, and Serre; the other direction asserts the converse. We expect that the converse to the theorem above also holds, see [5, §9].

The notion of a virtually small object carries over to any triangulated category, and the work in [4, 5] makes it clear that it would be worthwhile to investigate such objects. It is also useful to *quantify* the process of building one object from another. This is being investigated in [1], where it provides the technical tools to

bring to light an unexpected relationship between perfect complexes over a local ring and free summands of its conormal modules.

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## On well generated triangulated categories

BERNHARD KELLER

(joint work with Marco Porta)

In his book [4], A. Neeman introduced the class of *well generated triangulated categories*. He showed that this class has truly wonderful properties: It is stable under localizations (we always assume the kernel of a localization to be generated by a set of objects) and that the Brown representability theorem holds for each well generated triangulated category. The following characterization is due to H. Krause [2]: A triangulated category  $\mathcal{T}$  is well generated iff it admits arbitrary (set-indexed) coproducts and admits a *good set  $\mathcal{G}$  of generators*, *i.e.* a set of objects in  $\mathcal{T}$  which is stable under shifts in both directions and such that

- 1)  $\mathcal{G}$  *generates*, *i.e.* an object  $X$  of  $\mathcal{T}$  vanishes iff we have  $\mathrm{Hom}(G, X) = 0$  for each  $G$  in  $\mathcal{G}$ ;
- 2) there is a cardinal  $\alpha$  such that each object  $G$  of  $\mathcal{G}$  is  $\alpha$ -*small*, *i.e.* for each family  $(X_i)_{i \in I}$  of objects of  $\mathcal{T}$ , each morphism

$$G \rightarrow \coprod_{i \in I} X_i$$

factors through a subsum  $\coprod_{i \in J} X_i$  indexed by a subset  $J \subset I$  of cardinality strictly less than  $\alpha$ ;

- 3) if  $f_i : X_i \rightarrow Y_i$ ,  $i \in I$ , is a family of morphisms in  $\mathcal{T}$  such that the induced map

$$\mathrm{Hom}(G, X_i) \rightarrow \mathrm{Hom}(G, Y_i)$$

is surjective for each  $G \in \mathcal{G}$  and each  $i \in I$ , then the  $f_i$  induce a surjection

$$\mathrm{Hom}(G, \coprod_{i \in I} X_i) \rightarrow \mathrm{Hom}(G, \coprod_{i \in I} Y_i).$$

The category  $\mathcal{T}$  is *compactly generated* if in 2), we can take  $\alpha$  to be the cardinality of the set of natural numbers. In this case, condition 2) implies 3). Examples of well-generated categories abound: If  $X$  is a topological space, the (unbounded) derived category  $\mathcal{D}(\mathrm{Pre}(X))$  of presheaves of abelian groups on  $X$  is compactly generated, hence well generated, and thus the derived category  $\mathcal{D}(\mathrm{Sh}(X))$  of sheaves of abelian groups on  $X$  is well generated, since it is a localization of the derived category of presheaves. However, as shown in [3], the derived category of sheaves is not compactly generated in general: If  $X$  is a connected, non compact real manifold of dimension at least 1, all compact objects in  $\mathcal{D}(\mathrm{Sh}(X))$  vanish. Similarly, if  $\mathcal{A}$  is an arbitrary Grothendieck category, using the Popescu-Gabriel theorem [5] one can show that the (unbounded) derived category of  $\mathcal{A}$  is well-generated since it is the localization of the derived category of a category of modules. Another large class of examples is constructed as follows: Let  $\mathcal{B}$  be a small differential graded (=dg) category. Thus,  $\mathcal{B}$  is a category which is enriched over the category  $\mathcal{C}(\mathbb{Z})$  of complexes of abelian groups. A *dg  $\mathcal{B}$ -module* is a dg functor  $M : \mathcal{B}^{op} \rightarrow \mathcal{C}(\mathbb{Z})$ . The *derived category  $\mathcal{DB}$* , cf. e.g. [1], is the localization of the category of dg modules at the class of (pointwise) quasi-isomorphisms. By Neeman's theorem, any localization  $\mathcal{DB}/\mathcal{N}$ , where  $\mathcal{N}$  is a localizing subcategory generated by a set of objects, is well generated.

These examples lead one to ask whether any well generated triangulated category is a localization of a compactly generated category. We do not know the answer in general but the following theorem shows that it is positive if  $\mathcal{T}$  is an *algebraic* triangulated category, i.e. if it is triangle equivalent to a full triangulated subcategory of the category up to homotopy of complexes over some additive category (this allows one to construct an enriched Hom-functor  $\mathrm{RHom}$ ).

**Theorem 1.** *Let  $\mathcal{T}$  be an algebraic triangulated category. The following are equivalent:*

- (i)  $\mathcal{T}$  is well generated;
- (ii)  $\mathcal{T}$  is triangle equivalent to a localization of the derived category  $\mathcal{DB}$  of a small dg category  $\mathcal{B}$ .

Moreover, if these conditions hold and  $G$  is any generator of  $\mathcal{T}$ , i.e. an object  $X$  of  $\mathcal{T}$  vanishes iff we have  $\mathcal{T}(G, X[n]) = 0$  for all  $n \in \mathbb{Z}$ , then the functor

$$\mathrm{RHom}(G, ?) : \mathcal{T} \rightarrow \mathcal{DB}$$

is a localization, where  $B = \mathrm{RHom}(G, G)$ .

In the case where  $\mathcal{T}$  is compactly generated, one actually obtains a triangle equivalence  $\mathcal{T} \rightarrow \mathcal{DB}$  (cf. theorem 4.3 of [1]). The theorem can be considered



as an analogue of the Popescu-Gabriel theorem and then shows that among the algebraic triangulated categories, the well generated ones are analogous to the Grothendieck abelian categories.

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## On the Gabriel-Popescu theorem

WENDY LOWEN

An abelian category is called *Grothendieck* if it has a generator, arbitrary coproducts and exact filtered colimits. Let us start by stating the theorem in the title of the talk.

**Theorem 1.** [5] *Let  $\mathcal{C}$  be a Grothendieck category with a generator  $U$  and put  $A = \mathcal{C}(U, U)$ . The functors*

$$i : \mathcal{C} \longrightarrow \text{Mod}(A) : C \longmapsto \mathcal{C}(U, C)$$

*and the unique colimit preserving functor*

$$a : \text{Mod}(A) \longrightarrow \mathcal{C}$$

*extending the natural inclusion  $A \longrightarrow \mathcal{C}$  (where  $A$  is considered as a one object category) constitute a localization, i.e.  $i$  is fully faithful and its left adjoint  $a$  is exact.*

This theorem is reminiscent of a slightly older theorem of Giraud characterizing internally the localizations of presheaf categories  $\text{Pr}(\mathbf{u}) = \text{Fun}(\mathbf{u}^{\text{op}}, \text{Set})$  (where  $\mathbf{u}$  is a small category). Girauds proof realizes a category  $\mathcal{C}$  satisfying a certain list of axioms as a category of sheaves  $\text{Sh}(\mathbf{u}, \mathcal{T})$  for a Grothendieck topology  $\mathcal{T}$  on a full generating subcategory  $\mathbf{u}$ . In fact, Theorem 1 is a perfect additive analogue of Girauds theorem. Here additive means that we replace the base category  $\text{Set}$  by the category  $\mathbf{Ab}$  of abelian groups, we consider additive categories  $\mathfrak{a}$  (i.e. enriched in  $\mathbf{Ab}$ ), additive functor categories  $\text{Mod}(\mathfrak{a}) = \text{Add}(\mathfrak{a}^{\text{op}}, \mathbf{Ab})$ , additive topologies on  $\mathfrak{a}$ , and additive sheaves. As is shown more generally in [1], there is a one-one correspondence

$$\{\text{additive topologies on } \mathfrak{a}\} \longrightarrow \{\text{localizations of } \text{Mod}(\mathfrak{a})\} : \mathcal{T} \longmapsto \text{Sh}(\mathfrak{a}, \mathcal{T})$$

This raises the question, for a given Grothendieck category  $\mathcal{C}$ , which representations of  $\mathcal{C}$  as a category  $\text{Sh}(\mathfrak{a}, \mathcal{T})$  can occur. We will now present a theorem which characterizes those representations. We consider an additive functor  $a : \mathfrak{a} \rightarrow \mathcal{C}$  from a small additive category to a Grothendieck category. We will call a collection of morphisms  $f_i : A_i \rightarrow A$  in  $\mathfrak{a}$  *epimorphic* if the induced  $\coprod_i a(A_i) \rightarrow a(A)$  is an epimorphism in  $\mathcal{C}$ .

**Theorem 2.** [2] *Let  $a : \mathfrak{a} \rightarrow \mathcal{C}$  be as above. The following are equivalent:*

(1)  *$a$  induces a localization*

$$i : \mathcal{C} \rightarrow \text{Mod}(\mathfrak{a}) : C \mapsto \mathcal{C}(a(-), C)$$

(2)  *$a$  satisfies the following conditions:*

(G)  *$a(\mathfrak{a})$  generates  $\mathcal{C}$ .*

(F) *for every morphism  $c : a(A) \rightarrow a(A')$  in  $\mathcal{C}$ , there exists an epimorphic collection  $f_i : A_i \rightarrow A$  with  $ca(f_i) = a(g_i)$  for some  $g_i$ .*

(FF) *for every  $f : A \rightarrow A'$  in  $\mathfrak{a}$  with  $a(f) = 0$ , there exists an epimorphic collection  $f_i : A_i \rightarrow A$  with  $ff_i = 0$ .*

*Moreover, if these conditions are satisfied, the epimorphic collections define an additive topology  $\mathcal{T}_{\text{epi}}$  on  $\mathfrak{a}$  such that  $i$  factors over an equivalence  $\mathcal{C} \cong \text{Sh}(\mathfrak{a}, \mathcal{T}_{\text{epi}})$ .*

From Theorem 2, one easily deduces the following characterization of the Yoneda embedding:

**Theorem 3.** *Let  $a : \mathfrak{a} \rightarrow \mathcal{C}$  be an additive functor from an additive category to a Grothendieck category. The following are equivalent:*

(1)  *$a$  yields an equivalence  $\text{Mod}(\mathfrak{a}) \cong \mathcal{C}$ .*

(2)  *$a$  satisfies the conditions in Theorem 2 and  $\mathcal{T}_{\text{epi}}$  is the trivial topology, i.e. every covering subfunctor is representable.*

(3)  *$a$  is fully faithful and  $a(\mathfrak{a})$  consists of finitely generated projective generators of  $\mathcal{C}$ .*

Our motivation for Theorem 2 comes from the fact that certain standard additive sheaf representations do not fit into the setting of Theorem 1, because the restriction of the left adjoint  $\text{Mod}(\mathfrak{a}) \rightarrow \text{Sh}(\mathfrak{a}, \mathcal{T})$  to  $\mathfrak{a}$  is not fully faithful, or, equivalently, the representable functors in  $\text{Mod}(\mathfrak{a})$  fail to be sheaves. A situation where this occurs is the following. Let  $(X, \mathcal{O})$  be a ringed space and consider the categories  $\text{Mod}(\mathcal{O})$  of sheaves of  $\mathcal{O}$ -modules and  $\text{PMod}(\mathcal{O})$  of presheaves of  $\mathcal{O}$ -modules on  $X$ . The localization  $i : \text{Mod}(\mathcal{O}) \rightarrow \text{PMod}(\mathcal{O})$  can be described in terms of additive sheaves. Let  $\mathfrak{a}$  be the following additive category:  $\text{Ob}(\mathfrak{a}) = \text{Open}(X)$  and  $\mathfrak{a}(U, V) = \mathcal{O}(U)$  if  $U \subset V$  and zero otherwise. There is a natural fully faithful  $\mathfrak{a} \rightarrow \text{PMod}(\mathcal{O}) : U \mapsto P_U$  where  $P_U$  is the presheaf extension by zero of  $\mathcal{O}_U$ . This yields an equivalence of categories  $\text{Mod}(\mathfrak{a}) \cong \text{PMod}(\mathcal{O})$ . The additive topology  $\mathcal{T}_X$  on  $\mathfrak{a}$  for which we obtain an equivalence  $\text{Mod}(\mathcal{O}) \cong \text{Sh}(\mathfrak{a}, \mathcal{T}_X)$  is inherited directly from  $X$ : for  $U \subset V$ , put  $\delta_{U,V} = 1 \in \mathcal{O}(U) = \mathfrak{a}(U, V)$ . Then a covering of  $U$  in  $\mathfrak{a}$  has to contain  $\delta_{U_i, U}$  for a covering  $U_i \rightarrow U$  in  $X$ .

The additive sheaf representation we just discussed is useful in the context of deformations of ringed spaces. We will say a word about deformations of abelian categories. Let  $k$  be a field and  $k[\epsilon]$  the dual numbers.

**Definition 1.** [4] Let  $\mathcal{C}$  be an abelian  $k$ -linear category. An *infinitesimal abelian deformation* of  $\mathcal{C}$  is an abelian  $k[\epsilon]$ -linear category with an equivalence of categories  $\mathcal{C} \cong \mathcal{H}om_{k[\epsilon]}(k, \mathcal{D})$ , where  $\mathcal{H}om_{k[\epsilon]}(k, \mathcal{D})$  is the category of  $k[\epsilon]$ -linear functors from the one object category  $k$  into  $\mathcal{D}$ . In other words, it is the full  $k$ -linear subcategory of  $\mathcal{D}$  of objects annihilated by  $\epsilon$ .

If  $A$  is a  $k$ -algebra and  $B$  an infinitesimal algebra deformation of  $A$  (i.e. we have an isomorphism  $A \cong k \otimes_{k[\epsilon]} B$ ), then we naturally obtain that  $\text{Mod}(B)$  is an abelian deformation of  $\text{Mod}(A)$ . Moreover, with an appropriate notion of flatness, we get a 1-1 correspondence between

- algebra deformations of  $A$
- abelian deformations of  $\text{Mod}(A)$

It is our intension to understand the deformations of other abelian categories, for example to understand the deformations of  $\text{Mod}(\mathcal{O})$  in terms of certain algebraic deformations of  $\mathcal{O}$ .

To do so, we have a look at the module case first. If  $\mathcal{C}$  is an abelian deformation of  $\text{Mod}(A)$ , there is a natural functor  $k \otimes_{k[\epsilon]} - : \mathcal{C} \rightarrow \text{Mod}(A)$  along which we can lift certain objects. Lifting of  $k$ -flat objects is governed by an obstruction theory involving the second and first self-Ext groups. Since  $\text{Ext}^2(A, A) = \text{Ext}^1(A, A) = 0$ , there is a unique (up to isomorphism) lift  $\bar{A}$  of  $A$ , and  $\bar{A}$  is shown to be a finitely generated projective generator of  $\mathcal{C}$ , whence  $\mathcal{C} \cong \text{Mod}(\bar{A})$ .

When we start from a deformation of  $\text{Mod}(\mathcal{O})$ , things are more complicated for we have no control over  $\text{Ext}^{1,2}(\mathcal{O}, \mathcal{O})$ . However, if we consider the *stack*  $\mathcal{M}od(\mathcal{O})$  on  $X$  with  $\mathcal{M}od(\mathcal{O})(U) = \text{Mod}(\mathcal{O}_U)$ , we do get a certain analogy with the module case. Let  $\mathcal{M}od(\underline{k})$  be the stack of  $k$ -modules, with  $\mathcal{M}od(\underline{k})(U) = \text{Mod}(\underline{k}_U)$  the category of sheaves of  $k$ -modules on  $U$ . We get natural functors

$$\mathcal{E}xt^i(\mathcal{O}_U, -) : \text{Mod}(\mathcal{O}_U) \rightarrow \text{Mod}(\underline{k}_U)$$

where  $\mathcal{E}xt^i(\mathcal{O}_U, M)$  is the sheafification of  $\text{Ext}^i(\mathcal{O}_U, M)(V) = \text{Ext}^i(\mathcal{O}_V, M|_V)$ . Then we have that the  $\mathcal{O}_U$  are *locally finitely generated projective*, i.e  $\mathcal{E}xt^0(\mathcal{O}_U, -)$  preserves filtered colimits and  $\mathcal{E}xt^1(\mathcal{O}_U, -) = 0$ .

Now let  $\mathcal{C}(X)$  be a deformation of  $\text{Mod}(\mathcal{O})$ . There are induced deformations  $\mathcal{C}(U)$  of  $\text{Mod}(\mathcal{O})$  constituting a stack  $\mathcal{C}$  on  $X$ . We define a prestack  $\bar{\mathcal{O}}$  on  $X$  with

$$\bar{\mathcal{O}}(U) = \{(\text{flat}) \text{ lifts of } \mathcal{O}_U \text{ to } \mathcal{C}(U)\}$$

We have the following

**Theorem 4.** [3] *The map  $\bar{\mathcal{O}} \rightarrow \mathcal{C}$  yields an equivalence of stacks  $\mathcal{C} \cong \text{Mod}(\bar{\mathcal{O}})$ , where  $\text{Mod}(\bar{\mathcal{O}}) = \mathcal{H}om(\bar{\mathcal{O}}, \text{Mod}(\underline{k}[\epsilon]))$  is the stack of morphisms of prestacks.*

The proof of Theorem 4 is based upon a liftable characterization of ‘‘Yoneda’’ morphisms  $\mathcal{A} \rightarrow \mathcal{C}$  of prestacks yielding  $\mathcal{C} \cong \text{Mod}(\mathcal{A})$ . This characterization is twofold. First we need some assumptions on  $\mathcal{C}$ :  $\mathcal{C}$  has to be a stack of Grothendieck

categories and the restrictions  $\mathcal{C}(U) \rightarrow \mathcal{C}(V)$  for  $V \subset U$  have to be exact with a fully faithful right adjoint and an exact left adjoint. Next we need assumptions on  $\mathcal{A} \rightarrow \mathcal{C}$ . To formulate them we associate an additive category  $\mathfrak{a}_U$  to  $\mathcal{A}|_U$  for every  $U$ , in the same way that we associated  $\mathfrak{a}$  to  $\mathcal{O}$  earlier on, and we consider the additive topology  $\mathcal{T}_U$  on  $\mathfrak{a}_U$  for which  $\text{Mod}(\mathcal{A}|_U) \cong \text{Sh}(\mathfrak{a}_U, \mathcal{T}_U)$ . By the assumptions, we get morphisms  $\mathfrak{a}_U \rightarrow \mathcal{C}(U)$ .

**Theorem 5.** [3] *The following are equivalent:*

- (1)  $\mathcal{A} \rightarrow \mathcal{C}$  yields  $\mathcal{C} \cong \text{Mod}(\mathcal{A})$ .
- (2) every  $\mathfrak{a}_U \rightarrow \mathcal{C}(U)$  satisfies the conditions of Theorem 2 and  $\mathcal{T}_{\text{epi}} = \mathcal{T}_U$ .
- (3)  $(\mathfrak{a}_U, \mathcal{T}_U) \rightarrow \mathcal{C}(U)$  satisfies conditions (G), (F) and (FF) where in (F) and (FF) we use  $\mathcal{T}_U$ -coverings instead of epimorphic collections, and the objects of  $\mathcal{A}$  are mapped to locally finitely generated projectives in  $\mathcal{C}$ .

Clearly, Theorem 5 is a stack version of Theorem 3.

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## A question arising from a theorem of Rosicky

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Let  $\mathcal{T}$  be a triangulated category, and let  $\mathcal{S}$  be a triangulated subcategory. For any object  $t \in \mathcal{T}$  the representable functor  $\mathcal{T}(-, t)$  is a homological functor  $\mathcal{T}^{\text{op}} \rightarrow \text{Ab}$ . If we restrict to the subcategory  $\mathcal{S} \subset \mathcal{T}$  we obtain a homological functor  $\mathcal{S}^{\text{op}} \rightarrow \text{Ab}$ . We will denote this functor  $\mathcal{T}(-, t)|_{\mathcal{S}}$ . The first remarkable theorem, dealing with a situation of this form, was a result due to Brown and Adams:

**Theorem 1.** *Let  $\mathcal{T}$  be the homotopy category of spectra, and let  $\mathcal{S} \subset \mathcal{T}$  be the subcategory of finite spectra. Then the following three facts hold:*

- (i) *Every homological functor  $H : \mathcal{S}^{\text{op}} \rightarrow \text{Ab}$  is isomorphic to  $\mathcal{T}(-, t)|_{\mathcal{S}}$ , for some object  $t \in \mathcal{T}$ .*
- (ii) *Every natural transformation  $\mathcal{T}(-, t)|_{\mathcal{S}} \rightarrow \mathcal{T}(-, t')|_{\mathcal{S}}$  is induced by some morphism  $f : t \rightarrow t'$  in  $\mathcal{T}$ .*
- (iii) *If  $f : t \rightarrow t'$  is a morphism in  $\mathcal{T}$ , and the natural transformation*

$$\mathcal{T}(-, f)|_{\mathcal{S}} : \mathcal{T}(-, t)|_{\mathcal{S}} \longrightarrow \mathcal{T}(-, t')|_{\mathcal{S}}$$

*is an isomorphism of functors on  $\mathcal{S}$ , then  $f$  must be an isomorphism in  $\mathcal{T}$ .*

**Remark 1.** The proof of Theorem 1(i), in the special case where  $H(s)$  is assumed countable for any object  $s \in \mathcal{S}$ , was due to Brown [6]. The general statement may be found in Adams [1]. The theorem means that isomorphism classes of objects of  $\mathcal{T}$  may be identified with homological functors  $\mathcal{S}^{\text{op}} \rightarrow \mathcal{A}b$ . Given a homological functor  $H : \mathcal{S}^{\text{op}} \rightarrow \mathcal{A}b$ , Theorem 1(i) tells us that exists an object  $t \in \mathcal{T}$  and an isomorphism

$$H(-) \cong \mathcal{T}(-, t)|_{\mathcal{S}}.$$

If  $t$  and  $t'$  are two such objects, then we must have an isomorphism

$$\mathcal{T}(-, t)|_{\mathcal{S}} \longrightarrow \mathcal{T}(-, t')|_{\mathcal{S}};$$

Theorem 1(ii) says it must be induced by a morphism  $f : t \rightarrow t'$ , and Theorem 1(iii) establishes that this morphism is an isomorphism.

It is natural to ask whether Theorem 1 generalizes to other pairs  $\mathcal{S} \subset \mathcal{T}$ . This question was first asked in the case where  $\mathcal{T}$  is a compactly generated triangulated category and  $\mathcal{S} = \mathcal{T}^c$  is the subcategory of compact objects. The positive theorem says

**Theorem 2.** *Let  $\mathcal{T}$  be a compactly generated triangulated category, and let  $\mathcal{S} = \mathcal{T}^c$  be the subcategory of compact objects. Assume  $\mathcal{S}$  is essentially countable; that is,  $\mathcal{S}$  is equivalent to a category  $\mathcal{S}'$  with countably many objects and morphisms. Then the statements of Theorem 1(i), (ii) and (iii) are all true.*

**Remark 2.** It should be noted that Theorem 1 is a special case of Theorem 2; the homotopy category  $\mathcal{T}$  of spectra is compactly generated, the subcategory  $\mathcal{S}$  of finite spectra is the category of compacts in  $\mathcal{T}$ , and  $\mathcal{S}$  is essentially countable. Theorem 1 may be found in [8]. If  $\mathcal{S}$  is not countable there are counterexamples. The first example of a category in which Theorem 1(i) fails was found by Keller and myself, and appears in [8]. The first example where Theorem 1(ii) fails is due to Christensen, Keller and myself [7]. There is further work exploring this by Beligiannis [2], and by Benson and Gnacadja [4, 5] and Benson [3].

In other words, by now we understand pretty well what happens when  $\mathcal{T}$  is compactly generated and  $\mathcal{S} = \mathcal{T}^c$  is the subcategory of compact objects. It is natural to ask the question about other pairs  $\mathcal{S} \subset \mathcal{T}$ . Next we remind the reader how the theory of well generated triangulated categories provides a whole slew of such pairs. Let  $\mathcal{T}$  be a triangulated category which contains arbitrary small coproducts of its objects. Let  $\alpha$  be any regular cardinal. Then there is a recipe to produce a triangulated subcategory  $\mathcal{T}^\alpha \subset \mathcal{T}$ . We recall:

**Reminder 1.** The category  $\mathcal{T}^\alpha \subset \mathcal{T}$  is the largest triangulated subcategory  $\mathcal{S} \subset \mathcal{T}$  such that, if  $s$  is an object of  $\mathcal{S}$  and

$$f : s \longrightarrow \coprod_{\lambda \in \Lambda} t_\lambda$$

is any morphism, then there exist:

- (i) A subset  $\Lambda' \subset \Lambda$  of cardinality  $< \alpha$

(ii) For any  $\lambda \in \Lambda'$  there is an object  $s_\lambda \in \mathcal{S}$  and a morphism  $f_\lambda : s_\lambda \longrightarrow t_\lambda$ .

All of this data must be such that the morphism  $f$  factors as

$$s \longrightarrow \coprod_{\lambda \in \Lambda'} s_\lambda \xrightarrow{\coprod_{\lambda \in \Lambda'} f_\lambda} \coprod_{\lambda \in \Lambda'} t_\lambda \subset \coprod_{\lambda \in \Lambda} t_\lambda.$$

Note that the existence of a unique maximal  $\mathcal{S}$ , which we call  $\mathcal{T}^\alpha$ , is a theorem. It is also a theorem that the coproduct of  $< \alpha$  objects of  $\mathcal{T}^\alpha$  lies in  $\mathcal{T}^\alpha$ . The proofs of these facts may be found in [9].

We remind the reader also of one of the equivalent definitions of well generated triangulated categories

**Definition 1.** Let  $\mathcal{T}$  be a triangulated category closed under small coproducts. Let  $\alpha$  be a regular cardinal. The category  $\mathcal{T}$  is  $\alpha$ -compactly generated if

- (i)  $\mathcal{T}^\alpha$  is essentially small.
- (ii) Any non-zero object  $x \in \mathcal{T}$  admits a non-zero map  $s \longrightarrow x$ , with  $s \in \mathcal{T}^\alpha$ .

The category  $\mathcal{T}$  is *well generated* if it is  $\alpha$ -compactly generated for some regular cardinal  $\alpha$ .

It is natural to ask if some version of Theorem 1 holds in this context. That is assume  $\mathcal{T}$  is  $\alpha$ -compactly generated, and let  $\mathcal{S} = \mathcal{T}^\alpha$ . Do the conclusions of Theorem 1 hold for this pair  $\mathcal{S} \subset \mathcal{T}$ ?

To have a chance we must modify Theorem 1 a little bit. If  $t$  is any object of  $\mathcal{T}$ , then  $\mathcal{T}(-, t)|_{\mathcal{S}}$  is not just any homological functor  $\mathcal{S}^{\text{op}} \longrightarrow \mathcal{A}b$ . We know that  $\mathcal{S} = \mathcal{T}^\alpha$  is closed under the formation of coproducts of  $< \alpha$  objects, and the functor  $\mathcal{T}(-, t)|_{\mathcal{S}}$  must respect these products. We therefore make a definition.

**Definition 2.** Let  $(\mathcal{T}, \alpha)$  be a pair consisting of a triangulated category  $\mathcal{T}$  and a regular cardinal  $\alpha$ . Put  $\mathcal{S} = \mathcal{T}^\alpha$ . We call  $(\mathcal{T}, \alpha)$  a *Brown-Adams pair* provided

- (i)  $\mathcal{T}$  is  $\alpha$ -compactly generated.
- (ii) If  $H : \mathcal{S}^{\text{op}} \longrightarrow \mathcal{A}b$  is a homological functor, and  $H(\coprod_{\lambda \in \Lambda} s_\lambda) = \prod_{\lambda \in \Lambda} H(s_\lambda)$  for all coproducts of  $< \alpha$  objects of  $\mathcal{S}$ , then  $H$  is isomorphic to  $\mathcal{T}(-, t)|_{\mathcal{S}}$  for some  $t \in \mathcal{T}$ .
- (iii) Every natural transformation  $\mathcal{T}(-, t)|_{\mathcal{S}} \longrightarrow \mathcal{T}(-, t')|_{\mathcal{S}}$  is induced by some morphism  $f : t \longrightarrow t'$  in  $\mathcal{T}$ .

For the purpose of comparing with Theorem 1 note that, if  $f : t \longrightarrow t'$  is a morphism in  $\mathcal{T}$  and  $\mathcal{T}(-, f)|_{\mathcal{S}}$  is an isomorphism, then the fact that  $\mathcal{T}$  is  $\alpha$ -compactly generated allows one to prove quite easily that  $f$  must be an isomorphism.

There is a recent preprint by Rosicky proving the following. Let  $\mathcal{M}$  be a cofibrantly generated, locally presentable model category, and let  $\mathcal{T}$  be its stable homotopy category. Then for arbitrarily large cardinals  $\alpha$  the pair  $(\mathcal{T}, \alpha)$  is Brown-Adams. The question I want to ask is

**Problem 1.** Let  $\mathcal{T}$  be a well generated triangulated category. Is it true that there are arbitrarily large cardinals  $\alpha$  for which  $(\mathcal{T}, \alpha)$  is a Brown-Adams pair? Can one say more about the permissible  $\alpha$ 's?

**Remark 3.** Suppose  $\mathcal{M}$  is a cofibrantly generated, locally presentable model category and  $\mathcal{T}$  is its stable homotopy category. Rosicky's theorem tells us that there exist regular cardinals  $\alpha$  for which  $\mathcal{T}$  is  $\alpha$ -compactly generated; hence  $\mathcal{T}$  is well generated. From Keller's talk at this workshop we know that if  $\mathcal{T}$  is algebraic and well generated, then it is the stable homotopy category of a cofibrantly generated, locally presentable model category  $\mathcal{M}$ . If  $\mathcal{T}$  is algebraic, being well generated is therefore equivalent to having a cofibrantly generated, locally presentable model. When  $\mathcal{T}$  is algebraic the answer part (i) of Problem 1 is Yes, by Rosicky's theorem. It would be interesting to have a proof which does not appeal to models.

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