Mathematisches Forschungsinstitut Oberwolfach

Report No. 20/2006

Mini-Workshop: Feinstrukturtheorie und Innere Modelle

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April 30th – May 6th, 2006

Abstract. The main aim of fine structure theory and inner model theory can be summarized as the construction of models which have a canonical inner structure (a fine structure), making it possible to analyze them in great detail, and which at the same time reflect important aspects of the surrounding mathematical universe, in that they satisfy certain strong axioms of infinity, or contain complicated sets of reals. Applications range from obtaining lower bounds on the consistency strength of all sorts of set theoretic principles in terms of large cardinals, to proving the consistency of certain combinatorial properties, their compatibility with strong axioms of infinity, or outright proving results in descriptive set theory (for which no proofs avoiding fine structure and inner models are in sight).

Fine structure theory and inner model theory has become a sophisticated and powerful apparatus which yields results that are among the deepest in set theory.

Mathematics Subject Classification (2000): 03-06, 03-E45, 03-E55, 03-E35, 03-E15, 03-E60.

Introduction by the Organisers

The workshop *Fine Structure Theory and Inner Models*, organised by Ronald Jensen (Berlin), Menachem Magidor (Jerusalem) and Ralf Schindler (Münster) was held April 30th - May 6th, 2006. It was attended by most of the leading researchers in the area.

Fine structure theory was initiated by the first organizer, R. Jensen, in the 70ies. It has been exploited ever since for producing a series of spectacular results in set theory. One such is Jensen's Covering Lemma for Gödel's constructible hierarchy, L, which says, put informally, that the universe V of all sets either resembles L to a large extent or else is very different from L. Later on, various people (most of which were participants of this workshop) proved versions of the Covering Lemma for larger inner models.

The main goal of "Fine structure theory and inner model theory" is to construct fine structural inner models of set theory, i.e., definable transitive proper class-sized models of the standard axiom system ZFC of set theory, which reflect the large cardinal structure of the universe, but at the same time admit a fine structure that makes it possible to analyze them in great detail and prove various combinatorial properties in them. Other applications of such inner models are consistency strength investigations, and they can be used as a tool for proving implications which don't mention inner models at all, but for which no "direct" proof is in sight.

A large cardinal concept is one such that ZFC cannot prove that there is an incarnation of it. Our area is the key tool for uncovering the large cardinal structure which is implicit in many (not only set theoretic) hypotheses. In fact, often a given statement which doesn't mention large cardinals at all and a statement about the existence of models with large cardinals turn out to be two sides of the same coin. Breathtaking results by Martin, Steel, Woodin, and others in the 80ies and 90ies have shown that the large cardinal concept of a *Woodin cardinal* is a crucial one here.

The main issues of this area are the following.

- *Fine structure theory.* This is a general theory of the definability over the structures that form the building blocks of the inner models one wants to construct. In most cases, these structures are *p*remice, that is, models constructed from sequences of extenders which code fragments of elementary embeddings. The existence of such embeddings is the essence of the crucial large cardinal concepts.
- *Iterability.* That a (well-founded) structure be iterable means that we can keep taking ultrapowers of it (i.e., decoding the elementary embedding coded by some extender on the sequence of the structure, along with the target model, which, by elementarity, is again a model constructed relative to a sequence of extenders) without ever producing non-well-founded structures. In fact, what one needs for iterability is an iteration strategy for the given structure. The iterability of a premouse is the key property one needs in order to choose the next building block in the construction of an inner model in a canonical way. This eventually makes the resulting inner model *definable* in some reasonable way. Also, without iterability we wouldn't know how to prove key (fine structural) first order properties which we require of our premice and which are then inherited by the inner model we are about to construct. For instance, the fact that the (generalized) continuum hypothesis holds in the inner models we construct relies on iterability.

  It is important here to isolate criteria for the iterability of a premouse which are not too strong so that sufficiently many iterable premice can be shown to exist.

- *The model construction.* The construction of an inner model is done by recursion on its "building blocks". In order to verify that the construction doesn't trivialize one has to prove that sufficiently many premice meet the iterability criterion one works with.

  Also, one wants to show that a Covering Lemma holds for the inner model which was built.
- *Applications of inner models.* Woodin's core model induction makes use of "locally defined" inner models which are used for verifying inductively that (sufficiently iterable) models of ZFC plus there are such-and-such many Woodin cardinals exist. This induction can therefore be used for showing that a given (say, combinatorial) statement implies that definable sets of reals are determined. It turns out that in order to organize such an induction properly, one has to construct a new kind of "hybrid" premice which are constructed not only relative to a sequence of extenders, but also relative to iteration strategies for certain structures.

The conference had 16 participants. 13 talks were given, and they covered both pure and applied parts of inner model theory. Because this was a gathering of true specialists, there was no need for overview-style talks and we could concentrate on issues which are at the focus of current research. The talks came with intriguing results, but also with promising new perspectives for upcoming research. We had very lively discussions.

It was a fruitful workshop, and many of the ideas which were exchanged are sure to be further elaborated in the near future.

Mini-Workshop: Feinstrukturtheorie und Innere Modelle

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Abstracts

Supercomplete extenders and type 1 mice

Qi Feng
(joint work with Ronald B. Jensen)

Let M be a pre–mouse. Let $\tau < \kappa < ht(M)$.

τ is *strong upto* κ, denoted by $o^M(\tau) \geq \kappa$, if $\forall \beta < \kappa \exists \nu \leq ht(M)(\tau = crit(E^M_{\omega\nu}) \wedge lh(E^M_{\omega\nu}) \geq \beta)$.

κ is *of type* 0 in M iff there is a $\nu \leq ht(M)$ such that $\kappa = crit(E^M_{\omega\nu})$ and $\{\tau < \kappa \mid o^M(\tau) \geq \kappa\}$ is bounded in κ.

κ is *of type* ≥ 1 in M iff there is a $\nu \leq ht(M)$ such that $\kappa = crit(E^M_{\omega\nu})$ and $\{\tau < \kappa \mid o^M(\tau) \geq \kappa\}$ is unbounded in κ.

κ is *of type* ≥ 2 in M iff there is a $\nu \leq ht(M)$ such that $\kappa = crit(E^M_{\omega\nu})$ and

$$\{\tau < \kappa \mid o^M(\tau) \geq \kappa \wedge \tau \text{ is of type } \geq 1 \text{ in } M\}$$

is unbounded in κ.

κ is *of type* 1 in M if κ is of type ≥ 1 and κ is not of type ≥ 2.

For $\nu \leq ht(M)$, $E^M_{\omega\nu} \neq \emptyset$ is *of type* 0 (of type 1, or of type ≥ 2) if $crit(E^M_{\omega\nu})$ is of type 0 (of type 1, or of type ≥ 2).

A pre–mouse M is *of type* 0 iff for all $\nu \leq ht(M)$ if $E^M_{\omega\nu} \neq \emptyset$ then $crit(E^M_{\omega\nu})$ is of type 0 in M.

A pre–mouse M is *of type* 1 iff M is strongly acceptable and for all $\nu \leq ht(M)$ if $E^M_{\omega\nu} \neq \emptyset$ then $crit(E^M_{\omega\nu})$ is of type < 2 (i.e., not of type ≥ 2) in M.

Let $M = \langle J^E_\alpha, F \rangle$ be a J–structure. M is *strongly acceptable* if and only if M is acceptable and whenever $\tau < \alpha$, $\xi < \omega\tau$, $J^E_{\tau+1} \models \phi(\xi)$ and $J^E_\tau \models \neg\phi(\xi)$ for a Σ_1 formula ϕ, then $Card(\tau) \leq \max(\xi, \omega)$ in $J^E_{\tau+1}$.

The key properties of strongly acceptable structures are Σ_1–reflection between cardinals and it is preserved by Σ_1 embeddings. Also, mice are strongly acceptable.

Type 0 mice are those iterable premice whose iterations shall never result to infinite branching iteration trees. Hence iterations of type 0 premice are almost linear iterations, as studied by Schindler.

Type 1 mice are those iterable premice whose iterations may result infinite branching iteration trees but still enjoy certain finiteness character.

Here, we carry out a study of iteration trees of type 1 premice.

Our basic analysis gives us the following:

Let M be a type 1 premouse. Let $\mathcal{T} = \langle \langle N_i \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$ be a normal iteration of M. Then

1. For $i + 1 < lh(T)$, $T(i+1) \notin [0, i]_T$ if and only if there is a unique h such that $h + 1 \leq_T i$ and $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$.
2. For all i, if there is some $j > i$ such that $\kappa_i \leq \kappa_j < \lambda_i < \lambda_j$, then κ_i or E_{ν_i} is of type 1.
3. If κ_i is of type 1, then $T(i+1) \leq_T i$ and $\kappa_i < crit(\pi_{T(i+1)i})$.

In our basic analysis, we employ three sequences $\langle \lambda_{ij}, U_j, \kappa_{ij} \rangle$ that are naturally associated to a normal iteration tree of type 1 premice.

For $i < j$, we define $\lambda_{ij} = \min\{\kappa_h \mid i \leq h < j \ \wedge \kappa_h < \lambda_i \wedge \ \kappa_h \text{ is type } 0\}$ if there is some h such that $i \leq h < j$ and $\kappa_h < \lambda_i$ and κ_h is of type 0; and define $\lambda_{ij} = \lambda_i$ if otherwise.

Let $U_j = \{i < j \mid \lambda_{ij} > \sup_{l<i} \lambda_l\}$.

For each i, let $\kappa_{ii} = \lambda_i$.

For $j = h+1$, let $\xi = T(h+1)$. If $\xi \in U_j$, then set $\kappa_{\xi j} = \kappa_h$; if $i \in U_j \cap \xi$, then set $\kappa_{ij} = \kappa_{i\xi}$; and if $i \in U_j - (\xi+1)$, then set $\kappa_{ij} = \min(\kappa_h, \kappa_{ih})$.

For limit j, if $i \in U_j$, let $h <_T j$ be the least such that $i < h$, then we set $\kappa_{ij} = \kappa_{ih}$.

A Geometrical Theorem:

Theorem 1. *Assume that $i < j$ are two ordinals less than the length of a normal iteration tree $\mathcal{T}$ of type 1 premice. Set $T_\wedge(i,j) = \max\{m \mid m \leq_T i \ \ \& \ \ m \leq_T j\}$.*

1. *If $T(i+1) \notin [0,i]_T$, then $T_\wedge(T(i+1), i) = T(h(i)+1)$, where $h(i)$ is the unique h such that $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$.*
2. *If $i \in U_j$, then $T_\wedge(i,j) \in U_j$.*
3. *If $i \in U_j$, then $\kappa_{ij} = \min\{\kappa_{T_\wedge(i,j),i}, \kappa_{T_\wedge(i,j),j}\}$, hence,*
$$\kappa_{ij} = \min\{crit(\pi_{T_\wedge(i,j),i}), crit(\pi_{T_\wedge(i,j),j})\}.$$
4. *If $j = h+1$ and $T(j) < i < h$ and $i \in U_j$, then $T_\wedge(i,j) = T(j)$.*

Following this, we are able to derive the following Finiteness Lemma.

Lemma 2. *Let $\mathcal{T}$ be a normal iteration of type 1 premice. Then*

(1) *There is no infinite sequence $\langle i_m \mid m < \omega \rangle$ such that $i_m < i_{m+1}$ and $\kappa_{i_m} < \kappa_{i_{m+1}} < \lambda_{i_0}$ and all of these κ_{i_m} are of type 1.*
(2) *For each i, the set $\{\kappa_j \mid i < j \ \wedge \ \kappa_j < \lambda_i\}$ is finite.*

We are able to show the following existence and uniqueness of cofinal branch of normal iteration trees of type 1 premice.

Theorem 3. *Let $\mathcal{T} = \langle < M_i >, < \nu_i >, < \eta_i >, < \pi_{ij} >, T \rangle$ be a normal iteration of a type 1 premouse of limit length θ. Then*

(a) *$\mathcal{T}$ has at most one cofinal branch. In fact, let $b = b_{\mathcal{T}} = \{i \mid \forall k < \theta \, \exists \, j > k \ (i <_T j)\}$. Then b is a chain under the tree ordering and if $\mathcal{T}$ has a cofinal branch, then b is the unique cofinal branch of the tree.*
(b) *$\mathcal{T}$ has a cofinal branch.*

Let us now define supercomplete extenders.

Let F be an extender on $M = J^A_\alpha$. Let $\kappa = crit(F)$ and $\tau = (\kappa^+)^M$. Let $\pi : J^A_\tau \to_F J^{A'}_{\tau'}$. Let t_ξ be the ξ–th element of $J^{A'}_{\tau'}$. Let $\alpha(F, M)$ be the largest cardinal of M below $\pi(\kappa)+1$.

We say that F is *supercomplete on* M if and only if for every countable $X \subseteq lh(F)$, and every countable $W \subseteq P(\kappa) \cap J_\tau^A$, there is a *strong connection* $\delta : X \to \kappa$ such that

(a) $\prec \delta(\xi_1, \cdots, \xi_n) \succ \in Z \iff \prec \xi_1, \cdots, \xi_n \succ \in F(Z)$ for $Z \in W$ and $\xi_1, \cdots, \xi_n \in X$, and
(b) if $Y \subseteq X$ and $\bigcup_{\xi \in Y} t_\xi$ is a well–founded relation, then so is $\bigcup_{\xi \in Y} t_{\delta(\xi)}$.

We say that F is *supercomplete with respect to* M if and only if for every countable $X \subseteq lh(F)$, and every countable $W \subseteq P(\kappa) \cap J_\tau^A$, there is a *strong connection* $\delta : X \to \kappa$ such that

(a) $\prec \delta(\xi_1, \cdots, \xi_n) \succ \in Z \iff \prec \xi_1, \cdots, \xi_n \succ \in F(Z)$ for $Z \in W$ and $\xi_1, \cdots, \xi_n \in X$, and
(b) if $Y \subseteq X \cap \alpha(F, M)$ and $\bigcup_{\xi \in Y} t_\xi$ is a well–founded relation, then so is $\bigcup_{\xi \in Y} t_{\delta(\xi)}$.

We prove the following Iterability Theorem

Theorem 4. *Let M be a type 1 premouse such that every surviving extender is supercomplete with respect to M. Then M is uniquely simply normally iterable.*

Corollary 5. *Let M be a type 1 premouse such that every surviving extender is supercomplete with respect to M. Then M is a mouse.*

In fact, we prove the following Realization Theorem

Theorem 6. *Let $M = \langle J_\alpha^E, F \rangle$ be a type 1 premouse. Let $\sigma : N \to_{\Sigma^*} M$ be such that N is countable. Let $\mathcal{T} = \langle \langle N_i \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$ be a normal countable iteration of N. Assume that either $\mathcal{T}$ has no truncation and every surviving extender is supercomplete with respect to M or $\mathcal{T}$ has truncations and every surviving extender is supercomplete on M. Then there are $\sigma_i : N_i \to M$ and $\delta_i : \lambda_i \to \sigma_{T(i+1)}(\kappa_i)$ such that*

(a) *$\sigma_0 = \sigma$, $\sigma_j \pi_{ij} = \sigma_i$ for $i \leq_T j$;*
(b) *$\sigma_i(\kappa_i) \leq \sigma_{T(i+1)}(\kappa_i)$;*
(c) *if $\sigma_i(\kappa_i) = \sigma_{T(i+1)}(\kappa_i)$, then*

$$\delta_i : \lambda_i \to \sigma_{T(i+1)}(\kappa_i)$$

is a strong connection in that $\delta_i = g_i \sigma_i \restriction_{\lambda_i}$ and $g_i : \sigma_i[\lambda_i] \to \sigma_i(\kappa_i)$ is to witness the super completeness, and
(d) *$\sigma_{i+1}(\pi_{T(i+1),i+1}(f)(a)) = \sigma_{T(i+1)}(f)(\delta_i(a))$, where $f \in \Gamma(\kappa_i, N_{T(i+1)} || \eta_i)$, $a \in \lambda_i^{<\omega}$.*

(e) *Set* $n(0) = \omega$,

$$n(i+1) = \begin{cases} \omega & \textit{if } \sigma_{T(i+1)}(\kappa_i) < \omega\rho_M^\omega \\ n & \textit{if } \omega\rho_M^{n+1} \leq \sigma_{T(i+1)}(\kappa_i) < \omega\rho_M^n, \end{cases}$$

$n(i) = \min\{n(j) \mid j <_T i\}$ *for limit ordinal* i.
Then σ_i *is* $\Sigma_0^{(n(i))}$*–preserving and if* $n(i) = 0$*, then, in addition,* σ_i *is cardinal preserving.*

The outer model program

Sy-David Friedman

Outer model program. Show that any model with large cardinals has an L-like outer model with large cardinals.

Of course this depends on what one means by "large cardinals" and by "L-like". The large cardinals considered below are the following.

κ is *n-superstrong*, $n \leq \omega$, iff κ is the critical point of an elementary embedding $j : V \to M$ with $V_{j^n(\kappa)} \subseteq M$. κ is *hyperstrong* iff one instead requires $V_{j(\kappa)+1} \subseteq M$.

Here are some results in the outer model program (see my paper *Large cardinals and L-like universes*):

1. If κ is ω-superstrong then it remains so in some forcing extension satisfying GCH, the existence of a definable wellordering of the universe, $\diamondsuit_\kappa$ for all regular κ, the existence of a gap 1 morass at κ for all regular κ, $\Box$ on the singular cardinals and $\Box_\kappa$ restricted to ordinals of cofinality at most the least superstrong cardinal for each regular κ.

2. (Burke) If κ is superstrong then it remains so in some forcing extension satisfying $\Box_\kappa$. (Cummings-Schimmerling improve this to a 1-extendible. On the other hand, Jensen showed that $\Box_\kappa$ fails for hyperstrong κ.)

An interesting question is whether one can force the fundamental L-like properties of condensation and fine structure while preserving large cardinals. Below are some results concerning the former.

Gödel proved a strong form of condensation in L:

(a) $L = \bigcup_\alpha L_\alpha$, L_α transitive, $\mathrm{Ord}(L_\alpha) = \alpha$, $\alpha < \beta \to L_\alpha \in L_\beta$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit λ.
(b) For each α: $(M, \in) \prec (L_\alpha, \in) \to (M, \in) \simeq (L_{\bar\alpha}, \in)$ for some $\bar\alpha$.

This can be formulated axiomatically as follows:

Club Condensation:
(a) $V = \bigcup_\alpha M_\alpha$, M_α transitive, $\text{Ord}(M_\alpha) = \alpha$, $\alpha < \beta \to M_\alpha \in M_\beta$, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit λ.
(b) There are structures $\mathcal{A}_\alpha = (M_\alpha, \in, \ldots)$ for a countable language such that $\mathcal{A} \subseteq \mathcal{A}_\alpha \to \mathcal{A} \simeq \mathcal{A}_{\bar\alpha}$ for some $\bar\alpha$.

Proposition 1. *Club Condensation implies GCH; in fact, if $(M_\alpha \mid \alpha \in \text{Ord})$ witnesses (a) of Club Condensation then for all infinite cardinals κ, $H(\kappa) = M_\kappa$ has cardinality κ.*

Theorem 2. *If there is an ω_1-Erdős cardinal then Club Condensation fails.*

Are large cardinals consistent with weaker forms of Condensation? A natural weakening of Club Condensation is

Stationary Condensation:
(a) $V = \bigcup_\alpha M_\alpha$, M_α transitive, $\text{Ord}(M_\alpha) = \alpha$, $\alpha < \beta \to M_\alpha \in M_\beta$, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit λ.
(b) There are structures $\mathcal{A}_\alpha = (M_\alpha, \in, \ldots)$ for a countable language such that for each α and structure $\mathcal{B}_\alpha = (M_\alpha, \in, \ldots)$ for a countable language, there exists $(M, \in, \ldots) \subseteq \mathcal{B}_\alpha$ such that $\mathcal{A}_\alpha \restriction M \simeq \mathcal{A}_{\bar\alpha}$ for some $\bar\alpha$.

Club Condensation implies Stationary Condensation and Stationary Condensation implies the GCH.

Theorem 3. *If κ is ω-superstrong then it remains so in some forcing extension satisfying Stationary Condensation.*

In fact, the forcing used to prove Theorem 3 is quite simple: just add an α^+-Cohen subset of α^+ for each infinite cardinal α by an Easton support product.

Are large cardinals consistent with stronger forms of Condensation? The known fine-structural inner models for large cardinals obey much more than Stationary Condensation:

Large Condensation:
(a) $V = \bigcup_\alpha M_\alpha$, M_α transitive, $\text{Ord}(M_\alpha) = \alpha$, $\alpha < \beta \to M_\alpha \in M_\beta$, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit λ.
(b) There are structures $\mathcal{A}_\alpha = (M_\alpha, \in, \ldots)$ for a countable language such that for each α of uncountable cardinality κ and structure $\mathcal{B}_\alpha = (M_\alpha, \in, \ldots)$ for a countable language, there exists a continuous chain $(\mathcal{B}_\gamma \mid \omega \leq \gamma < \kappa)$ of substructures of $\mathcal{B}_\alpha$ with union $\mathcal{B}_\alpha$, where each $\mathcal{B}_\gamma = (M_\gamma, \in, \ldots)$ has size card γ and $\mathcal{A}_\alpha \restriction M_\gamma \simeq \mathcal{A}_{\bar\alpha}$ for some $\bar\alpha$.

Club Condensation implies Large Condensation, which in turn implies Stationary Condensation. Large Condensation implies, for example, that for any α of cardinality ω_2 there is a club of x in $P_{\omega_2}(\alpha)$ for which condensation holds on a club of y in $P_{\omega_1}(x)$.

Theorem 4. *In known $L[E]$ models, Large Condensation holds.*

Theorem 5. *If κ is ω-superstrong then it remains so in some forcing extension satisfying Large condensation.*

Theorem 5 is proved by a construction very reminiscent of Jensen coding, however Easton support must be used at inaccessibles for the sake of large cardinal preservation.

Degrees of Rigidity for Souslin Trees

Gunter Fuchs
(joint work with Joel Hamkins)

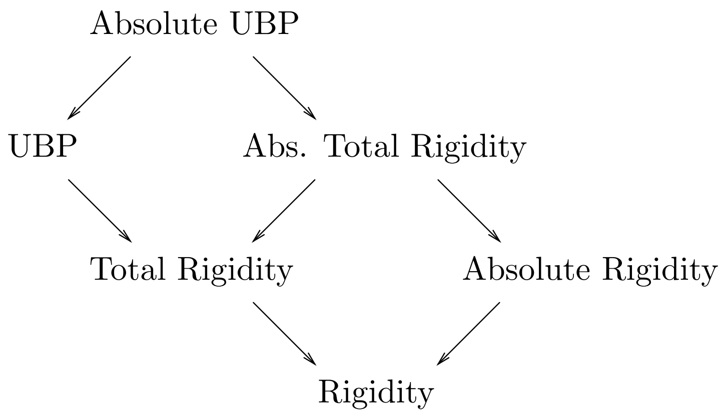
A Souslin tree T is *r*igid if it has no nontrivial automorphism. Consider the following notions of rigidity.

Definition 1. T is *totally rigid* if whenever $p, q \in T$, $p \neq q$, then T_p is not isomorphic to T_q. Here, T_p is the tree obtained by restricting the tree order of T to nodes which lie above p; cf. [1].

T has the unique branch property (UBP) if forcing with T adds precisely one branch to T.

We also consider the (T-)absolute forms of these notions. Namely, T has a property T-absolutely if after forcing with T, T still has that property.

Our main result concerning these notions of rigidity is that, assuming the combinatorial principle $\diamondsuit$, the following diagram exhibits all the implications between them.



The construction of a tree that witnesses that the unique branch property does not imply absolute rigidity results in a tree that has the UBP but has the property that branches through it code automorphisms of some other part of the tree (so that the automorphism fixes the branch that codes it). The construction can be extended to give a result on the automorphism tower problem in group theory (see [3] and [4]).

Namely, if G is a centerless group, then the map sending $g \in G$ to the inner automorphism i_g is injective, and so G can be viewed as a normal subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G. Moreover, Aut is again a centerless group, and so we can iterate the process of passing to the automorphism group of a centerless group, viewing the old group as a normal subgroup of its automorphism group. At limit stages, we can just take unions, by this identification. The sequence of groups obtained in this way is called the automorphism tower of G. Its height is the least α such that the α^{th} group in the tower is isomorphic to its automorphism group.

Using methods of Hamkins and Thomas ([2]), we prove the following result.

Theorem 2. *If $\diamondsuit$ holds, then for any $m < \omega$ there is a group G_m, whose automorphism tower has height m, but for any $n \in [1, \omega)$, there is a $<\omega_1$-distributive notion of forcing, so that in the corresponding extensions, the automorphism tower of the same group has height n.*

Actually, the $<\omega_1$-distributive notions of forcing in the theorem are Sousling trees. Carrying out the construction at higher cardinalities κ, i.e., constructing κ^+-Souslin trees instead of ω_1-Souslin trees, we obtain the following.

Theorem 3. *If $\diamondsuit_{\kappa^+}(\mathrm{cof}_\kappa) + 2^{<\kappa} = \kappa$ holds, then for any $\alpha < \kappa$ there is a group whose automorphism tower has height α, but for any $\beta \in [1, \kappa)$, there is a $<\kappa^+$-distributive notion of forcing such that after forcing with it, the automorphism tower of the same group has height β.*

Again, the heights of the automorphism towers of the abovementioned groups are changed by forcing with Souslin trees, κ^+-Souslin trees this time.

Thus we get:

Theorem 4. *In L, the following is true. If κ is an arbitrary cardinal and $\alpha < \kappa$, then there is a group the automorphism tower of which has height α and is such that for any $\beta \in [1, \kappa)$, there is a $<\kappa$-distributive notion of forcing such that after forcing with it, the height of the automorphism tower of the same group is β.*

Simon and Thomas [2] showed that this statement holds in a generic extension obtained by doing a class size iterated forcing construction.

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Robust Extenders and Coherent Realizability

Ronald B. Jensen

Generalizing the notions of "supercomplete", developed jointly with Qi Feng, we define:

Definition 1. Let $N = \langle J^E_\alpha, F\rangle$ be an active premouse. F is *robust* on N iff whenever $W \subset \mathfrak{P}(\kappa) \cap N$, where $\kappa = crit(F)$ and $U \subset \lambda = F(\kappa)$ are countable, there is a $g : U \to \kappa$ s.t.

(a) $\prec \vec{\alpha} \succ \in F(X) \iff \prec g(\vec{\alpha}) \succ \in X$ for $\alpha_1, \ldots, \alpha_n \in U, X \in W$.

(b) Let $u_1, \ldots, u_n \subset U$. Then if φ is a Σ_1–formula, we have:

$$C^E_{c,\infty} \models \varphi[\vec{u}] \leftrightarrow C^E_{\bar{c},\kappa} \models \varphi[g''\vec{u}],$$

where $c = \sup U, E = \sup g''U$.

Here, the hierarchy $C^E_{\tau,\nu} (\nu \leq \infty)$ is defined by:

$$\begin{aligned} C_0(e) &= TC(\{e\}), \\ C_{\alpha+1}(e) &= \mathrm{Def}(C_\alpha(e)) \cup [\alpha]^\omega, \\ C_\lambda(e) &= \bigcup_{\nu<\lambda} C_\nu(e), \text{for } \lambda \leq \infty \text{ a limit.} \end{aligned}$$

(This is the Chang–hierarchy over the set e.) We then set:

$$\tilde{C}^E_{\tau,\eta} = C_\eta(\langle L_\tau[E], E \cap L_\tau[E]\rangle)$$

for $E \subset V$. Finally

$$C^E_{\tau,\eta} = \langle \tilde{C}^E_{\tau,\eta} = \langle \tilde{C}^E_{\tau,\eta}, \langle C^E_{\tau,\xi} | \xi < \eta\rangle\rangle.$$

In order to apply this to the problem of coherent iterability we set:

$$\hat{C}^E_{\tau,\eta} = C^{(E\times\{0\})\cup(e\times\{1\})}_{E,\eta} \text{ where } \bar{e} : \gamma \leftrightarrow V_{\omega_2}$$

and $e = \{\langle \nu, \tau\rangle | \bar{e}(\nu) \in \bar{e}(\tau)\}$. We define the notion of *e–robustness* be replacing $C^E_{\tau,\eta}$ by $\hat{C}^E_{\tau,\eta}$ in the above definition.

Now let $\langle N_\xi | \xi \leq \Theta\rangle$ $(\Theta \leq \infty)$ be a Steel array in which e–robustness is the criterion for adding extenders. Let $\sigma_0 : P \prec N_\xi$, where P is countable. Let $I = \langle\langle P_i\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a countable normal iteration of P. If I has no truncation, we define a *coherent realization* of I wrt. σ_0 to be a sequence $\langle \sigma_i\rangle$ st. $\sigma_i : P_i \to N_\xi$ (with appropriate preservation) and $\sigma_j \pi_{ij} = \sigma_i$ if iTj. (If I has truncations the definition is modified accordingly.) We prove

(1) A coherent realization exists if there is no *interlocking chain* in I. [By an interlocking chain we mean a sequence $\langle i_n | n < \omega \rangle$ st. $i_n < i_{n+1}$ and $T(i_n + 1) < T(i_{n+1} + 1) \leq i_n$ for $n < \omega$ (where $T(j+1) =$ the immediate predecessor of $j+1$ in T). (We can also weaken the hypothesis to the assumption that there is no "severely interlocking chain".)]

The set of mice which permit an iteration with an interlocking chain can be characterized as follows:

(2) Let M be a mouse. The following are equivalent:
  (a) Some $M || \xi$ has a normal iteration with an interlocking chain
  (b) Some $M || \xi$ exists which sees arbitrarily large Σ_2–strong cardinals. (The definition of a Σ_n–strong cardinal in a premouse is the same as in ZFC, except that the extenders which verify Σ_n–strongness must be in the sequence.)

Open Question: What is the first N_ξ which does not permit coherent realization for some σ_0, P, I as above? Does this N_ξ have an iteration with an alternating chain?

We also applied the notion of robustness to the problem of realizability in Steel's sense. We get:

(3) Let $\langle N_\xi | \xi < \Theta \rangle$ be a Steel array in which robustness in the criterion for adding extenders. Let $\sigma_0 : P \prec N_\xi$, I be as above. Then I has a maximal realizable branch wrt. σ_0.

Thus robustness has (as far as known) the same efficacy as Steels "background certifiability", although it is weaker. Using (3) and methods of Mitchell and Schindler we prove:

(4) (ZFC) Assume that there is no inner model with a Woodin cardinal. Form K^c, using robustness as the criterion for adding extenders. Then K^c is universal.

An inner models proof of the Kechris-Martin theorem

Itay Neeman

A *code* for an ordinal $\alpha \in [\omega_n, \omega_{n+1})$ is a pair $\langle x^\#, t\langle v_1, \ldots, v_n \rangle \rangle$ such that $t^{L[x]}[\omega_1, \ldots, \omega_n]$ is equal to α. An ordinal *belongs to* Γ if it has a code in Γ. $A \subseteq [\omega_n, \omega_{n+1})$ *belongs to* Γ if the set D of codes for ordinals in A belongs to Γ.

The following classical results are due to Kechris and Martin:

Lemma 1. *Assume* AD. *Let* $A \subseteq [\omega_n, \omega_{n+1})$ *be* Σ^1_3 *and bounded below* ω_{n+1}. *Then* A *has a* Δ^1_3 *bound.*

Theorem 2. *Assume* AD. *Every* Π^1_3 *subset of* $[\omega_n, \omega_{n+1})$ *has a* Δ^1_3 *member.*

Corollary 3. *Assume* AD*. Let* ψ *be* Σ_1*. For* $x \in \mathbb{R}$ *let* α_x *be least such that* $L_{\alpha_x}(T_2, x)$ *is admissible. Then the set* $\{x \mid L_{\alpha_x}(T_2, x) \models \psi[T_2, x]\}$ *is* Π^1_3*.*

We prove the corollary using inner models and genericity iterations rather than determinacy. The inner models proof may be easier to generalize to higher levels, but at the moment, this is not known.

Problems related to $I[\lambda]$

William J. Mitchell

The argument which this talk attempted to describe does not seem to be valid. In the remainder of this abstract we will describe the aim of the investigation and state some open problems.

The investigation concerned Shelah's approachability ideal $I[\lambda]$ [3]:

Definition 1. We say that an ordinal α is *approachable* via the sequence $\mathbf{a} = \langle a_\nu : \nu < \lambda \rangle$ if there is a cofinal set $c \subset \alpha$ with $\operatorname{otp} c = \operatorname{cf}(\alpha)$ such that $c \cap \beta \in \{ a_\nu : \nu < \alpha \}$ for all $\beta < \alpha$.

A set $x \subseteq \lambda$ is in $I[\lambda]$ if there is a sequence $\mathbf{a}$ such that every ordinal in λ except for a nonstationary set is approachable via $\mathbf{a}$.

I earlier answered [2] a question of Shelah by showing that

Theorem 2. *If* κ *is* κ^+*-Mahlo then there is a generic extension in which there is no nonstationary set* S *of ordinals of cofinality* ω_1 *in* $I[\omega_2]$*.*

This theorem easily extends to $I[\kappa^+]$ whenever κ is the successor of a regular cardinal. The program discussed in this talk aims to understand the remaining cases. The most interesting case in that in which κ is singular, in which case Shelah has shown that the statement of theorem 2 cannot hold of $I[\kappa^+]$ for sets of ordinals of any cofinality $\gamma < \kappa$. We would like to show that this result cannot be strengthened:

Conjecture 3. *Under an appropriate large cardinal hypothesis, there in a generic extension with a singular cardinal* κ *such that for each uncountable cardinal* $\gamma < \kappa$ *there is a set* $S \subset \kappa$ *of ordinals of cofinality* γ *which is not in* $I[\kappa^+]$*.*

A proof of conjecture 3 could be expected to involve an iteration of the forcing of theorem 2, combined with Prikry style forcing at κ to make κ singular. We are far from knowing how to do this, and the progress described in the present talk is limited to the problem of a finite iteration of the forcing.

Conjecture 4. *It is consistent that (under a suitable large cardinal hypothesis) there is a generic extension in which* $I[\omega_2]$ *has the property of theorem 2, and in addition* $I[\omega_3]$ *satisfies the same property for sets of ordinals of confinality* ω_2*.*

In fact the current work does not deal with $I[\omega_2]$, but instead attempts to use the basic technique of this forcing to reprove a known result:

Theorem 5. *[Uri Abraham [1]] If $\kappa < \lambda$ are cardinals such that κ is λ-supercompact and λ is weakly compact then there is a generic extension in which there are no $\aleph_2-$ or $\aleph_3$-Aronszajn trees.*

Question 6. *Can this technique be used to reprove theorem 5, or and least construct a model with no special Aronszajn trees on $\aleph_2$ or $\aleph_3$?*

The question about Special Aronszajn trees is likely to be less difficult than that for Aronszajn trees — assuming either is possible — and would probably be the relevant problem towards the extension to $I[\omega_2]$.

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Threads

Ernest Schimmerling

The combinatorial principle $\square(\lambda)$ says that all coherent sequences of clubs of length λ can be threaded. If $\lambda = \kappa^+$, then the related principle $\square_\kappa$ implies $\square(\lambda)$. Square principles such as these were isolated by Jensen and are key technical links between several parts of set theory. They have also been applied in other areas such as model theory and topology. Jensen and Solovay showed that the failure of $\square_\kappa$ for some regular $\kappa \geq \aleph_1$ is equiconsistent with the existence of a Mahlo cardinal. Velickovic showed that the failure of $\square(\lambda)$ for some regular $\kappa \geq \aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. My results combined with those of several others show the following. If $\kappa \geq \aleph_2$ and both $\square_\kappa$ and $\square(\kappa)$ fail, then there is an inner model with a proper class of strong cardinals. If $\kappa \geq \max(\aleph_2, 2^{\aleph_0})$ and both $\square_\kappa$ and $\square(\kappa)$ fail, then all sets of reals in $L(\mathbb{R})$ are determined. If κ is a singular cardinal, then the hypothesis that $\square(\kappa)$ fails is not needed. What is new here is the case in which κ is a regular cardinal. As a corollary, the Proper Forcing Axiom for posets of cardinality $(2^{\aleph_0})^+$ then all sets of reals in $L(\mathbb{R})$ are determined. I spoke on these results on the morning of Friday, May 5, 2006, explained credit and sketched the proofs.

The self-iterability of $L[E]$ and $\diamondsuit^*_{\kappa,\lambda}$

Ralf Schindler
(joint work with John Steel)

Let $L[E]$ be an iterable tame fine structural model, and let Σ be an iteration strategy for $L[E]$. We analyze to which extent $L[E]$ knows fragments of Σ.

Definition 1. Let γ be either a cardinal of $L[E]$, or else $\gamma = \infty$. An ordinal $t < \gamma$ is called a *transition point of* $L[E]$ *below* γ iff t is a cutpoint in $L[E]$ and for every $\bar{\gamma} < \gamma$, $L[E] \models$ "$J_{\bar{\gamma}}[E]$ is γ-iterable above t, as witnessed by $\Sigma \restriction X \in L[E]$", where X is the collection of all trees $\mathcal{T}$ on $J_{\bar{\gamma}}[E]$ which are above t, have length less than γ, and are in $L[E]$.

Theorem 2. *For every $\gamma > \omega$ such that either γ is a cardinal in $L[E]$ or else $\gamma = \infty$, there is a transition point of $L[E]$ below γ.*

The proof of Theorem 2 exploits trying to make an initial segment of $L[E]$ generic over the common part model of the tree $\mathcal{T}$ for which $L[E]$ is in search of the right branch, so that then $L[E]$ can serve as a certificate for a Q-structure for $\mathcal{T}$. Theorem 2 is in part motivated by [4]. It is also motivated by the question whether any iterable tame fine structural model $L[E]$ satisfies "there is $x \in \mathbb{R}$ and an OD_x-well-ordering of the reals," which was recently settled in the affirmative by Steel for all ω-small $L[E]$.

Definition 3. Let $\kappa \leq \lambda$ be cardinals. The principle $\diamondsuit^*_{\kappa,\lambda}$ denotes the following statement. There is a function $F\colon \mathcal{P}_\kappa(\lambda) \to V$ such that for every uncountable $X \in \mathcal{P}_\kappa(\lambda)$, $F(X)$ is a subset of $\mathcal{P}(X)$ of size at most $\mathrm{Card}(X)$, and for all $A \subset \lambda$, there is a club $C \subset \mathcal{P}_\kappa(\lambda)$ such that for all $X \in C$, $X \cap A \in F(X)$.

This principle was isolated by Jensen who showed that if $\kappa < \lambda$, then $\diamondsuit^*_{\kappa,\lambda}$ holds in L. In fact, Jensen's original formulation of $\diamondsuit^*_{\kappa,\lambda}$ results from the one given above by deleting "uncountable," and he proved that if $\kappa < \lambda$, then this stronger form of $\diamondsuit^*_{\kappa,\lambda}$ holds in L.

We combine the proof of Theorem 2 with the argument of [3] and [2] to show the following, where $L[E]$ is still an iterable tame fine structural model. The reason for having included "uncountable" in the above formulation of $\diamondsuit^*_{\kappa,\lambda}$ is that the covering argument does not apply to countable substructures of initial segments of $L[E]$.

Theorem 4. *If $\kappa < \lambda$, then $\diamondsuit^*_{\kappa,\lambda}$ holds in $L[E]$.*

It turns out that as a matter of fact a version of the strengthening $\diamondsuit^+_{\kappa,\lambda}$ of $\diamondsuit^*_{\kappa,\lambda}$ (cf. [1]) also holds in $L[E]$ (provided that $\kappa < \lambda$).

Our results will appear in [5].

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The global mouse set conjecture implies its local refinement

John R. Steel

The mouse set conjecture is one of the central open problems in inner model theory. There is one statement of it:

Definition 1. Let $\varphi(v) = \exists A \subseteq \mathbb{R}\psi(A,v)$ be a Σ_1^2–formula, where ψ has k real quantifiers, and let x be a real. A $\langle\varphi, x\rangle$*–witness* is a countable, transitive N such that $x \in N$ and

(i) $N \models ZFC + \delta_0 < \ldots < \delta_k$ are Woodin cardinals,
(ii) $N \models \exists A \in Hom_{\delta_k}\ \psi(A,x)$, and
(iii) N is ω_1–iterable.

Here Hom_k is the collection of κ–homogeneously Suslin–sets of reals. The notion of $\langle\varphi, x\rangle$–witness as defined here is meant to be considered in models of AD, where ω_1–iterability implies $\omega_1 + 1$–iterability, and it is easily shown that

Proposition 2. *Assume AD, and let N be a mouse over x which is a $\langle\varphi, x\rangle$ witness; Then $\varphi(x)$ is true.*

It is important here that N be a *mouse* (a fine structural object). This implies that N will have an iteration strategy with the Dodd–Jensen–property. That can be used to blow up the $Hom^N_{\delta_k}$ set A such that $N \models \psi[A,x]$, using genericity iterations, to an A^* such that $V \models \psi[A^*, x]$. The mouse–set–conjecture (MSC) is that the converse to the proposition holds:

MSC: Assume AD^+, and that there is no ω_1–iteration strategy for a mouse with a superstrong cardinal. Let $\varphi(v)$ be a Σ_1^2–formula, x a real, and suppose $\varphi(x)$ is true. Then there is a mouse N over x such that N is a $\langle\varphi, x\rangle$–witness.

Hugh Woodin has proved MSC under the stronger hypothesis that AD^+ holds, and there is no boldface pointclass Γ such that $L(\Gamma,\mathbb{R})$ models $AD + \Theta = \Theta_{\omega_1}$. Neeman and the author have made some incremental improvements to this result.

The natural attempt to prove MSC involves an induction on the Wadge–hierarchy. Let P_β = the β^{th} pointclass closed under $\neg$, $\exists\mathbb{R}$, and Wadge–reducibility. (Thus P_0 is the class of projective sets.) One would try to show

Local MSC: Assume AD^+; Then for any β, $P_\beta \models$ MSC. That is, if $\exists A \in P_\beta\ \psi(A,x)$ is true, where ψ involves only real quantifiers, then there is a $\langle\varphi,x\rangle$–witness N such that N has an ω_1–iteration strategy in P_β.

We show

Theorem 3. *Assume AD^+ and MSC; Then Local MSC holds.*

On the consistency strength of the inner model hypothesis

Philip Welch
(joint work with Sy-David Friedman and Hugh W. Woodin)

The inner model hypothesis (IMH) and the strong inner model hypothesis (SIMH) were introduced by S-D. Friedman in [2]. We describe here some recent consistency strength computations. This is joint work with S-D. Friedman, and W.H. Woodin.

Definition 1. Let a first order sentence $\sigma \in \mathcal{L}_{\dot\in}$ be called *internally consistent* if it holds in some inner model (IM) (not necessarily proper) of ZFC set theory.

We treat with models of the form $\mathcal{V} = \langle V, \in, \mathcal{C}_V\rangle$ where
(a) V is a countable transitive inner model of ZFC.
(b) $\mathcal{C}_V$ is a countable set of *classes* over V.
We assume that the classes C in $\mathcal{C}_V$ are ZFC-preserving in that $\langle V, \in, C\rangle$ is also a ZFC model in the appropriately widened language. We assume that $\mathcal{C}_V$ contains at least the definable classes over $\langle V, \in\rangle$. $\mathcal{C}_V$ will also be deemed to contain as a minimum, the inner models of V

Definition 2. $\mathcal{V}^* = \langle V^*, \in, \mathcal{C}_{V^*}\rangle$ is an *outer model* of $\mathcal{V}$ if it satisfies (a) and (b) above and:
(i) $\mathcal{V}^* \supseteq \mathcal{V}$
(ii) $\mathrm{On} \cap V^* = \mathrm{On} \cap V$;
(iii) $\mathcal{C}_V \subseteq \mathcal{C}_{V^*}$.

Definition 3. (IMH) The *inner model hypothesis* holds of $\mathcal{V}$, if, for any sentence $\sigma \in \mathcal{L}_{\dot\in}$, if it is internally consistent in an outer model $\mathcal{V}^*$ of $\mathcal{V}$, then it is already internally consistent in $\mathcal{V}$.

- Note that IMH easily implies the Σ^1_3-correctness of V in its outer models.

Utilising a coding result of David and of Beller (*cf* [1]) the following was known:

Theorem 4. ([1]) *The inner model hypothesis implies that for some real r, for any ordinal $\alpha \in V$, $L_\alpha[r] \not\models$ ZFC. In particular, there are no inaccessible cardinals and the reals are not closed under the #-operation.*

As was the following:

- (S-D.Friedman) IMH implies the existence of $O^{\#}, O^{\#\#}, \ldots, O^{\#(n)}, \ldots$ in V.

We improve this to:

Theorem 5. *IMH implies that in the Core Model K, for any ordinal δ, there is a measurable cardinal κ with mitchell order equal to δ: $o^K(\kappa) = \delta$.*

We note that this is rather weak (in view of the theorem to come): we do not even claim there is a measurable with order $o(\kappa) = \kappa$.

As an upper bound we have, using a result of Kechris and Solovay, *cf* [3]:

Theorem 6. *Suppose there is an inaccessible cardinal and Δ^1_2-Determinacy (lightface) holds. Then there is a model $\mathcal{V} = \langle V, \in, \mathcal{C}_V \rangle$ satisfying (a) and (b) above, for which IMH holds.*

One may try to strengthen the IMH by allowing parameters in the definition. This can quickly lead to inconsistency. One such principle which we do not know to be consistent, but which is not obviously inconsistent can be obtained as follows.

Definition 7. A set p is *(globally) absolute* if there is $\varphi(v_0) \in \mathcal{L}_{\dot{\in}}$, parameter free, so that for all $\mathcal{V}^* \supseteq \mathcal{V}$:

$(*)$ $\forall\alpha(V \models$"$\text{card}(\alpha) \wedge \overline{\overline{\text{TC}(p)}} \geq \alpha$"$\longrightarrow V^* \models$"$\text{card}(\alpha) \wedge \overline{\overline{\text{TC}(p)}} \geq \alpha$")

implies that $\varphi(v_0)$ uniquely defines p in V and in V^*.

We thus require absoluteness of the definition of a parameter between V and any outer model which has the same cardinals below $\text{TC}(p)$.

Definition 8. (SIMH) If $p \in V$ is absolute, and $\psi(v_0)$ and formula, and for any $\mathcal{V}^* \supseteq \mathcal{V}$ satisfying the antecedent $(*)$ above, if $\psi(p)$ holds in an inner model of $\mathcal{V}^*$, then it holds in an inner model of $\mathcal{V}$.

Note: we are not requiring that the cardinals below that of $TC(p)$ in the *inner* model be precisely those of V, only that p is properly identified in the same way as it is in V. As a lower bound on the strength of this principle we do have:

Theorem 9. *SIMH implies that there is an inner model with a strong cardinal.*

Question: Is SIMH consistent, relative to some large cardinal hypothesis?

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Beyond ω-huge

Hugh W. Woodin

The main topic for this series of lectures is the identification and analysis of an $\mathsf{AD}_{\mathbb{R}}$-like axiom at the level of $V_{\lambda+1}$. We also develop the theory of such axioms, isolating a key conjecture: *The Weak Uniqueness of Square Roots at* λ.

The basic thesis is that inner models, $L(N)$, such that

$$N = L(N, V_{\lambda+1}) \cap V_{\lambda+2}$$

and for which there is an elementary embedding,

$$j : L(N) \to L(N),$$

with critical point below λ, are analogs at λ of inner models $L(\Gamma)$ where $\Gamma \subset \mathcal{P}(\mathbb{R})$,

$$\Gamma = L(\Gamma, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}),$$

and where $L(\Gamma, \mathbb{R}) \models \mathsf{AD}$. Thus the existence of an elementary embedding,

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with critical point below λ is the analog at λ of $L(\mathbb{R}) \models \mathsf{AD}$.

Suppose

$$V_{\lambda+1} \subset N \subset V_{\lambda+2}$$

and that $N = L(N) \cap V_{\lambda+2}$. Then Θ^N denotes the supremum of the ordinals α such that there exists a surjection,

$$\pi : V_{\lambda+1} \to \alpha$$

with $\pi \in L(N)$.

Definition 1. Suppose that N is transitive,

$$V_{\lambda+1} \subset N \subset V_{\lambda+2},$$

$N = L(N) \cap V_{\lambda+2}$, and that

$$j : L(N) \to L(N)$$

is an elementary embedding with $\mathrm{CRT}(j) < \lambda$. Then:

(1) j is *weakly proper* if $L(N) = \{j(F)(j|V_\lambda) \mid F \in L(N)\}$;
(2) j is *proper* if j is weakly proper and for all $X \in N$,

$$\langle X_i : i < \omega \rangle \in L(N, V_{\lambda+1}),$$

where $X_0 = X$ and for all $i < \omega$, $X_{i+1} = j(X_i)$. □

There is an analog of of $\mathsf{ZF} + \mathrm{DC} + \mathsf{AD}_{\mathbb{R}}$ at the level of $V_{\lambda+1}$, the definition is motivated by the fact that assuming $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$, for each set $A \subset \mathbb{R}$ there exists $\eta < \Theta$ such that there is no surjection, $\rho : \mathbb{R} \to \eta$, which is OD from A, here Θ denotes the supremum of the ordinals η for which there is a surjection of $\mathbb{R}$ onto η.

Definition 2. Suppose that

$$V_{\lambda+1} \subset N \subset V_{\lambda+2}$$

and $N = L(N, V_{\lambda+1}) \cap N$. Then N is an $\mathsf{AD}_{\mathbb{R}}$-*like model at* λ if:

(1) $\mathrm{cf}(\Theta^N) > \lambda$,
(2) There is a proper elementary embedding, $j : L(N) \to L(N)$;
(3) For all $X \in N$ there exists $\eta < \Theta^N$ such that there is no surjection,

$$\rho : V_{\lambda+1} \to N$$

such that ρ is OD in $L(N)$ from X. □

Suppose that there is an inner model of $\mathsf{ZF} + \mathrm{DC} + \mathsf{AD}_{\mathbb{R}}$ (containing $\mathbb{R} \cup \mathrm{Ord}$) and let

$$\Gamma_0 = \cap\{\Gamma \subset \mathcal{P}(\mathbb{R}) \mid \Gamma = L(\Gamma, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) \text{ and } L(\Gamma, \mathbb{R}) \models \mathsf{ZF} + \mathrm{DC} + \mathsf{AD}_{\mathbb{R}}\}.$$

Then $L(\Gamma_0) \models \mathsf{ZF} + \mathrm{DC} + \mathsf{AD}_{\mathbb{R}}$. This generalizes to $\mathsf{AD}_{\mathbb{R}}$-like models at λ.

Theorem 3. *Suppose that there is an* $\mathsf{AD}_{\mathbb{R}}$*-like model at* λ *and let*

$$N_0 = \cap\{N \mid N \text{ is an } \mathsf{AD}_{\mathbb{R}}\text{-like model at } \lambda\}.$$

Then N_0 *is an* $\mathsf{AD}_{\mathbb{R}}$*-like model at* λ. □

The analysis of $\mathsf{AD}_{\mathbb{R}}$-like models at λ leads naturally to the following definitions and conjecture.

Definition 4. Suppose $X \subset V_\lambda$ and

$$j, k : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

are proper elementary embeddings with critical point below λ. Then $j = k(k)$ if

$$j|V_{\lambda+2} = \cup\{k(k|Z) \mid Z \subset V_{\lambda+2}, (Z, k|Z) \in L(X, V_{\lambda+1})\}. \qquad \square$$

This the natural definition of $k(k)$. Notice that if j is proper and if $k(k) = j$ then necessarily k is proper.

Definition 5. Suppose $X \subset V_\lambda$ and

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

is a proper elementary embedding. An elementary embedding

$$k : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

is a *square root* of j if $k(k) = j$. □

We now come to our main structural conjecture.

Conjecture 6. (Weak Uniqueness of Square Roots at λ) *Suppose λ is a limit of supercompact cardinals. For all $X \subset V_{\lambda+1}$, if*

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

is a proper elementary embedding and if k_1 and k_2 are each square roots of j such that

(1) $k_1|V_\lambda = k_2|V_\lambda$,
(2) $k_1(L_\omega(X, V_{\lambda+1})) = k_2(L_\omega(X, V_{\lambda+1})) = L_\omega(X, V_{\lambda+1})$,

then $k_1|\Theta^{L(X,V_{\lambda+1})} = k_2|\Theta^{L(X,V_{\lambda+1})}$. □

This conjecture if provable would probably yield a proof of the following conjecture.

Conjecture 7. (Minimum Model Conjecture at λ) *Suppose that*

$$V_{\lambda+1} \subset M \subset V_{\lambda+2}$$

and that M is an $\mathsf{AD}_{\mathbb{R}}$-like model at λ. Let

$$M_0 = \cap\{N \mid N \textit{ is an } \mathsf{AD}_{\mathbb{R}}\textit{-like model at } \lambda \textit{ with } \Theta^N = \Theta^M\}.$$

Then M_0 is an $\mathsf{AD}_{\mathbb{R}}$-like model at λ and $\Theta^{M_0} = \Theta^M$. □

A sentence ϕ is Ω-valid from ZFC if for all complete Boolean algebras, $\mathbb{B}$, and for all $\alpha \in \mathrm{Ord}$ if

$$V_\alpha^{\mathbb{B}} \models \mathrm{ZFC}$$

then $V_\alpha^{\mathbb{B}} \models \phi$. This definition is in the context of ZF.

Another interesting consequence the conjecture on weak square roots is given in the following theorem where for a nontrivial elementary embedding,

$$j : V_\lambda \to V_\lambda,$$

$\kappa_\omega(j)$ denotes the supremum of the critical sequence of j, this is the sequence

$$\langle \kappa_i : i < \omega \rangle$$

where κ_0 is the critical point of j and for all $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$.

Theorem 8. (ZF) *Suppose that the weak uniqueness of square roots at λ is Ω-valid from ZFC and that*

$$V_\lambda \prec_{\Sigma_4} V.$$

Then there is no nontrivial elementary embedding

$$j : V_{\lambda+3} \to V_{\lambda+3}$$

such that $\lambda = \kappa_\omega(j)$. □

The main open problems are the two conjectures:

(1) *Weak Uniqueness of Square Roots at λ*
(2) *Minimum Model Conjecture at λ*

Constructing global square sequences in extender models

Martin Zeman

Various types of square sequences were introduced by Jensen in his seminal paper on fine structure of the constructible hierarchy. Given a class $\mathcal{S}$ of singular limit ordinals, a global square sequence on $\mathcal{S}$, briefly a global $\square^{\mathcal{S}}$-sequence is a sequence of sets $\langle C_\alpha \mid \alpha \in \mathcal{S}\rangle$ satisfying the following properties:

(a) Each C_α is a closed unbounded subset of α, and if α is of uncountable cofinality then $C_\alpha \subseteq \mathcal{S}$.
(b) (Coherency.) If $\bar{\alpha}$ is a limit point of C_α then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
(c) For each $\alpha \in \mathcal{S}$, the order type of C_α is strictly smaller than α.

We write briefly $\square$ if $\mathcal{S}$ is the set of all singular ordinals. Square sequences are ubiquitous in set theory, as they provide a combinatorial structure that enables to run certain type of constructions by transfinite recursion. Classical applications of square sequences involve constructions of Suslin trees, special Aronszajn trees and nonreflecting stationary sets. Jensen showed that various kinds of square sequences, among others also global square sequences $\square$, exist in Gödel's constructible universe $\mathbf{L}$. In the absence of $0^\#$, square sequences constructed in $\mathbf{L}$ can often be turned into square sequences in the sense of the actual universe $\mathbf{V}$, which is crucial in most of the applications. It is therefore desirable to look for stronger results where the non-existence of $0^\#$ is replaced by weaker hypotheses. In such results, Gödel's $\mathbf{L}$ has to be replaced by a more general type of model, a so-called extender model. These models are of the form $\mathbf{L}[E]$ where E is a predicate that codes a coherent extender sequence. The literature on construction varisus square sequences in such models is quite extensive. The ultimate result on $\square_\kappa$ in fine structural extender models where extenders on the E-sequence are of a superstrong type or shorter is due to Schimmerling and Zeman and states that a $\square_\kappa$-sequence exists in $\mathbf{L}[E]$ just in case that κ is not subcompact, i.e. when $\{x \in \mathcal{P}_\kappa(\kappa^+) \mid \mathrm{otp}(x) \text{ is a cardinal}\}$ is non-stationary.

The main result I presented is a construction of a global square sequence $\square$. By a result of Jensen, such a sequence exists just in case that there is a $\square_\kappa$-sequence for all cardinals κ and also a $\square^{\mathrm{SC}}$-sequence where SC is the class of all singular cardinals. Building on the result of Schimmerling-Zeman, the heart of the construction of a $\square$-sequence is a construction of a $\square^{\mathrm{SC}}$-sequence. It turns out that such a senquence exists regardless of what large cardinals live in our extender model. Although this might look a bit surprising, it is consistent with earlier results that show the existence of a global square sequence in the presence of supercompact cardinals; these results are forcing constructions. The main theorem thus reads:

Theorem. If W is arbitrary extender models whose initial levels are weakly iterable then $W \models \square^{\mathrm{SC}}$.

Here "weakly iterable" means that any countable premouse elementarily embeddable into the structure in question is $(\omega_1 + 1, \omega_1)$-iterable. This amount of iterability enables to show that W admits the necessary fine structure theory needed in our construction. The construction itself follows the original Jensen's construction in $\mathbf{L}$, but involves a new important feature, namely that it has to be carried out on two disjoint subclasses of SC which we denote $\mathcal{S}^0$ and $\mathcal{S}^1$. On $\mathcal{S}^0$ we imitate Jensen's construction using levels of W instead of levels of $\mathbf{L}$. On $\mathcal{S}^1$ we also imitate Jensen's construction, but this time we are forced to use structures called *protomice* that code levels of W in a canonical way, but are not levels of any extender models, as their top extender fails to be total. The basic analysis of the situation was done by Schimmerling-Zeman in connection with their construction of a $\square_\kappa$-sequence. It has turned out that to each level corresponding to an ordinal from $\mathcal{S}^1$ we can assign a protomouse in a canonical way and that this assignment is robust enough that it is preserved under manipulations arising in the construction. In the construction of a global $\square^{\mathrm{SC}}$ sequence, there are three new issues that has to be addressed and which do not occur in the consctuction of a $\square_\kappa$-sequence.

First, to each $\alpha \in \mathcal{S}^1$ we assign the singularizing W-level N_α. This level projects to α, but the first n satisfying $\omega\varrho^n_{N_\alpha} \leq \alpha$ might differ from the least n such that α is definably singularized over N_α via some good $\mathbf{\Sigma}_1^{(n)}$-function. This causes serious difficulties in the translation of fine structural information between N_α and the corresponding protomouse M_α. The key here is to restrict ourselves to those $\alpha \in \mathrm{SC}$ where there is no difference between the two values of n; we call such levels N_α *exact.* I prove that exact levels constitute sufficiently large class of W-levels that makes the main construction go through.

The second new feature is a condensation lemma for protomice, which has to be formulated with more caution than the one used for the construction of a $\square_\kappa$-sequence. Roughly speaking, the condensation lemma asserts that if $\sigma : \bar{M} \to M$ is a Σ_0-preserving map from a sound and solid coherent structure $\bar{M}$ into a protomouse M and $\alpha \stackrel{\mathrm{def}}{=} \mathrm{cr}(\sigma) \geq \omega\varrho^1_{\bar{M}}$ then $\bar{M}$ is a protomouse, and if $\bar{M}$ singularizes α then $\bar{M}$ codes the singularizing W-level for α. This condensation lemma is more subtle than the one mentioned above, and I am confident that it can be also used for construction of $\square(E)$-sequences and thus used for characterization of stationary reflection at inaccessibles in W.

The third problem that had to be solved concerns the fact that the choice of a canonical protomouse is preserved under direct limits that are used in the proof that C_α is closed. In general, if we consider arbitrary direct limits, the preservation might fail. In order to guarantee that the direct limit protomouse is the canonical one, we have to impose an additional restriction on the embeddings used in the main construcion on $\mathcal{S}^1$. Surprisingly, this can be done by stipulating that the largest "satisfiable" ordinal α^* is in the range of any such embedding that is used in the construction of C_α, an idea used by Jensen in his original construction of a global square sequence in $\mathbf{L}$, but for a **completely different** purpose. Here "satisfiable" means that $H_{\alpha^*} \cap \alpha = \alpha^*$ where H_{α^*} is the Σ_1-Skolem hull of $\alpha^* \cup \{p_{M_\alpha}\}$ over M_α and p_{M_α} is the standard parameter of M_α.

Σ^1_2 Sets and Weak Capturing

Stuart Zoble

Consider the following two properties of a set of reals $A \subset \omega^\omega$ at some infinite cardinal κ.

(1) For every continuous $f : \kappa^\omega \to \omega^\omega$ there is a dense set of $p \in \kappa^{<\omega}$ such that either $f^{-1}(A)$ is meager or comeager below p.
(2) For every continuous $f : \kappa^\omega \to \omega^\omega$ there is a dense set of $p \in \kappa^{<\omega}$ such that either $f^{-1}(A) \cap \sigma^\omega$ is meager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$ or comeager below p in σ^ω for a club of $\sigma \in [\kappa]^\omega$.

The first asserts that A is κ-Universally Baire and the second that A is weakly captured at κ. The following is a reformulation of property (2) involving forcing terms.

> There is a $Col(\omega, \kappa)$-term $\dot{A}$ such that for sufficiently large θ, for a club of countable $H \prec H(\theta)$, and for a comeager set of $g : \omega \to otp(H \cap \kappa)$,
>
> $$\pi_H(\dot{A})_g = A \cap H[g],$$
>
> where $otp(H \cap \kappa)$ is the order type of $H \cap \kappa$ and π_H is the transitivization map.

If the phrase "comeager set of g" is replaced by "all H-generic g" then an equivalent version of (1) is obtained (see [6]). We use Γ^{UB}_κ to denote the pointclass of sets satisfying property (1) and Γ^{WC}_κ for property (2). If a set $A \in \Gamma^{WC}_\kappa$ and both A and its complement have semiscales whose norms belong to Γ^{WC}_κ then $A \in \Gamma^{UB}_\kappa$ (see [6]). It is also shown in [6] that $\Gamma^{WC}_{\omega_1} = \Gamma^{WC}_\kappa$ under $WRP_{(2)}(\kappa)$. Thus under MM, self-justifying systems which are ω_1-Universally Baire are Universally Baire. On the other hand, it is shown in [4] that any set of reals of size ω_1 belongs to $\Gamma^{UB}_{\omega_1} \setminus \Gamma^{UB}_{\omega_2}$ under MM. Thus $\Gamma^{WC}_{\omega_2} \setminus \Gamma^{UB}_{\omega_2} \neq \emptyset$ under MM. It is also shown in [4] that $\Gamma^{UB}_{\omega_1} = \Gamma^{UB}_\infty$ if ω_2 is generically supercompact. This paper was motivated by a desire to find a definable set of reals which is weakly captured at some uncountable cardinal but not fully captured at that cardinal. The scenario suggested by [6] would involve arranging that ω_2 has some generic weak compactness in a minimal model for $\Sigma^1_2 \subseteq \Gamma^{UB}_{\omega_1}$. Then it could be argued that $\Sigma^1_2 \subseteq \Gamma^{WC}_{\omega_2}$ by the result of [6] but that Σ^1_2 sets are not ω_2-Universally Baire on consistency strength grounds. The relevant global result is due to Feng, Magidor, and Woodin (see [1]) where it is shown that Σ^1_2 sets being Universally Baire is equivalent to assertion that every set has a sharp which is in turn equivalent to two-step Σ^1_3 generic absoluteness. The proof of this theorem is not local, and in fact only shows that $\Sigma^1_2 \subset \Gamma^{UB}_{\omega_{\omega+1}}$ implies $0^\#$. Using the full strength of covering for L we can reduce this to ω_3. The following is a convenient reformulation of $\Sigma^1_2 \subseteq \Gamma^{UB}_\kappa$ for consistency strength arguments. It follows from arguments in [1].

Theorem 1. *The following are equivalent for a cardinal κ.*

(1) Σ^1_2 *sets are κ-Universally Baire*
(2) *For all sufficiently large θ, there is a club of countable $X \prec H(\theta)$ such that $X[g]$ is Σ^1_2 elementary in V for every X-generic $g \subset Col(\omega, \kappa \cap X)$.*

Theorem 2. *If Σ^1_2 sets are ω_3-Universally Baire then $0^\#$ exists.*

The desired minimal model for Σ^1_2 sets ω_1-Universally Baire is given by the following theorem which is joint with Woodin whose idea it was to force over L using a fragment of $0^\#$.

Theorem 3. *The following are equiconsistent.*

(1) Σ^1_2 *sets are ω_2-cc-Universally Baire*
(2) *There are ordinals $\kappa < \lambda$ such that λ is weakly compact in L and for every $\alpha < \lambda$ there is an elementary $j : L_\alpha \to L_\beta$ with critical point κ such that $j(\kappa) \geq \alpha$*

Note that condition (2) does not imply that $0^\#$ exists. Ralph Schindler pointed out to the author that (2) is equiconsistent with the existence of a cardinal which is remarkable to a weakly compact. The model for (1) is a forcing extension of L in which CH holds. Thus two-step Σ^1_3 absoluteness holds in this model for iterations that satisfy the $(2^\omega)^+$-chain condition. Further it can be argued that $WRP_{(2)}(\omega_2)$ holds in the model so that $\Sigma^1_2 \subset \Gamma^{WC}_{\omega_2}$. On the other hand, Σ^1_2 sets cannot be ω_2-Universally Baire in this model as the argument of Theorem 1.2 would show that there is a club of $\alpha < \omega_2$ which are regular in L, an impossibility by the nature of the forcing used to obtain the model. Thus the motivating question is answered.

Theorem 4. *It is consistent that Σ^1_2 sets are weakly captured but not fully captured at ω_2.*

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