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Mini-Workshop: Shape Analysis for Eigenvalues

Organised by
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April 8th – April 14th, 2007

Abstract. The main goal of the meeting was to bring together two mathematical communities working on the shape analysis of eigenvalues by quite different methods. On the one hand explicit solutions to optimal shape problems for eigenvalues are searched by means of direct estimations, symmetrizations, rearrangements; on the other hand recent techniques of variational type have been developed to prove the existence of an optimal shape and intensive research is carried out to prove the regularity of the free boundary and to analyze the optimality

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Introduction by the Organisers

The mini workshop *Shape analysis for eigenvalues*, organised by Dorin Bucur (Chambéry), Giuseppe Buttazzo (Pisa) and Antoine Henrot (Nancy) was held April 8th–April 14th, 2007. This meeting was attended by 18 participants.

The question of localizing or optimizing the eigenvalues differential operators has applications in several domains like acoustics, quantum mechanics, visualization, solid or fluid mechanics and bio-mathematics. Those questions have also a specific mathematical interest since they melt geometrical questions with analysis of partial differential equations and provide model problems for general and applied shape optimization problems.

Despite (or because) their false simplicity, several problems are still open, although formulated hundreds year ago (see for instance the recent books by Bucur & Buttazzo *Variational Methods in Shape Optimization Problems*, Birkhäuser-Verlag, 2005 and Henrot *Extremum problems for eigenvalues of elliptic operators*, Birkhäuser, 2006) and give rise to important debates within the international scientific community. Among them, we can cite

- minimization of the eigenvalues of the Dirichlet Laplacian
- maximization of the eigenvalues of the Neumann Laplacian
- estimates of the fundamental gap for the Laplace or Schrödinger operators
- the hot spot or the nodal line conjecture
- minimization of the buckling load

Each of these problems is associated to one or several constraints for the geometries, which may be local or not local (convexity, connectedness).

One of the goals of the proposed mini-workshop was to bring together two mathematical communities working on the topics above by quite different methods. On the one hand explicit solutions to optimal shape problems for eigenvalues are searched by means of direct estimations, symmetrizations, rearrangements; on the other hand recent techniques of variational type have been developed to prove the existence of an optimal shape and intensive research is carried out to prove the regularity of the free boundary and to analyze the optimality conditions.

A second main issue was to present the state of the art on some famous open problems and conjectures, and try to make a step forward in the direction of solving them.

The organization of the mini-workshop was the following: mornings were devoted to lectures by participants while each afternoon began by an open problems session and smaller working groups (4-5 persons) were organized to work on these problems.

Mini-Workshop: Shape Analysis for Eigenvalues

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Abstracts

Isoperimetric and Universal Inequalities for the Eigenvalues of the Laplacian and Related Operators

Mark S. Ashbaugh

This talk will survey known isoperimetric and universal inequalities for the eigenvalues of the Laplacian, centering on several of the classical results and more recent developments stemming from them. The discussion will begin with the eigenvalues of the Dirichlet Laplacian on domains in Euclidean space and will branch out from there to include other spaces and other eigenvalue problems, such as those for the buckling and vibration of a clamped plate. Problems concerning the asymptotics of eigenvalues and various open problems may also be discussed.

Some references appear below. In general, [1], [5], [6], [8], [9], [10], [12], and [13] are general references and survey articles (especially recommended are the books [6], [8]), while [2], [7], [11], and [14] concern universal inequalities for eigenvalues and [3] and [4] concern isoperimetric inequalities for eigenvalues (especially eigenvalue ratios). This list is far from complete, but should help serve to give entry to the topics discussed in the talk. Further references, to such things as the Faber-Krahn inequality, can be found in the books and survey articles referred to below.

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Optimization problems for functions of eigenvalues

Giuseppe Buttazzo

We consider shape optimization problems of the form

$$\min\Big\{F(A)\ :\ A\subset\Omega,\ |A|\le m\Big\}$$

where F is a given mapping and $m>0$ is fixed. We deal with the problem of the existence of an optimal solution in the class of quasi-open sets (see [1, 5]).

Theorem. *Assume that:*

- *F is lower semicontinuous for the γ-convergence on quasi-open sets;*
- *F is nonincreasing with respect to the set inclusion.*

Then there exists a solution to the minimization problem above.

When F does not satisfy the monotonicity assumption above in general the existence of a domain solution may fail and only relaxed solutions exist (see [1, 4]), that are nonnegative Borel measures, possibly $+\infty$ valued, that vanish on all sets of capacity zero. We denote by $\mathcal{M}_0$ the class of such measures.

An interesting case occurs when

$$F(A)=\Phi(\lambda_1(A),\lambda_2(A),\dots)$$

where Φ is a given continuous function and $\lambda_k(A)$ are the Dirichlet eigenvalues of the Laplacian on A. If Φ is nondecreasing in each variable, due to the natural monotonicity of eigenvalues with respect to the domain, we are in the framework of the theorem above and a domain solution exists; otherwise in general the optimum is only a measure $\mu\in\mathcal{M}_0$.

The case $\Phi(\lambda_1,\lambda_2)$ where the cost depends only on the first two eigenvalues is very particular (see [2]); in fact in this case the existence of an optimal domain occurs for any continuous function Φ independently of the monotonicity assumption.

Similarly, we consider the problem of finding an optimal partition of Ω

$$\min\big\{F(A_1,\dots,A_k)\ :\ A_i\subset\Omega,\ A_i\cap A_j=\emptyset\text{ for }i\ne j\big\}.$$

Again, when F is nonincreasing in each variable, an optimal partition made by quasi-open domains exists. The case of eigenvalues

$$F(A_1,\dots,A_k)=\Phi(\lambda_{i_1}(A_1),\dots,\lambda_{i_k}(A_k))$$

falls in the framework above, for any choice of indices $i_1, \dots, i_k$.

A number of results and of open questions will be presented.

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One-dimensionality in some vector–valued elliptic problems

Friedemann Brock

(joint work with Raul Manasevich)

Let Ω be a bounded domain in R^N and $n \in N$. For any weakly differentiable vector-valued function $u = (u_1, \dots, u_n)^T$ let ∇u denote the gradient of u and $|\nabla u|$ its Euclidean norm. Consider the eigenvalue problem for the vector-valued p-Laplacian operator, $(p > 1)$,

$$(\mathbf{E}) \quad -\Delta_p u \equiv \text{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega.$$

The ODE case $N = 1$ has been extensively studied - also subject to other boundary conditions - in [3] and [4]. It turned out that the components of the eigenvectors are merely eigenfunctions of an associated scalar problem.

Our main result in [2] is:

Theorem 1: *Let $n \geq 2$. Then the first eigenvalue λ_1 of problem $(\mathbf{E})$ is equal to the first eigenvalue of the corresponding scalar problem. Moreover, if u is an eigenfunction for λ_1, then any component of u is an eigenfunction of the corresponding scalar problem.*

The proof of Theorem 1 relies on some convex functional inequalities.

In recent years also some homogeneous anisotropic operators which are related to the p-Laplacian have received some attention (see [1]). These operators permit natural generalizations to the vector-valued case, and we have shown some results similar to Theorem 1 for these operators.

Our research on this subject is still ongoing. Recently we found that our method can also be applied to positive - non-minimizing - solutions of some vector–valued problems. Below we give an example which is unpublished.

Theorem 2: *Let* $F \in C^1((0,+\infty)) \cap C([0,+\infty))$, *with* $F(0) = 0$, *and let* $u \in C^1(\overline{\Omega}, R^n) \cap W_0^{1,p}(\Omega, R^n)$ *be a critical point with* $u_i \geq 0$, $(i = 1, \ldots, n)$, *of*

$$H_n(v) := \int_\Omega ((1/p)|\nabla v|^p - F(|v|))\, dx, \quad v \in W_0^{1,p}(\Omega, R^n).$$

Then $u_i = t_i U$, *with* $t_i \geq 0$, $(i = 1, \ldots, n)$, *where* U *is a nonnegative critical point of*

$$H_1(V) := \int_\Omega ((1/p)|\nabla V|^p - F(|V|)\, dx, \quad V \in W_0^{1,p}(\Omega).$$

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A variational approach to shape optimization of eigenvalues

Dorin Bucur

A tentative way for dealing with isoperimetric problems for eigenvalues is to use the direct methods of the calculus of variations (see [2]):

- prove the existence of a solution (which should be an open set)
- investigate its regularity (i.e. smoothness of the boundary)
- write the optimality conditions (get an overdetermined problem and extra information on the solution)

In this talk, we discussed the global existence question and referred to the third eigenvalue of the Dirichlet Laplacian (see [3]). This result states the existence of a quasi-open set which minimizes the third eigenvalue of the Dirichlet Laplacian among all (quasi)-open sets of prescribed measure of R^N. Moreover, if *bounded* quasi-open sets minimize $\lambda_3, ..., \lambda_k$ among sets of prescribed measure, then a minimizer also exists for λ_{k+1}. The main tool for proving global existence results for shape functionals which are not of energy type is related to a concentration-compactness result for the resolvent operators (see [1]).

Some recent work on the eigenvalues of the Neumann Laplacian was also reported (see [4]).

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About regularity of optimal eigenfunctions for the Dirichlet-Laplacian operator

Michel Pierre

Given D a bounded open subset of $\mathbb{R}^d$ and $m \in (0, |D|)$ (where $|D|$ denotes the Lebesgue measure of D), there exists a quasi-open set $\hat{\Omega}$ solution of the following minimization problem

$$\hat{\Omega} \subset D,\ |\hat{\Omega}| = m,\ \lambda_k(\hat{\Omega}) = \min\{\lambda_k(\Omega); \Omega \subset D \text{ measurable},\ |\Omega| = m\},$$

where $\lambda_k(\Omega)$ denotes the k-th eigenvalue of the Laplacian operator on Ω with homogeneous Dirichlet boundary conditions (see [3] for a proof).

The question which is discussed is the regularity of $\hat{\Omega}$ and of the associated eigenfunctions.

For $k = 1$, one knows that, if there is 'enough room' in D so that it contains a ball of volume m, then this ball is the unique optimal shape (see e.g. in [4]). In other cases, the following is true:
(i) all eigenfunctions are locally Lipschitz continuous in D. As a consequence, there exists at least one optimal set $\hat{\Omega}$ which is *open*. They are all open if D is connected (see [2]).
(ii) If D is not connected, optimal sets are not necessarily regular as shown by easy examples. But, if D is connected, all $\hat{\Omega}$ has finite perimeter and:
- the reduced boundary $\partial^*\hat{\Omega}$ is a regular manifold
- $H^{d-1}(\partial\hat{\Omega} \setminus \partial^*\hat{\Omega}) = 0$. If $d = 2$, the boundary $\partial\hat{\Omega}$ itself is regular (see [1]).

For $k = 2$, the situation is not so clear. If there is 'enough room' in D for two disjoint balls of volume $m/2$ each, then it is known that their union is the unique optimal shape (see e.g. [4]). In other cases, the following is known:

(i) if D is not connected, optimal sets and eigenfunctions may be irregular as seen by easy examples. However, if D is connected, $\hat{\Omega}$ is either (quasi-)connected or the union of two (quasi-)connected quasi-open sets Ω_1, Ω_2 with capacity$(\Omega_1 \cap \Omega_2) = 0$ and the following regularity holds [5]:

(ii) if $\hat{\Omega}$ is (quasi-)connected, then the corresponding eigenfunction is locally Lipschitz continuous in D

(iii) it is also the case if $capacity(\widetilde{\Omega_1} \cap \widetilde{\Omega_2}) = 0$ where $\widetilde{\Omega_i}$ denotes the fine closure of Ω_i
(iv) it remains to understand the other cases.

Among other open problems:

- for $k = 1$: how much can one improve the estimate on the size of the singular part of the boundary of $\hat{\Omega}$?
- for $k = 2$: can one prove symmetry of $\hat{\Omega}$ when, say, $d = 2$ and D is a rectangle?
- for $k \geq 3$, can one prove (or disprove) the (Lipschitz-)continuity of the eigenfunctions (assuming D is connected)?

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Optimization of eigenmodes with respect to the domain

Édouard Oudet
(joint work with Antoine Henrot)

1.1. **Historical background.** The first result in that area has been obtained by Faber and Krahn which proved that the ball minimizes the first eigenmode of the laplacian among sets of constant volume (see [2] and [5]) . Quite in the same time, Polya and Szegö [7] established that the union of two disconnected balls of the same volume minimizes the second eigenmode. More recently, Buttazzo and Dal Maso obtained in [1] a general existance result for this kind of shape optimization problem.

In 1973, Troesch gave in [8] numerical values of the second eigenmode of the laplacian for some convex shapes in dimension 2 and raised the question :

Does the stadium minimizes the the second eigenmode of the laplacian among convex sets of given volume ?

This question is the starting point of our study. We first address some theoretical results directly related to this question. Then, we study the general problem of minimizing numerically one eigenmode of the laplacian among sets of constant volume.

1.2. **Some theoretical results related to the problem of Troesch ([3], [4]).**
• Regularity of an optimal shape Ω^* :
Proposition 1 Ω^* is at least C^1.
Proposition 2 Ω^* can not be of regularity $C^{2,\alpha}$ with $\alpha > 0$.

In the following, we assume that Ω^* of class $C^{1,1}$.

• Simplicity of $\lambda_2(\Omega^*)$:
Proposition 3 $\lambda_2(\Omega^*) \neq \lambda_3(\Omega^*)$.

• Geometry properties of the boundary of Ω^* :
Proposition 4 $\partial\Omega^*$ contains exactly two flat parts.

• The answer to Troesch's question :
Proposition 5 The stadium does not minimize λ_2 among convex sets of given volume.

1.3. **Numerical optimization of the eigenmodes of the Laplace operator [6].** In this part we present new techniques enabling to approximate numerically the solutions of the following problems :

$$\min\{\lambda_2(\Omega), \Omega \subset \mathbb{R}^2,\ \Omega \text{ convex}, |\Omega| = 1\} \tag{1}$$

$$\min\{\lambda_k(\Omega), \Omega \subset \mathbb{R}^2,\ |\Omega| = 1\} \text{ for } k \geq 3. \tag{2}$$

The method we are presenting combines two approaches that were generated in the last twenty years, respectively the homogeneization method and the level set method. We give a short description of the three main numerical methods in shape optimization, namely the boundary variation, the homogeneization and the level set methods. For each of them we shall underline the drawbacks when applying those techniques to minimize the eigenmodes of the Laplace operator. In consequence we shall develope a new process.

In conclusion we shall report numerical results. On one side we improve the values published in [9] and on the other side we propose below a geometrical description of the ten first optimal sets for the problem (2).

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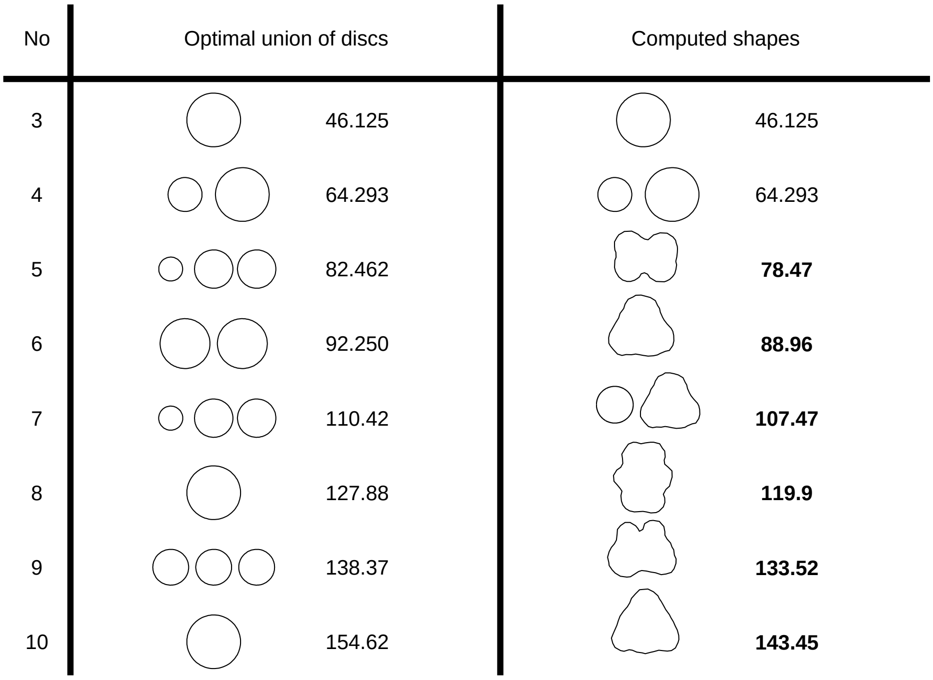

No	Optimal union of discs	Computed shapes
3	46.125	46.125
4	64.293	64.293
5	82.462	**78.47**
6	92.250	**88.96**
7	110.42	**107.47**
8	127.88	**119.9**
9	138.37	**133.52**
10	154.62	**143.45**

FIGURE 1. Best-known shapes

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The Faber-Krahn inequality for Robin problems

DANIEL DANERS

The talk is based on [4] and joint work with James Kennedy [5] resolving an old conjecture explicitly stated in [3, 7], but going back much further, with a weaker result in [8]. The aim is to prove a Faber-Krahn inequality for the Laplacian with Robin rather than Dirichlet boundary conditions. This means we replace the fixed membrane by an elastically supported membrane and prove that amongst all membranes of the same measure, the disc has the lowest ground frequency. More precisely, we consider the first eigenvalue $\lambda_1(\Omega)$ of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is bounded, ν the outward pointing unit normal and $\beta > 0$ constant. We compare it to the first eigenvalue of the corresponding problem on a ball B of the same measure. The problem has a simple first eigenvalue $\lambda_1(\Omega) > 0$ with eigenfunction $\psi > 0$ normalised by $\|\psi\|_\infty = 1$. Building on ideas from Bossel [1], treating the problem for $N = 2$, we establish the following theorem.

Theorem 1 *Suppose $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded Lipschitz domain and B is a ball with the same measure as Ω. Then $\lambda_1(B) \leq \lambda_1(\Omega)$. Moreover, if Ω is a C^2-domain and $\lambda_1(B) = \lambda_1(\Omega)$, then Ω is a ball.* The proof avoids symmetrisation methods. It makes use of the functional

$$H_\Omega(\varphi, U_t) := \frac{1}{|U_t|}\Big(\int_{S_t} \varphi \, d\sigma + \int_{\Gamma_t} \beta \, d\sigma - \int_{U_t} \varphi^2 \, dx\Big),$$

where $U_t := \{x \in \Omega : \psi(x) > t\}$, $|U_t|$ its Lebesgue measure, $S_t := \{x \in \Omega : \psi(x) = t\}$ and $\Gamma_t := \partial\Omega \cap \partial U_t$. It is defined for $\varphi \in C(\Omega)$ and $m := \min_{x\in\bar{\Omega}} \psi(x) < t < 1$. The key to prove Theorem is the following theorem valid for C^2-domains.

Theorem 2 *Let $\varphi \in C(\bar{\Omega})$ with $0 \leq \varphi \leq \beta$. If $\varphi \neq |\nabla\psi|/\psi$, then there exists a set $S \subset (m, 1)$ of positive measure such that $H_\Omega(\varphi, U_t) < \lambda_1(\Omega)$ for all $t \in S$. Moreover, $H_\Omega(|\nabla\psi|/\psi, U_t) = \lambda_1(\Omega)$ for almost all $t \in (m, 1)$.* We construct $\varphi \in C(\Omega)$ by rearrangement of $\varphi^* := |\nabla\psi^*|/\psi^*$, where ψ^* is the first eigenfunction on B. The isoperimetric inequality yields $H_B(\varphi^*, B_{r(t)}) \leq H_\Omega(\varphi, U_t)$ for all $t \in (m, 1)$, where $B_{r(t)}$ is a ball of the same measure as U_t. Then Theorem follows from Theorem and a uniqueness property in the isoperimetric inequality. For Lipschitz domains approximation results from [2, 6] are used.

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Recent progress on an optimal shape problem for the first eigenvalue of the buckling plate

Alfred Wagner

An old conjecture of Polya states that among all planar domain the disc minimizes the first buckling eigenvalue for the plate. This eigenvalue is defined as

$$\Lambda(\Omega) = \min\{\frac{\int\limits_{\Omega} |\Delta u(x)|^2\, dx}{\int\limits_{\Omega} |\nabla u(x)|^2\, dx} \ :\ u \in H_0^{2,2}(\Omega)\}.$$

Thus the conjecture states, that there exists a domain $\Omega^* \subset \mathbb{R}^2$ such that

$$\Lambda(\Omega^*) = \min\{\Lambda(\Omega) : \Omega \subset \mathbb{R}^2, |\Omega| = 1\}$$

and that Ω^* is a disc of area 1. Weinberger and Willms proved this conjecture under the following assumptions:

- there exists an optimal set Ω^* which is smooth;
- Ω^* is connected and simply connected.

Their proof can be found in [5] (e.g.). In [4] the authors proved the existence of an optimal domain in the class of simply connected domains.

In this contribution we reformulate the problem as a free boundary problem, which involves a penalization term for the control of the measure of the support of the admissible functions. More precisely we consider the functional

$$J_\epsilon(u) = \frac{\int\limits_{B} |\Delta u(x)|^2\, dx}{\int\limits_{B} |\nabla u(x)|^2\, dx} + f_\epsilon(\Omega(u)) \qquad \text{for} \quad u \in H_0^{2,2}(B),$$

where

$$f_\epsilon(\Omega(u)) = \frac{1}{\epsilon}\left(|\Omega(u)| - 1\right) \qquad \text{if} \quad |\Omega(u)| \geq 1$$

and zero otherwise. $\Omega(u)$ denotes the support of u. The set $\partial\Omega(u)$ is called free boundary. This formulation is very much in the spirit of the work of H. Alt, L. Caffarelli and their coauthors (see e.g. [1] - [3]). The aim of this talk is to present a proof for the optimal regularity of u.

Strategy:

- Prove the existence of a minimizer u in $X := \{u : \|u\|_{C^{1,\alpha}(B)} \leq K\}$;
- Prove that $\Delta^2 u + \Lambda\Delta u \leq 0$ in $\{u \geq 0\}$ and $\Delta^2 u + \Lambda\Delta u \geq 0$ in $\{u \leq 0\}$ in the sense of distributions;
- Prove $C^{1,\alpha}$ - regularity for any minimizer u;
- Prove $\Delta u \in L^\infty_{loc}(B)$;
- Prove $u \in C^{1,1}_{loc}(B)$.

There are various implications for the regularity of the free boundary.

- Density estimates from above an below which are uniform in free boundary points;
- Rectifiability of the free boundary.

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On Anti-Eigenvalues for Elliptic Systems and a Question of McKenna and Walter

Bernd Kawohl
(joint work with Guido Sweers)

Imagine a simply supported horizontal beam in an elastic ambient medium. Under an upward load $f \geq 0$ it will bend upwards but be pushed down by a restoring force proportional to its deformation u, so $u^{(4)} = f - bu$ in $(=, 1)$, say, with $u = u'' = 0$ at the boundary points 0 and 1. For $b = 0$ the solution will be concave and as b increases it may loose first concavity and later even positivity. This b will be called critical. One can also study this problem for negative b and finds out that at $b = -\lambda_1^2$, where λ_1 denotes the first eigenvalue of the Dirichlet Laplacian, the **p**ositivity **p**reserving **p**roperty (ppp for short) fails again.

If we extrapolate the problem to higher dimensions, we expect the equation $(\Delta)^2 u = f(x) - bu(x)$ in a bounded and connected domain Ω under $u = \delta u = 0$ as boundary conditions to satisfy a ppp as long as $b \in (-\lambda_1^2(\Omega), b_c(\Omega)]$, where b_c depends on Ω. Notice that one can reduce the fourth order equation to an elliptic system (A) given by $-\Delta u = f - bv$ and $-\Delta v = u$ in Ω with $u = v = 0$ on $\partial\Omega$. McKenna and Walter conjectured that among all domains of given volume, the shape function $b_c(\Omega)$ attains its maximum for the ball Ω^*, in other words $b_c(\Omega) \leq b_c(\Omega^*)$.

Guido Sweers and I were able to disprove this conjecture in [1] by relating the problem to the elliptic system (B) $-\Delta u = f - \lambda v$ and $-\Delta v = f$ in Ω with $u = v = 0$ on $\partial\Omega$, in which $u \geq 0$ for $f \geq 0$, provided $\lambda \leq \lambda_c(\Omega)$. In fact, $\lambda_c(\Omega)^2 \leq b_c(\Omega)$ and $\lambda_c(\Omega)$ can go to infinity when Ω has the shape of an amoebae with thin tentacles.

We also investigated the conjecture if the maximum of $\lambda_c(\Omega)$ among plane convex domains is attained for the disk. The answer is again negative. For the proof we had to investigate the ratio of an iterated Greens function $G_2(x, y) = \int_\Omega G(x, z)G(z, y)\, dz$ over $G(x, y)$. The L^∞ norm of this ratio depends on Ω and has a probabilistic interpretation. Its inverse equals $\lambda_c(\Omega)$. For a particular domain S we were able to estimate it in [2] and show that $\lambda_c(S) > \lambda_c(S^*)$, which disproves this conjecture as well. Estimating the supremum of G_2/G is extremely delicate because G vanishes at the boundary. Therefore only a short outline of the estimate was published in [2], while the actual proof requires some 85 pages and can be downloaded from www.mi.uni-koeln.de/˜kawohl.

The question as to which convex domains of given volume maximize $b_c(\Omega)$ and $\lambda_c(\Omega)$ remains open. Numerical experiments suggest that $b_c(\Omega)$ might become maximal for a regular pentagon.

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A fourth order Steklov eigenvalue problem

Filippo Gazzola

(joint work with Alberto Ferrero, Tobias Weth)

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^2$, let $d \in \mathbf{R}$ and consider the boundary eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Delta u = d u_\nu & \text{on } \partial\Omega\ , \end{cases} \tag{1}$$

where u_ν denotes the outer normal derivative of u on $\partial\Omega$. I am interested in studying the eigenvalues d of (1). Problems with eigenvalues in the boundary conditions are called Steklov problems from their first appearance in [7]. A solution of (1) is a function $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx = d \int_{\partial\Omega} u_\nu v_\nu \, dS \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega). \tag{2}$$

Taking $v = u$ in (2) shows that all the eigenvalues of (1) are strictly positive. Let

$$d_1 = d_1(\Omega) \ := \ \min \left\{ u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega); \ \frac{\int_\Omega |\Delta u|^2}{\int_{\partial\Omega} u_\nu^2} \right\} .$$

It represents the least positive eigenvalue and, as pointed out by Kuttler [6], it is the sharp constant for a priori estimates for Laplace equation under nonhomogeneous Dirichlet boundary conditions. This follows from Fichera's principle of duality [4]. Moreover, d_1 also plays a crucial role in the positivity preserving property for the biharmonic operator under Steklov boundary conditions, see [1, 5]. In the talk, I describe the spectrum of (1), study some isoperimetric properties of d_1 and show a generalized version of Fichera's principle, see [2, 3] for the details.

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Open Problems suggested by F. Gazzola concerning his talk.

Concerning the first eigenvalue $d_1(\Omega)$ of the biharmonic Steklov boundary problem, I suggest to following questions:

1) The monotonicity property: is it true that if $\Omega_1 \subset \Omega_2$ then $d_1(\Omega_2) \leq d_1(\Omega_1)$? I believe this is not true.

2) Does there exist an optimal *convex* set of given measure or perimeter which minimizes d_1? I believe the answer is yes.

3) Is it possible to perform some numerical computations and/or to show a kind of Babuska paradox?

Optimimization problem for weighted Sobolev constants

Catherine Bandle
(joint work with Alfred Wagner)

Let $D \subset \mathbf{R^N}$ be a bounded open set, let $a(x), b(x)$ be two positive, Lipschitz continuous weights and consider for $p > 1$ the following *Sobolev constant*

$$S_p(D) = \inf_v \int_D a(x)|\nabla v|^p \, dx, \; v \in \mathcal{K}(D) \text{ where} \tag{1}$$

$$\mathcal{K}(D) = \{w \in W_0^{1,p}(D) : w \geq 0 \text{ a.e.}, \int_D b(x) w \, dx = 1\}.$$

The optimization problem addressed in this talk is:

$$s_p(m) = \inf_D S_p(D) \;\; \text{where } D \subset B(\text{fixed fundamental domain}), \qquad \text{and } \int_D b(x)dx \leq m. \tag{2}$$

The strategy is to solve the unconstrained variational problem

$$\mathcal{J}_{\epsilon,m} = \inf_{\mathcal{K}(B)} \int_B a(x)|\nabla v|^p \, dx + f_\epsilon\Big(\int_{\{v>0\}} b(x)\, dx\Big), \tag{3}$$

where

$$f_\epsilon(s) = \begin{cases} \frac{1}{\epsilon}(s-m) & : \quad s \geq m \\ 0 & : \quad s \leq m. \end{cases}$$

We prove

(1) $\mathcal{J}_{\epsilon,m}$ is attained.
(2) $\mathcal{J}_{\epsilon_1,m} \leq \mathcal{J}_{\epsilon_2,m}$ for $\epsilon_1 \geq \epsilon_2$.
(3) From an argument given in [2] it follows that there exists ϵ_0 such that for all $\epsilon < \epsilon_0$

$$\mathcal{J}_{\epsilon,m} = \mathcal{J}_{\epsilon_0,m} \leq s_p(m).$$

We then study the regularity of the minimizers u_ϵ. It turns out that under an additional assumption on $b(x)$ which is needed for technical reasons and is satisfied for instance for $|x|^q$, $\frac{1}{(1+]x]^q)^s}$ and exponentials, we have

(1) u_ϵ is Hölder continuous,
(2) for $p \geq 2$, the minimizers are Lipschitz continuous.

Hence the following main result cf. [1] holds true:
There exists an optimal domain $D_0 \subset B$ such that $s_p(m) = S_p(D_0)$. If $p \geq 2$ then D_0 is a Lipschitz domain.

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Sharp *dynamic* bounds for eigenvalues of the Laplacian

Pedro Freitas

The purpose of this talk is twofold. On the one hand, we wish to present a line of research for the study of low eigenvalues of the Laplace operator on bounded Euclidean domains, based upon recent numerical results and conjectures [1, 2]. On the other hand, we shall report on some recent results which were obtained in this direction.

We are mainly interested in bounds for eigenvalues giving equality for some domain Ω_0, and containing a correction term which takes into account the deviation from Ω_0. As examples of such results, we present some new bounds for triangles and $n-$dimensional star–shaped domains [3, 4, 5]. From the last of these, it is possible to derive new relations between geometric and spectral properties of a domain, such as the following lower bound for the isoperimetric constant of a convex domain [5]:

Theorem 1 *Let Ω be a bounded convex domain of $\mathbb{R}^n$. Then*

$$\frac{|\partial\Omega|}{|\Omega|^{1-1/d}} \geq \frac{|\partial B_1|}{|B_1|^{1-1/d}} \frac{\pi}{2\sqrt{\lambda_1(B_1)}} \sqrt{\frac{\lambda_1(\Omega)}{\lambda_1(B)}},$$

where $\lambda_1(X)$ denotes the first Dirichlet eigenvalue of the domain X and B and B_1 are the balls of volume $|\Omega|$ and of unit radius, respectively.

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The Best Damped Disk

Steven J. Cox
(joint work with Mark Embree)

We consider the wave equation on the unit disk with radial damping subject to Dirichlet bounday conditions,

$$u_{tt}(r,\theta,t) - \Delta u(r,\theta,t) + 2a(r)u_t(r,\theta,t) = 0$$
$$u(1,\theta,t) = 0, \quad u(r,\theta,0) = u_0(r,\theta), \quad u_t(r,\theta,0) = v_0(r,\theta)$$

It is useful to take $U(t) = [u(t)\ u_t(t)]$ and interpret our pde as $U_t = A(a)U$ where

$$A(a) = \begin{pmatrix} 0 & I \\ \Delta & -2a \end{pmatrix}.$$

From here one goes on to study the eigenvalues and eigenfunctions of $A(a)$. The best damped drum is the a for which $A(a)$ has the least spectral abscissa. If $V = [u\ v]$ is an eigenfunction associated with the eigenvalue λ then $v = \lambda u$ and $\Delta u - 2av = \lambda v$, or

$$r(ru_r(r,\theta))_r + u_{\theta\theta}(r,\theta) - 2\lambda r^2 a(r)u(r,\theta) = \lambda^2 r^2 u(r,\theta)$$

subject to $u(1,\theta) = 0$. We now separate variables by writing $u(r,\theta) = R(r)T(\theta)$. Inserting this into our pde and dividing by RT gives

$$r(rR')'/R + T''/T - 2\lambda r^2 a = \lambda^2 r^2$$

It follows that T''/T is constant, say $-\gamma$, i.e.,

$$T''(\theta) + \gamma T(\theta) = 0, \quad T(0) = T(2\pi) \quad T'(0) = T'(2\pi)$$

and so

$$\gamma_n = n^2 \quad \text{and} \quad T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 0, 1, 2, \ldots$$

and so we have a one parameter family of equations in r,

$$r(rR')' - n^2R - 2\lambda r^2 aR = \lambda^2 r^2 R, \quad R(1) = 0. \tag{1}$$

The central question then is: How do the eigenvalues, λ, of (1) vary with n and a?

Following the methods of Castro and Cox [2] we construct, for each n, a one parameter family of dampings for which the associated spectral abscissa approaches $-\infty$. Regarding the full operator, $A(a)$, we show that our designs are asymptotically optimal (in the sense of Asch and Lebeau [1]) but suffer from small real eigenvalues.

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Geodesics between probability measures and the dimensional distance

Qinglan Xia

In mathematics, there are at least two different but very important types of optimal transportation: Monge-Kantorovich problem and ramified transportation (dealing with tree-type branching structures). In [1], Buttazzo etc. give a very nice approach in attempting to unify these two theories by considering path functionals in Wasserstein spaces. The length of a function $f : [0,1] \to (P(X), W_2)$ is given by $\int J(f(t))|\dot{f}(t)|_{W_2} dt$. After picking suitable functionals J on the space of probability measures, they got both types of transportation. Their approach agrees well with Monge-Kantorovich problem. However, as for ramified transportation, their approach is similar but still a little bit different to optimal transport paths studied in [3]. In this talk, I fill in this gap by considering geodesic problems in the space of probability measures under different (semi-)metrics J. The length of a curve $f : [0,1] \to P(X)$ will be $\int |\dot{f}(t)|_J dt$, where $|\dot{f}(t)|_J$ denotes the (semi-)metric derivative. By choosing suitable (semi-)metrics, we will get both types of transportation. This approach agrees well with both of them. A special kinds of (semi-)metric may be induced from suitable functionals on "transport plans", which are probability measures in product spaces. Under some suitable functionals on "transport plans", the length of optimal transport paths (i.e. geodesics) between any two probability measures will determine a distance between measures. We will mainly consider a special family of functionals determined by a parameter α. For any given two probability measures, its distance will be finite whenever the parameter α is less than a critical value. What interesting is that this critical value itself determines another distance between these two measures. I will call this distance "dimensional distance" because it contains mainly the dimensional information about these measures. It generalizes the "irrigation dimension" studied by [2] to the case "$a < 0$", which corresponds to the self-similar dimension of fractals including cantor sets and others.

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New optimization problems for the second eigenvalue of the Laplacian

Antoine Henrot

(joint work with Dorin Bucur)

Let $\lambda_2(\Omega)$ be the second eigenvalue of the Laplace operator on the domain Ω with Dirichlet boundary conditions. The problem of minimization of $\lambda_2(\Omega)$ with a volume constraint is now well-known (with or without convexity assumption), see e.g. [2] or [1]. In this talk, we investigate two other problems of minimization for $\lambda_2(\Omega)$ in two dimensions:

(1) with a perimeter constraint,
(2) with a diameter constraint.

More precisely, for the first problem, we prove:
Theorem 1 :
There exists an optimal plane domain Ω^* which minimizes $\lambda_2(\Omega)$ among domains of given perimeter. Moreover Ω^* has the following properties:

- it is convex,
- its boundary is C^∞,
- it has (at least) one axis of symmetry,
- its boundary contains neither segment, nor arc of circle.

For the second problem, we prove:
Theorem 2 :
There exists an optimal plane domain Ω^* which minimizes $\lambda_2(\Omega)$ among domains of given diameter. The set Ω^* is convex and is a body of constant width. The disk is a local minimizer.

The conjecture, supported by the last assertion of Theorem 2 and some numerical evidence is that the disk is the global minimizer for that problem. Let us remark that it was quite unexpected for such a problem of minimization of the second eigenvalue.

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Partially overdetermined elliptic problems

Ilaria Fragalà

(joint work with Filippo Gazzola)

We consider an elliptic equation of the kind $-\Delta(u) = f(u)$ on a bounded domain Ω in R^n, and we complement this equation with two boundary data: a homogeneous Dirichlet condition $u = 0$ and a constant Neumann condition $|\nabla u| = c$, which are required to hold either the former on $\partial\Omega$ and the latter on a proper subset $\Gamma \subset \partial\Omega$, or the viceversa. We investigate symmetry of domains Ω where the resulting boundary value problem, that we call "partially overdetermined", admits a solution. The "totally overdetermined" case when $\Gamma \equiv \partial\Omega$ has been studied in the seminal paper [2].

We give some positive symmetry results, which hold under different kind of assumptions on Γ, f, and c; some of these results can be extended also to the case of exterior partially overdetermined problems.

More precisely, we analyze the problem when some further information is available in one of the following aspects:

(I) regularity of Γ;
(II) maximal mean curvature of Γ;
(III) geometry of Γ.

For each of these situations, our approach is completely different. In case (I) we treat partially overdetermined problems as *initial* value problems in the spirit of Cauchy-Kowalewski Theorem; in cases (II) and (III) we take advantage respectively of the P-function and the moving planes methods already existing in literature, adapting them to our framework with some suitable modifications. The only common feature is that, in any of the cases (I), (II), (III), our proof strategy consists in showing that the partially overdetermined problem can be turned into a totally overdetermined one.

It remains essentially open to establish whether symmetry continues to hold under weaker requirements: the problem of finding the minimal assumptions which ensure symmetry deserves further investigation.

In the last part of the talk we discuss the possibility that, without any kind of additional assumptions with respect to the totally overdetermined case, counterexamples to symmetry for partially overdetermined problems can be constructed in the framework of shape optimization. We address two possible shape optimization problems which may lead to a counterexample. One of them concerns the shape optimization for the second Laplace-Dirichlet eigenvalue among all convex planar domains with a given area.

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On Riesz and Carleman Means of Eigenvalues

Lotfi Hermi

(joint work with Evans M. Harrell II)

The results described in this report are based on two preprints [4] [5], both of which are concerned with inequalities for Riesz and Carleman means and consequences thereof. Riesz means are "smoothed" averages of the eigenvalues, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, of the fixed membrane problem

$$\begin{aligned} -\Delta u = \lambda\ u \quad &in \quad \Omega \subset \mathbb{R}^n \\ u = 0 \quad &on \quad \partial\Omega. \end{aligned}$$

They are generalizations of the Weyl counting function for the eigenvalues, $N(\lambda)$. The Riesz mean of order $\sigma > 0$ is defined by

$$R_\sigma(\lambda) = \sum_k (\lambda - \lambda_k)_+^\sigma$$

where $x_+ := \max(0, x)$. In the same vein, the Carleman mean of order (σ, μ) is defined by

$$C_\sigma^\mu(\lambda) = \sum_k \frac{(\lambda - \lambda_k)_+^\mu}{\lambda_k^\sigma}.$$

These means are to be interpreted in the obvious way when σ or $\mu \to 0+$. It has been known since Weyl [9] that, as $\lambda \to \infty$,

$$N(\lambda) \sim L_{0,n}^{cl}\ |\Omega| \lambda^{n/2} \tag{1}$$

and

$$R_\sigma(\lambda) \sim L_{\sigma,n}^{cl}\ |\Omega|\ \lambda^{\sigma+n/2}.$$

Here $L_{\sigma,n}^{cl}$ is the classical constant given by

$$L_{\sigma,n}^{cl} = \frac{\Gamma(1+\sigma)}{(4\pi)^{n/2}\,\Gamma(1+\sigma+n/2)}.$$

When $\sigma \geq 1$, Laptev-Weidl proved [8]

$$R_\sigma(\lambda) \leq L_{\sigma,n}^{cl}\ |\Omega|\ \lambda^{\sigma+n/2}. \tag{2}$$

Note that (1) is equivalent to, when $k \to \infty$,

$$\lambda_k \sim \frac{k^{2/n}}{\left(L^{cl}_{0,n}\,|\Omega|\right)^{2/n}}.$$

The thrust behind many of these inequalities is to prove the Pólya conjecture

$$\lambda_k \geq \frac{k^{2/n}}{\left(L^{cl}_{0,n}\,|\Omega|\right)^{2/n}}.$$

The results described in this report focus on four theorems from [4]. Connections between the universal inequalities of Harrell-Stubbe [6] [3] (see also [2]) and the domain-dependent Berezin-Li-Yau inequality (2) are also made, via a host of integral transforms such as the Laplace, Weyl, and Riemann-Liouville fractional transforms, adding new tools to an already rich class of convexity and Legendre transform methods (see [8], [1], [3] to cite a few examples). The first result is a new monotonicity principle, to wish two independent proofs are produced in [4] and [5].

Theorem 1. The function

$$\lambda \mapsto \frac{R_\sigma(\lambda)}{\lambda^{\sigma+n/2}}$$

is a nondecreasing function of λ*, for* $\lambda \geq \lambda_0$*, for a fixed* $\lambda_0 > 0$ *and* $\sigma \geq 2$*.*

This theorem is central to the proof of the following.

Theorem 2. When $\sigma \geq 2$*, the Berezin-Li-Yau inequality (2) is equivalent to the classical inequality of Kac* [7]*,*

$$Z(t) := \sum_{k=1}^{\infty} e^{-\lambda_k t} \leq \frac{|\Omega|}{(4\pi t)^{n/2}}. \tag{3}$$

Theorem 3. For $0 \leq \sigma < n/2$*,* $\mu > 0$*, let* $M_{n,\sigma} = \dfrac{e^{\frac{n}{2}-\sigma}}{\left(\frac{n}{2}-\sigma\right)^{\frac{n}{2}-\sigma}}$*, then*

$$C^{\mu}_{\sigma}(\lambda) \leq M_{n,\sigma}\,\frac{\Gamma(\frac{n}{2}-\sigma)}{\Gamma(\frac{n}{2})}\,\frac{\Gamma(\mu+1)\Gamma(\frac{n}{2}-\sigma+1)}{\Gamma(\mu+\frac{n}{2}-\sigma+1)}\,\frac{|\Omega|}{(4\pi)^{n/2}}\,\lambda^{\mu+\dfrac{n}{2}-\sigma}. \tag{4}$$

This theorem is in fact a corollary to the Kac's inequality (3). One would hope to prove this result with the sharp constant expected from semiclassical considerations.

Theorem 4. For $\sigma \geq 1$

$$R_\sigma(\lambda) \geq H_n^{-1} \lambda_1^{-n/2} \, \frac{\Gamma(1+\sigma)\Gamma(1+n/2)}{\Gamma(1+\sigma+n/2)} \, (\lambda - \lambda_1)_+^{\sigma+n/2} \, .$$

Here

$$H_n = \frac{2 \; n}{j_{n/2-1,1}^2 J_{n/2}^2(j_{n/2-1,1})},$$

where $J_\alpha(x)$ denotes the Bessel function of order α and $j_{\alpha,p}$ is its $p-th$ zero. Results stronger than Theorem 4 appear in [5], resulting in universal Weyl-type upper bounds for λ_k and $\sum_{j=1}^{k} \lambda_k$ in terms of λ_1.

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The phase–field method in optimal design

Blaise Bourdin

The phase field method is a versatile, robust, and rigorous framework for topology optimization problems. It is based on the penalization of the variation of the properties of designs (i.e. perimeter penalization), and its variational approximation. It uses a smooth function, the phase-field, to represent the materials involved in the device or the system.

Consider the following optimal design problem of finding p materials occupying p disjoint regions $D_1, \ldots, D_p$ of a ground domain Ω, and minimizing the objective function F, under a perimeter constraint:

$$(\mathcal{P}) : \inf_{D_1,\ldots,D_p \text{ admissible}} F(D_1, \ldots, D_p) + \sum_{1 \leq i \leq j \leq p} length(\partial D_i \cap \partial D_j).$$

In the phase field framework, one uses a single differentiable function $\rho = (\rho_1, \dots, \rho_p)$ to represent all the materials. For any $\varepsilon > 0$, one defines the following problem:

$$(\mathcal{P}_\varepsilon): \inf_\rho F_\varepsilon(\rho) + \int_\Omega \frac{1}{\varepsilon} W(\rho) + \varepsilon |\mathrm{D}\rho| \, dx,$$

where W is a p–wells function, such that $W(\rho) = 0$ if one and only one of the components of ρ is equal to 1, and strictly positive otherwise. Then, under some technical conditions on F, one can prove that when $\varepsilon \to 0$, the solutions of problem $(\mathcal{P}_\varepsilon)$ converge in some sense to that of $(\mathcal{P})$. Moreover, since $(\mathcal{P}_\varepsilon)$ is a well-posed problem, whose arguments are classical differential functions, the convergence result suggests a numerical algorithm, that is to solve $(\mathcal{P}_\varepsilon)$ for a "small enough" ε.

This framework has already been applied to several problems in structural optimization, including the stiffness optimization of pressurized structures, as illustrated in the following figure.

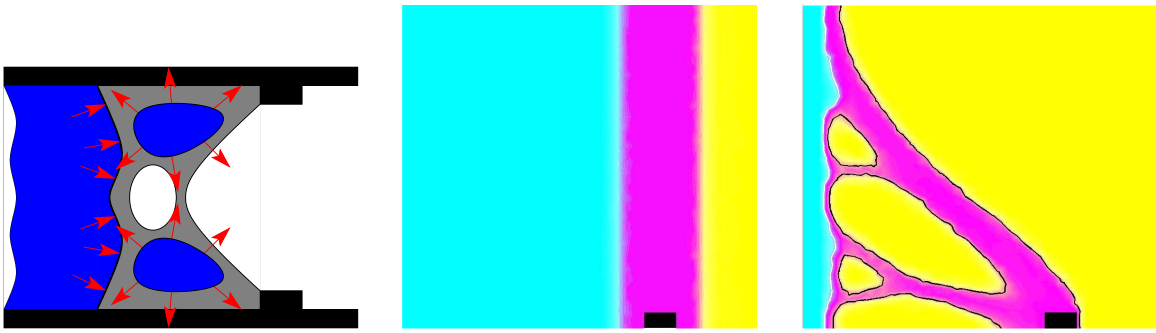

FIGURE 1. Optimal design of pressurized structure. From left to right: schematic of the problem, phase field ρ for the initial design on a half domain (blue corresponds to a liquid under pressure, magenta to some elastic material and yellow to the void), and the final design.

Open problems proposed by Mark Ashbaugh

1. **The fundamental gap problem, or van den Berg's conjecture.** One looks at the gap $\lambda_2 - \lambda_1$ between the first two eigenvalues of the Dirichlet Laplacian $-\Delta$ on a bounded convex domain Ω, or at the same quantity for the Schrödinger operator $-\Delta + V$ where V is a potential defined and convex on Ω. The conjecture is

$$\lambda_2 - \lambda_1 \geq 3\pi^2/d^2$$

where d is the diameter of the domain (*sup* of the distance between any two points of the domain). This result was conjectured by van den Berg [8] in 1983. In 1985

Singer, Wong, Yau, and Yau [12] obtained the lower bound $\pi^2/4d^2$. The best result to date is the lower bound π^2/d^2 of Yu and Zhong [13].

For more information on this problem, see the write-up “The Fundamental Gap” for the AIM workshop, “Low eigenvalues of Laplace and Schrödinger operators,” from May 2006. This can be found at the website http://www.aimath.org /past-workshops/loweigenvalues.html. Further material on the problem can be found at the same site under “open problems”.

2. $\lambda_{k+1}/\lambda_k(\Omega) \leq \lambda_2/\lambda_1|_{n-\mathbf{ball}}$ where Ω is a domain in $\mathbb{R}^n$. This optimal inequality for λ_{k+1}/λ_k was conjectured by Payne, Pólya, and Weinberger (henceforth PPW) in their 1956 paper [11]. If this bound holds, then it is expected that a saturating case would be that of a domain that in the limit approaches k identical disconnected n-balls.

The conjectured bound is known for $k = 1, 2, 3$ (see [3], [4]). For all other cases the best bound known is only $1 + 4/n$, proved in the original 1956 paper of PPW.

3. $\lambda_{2m}/\lambda_m(\Omega) \leq \lambda_2/\lambda_1|_{n-ball}$ where Ω is a domain in $\mathbb{R}^n$. If this bound holds, then it is expected that a saturating case would be that of a domain that in the limit approaches m identical disconnected n-balls.

The conjectured bound is known for $m = 1, 2$ (see [3], [4]). Obviously, the PPW conjecture listed in item 2 above would follow from this conjecture. No weaker bounds of this form that are suggestive of this bound are known. However, there are bounds for λ_k/λ_1 and λ_k/λ_2 that “accumulate” only according to the powers of 2 in k, so suggestive of the λ_{2m}/λ_m conjecture at least covering the worst case, “on average”. Thus, one has [5]

$$\lambda_{2^k}/\lambda_1(\Omega) \leq (\lambda_2/\lambda_1|_{n-ball})^k$$

and a similar bound for λ_{2^k}/λ_2.

4. **The Pólya conjectures.** No list of this kind would be complete without the Pólya conjectures for the Dirichlet and Neumann eigenvalues of the Laplacian for domains in Euclidean space. Here we denote the Dirichlet and Neumann eigenvalues of the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$ by $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=0}^{\infty}$, respectively (note that we have purposely indexed the Dirichlet eigenvalues from 1, and the Neumann eigenvalues from 0). In two dimensions the conjectured bounds read

$$\lambda_k \geq 4\pi k/A \geq \mu_k,$$

where A represents the area of Ω. In n dimensions the general inequalities read

$$\lambda_k \geq 4\pi^2 k^{2/n}/(C_n|\Omega|)^{2/n} \geq \mu_k,$$

where $C_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the ball of unit radius in $\mathbb{R}^n$ and $|\Omega|$ is the volume of Ω. For further discussion of the Pólya conjectures, including references to the literature, see, for example, [6] and/or [5].

Other References. More on these and related open problems can be found in [1], [2], [6], [5], [9], and [8].

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Conjecture concerning a Faber-Krahn inequality for Robin problems

Open problem proposed by Friedemann Brock and Daniel Daners

For $\beta > 0$ it was proved in [2, 3] that amongst all domains Ω of equal measure, the ball minimises the first eigenvalue of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \partial\Omega.$$

If $\beta < 0$ we conjecture that the ball *maximises* the first eigenvalue. The conjecture is supported by partial results in [1].

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On a long-standing conjecture by Pólya-Szegö

Open problem proposed by Ilaria Fragala

The electrostatic capacity of a convex body is usually not simple to compute. We discuss a possible approximations of it, which is related to a long-standing conjecture by Pólya-Szegö. It states that, among all convex bodies, the "worst shape" for the approximation exists and is the planar disk. The first part of this conjecture was proved in [1], where we established some related results which give further evidence for the validity of the second part. We also suggest a complementary conjecture related to some overdetermined boundary value problems.

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