Mathematisches Forschungsinstitut Oberwolfach

Report No. 32/2007

Differentialgeometrie im Grossen

Organised by
Olivier Biquard (Strasbourg)
Bruce Kleiner (New Haven)
Bernhard Leeb (München)
Gang Tian (Princeton)

July 1st – July 7th, 2007

Abstract.
The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Global Riemannian geometry with its connections to topology, geometric group theory and geometric analysis remained an important focus of the conference. Special emphasis was given to the Ricci flow and its applications and to the geometry of metric spaces with curvature conditions.

Mathematics Subject Classification (2000): 53Cxx, 53Dxx, 32Qxx, 57xx, 58Jxx, 83Cxx.

Introduction by the Organisers

The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Traditionally, the conference series covers a wide scope of different aspects of global differential geometry and its connections with topology, geometric group theory and geometric analysis. The Riemannian aspect is emphasized, but the interactions with the developments in complex geometry, symplectic/contact geometry/topology and mathematical physics play also an important role. Within this spectrum each particular conference gives special attention to two or three topics of particular current relevance.

The scientific program consisted of 22 (almost) one hour talks leaving ample time for informal discussions.

This time, a main focus of the workshop were *geometric evolution equations.* 6 talks discussed the Ricci flow and its applications to the geometrization of 3-manifolds, the Ricci flow in arbitrary dimension and applications to Riemannian manifolds of positive curvature, the Kähler-Ricci flow and the (inverse) mean curvature flow.

A second focus was the *geometry of singular spaces* (5 talks), that is, metric spaces with sectional curvature bounds (in the sense of Aleksandrov), Gromov-hyperbolic spaces and Carnot spaces with connections to geometric group theory. One of the talks discussed the theory of collapse with lower curvature bound which is another ingredient (independent of Ricci flow) in the argument for geometrization in dimension 3.

Other talks covered results about *geometric structures on manifolds* (hyperbolic geometry and representation varieties), from *geometric analysis* (Dirac operators, metrics of positive scalar curvature), *symplectic and contact geometry* (open book decompositions, confoliations, construction of special Lagrangian submanifolds with isolated conical singularities), and *complex geometry* (extremal metrics on Kähler manifolds, Kähler-Ricci flow).

There were 52 participants from 8 countries, more specifically, 23 participants from Germany, 12 from France, 6 from other European countries, and 11 from North-America. 6% of the participants were women. More than half of the participants (about 27) were young researchers (less than 10 years after diploma or B.A.), both on doctoral and postdoctoral level.

The organizers would like to thank the institute staff for their great hospitality and support before and during the conference. The financial support of the European Union (in particular for young participants) is gratefully acknowledged.

Workshop: Differentialgeometrie im Grossen

Table of Contents

Abstracts

Dimensional reduction and the long-time behavior of Ricci flow

John Lott

Let M be a compact orientable 3-manifold. Let $(M, g(\cdot))$ denote a Ricci flow with surgery on M, in the sense of Perelman, defined for $t \in [1, \infty)$. In between surgeries, $g(\cdot)$ satisfies the Ricci flow equation $\frac{dg}{dt} = -2Ric$. Put $\widehat{g}(t) = \frac{g(t)}{t}$. Perelman showed that for large t, $(M, \widehat{g}(t))$ decomposes into a nearly hyperbolic piece and a graph manifold piece. Along with the incompressibility of the cuspidal tori, this was enough to prove the geometrization conjecture. However, it is an open question whether by running the Ricci flow, one actually sees the asymptotic appearance of the geometric pieces in Thurston's geometric decomposition.

In the known examples of Ricci flow, there is a finite number of surgeries and after the surgeries are over, the Ricci flow evolves smoothly with sectional curvatures that decay like $O(t^{-1})$. We show that with these assumptions, and a diameter bound, the Ricci flow does become more and more homogeneous as time evolves.

To state the result, we define certain special Ricci solutions on $\mathbf{R}^3$, which are expanding solitons.

1. $(\mathbf{R}^3)$ $g(t) = g_{flat}$.
2. $(\mathbf{H}^3)$ $g(t) = 4tg_{hyp}$.
3. $(\mathbf{H}^2 \times \mathbf{R})$ $g(t) = 2tg_{hyp} + g_{\mathbf{R}}$.
4. $(\mathbf{Sol})$ $g(t) = e^{-2z}dx^2 + e^{2z}dy^2 + 4tdz^2$.
5. $(\mathbf{Nil})$ $g(t) = \frac{1}{3t^{1/3}}(dx + \frac{1}{2}ydz - \frac{1}{2}zdy)^2 + t^{1/3}(dy^2 + dz^2)$.

Given a Ricci flow solution $g(\cdot)$ and a parameter $s > 0$, define a rescaled Ricci flow solution $g_s(\cdot)$ by $g_s(t) = \frac{1}{s}g(st)$.

Theorem 1. *Let $(M^3, g(\cdot))$ be a smooth Ricci flow solution on a compact orientable 3-manifold M, defined for all $t \in [1, \infty)$. Suppose that $|Rm(g(t))|_\infty = O(t^{-1})$ and $diam(M, g(t)) = O(t^{1/2})$. Then for any sequence s_j tending toward infinity, $\lim_{j\to\infty} \widetilde{g}_{s_j}(\cdot)$ exists on the universal cover $\widetilde{M}$ and is one of the above expanding soliton solutions, provided that the Thurston type of M is $\mathbf{R}^3$, $\mathbf{H}^3$, $\mathbf{H}^2 \times \mathbf{R}$, Sol or Nil. If the Thurston type of M is $\widetilde{SL_2}(\mathbf{R})$ then there is some sequence s_j tending toward infinity so that $\lim_{j\to\infty} \widetilde{g}_{s_j}(\cdot)$ is the $\mathbf{H}^2 \times \mathbf{R}$ expanding soliton solution.*

When defining $\lim_{j\to\infty} \widetilde{g}_{s_j}(\cdot)$, we allow s-dependent pointed diffeomorphisms to be performed. The role of the diameter assumption $diam(M, g(t)) = O(t^{1/2})$ is to ensure that there is only one piece in the Thurston decomposition of M.

Ricci flow, simplicial volume and aspherical 3-manifolds

Sylvain Maillot

(joint work with Laurent Bessières, Gérard Besson, Michel Boileau, Joan Porti)

G. Perelman [10, 11, 12] recently proved W. Thurston's geometrization conjecture. Various authors [7, 8, 9, 6] have given more detailed expositions of some of Perelman's arguments or alternative arguments for various parts of the proof. My talk was based on the preprint [2], which is to be included as the last part of a forthcoming book [3]. The purpose was to outline a proof of the geometrization conjecture in the aspherical case assuming a black box containing most of Perelman's results on the Ricci flow.

Modulo known results, the geometrization conjecture can be stated as follows: if M is a closed, orientable, irreducible smooth 3-manifold, then M is hyperbolic, Seifert-fibered, or contains an incompressible torus. The bulk of the proof consists in showing that for any riemannian metric g_0 on M, one can construct a 1-parameter family $\{g(t)\}_{t\in I}$ of riemannian metrics on M which satisfies R. Hamilton's Ricci flow equation $\frac{dg}{dt} = -2\mathrm{Ric}_{g(t)}$ in a weak sense, and such that $g(0) = g_0$. We shall call those 1-parameter families *surgical solutions.*

When M is aspherical, one shows that for every initial condition there is a surgical solution defined on $I = [0; +\infty)$ satisfying various geometric conditions. In particular, when one picks any sequence of times t_n going to infinity, and defines a sequence g_n of riemannian metrics on M by putting $g_n := (4t_n)^{-1}g(t_n)$ for $n \geq 1$, then the sequence $\{g_n\}_{n\geq 0}$ (where g_0 is the initial condition we started with) has four properties, which can be loosely described as follows : the volume-rescaled minimum of the scalar curvature $\hat{R} := (\min R)\mathrm{vol}^{2/3}$ is nonpositive and nondecreasing; the volume is uniformly bounded; if one can find a thick sequence of basepoints x_n in (M, g_n), then the pointed sequence (M, g_n, x_n) subconverges to a pointed hyperbolic manifold of finite volume; lastly, the curvature is locally controlled (for precise definitions, see [2].)

In the talk I explained a way to deduce the conclusion of the geometrization conjecture from this information. This part of the proof does not require any knowledge of Perelman's results on the Ricci flow, and can be summarized as Theorem 1 below. Let $V_0(M)$ denote the minimal volume of a hyperbolic link complement in M. Such links are known to exist by a theorem of Myers, and since the set of volumes of hyperbolic 3-manifolds is well-ordered, the minimum is attained.

Theorem 1. *Let M be a closed, orientable, irreducible, aspherical 3-manifold. Assume that M carries a sequence of riemannian metrics g_n such that:*

(1) $\liminf \hat{R}(g_n) > -6V_0(M)^{2/3}$;
(2) $\sup \mathrm{vol}(g_n) < \infty$;
(3) *For every thick sequence of basepoints there is a hyperbolic pointed limit;*
(4) *The curvature is locally controlled.*

Then M is Seifert-fibered or contains an incompressible torus.

To deduce the geometrization of aspherical 3-manifolds from Theorem 1, we argue as follows: if M is not hyperbolic, let H_0 be a minimal volume hyperbolic link complement. Following [1], put a metric on M with $\hat{R}(g_\epsilon) > \hat{R}(H_0)$. Then by (a slight variant of) Perelman's construction produce a surgical solution to the Ricci flow equation with initial metric $g_0 = g_\epsilon$. As explained above, extract a sequence of metrics g_n from the rescaled solution. By monotonicity of $\hat{R}$, this sequence will satisfy the hypotheses of Theorem 1, which gives the required conclusion.

To prove Theorem 1, we first cover the thick part by (compact cores of) hyperbolic manifolds. The two main issues are: (1) the question whether the tori arising as cusp cross-sections of those hyperbolic manifolds are incompressible or compressible in M; (2) to recognize the topology of the thin part. In particular, if the manifold (M, g_n) becomes thinner and thinner, hypothesis (3) of Theorem 1 is vacuous, and one needs to show that M is a graph manifold. This is sometimes referred to as the *collapsing case*.

To deal with the incompressibility of tori, we use hypothesis (1) connecting $\hat{R}(g_n)$ and $V_0(M)$. This approach stems from [11, Section 8], where Perelman's invariant $\hat{\lambda}$ is used instead of $\hat{R}$. The idea of replacing $\hat{\lambda}$ by $\hat{R}$, which is technically simpler to work with, is apparently due to M. Anderson. Two versions of the argument, somewhat different to ours, can be found in [8, Section 93][1].

To study the topology of the thin part, we use a method developed by Boileau-Porti [4] and Boileau-Leeb-Porti [5] in their proof of the orbifold theorem. This approach relies on M. Gromov's simplicial volume and W. Thurston's hyperbolization theorem for Haken manifolds. A different approach to collapsing 3-manifolds has been proposed by Shioya-Yamaguchi [13] using Alexandrov space theory.

References

[1] M. Anderson. Scalar curvature and the existence of geometric structures on 3-manifolds. I. *J. Reine Angew. Math.*, 553, 2002.

[2] L. Bessières, G. Besson, M. Boileau, S. Maillot, and J. Porti. Suites de métriques extraites du flot de Ricci sur les variétés asphériques de dimension 3. ArXiv: 0706.2065, 14 Jun 2007.

[3] L. Bessières, G. Besson, M. Boileau, S. Maillot, and J. Porti. Géométrisation des variétés de dimension 3. Book in preparation.

[4] M. Boileau and J. Porti. Geometrization of 3-orbifolds of cyclic type. *Astérisque*, 272:208, 2001. Appendix A by Michael Heusener and Porti.

[5] M. Boileau, B. Leeb, and J. Porti. Geometrization of 3-dimensional orbifolds. *Ann. of Math. (2)*, 162(1):195–290, 2005.

[6] H.-D. Cao and X.-P. Zhu. A Complete Proof of the Poincaré and Geometrization Conjectures - application of the Hamilton-Perelman theory of the Ricci flow. *Asian Journal of Mathematics*, 10(2):165–492, 2006.

[7] T. Colding and W. Minicozzi. Estimates for the extinction time for the Ricci flow on certain three-manifolds and a question of Perelman. *Journal of the A.M.S.*, 18(3):561–569, 2005.

[1] As remarked in [11, Section 7], it is also possible to work more on the Ricci flow part of the proof and prove incompressibility of the tori by adapting an argument of R. Hamilton involving minimal surfaces. For a detailed exposition of this approach, see e.g. [8].

[8] B. Kleiner and J. Lott. Notes on Perelman's papers. ArXiv: math.DG/0605667v2, 21 Feb 2007.
[9] J. Morgan and G. Tian. Ricci Flow and the Poincaré Conjecture. ArXiv: math/0607607v2, 21 Mar 2007.
[10] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. ArXiv : math.DG/0211159, 11 Nov 2002.
[11] G. Perelman. Ricci flow with surgery on three-manifolds. ArXiv : math.DG/0303109, 10 March 2003.
[12] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. ArXiv : math.DG/0307245, 17 Jul 2003.
[13] T. Shioya and T. Yamaguchi. Volume collapsed three-manifolds with a lower curvature bound. *Math. Ann.*, 333(1):131–155, 2005.

Gluing constructions of special Lagrangian cones

Mark Haskins

Our main theorem is the following existence result for special Lagrangian cones in $\mathbb{C}^3$.

Theorem 1 (Haskins-Kapouleas, 2007). *For any odd genus $g = 2d + 1$ there exist infinitely many special Lagrangian cones C in $\mathbb{C}^3$, whose link $C \cap S^5(1)$ is an orientable surface of genus g.*

These are currently the only known examples of special Lagrangian cones with links of genus $g > 1$.

We use a gluing method to prove this result. The basic building blocks of the construction are SO(2)–invariant special Lagrangian cones in $\mathbb{C}^3$ studied previously by the author. The construction uses these invariant SL-cones, close to a singular limit in which the link is the union of round 2–spheres. Close to this singular limit the link of the cone is a long torus consisting of many almost spherical regions connected to its neighbours by a small highly curved neck.

The construction is technically challenging, because for geometric reasons the linearization of the nonlinear equation to be solved has many small eigenvalues. Understanding all the small eigenvalues of the linearization and how to compensate for this problem is the main technically difficult part of the proof.

Gromov hyperbolic spaces and the sharp isoperimetric constant

Stefan Wenger

The goal of this work is to find optimal characterizations of Gromov hyperbolicity in terms of isoperimetric inequalities and filling radius inequalities. The theory of δ-hyperbolic spaces was first developed by Gromov [5] in the context of geometric group theory.

A geodesic metric space is said to be δ-hyperbolic if every geodesic triangle in X is δ-slim, i.e., if each side of the triangle is contained in the δ-neighborhood of the union of the other two sides. In order to state our first result recall the classical isoperimetric inequality in the Euclidean plane $\mathbb{E}^2$ which asserts that the area A enclosed by a closed curve γ in $\mathbb{E}^2$ satisfies

$$A \leq \frac{1}{4\pi}\,\mathrm{length}(\gamma)^2,$$

with equality if and only if γ parameterizes a circle. Our first theorem shows that a geodesic metric space cannot have a quadratic isoperimetric inequality for long curves with isoperimetric constant strictly smaller than $\frac{1}{4\pi}$ unless it is Gromov hyperbolic.

Theorem 1. *Let X be a geodesic metric space and suppose there exists $\varepsilon > 0$ such that every sufficiently long Lipschitz loop γ in X bounds a singular Lipschitz disc Σ in X of area*

$$\mathrm{Area}(\Sigma) \leq \frac{1-\varepsilon}{4\pi}\,\mathrm{length}(\gamma)^2. \tag{1}$$

Then X is Gromov hyperbolic and thus admits a linear isoperimetric inequality for sufficiently long loops.

A singular Lipschitz disc in X is (the image of) a Lipschitz map $\Sigma : D^2 \to X$, where $D^2 \subset \mathbb{E}^2$ denotes the unit disc. Furthermore, $\mathrm{Area}(\Sigma)$ is the 'parameterized' 2-dimensional Hausdorff measure of Σ. If Σ is one-to-one on a set of full measure then $\mathrm{Area}(\Sigma) = \mathcal{H}^2(\Sigma(D^2))$, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure on X. Finally, X is said to admit a linear isoperimetric inequality for long loops if there exists $D > 0$ such that every sufficiently long Lipschitz loop γ in X bounds a singular Lipschitz disc Σ with

$$\mathrm{Area}(\Sigma) \leq D\,\mathrm{length}(\gamma).$$

Clearly, the constant $\frac{1}{4\pi}$ appearing in (1) is optimal as follows from the classical isoperimetric inequality in $\mathbb{E}^2$. Theorem 1 is new even in the setting of Riemannian manifolds and was previously only known in the special case when X is a CAT(0)-space (for which it was observed by Gromov). In the setting of Riemannian manifolds the best constant previously established was $\frac{1}{16\pi}$, again by Gromov [5]. Indeed, using conformal mappings Gromov proved that a 'reasonable' Riemannian manifold M is δ-hyperbolic provided (1) holds with $\varepsilon := \frac{3}{4}$, i.e. if every sufficiently long Lipschitz loop γ in M bounds a singular Lipschitz disc Σ in M of area

$$\mathrm{Area}(\Sigma) \leq \frac{1}{16\pi}\,\mathrm{length}(\gamma)^2.$$

For the meaning of 'reasonable' see [5, p. 176]. For example, the universal covering of a closed Riemannian manifold is 'reasonable'. A detailed account of Gromov's proof appears in [2]. Gromov furthermore showed that the same conclusion holds for geodesic metric spaces provided (1) is satisfied with $\varepsilon \in (0,1)$ close enough to 1. Similar results and alternative proofs of the latter were later given by Olshanskii [7], Short [10], Bowditch [1], Papasoglu [8], and Druţu [3]. Finally, the fact that Gromov hyperbolic metric spaces admit a (coarse) linear isoperimetric inequality follows from a well-known argument essentially going back to Dehn.

We now describe a generalization of Theorem 1 which accounts also for spaces which in general do not admit non-trivial Lipschitz discs (such as Cayley graphs of groups). For this recall that every metric space X isometrically embeds into $L^\infty(X)$, the space of bounded functions on X, by the Kuratowski embedding. We have:

Lemma 2. *Let X be a metric space and γ a Lipschitz loop in X. Let Y be a metric space which isometrically contains X and let Σ be a Lipschitz disc in Y which γ bounds. Then γ bounds a Lipschitz disc Σ' in $L^\infty(X)$ for which*

$$\mathrm{Area}(\Sigma') \leq \mathrm{Area}(\Sigma).$$

The lemma thus asserts that the area needed to fill γ in $L^\infty(X)$ is smaller or equal to that in Y, for any Y in which X isometrically embeds.

Definition 3. *A metric space X is said to be admissible if there exists a complete metric space X' which isometrically contains X, which is at finite Hausdorff distance from X and for which there exists C such that each Lipschitz loop γ in X' bounds a Lipschitz disc Σ in X' satisfying*

$$\mathrm{Area}(\Sigma) \leq C\,\mathrm{length}(\gamma)^2.$$

Length spaces admitting a coarse homological quadratic isoperimetric inequality for curves are for example admissible. This includes in particular Cayley graphs of finitely presented groups with quadratic Dehn function. The generalization of Theorem 1 can now be stated as follows:

Theorem 4. *Let X be an admissible geodesic metric space and suppose that there exists $\varepsilon > 0$ such that every sufficiently long Lipschitz loop γ in X bounds a singular Lipschitz disc Σ in $L^\infty(X)$ with*

$$\mathrm{Area}(\Sigma) \leq \frac{1-\varepsilon}{4\pi}\,\mathrm{length}(\gamma)^2. \tag{2}$$

Then X is Gromov hyperbolic and, in particular, has a thickening which admits a linear isoperimetric inequality for curves.

A similar result holds for the filling radius inequality: Given a Lipschitz loop γ in X denote by $\mathrm{Fill\,Rad}_{L^\infty(X)}(\gamma)$ the infimum of $r \geq 0$ such that γ bounds a 2-chain in $L^\infty(X)$ with support in the r-neighborhood $N_r(\gamma)$ of γ. Note that $N_r(\gamma)$ is the set of points in $L^\infty(X)$ which lie at a distance at most r from the image of γ. Next, let α_0 be the largest number such that in any 2-dimensional normed space V there is a Lipschitz loop $\gamma : S^1 \to V$ with $\mathrm{length}(\gamma) = 1$ and

$$\mathrm{Fill\,Rad}_{L^\infty(V)}(\gamma) \geq \alpha_0.$$

It can be shown that $\frac{3}{32} \leq \alpha_0 \leq \frac{1}{8}$. We then have:

Theorem 5. *Let X be an admissible geodesic metric space and suppose there exist $\varepsilon > 0$ such that for every sufficiently long Lipschitz loop γ in X*

$$\operatorname{Fill Rad}_{L^\infty(X)}(\gamma) \leq (1-\varepsilon)\alpha_0 \operatorname{length}(\gamma).$$

Then X is Gromov hyperbolic and, in particular, has a thickening which admits a logarithmic filling radius inequality for curves.

The theorem is clearly optimal in the class of admissible metric spaces, as follows from the definition of α_0. It generalizes results in [5], [3], [9] and improves the best known constant $\frac{1}{73}$ obtained by Papasoglu [9]. The optimal value for the intrinsic filling radius inequality is conjectured to be $\frac{1}{8}$, see [9]. At present we do not know the exact value of α_0.

References

[1] B. H. Bowditch, *A short proof that a subquadratic isoperimetric inequality implies a linear one*, Michigan Math. J. **42** (1995), no. 1, 103–107.

[2] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics, 1441, Springer-Verlag, Berlin, 1990.

[3] C. Druţu, *Cônes asymptotiques et invariants de quasi-isométrie pour des espaces métriques hyperboliques*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1, 81–97.

[4] C. Druţu, *Quasi-isometry invariants and asymptotic cones*, International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000). Internat. J. Algebra Comput. **12** (2002), no. 1-2, 99–135.

[5] M. Gromov, *Hyperbolic groups*, Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[6] M. Katz, *The filling radius of two-point homogeneous spaces*, J. Differential Geom. **18** (1983), no. 3, 505–511.

[7] A. Yu. Olshanskii, *Hyperbolicity of groups with subquadratic isoperimetric inequality*, Internat. J. Algebra Comput. **1** (1991), no. 3, 281–289.

[8] P. Papasoglu, *On the sub-quadratic isoperimetric inequality*, Geometric group theory (Columbus, OH, 1992), 149–157, Ohio State Univ. Math. Res. Inst. Publ., 3, de Gruyter, Berlin, 1995.

[9] P. Papasoglu, *Quasi-flats in semihyperbolic groups*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1267–1273.

[10] H. Short (editor and contributor), *Notes on word hyperbolic groups*, Group theory from a geometrical viewpoint (E. Ghys, A. Haefliger, A. Verjovsky eds.), World Sci. Publ., River Edge, NJ, 1991.

Ptolemy's Inequality and Non-Positive Curvature

Thomas Foertsch
(joint work with Alexander Lytchak and Viktor Schroeder)

We examine certain features of geodesic Ptolemy metric spaces, focusing on the question of how the Ptolemy four point property relates to notions of non-positive curvature.

Definition 1. *A metric space X is called a Ptolemy metric space if*
$$|xy|\,|zw| \;\leq\; |xz|\,|yw| \;+\; |xw|\,|yz| \qquad \forall\, x,y,z,w \in X.$$

Note that the property of being Ptolemy is a Möbius invariant!

Certain aspects of Ptolemy metric spaces have occasionally been studied in the past (see, for instance, [4], [5] and [6]).

Examples: (1) The n-dimensional Euclidean space $\mathbb{E}^n$ is Ptolemy.
(2) CAT(0)-spaces are Ptolemy.
(3) A C^2-Riemannian manifold (M,g) is locally Ptolemy if and only if it is of non-positive sectional curvature, $K_g \leq 0$.
(4) A normed vector space is Ptolemy if and only if it is Euclidean.
(5) A locally Ptolemy Finsler manifold is Riemannian.

Our interest in these spaces originates from the following observation.

Theorem 2. *(Theorem 1.1. in* [3]*) Let Y be the boundary of a* CAT(-1)*-space endowed with a Bourdon- or Hamenstädt metric $|\cdot|$. Then for all $y_1, y_2, y_3, y_4 \in Y$ it holds*
$$|y_1y_3|\,|y_2y_4| \;\leq\; |y_1y_2|\,|y_3y_4| \;+\; |y_1y_4|\,|y_2y_3|.$$
Moreover, equality holds if and only if the convex hull of the four points is isometric to an ideal quadrilateral in the hyperbolic plane $\mathbb{H}^2$, where $\overline{y_1y_3}$ and $\overline{y_2y_4}$ are the diagonals.

Recall that for a basepoint o in a CAT(-1)-space, the associated Bourdon metric ρ_o is given by $\rho_o(\xi,\xi') := e^{-(\xi\cdot\xi')_o}$ for all $\xi,\xi' \in \partial_\infty X$, while, for an additional basepoint $\omega \in \partial_\infty X$, the associated Hamenstädt metric $\rho_{\omega,o}$ writes as
$$\rho_{\omega,o}(\xi,\xi') \;=\; \frac{\rho_o(\xi,\xi')}{\rho_o(\xi,\omega)\rho_o(\xi',\omega)} \qquad \forall\, \xi,\xi' \in \partial_\infty X \setminus \{\omega\},$$
i.e., the Hamenstädt metric is obtained from the corresponding Bourdon metric by involution at the point ω.

From the rigidity part of this theorem one easily obtains the

Corollary 3. *(see [2]) Let X be a* CAT(κ)*-space, $\kappa < 0$, $x \in X$ and $\omega \in \partial_\infty X$ such that $H := (\partial_\infty X \setminus \{\omega\}, \rho_{\omega,o}^{\sqrt{-\kappa}})$ is geodesic, then H is* CAT(0)*. If, moreover, the space X is ω-visual, i.e. that every geodesic ray in X to ω can be extended to all of $\mathbb{R}$, then X is isometric to the metric warped product $\mathbb{R} \times_{e^{-\sqrt{-\kappa}t}} H$.*

On the other hand, for every $\kappa < 0$, every CAT(0)-space H can be realized as the boundary at infinity of a CAT(κ)-space X when endowed with a Hamenstädt metric; namely let X be the metric warped product $X = \mathbb{R} \times_{e^{-\sqrt{-\kappa}t}} H$.

These observations and the Examples (1)-(5) above raise the

Question: How does the Ptolemy inequality relate to concepts of non-positive curvature?

We answer this question by proving the following three theorems:

Theorem 4. *(Theorem 1.1. in [1]) Every Ptolemy metric space admits an isometric embedding into a complete, geodesic, Ptolemy metric space.*

Note that it follows easily that, in general, a geodesic Ptolemy metric space need not be uniquely geodesic. The situation changes completely, if one assumes the space to be locally compact.

Theorem 5. *(Theorem 1.2. in [1]) A locally compact, geodesic, Ptolemy metric space is uniquely geodesic.*

Since, as mentioned above, general geodesic Ptolemy metric spaces need not be uniquely geodesic, it is clear that the Ptolemy condition is not a suitable kind of non-positive curvature condition itself. However, it turns out that the Ptolemy condition precisely distinguishes between the two most common non-positive curvature conditions; namely between those due to Alexandrov and Busemann, respectively.

Theorem 6. *(Theorem 1.3. in [1]) A metric space is* CAT(0) *if and only if it is Ptolemy and Busemann convex.*

References

[1] T. Foertsch, A. Lytchak, V. Schroeder, *Non-positive curvature and the Ptolemy inequality*, preprint

[2] T. Foertsch, K. Radke, *Characterizing complete* CAT(κ)*-spaces with geodesic boundaries*, preprint

[3] T. Foertsch, V. Schroeder, *Hyperbolicity,* CAT(-1)*-spaces and the Ptolemy inequality*, preprint

[4] D.D. Kay, *The Ptolemaic inequality in Hilbert geometries*, Pacific J. Math. **21** (1967), 389–392

[5] D.D. Kay, Michigan State University, Doctoral Dissertation, 1963

[6] J.D. Schoenberg, *A remark on M. M. Day's characterization of inner-product spaces and a conjecture of L. M. Blumenthal*, Proc. Amer. Math. Soc. **3** (1952), 961–964

New Ricci flow invariant curvature conditions in large dimensions

Burkhard Wilking
(joint work with Christoph Böhm)

This is a report on work in progress. For a compact Riemannian manifold (M,g) the curvature operator R_p of M at p is a selfadjoint endomorphism of the second exterior product $\Lambda^2 T_pM$. By the second Bianchi identity it is contained in a subspace $S^2_B(\Lambda^2 T_pM)$ of the vectorspace of all selfadjoint endomorphisms. In dimensions above three $S^2_B(\Lambda^2 T_pM)$ decomposes under the natural representation of the orthogonal group $\mathrm{O}(T_pM)$ into three subspaces.

$$\begin{aligned} S^2_B(\Lambda^2 T_pM) &= \langle I\rangle \oplus \langle \mathrm{Ric}_0\rangle \oplus \langle \mathrm{W}\rangle. \\ R &= R_I + R_{\mathrm{Ric}_0} + R_W \end{aligned}$$

We view any $\mathrm{O}(n)$-invariant subset $C \subset S^2_B(\Lambda^2\mathbb{R}^n)$ as a curvature condition. A manifold is said to satisfy the condition C if the curvature operator at any point is contained in C. We also recall that condition C is said to be invariant under the Ricci flow if for any compact Riemannian (M,g) satisfying C for all $t \in [0,T)$, the manifold (M,g_t) also satisfies C, where (M,g_t) is the solution to the Ricci flow

$$\tfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}\,.$$

The aim of this project is to find Ricci flow invariant curvature conditions that do not depend on R_{Ric_0}.

If C is a convex curvature condition then, by Hamilton's maximum principle, C defines a Ricci flow invariant curvature condition provided that it is invariant as a set under the ODE

$$R' = R^2 + R^\# \tag{1}$$

where $\mathrm{ad}\colon \Lambda^2(\mathfrak{so}(n)) \to \mathfrak{so}(n)$ is the adjoint representation and where we have identified $\Lambda\mathbb{R}^n$ with $\mathfrak{so}(n)$.

Theorem 1. *There is a positive integer n_0 such that in all dimensions $n \geq n_0$ the following holds. Consider for $c > 0$ the convex curvature condition*

$$\|R_W\|^2 \leq c\|R_I\|^2 \text{ and } \mathrm{scal}(R) \geq 0. \tag{2}$$

- a) *If n is even the condition* (2) *is invariant under the ODE* (1) *if and only if $c = \frac{n}{n-2}$.*
- b) *If n is even the condition* (2) *is invariant under the ODE* (1) *if and only if $c = \left[\frac{n}{n-2} - \delta_n, \frac{n}{n-2} + \varepsilon_n\right]$ where δ_n and ε_n are sequences of positive numbers converging to* 0.

Corollary 2. *Let $n \geq n_0$. Let (M^n,g) be a simply connected compact Einstein manifold with a positive Einstein constant and $\|R_W\|^2 \leq \frac{n}{n-2}\|R_I\|^2$. Then (M,g) is isometric to $\mathbb{S}^n$ or $\mathbb{S}^{n/2} \times \mathbb{S}^{n/2}$.*

The constant $c = \frac{n}{n-2}$ is chosen such that for even n equality is attained for the manifold $\mathbb{S}^{n/2} \times \mathbb{S}^{n/2}$. We can also show in all dimensions that a positive compact

Einstein manifold with $\|R_W\| \leq \|R_I\|$ has constant curvature unless $n = 4$ and M is locally isometric to $\mathbb{CP}^2$.

Finally we give a method of constructing examples of such manifolds.

Theorem 3. *Let F be a spherical space form and $F \to M \to B$ a fiber bundle with structure group contained in* $\mathrm{Iso}(F)$. *Assume* $\dim(F) > \dim(B)$. *Then M admits a metric with*

$$\|R_W\|^2 < \tfrac{n}{n-2}\|R_I\|^2 \text{ and } \mathrm{scal}(R) > 0.$$

It is also known that connected sums of spherical space forms admits conformally flat metrics with positive scalar curvature.

The constant n_0 in Theorem 1 is at least 12 and we conjecture that equality holds.

It is well known that positive isotropic curvature in dimension 4 is a Ricci flow invariant curvature condition which is independent of R_{Ric_0}. By recent work of Nguyen and independently by Brendle and Schoen, positive isotropic curvature defines a Ricci flow invariant curvature condition in all dimensions. However, only in dimension 4 this condition is independent of R_{Ric_0}. Nevertheless, in view of Theorem 1 it seems reasonable to expect that there should be a Ricci flow invariant curvature condition independent of R_{Ric_0} in all dimensions.

References

[BW] C. Böhm and B. Wilking, *Manifolds with positive curvature operators are space forms*, to appear in Annals of Math., 15p.

[BS] S. Brendle and R. Schoen, *1/4-pinched manifolds are space forms,* preprint, 25 p.

[H1] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255-306.

[H2] R. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153-179.

[Ng] H. Nguyen, *Invariant curvature cones and the Ricci flow,* preprint 16 p.

Shape of large balls in groups of polynomial growth and sub-Riemannian geometry

Emmanuel Breuillard

Let G be a locally compact and compactly generated group. The group G is said to have polynomial growth if for some (hence all) compact generating set Ω there are positive constants c and k such that $vol_G(\Omega^n) \leq c{\cdot}n^k$ for all $n \in \mathbb{N}$, where vol_G is a left (or right) Haar measure on G. A celebrated theorem of Gromov [2] says that if G has polynomial growth and is discrete then G is commensurable to a nilpotent group. We first extend this to locally compact groups and show:

Theorem 1. *Any locally compact group G of polynomial growth is weakly commensurable to a connected and simply connected solvable Lie group S of polynomial growth, namely there is a compact normal subgroup K in G and a closed co-compact subgroup H of G containing K such that H/K embeds as a closed and co-compact subgroup of S. We call S a Lie shadow of G.*

It is worth noting that solvable cannot be replaced by nilpotent in the above result.

Our goal is to study natural distances ρ on G and determine the asymptotic cone of (G,ρ). We require the distance ρ to be coarsely geodesic (i.e. there is $C>0$ such that any two points in G can be joined by a C-coarse geodesic, i.e. a path $\phi(t)$ for $t\in[a,b]$ with $|\rho(\phi(s),\phi(t))-|t-s||<C$), and periodic (i.e. left invariant under some co-compact subgroup H). Examples of such distances include word metrics $\rho_\Omega(x,y)=\inf\{n\in\mathbb{N},\ x^{-1}y\in\Omega^n\}$ induced by a compact symmetric generating set Ω, Riemannian or sub-Riemannian distances on G in case G is a connected Lie group, or lifts of a Riemannian metric on a compact homogeneous space of G.

From Theorem 1 we get that there is a periodic metric ρ_S on S such that (G,ρ) is coarsely isometric (i.e. $(1,C)$-quasi-isometric) to (S,ρ_S). So the coarse geometry of G reduces to that of S. Let $S_{nil}=(S,*)$ be the nilshadow of S. It is a simply connected nilpotent Lie group, obtained from S by keeping the same underlying manifold and modifying the Lie product (see [1] for a construction). Although the solvable Lie shadow S obtained from G by Theorem 1 may not be unique, its nilshadow S_{nil} is uniquely determined by G.

To describe the asymptotic cone of (G,ρ) we also need to introduce the graded nilshadow $S_\infty=(S,*_{gr})$ which is obtained by further modifying the Lie product on S_{nil} in order to make it a graded nilpotent Lie group (also called Carnot group). This group comes with a 1-parameter group of dilations $(\delta_t)_{t>0}$, which are automorphisms of S_∞.

A Carnot-Caratheodory-Finsler metric on S_∞ is a left invariant sub-Riemannian metric induced by a norm on a horizontal vector subspace of $Lie(S_\infty)$ (i.e. transverse to the commutator ideal).

We obtain

Theorem 2. *Let ρ be a coarsely geodesic periodic distance on a simply connected solvable Lie group of polynomial growth S. Then there exists a left invariant Carnot-Caratheodory-Finsler metric d_∞ on S_∞ such that $\frac{\rho(e,x)}{d_\infty(e,x)}\to 1$.*

In particular, if $B_\rho(n)$ is the ρ-ball of radius n centered at the identity e, then the renormalized balls $\delta_{\frac{1}{n}}(B_\rho(n))$ converge in the Hausdorff topology to $B_{d_\infty}(1)$ the d_∞-unit ball centered at e.

This theorem extends the work of P. Pansu [6], who showed the same result when S was assumed to be nilpotent and the Malcev closure of some finitely generated nilpotent group. See also [4] for a result of a similar flavor in the context of reductive groups.

Corollary 3. *Let G be a locally compact group of polynomial growth and ρ a coarsely geodesic periodic distance on G, then $(G,\frac{\rho}{n})$ converges in the Gromov-Hausdorff topology to (S_∞,d_∞). And*

$$\frac{vol_G(B_\rho(n))}{n^{d(G)}}\to vol_{S_\infty}(B_{d_\infty}(1))$$

where $d(G)$ is given by the Bass-Guivarc'h formula

$$d(G) = d(S_\infty) = \sum_{k\geq 1} k \dim(C^k(S_\infty)/C^{k+1}(S_\infty)).$$

Corollary 4. *The asymptotic cone of (G, ρ) is uniquely defined (independent of the ultrafilter) and isometric to (S_∞, d_∞).*

Corollary 5. *The sequence of balls $B_\rho(n)$ is a Folner sequence.*

This in turn yields (see [5]):

Corollary 6. *The pointwise ergodic theorem for a measure preserving G-action holds for ball averages along any sequence of centered balls of radii tending to infinity.*

Remark: Theorem 2 above also enables us to answer negatively a question of Burago and Margulis (see [4] Conj. (a)) about asymptotic metrics on finitely generated groups.

Outline of the proof: Theorem 1 follows from Losert's extension of Gromov's theorem to locally compact groups together with an embedding theorem for solvable groups due to H.C. Wang. In Theorem 2 the strategy is to construct another distance ρ_K (by some averaging procedure) which will be asymptotic to the original distance ρ and periodic with respect to the new Lie structure on S, i.e. the nilshadow product. The key point is that at a large scale ρ becomes invariant under the semisimple part of the action of S on itself by conjugation. We hence reduce the problem to S_{nil}. Then the methods from [6] can be carried out to show that the ρ_K is asymptotic to the desired sub-Riemannian metric d_∞ on S_∞.

References

[1] E. Breuillard, Geometry of locally compact groups of polynomial growth and shape of large balls, preprint arXiv:0704.0095.

[2] M. Gromov, Groups of polynomial growth and expanding maps, Publications Mathématiques de l'IHES, no 53 (1981), 53-73.

[3] V. Losert, On the structure of groups with polynomial growth, Math. Z. 195 (1987), no 1, 109-117.

[4] G.A. Margulis, Metrics on reductive groups, in Oberwolfach Report 2006-33, Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry, July 23rd - July 29th, 2006.

[5] A. Nevo, Pointwise ergodic theorems for actions of connected Lie groups, Handbook of Dynamical Systems, Eds. B. Hasselblatt and A. Katok, to appear.

[6] P. Pansu, Croissance des boules et des géodésiques fermées dans les nilvariétés, Ergodic Theory Dynam. Systems 3 (1983), no. 3, 415-445.

Solvable Kähler groups

Thomas Delzant

We study the generalized Green–Lazarsfeld set of a group. As an application, it is shown that if a Kähler group is solvable, then it is virtually nilpotent.

Boundary value problems for Dirac operators

Werner Ballmann
(joint work with Jochen Brüning and Gilles Carron)

In their work on the signature theorem for compact Riemannian manifolds with boundary, Atiya, Patodi, and Singer observed that there is a correction term to the signature integral which is a spectral invariant of the boundary [1]. This initiated the study of boundary value problems for Dirac operators, compare [5].

Let M be a Riemannian manifold with compact boundary N and $E \to M$ be a Dirac bundle over M in the sense of Gromov and Lawson [6]. Consider the Dirac operator D of E as an unbounded operator on the space $L^2(M,E)$ of square-integrable sections of E: Let $D_{0,c}$ be the restriction of D to $\operatorname{dom} D_{0,c} := C^\infty_{0,c}(M,E)$, the space of smooth sections of E with compact support which vanish along N. Set $D_{\max} := (D_{0,c})^*$, the $L^2(M,E)$-dual operator of $D_{0,c}$. Since $D_{0,c}$ is symmetric, $D_{0,c} \subseteq D_{\max}$. We call $D_{\max}$ the maximal extension of $D_{0,c}$. The dual operator $D_{\min}$ of $D_{\max}$ is the minimal extension of $D_{0,c}$ as a closed operator in $L^2(M,E)$. The work reported on is concerned with closed extensions of $D_{0,c}$ between $D_{\min}$ and $D_{\max}$ and with their regularity properties.

By interior elliptic regularity, sections in the domain $\operatorname{dom} D_{\max}$ of the maximal extension are contained in the Sobolev space $H^1(M,E)$ locally in $M \setminus N$. The regularity of elements in $\operatorname{dom} D_{\max}$ is, therefore, connected to their boundary regularity.

Let $H := L^2(N, E|N)$ and A be the induced Dirac operator in H. Let H^s, $s \in \mathbb{R}$, be the domain of definition of $(I + A^2)^{s/2}$. It turns out that the space of boundary values of elements in $\operatorname{dom} D_{\max}$ is the hybrid Sobolev space $\check{H} := H^{1/2}_{\le} \oplus H^{-1/2}_{>}$, where the indices refer to the corresponding spectral projections of A. Closed linear subspaces of $\check{H}$ are called boundary conditions. For a boundary condition $B \subset \check{H}$, we set $\operatorname{dom} D_{B,\max} := \{\sigma \in \operatorname{dom} D_{\max} : \sigma|N \in B\}$ and $D_{B,\max} := D_{\max}|\operatorname{dom} D_{B,\max}$.

Proposition 1. *The closed extensions of $D_{0,c}$ contained in $D_{\max}$ are precisely the operators of the form $D_{B,\max}$, where $B \subset \check{H}$ is a boundary condition.*

We say that a boundary condition B is regular, if $\operatorname{dom} D_{B,\max} \subset H^1(M,\mathbb{R})$. From the explicit description of boundary regularity in the case where M is a half-cylinder $\mathbb{R}_+ \times N$ over N and the data for D do not depend on the $\mathbb{R}_+$-variable, it follows that B is regular if and only if B is a closed subspace of $\check{H}$ which is contained in $H^{1/2} \subset \check{H}$. We say that a boundary condition is elliptic if B and the boundary condition B^a of the adjoint operator $(D_{B,\max})^*$ are regular. We obtain an explicit characterization of elliptic boundary conditions, and it turns out that they coincide with the boundary condition which were considered in [2]. The explicit description of elliptic boundary conditions shows that they are very flexible. For example, they are well-suited for deformations, compare [2].

Theorem 2. *If M is compact and B is an elliptic boundary condition, then $D_{B,\max}$ is a Fredholm operator.*

This theorem holds also under the more general assumption that B is elliptic and that there is a constant C such that the L^2-norm of smooth sections σ of E with compact support in $M \setminus N$ is bounded from above by C times the L^2-norm of $D\sigma$. For details we refer to [3], applications will appear in [4]. From a different angle, the theory of elliptic boundary conditions will also be partly detailed in [2].

References

[1] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
[2] W. Ballmann and C. Bär, *In preparation.*
[3] W. Ballmann, J. Brüning, and G. Carron, *On the index and regularity theory of Dirac-Schrödinger systems*, Preprint MPI **07–24**.
[4] W. Ballmann, J. Brüning, and G. Carron, *In preparation.*
[5] B. Booß-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1993. xviii+307 pp. ISBN: 0-8176-3681-1
[6] M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 83–196 (1984).

Path integrals on manifolds by finite dimensional approximation

Christian Bär
(joint work with Frank Pfäffle)

This talk is based on the results in [2]. Let M be a compact Riemannian manifold without boundary, let $E \to M$ be a Riemannian or Hermitian vector bundle with compatible connection ∇. We study selfadjoint generalized Laplace operators, i. e. operators of the form $H = \nabla^*\nabla + V$ where V is a potential (symmetric endomorphism field on E). For simplicity of notation we restrict ourselves to scalar potentials even though this is not necessary.

Our main result can formally be stated as follows: The solution to the heat equation

$$\frac{\partial U}{\partial t} + HU = 0$$

with initial condition

$$U(0,x) = u(x)$$

is given by the path integral

$$U(t,x) = \frac{1}{Z}\int_{\mathfrak{C}_x(M,t)} \exp\left(-\frac{1}{2}\mathrm{E}(\gamma) + \int_0^t \left(\frac{1}{3}\mathrm{scal}(\gamma(s)) - V(\gamma(s))\right) ds\right) \tau(\gamma)_t^0\, u(\gamma(t))\, \mathcal{D}\gamma.$$

Here $\mathfrak{C}_x(M,t)$ is the space of all continuous paths $\gamma : [0,t] \to M$ emanating from x, $\mathrm{E}(\gamma)$ denotes the energy of the path γ, $\tau(\gamma)$ is parallel translation along γ, $\mathcal{D}\gamma$ is a formal measure on $\mathfrak{C}_x(M,t)$ and Z is a normalizing constant.

Such formulas are very common in the physics literature but there are various problems with a rigorous mathematical interpretation:

- $\mathfrak{C}_x(M,t)$ is an infinite dimensional space and the meaning of the measure $\mathcal{D}\gamma$ is unclear,
- $\mathrm{E}(\gamma)$ and $\tau(\gamma)$ are not defined for continuous paths without differentiability properties,
- Z is infinite.

It is well-known that $\frac{1}{Z}\exp\left(-\frac{1}{2}\mathrm{E}(\gamma)\right)\mathcal{D}\gamma$ yields a well-defined measure on path space $\mathfrak{C}_x(M,t)$, the *Wiener measure*. Parallel transport $\tau(\gamma)$ can be treated using stochastic differential equations. This then generalizes the *Feynman-Kac formula*, see e. g. [4].

We follow a different approach. We approximate $\mathfrak{C}_x(M,t)$ by finite dimensional spaces of geodesic polygons. It turns out that the formally identical integrals over these finite dimensional space approximate the solution to the heat equation. The necessary analysis can be organized nicely using a classical theorem of Chernoff's [3]. The short time asymptotics of the heat kernel also play an important role.

Our technique allows us to derive different versions of the path integral formula. For example, one can remove the scalar curvature term if one uses another measure on the approximating spaces of geodesic polygons. This clarifies a discussion in [1] where our main result has been proved by different methods in the special case of the Laplace-Beltrami operator acting on functions.

As an application we find a very simple and natural proof of the Hess-Schrader-Uhlenbrock estimate for the heat kernel by the kernel of a scalar comparison operator, see [5]. Moreover, we can express the trace of the heat operators by a path integral. Formally,

$$\mathrm{Tr}(e^{-tH}) = \frac{1}{Z}\int_{\mathfrak{C}_{\mathfrak{cl}}(M,t)} \exp\left(-\frac{1}{2}\mathrm{E}(\gamma) + \int_0^t \left(\frac{1}{3}\mathrm{scal}(\gamma(s)) - V(\gamma(s))\right) ds\right) \mathrm{tr}(\mathrm{hol}(\gamma))\, \mathcal{D}\gamma.$$

Here $\mathfrak{C}_{\mathfrak{cl}}(M,t)$ denotes the space of closed continuous loops in M, parameterized on $[0,t]$, and $\mathrm{hol}(\gamma)$ is the holonomy of such a loop γ.

References

[1] L. Andersson, B. Driver, *Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds*, J. Funct. Anal. **165** (1999), 430–498.

[2] C. Bär, F. Pfäffle, *Path integrals on manifolds by finite dimensional approximation*, arXiv: `math.AP/0703272`.
[3] P. R. Chernoff, *Note on product formulas for operator semigroups*, J. Funct. Anal. **2** (1968), 238–242.
[4] B. Driver, A. Thalmaier, *Heat equation derivative formulas for vector bundles*, J. Funct. Anal. **183** (2001), 42–108.
[5] H. Hess, R. Schrader, D. A. Uhlenbrock, *Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifolds*, J. Diff. Geom. **15** (1980), 27–37.

A link between Willmore energy, isoperimetric inequalities and inverse mean curvature flow

Gerhard Huisken

Consider a complete noncompact 3–manifold (M^3, g) without boundary and satisfying one of two curvature conditions:

(i) non–negative scalar curvature or
(ii) non–negative Ricci–curvature.

The first curvature condition is natural for asymptotically flat manifolds arising in General Relativity, the second is natural in the context of manifolds that are asymptotically cone–like.

Let ϕ be the isoperimetric profile of the corresponding model spaces (Riemannian Schwarzschild and cone respectively), which assigns to a given area of boundary 2–spheres the maximal volume enclosable by such spheres in (M^3, g). The lecture explains that the difference

$$\phi(|\Sigma_t^2|) - \mathrm{Vol}(\Omega_{\Sigma_t})$$

is monotone under (weak) mean curvature and inverse mean curvature flow, leading to optimal isoperimetric inequalities in the two cases above. The results rely on the properties of weak mean curvature flow (Giga–Goto, Evans–Spruck, White) and weak inverse mean curvature flow (Ilmanen–H.) and extend earlier work on isoperimetric inequalities by Kleiner, Bray, Morgan, Schulze and Topping.

Refinement of Perelman's Stability Theorem

Vitali Kapovitch

A fundamental observation of Gromov says that the class of complete n-dimensional Riemannian manifolds with fixed lower curvature and upper diameter bounds is precompact in Gromov-Hausdorff topology. The limit points of this class are Alexandrov spaces of dimension $\leq n$ with the same lower curvature and upper diameter bounds. Given a sequence of manifolds M_i in the above class converging to an Alexandrov space X it's interesting to know what can be said about the relationship between topology of the limit and elements of the sequence.

Our main purpose is to give a careful proof of the following Theorem of Perelman which answers this question in the situation when $\dim X = n$.

Theorem 1. *Let X^n be a compact n-dimensional Alexandrov space of* curv $\geq \kappa$. *Then there exists an $\epsilon = \epsilon(X) > 0$ such that for any n-dimensional Alexandrov space Y^n of* curv $\geq \kappa$ *with $d_{G-H}(X, Y) < \epsilon$, Y is homeomorphic to X.*

A proof of the Stability Theorem was given in [1]. However, that paper is very hard to read and is not easily accessible. We aim to provide a comprehensive and hopefully readable reference for Perelman's result.

It is also worth pointing out that the Stability Theorem in dimension 3 plays a key role in the classification of collapsing of 3-manifolds with a lower curvature bound by Shioya and Yamaguchi [4, 5] which in turn plays a role in Perelman's work on the geometrization conjecture. However, as was communicated to the author by Kleiner, for that particular application, if one traces through the proofs of [4, 5] carrying along the additional bounds arising from the Ricci flow, then one finds that in each instance when a 3-dimensional Alexandrov space arises as a Gromov-Hausdorff limit of smooth manifolds, it is in fact smooth, and after passing to an appropriate subsequence, the convergence will also be smooth to a large order. For such convergence the stability theorem is very well known and easily follows from Cheeger-Gromov compactness.

Besides the original proof of Perelman, we also present a different proof of the Stability Theorem for limits of Riemannian manifolds based on techniques of controlled homotopy theory. In its current form this proof for dim = 3 uses the 3-dimensional Poincare conjecture (the proof of which does not require stability). While the use of the Poincare conjecture doesn't seem very satisfactory, the author fully believes that the proof can be modified so that it doesn't rely on the Poincare conjecture at all.

Perelman showed in [2] that any n-dimensional Alexandrov space possesses a canonical topological stratifications where its i-th strata is an i-dimensional topological manifold. Any homeomorphism between two Alexnadrov spaces clearly has to preserve their topological stratifications.

It turns out [3], that the above mentioned topological stratification is a part of a finer geometric stratification of an Alexandrov space into *extremal* subsets which plays an important role in the study of Alexandrov spaces.

Being geometric rather than topological, this stratification need not be preserved by arbitrary homeomorphisms between Alexandrov spaces. Nevertheless we show that the stability homeomorphisms in the context of the Stability theorem can be chosen to preserve this stratification. More precisely, we show

Theorem 2 (Relative Stability Theorem)**.** *Let $X_i^n \underset{i\to\infty}{\overset{G-H}{\longrightarrow}} X^n$ be a noncollapsing sequence of compact Alexandrov spaces with* curv $\geq \kappa$ *and* diam $\leq D$. *Let $E_i \subset X_i$ be a sequence of extremal subsets converging to an extremal subset E in X. Then for all large i there exist homeomorphisms $\theta_i \colon (X, E) \to (X_i, E_i)$, close the original Hausdorff approximations $X_i \to X$.*

References

[1] G. Perelman. Alexandrov spaces with curvatures bounded from below II. preprint, 1991.
[2] G. Perelman. Elements of Morse theory on Aleksandrov spaces. *St. Petersbg. Math. J.*, 5(1):205–213, 1993.
[3] G. Perelman and A. M. Petrunin. Extremal subsets in Aleksandrov spaces and the generalized Liberman theorem. *Algebra i Analiz*, 5(1):242–256, 1993.
[4] T. Shioya and T. Yamaguchi. Collapsing three-manifolds under a lower curvature bound. *J. Differential Geom.*, 56(1):1–66, 2000.
[5] T. Shioya and T. Yamaguchi. Volume collapsed three-manifolds with a lower curvature bound. *Math. Ann.*, 333(1):131–155, 2005.

Curvature explosion in quotients and applications

Alexander Lytchak

(joint work with Gudlaugur Thorbergsson)

Let M be a complete Riemannian manifold and G a closed group of isometries of M. The quotient space $B = M/G$ is an Alexandrov space with curvature locally bounded below. It is stratified by smooth strata B_K corresponding to conjugacy classes K of isotropy groups in G. The main stratum B^0 is open, dense and convex in B. Usually, the supremum of sectional curvatures $\bar{k}(y)$ at points $y \in B^0$ goes to infinity as the points y approach the boundary of B^0. On the other hand, in many important examples the curvature is uniformly bounded on B^0. For instance, it happens in the case of polar actions, like the isotropy representation of a symmetric space, i.e. actions with identically vanishing O'Neill tensor in the regular part. Our first result gives a precise description of the explosion of curvature.

Theorem 1. *Let $x \in M$ be a point and let $\bar{x}$ denote its projection in B. Then the following are equivalent:*

(1) $\limsup_{y\to\bar{x}, y\in B^0} \bar{k}(y) < \infty$;
(2) $\limsup_{y\to\bar{x}, y\in B^0} \bar{k}(y) \cdot d^2(\bar{x}, y) = 0$;
(3) *The isotropy representation at the point x is polar;*
(4) *A neighborhood of $\bar{x}$ in B is isometric to a smooth Riemannian orbifold.*

The equivalence of (1) and (2) in the above theorem says that if the explosion occurs then only at a prescribed rate, namely, proportional to the inverse of the square of the distance. On the other hand, the theorem says that large parts of B consists of orbifold points, that allows us to use differential geometric methods in B. For instance, the set of orbifold points contains all strata of codimension at most 2. As a consequence we deduce:

Corollary 2. *The geodesic flow in the quotient B preserves the Liouville measure on the unit tangent bundle of B.*

If B is compact, the Liouville measure is finite and we get a recurrence theorem, that is very useful in applications concerning conjugate points along horizontal geodesics.

We say that G is infinitesimally polar at x if the isotropy representation at x is polar. Our motivation for studying such points was an observation that can be loosely formulated as follows: G is infinitesimally polar at x if and only if in a neighborhood of x indices of horizontal geodesics have continuous vertical and horizontal parts. The horizontal index is the index that corresponds to the index of the projected quasi-geodesic in the orbit space and can be defined using the work of Wilking ([7]). The vertical index is the part of the index that cannot be seen below and that can be easily read off in the total space. It was implicitly used by Bott and Samelson([2]) in studying the topology of (loop spaces of) symmetric spaces.

To make the idea more precise we give two definitions and two theorems related to them. We call a horizontal geodesic γ regular if it starts and terminates in a regular point. Such a geodesic crosses singular orbits only at finitely many points $\gamma(t_i)$ and we define the crossing number $c(\gamma)$ to be the sum of $r - \dim(\gamma(t_i))$, where r is the dimension of a regular orbit. This crossing number is precisely the vertical index mentioned above. The above statement in combination with the first theorem can now be stated as follows.

Theorem 3. *The crossing number function is continuous on the space of all regular geodesics if and only if the quotient space B is a smooth Riemannian orbifold.*

Finally our results can be used to obtain a description of variationally complete actions, that were introduced by Bott in [1]. The action is variationally complete if the vertical index of each regular geodesics coincides with its focal index, with respect to the orbit through its starting point. Variationally complete actions are very useful in topology (c.f. [1],[2]). They have been studied by many authors by very different means (cf. [3], [4], [5] [6]). We prove:

Theorem 4. *The action of G on M is variationally complete if and only if the quotient B is isometric to the quotient of a smooth Riemannian manifold N without conjugate points by a discrete group Γ of isometries of N.*

All results explained above are proven in the much more general case of singular Riemannian foliations.

References

[1] R. Bott *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956),251-281

[2] R. Bott, H. Samelson *An application of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964-1029

[3] L. Conlon *A class of variationally complete representations*, J. Differential Geom. **7** (1972), 149-160

[4] C. Gorodski, G. Thorbergsson, *Variationally complete actions on compact symmetric space*, J. Differential Geom. **62**, (2002), 39-48

[5] A.J. Di Scala, C. Olmos, *Variationally complete representations are polar*, Proc. Amer. Math. Soc. **129** (2001), 3445-3446

[6] A. Lytchak, G. Thorbergsson, *Variationally complete actions on nonnegatively curved manifolds*, to appear in Ilinois J.Math.

[7] B. Wilking *A duality theorem for Riemannian foliation in nonnegative curvature*, to appear in GAFA

Reeb vector fields and open book decompositions

Vincent Colin
(joint work with Ko Honda)

Let M be an oriented closed three manifold. An open book decomposition of M is a pair (K, θ) where K is a link in M and $\theta : M \setminus K \to S^1$ is a fibration that coincides with the function given by the normal angular coordinate in a trivialized neighbourhood of K. The link K is called the binding, and the fibers of θ are called the pages. They are both canonically oriented. Such an open book decomposition can be entirely described by a small compact retraction of one page S and the monodromy of the fibration θ, viewed as a diffeomorphism of S which is the identity along ∂S.

It is known by a work of Giroux [Gi], that isotopy classes of positive contact structures on M are in one-to-one correspondence with isotopy classes of open book decompositions of M, considered modulo stabilization. The stabilization procedure is the plumbing of (K, θ) by a Hopf band (it can be applied several times). More precisely, a contact structure ξ is said to be carried by an open book decomposition (K, θ) if it admits a Reeb vector field which is (positively) tangent to the binding and (positively) transversal to the pages. Using this definition, Giroux's theorem states that every contact structure is carried by some open book; that two contact structures carried by the same open book are isotopic, and that if ξ and ξ' are carried by (K, θ) and (K', θ'), then ξ is isotopic to ξ' if and only if (K, θ) and (K', θ') have isotopic stabilizations.

Our initial motivation in what follows is to give a proof the Weinstein conjecture in dimension 3 ("every Reeb vector field has a periodic orbit") and to analyze the links between the dynamical behavior of Reeb vector fields and the topology of the ambient manifold. A complete proof of the 3-dimensional Weinstein conjecture has been given recently by Taubes [Ta].

On our side, we prove the following:

Theorem 1. *If a contact structure ξ is carried by an open book decomposition whose monodromy is isotopic to a periodic diffeomorphism, then the Weinstein conjecture holds for ξ: every Reeb vector field for ξ has a periodic orbit.*

In fact we have that [CH]:

Theorem 2. *Every contact structure is carried by an open book whose monodromy is isotopic to a pseudo-Anosov diffeomorphism and whose binding is connected.*

In this case, the isotopy between the monodromy h and its pseudo-Anosov representative ψ is not the identity along ∂S. If we follow the trace of this isotopy on ∂S, we get a rotation number $c \in \mathbb{R}$, which is a rational number of the form

k/n. Here, n is the number of singularities of the stable invariant foliation of ψ that sit on ∂S.

By a theorem of Honda, Kazez and Matiĉ [HKM], if $c \leq 0$ then the contact structure ξ is overtwisted, and thus, applying a result of Hofer [Ho], the Weinstein conjecture holds for ξ.

The following result is still a work in progress:

Let ξ be the contact structure carried by an open book decomposition whose monodromy is isotopic to a pseudo-Anosov diffeomorphism with rotation number k/n. If $k \geq 2$, then ξ is tight, $\pi_2(M) = 0$, the universal cover of M is not S^3, and for every Reeb vector field R associated to a contact form α for ξ, the number of periodic orbits γ of R of actions $\int_\gamma \alpha$ less than L grows exponentially with L.

The proofs of these results involve computations in contact homology [EGH].

We say that a contact structure ξ is *dynamically hyperbolic* if every Reeb vector field associated with ξ has an exponentially growing set of periodic orbits with respect to their actions. We make the conjecture that every universally tight contact structure on a hyperbolic three manifold is dynamically hyperbolic.

References

[CH] V. Colin, K. Honda, *Stabilizing the monodromy of an open book decompositon*, to appear in Geom. Dedicata.

[EGH] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part II, 560–673.

[Gi] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

[Ho] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), 515–563.

[HKM] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary I*, preprint 2005. `ArXiv:math.GT/0510639`.

[Ta] C. Taubes, *The Seiberg-Witten equations and the Weinstein conjecture*, preprint 2006.

The Thurston-Bennequin inequality for tight confoliations

Thomas Vogel

In this talk we discuss a generalization of a result of Eliashberg (for tight contact structures) respectively Thurston (for Reebless foliations). Throughout this talk M will be an oriented manifold of dimension 3.

Definition 1. *A* positive confoliation *on M is a plane field on M which is locally defined by a 1-form α such that $\alpha \wedge d\alpha \geq 0$.*

Contact structures are defined by the condition that $\alpha \wedge d\alpha > 0$ while for foliations the equality $\alpha \wedge d\alpha = 0$ holds.

Contact structures are divided into two classes according to whether they are tight or not (the definition of tightness can be found below). It turns out that only tight contact structures reflect properties of the underlying manifold. The following generalization of tightness to confoliations was suggested in [ET].

Definition 2. *A confoliation ξ on M is* tight *if for every embedded disc $D^2 \subset M$ such that*

(i) *∂D is tangent to ξ,*
(ii) *TD^2 and ξ are transverse along ∂D^2*

there is an embedded disc D' satisfying the following requirements

(i) *$\partial D = \partial D'$,*
(ii) *D' is everywhere tangent to ξ,*
(iii) *$e(\xi)[D \cup D'] = 0$.*

If ξ is a contact structure, then there are no surfaces tangent to ξ and Definition 2 reduces to a definition of tightness. In the case when ξ is a foliation, then Definition 2 corresponds to the absence of so called vanishing cycles. By a theorem of Novikov [No] the absence of vanishing cycles on a closed manifold is equivalent to the absence of Reeb components and foliations without Reeb components have turned out to be much more interesting than foliations with Reeb components. Thus Definition 2 interpolates between tight contact structures and Reebless foliations.

As is pointed out in [ET] there is an interesting inequality imposing restrictions on the Euler class of ξ when ξ is either a tight contact structures or a Reebless foliation. In this article we generalize this inequality to the case when ξ is a tight confoliations. This confirms conjecture 3.5.4. from [ET] and provides additional evidence that Definition 2 is the correct generalization of the notion of tightness to confoliations. Before we can state the inequalities mentioned above we need one more definition.

Definition 3. *Let γ be a nullhomologous knot in a confoliated manifold (M, ξ) which is transverse to ξ. For each choice F of an oriented Seifert surface of γ we define the* self linking number *$l(\gamma, \xi)$ of γ as follows. Choose a nowhere vanishing section X of $\xi|_F$ and let γ' be the knot obtained by pushing γ off itself by X. Then*

$$l(\gamma, F) = \gamma' \cdot F \ .$$

We orient γ using the coorientation of ξ (usually a coorientionof ξ is obtained from a form α such that $\ker(\alpha) = \xi$). In [Be] D. Bennequin proved the following inequality between $l(\gamma)$ and the Euler number $\chi(F)$ of F.

Theorem 4. *Let γ be a transverse knot in the standard contact structure $\xi = \ker(dz - y\,dx)$ on $\mathbb{R}^3$. Then*

$$l(\gamma) \leq -\chi(F). \tag{1}$$

Eliashberg showed in [El] that (1) holds for all tight contact structures and he also showed the other inequalities in the context of tight contact structures. On the other hand, it follows from Thurston's work in [Th] that the same inequalities hold for surfaces in foliations without Reeb components.

We discuss a possible generalization of these results to tight confoliations, for example we prove that for every embedded sphere in a manifold with tight confoliations satisfies $e(\xi)[F] = 0$.

References

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Thisd Schnepfenried geometry conference, Vol. 1 (1982), Astérisque 107–108, 87–161, Soc. Math. France, 1983.

[Co] V. Colin, *Recollement de variétés de contact tendues*, Bull. Soc. Math. France 127 (1999), no. 1, 43–69.

[El] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier 42, 1-2 (1992), 165-192.

[ET] Y. Eliashberg, W. Thurston, *Confoliations*, University Lecture Series Vol. 13.

[Gi1] E. Giroux, *Convexité en topologie de contact*, Comm. Math. Helv. 66 (1991), 637–677.

[Gi2] E. Giroux, *Structures de contact en dimension trois et biffurcations des feuilletages de surfaces*, Invent. Math. 141 (200), no. 3, 615–689.

[No] S. Novikov, *Topology of foliations*, Trans. of the Moscow Math. Soc. 14 (1963), 268-305.

[Th] W. Thurston, *Norm on the homology of 3-manifolds*, Memoirs of the AMS 339 (1986), 99-130.

Geometry of hyperbolic 3-manifolds and rank of the fundamental group

Juan Souto

(joint work with Ian Biringer)

We consider the relation between the geometry of closed orientable hyperbolic 3-manifolds and the rank of their fundamental group. Recall that the rank of a finitely generated group is the minimal number of elements needed to generate it. Recall also that the Heegaard genus $g(M)$ of a 3-manifold is the smallest genus of a Heegaard splitting of M.

Observe that for every 3-manifold M one has $rank(\pi_1(M)) \leq g(M)$. Waldhausen asked whether equality always holds and this question was answered in the negative by Boileau-Zieschang [1] who constructed examples with rank 2 and genus 3. Other examples are due to Schultens-Weidmann. However, the following conjecture is still possible, and in the opinion of the authors even plausible:

Conjecture. *For all k there is g_k such that for every hyperbolic 3-manifold M with $rank(\pi_1(M)) = k$ one has $g(M) \leq g_k$.*

We prove:

Theorem 1. *For all k and ϵ there is $g_{k,\epsilon}$ such that every ϵ-thick hyperbolic 3-manifold with $rank(\pi_1(M)) = k$ then $g(M) \leq g_{k,\epsilon}$.*

This result extends an unfortunately unavailable theorem of Agol in the case that $rank(M) = 2$. While Theorem 1 asserts that the Heegaard genus is bounded in terms of the rank and the injectivity radius, the following implies that the rank and the genus can only differ if the minimal genus Heegaard splittings of M have small distance in the sense of Hempel [2]:

Theorem 2. *For all g and ϵ there is $d_{k,\epsilon}$ such that every ϵ-thick hyperbolic 3-manifold M which admits a genus g Heegaard splitting with at least distance $d_{g,\epsilon}$ has $rank(\pi_1(M)) = g$.*

We deduce Theorem 1 and Theorem 2 directly from the first claim of the following result:

Theorem 3. *For all k and ϵ there is a finite collection of compact 3-manifolds $N_1, \ldots, N_r$ and a number n such that the following holds:*

- *Every closed hyperbolic 3-manifold with $inj(M) > \epsilon$ and $rank(\pi_1(M)) = k$ can be obtained by gluing along the boundary at most n of the manifolds $N_1, \ldots, N_r$.*
- *If (M_i, p_i) is a pointed sequence of pairwise distinct closed hyperbolic 3-manifolds with $inj(M_i) \geq \epsilon$ and $rank(\pi_1(M_i)) = k$ which converges geometrically to a manifold M_G, then M_G is homeomorphic to one of the manifolds N_i and every end of M_G is degenerate.*

Combining the second part of the main theorem with a result of Vigneras [3] we obtain the following finiteness results for arithmetic manifolds:

Theorem 4. *For all k and ϵ there are only finitely many closed, arithmetic, hyperbolic 3-manifolds M with $inj(M) \geq \epsilon$, $rank(\pi_1(M)) = k$ and $H_1(M; \mathbf{Z}) = 0$.*

Theorem 5. *For all k and ϵ there are only finitely many commensurability classes of closed, arithmetic, hyperbolic 3-manifolds M with*

$$inj(M) \geq \epsilon \text{ and } rank(\pi_1(M)) = k.$$

Before concluding, recall that the geometric form of the Lehmer conjecture asserts that there is some ϵ such that every closed arithmetic hyperbolic 3-manifold has at least injectivity radius ϵ. Should this conjecture hold, then the finiteness results of Theorem 3 and Theorem 4 depend only on the genus.

References

[1] M. Boileau, and H. Zieschang, *Heegaard genus of closed orientable Seifert* 3*-manifolds*, Invent. Math. **76** (1984), no. 3, 455–468.

[2] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology **40** (2001), no. 3, 631–657.

[3] M.-F. Vignéras, *Quelques remarques sur la conjecture* $\lambda_1 \geq 1/4$, Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), 321–343, Progr. Math., 38, Birkhäuser, 1983.

Extremal metrics on blow ups

Frank Pacard

(joint work with C. Arezzo, M. Singer)

In [3], [4] Calabi has proposed, as best representatives of a given Kähler class $[\omega]$ of a complex compact manifold (M, J), a special type of metrics (called *extremal metrics*) which are critical points of the L^2-square norm of the scalar curvature $\mathbf{s}$. The corresponding Euler-Lagrange equation reduces to the fact that the vector field

$$\Xi_{\mathbf{s}} := J\,\nabla \mathbf{s} + i\,\nabla \mathbf{s},$$

is a holomorphic vector field on M. The set of extremal metrics clearly contains the set of constant scalar curvature Kähler ones.

Let (M, J, g, ω) be a Kähler manifold with complex structure J and Kähler form ω and let g denote the metric associated to the Kähler form ω, so that

$$\omega(X, Y) = g(J\,X, Y).$$

We further assume that g is an extremal metric. Let K be a compact subgroup $K \subset \mathrm{Isom}(M, g)$ and let K_0 be the identity component of K. If $\mathfrak{k}$ denotes the Lie algebra associated to K_0, we assume that

$$J\,\nabla \mathbf{s} \in \mathfrak{k}.$$

Assume that we are given points

$$p_1, \ldots, p_n \in \mathrm{Fix}\,(K_0),$$

such that $\{p_1, \ldots, p_n\}$ is globally invariant under the action of K. In order to produce extremal metrics on the blow up of M at the points $p_1, \ldots, p_n$, we have to identify, among all C^∞ functions on the blown up manifold, those who generate real-holomorphic vector fields, since these can arise as scalar curvatures of extremal metrics. To this aim, we define $\mathfrak{h}$ to be the vector space of K-invariant Killing vector fields on M which vanish somewhere on M. The correspondence between the elements of $\mathfrak{h}$ and the scalar functions on M is given by the moment map

$$\xi_\omega \colon M \to \mathfrak{h}^*,$$

which is uniquely determined by the fact that the function $f := \langle \xi_\omega, X\rangle$ associated to the vector field $X \in \mathfrak{h}$ is the unique solution of

$$-\mathrm{d}f = \omega(X, -),$$

whose mean value over M is 0.

There is a natural orthogonal decomposition

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'',$$

where $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{k}$ and where the scalar product is taken to be

$$(X, \tilde{X})_{\mathfrak{h}} := \int_M \langle \xi_\omega, X\rangle\,\langle \xi_\omega, \tilde{X}\rangle\, dvol_g.$$

Using this setup, our main result reads :

Theorem 1. *Assume that :*

(i) *There exists $a_1, \ldots, a_n > 0$ satisfying*

$$\sum_{j=1}^{n} a_j^{m-1} \, \xi_\omega(p_j) \in \mathfrak{h}'^{\,*},$$

and $a_{j_1} = a_{j_2}$ if p_{j_1} and p_{j_2} belong to the same K-orbit.

(ii) *The projections of $\xi_\omega(p_1), \ldots, \xi_\omega(p_n)$ over $\mathfrak{h}''^{\,*}$ span $\mathfrak{h}''^{\,*}$.*

(iii) *There is no nontrivial element of $\mathfrak{h}''$ vanishing at $p_1, \ldots, p_n$.*

Then, there exists $\varepsilon_0 > 0$ and, for all $\varepsilon \in (0, \varepsilon_0)$, there exists a K-invariant extremal Kähler metric g_ε on $\tilde{M}$, the blow up of M at $p_1, \ldots, p_n$, whose associated Kähler form ω_ε lies in the class

$$\pi^*[\omega] - \varepsilon^2 \, \left(a_1 \, PD[E_1] + \ldots + a_n \, PD\,[E_n]\right).$$

Here $\pi \colon \tilde{M} \to M$ is the standard projection map and $PD[E_j]$ are the Poincaré duals of the $(2m-2)$-homology classes of the exceptional divisors of the blow up at p_j.

Finally, the sequence of metrics $(g_\varepsilon)_\varepsilon$ converges to g (in smooth topology) on compacts, away from the exceptional divisors.

Observe that, in the case where K is connected, and $\mathfrak{h} \subset \mathfrak{k}$ (so that $\mathfrak{h}' = \mathfrak{h}$ and $\mathfrak{h}'' = \{0\}$), conditions (i), (ii) and (iii) become vacuous.

Extremal *versus* **constant scalar curvature metrics** If the metric g we start with has constant scalar curvature, it might well be that the extremal metrics we obtain have in fact constant scalar curvature. There is a simple criterion involving the points $p_1, \ldots, p_n$ and the parameters $a_1, \ldots, a_n$, which ensures that this is not the case. Under the assumptions of Theorem 1, if we further assume that the metric g has constant scalar curvature and if the points and weights are chosen so that

$$\sum_{j=1}^{n} a_j^{m-1} \, \xi_\omega(p_j) \neq 0,$$

then the metrics we obtain on $\tilde{M}$ are extremal with non-constant scalar curvature. In [1] is treated the case where g is a constant scalar curvature Kähler metric, $K = \{Id\}$ and $\mathfrak{h} = \{0\}$, while in [2] is treated the case where g is a constant scalar curvature Kähler metric, K is a discrete subgroup of $\mathrm{Isom}\,(M, g)$, $\mathfrak{h}' = \{0\}$, and $\mathfrak{h}''$ is not necessarily trivial. Theorem 1 is therefore a generalization of the constructions given in [1] and [2].

The case of projective spaces When (M, J, g, ω) is the projective space $\mathbb{P}^m$ endowed with the Kähler form ω_{FS} associated to a Fubini-Study metric, we consider the group $K = S^1 \times \ldots \times S^1$, to be the maximal compact subgroup of $PGL(m+1)$, whose action is given by

$$\begin{array}{ccc} K \times \mathbb{P}^m & \longrightarrow & \mathbb{P}^m \\ \big((\alpha_1, \ldots, \alpha_{m+1}), [z^1 : \ldots : z^{m+1}]\big) & \longmapsto & [\alpha_1 z^1 : \ldots : \alpha_{m+1} z^{m+1}], \end{array}$$

(where $(z^1, \dots, z^{m+1})$ are complex coordinates in $\mathbb{C}^{m+1}$)and we consider the set of fixed points of K

$$p_1 := [1 : 0 : \dots : 0], \quad \dots, \quad p_{m+1} := [0 : \dots : 0 : 1].$$

In this case, the space $\mathfrak{h}$ is spanned by vector fields of the form $\Re\left(z^j\,\partial_{z^j} - z^k\,\partial_{z^k}\right)$, and we have $\mathfrak{k} = \mathfrak{h} = \mathfrak{h}'$ and $\mathfrak{h}'' = \{0\}$. As a consequence of the result of Theorem 1, we obtain extremal Kähler metrics on the blow up of $\mathbb{P}^m$ at the points $p_1, \dots, p_n$, for any $n = 1, \dots, m+1$. In addition, the Kähler metrics we obtain have non-constant scalar curvature if $n < m+1$.

The case corresponding to $n = 1$ was already obtained by Calabi [3] in more generality (i.e. for all Kähler classes) and the case where $\mathbb{P}^m$ is blown up at $m+1$ linearly independent points $q_1, \dots, q_{m+1}$ and $a_1 = \dots = a_{m+1}$ was already studied in [2] where constant scalar curvature metrics were obtained.

More generally, if (M, J, g, ω) is a m-dimensional toric variety whose associated metric is extremal, one can take K to be the maximal torus giving the torus action. In this case $\mathfrak{h} = \mathfrak{k}$, $\mathfrak{h}'' = \{0\}$, and it follows from Theorem 1 that, given $p_1, \dots, p_n \in \mathrm{Fix}\,(K)$, there exists an extremal Kähler metric on the blow up of M at $p_1, \dots, p_n$. Since blowing up a toric variety at such points preserves the toric structure, one can apply inductively Theorem 1 in this setting.

References

[1] C. Arezzo and F. Pacard, *Blowing up and desingularizing Kähler orbifolds with constant scalar curvature*, Acta Mathematica **196**, no 2, (2006) 179-228.

[2] C. Arezzo and F. Pacard, *Blowing up Kähler manifolds with constant scalar curvature II*, to appear in Annals of Math. math.DG/0504115.

[3] E. Calabi, *Extremal Kähler metrics*, Annals of Math. Studies, **102** , Princeton Univ. Press, 1982, 269-290.

[4] E. Calabi, *Extremal Kähler metrics II*, in Differ. Geometry and its Complex Analysis, edited by I. Chavel and H.M. Farkas, Springer, 1985.

[5] S. Simanca, *Kähler metrics of constant scalar curvature on bundles over* $C\mathrm{P}^{n-1}$, Math. Ann. **291** (1991), 239-246.

Kähler-Ricci flow: a finite dimensional approach

Julien Keller

In this talk we discuss a natural way to approach the Kähler-Ricci flow on a projective manifold M with $c_1(M) > 0$ or $c_1(M) < 0$. First, we give a natural discretization of the Kähler Ricci flow in the infinite dimensional world of Kähler potentials. This leads to define the operator on Kähler forms

$$Ric^{-1}$$

using Calabi-Yau Theorem [Ya]. The iterations of this operator leads us to consider a natural dynamical system, as in [Na]. For instance, we prove, using the decrease of the Mabuchi functional under an iteration, that one cannot expect non trivial periodic points Ric^{-1} when M is a Fano manifold. When M is a Kähler-Einstein Fano manifold then, under some strong assumptions, we can prove the convergence of these iterations towards the Kähler-Einstein metric. On the other hand, in the case of a Fano toric manifold we get the convergence under no extra-assumption towards a Kähler-Ricci soliton (X, ω_{KS}). Note that in that context, the holomorphic vector field X is unique and given in terms of the toric geometry. The Kähler-Ricci soliton ω_{KS} is unique up to holomorphic automorphisms and its existence has been proved in [WZ, Zh] by flow or continuity methods. We reprove Wang and Zhu's results by showing

Theorem 1. *Let M be a Fano toric manifold. We consider the sequence of metrics ω_j given by*

$$Ric(\omega_{j+1}) = L_X \omega_{j+1} + \omega_j$$

where ω_0 is any Kähler form invariant under the maximal compact subgroup of the acting torus on M. Then ω_j converges smoothly to a Kähler-Ricci soliton when $j \to +\infty$. In particular, if M is Kähler-Einstein, the sequence of metrics ω_j defined by

$$Ric(\omega_{j+1}) = \omega_j$$

converges to a Kähler-Einstein metric.

Denote $\mu = 1$ if $c_1(M) > 0$ and $\mu = -1$ if $c_1(M) < 0$. Now, using the notion of ν-balanced metrics introduced by Donaldson [Do3], we are able to derive a procedure in a finite dimensional setting that produces natural metrics on $H^0(M, K_M^{-\mu k})$ close to the Kähler-Einstein metric. More precisely, for k large enough and given $H_0 \in Met(H^0(K_M^{-k}))$, we can define a new metric on $H^0(K_M^{-\mu k})$, by

$$H_1(s_i, s_j) = \widetilde{T}(H_0)(s_i, s_j) = \int_M \frac{s_i \otimes \overline{s_j}}{(\sum_{i=1}^N S_i \otimes \overline{S_i})^{1+\mu/k}}$$

where $(S_i)_{i=1,..,N} \in H^0(K_M^{-\mu k})$ form an orthonormal basis with respect to H_0, and $N = \dim H^0(K_M^{-\mu k})$. The fixed points of the $\widetilde{T}$ operator (if they do exist) are called *canonically balanced* metrics. Let's now set $FS : Met(H^0(K_M^{-\mu k})) \to Met(K_M^{-\mu k})$ the Fubini-Study map.

Theorem 2. *Let M be a Kähler-Einstein manifold with $c_1(M) > 0$ and no non-trivial holomorphic vector field or $c_1(M) < 0$. Then there exists a sequence of canonically balanced metrics $\widetilde{H}_k \in Met(H^0(K_M^{-\mu k})$ and $c_1(FS(\widetilde{H}_k)^{1/k})$ converge to the Kähler-Einstein metric when $k \to \infty$. Furthermore, the operator $\widetilde{T}$ has an attractive fixed point.*

This is this technique that we use in order to compute an approximation of the Kähler-Einstein metric on the Fano toric manifold given by $\mathbb{CP}^2$ blown up in 3 (generic) points. From a technical point of view, this algorithm is particularly efficient.

Finally, a slight modification of our procedure in the case $c_1(M) < 0$ gives us back an iterative scheme studied recently by Tsuji [Ts]. In that case, we prove that the sequence of induced metrics has exponential speed of convergence (towards the Kähler-Einstein metric), improving slightly [SW].

In the case of Fano manifolds, we expect that the existence of canonically balanced metrics is related to an algebraic notion of stability in G.I.T sense of the manifold.

References

[Bi] O. Biquard, *Métriques kählériennes à courbure scalaire constante*, Astérisque, 938 (2004).

[Do1] S.K. Donaldson, *Scalar curvature and projective embeddings I*, J. Diff. Geom. 59 (2001).

[Do2] S.K. Donaldson, *Scalar curvature and projective embeddings II*, Quaterly Jour. Math 56 (2005).

[Do3] S.K. Donaldson, *Some numerical results in complex differential geometry*, arXiv (2005).

[Ma] T. Mabuchi, *K-energy maps integrating Futaki invariants*, Tôhuku Mathematical Journal 38 (1986).

[Na] A. Nadel, *On the Absence of Periodic Points for the Ricci Curvature Operator Acting on the Space of Kahler Metrics*, in Modern Methods in Complex Analysis: The Princeton Conference in Honor of Gunning and Kohn, Annals of Mathematics Studies 137 (1995).

[SW] J. Song & B. Weinkove, *Constructions of Kähler-Einstein metrics with negative scalar curvature*, preprint, arxiv:math.DG/0704.1005 (2007).

[Th] R. Thomas, *Notes on GIT and symplectic reductions for bundles and varieties*, Surveys in Differential Geometry, Vol. X (2006).

[Ti] G. Tian, *Canonical Metrics in Kähler Geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser, (2000).

[Ts] H. Tsuji, *Dynamical constructions of Kähler-Einstein metrics*, Arxiv:math.AG/0606626 (2006).

[WZ] X-J. Wang & X. Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Advances in Math. 188 (2004).

[Ya] S-T. Yau, *On the Ricci Curvature of a Compact Kähler manifold and the Complex Monge-Ampère equation*, I, Communications in Pure and Applied Mathematics 31 (1978).

[Zh] X. Zhu, *Kähler-Ricci flow on a toric manifold with positive first Chern class*, arXiv:math.DG/0703486 (2007).

Cross Ratios and Identities for Higher Thurston Theory

François Labourie

(joint work with Gregory Mc Shane)

1. Introduction

1.1. **Identities for lengths of simple closed geodesics in hyperbolic geometry:** In [9], the second author established an identity for lengths of simple closed geodesics on punctured hyperbolic surfaces. To simplify the exposition in this talk we consider the case where Σ denotes a complete hyperbolic surface with a single cusp. If C is a closed curve then we denote by $\ell(C)$ the infimum of the set of lengths of curves freely homotopic to C with respect to the hyperbolic metric; this extends naturally to finite set of curves. With this notation Mc Shane's identity for Σ with a single cusp is

$$1 = \sum_{P \in \mathcal{P}} \frac{1}{e^{\frac{\ell(\partial P)}{2}} + 1}, \tag{1}$$

M. Mirzakhani [10] extended this identity to hyperbolic surfaces with geodesic boundary : Let Σ be a complete hyperbolic surface with a single totally geodesic boundary component $\partial\Sigma$ then Mirzakhani's identities are

$$\ell(\partial\Sigma) = \sum_{P\in\mathcal{P}} \log\left(\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial\Sigma)}}{e^{\frac{\ell(\partial P)}{2}} + 1}\right), \tag{2}$$

where $\mathcal{P}$ is the set of embedded pants (with marked boundary) up to homotopy such that first the boundary component of the pair of pants is $\partial\Sigma$.

We show that the identity above has a natural formulation in terms of (generalised) cross ratios. Then, using this formulation, we study identities arising from the cross ratios constructed for representations in $PSL(n,\mathbb{R})$ by the first author [6].

1.2. **Cross ratio and periods:** Let Σ be a closed surface. Let $\partial_\infty\pi_1(\Sigma)$ be the boundary at infinity of the fundamental group $\pi_1(\Sigma)$ of Σ. A *cross ratio* on $\partial_\infty\pi_1(\Sigma)$ is a $\pi_1(\Sigma)$-invariant Hölder function on

$$\partial_\infty\pi_1(\Sigma)^{4*} = \{(x,y,z,t) \in \partial_\infty\pi_1(\Sigma)^4 \ x \neq t, \text{ and } y \neq z\},$$

satisfying some rules (compare with Otal's original definition in [11]), the most significant being the "multiplicative cocycle type" identities

$$b(x,y,z,t) = b(x,y,z,w)b(x,w,z,t), \qquad b(x,y,z,t) = b(x,y,w,t)b(w,y,z,t).$$

To every non trivial element γ of the group $\pi_1(\Sigma)$, we associate a positive number, $\ell_b(\gamma)$, called the *period* of γ defined by

$$\ell_b(\gamma) = \log b(\gamma^-, \gamma y, \gamma^+, y),$$

where γ^+ and γ^- are respectively the attractive and repulsive fixed points of γ in $\partial_\infty\pi_1(\Sigma)$ and where y is any point of $\partial_\infty\pi_1(\Sigma)$ such that $\gamma(y) \neq y$. Observe that a complete hyperbolic metric on Σ gives rise to an identification of $\partial_\infty\pi_1(\Sigma)$ with the real projective line, hence to a cross ratio on $\partial_\infty\pi_1(\Sigma)$ such that the period of γ is just the hyperbolic length of the closed geodesic freely homotopic to γ.

1.3. **Pant gap function and the generalized formula:** Given a cross ratio on $\partial_\infty\pi_1(\Sigma)$, we now define the *pant gap function* which takes a homotopy class of immersed pair of pants with marked boundary α to a positive number. If P is such a homotopy class of immersions of pants then, by considering three loops going round the boundary components of some representative, this corresponds to a triple (α,β,γ) of elements of $\pi_1(\Sigma)$. The triple is well defined up to conjugation and such that $\alpha\gamma\beta = 1$. We define the value of pant gap function at P to be the positive number

$$G_b(P) = \log(b(\alpha^+, \gamma^-, \alpha^-, \beta^+).$$

We shall prove

Theorem 1. *Let Σ be closed surface. Let b be a cross ratio on $\partial_\infty \pi_1(\Sigma)$. Let α be a non trivial element of $\pi_1(\Sigma)$ which corresponds to an essential separating closed curve. Let $\mathcal{P}$ be the space of homotopy classes of pair of pants with marked boundary in Σ whose first boundary component is α, then*

$$\ell_b(\alpha) = \sum_{P \in \mathcal{P}} G_b(P).$$

Moreover, the theorem generalizes to open surfaces of finite type after a suitable extension of the notion of cross ratio in this context. It also generalizes "at a cusp" in order to cover the case of Formula (1)

1.4. **Cross ratios and hyperbolic geometry:** The case of hyperbolic geometry is special in that the pant gap function can be computed in terms of the lengths of just the boundary components. Using Thurston's *shear coordinates* [2] and elementary manipulations involving the classical cross ratio – as opposed to hyperbolic trigonometry in the original proofs – we recover Mirzakhani-Mc Shane's formulae (1) and (2) for the pant gap function.

1.5. **Cross ratios and $PSL(n, \mathbb{R})$:** In [7], the first author gives an interpretation of *the Hitchin representations*, a connected component of the space of representations of the $\pi_1(\Sigma)$ in $PSL(n, \mathbb{R})$, as the space of cross ratios on $\partial_\infty \pi_1(\Sigma)$ satisfying an extra functional identity the form of which depends on n. Unfortunately, for $n \geq 3$ the pant gap function G_b is no longer only determined by the monodromies of three boundary components of the pants: it also depends on "internal parameters" which we describe in the following paragraphs.

1.6. **Hitchin representation for open surfaces.** In a series of articles [8], [6] and [7], the first author has shown that Hitchin representations are discrete and faithful, that every non trivial element is purely loxodromic. V. Fock and A. Goncharov [4] introduced coordinates – actually $(n!)^3$ set of coordinates – to describe a moduli space related to Hitchin representations which are far reaching generalizations of Thurston's shear coordinates. We then show that for a suitable choice of coordinates, the pant gap function has a nice expression. On the other hand, using a computer algebra software and the explicit description of the holonomies given by V. Fock and A. Goncharov in [5], we show that even in the case of $n = 3$, the pant gap function has a very complicated expression.

1.7. **Possible applications and conclusion:** Using her identities, M. Mirzakhani gives a recursive formula for the volume of moduli space of hyperbolic structure, *i.e* the quotient of Teichmüller space by the mapping class group. From the work of the author in [6], it follows that the mapping class group acts properly on the moduli space of Hitchin representations. It is quite possible that the formula obtained in Theorem 1 combined with the use of Fock-Goncharov coordinates can help to compute geometric quantities associated to the corresponding quotient.

However the volume is not the right thing to compute since for $n \geq 3$, one can show it is infinite.

We conclude by saying that it is a striking fact that so many of the familiar ideas from the world of hyperbolic geometry translate naturally to the world of Hitchin representations. So much so that one is tempted to call the latter *a higher (rank) Thurston theory.*

References

[1] Hirotaka Akiyoshi, Hideki Miyachi, and Makoto Sakuma, *Variations of McShane's identity for punctured surface groups*, Spaces of Kleinian groups, London Math. Soc. Lecture Note Ser., vol. 329, Cambridge Univ. Press, Cambridge, 2006, pp. 151–185.

[2] Francis Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. (6) **5** (1996), no. 2, 233–297.

[3] Brian H. Bowditch, *A proof of McShane's identity via Markoff triples*, Bull. London Math. Soc. **28** (1996), no. 1, 73–78.

[4] Vladimir V. Fock and Alexander B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Inst. Hautes Études Sci. Publ. Math. (2006), no. 103, 1–211.

[5] ______, *Moduli spaces of convex projective structures and surfaces*, Adv. in Math. **208** (2007), 249–273.

[6] François Labourie, *Cross ratios, anosov representations and the energy functional on teichmuller space.*, ArXiv math.DG/0512070, 2005.

[7] ______, *Cross ratios, surface groups, $sl(n,\mathbb{R})$ and diffeomorphisms of the circle*, ArXiv math.DG/0512070, 2005.

[8] ______, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

[9] Gregory McShane, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math. **132** (1998), no. 3, 607–632.

[10] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222.

[11] Jean-Pierre Otal, *Le spectre marque des longueurs des surfaces a courbure negative*, Ann. of Math. (2) **131** (1990), no. 1, 151–162.

[12] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *Mc shane's identities for classical schottky groups*, ArXiv: math.GT/0411628, 2004.

Foliated projective structures and the Hitchin component for PSL(4, R)

Anna Wienhard

(joint work with Olivier Guichard)

Let Σ be a connected oriented closed surface of genus $g \geq 2$. The Teichmüller space $\mathcal{T}(\Sigma)$ can be realized as the space of hyperbolic structures

$$\mathcal{T}(\Sigma) = \{(M,f) \text{ hyperbolic structure on } \Sigma\}/\sim,$$

where a hyperbolic structure (M,f) consists of a hyperbolic surface M and an orientation preserving homeomorphism $f : \Sigma \to M$. Two hyperbolic structures (M,f) and (M',f') on Σ are said to be *equivalent*, $(M,f) \sim (M',f')$, if there exists an isometry $i : M \to M'$ such that $i \circ f$ is isotopic to f'.

Associating to a hyperbolic structure (M, f) the holonomy homomorphism $f_* : \pi_1(\Sigma) \to \pi_1(M) < \mathrm{PSL}(2, \mathbf{R})$ defines an embedding of the Teichmüller space into the space of representations

$$\mathcal{T}(\Sigma) \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, \mathbf{R}))/\mathrm{PSL}(2, \mathbf{R}).$$

The image is a connected component which is homeomorphic to $\mathbf{R}^{6g-6}$ and consists entirely of discrete and faithful representations.

In [4] Hitchin discovered a special connected component, the "Teichmüller component", in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(n, \mathbf{R}))/\mathrm{PSL}(n, \mathbf{R})$. He showed that the Teichmüller component, now usually called "Hitchin component", is diffeomorphic to a ball of dimension $(2g-2)(n^2-1)$. Recently, Labourie [5] proved that the Hitchin component consists entirely of discrete and faithful representation which in addition are loxodromic.

Considering the nice properties of representations in the Hitchin component it is natural to ask whether there is a moduli space of geometric structures realizing the Hitchin component. We show in [3] that the Hitchin component for $\mathrm{PSL}(4, \mathbf{R})$ can indeed be interpreted as moduli space of certain locally homogeneous geometric structures.

Theorem 1. *The Hitchin component for* $\mathrm{PSL}(4, \mathbf{R})$ *is naturally homeomorphic to the moduli space of (marked) properly convex foliated projective structures on the unit tangent bundle of* Σ.

Convex foliated projective structures are locally homogeneous $(\mathrm{PSL}(4, \mathbf{R}), \mathbf{RP}^3)$-structures on the unit tangent bundle M of the surface Σ satisfying the following additional conditions:

- every orbit of the geodesic flow on M is locally a projective line,
- every stable leaf of the geodesic flow is locally a projective plane and the projective structure on the leaf obtained by restriction is convex.

There is a natural map from the moduli space of projective structures to the variety of representation $\pi_1(M) \to \mathrm{PSL}(4, \mathbf{R})$. The restriction of this map to the moduli space of properly convex foliated projective structures is a homeomorphism onto the Hitchin component; in particular, the holonomy representation of a properly convex foliated projective structure factors through the projection $\pi_1(M) \to \pi_1(\Sigma)$.

Our result relies on the following geometric characterization of representations inside the Hitchin component of $\mathrm{PSL}(n, \mathbf{R})$.

Theorem 2 (Labourie [5], Guichard [2])**.** *A representation* $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(n, \mathbf{R})$ *lies in the Hitchin component if and only if there exists a continuous* ρ*-equivariant convex curve* $\xi : \partial\pi_1(\Sigma) \to \mathbf{RP}^{n-1}$.

A curve $\xi : \partial\pi_1(\Sigma) \to \mathbf{RP}^{n-1}$ is said to be *convex* if for every n-tuple of pairwise distinct points in $\partial\pi_1(\Sigma)$ the corresponding lines are in direct sum. Convex curves into $\mathbf{RP}^2$ are exactly injective maps whose image bounds a strictly convex domain in $\mathbf{RP}^2$.

It is easy to prove that the existence of such a curve for $\mathrm{PSL}(2, \mathbf{R})$ implies that the representation is in the Teichmüller space. Let us indicate how this characterization implies a result of S. Choi and W. Goldman [1] that the representations in the Hitchin component for $\mathrm{PSL}(3, \mathbf{R})$ are precisely the holonomy representations of convex real projective structure on Σ.

A convex real projective structure on Σ is a pair (N, f), where N is a convex real projective manifold, that is N is the quotient Ω/Γ of a strictly convex domain Ω in $\mathbf{RP}^2$ by a discrete subgroup Γ of $\mathrm{PSL}(3, \mathbf{R})$, and $f : \Sigma \to N$ is a diffeomorphism. Given a representation $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(3, \mathbf{R})$ in the Hitchin component for $\mathrm{PSL}(3, \mathbf{R})$, let $\Omega_\xi \subset \mathbf{RP}^2$ be the strictly convex domain bounded by the convex curve $\xi^1(\partial\pi_1(\Sigma)) \subset \mathbf{RP}^2$. Then $\rho(\pi_1(\Sigma))$ is a discrete subgroup of the group of Hilbert isometries of Ω_ξ and hence acts freely and properly discontinuously on Ω_ξ. The quotient $\Omega_\xi/\rho(\pi_1(\Sigma))$ is a real projective convex manifold, diffeomorphic to Σ. Conversely given a real projective structure on Σ, we can ρ-equivariantly identify $\partial\pi_1(\Sigma)$ with the boundary of Ω and get a convex curve $\xi^1 : \partial\pi_1(\Sigma) \to \partial\Omega \subset \mathbf{RP}^2$.

To associate a geometric structure to a representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(4, \mathbf{R})$ lying in the Hitchin component we consider domains of discontinuity for the action of $\rho(\pi_1(\Sigma))$ on $\mathbf{RP}^3$, similar to Ω_ξ for $\mathrm{PSL}(3, \mathbf{R})$. For this it is useful to consider the surface Δ formed by the tangent lines of the convex curve $\xi : \partial\pi_1(\Sigma) \to \mathbf{RP}^3$. The complement $\mathbf{RP}^3 - \Delta$ decomposes into two connected components Ω and Λ. The action of $\rho(\pi_1(\Sigma))$ on Ω and on Λ is properly discontinuous. The quotient $\Omega/\rho(\pi_1(\Sigma))$ is a projective manifold homeomorphic to the unit tangent bundle M of Σ and induces a properly convex foliated projective structure on M.

This construction gives rise to a map from the Hitchin component to the moduli space of properly convex foliated structures on M. The proof of the converse direction is more involved. Starting with a properly convex foliated projective structure on M we construct an equivariant convex curve and show that the projective structure is obtained by the above construction.

We expect that the interpretation of the Hitchin component for $\mathrm{PSL}(4, \mathbf{R})$ in terms of geometric structures can be used to obtain Fenchel-Nielsen type coordinates for the Hitchin component. We also hope that the geometric description will be helpful to understand the structure of the quotient of the Hitchin component by the mapping class group. Conjecturally this quotient is a vector bundle over the Riemannian moduli space. For $\mathrm{PSL}(3, \mathbf{R})$ this was proven by Labourie [6] (see also Loftin [7]) using the geometric description of Choi and Goldman.

References

[1] S. Choi and W. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 657–661.

[2] O. Guichard, *Composantes de Hitchin et représentations hyperconvexes de groupes de surface*, to appear in J. Differential Geometry.

[3] O. Guichard and A. Wienhard, *Convex Foliated Projective Structures and the Hitchin component for* $\mathrm{PSL}(4, \mathbf{R})$, preprint 2007.

[4] N. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473.

[5] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.
[6] F. Labourie, *Flat projective structures on surfaces and cubic holomorphic differentials*, Pure and Applied Mathematics Quaterly (To be published in a special issue dedicated to G. Margulis).
[7] J. Loftin, *Affine Spheres and Convex* $\mathbf{RP}^n$*-Manifolds*, Amer. J. Math. **123** (2001), no. 2, 255–274.

Obstructions to positive scalar curvature using codimension 2 submanifolds

Thomas Schick
(joint work with Bernhard Hanke)

Let M be a closed smooth spin manifold of dimension m. There are important obstructions to the existence of a Riemannian metric with positive scalar curvature on M, based on the Dirac operator, the Weitzenböck formula and index theory.

Gromov and Lawson define a particular such obstruction, based on submanifolds of codimension 2, which applies in particular to circles in 3-manifolds:

Theorem 1 (Gromov/Lawson)**.** *Assume that M is an aspherical closed spin manifold, $N \subset M$ is a submanifold of codimension 2 with trivial normal bundle and $\pi_1(N)$ injects into $\pi_1(M)$ as a proper subgroup.*

If N is enlargeable, then M does not admit a metric with positive scalar curvature.

Here, by definition a manifold is enlargeable, if it admits a sequence of coverings M_k, together with maps $f_k\colon M_k \to S^n$ which have non-zero degree but are $1/k$-contracting (i.e. the differential at every point has norm $\leq 1/k$, for metrics which are pulled back from a fixed Riemannian metric on M).

On the other hand, a very powerful obstruction among the "Dirac operator obstructions" is the K-theoretic index in the (real) K-theory of the (real) maximal C^*-algebra of the fundamental group, $\alpha_{max}(M) \in KO^m(C^*_{max}\pi_1(M))$, a certain completion of the group ring $\mathbb{R}\pi_1 M$, constructed in [8, 6, 7].

Question: How much information is contained in $\alpha_{max}(M)$. In particular, what is the relation between the codimension 2-obstructions and $\alpha(M)$. Is $\alpha(M) \neq 0$ for manifolds to which the codimension-2-method applies?

The goal of the talk is, to present a generalization of the theorem of Gromov-Lawson, which at the same time is much more in the spirit of the index theory obstruction $\alpha(M)$. However, this new formulation is still not strong enough to give an answer to the question above.

Theorem 2. *Assume that $W \subset M$ is a submanifold of codimension 0 with boundary ∂M. Assume furthermore*

(1) $\pi_0(\partial W) = 0$
(2) $\pi_2(M) = 0$
(3) $\pi_1(W) \hookrightarrow \pi_1(M)$ *is injective, but the image has infinite index in* $\pi_1(M)$*.*

(4) *For* $i_*\colon \pi_1(\partial W) \to \pi_1(W)$*, the composition* $\ker(i_*) \hookrightarrow \pi_1(\partial W) \to H_1(\partial W)$ *is injective.*

In this situation, if $scal_M > 0$*, then the Rosenberg index*

$$ind(D_{\partial W, C^*\pi_1 \partial W}) = 0 \in K_*(C^*\pi_1(\partial W)).$$

Given N and M as in the theorem of Gromov-Lawson (and N connected), we let W be the closed normal bundle of N, i.e. $W \cong N \times D^2$, with $\partial W = N \times S^1$. Then all the conditions are satisfied, and therefore the result of Gromov-Lawson is a special case of our theorem.

We now outline the proof of the theorem.

First, let $p\colon \overline{M} \to M$ be the covering of M with fundamental group $\pi_1(W)$. This implies that the inverse image $p^{-1}(W)$ is the disjoint union $\pi_1(M)/\pi_1(W) \times W$ (note that the covering will in general not be a normal covering with transitive action of the deck transformation group on the fibers). Fix one component/copy $\{1\} \times W \subset \overline{M}$ of $p^{-1}(W)$, and set $\overline{X} := \overline{M} \setminus \{1\} \times W$. Consequently, $\partial \overline{X} = \partial W$.

Lemma 3. *The inclusion induces a* split *injection* $\pi_1(\partial W) \hookrightarrow \pi_1(\overline{X})$.

Consider now the manifold $\overline{Y} := \overline{X} \cup_{\partial W} \overline{X}$. By the van Kampen theorem and the above Lemma, we again have a split injection $\pi_1(\overline{Y}) \hookrightarrow \pi_1(\partial W) \xrightarrow{s} \pi_1(\partial W)$, the split being compatible with both inclusions $\overline{X} \hookrightarrow \overline{Y}$.

We put a complete Riemannian metric on $\overline{Y}$ by lifting the metric of M and smoothing out in a (compact) neighborhood of ∂W. In particular, if M has a metric with scal > 0, then the scalar curvature of $\overline{Y}$ is uniformly positive outside a compact neighborhood of ∂W.

Using a suitable twist bundle for the Dirac operator on $\overline{Y}$ and a suitable coarse C^*-algebra $C^*(\overline{Y}, \pi)$, we define a Roe index

$$ind(D_{\overline{Y},\pi}) \in K_*(C^*(\overline{Y}, \pi)).$$

This is inspired by [5].

If the Riemannian metric on $\overline{Y}$ has uniformly positive scalar curvature outside a compact subset of $\overline{Y}$, then $ind(D_{\overline{Y},\pi}) = 0$

Theorem 4. *In the given twisted situation, there is a partitioned manifold index theorem (like in* [5]*), with a (boundary) map* $K_*(C^*(\overline{Y}, \pi)) \to K_{*-1}(C^*\pi)$ *given by the partition, and this map sends* $ind(D_{\overline{Y},C^*\pi})$ *to* $ind(D_{\partial W}, C^*\pi)$.

Under our conditions on the homotopy groups, $j^*\pi = \pi_1(\partial W)$, so that the image under the boundary map is exactly the Mishchenko-Fomenko index

$$ind(D_{\partial W, \pi_1(\partial W)}) \in K_{*-1}(C^*\pi_1(\partial W)).$$

As a corollary, if N is a closed spin manifold with $ind(D_N, \pi_1(N)) \neq 0 \in K_{*-2}(C^*\pi_1(N))$ then no N-bundle over a surface different from S^2 or $\mathbb{R}P^2$ admits a Riemannian metric with scal > 0.

By the prove of the stable Gromov-Lawson-Rosenberg conjecture due to Stephan Stolz [10], a manifold with $\alpha(M) = 0$ but to which the obstruction of our main theorem would apply is a counterexample to the strong Novikov conjecture, and therefore is probably hard to come by, a counterexample to the unstable conjecture like [9, 1] can never work.

Other index theoretic obstructions to positive scalar curvature based on enlargeability indeed are contained in $\alpha(M)$, as is shown in [3, 4, 2].

Question.

(1) Is it possible to work with submanifolds of even higher codimension (using more conditions on the ambient manifold, e.g. on vanishing of higher homotopy groups)?
(2) In some sense, the proof of the theorem uses the ends of suitable coverings of the manifold in question and pins down the obstruction to positive scalar curvature near those. One should try to formalize this and consider this approach much more systematically.

References

[1] William Dwyer, Thomas Schick, and Stephan Stolz. Remarks on a conjecture of Gromov and Lawson. In *High-dimensional manifold topology*, pages 159–176. World Sci. Publishing, River Edge, NJ, 2003.

[2] Bernhard Hanke, Dieter Kotschick, John Roe, and Thomas Schick. Coarse topology, enlargeability, and essentialness. arXiv:0707.1999.

[3] Bernhard Hanke and Thomas Schick. Enlargeability and index theory. *J. Differential Geometry*, 74(2):293–320, 2006. arXiv:math.GT/0403256.

[4] Bernhard Hanke and Thomas Schick. Enlargeability and index theory: infinite covers. arXiv:math.GT/0604540, to appear in K-theory, 2006.

[5] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[6] J. Rosenberg. C^*-algebras, positive scalar curvature and the Novikov conjecture. II. In *Geometric methods in operator algebras (Kyoto, 1983)*, volume 123 of *Pitman Res. Notes Math. Ser.*, pages 341–374. Longman Sci. Tech., Harlow, 1986.

[7] Jonathan Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212 (1984), 1983.

[8] Jonathan Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. III. *Topology*, 25(3):319–336, 1986.

[9] Thomas Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998.

[10] Stephan Stolz. Manifolds of positive scalar curvature. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 661–709. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

Localization of Perelman's reduced volume monotonicity

Klaus Ecker

(joint work with Dan Knopf, Lei Ni and Peter Topping ([EKNP]))

We consider a time-dependent Riemannian manifold $((M, g(t))$ (closed or complete) satisfying

$$\frac{\partial g}{\partial t} = 2h$$

and two functions $u, \Psi : M \times I \to \mathbf{R}$.

For $r > 0$ we denote by E_r the set in $M \times I$ where $\Psi > r^{-n}$. We prove that if the functions u and Ψ satisfies certain conditions which ensure that all expressions appearing below are well-defined such as for instance relative compactness of the sets E_r for small enough r then the following formula holds:

$$\frac{d}{dr}\left(\frac{1}{r^n}\int_{E_r}\left(|\nabla \log \Psi|^2 - (tr_g h)\log(r^n\Psi)\right) u \, d\mu_t \, dt\right)$$
$$= -\frac{n}{r^{n+1}}\int_{E_r}\left(\log(r^n\Psi)\left(\frac{\partial}{\partial t} - \Delta\right) u + \Psi^{-1} u \left(\frac{\partial}{\partial t} + \Delta + tr_g h\right)\Psi\right) d\mu_t \, dt$$

Integrating with respect to r leads to a local mean value formula localising for example representation formulas the solutions of the heat equation.

Admissible functions Ψ are for instance fundamental solutions for the backward heat operator on time-dependent manifolds. This generalizes results of Watson ([W]), Evans and Gariepy ([EG]) in the case of Euclidean space with the standard metric and of Fabes and Garofalo ([FG]) in the case of Riemannian manifolds with a fixed metric and for Ψ being the backward heat kernel.

In the case of mean curvature flow this formula was obtained in [E], localizing Huisken's monotonicity formula in [H]. The above formula also applied to Ricci flow, that is for $h = -Ric$ so that $tr_g h = -R$, the scalar curvature of g. In fact, let ℓ be Perelman's space-time distance function see [P] with respect to some base point $(x_0, t_0) \in M \times I$. The quantity

$$\Psi = v = \frac{e^{-\ell}}{(4\pi\tau)^{\frac{n}{2}}}$$

then satisfies the inequality

$$\left(\frac{\partial}{\partial t} + \Delta - R\right) v \geq 0.$$

Inserting this and $u \equiv 1$ into the above formula leads to a local version of Perelman's reduced volume monotonicity formula. Moreover, taking as u the scalar curvature which satisfies the equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) R = 2|Ric|^2$$

we obtain (after integration with respect to r) the Harnack type inequality

$$R(x_0, t_0) \geq \frac{1}{r^n} \int_{E_r} \left(|\nabla \ell|^2 + R \log(r^n v)\right) \, d\mu_t \, dt + \int_0^r \frac{2n}{s^{n+1}} \int_{E_s} \log(s^n v) |Ric|^2 \, d\mu_t \, dt \, ds$$

as long as $R \geq 0$ in the set $E_r = \{\log(r^n v) > 0\}$.

References

[E] K. Ecker, A local monotonicity formula for mean curvature flow, Ann. Math. **154** (2001) 503-525

[EKNP] K. Ecker, D. Knopf, L. Ni and P. Topping, Local monotonicity and mean value formulas for evolving metrics, to appear in Journal für die Reine und Angewandte Mathematik

[EG] L.C. Evans, R.F. Gariepy, Wiener's criterion for the heat equation, Arch. Rational. Mech. Anal. **78** (1982), 293-314

[FG] E.B. Fabes, N. Garofalo, Mean value properties of solutions to parabolic equations with variable coefficients, J. Math. Anal. appl. **121** (1987), 305-316

[H] G. Huisken, Asymptotic behavior for singularities of mean curvature flow, J. Diff. Geom. **31** (1990), 285-299

[P] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arxiv:math.DG/0211159

[W] N.A. Watson, A theory of subtemperatures in several variables, Proc. London. Math. Soc. **26** (1973) 385-417

Reporter: Thomas Vogel

Participants

Prof. Dr. Bernd Ammann
Universite Henri Poincare
Nancy 1
Institut Elie Cartan
F-54506 Vandoeuvre-les-Nancy

Prof. Dr. Christian Bär
Fachbereich Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Prof. Dr. Werner Ballmann
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Richard Bamler
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Laurent Bessieres
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Gerard Besson
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Olivier Biquard
I.R.M.A.
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg -Cedex

Dr. Christoph Böhm
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Marc Bourdon
Mathematiques
UMR 8524 CNRS
Universite de Lille 1
F-59655 Villeneuve d'Ascq.

Prof. Dr. Emmanuel Breuillard
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Prof. Dr. Vincent Colin
Laboratoire de Mathematiques
Universite de Nantes
2 rue de la Houssiniere
F-44322 Nantes Cedex 03

Prof. Dr. Thomas C. Delzant
Institut de Recherche
Mathematique Avancee
ULP et CNRS
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Jonathan Dinkelbach
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Klaus Ecker
Fachbereich Mathematik
und Informatik
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin

Dr. Thomas Foertsch
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Dr. Nicolas Ginoux
Institut für Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Galina Guzhvina
Fachbereich 10 - Mathematik und
Informatik
Universität Münster
Einsteinstr. 62
48149 Münster

Florian Hanisch
Institut für Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Dr. Mark Haskins
Department of Pure Mathematics
Imperial College London
South Kensington Campus
GB-London SW7 2AZ

Hans-Joachim Hein
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Van Le Hong
Mathematical Institute
ASCR
Zitna 25
115 67 Praha 1
Czech Republic

Prof. Dr. Gerhard Huisken
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Julien Keller
Imperial College London
Institute for Mathematical Sciences
53 Prince's Gate
GB-London SWJ 2PG

Prof. Dr. Marcus Khuri
Institute of Mathematical Science
SUNY at Stony Brook
Stony Brook , NY 11794-3660
USA

Prof. Dr. Bruce Kleiner
Yale University
Mathematics Department
PO Box 208283
New Haven , CT 06520-8283
USA

Robert Kremser
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Francois Labourie
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Bernhard Leeb
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Joachim Lohkamp
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. John Lott
Department of Mathematics
University of Michigan
East Hall, 525 E. University
Ann Arbor , MI 48109-1109
USA

Dr. Alexander Lytchak
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Prof. Dr. Sylvain Maillot
I.R.M.A.
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg -Cedex

Vincent Minerbe
Mathematiques
Universite de Nantes
2, Chemin de la Houssiniere
F-44072 Nantes Cedex 03

Dr. Olaf Müller
Instituto de Matematicas UNAM
Antigua Carretera a Patzcuaro Km 8
C.P. 58190
Morelia
Michoacan
MEXICO

Dr. Ignasi Mundet i Riera
Facultat de Matematiques
Universitat de Barcelona
Departament d'Algebra i Geometria
Gran Via 585
E-08007 Barcelona

Prof. Dr. Frank Pacard
Centre de Mathematiques
Faculte de Sciences et Technologie
Universite Paris XII - Val de Marne
61 Avenue du General de Gaulle
F-94010 Creteil Cedex

Carlos Ramos Cuevas
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Yann Rollin
Dept. of Mathematics
Imperial College of Science and
Technology, Huxley Building
180 Queen's Gate
GB-London SW7 2AZ

Prof. Dr. Thomas Schick
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Prof. Dr. Viktor Schroeder
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Juan Souto
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Prof. Dr. Gudlaugur Thorbergsson
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln

Dr. Thomas Vogel
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Gregor Weingart
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Dr. Hartmut Weiss
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Stefan Wenger
Courant Institute
Warren Weaver Hall
251 Mercer Street
New York N.Y. 10012
USA

Dr. Anna Wienhard
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago , IL 60637-1514
USA

Prof. Dr. Burkhard Wilking
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Andreas Wörner
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster