MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 44/2007

Nonlinear Waves and Dispersive Equations

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September 9th – September 15th, 2007

ABSTRACT. The aim of the workshop was to discuss current developments in nonlinear waves and dispersive equations from a PDE based view. Asymptotic properties of solutions (including multi soliton solutions und singular solutions), the initial value problem in critical spaces and dispersive estimates for linear equations with variable coefficients were the central topics of the workshop.

Mathematics Subject Classification (2000): 35xx.

Introduction by the Organisers

Dispersive equations occur as asymptotic models for the propagation of linear and nonlinear waves. Mathematically they display an interplay between linear dispersion and nonlinear focusing and defocusing effects. They are linked to diverse areas of mathematics and physics, ranging from nonlinear optics over oscillatory integrals to integrable systems. The workshop focused on a PDE based approach to dispersive equations.

Current activities covered by the workshop include:

(1) Progress in understanding critical nonlinear Schröder equations. Generalized Morawetz estimates and an induction on energy improved our understanding on the interaction of waves for nonlinear Schrödinger equations, which led to a treatment of many such critical equations.

(2) The study of dispersive equations with variable coefficients. During the last few years our understanding of wave propagation in nonhomogeneous situations improved considerably and there are now several interesting results on general relativity linearized at the Schwarzschild metric. This may be seen as a step towards nonlinear stability of the Schwarzschild metric.

(3) The large time asymptotics of solutions, including asymptotics of multisoliton solutions in the nonintegrable case and sharp asymptotics of singular solutions.

There is a large number of promising young mathematicians working in this area. The meeting was attented by 45 participants. The organizers gave a strong preference to talks by young researchers.

Workshop: Nonlinear Waves and Dispersive Equations

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Abstracts

Scattering for the Gross-Pitaevskii equation
Kenji Nakanishi

(joint work with Stephen Gustafson and Tai-Peng Tsai)

We study asymptotic behavior of solutions for the nonlinear Schrödinger equation:

$$(1) \qquad i\psi_t + \Delta\psi = \pm|\psi|^2\psi, \quad \psi(t,x) : \mathbb{R}^{1+d} \to \mathbb{C},$$

as $t \to \infty$. There is a vast literature on this problem especially in the scattering theory when the solution satisfies

$$(2) \qquad \psi(t,x) \to 0 \quad (|x| \to \infty),$$

in such a sense as $\psi(t) \in L_x^2(\mathbb{R}^d)$ or $H_x^1(\mathbb{R}^d)$. Since the spatial decay is intimately connected with the time decay through the dispersive nature of the equation, it is often assumed, to make the problem easier, that the solutions or the initial data decay sufficiently fast at the spatial infinity. That may be natural if one regards the equation as a model to describe instantaneous reaction to spatially localized disturbance, where the time and space infinity refers in reality to the region far away from the disturbance.

But why should one start with the 0 solution to perturb? The second simplest and natural, but more general choice is the plane wave, which is also uniform in space-time with constant $|\psi|$. Then the boundary condition for perturbed solutions becomes

$$(3) \qquad |\psi(t,x)| \to C \neq 0 \quad (|x| \to \infty).$$

In fact, it is more natural than the zero boundary condition in various physical contexts such as superfluid, Bose-Einstein condensation and nonlinear optics.

By using the scaling and Galilean invariance of the equation, we can reduce the question to the case $\psi = e^{\mp it}$. Furthermore, by changing $\psi \mapsto e^{\mp it}\psi$ we can reduce to the case $\psi = 1$ for the equation

$$(4) \qquad i\psi_t + \Delta\psi = \pm(|\psi|^2 - 1)\psi, \quad \psi(t,x) : \mathbb{R}^{1+d} \to \mathbb{C},$$

with the boundary condition

$$(5) \qquad |\psi(t,x)| \to 1 \quad (|x| \to \infty).$$

This equation is often called Gross-Pitaevskii equation in those physical contexts. By decomposing the solution into the background constant and the perturbation $\psi = 1 + u$, we get

$$(6) \qquad iu_t + \Delta u \mp 2u_1 = \pm(2|u|^2 + u^2 + |u|^2 u),$$

where $u = u_1 + iu_2$, $(u_1, u_2) \in \mathbb{R}^2$.

Now it is easy to observe in the focusing case (the $-$ sign), the linearized operator has exponentially growing mode and so there is no chance to have any

stability. Hence we are naturally forced to assume the defocusing nonlinearity with the sign $+$ in the rest. By further change of variable

$$(7) \qquad\qquad u = Uv_1 + iv_2, \quad U := \sqrt{-\Delta/(2-\Delta)},$$

we can transform the linear part into a complex linear form. Then we have the equation for v

$$(8) \qquad\qquad iv_t - Hv = 3u_1^2 + u_2^2 + |u|^2 u_1 + iU^{-1}(2u_1 u_2 + |u|^2 u_2),$$

where $H = \sqrt{-\Delta(2-\Delta)}$.

Then it seems natural to approximate the solution by the linear evolution e^{-iHt}. For large time asymptotic, it means the scattering problem for the equation of v. This problem appears to have several difficulties compared with the zero boundary condition.

(1) The linear part contains a nonlocal operator, which is inconvenient to exploit spatial decay property.
(2) The nonlinearity contains quadratic terms which are supposed to decay slower than the cubic ones. When considering perturbation of solitary wave (with the zero boundary condition), we also get quadratic terms, but they are usually easy to treat because they are multiplied with the solitary wave, which decays exponentially in space. That is not the case for our background solution with no decay.
(3) Some nonlinear terms contain the operator U^{-1}, which is singular at the zero frequency. It is especially bad for time decay, because those quadratic terms are resonant mostly at the zero frequency. Namely the operator would enhance those interactions which does not decay well.
(4) Our boundary condition destroys those invariance such as scaling and Galilean, which play important roles to derive a priori dispersive estimates in the zero boundary condition.

In addition, one should keep in mind that there exist global solutions which do not disperse, such as traveling waves and stationary vortexes.

Despite of those apparent difficulties, we have succeeded in getting scattering results for $d \geq 2$ in [5, 6]. Roughly speaking, those results are as follows:

(1) For $d \geq 4$, if the initial data is sufficiently small in $H^{d/2-1}$, then the solution v approaches a linear solution $e^{-iHt}\varphi$ in the same space as $t \to \infty$. one can also start with a prescribed asymptotic φ.
(2) For $d = 3$, we can construct unique solution v which is asymptotic to a given profile $e^{-iHt}\varphi$ if $\varphi \in H^1 \cap H_p^1$ with some $p < 3/2$.
(3) For $d = 2$, we can construct global solution for a prescribed asymptotic profile, if φ is sufficiently smooth and decays fast, but the asymptotic behavior should be modified as follows:

$$(9) \qquad v + H^{-1}|u|^2 \sim v_+ - i \int_\infty^t e^{-iH(t-s)} |Uv_+(s)|^2 ds, \quad v_+ = e^{-iHt}\varphi.$$

Those results are based on the nonlinear transform

$$(10) \qquad z = U^{-1}u_1 + H^{-1}|u|^2 + iu_2,$$

which magically removes the singularity and further brings decay at $\xi = 0$ in all the quadratic terms. Actually one can observe that the above transform is somehow canonical in view of the conserved energy:

$$(11) \qquad
\begin{aligned}
&\int |\nabla\psi|^2 + \frac{(|\psi|^2 - 1)^2}{2}\,dx \\
&= \int |\nabla u|^2 + 2|u_1|^2 + 2|u|^2 u_1 + \frac{|u|^4}{2}\,dx \\
&= \int |\nabla u_2|^2 + \left[\sqrt{2 - \Delta}\,u_1 + \frac{|u|^2}{\sqrt{2 - \Delta}}\right]^2 - \left[\frac{|u|^2}{\sqrt{2 - \Delta}}\right]^2 + \frac{|u|^4}{2}\,dx \\
&= \int |\nabla z|^2 + \frac{(U|u|^2)^2}{2}\,dx.
\end{aligned}$$

Thus the linear transform $u \mapsto v$ is the unique one which removes the quadratic part in the nonlinear energy, and the quadratic transform $u \mapsto z$ is the unique one which removes the cubic term as well. In particular, finite energy implies uniform bound in $\dot{H}^1$ of z, but not of v. Using this fact, we can prove

Theorem 1. *Let $d = 3$. Then for any $\varphi \in \dot{H}^1$, there exists a global solution $\psi = 1 + u$ of (4) such that $z = U^{-1}u_1 + H^{-1}|u|^2 + iu_2$ satisfies*

$$(12) \qquad \lim_{t\to\infty} \|z(t) - e^{-iHt}\varphi\|_{\dot{H}^1} = 0.$$

The same asymptotic does not necessarily hold for v, because $H^{-1}|u|^2 \notin \dot{H}^1$ for general finite energy solutions. We cannot claim the uniqueness of z for given φ, because we use the compactness argument.

There is a lower bound on the energy of traveling waves on $\mathbb{R}^3$ [1], and it is conjectured [2] that solutions with energy below that bound should disperse for large time. The above theorem suggests a natural embodiment for it, namely the scattering in $\dot{H}^1$ for z, showing at least that there are plenty of solutions with such asymptotic behavior.

The proof is based on a uniform estimate on the nonlinearity in the equation

$$(13) \qquad iz_t - Hz = 2u_1^2 + |u|^2 u_1 - 4iH^{-1}\nabla \cdot (u_1\nabla u_2) + iU(|u|^2 u_2)$$

in the sum of Lorentz spaces

$$(14) \qquad L_x^{6/5,1}(\mathbb{R}^3) + L_x^1(\mathbb{R}^3),$$

and the decay estimates

$$(15) \qquad \|e^{-iHt}\varphi\|_{B^0_{\infty,2}} \lesssim |t|^{-3/2}\|\varphi\|_{B^0_{1,2}}, \quad \|e^{-iHt}\varphi\|_{L^1_t(L^\infty_x + L^1_x)} \lesssim \|\varphi\|_{L^{6/5,1}},$$

combined with the compactness argument using the conservation law to derive the strong convergence (cf. [7] for the zero boundary condition.) More precisely, we take the weak limit $T \to \infty$ of the sequence of solutions z_T with the initial

data $z_T(T) = e^{-iHT}\varphi$, for which we may assume smallness of the L_x^6 norm, for appropriate sequence of T, to ensure invertibility of the transform $u \mapsto z$. We use the global wellposedness in the finite energy class [3]. The strong convergence follows from the convergence of the energy as $T \to \infty$ and $t \to \infty$.

Surprisingly, a similar result for the zero boundary condition is not available in the same setting, because in that case the L_x^2 bound (conservation) plays an essential role (cf. [4] for the definitive result in H_x^1). As far as we know, there is no scattering result in that case assuming only the finiteness of energy, even starting from the asymptotic data.

In this sense, we might expect that the nonzero constant has stronger stability than the trivial solution. However proving the above conjecture looks extremely difficult, for example from the scaling speculation. It seems more feasible and still interesting to get a scattering result starting from initial data with some spatial decay in three dimensions.

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Strichartz estimates on Schwarzschild space-times

Jason Metcalfe

(joint work with Daniel Tataru, Mihai Tohaneanu)

Two of the more robust ways of measuring dispersion for solutions to the wave equation are the localized energy estimates and the Strichartz estimates. For the wave equation on Minkowski space-time $\mathbb{R} \times \mathbb{R}^n$, these say, respectively, that solutions u to $\Box u = (\partial_t^2 - \Delta)u = 0$ satisfy

$$(1) \qquad \sup_j \|\langle x \rangle^{-1/2} \nabla u\|_{L_{t,x}^2(\mathbb{R} \times \{|x| \approx 2^j\})} \lesssim \|\nabla u(0, \cdot)\|_2, \quad n \geq 3$$

and

$$(2) \qquad \||D|^{-\rho} \nabla u\|_{L_t^p L_x^q(\mathbb{R}_+ \times \mathbb{R}^n)} \lesssim \|\nabla u(0, \cdot)\|_2.$$

For the latter, we require that (p, q, ρ) be Strichartz admissible which we define to mean that $2 \leq p, q \leq \infty$,

$$(3) \qquad \rho = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}, \quad \frac{2}{p} \leq \frac{n-1}{2}\left(1 - \frac{2}{q}\right),$$

and $(p, q, \rho) \neq (2, \infty, 1)$ when $n = 3$. We say that the Strichartz exponents are sharp if equality holds in the second equation in (3).

For simplicity of exposition, we shall only explore estimates for solutions to homogeneous equations here. In the flat case, there are also well known estimates for the inhomogeneous equation where the forcing term is in a dual space. The estimates below also have similar analogs.

We wish to explore to what extent these estimates carry over to variable coefficient settings and, in particular, to Schwarzschild space-times. While Strichartz estimates for variable coefficient wave equations are known to hold locally in time, there is still relatively little known concerning global estimates.

An outgoing paramatrix was constructed in [3] which allows one to roughly say

global-in-time localized energy $\implies$ global-in-time Strichartz.

This proof is based on the earlier construction [5] for Schrödinger equations. After reducing to a half-wave equation, we conjugate by a time-dependent FBI transform. A second order term in the asymptotic expansion is nontrivial, and we are left with a degenerate parabolic equation. The bounds from [5], which are based on the maximum principle, can then be referenced. Moreover, if the perturbation is small, the necessary frequency-localized versions of the localized energy estimates are shown using a positive commutator argument, which in turn yields the Strichartz estimates.

As an example of an asymptotically flat perturbation which is not small, we examine the wave equation on Schwarzschild space-times. The Schwarzschild space-time is the simplest solution to Einstein's equations in vacuum which contains a black hole. We restrict our analysis to the exterior of the black hole $(t, r, \omega) \in \mathbb{R} \times (2M, \infty) \times S^2$, and the line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\omega^2.$$

A key obstacle, which is not encountered for small perturbations, is the existence of trapped rays. For the Schwarzschild space-time, trapping occurs at the event horizon $r = 2M$ and the photon sphere $r = 3M$.

Solutions to the homogeneous wave equation $\Box_g \phi = 0$, where

$$\Box_g = -\left(1 - \frac{2M}{r}\right)^{-1}\partial_t^2 + r^{-2}\partial_r\left(1 - \frac{2M}{r}\right)r^2\partial_r + r^{-2}\Delta_{S^2},$$

have a conserved energy

$$E[\phi](t) = \int \left[\left(1 - \frac{2M}{r}\right)^{-1}(\partial_t \phi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r \phi)^2 + |\nabla \phi|^2\right] dx,$$

where $\overline{\nabla}$ denotes the angular derivatives. In analogy to (1), we prove

(4)
$$\int_{t>0} \int \left[c_r^0\left(1 - \frac{2M}{r}\right)(\partial_r\phi)^2 + c_t^0\left(1 - \frac{2M}{r}\right)^{-1}(\partial_t\phi)^2 + c_\omega^0|\overline{\nabla}\phi|^2 + c^0\phi^2 \right] dx\, dt$$
$$\lesssim E[\phi](0),$$

where $dx = r^2\, dr\, d\omega$. Here

$$c_r^0 = \frac{1}{r^2\left(1 - \ln\left(1 - \frac{2M}{r}\right)\right)^2}, \quad c_t^0 = \frac{\left(1 - \frac{3M}{r}\right)^2}{r^2\left(1 - \ln\left(1 - \frac{2M}{r}\right)\right)^2}$$

$$c_\omega^0 = \frac{1}{r}\left(1 - \frac{3M}{r}\right)^2, \quad c^0 = \frac{1}{r^4\left(1 - \frac{2M}{r}\right)(1 - \ln\left(1 - \frac{2M}{r}\right))^2}.$$

Estimates of this form have been previously shown in [1] and [2]. These works proceed by expanding into spherical harmonics and choosing a different multiplier on each harmonic. We instead focus on finding a single multiplier which permits us to avoid the spherical harmonic decomposition. Our multiplier roughly looks like $X = f(r)\left(1 - \frac{2M}{r}\right)\partial_r$ where

$$f(r) = \begin{cases} \frac{1}{r^2}\left[\left(\frac{r^2}{3} + 2Mr + 10M^2\right)(r - 3M) + 8M^3\ln\left(\frac{r-2M}{M}\right)\right], & r \in (2M, 3M), \\ \frac{9M}{2}\frac{r-3M}{r-\frac{3M}{2}}, & r \in [3M, \infty). \end{cases}$$

This is not quite sufficient as it is not bounded near $r = 2M$, where one needs to "smooth it out". While relatively elementary, the estimate (4) plays an important role in the analysis. Since it is lossless away from $r = 2M$, $r = 3M$, and $r = \infty$, it allows us to glue together analyses which are performed separately in the regions surrounding these points.

In the estimate (4), notice, in particular, the vanishing of the coefficients at $r = 2M$ and $r = 3M$ which is a result of the trapping. We hope to be able to take the losses at these points to be logarithmic in both cases. Roughly, we prove that the coefficients c_ω^0 and c_t^0 can be replaced by

$$c_\omega = \frac{1}{r}\left(1 - \ln\left|1 - \frac{3M}{r}\right|\right)^2, \quad c_t = \frac{\left(1 - \ln\left|1 - \frac{3M}{r}\right|\right)^2}{r^2\left(1 - \ln\left(1 - \frac{2M}{r}\right)\right)^2}$$

respectively. To prove this, we may localize to a neighborhood of $r = 3M$, take the Fourier transform in t, and expand in spherical harmonics (indexed with λ). The estimates are easy unless λ and the time frequency variable have a delicate balance. In this case, the following serves as a fairly representative model problem:

$$u'' + \lambda^2(x^2 \pm \varepsilon)u = f, \quad \text{near } r = 0.$$

Estimates for solutions to this equation are proved using a WKB approximation.

Finally, we prove global Strichartz estimates which are roughly of the form

$$\left\|\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q} + \frac{1}{2}\right)} |D|^{1-\rho}\phi\right\|_{L^p L^q} \lesssim E[\phi](0)$$

for nonsharp Strichartz exponents. Here $|D|$ is a pseudodifferential operator which looks roughly like the derivatives occurring in the energy. There are also estimates available for sharp Strichartz exponents, but these necessitate a logarithmic loss.

Using (4), we may analyze the regions near $r = 2M, 3M, \infty$ separately. In the bounded regions near $r = 2M$ and $r = 3M$, the global Strichartz estimates follow, via fairly standard arguments, from the local-in-time Strichartz estimates (see, e.g., [4]) and the localized energy estimates (with the improved coefficients near $r = 3M$). Outside of a sufficiently large ball, the perturbation becomes small, and thus, in a region near ∞, the analysis from [3] can be referenced.

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The Schrödinger equation with a large magnetic potential

Michael Goldberg

(joint work with M. Burak Erdoğan, Wilhelm Schlag)

We prove Kato smoothing bounds and a full range of Strichartz estimates for the Schrödinger equation in $\mathbb{R}^n$, $n \geq 3$, with a time-independent first order self-adjoint perturbation [2]. In other words, propagation of solutions is given by e^{itH} with H taking the form

$$H = -\Delta + L = -\Delta + i(A(x) \cdot \nabla + \nabla \cdot A(x)) + V(x).$$

The coefficients of the potential must decay faster at infinity than the natural scaling rates of the equation ($|x|^{-1}$ for each $A_j(x)$ and $|x|^{-2}$ for $V(x)$), however no smallness condition governing the size of the potential is assumed.

Because of the large size of the perturbation, bound-state solutions certainly may exist. These are confined to the nonpositive spectrum of H, and an additional assumption is imposed that zero energy should neither be an eignvalue nor a

resonance. It follows that the bound states of H form a finite-dimensional subspace of $L^2(\mathbb{R}^n)$. Our result asserts the validity of the propagator bounds

$$\|\langle x \rangle^{-\frac{1}{2}-\epsilon} |\nabla|^{\frac{1}{2}} e^{itH} f\|_{L_t^2 L_x^2} \leq C\|f\|_2$$
$$\|e^{itH} f\|_{L_t^p L_x^q} \leq C\|f\|_2, \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \ p \in (2, \infty]$$

for all initial data which are orthogonal to the space of bound states.

These estimates for the free equation [3], [7] are derived from $L^1 \to L^\infty$ mapping estimates for the convolution operator $e^{-it\Delta}$, $t \neq 0$. The same dispersive bounds do not hold in general for e^{itH}, even after projecting onto its continuous spectrum. Counterexamples with $A \equiv 0$ are known in dimensions $n \geq 4$ [5] and the case of nontrivial magnetic potentials is entirely open.

There is an alternative path toward proving Strichartz estimates, described in [8], which uses Kato smoothing as the intermediate step. Precisely, one requires that the perturbation L be factorized as a finite sum of terms $Z_j^* W_j$, where each W_j is H-smooth and each Z_j is smooth with respect to the Laplacian. So long as $|\nabla|^{\frac{1}{2}} A$ is bounded, terms like $A \cdot \nabla$ can be split so that each factor possesses half of a derivative and pointwise decay at the rate $|x|^{-\frac{1}{2}-\epsilon}$. Such operators are well known to be Δ-smooth [6].

The criterion of choice for testing whether they are also H-smooth is to determine whether the resolvents $R(\lambda^2) = (H - (\lambda + i0)^2)^{-1}$ are bounded from the weighted Sobolev space $\langle x \rangle^{-\frac{1}{2}-\epsilon} H^{\frac{1}{2}}$ to its dual, uniformly over all $\lambda \in \mathbb{R}$. Once again this is well known for the free resolvent $R_0(\lambda^2) = (-\Delta - (\lambda + i0)^2)^{-1}$. The perturbation appears here in the form of a multiplicative correction

$$R(\lambda^2) = (I + R_0(\lambda^2)L)^{-1} R_0(\lambda^2)$$

thus it suffices to prove boundedness of $(I + R_0(\lambda^2)L)^{-1}$ uniformly over $\lambda \in \mathbb{R}$.

At each value of λ, existence of the operator inverse can be established via the Fredholm Alternative and Agmon's bootstrapping method [1]. By continuity this process gives a uniform bound when λ is restricted to a compact set in $\mathbf{R}$. Recent results [4] suggest that eigenvalues at zero energy need not be excluded so long as the associated eigenfunction belongs to $\langle x \rangle^{-1} L^2(\mathbb{R}^n)$. In very high dimensions ($n \geq 7$) no zero-energy assumption would be needed at all. In three dimensions, however, extra decay of L^2 static solutions is not generic.

The argument for large λ proceeds by expanding the inverse operator as a Neumann series, with a significant complication arising from the fact that (unless $A \equiv 0$) the norm of $R_0(\lambda^2)L$ is bounded from below by a large constant as $\lambda \to \infty$. The Neumann series still converges thanks to a key estimate on higher powers of $R_0(\lambda^2)L$. For any $r > 0$ there exists an exponent $m < \infty$ so that

$$\limsup_{\lambda \to \infty} \|(R_0(\lambda^2)L)^m\| \leq (2r)^m.$$

The free resolvent $R_0(\lambda^2)$ consists of convolution against a kernel $|x|^{2-n} K(\lambda|x|)$, with function $K(r) \sim e^{1\lambda r} r^{(n-3)/2}$ as $r \to \infty$. Let $\{\Omega_j\}_{j=1}^N$ be a partition of unity

on the sphere. Decomposing $R_0(\lambda^2)$ into convolutions with a directed kernel

$$|x|^{2-n}K(\lambda|x|)\Omega_j(\tfrac{x}{|x|})$$

leads to an N^m-fold decomposition of the iterated integral operator $(R_0(\lambda^2)L)^m$. Terms where two of the chosen functions Ω_j have disjoint support are shown to be small by a variation of the Riemann-Lebesgue Lemma. Terms where the supports overlap gain no benefit from oscillation, however their strong directionality eventually leads to norm improvement (similar to iteration of a Volterra operator). Because m is not determined *a priori*, the partition of unity may require sets of very small diameter. It is important that estimates for the directed resolvent be uniform with respect to this size parameter. We are fortunately able to control the norm of this type of oscillatory integral using Hörmander's variable-coefficient Plancherel theorem.

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A uniqueness property of the Kerr spaces

Alexandru D. Ionescu

(joint work with Sergiu Klainerman)

My talk was concerned with several models related to proving "no hair" theorems for smooth black holes. Our first model theorem is in the Minkowski space $(\mathcal{M} = \mathbb{R} \times \mathbb{R}^d, \mathbf{m})$ of dimension $d+1$. We define the subsets of $\mathcal{M}$

$$(1) \qquad \mathcal{E} = \{(t,x) \in \mathcal{M} : |x| > |t| + 1\},$$

and

$$(2) \qquad \mathcal{H} = \delta(\mathcal{E}) = \{(t,x) \in \mathcal{M} : |x| = |t| + 1\}.$$

Let $\overline{\mathcal{E}} = \mathcal{E} \cup \mathcal{H}$. We start with a uniqueness property of solutions of systems of wave equations on $\mathcal{E}$.

Theorem 1. *(Uniqueness in the Minkowski spaces) Assume $N \geq 1$, $\phi_I \in C^2(\mathcal{M})$, and $A_I^J, B_I^{J,l} \in C^0(\mathcal{M})$ for any $I, J = 1, \ldots N$ and $l = 0, \ldots, d$. Assume that, for any $I = 1, \ldots, N$,*

$$(3) \qquad \Box(\phi_I) = \sum_{J=1}^{N} A_I^J \cdot \phi_J + \sum_{J=1}^{N} \sum_{l=0}^{d} B_I^{J,l} \cdot \partial_l(\phi_J) \quad on \ \ \mathcal{E}.$$

Assume that $\phi_I \equiv 0$ on $\mathcal{H}$ for any $I = 1, \ldots, N$. Then $\phi_I \equiv 0$ on $\overline{\mathcal{E}}$ for any $I = 1, \ldots, N$.

As a corollary, we have a uniqueness property of solutions of systems of nonlinear wave equations on $\overline{\mathcal{E}}$.

Corollary 2. *Assume $\Phi^{(i)} = (\phi_1^{(i)}, \ldots, \phi_N^{(i)})$, $\phi_I^{(i)} \in C^2(\mathcal{M})$, $i = 1, 2$, $I = 1, \ldots, N$, are solutions of the system of nonlinear wave equations on $\mathcal{E}$*

$$(4) \qquad \Box(\phi_I) = \Gamma_I(t, x, \Phi, \partial_0 \Phi, \ldots, \partial_d \Phi) \ \ for \ I = 1, \ldots, N.$$

If $\Gamma_1, \ldots, \Gamma_N \in C^1(\mathcal{M} \times \mathbb{R}^N \times \ldots \times \mathbb{R}^N)$ and $\Phi^{(1)} \equiv \Phi^{(2)}$ on $\mathcal{H}$, then

$$\Phi^{(1)} \equiv \Phi^{(2)} \ on \ \overline{\mathcal{E}}.$$

We remark that Corollary 2 is a pure uniqueness statement. The corresponding initial-value problem is ill-posed. Our proof of Theorem 1 is based on the method of Carleman estimates.

We consider now a similar model in the Kerr spaces. The Kerr spaces are the only known explicit solutions that model rotating black holes in vacuum. They depend on two parameters: m (the mass of the black hole) and J (the angular momentum of the black hole). We assume $m > 0$ and $a = J/m \in [0, m)$. In standard Boyer-Lindquist coordinates $(r, t, \theta, \phi) \in (r_+, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$, $r_+ = m + (m^2 - a^2)^{1/2}$, the metric in the exterior region $\mathbf{E}^4$ of the Kerr space $\mathbf{K}^4(m, a)$ is (see [2])

$$(5) \qquad -\frac{\rho^2 \Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\Big(d\phi - \frac{2amr}{\Sigma^2}dt\Big)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2,$$

where

$$(6) \qquad \begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

Let $\mathbf{H} = \delta\mathbf{E}^4$ denote the event horizon of the Kerr space $\mathbf{K}^4(m, a)$ and $\xi = \partial_t$ the Killing vector field on $\mathbf{K}^4(m, a)$.

Theorem 3. *(Uniqueness in the Kerr spaces) Assume W, A, B, C are smooth tensor fields in the Kerr space $\mathbf{K}^4(m, a)$, and*

$$(7) \qquad \begin{cases} \Box_{\mathbf{g}} W_{\alpha_1\ldots\alpha_k} = A^{\beta_1\ldots\beta_k}{}_{\alpha_1\ldots\alpha_k} W_{\beta_1\ldots\beta_k} + B^{\beta_{k+1}\beta_1\ldots\beta_k}{}_{\alpha_1\ldots\alpha_k} \mathbf{D}_{\beta_{k+1}} W_{\beta_1\ldots\beta_k}; \\ \mathcal{L}_\xi W_{\alpha_1\ldots\alpha_k} = C^{\beta_1\ldots\beta_k}{}_{\alpha_1\ldots\alpha_k} W_{\beta_1\ldots\beta_k}, \end{cases}$$

in $\mathbf{E}^4$. If $W \equiv 0$ on $\mathbf{H}$ then $W \equiv 0$ on $\mathbf{E}^4 \cup \mathbf{H}$.

Our proof of Theorem 3 is also based on the method of Carleman estimates. This theorem is an important ingredient in the proof of a conditional "no hair" theorem for smooth manifolds.

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The Kadomtsev-Petviashvili-II equation in critical spaces

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(joint work with Martin Hadac, Herbert Koch)

We report on the results obtained in the recent preprint [3] concerning the Cauchy problem for the Kadomtsev-Petviashvili-II equation

(KP-II)
$$(u_t + u_{xxx} + uu_x)_x + u_{yy} = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^2$$
$$u|_{t=0} = u_0$$

for initial data u_0 in the non-isotropic Sobolev spaces $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ and $H^{-\frac{1}{2},0}(\mathbb{R}^2)$. These are spaces of tempered distributions, defined via

$$\|u_0\|_{\dot{H}^{-\frac{1}{2},0}} := \left(\int_{\mathbb{R}^2} |\xi|^{-1} |\widehat{u_0}(\xi,\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} < \infty \quad \text{and}$$

$$\|u_0\|_{H^{-\frac{1}{2},0}} := \left(\int_{\mathbb{R}^2} (1+\xi^2)^{-\frac{1}{2}} |\widehat{u_0}(\xi,\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} < \infty,$$

respectively. The equation (KP-II) models the propagation of weakly transverse water waves in the long wave regime with small surface tension. The homogeneous space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ is a natural data space in the sense that its norm is invariant under the symmetries of the equation, namely *translation*, *scaling*, and *Galilean invariance*.

The Cauchy problem (KP-II) has attracted a lot of attention. We mention only a few previous results and refer the reader to these works for further references: It has been shown by J. Bourgain [1] that (KP-II) is globally well-posed in $L^2(\mathbb{T}^2; \mathbb{R})$ and $L^2(\mathbb{R}^2; \mathbb{R})$. Later, H. Takaoka [6] proved local well-posedness in the homogeneous spaces $\dot{H}^{s,0}(\mathbb{R}^2)$ in the full subcritical range $s > -\frac{1}{2}$. P. Isaza–J. Mejía [4]

derived global well-posedness in $H^{s,0}(\mathbb{R}^2;\mathbb{R})$ for $s > -\frac{1}{14}$. Recently, it has been proved by M. Hadac [2] that (KP-II) is locally well-posed in the inhomogeneous spaces $H^{s,0}(\mathbb{R}^2)$ in the full subcritical range $s > -\frac{1}{2}$.

We are interested in well-posedness and scattering for the (KP-II) equation in critical spaces. In [3] the following small data global well-posedness result is obtained:

Theorem 1 (cp. [3]). *There exists a space* $\dot{Z}^{-\frac{1}{2}}([0,\infty)) \subset C([0,\infty); \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2))$ *and* $\delta > 0$, *such that for all initial data*

$$u_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2) \text{ satisfying } \|u_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta$$

there exists a unique global solution

$$u \in \dot{Z}^{-\frac{1}{2}}([0,\infty))$$

of (KP-II). *Moreover, the flow map* $u_0 \mapsto u$ *is analytic.*

Additionally, local well-posedness of the (KP-II) equation is proved:

Theorem 2 (cp. [3]). *The equation* (KP-II) *is locally well-posed for arbitrarily large initial data, both in* $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ *and in* $H^{-\frac{1}{2},0}(\mathbb{R}^2)$. *The time of existence may depend on the frequency profile of the data.*

Moreover, it is shown that for small data the global solutions from Theorem 1 scatter to free solutions:

Theorem 3 (cp. [3]). *There exists* $\delta > 0$, *such that for initial data*

$$u_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2) \text{ satisfying } \|u_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta$$

there exists $u_+ \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ *with the property that the unique solution*

$$u \in \dot{Z}^{-\frac{1}{2}}([0,\infty)) \subset C([0,\infty); \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2))$$

with $u(0) = u_0$ *of the* (KP-II) *satisfies*

$$\|u(t) - e^{-t(\partial_x^3 + \partial_x^{-1}\partial_y^2)}u_+\|_{\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)} \to 0 \quad (t \to \infty).$$

The proofs of the above Theorems 1, 2 and 3 are based on sharp bilinear estimates and the contraction mapping principle. It is crucial to construct suitable function spaces which

- have the correct behaviour with respect to scaling,
- contain functions which are close to free solutions, at least in the sense that all elements satisfy the linear and bilinear Strichartz estimates,
- and allow us to take into account the bilinear structure of the nonlinearity, e.g. in terms of the resonance identity,

such that eventually we are able to close bilinear estimates on the Duhamel term in these spaces. Following H. Koch–D. Tataru [5], we use the atomic space U^2 and the space V^2 of bounded 2-variation as building blocks for the definition of our solution spaces and the linear theory. We examine the duality, continuous embeddings into various function spaces, and interpolation type properties of these

spaces and provide linear and bilinear Strichartz estimates. Finally, we show how the bilinear estimates can be derived by exploiting these properties.

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The cubic nonlinear Schrödinger equation in two space dimensions

Monica Visan

(joint work with Rowan Killip and Terence Tao)

We consider the Cauchy problem for the cubic nonlinear Schrödinger equation (NLS)

$$(1) \qquad iu_t + \Delta u = \pm|u|^2 u$$

in two space dimensions with L_x^2 initial data. When the nonlinearity appears with the '+' sign, we refer to it as *defocusing*, while the '−' sign corresponds to the *focusing* case.

The cubic nonlinearity is the most common nonlinearity in applications. It arises as a simplified model for studying Bose–Einstein condensates [3, 4, 11], Kerr media in nonlinear optics [6, 13], and even freak waves in the ocean [2, 5].

From a mathematical point of view, the cubic NLS in two dimensions is remarkable for being mass-critical. The name is a testament to the fact that there is a scaling symmetry

$$u(t, x) \mapsto \lambda u\big(\lambda^2 t, \lambda x\big), \quad \lambda > 0$$

that leaves both the equation and the mass invariant. Mass is a term used in physics to represent the square of the L_x^2-norm:

$$M(u(t)) := \int_{\mathbb{R}^2} |u(t, x)|^2 \, dx.$$

For (1), this is a conserved quantity.

Our main result is to construct global strong $L_x^2(\mathbb{R}^2)$ solutions to (1) for spherically symmetric initial data. More precisely, we prove

Theorem 1 ([9]). *Let $u_0 \in L_x^2(\mathbb{R}^2)$ be spherically-symmetric; in the focusing case assume also that $M(u) < M(Q)$. Then there exists a unique global strong solution u to* (1). *Moreover,*

$$\int_{\mathbb{R} \times \mathbb{R}^2} |u(t,x)|^4 \, dx \, dt \leq C(M(u))$$

and there exist $u_\pm \in L_x^2(\mathbb{R}^2)$ such that

$$\|u(t) - e^{it\Delta} u_\pm\|_2 \to 0 \quad as \ t \to \pm\infty.$$

This result has recently been extended to treat the corresponding mass-critical equations in all higher dimensions; see [10].

The ground state Q in the statement of Theorem 1 is the unique positive radial Schwartz solution to the elliptic equation

$$\Delta Q + Q^3 = Q.$$

Note that $u(t,x) := e^{it}Q(x)$ is a solution to (1) and hence Theorem 1 is sharp in the focusing case in the sense that solutions with mass equal to that of the ground state may have infinite $L_{t,x}^4$-norm. In fact, the pseudoconformal symmetry, that our equation enjoys, allows us to construct solutions with mass equal to that of the ground state but which blow up in finite time, for example,

$$u(t,x) := |t|^{-1} e^{-\frac{i|x|^2}{4t} + \frac{i}{t}} Q(x/t),$$

which blows up at time $t = 0$.

The local theory for (1) was worked out by Cazenave and Weissler [1]. They constructed local-in-time solutions for arbitrary initial data in $L_x^2(\mathbb{R}^2)$; however, due to the critical nature of the equation, the resulting time of existence depends on the profile of the initial data and not merely on its L_x^2-norm. Cazenave and Weissler also constructed global solutions for small initial data.

To attack the question of global existence for large data, we follow the approach of Kenig and Merle [7]. More precisely, using a concentration-compactness technique based on a linear profile decomposition of Keraani [8] (see also [14]), we reduce matters to studying a very special class of solutions, that is, solutions which are almost periodic modulo scaling (see the definition below) and which blow up in both time directions in the sense of infinite $L_{t,x}^4$-norm.

Definition 1 (Almost periodicity modulo scaling). A spherically-symmetric solution u with lifespan I is said to be *almost periodic modulo scaling* if there exist a function $N : I \to \mathbb{R}^+$ and a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x| \geq C(\eta)/N(t)} |u(t,x)|^2 \, dx \leq \eta \quad \text{and} \quad \int_{|\xi| \geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the *frequency scale function* for the solution u and to C as the *compactness modulus function*.

Further analysis allows us to reduce the proof of Theorem 1 to precluding the following three scenarios:

- *soliton-like* solution, that is, a global solution such that $N(t) = 1$ for all $t \in \mathbb{R}$;
- *double low-to-high frequency cascade*, that is, a global solution such that

$$\sup_{t \in \mathbb{R}} N(t) < \infty \quad \text{and} \quad \liminf_{t \to -\infty} N(t) = \liminf_{t \to +\infty} N(t) = 0;$$

- *self-similar* solution, that is, a solution defined on $(0, \infty)$ such that $N(t) = t^{-1/2}$ for all $t > 0$.

In all three scenarios, the key step is to prove that u has additional regularity, indeed, more than one derivative in L_x^2. At first glance, additional regularity may seem unreasonable since u is *a priori* only known to have finite mass and dispersive equations such as (1) do not exhibit global smoothing properties. As such, additional regularity should be viewed as an expression of the fact that in each of the three scenarios, the solution u is a *minimal-mass* blowup solution.

Additional regularity for the self-similar solution is proved by iterating various versions of the Strichartz inequality (including a recent refinement of that inequality in the spherically symmetric case due to Shao [12]) and taking full advantage of the self-similarity to control the motion of mass between frequencies. The existence of the self-similar solution is disproved by noting that H_x^1 solutions are global (see [16] for this result in the focusing case), while the self-similar solution is obviously not.

Higher regularity in the remaining two cases is obtained by exploiting the global existence together with the almost periodicity modulo scaling and the Duhamel formula both in the past and in the future. If done naively, neither of these Duhamel integrals are absolutely convergent. However, using the decomposition into incoming waves (which we propagate backwards in time) and outgoing waves (which we propagate forward in time), we can successfully exploit the radial symmetry of the solution. In this way, we obtain convergent integrals and regularity is then obtained by a simple iteration argument. To preclude the double high-to-low frequency cascade we use the additional regularity together with the conservation of energy, while in order to disprove the existence of soliton-like solutions we use a truncated version of the virial identity.

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Mathematical results related to dispersion management in nonlinear optical fibers

Markus Kunze

1. Introduction

Traditionally the transmission of optical pulses in nonlinear fiber optics was intimately connected to the classical soliton solution of the NLS equation that arises as a ground state of the equation, after averaging out the rapid oscillations of the power. This standard optical soliton decays like $\sim e^{-|x|}$ and preserves its shape during propagation by compensating the constant dispersion in the fiber through the nonlinearity. Starting at about 1995, however the concept of dispersion-managed optical solitons (DM solitons) was introduced. The basic set-up for these devices consists in two optical fibers of opposite dispersions that are concatenated into a line. Furthermore, a periodic chain of amplifiers is used to compensate for the fiber losses. It turned out that in real-world applications DM solitons could be used for highly efficient data transmission, in particular leading to an excellent performance in systems that are designed with a large variation of the dispersions in two adjacent pieces of the line, along with a low average.

To motivate the equation that will be of interest to us in this short survey, consider an optical fiber extended in the z-direction of $\mathbb{R}^3$. It is assumed that the fiber has a constant circular cross-section in the transversal x, y-directions. A z-segment of length L^+ and dispersion $\beta^+ > 0$ is followed by a segment of length L^- and dispersion $\beta^- < 0$. Then the piece $[0, L^+] \cup [L^+, L^+ + L^-]$ is periodically repeated along the z-axis. In order to make some simplifying assumptions it is

supposed that the fiber is unimodal and supports a monochromatic wave. Denoting ω_1 a fixed frequency and k_1 the associated wave number, the ansatz $E(x, y, z, t) = \kappa A(z, t)\Phi(x, y)e^{i(k_1 z - \omega_1 t)}$ is made for the electromagnetic field E. Here Φ is a (transversal) eigenfunction. To lowest order in κ the equation

$$iA_z + \beta_2(z)A_{tt} + i\beta_3(z)A_{ttt} + |A|^2 A = 0$$

formally arises [10] from the Maxwell equation for E, where $\beta_2(z) = \beta^+$ in $[0, L^+]$ and $\beta_2(z) = \beta^-$ in $]L^+, L^+ + L^-[$, and also a non-constant third order dispersion function $\beta_3(z)$ has been included. Due to the periodic change of the dispersion and the periodic amplification, the system will exhibit rapid oscillations of the pulse width and power. This fast dynamics is averaged out by replacing $\beta_j(z)$ with $\varepsilon^{-1}\beta_j(\varepsilon^{-1}z)$ and performing a formal averaging over ε on one segment. Renaming $z \to t$, $t \to x$, $A \to u$, the resulting propagation equation for the slow dynamics is found to be

$$(1) \qquad iu_t + \beta_2 u_{xx} + i\beta_3 u_{xxx} + \langle Q \rangle(u) = 0,$$

where $\langle Q \rangle(u) = \int_0^1 T(-t)(|T(t)u|^2 T(t)u)\, dt$ is the averaged nonlinearity for the function $u(t, x) = (T(t)u)(x)$ solving $iu_t + \beta_2 u_{xx} + i\beta_3 u_{xxx} = 0$ and $T(0)u = u$. The constants $\beta_j \geq 0$ denote the residual dispersions; for instance, $\beta_2 = L^+\beta^+ + L^-\beta^-$ for $\beta_2(z)$ as described above.

2. Summary of Results

First we consider the case where $\beta_3 = 0$. In [18] it was shown that the averaging outlined above is mathematically justified. Hence it is reasonable to look for ground state solutions, i.e., minimizers of the functional $H(u) = \alpha \int_{\mathbb{R}} |u_x|^2\, dx - \int_0^1 \int_{\mathbb{R}} |T(t)|^4\, dx\, dt$ under the constraint $\int_{\mathbb{R}} |u|^2\, dx = 1$. Note that a minimizer u_* leads to the periodic solution $u(t, x) = e^{i\omega t}u_*(x)$ of (1) for some ω, and hence to a nearly stable pulse for the non-averaged equation. If $\alpha > 0$, then H has a minimizer; see [21, 4]. The regularity of such minimizers and further properties are investigated in [19]. If $\alpha = 0$, then H still has a minimizer [7], and such minimizers also arise as the singular limit $\alpha \to 0^+$ of minimizers u_α for $\alpha > 0$ [6]. From a technical viewpoint, the difficulty of such a result is due to the invariances of the functional, and furthermore it is owed to the fact that there a no bounds on minimal sequences $(u_j)_{j \in \mathbb{N}}$ in spaces different from $L^2(\mathbb{R})$. In [7] a new and general method was devised that relies on applying the concentration compactness principle to both unit-mass sequences $(u_j)_{j \in \mathbb{N}}$ and $(\hat{u}_j)_{j \in \mathbb{N}}$ ('two-level concentration compactness'). The paper [16] reproved the existence of a minimizer u_* with different methods (using $X^{s,b}$-spaces) that also allowed to show that $u_* \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is smooth. For $x \in \mathbb{R}^2$, there is no minimizer, and there is also no minimizer for $x \in \mathbb{R}$, if $|T(t)|^4$ is replaced by $|T(t)|^6$ in H; see [16].

Closely related to H at $\alpha = 0$ is the functional $H_s(u) = -\int_{\mathbb{R}} \int_{\mathbb{R}} |T(t)|^6\, dx\, dt$. Refining the method from [7], it was proved in [8] that the constraint variational problem for H_s admits a minimum, i.e., the best constant $S > 0$ in the Strichartz inequality $\|u\|_{L^6_{tx}(\mathbb{R} \times \mathbb{R})} \leq S\|u_0\|_{L^2(\mathbb{R})}$ is attained. In [1], this result was reproved

by a more elementary method that relies on interpreting the space-time Fourier transform $\widetilde{u^3}(\tau, \xi)$ in a clever way as a (τ, ξ)-dependent inner product and the Strichartz estimate as an application of the Cauchy-Schwarz inequality to this inner product. In particular, as the cases of equality in the Cauchy-Schwarz inequality are known, the best constant could be evaluated to be $S = 12^{-1/12}$ with corresponding minimizer $u_*(x) = e^{-|x|^2}$ (and all orbits thereof under the symmetry groups). In two dimensions, $x \in \mathbb{R}^2$, and $|T(t)|^6$ in H_s replaced by $|T(t)|^4$, the best constant is $2^{-1/2}$ and obtained from the same minimizer. Furthermore, [1] contains similar results for some Strichartz inequalities for wave equations. Yet by another method, without making use of the Fourier transform at all, similar results are obtained in [2].

For the case $\beta_3 > 0$ in (1), i.e., higher-order dispersion, the existence of a minimizer is due to [11, 12] in the case of non-zero average dispersion. For zero average dispersion see [9], where also certain dispersion relations of order higher than three could be included; once again, this paper relies on the method of two-level concentration compactness.

Further references related to the subject of dispersion management include [3, 5, 14, 15, 20, 22, 23].

Quite recently, also so-called diffraction-managed optical fibers attracted some interest [13, 17]. Mathematically, here the continuous problem for $x \in \mathbb{R}$ has to be replaced by a discrete version.

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Loss of regularity for super-critical nonlinear Schrödinger equations

RÉMI CARLES

(joint work with Thomas Alazard)

We consider the nonlinear Schrödinger equation with defocusing, smooth, nonlinearity:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = |\psi|^{2\sigma}\psi, \quad \sigma \in \mathbb{N}, \ x \in \mathbb{R}^n.$$

The critical index given by scaling arguments is

$$s_c = \frac{n}{2} - \frac{1}{\sigma}.$$

We assume $s_c > 0$ (the nonlinearity is L^2 super-critical). If $\psi_{|t=0} \in H^s$ with $0 < s < s_c$, it is known that the Cauchy problem is ill-posed in H^s [5]. We show that this is even worse: there is a loss of regularity (in any space dimension). A consequence of this result is easy to state for energy super-critical problems: assume $n \geqslant 3$ and $\sigma > 2/(n-2)$. We can find a sequence of initial data $(\varphi^\lambda)_{0<\lambda\leqslant 1}$ in the Schwartz class, and a sequence of time $t^\lambda \to 0$, such that the mass and the nonlinear energy of φ^λ go to zero as $\lambda \to 0$, and

$$\left\|\psi^\lambda\left(t^\lambda\right)\right\|_{H^s} \to +\infty \text{ as } \lambda \to 0, \quad \forall s > 1,$$

where ψ^λ is the solution to the nonlinear Schrödinger equation with data φ^λ. Since for strong solutions, the energy is conserved, and for weak solutions, it is at

most the initial energy, this result is sharp. This result is in the same spirit of the pioneering work of G. Lebeau [6] for the wave equation. However, it seems that the method of G. Lebeau does not work so nicely in the case of Schrödinger equation; our proof follows a different approach, which is inspired by WKB analysis and fluid mechanics. This both simplifies and generalizes the proof in [4], which treated only the case $\sigma = 1$.

The proof proceeds in three steps. First, we reduce the problem to the study of the nonlinear Schrödinger equation in a high frequency régime:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma}\, u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0(x),$$

where a_0 is any non-trivial function in the Schwartz class, independent of the semi-classical parameter ε. The main result then follows from the fact that for t of order 1 (as $\varepsilon \to 0$), u^ε is exactly ε-oscillatory. The rest of the analysis consists in establishing this fact.

Second, we consider the expected limiting system: seeking $u^\varepsilon \approx ae^{i\phi/\varepsilon}$, let

(1)
$$\begin{cases} \partial_t\phi + \dfrac{1}{2}|\nabla\phi|^2 + |a|^{2\sigma} = 0 \quad ; \qquad \phi_{|t=0} = 0. \\[2mm] \partial_t a + \nabla\phi \cdot \nabla a + \dfrac{1}{2}a\Delta\phi = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases}$$

In terms of $(\nabla\phi, |a|^2)$, it is a compressible, isentropic Euler equation. Because of the possible presence of vacuum, this problem is not directly hyperbolic. Using an intermediate nonlinear change of unknown function due to T. Makino, S. Ukai and S. Kawashima [7], we show that this system is well-posed in Sobolev space, with a loss of at most one derivative.

The last step consists in proving a mild convergence of u^ε to the Euler type system, using a modulated energy functional *à la* Y. Brenier [3]. By mild convergence, we mean that we do not need to describe the asymptotic of u^ε in L^2 (for instance). Essentially, we need to know the behavior of $|u^\varepsilon|$ and $|\nabla u^\varepsilon|$ only:

$$\left\|(\varepsilon\nabla - i\nabla\phi)u^\varepsilon\right\|^2_{L^\infty([0,T];L^2)} + \left\|\left(|u^\varepsilon|^2 - |a|^2\right)^2 \left(|u^\varepsilon|^{2\sigma-2} + |a|^{2\sigma-2}\right)\right\|_{L^\infty([0,T];L^1)}$$
$$= \mathcal{O}(\varepsilon^2).$$

Using Hölder's inequality, we give a rigorous meaning to the approximations:

$$\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2} \approx \|u^\varepsilon(t)\nabla\phi(t)\|_{L^2} \approx \|a(t)\nabla\phi(t)\|_{L^2}.$$

Using small time properties of the solution to (1), we see that there exists $\tau > 0$ independent of ε such that the last term is positive at time $t = \tau$. This shows that u^ε has become ε-oscillatory at time τ, hence the result.

Note that the study of (1) does not suffice to infer the limiting behavior of the wave function itself, due to more subtle modulation phenomena:

$$\left\|u^\varepsilon(t) - a(t)e^{i\phi^{(1)}(t)}e^{i\phi(t)/\varepsilon}\right\|_{L^2} \to 0 \text{ as } \varepsilon \to 0, \text{ for } t = \mathcal{O}(1),$$

where $\phi^{(1)}(t,x) = \mathcal{O}(t)$ is L^∞, and is non-trivial in general. Also, it is not possible to prove the above mentioned convergence by applying the Gronwall lemma for Schrödinger equations. In view of this aspect, the proof proposed to show the loss of regularity phenomenon is rather cheap: we establish the minimal information needed to conclude (we do not need to consider $\phi^{(1)}$). See [1] for the proof. Note also that this proof allows to consider weak solutions of the nonlinear Schrödinger equation, even if we have proved in a subsequent work [2] that, at least when $n \leqslant 3$, the solution u^ε remains a strong solution on the time interval that we consider.

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On global well-posedness for defocusing L^2-critical NLS in 1D

Nataša Pavlović

(joint work with Daniela De Silva, Gigliola Staffilani, Nikolaos Tzirakis)

1. Introduction

In the talk we presented our recent result [13] on global well posedness for the following Cauchy problem for a defocusing nonlinear Schrödinger (NLS) equation:

$$(1) \qquad iu_t + \Delta u = |u|^{p-1}u,$$

$$(2) \qquad u(x,0) = u_0(x) \in H^s(\mathbb{R}^n), \ t \in \mathbb{R},$$

with $p = 5$ and $n = 1$.

Some of the important attributes of the NLS such as conserved or monotone quantities are at low regularities, and to utilize them one needs to establish existence theory at low regularities. Before we summarize local well-posedness results for (1)-(2), a scaling property of (1) is recalled. Precisely, if $u(x,t)$ solves (1) then $u_\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ is a solution of (1) too and

$$\|u_\lambda(x,0)\|_{\dot{H}^{s_c}} = \|u(x,0)\|_{\dot{H}^{s_c}},$$

where s_c denotes the scaling invariant Sobolev regularity $s_c = -\frac{2}{p-1} + \frac{n}{2}$. The $\dot{H}^{s_c}$ scaling invariance inspires the heuristic that one should expect to have local well-posedness in H^s with $s \geq s_c$. Indeed, Cazenave and Weissler [3] proved that (1)-(2) is locally well-posed in $H^s(\mathbb{R}^n)$ for $s > s_c$. A more general version of local well-posedness, in the sense that the time of existence depends on the profile of u_0, is obtained for H^{s_c} initial data [20, 4].

The problem (1)-(2) with $p = \frac{4}{n} + 1$ is referred to as the L^2-critical, since in that case $s_c = 0$. Although the question addressing local well-posedness for the L^2-critical NLS is settled there are many issues to be addressed among which is global well-posedness. In order to recall known global well-posedness result we first look at the following conservation laws of (1):

- Mass conservation: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.
- Energy conservation: $E(u)(t) = \frac{1}{2} \int |\nabla_x u(x,t)|^2 dx + \frac{1}{p+1} \int |u(x,t)|^{p+1} dx = E(u_0)$.

In the case of the L^2-critical problem the energy conservation combined with the local well-posedness of the problem in $H^s(\mathbb{R}^n)$ with $s \geq 0$ implies that the local in time solution can be extended to a global solution for initial data in $H^s(\mathbb{R}^n)$ with $s \geq 1$. It is conjectured that the problem is globally well-posed in time for all data for which the local theory is valid. We note that global well-posedness and scattering for the L^2-critical problem in $L^2(\mathbb{R}^n)$ has been proved recently [19, 18] in all dimensions $n \geq 2$ for spherically symmetric data. As in the case of energy critical problem [2, 11, 22, 17], the proof relies upon a combination of several sophisticated tools. It remains a subtle problem to establish global existence of solutions to L^2-critical NLS corresponding to arbitrary infinite energy initial data.

Below we describe how we combine the I-method with a Morawetz type estimate to improve global well-posedness for (1)-(2) with $p = 5$ and $n = 1$.

2. Global existence for the NLS on $\mathbb{R}$

The following Cauchy problem is considered:

$$iu_t + \Delta u = |u|^4 u, \tag{3}$$

$$u(x,0) = u_0(x) \in H^s(\mathbb{R}), \ t \in \mathbb{R}, \tag{4}$$

As a special case of the initial value problem (1)-(2), it is known that (3)-(4) is locally well-posed in $H^s(\mathbb{R})$ with $s \geq s_c = 0$. Existence of global solutions to (3)-(4) corresponding to initial data below the energy threshold was first obtained in [7], [8] using the I-method. The gap between known local and global well-posedness was further filled out in [21], where global well-posedness was obtained in $H^s(\mathbb{R})$ with $s > 4/9$.

In [13] we proved the following result:

Theorem 2.1. *The initial value problem* (3)-(4) *is globally well-posed in* $H^s(\mathbb{R})$, *for any* $1 > s > \frac{1}{3}$. *Moreover the solution satisfies*

$$(5) \qquad \sup_{t \in [0,T]} \|u(\cdot, t)\|_{H^s(\mathbb{R})} \leq C(1+T)^{\frac{s(1-s)}{2(3s-1)}}$$

where the constant C *depends only on* s *and* $\|u_0\|_{L^2}$.

We prove Theorem 2.1 by combining the I-method with an interaction Morawetz type estimate for the smoothed out version Iu of the solution. Such a Morawetz estimate for an almost solution, that we call "almost Morawetz", is the main novelty of our approach. We remark that recently a similar approach has been used in the L^2-critical case in 2D [5].

Before we outline the main steps of the proof, we say a few words about the above mentioned tools: the I method and the "almost Marawetz estimate". An important contribution in understanding evolution of rough initial data was obtained by introduction of the I method, that can be thought of as a refinement of Bourgain's Fourier truncation method [1]. The first steps towards the formalization of the I-method appear in the context of nonlinear wave maps, see Keel and Tao [15], [16]. In its more sophisticated and current version the I-method was first introduced by Colliander et al (see, for example, [7, 8, 9]). The idea is to control the behavior in time of a rough solution by controlling the energy of a smoothed out solution. More precisely, one replaces the conserved quantity $E(u)$, which is no longer available for $s < 1$, with an "almost conserved" variant $E(Iu)$ where I is a smoothing operator of order $1-s$. However Iu is not a solution to (1) and hence we expect an energy increment. This increment is quantifying $E(Iu)$ as an "almost conserved" energy. The approach of combining the I-method with an interaction Morawetz estimate goes back to [10] where authors derived and used a two-particle Morawetz estimate to improve global well-posedness of (1)-(2) with $p = 3$ in $H^s(\mathbb{R}^3)$. Fang and Grillakis [14] obtained a local in time interaction Morawetz estimate in $\mathbb{R}^2$ and combined it with I-method to improve global well-posedness for (1)-(2) with $p = 3$ in $H^s(\mathbb{R}^2)$. A similar approach based on a local in time interaction Morawetz estimate yielded progress in global well-posedness for the L^2-critical problem in higher dimensions [12]. However until recent work [6] there was no available interaction Morawetz inequality in 1D. The authors in [6] found an elegant way to overcome a dimensional obstacle and obtained the following four-particle interaction Morawetz estimate in 1D:

$$(6) \qquad \|u\|_{L^8_{t \in [0,T]} L^8_x}^8 \lesssim \sup_{t \in [0,T]} \|u\|_{\dot{H}^{1/2}}^2 \|u\|_{L^2}^6.$$

We remark that for initial data below $H^{1/2}$ the estimate (6) is not useful anymore. In order to overcome that difficulty we introduced an interaction Morawetz estimate for the smoothed out solution that we call "almost Morawetz" estimate,

which is an a priori estimate of the form

$$\|Iu\|_{L^8_{t\in[0,T]}L^8_x}^8 \lesssim \sup_{t\in[0,T]} \|Iu\|_{\dot{H}^1}\|Iu\|_{L^2}^7 + Error.$$

Using harmonic analysis estimates of Coifmann and Meyer type we show that the the *Error* terms are negligible in some sense.

In order to prove the global well-posedness result stated in Theorem 2.1 we combine the above mentioned tools via the following steps (for details, see [13]):

Step 1: Establish local well-posedness for the I system.

Step 2: Prove almost conservation of the modified energy.

Step 3: Obtain Global well-posedness. Here the main idea is to interpolate the information on Iu coming from the almost Morawetz estimate with apriori bounds on Iu that are based on conservation of energy, with a goal to obtain the information about Iu in the Strichartz space $L^6_t L^6_x$ which controls the local well-posedness. Then one glues the intervals of local well-posedness to obtain a global solution taking advantage of the I-method and rescaling.

We believe that this new "almost Morawetz" estimate (which shall be modified to suit the new nonlinear term) can be used together with I-method to improve global well-posedness for (1)-(2) with monomial nonlinearities in 1D.

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Blow up for critical nonlinear wave equations

WILHELM SCHLAG

We will discuss blow-up for nonlinear hyperbolic equations of the critical type. More precisely, we study the energy critical wave map equation from $2+1$ dimensions into S^2 as well as the quintic semilinear focusing equation in $3+1$ dimensions. Blow up solutions are constructed through a rescaling procedure starting from special stationary solutions. In the wave map case, these are ground state harmonic maps, and in the semilinear case they are the Talenti-Aubin solutions. In both cases, the rescaling is prescribed, in contrast to the modulational approach in which solutions are constructed via a process that finds an appropriate ODE for the scaling law.

Solutions to the generalizwed Korteweg-de Vries equations with a prescribed asymptotic behavior

RAPHAËL CÔTE

Given $p > 1$, we consider the generalized Korteweg-de Vries equations

$$(\text{gKdV}) \qquad \begin{cases} u_t + (u_{xx} + |u|^p)_x = 0, & t, x \in \mathbb{R}, \\ u(t = 0) = u_0, \end{cases}$$

From [4], these equations are locally well-posed in H^1, and even in L^2 in L^2 critical case $p = 5$ (which we denote (cKdV)). A fundamental feature of (gKdV) is the

existence of explicit travelling wave solutions : the solitons. Let Q be the unique profile (up to translation) to

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \text{ and } Q_{xx} + Q^p = Q.$$

Then the soliton $R_{c,x_0} = Q_c(x - x_0 - ct) = c^{\frac{1}{p-1}} Q(\sqrt{c}(x - x_0 - ct))$ is a solution to (gKdV). Notice that solitons are exponentially decaying and travel to the right.

Denote $U(t)$ the linear group, that is $U(t)\phi$ is the unique solution to $u_t + u_{xxx} = 0$, $u(t = 0) = \phi$, or explicitly $\widehat{U(t)\phi} = e^{it\xi^3}\hat{\phi}$.

We construct solutions to (gKdV), defined for large enough times, which, as times goes to $+\infty$, behave (in H^1) as the sum of a linear solution and of N solitons.

Theorem 1 (Non-linear wave operator, subcritical case [1]). *Let $p = 4$. Let $V \in H^{5,1} \cap H^{2,2}$ be such that $x_+^{4/3} D_x^5 V \in L^2$ and $x_+^8 V \in H^1$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$, we introduce the N solitons $R_j(t, x) = Q_{c_j}(x - x_j - c_j t)$. Then there exists $u^* \in C([T_0, +\infty[, H^4)$, for some $T_0 \in \mathbb{R}$, solution to (gKdV) (with $p = 4$), such that :*

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^{N} R_j(t) \right\|_{H^4} \leq Ct^{-1/3}.$$

Theorem 2 (Non-linear wave operator, critical case [2]). *Let $p = 5$. Let $V \in H^1$ be such that $x_+^{2+\delta_0} V \in L^2$ for some $\delta_0 > 0$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$, we introduce the N solitons $R_j(t, x) = Q_{c_j}(x - x_j - c_j t)$. Then there exists $u^* \in C([T_0, +\infty[, H^1)$, for some $T_0 \in \mathbb{R}$, solution to (cKdV), such that :*

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^{N} R_j(t) \right\|_{H^1} \to 0 \quad as \quad t \to \infty.$$

The decay on the right condition on V corresponds to small interaction of the linear term $U(t)V$ with the solitons. In the critical case, the requirements seem almost optimal for the method.

The proof of these results follows the following scheme. We introduce a sequence of time $S_n \to \infty$ as $n \to \infty$, and the solutions u_n, which have exactly the desired profile at time S_n : $u_n(S_n) = U(S_n)V + \sum_{j=1}^{N} R_j(S_n)$. Our goal is to obtain uniform estimates on the error term $u_n(t) - U(t)V + \sum_{j=1}^{N} R_j(t)$ on some interval with fixed lower bound $[T_0, S_n]$.

The proof of the uniform estimates goes in two main steps. First, we rely on the work of Martel, Merle and Tsai [5] regarding the stability of a sum of solitons, and we obtain a control on the right. Introduce the cut-off function between the solitons and the linear term : $\psi(x) = 1 - \frac{2}{\pi}\arctan(\exp x)$ and

$$\psi_0(t, x) = \psi(x - \sigma_0 t), \quad \text{where} \quad \sigma_0 = \min\{c_1, c_2 - c_1, \ldots, c_N - c_{N_1}\}/2.$$

Then we have

$$(1) \qquad \|w_n(t)\|_{L^2(1-\psi_0(t))} \leq \int_t^{S_n} \|U(t)V\|_{L^2(1-\psi_0(t))}dt + ...$$

The term on the right hand side should be understood as interaction between the linear term and the solitons. Using our decay assumptions on V, we get a polynomial decay with arbitrary order.

The second step is to obtain global estimates. For this, we relie on two result of linear scattering for small data : the work Hayashi and Naumkin [3] in the case $p = 4$, and the work of Kenig Ponce and Vega [4] in the critical case $p = 5$.

In the critical case, the linear estimates of [4], along with (1) allows to conclude that for some fixed function $\eta(t) \to 0$ as $t \to \infty$,

$$\|w_n(t)\|_{L^2} \leq \eta(t),$$

which is the desired uniform decay estimates.

In the case $p = 4$, the solitons prevent a nice cancellation which was at the heart of the estimates in [3]. Hence we need to strenghten the settings : we derived H^4 uniform decay bounds, using "almost conservation" laws. We can then bootstrap the estimates, and obtain :

$$\|w_n(t)\|_{H^4} \leq C/t^{1/3}.$$

The proof of the Theorems then follows form a compactness argument on u_n.

The uniqueness of $u^*(t)$ is unclear, although one has uniqueness in the cases of a pure solitons behavior (see [6]) or pure linear behavior. A second question is the restriction to $p = 4$: from [3], one could expect a cosntruction of a non linear wave operator in the whole range $p \in (3, 5)$.

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Strichartz estimates for the Schrödinger equation with time dependent magnetic potentials and applications
Atanas Stefanov

This talk is based on the series of works [6], [7] and [13], which were motivated by concrete systems of PDE's arising in mathematical physics. More precisely, the concern is to address the well-posedness and scattering properties of the Modified Schrödinger map (MSM) system and the Maxwell-Schrödinger system.

The MSM system is roughly in the form

$$(1) \qquad \partial_t u - i\Delta u + A(u) \cdot \nabla u = N(u) \qquad (t,x) \in \mathbf{R}^{1+n},$$

where $A \sim |\nabla|^{-1} Q(u, \bar{u})$, $N(u)$ behaves like a power nonlinearity, so that $N(u) = O(u^3)$. The local well-posedness and the global regularity issues for the original Schrödinger map problem has been largely settled[1] in [1], [2], [3], [8], [9].

The other basic example is the Maxwell-Schrödinger system, which in Coulomb gauge is represented by

$$(2) \qquad \left| \begin{array}{l} i\partial_t \psi + \Delta_A \psi = (-\Delta)^{-1}(|\psi|^2)\psi, \quad (t,x) \in \mathbf{R}^{1+3} \\ \Box A = \mathbf{P} Im(\bar{\psi}\nabla_A \psi) \end{array} \right.$$

where the magnetic Laplacian $\Delta_A = \sum_{j=1}^{n} (\partial_j - iA_j(t, \cdot))^2$ and $\mathbf{P}$ is the Leray projection. The problem for well-posedness has been considered in [11], [12], and in[7], with the last paper establishing existence and uniqueness for $(\psi, A) \in H^{7/8+} \times H^{1+}$. A recent work [4] establishes local well-posedness for data in the energy space $(\psi, A) \in H^1 \times H^1$.

As one sees, both of these PDE's exhibit a Schrödinger equation with first order perturbation structure. This prompts the natural question: under what conditions on $A(t,x)$, the "magnetic" Schrödinger equation with time dependent potential

$$(3) \qquad u_t - i\Delta u + A(t,x) \cdot \nabla u = 0,$$

one has Strichartz estimates for u? A lot of important results have been achieved in that direction.

- (Georgiev-Stefanov-Tarulli, [6]) Let $d \geq 3$ and

$$\|\nabla A\|_{L^\infty L^{d/2}} + \sup_k \sum_m 2^m \|A_{<k}\|_{L^\infty_{tx}(|x| \sim 2^m)} \leq \varepsilon << 1.$$

  Then Strichartz estimates hold in the form $\|u\|_{L^q_t L^r(\mathbf{R}^d} \leq C\|u(0)\|_{L^2}$, where $q, r \geq 2 : 2/q + d/r = d/2$.
- (Erdogan-Goldberg-Schlag, [5]) Let $d \geq 3$, $A = A(x)$. Assume[2]

$$<x>^{1+} |A(x)| + <x>^{2+} |\nabla A(x)| \leq C.$$

  and 0 is not a resonance nor eigenvalue. Then Strichartz estimates hold true.

---

[1]With the notable exception of the global regularity for small data in dimensions $n = 2$ and the case of small initial data in the critical Sobolev class $\dot{H}^{3/2}(\mathbf{R}^3)$.

[2]Note that smallness is not required here.

- (Tataru, [14]; Marzuola-Metcalf-Tataru, [10]) Let $d \geq 3$. If[3]

$$\sum_m 2^m \|A\|_{L^\infty_{tx}(|x| \sim 2^m)} \leq \varepsilon << 1.$$

then Strichartz estimates hold true.

The common pitfall of all these results for the applications is that they require essentially a pointwise bound in the form "$|A(t,x)| \leq \varepsilon |x|^{-1-}$", which is usually unavalaible for the concrete magnetic potentials arising in the nonlinear problems, such as (MSM) or (MS).

As an attempt to remedy this problem, we refer to [13][4]. Scale invariant conditions of L^p type are found under which the Strichartz estimates hold true. Just to give a flavor of these, it is required among other things that

$$\sum_k 2^{k(d+3)/2} \sup_{U \in SU(d)} \sup_x \|A_k(t, x + Uz)\|_{L^1_t L^2_{z_2,\ldots,z_d} L^1_{z_1}} +$$

$$+ 2^{k(d-1)/2} \sup_{U \in SU(d), x(t)} \|(|\partial^2 A_k| + |\partial_t A_k|)(t, x(t) + Uz)\|_{L^1_t L^2_{z_2,\ldots,z_d} L^1_{z_1}}) << 1.$$

Note that the conditions are reminiscent of the well-known Mizohata necessary condition for well-posedness of (3) $\sup_{t,x,\theta \in \mathbf{S}^{n-1}} Im \int A(t, x + z\theta) \cdot \theta dz| < \infty$.

One of the main results in [13] is that one can actually solve the (MSM) with small Cauchy data in high dimensions $d \geq 6$. The dimensional restrictions come as a consequence of the L^1_t requirements on the magnetic potentials displayed above.

Next, we give an outline of the parametrix construction in [13]. It is standard that one needs v, so that $v : \|v(0,x) - f_k\|_{L^2} \leq \varepsilon \|f_k\|_{L^2}, \|v\|_{L^q_t L^r_x} \leq C \|f_k\|_{L^2}, \|v_t - i\Delta v + A_{<k} \nabla v\|_{L^1 L^2} \leq \varepsilon \|f_k\|_{L^2}$. Modulo some minor correction terms, we choose

$$(4) \qquad v(t,x) = \int e^{i\sigma(t,x,\xi/|\xi|)} e^{-4\pi^2 t|\xi|^2} e^{2\pi i \langle \xi, x \rangle} \hat{f}(\xi) d\xi$$

where (modulo a technical correction term)

$$\sigma(t, x, \xi/|\xi|) = \int_0^\infty \langle A_{<k}(t, x + z\xi/|\xi|), \xi/|\xi| \rangle dz.$$

We then employ the following decomposition in Fourier space: for fixed $l < k$, consider the sets $Z^l_j := \{\xi : |\xi/|\xi| - \theta^l_j| < 2^{l-k}\}$, $j = 1, \ldots, 2^{(k-l)(d-1)}$. Then observe that

$$\sigma \sim \sum_{l<k} \sum_j \int_0^\infty \langle A_l(t, x + z\theta^l_j), \theta^l_j \rangle dz.$$

It is important to note that the Fourier support properties of v hold relatively nicely under $e^\sigma = 1 + \sigma + \ldots + \sigma^n/n! + \ldots$.

---

[3]This is just a particular case of much more general theorem. Also in the same theorem, the smallness assumption may be removed if one is willing to place an extra $\|u\|_{L^2_t L^2_x(|x|<M)}$ for some $M >> 1$ of the right hand side of the Strichartz estimates.

[4]The approaches in [1], [2], [3], [8], [9] are somewhat similar. The main difference is that the specific structure of the Schrödinger maps plays a major role, while the approach of [13] is to work with general A.

After some computations, one can see that $v(0,x) = f + O(A)$ and by a combination of energy and dispersive estimates, one shows $\|v\|_{L_t^q L_x^r} \leq C\|f_k\|_{L^2}$. This is all done under smallness, but no temporal decay requirements for A. Finally, to establish $v_t - i\Delta v + A_{<k}\nabla v \in L^1 L^2$ requires placing A in $L_t^1(X)$ spaces as indicated above. One could of course imagine placing the error term $v_t - i\Delta v + A_{<k}\nabla v \in L_t^2 L_x^{2d/(d+2)}$ (or a combination of other dual Strichartz space or even $X^{0,-1/2,\infty}$) in order to weaken the temporal requirements of A, but the techniques of [13] at this point rely too strongly on L_x^2 based methods. In any case, it is hoped that the parametrix construction (4) will lend itself usefull in any further attempts at lowering the temporal decay requirements on the magnetic potential A and the subsequent applications to the nonlinear problems at hand.

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On low regularity local well-posedness of the Derivative Nonlinear Schrödinger Equation with periodic initial data

Axel Grünrock

(joint work with Sebastian Herr)

This is an account on joint work with S. Herr, see [GH07].

The Cauchy problem for the derivative nonlinear Schrödinger equation with periodic initial value

$$
(1) \qquad
\begin{aligned}
i\partial_t u(t) + \partial_x^2 u(t) &= i\partial_x(|u|^2 u)(t) \,, \quad t \in (-T, T) \\
u(0, x) &= u_0(x) \,, \quad x \in \mathbb{T}
\end{aligned}
$$

is considered for data u_0 in the function spaces $\widehat{H}_r^s(\mathbb{T})$, defined by the norms

$$
\|u_0\|_{\widehat{H}_r^s(\mathbb{T})} = \|\langle\xi\rangle^s \widehat{u}_0\|_{\ell_\xi^{r'}}, \qquad \frac{1}{r} + \frac{1}{r'} = 1.
$$

Local well-posedness of (1) is shown in the parameter range

$$
s \geq \frac{1}{2}, \qquad 2 \geq r > \frac{4}{3},
$$

thus we obtain a generalisation of Herr's $H^{\frac{1}{2}}(\mathbb{T})$ - result [H06]. For related results concerning the nonperiodic case see [G05].

Important elements of the proof are:

- an adaption of the gauge transform (GT) to the periodic setting,
- essential cancellations caused by certain correction terms, which arise naturally from this variant of the GT,
- new function spaces of $X^{s,b}$-type, which are based on *mixed L^p-spaces* on Fourier side, and, finally,
- elementary geometrical refinements of the number of divisors argument, which serve as a substitute for the L^6 Strichartz type estimate.

Concerning the second parameter r, we must leave open the question, whether or not there is local well-posedness for $r \leq \frac{4}{3}$. Nonetheless we present a counterexample showing that our result is optimal within the framework we use here.

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Bilinear virial identities

FABRICE PLANCHON

(joint work with Luis Vega)

We present bilinear virial identities for the nonlinear Schrödinger equation, which are extensions of the Morawetz interaction inequalities ([5]). We recover and extend known bilinear improvements to Strichartz inequalities and give applications to various nonlinear problems, most notably on domains with boundaries.

Let $n \geq 1$, $p \in \mathbb{R}$, $p \geq 1$, $\varepsilon \in \{-1, 1\}$, and u, v solutions to

$$(1) \qquad i\partial_t u + \Delta u = \varepsilon |u|^{p-1} u \quad \text{and} \quad i\partial_t v + \Delta v = \varepsilon |v|^{p-1} v.$$

We introduce several quantities which will play a key role: given a function f, its Radon transform is

$$(2) \qquad R(f)(s, \omega) = \int_{x \cdot \omega = s} f \, d\mu_{s,\omega},$$

where $\mu_{s,\omega}$ is the induced measure on the hyperplane $x \cdot \omega = s$. We set

$$(3) \qquad I_\omega(\varepsilon, u, v) = \int_{x \cdot \omega > y \cdot \omega} (x \cdot \omega - y \cdot \omega)|u|^2(x)|v|^2(y) \, dxdy,$$

A simple computation leads to

$$(4) \quad \dot{I}_\omega(\varepsilon, u, v) = \partial_t I_\omega(\varepsilon, u, v)$$
$$= i \left(\int_{x \cdot \omega > y \cdot \omega} (u \nabla_x \bar{u} - \bar{u} \nabla_x u)(x)|v(y)|^2 - (v \nabla_y \bar{v} - \bar{v} \nabla_y v)(y)|u(x)|^2 \, dy \, dx \right).$$

We may now state our first result.

Theorem 1. *Let $\omega \in \mathbb{S}^{n-1}$, u solution to (1). Let $x = (x^\perp, x_\omega)$ with $x_\omega = x \cdot \omega$. Then*

$$(5) \quad \int_s |\partial_s(R(|u|^2))(s, \omega)|^2 \, ds + \varepsilon \frac{p-1}{p+1} \int_s R(|u|^2) R(|u|^{p+1}) \, ds$$
$$+ \int_s \int_{x \cdot \omega = s} |u(x^\perp, x_\omega) \partial_{x_\omega} u(y^\perp, x_\omega) - u(y^\perp, x_\omega) \partial_{x_\omega} u(x^\perp, x_\omega)|^2 \, dx^\perp dy^\perp dx_\omega ds$$
$$= \frac{1}{4} \partial_t \dot{I}_\omega(\varepsilon, u, u) = \frac{1}{4} \partial_t^2 I_\omega(\varepsilon, u, u).$$

In other words, $I_\omega(\varepsilon, u, u)$ is a convex function in time.

In the specific $1D$ case, one has actually the following bilinear identity.

Theorem 2. *Let $n = 1$, u, v two solutions to (1), then*

$$(6) \quad 4 \int_x (\partial_x(u\bar{v}))^2 \, dx + 2\varepsilon(1 - \frac{2}{p+1}) \int_x |u|^2|v|^{p+1} + |v|^2|u|^{p+1} \, dx = \partial_t^2 I(\epsilon, u, v).$$

In order to turn these bounds into useful nonlinear control, we use

Proposition 1. *Let ω be fixed, then*

$$(7) \qquad |I_\omega| \leq \|u\|^2_{L^2_x}\|v\|^2_{\dot{H}^{\frac{1}{2}}} + \|v\|^2_{L^2_x}\|u\|^2_{\dot{H}^{\frac{1}{2}}}.$$

From Theorem 1 and Proposition 1, one may deduce the following bound (which we learned at Oberwolfach has been obtained simultaneously and independently by J. Colliander, M. Grillakis and N. Tzirakis, [3]).

Proposition 2. *Let u be a solution to (1), with $\varepsilon = 0, 1$. Then*

$$(8) \qquad \int_0^T \||\nabla_x|^{\frac{3-n}{2}}(|u|^2)\|^2_{L^2_x}\,dt \lesssim \|u\|^2_{L^2_x}\|u\|^2_{\dot{H}^{\frac{1}{2}}}.$$

This follows readily from averaging the previous bounds on ω.

Theorem 1 may be used in a different direction, recovering a known bound for the linear equation (see [1]).

Proposition 3. *Let u and v be two solutions to (1), with $\varepsilon = 0$ and data u_0, v_0. Assume moreover that* $\operatorname{supp}\hat{u}(\xi) \subset \{|\xi| \leq 2^k\}$ *and* $\operatorname{supp}\hat{v}(\xi - \xi_0) \subset \{|\xi| \leq 2^k\}$, *with $|\xi_0| \sim 2^j$ and $k << j$ (hence, the Fourier supports are separated and at distance roughly 2^j). Then*

$$(9) \qquad \|u\bar{v}\|^2_{L^2_{t,x}} \lesssim 2^{(n-1)k-j}\|u_0\|_{L^2_x}\|v_0\|_{L^2_x}.$$

Now, let $\Omega \subset \mathbb{R}^n$ be a domain with a smooth boundary $\partial\Omega$, and u the solution to

$$(10) \qquad i\partial_t u + \Delta u = \varepsilon|u|^{p-1}u, \text{ with } u_{|\partial\Omega} = 0.$$

Define

$$(11) \qquad R(f)(s,\omega) = \int_{x\cdot\omega=s\cap\Omega} f\,d\mu_{s,\omega}.$$

and

$$(12) \qquad I_\rho = \int_{x,y\in\Omega} \rho(x-y)|u|^2(x)|u|^2(y)\,dxdy.$$

We may now state our result.

Theorem 3. *Let $\omega \in \mathbb{S}^n$, $\rho_\omega(z) = |z\cdot\omega|$, and u solution to (10). Then*

$$(13) \quad \int_s |\partial_s(R(|u|^2))(s,\omega)|^2\,ds + \varepsilon\frac{p-1}{p+1}\int_s R(|u|^2)R(|u|^{p+1})\,ds$$

$$+ \int_s\int_{x\cdot\omega=s} |u(x^\perp,x_\omega)\partial_{x_\omega}u(y^\perp,x_\omega) - u(y^\perp,x_\omega)\partial_{x_\omega}u(x^\perp,x_\omega)|^2\,dx^\perp dy^\perp dx_\omega ds$$

$$- \int_{x\in\partial\Omega,y\in\Omega} |u|^2(y)\partial_n\rho_\omega(x-y)|\partial_n u|^2(x)\,dS_x dy = \partial_t^2 I_{\rho_\omega}.$$

We now illustrate how to obtain useful estimates from Theorem 3 when one has control of the boundary term.

Proposition 4. *Let Ω be $\mathbb{R}^n \setminus \Sigma$, where Σ is star-shaped. Assume moreover $\varepsilon = 0, 1$ (linear or defocusing). Then*

$$(14) \qquad \int_0^T \int_{x \in \partial\Omega} |\partial_n u|^2 \, dS_x \lesssim \sup_{t \in [0,T]} \|u\|^2_{\dot{H}_0^{\frac{1}{2}}(\Omega)}.$$

Note that, more generally, for the linear equation, the result of Proposition 4 holds for unbounded domains, assuming one does not have any trapped rays. In fact, for such domains, the local smoothing estimate holds ([2]), and a simple integration by part argument (with a normal vector field, close to the boundary) yields control of the boundary term. As such, one obtains

Theorem 4. *Let $\omega \in \mathbb{R}^n$ be an unbounded domain with no trapped rays, u solution to the linear equation* (10) *($\varepsilon = 0$). Then*

$$(15) \qquad \||\nabla_x|^{\frac{3-n}{2}} (|u|^2)\|_{L^2_{t,x}} \lesssim \|u_0\|^2_{L^2(\Omega)} \sup_{t \in [0,T]} \|u\|^2_{\dot{H}_0^{\frac{1}{2}}(\Omega)}.$$

A typical application of our result is to obtain scattering for the defocusing nonlinear Schrödinger equation, with $1 + 4/n < p < 1 + 4/(n-2)$ (L^2-supercritical and $\dot{H}^1$-subcritical). This was already observed in [5] for $n = 3$ and in [4] for $n = 1$. One should point out that, unlike in the original proofs of [6]($n \geq 3$) and [7]($n \leq 2$), one obtains polynomial bounds on space-time norms in term of the mass and energy.

On the exterior domain with no trapping condition, one may obtain scattering results as well, but in a restricted range (due to the lack of Strichartz estimates for the linear equation).

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Global solution for the Maxwell-Schrödinger system in the energy space

Ioan Bejenaru

(joint work with Daniel Tataru)

The Maxwell-Schrödinger system in $\mathbb{R}^{3+1}$ describes the classical approximation to the quantum field equations for an electrodynamical nonrelativistic many body system. It has the form

$$(1) \qquad \begin{cases} iu_t - \Delta_A u = \phi u \\ -\Delta \phi + \partial_t \,\mathrm{div} A = \rho, \qquad\quad \rho = |u|^2 \\ \Box A + \nabla(\partial_t \phi + \mathrm{div} A) = J, \qquad J = 2Im(\bar{u}, \nabla_A u) \end{cases}$$

where A represents the magnetic potential and u is the complex scalar field of nonrelativistic charged particles,

$$(u, A, \phi) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} \times \mathbb{R}^3$$

and $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$.

The system is invariant under the gauge transform:

$$(u', \phi', A') \to (e^{i\lambda}u, \phi - \partial_t \lambda, A + \nabla \lambda)$$

where $\lambda : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. To remove this degree of freedom we need to fix the gauge. In this article we choose to work in the Coulomb gauge

$$(2) \qquad\qquad\qquad\qquad\qquad \mathrm{div} A = 0$$

Under this assumption, the system can be rewritten as:

$$(3) \qquad \begin{cases} iu_t - \Delta_A u = \phi u \\ \Box A = PJ \end{cases}$$

where $\phi = (-\Delta)^{-1}(|u|^2)$ and $P = 1 - \nabla \mathrm{div} \Delta^{-1}$ is the projection on the divergence free vectors functions - also called Helmholtz projection. We consider the above system with a set of initial data chosen in Sobolev spaces:

$$(u(0), A(0), A_t(0)) = (u_0, A_0, A_1) \in H^s \times H^\sigma \times H^{\sigma-1}$$

The gauge condition (2) is conserved in time provided the initial data (A_0, A_1) satisfies it due to the form of the second equation in (3).

The charge and energy associated to the system are

$$Q(u) = \int_{\mathbb{R}^3} |u|^2 dx$$

$$E(u) = \int_{\mathbb{R}^3} |\nabla_A u|^2 + \frac{1}{2}(|A_t|^2 + |\nabla_x A|^2) + \frac{1}{2}|\nabla \phi|^2 dx$$

The local well-posedness of the system in various Sobolev spaces above the energy level is known, see [3], [2] as well as the existence of weak energy solutions [1]. The main outstanding problem which we seek to address is the *well-posedness in the energy space.*

Our main result is

Theorem 1. *The Maxwell-Schrödinger system* (3) *is globally well-posed in the energy space* $H^1 \times H^1 \times L^2$.

Well-posedness in our context is meant to be in the standard way. We prove that for N be sufficiently large and for each initial data

$$(u_0, A_0, A_1) \in H^N \times H^N \times H^{N-1}$$

there exists an unique global solution

$$(u, A) \in C(\mathbb{R}, H^N) \times C(\mathbb{R}, H^N) \cap C^1(\mathbb{R}, H^{N-1}).$$

Then we show that for rough initial data

$$(u_0, A_0, A_1) \in H^1 \times H^1 \times L^2$$

there exists a global solution

$$(u, A) \in C(\mathbb{R}, H^1) \times C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2).$$

which is the unique strong limit of the smooth solutions above.

Finally we establish that the rough solutions (u, A) depend continuously on the initial data.

The nonlinearities on the right hand side of both equations in (3) are fairly mild. Indeed, if Δ_A were replaced by Δ then it would be quite straightforward to iteratively close the argument in $X^{s,b}$ or Strichartz spaces. Thus the main difficulty stems from the linear magnetic Schrödinger equation.

For the magnetic potential A it is quite reasonable to hope to obtain an $X^{s,b}$ type regularity. Consequently, most of our analysis is devoted to the study of the linear equation

$$(4) \qquad\qquad iu_t - \Delta_A u = f, \qquad u(0) = u_0$$

Previous approaches establish Strichartz estimates with a loss for this equation in a perturbative manner, starting from the free Schrödinger equation. This fails for A in the energy space, and instead one needs to study directly the dispersive properties for the linear magnetic Schrödinger equation.

For a short frequency dependent time we produce a direct wave packet parametrix which equals the free flow modulo a phase correction. For larger time scales we instead obtain a weaker generalized wave packet decomposition. Together, these structures provide enough information in order to establish the required linear and multilinear dispersive estimates.

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Wave packet parametrices for evolutions governed by pdo's with rough symbols

JEREMY LOUIS MARZUOLA

(joint work with Jason L. Metcalfe, Daniel I. Tataru)

In this talk we consider evolution equations of the form

(1)
$$\begin{cases} (D_t + a^w(t,x,D) + ib^w(t,x,D))u = f, & \text{in } \mathbb{R}^+ \times \mathbb{R}^n \\ \\ u(0) = u_0, & \text{in } \mathbb{R}^n \end{cases}$$

where $a(t,x,\xi)$ and $b(t,x,\xi)$ are real symbols which are continuous in t and smooth with respect to x and ξ. The following abstract is mostly taken from the introduction given in [5].

The operator $a^w(t,x,D)$ is selfadjoint; if $b = 0$ then this formally guarantees that the above evolution is L^2 well-posed and the corresponding evolution operators $S(t,s)$ are L^2 isometries. The b^w term rougly contributes to the growth or decay of energy along the flow, depending on whether b is negative or positive.

We are interested in the phase space localization properties of the evolution operators $S(t,s)$. These are best described in terms of the Bargman transform,

(2)
$$(Tf)(x,\xi) = 2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}} \int e^{-\frac{1}{2}(x-y)^2} e^{i\xi(x-y)} f(y) \, dy,$$

which is an isometry from $L^2(\mathbb{R}^n)$ to the subspace of $L^2(\mathbb{R}^{2n})$ of functions satisfying the Cauchy-Riemann type relation

(3)
$$i\partial_\xi Tf = (\partial_x - i\xi)Tf.$$

The inversion formula is

(4)
$$f(y) = 2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}} \int e^{-\frac{1}{2}(x-y)^2} e^{i\xi(y-x)} (Tf)(x,\xi) \, dxd\xi.$$

One would like to describe the phase space localization of $S(t,s)$ relative to the Hamilton flow corresponding to (1). This is given by

(5)
$$\begin{cases} \dot{x} = a_\xi(t,x,\xi) \\ \dot{\xi} = -a_x(t,x,\xi) \end{cases}$$

We denote by $\chi(t,s)$ the corresponding family of cannonical transformations, and by

$$t \to (x^t, \xi^t)$$

the trajectories of the Hamilton flow.

This problem has already been considered in [9], [2]. There, they study the class $S_{00}^{0,(k)}$ of symbols which satisfy the bounds

(6)
$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le c_{\alpha\beta}, \qquad |\alpha| + |\beta| \ge k.$$

The main result has the form

Theorem 1. [9],[2] *Assume that the symbol $a(t,x,\xi)$ satisfies $a(t,x,\xi) \in S_{00}^{0,(2)}$ uniformly with respect to t. Then*

   a) The Hamilton flow is bilipschitz.

   b) The kernel $K(t,s)$ of the phase space operator $T^*S(t,s)T$ decays rapidly away from the graph of the Hamilton flow,

$$(7) \qquad |K(t,x,\xi,s,y,\eta)| \lesssim (1 + |(x,\xi) - \chi(t,s)(y,\eta)|)^{-N}.$$

However, for applications to nonlinear evolution equations one would like to relax the above class of symbols and replace uniform bounds by an integrability condition. For instance, in the context of the wave equation related results have been obtained in [8] under assumptions which correspond to replacing the L^∞ bounds in (6) with $L_t^1 L_x^\infty$.

In this article we go one step further and restrict the time integrability to the bicharacteristic rays. This is a much more natural condition from the point of view of applications. One motivation for this already appears in early works of Mizohata [6, 7] which is concerned with the b^w type terms. They consider the equation

$$Lu := \partial_t - i\Delta + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + c(x,t)u = f(x,t),$$

and show that a necessary condition for L to be well-posed in H^∞ is the bound

$$(8) \qquad \sup_{x \in \mathbb{R}^n, \omega \in \mathbb{S}^{n-1}, R>0} \left| Im \int_0^R b_1(x + r\omega) \cdot \omega \, dr \right| < \infty.$$

A slightly stronger version of (8) was shown to be sufficient for L^2 wellposedness in [1]. In the case where Δ is replaced by the variable coefficient operator $a_{jk}(x,t)\partial_j\partial_k$, then our assumptions are a natural extension of this condition.

Another motivation for this work comes from the study of general quasilinear Schrödinger equations. In [3] and [4], well-posedness is established in highly regular Sobolev spaces by using estimates for the corresponding linear equation.

Given a symplectic flow χ in $\mathbb{R} \times \mathbb{R}^{2n}$ we introduce the symbol class $S^{(k)}L_\chi^1$ of symbols q, which are smooth in (x,ξ), continuous in t and satisfy

$$(9) \qquad \sup_{x,\xi} \int_0^1 |\partial_x^\alpha \partial_\xi^\beta q(t,\chi(t,0)(x,\xi))| dt \le c_{\alpha\beta}, \qquad |\alpha| + |\beta| \ge k.$$

Then our condition for the symbol a is implicit, namely $a \in S^{(2)}L_\chi^1$ where χ is the Hamilton flow of a defined by (5). For the symbol b we will assume that $b \in S^{(1)}L_\chi^1$. Given such a and b we introduce the notation

$$(10) \quad \kappa_N = \max_{2 \le |\alpha|+|\beta| \le N} c_{\alpha\beta}^a + \max_{1 \le |\alpha|+|\beta| \le N} c_{\alpha\beta}^b, \qquad \kappa_0 = \max_{|\alpha|+|\beta|=2} c_{\alpha\beta}^a + \max_{|\alpha|+|\beta|=1} c_{\alpha\beta}^b$$

where $c_{\alpha\beta}^a$ and $c_{\alpha\beta}^b$ are as in (9) corresponding to the symbols a and b.

The other important parameter in our analysis corresponds to (8). We set

$$(11) \qquad M = \sup_{x,\xi} \; \sup_{0 \le t_0 \le t_1} \int_{t_0}^{t_1} b(t, x^t, \xi^t) dt$$

and assume that M is finite. Then our main result is

Theorem 2. [5] *a) Assume that the symbol $a(t, x, \xi)$ satisfies $a(t, x, \xi) \in S^{(2)} L^1_\chi$. Then the Hamilton flow defined by (5) is globally well defined and bilipschitz.*

b) Assume in addition that b is a symbol in $S^{(1)} L^1_\chi$ so that M given by (11) is finite and the following relation holds for some large N:

$$e^{2M} \kappa_0 \kappa_{4N} \ll 1.$$

Then the kernel $K(t, s)$ of the phase space operator $T^* S(t, s) T$ decays rapidly away from the graph of the Hamilton flow,

$$(12) \qquad |K(t, x, \xi, s, y, \eta)| \lesssim (1 + |(x, \xi) - \chi(t, s)(y, \eta)|)^{-N}.$$

The proof of part a) involves a simple application of Gronwall's inequality and the boundedness of the derivatives of a. For part b), the result follows by taking linearizations of a and b, which gives us an ODE along the flow for a function $v = Tu$

$$(13) \qquad (D_t + a + i(a_x \partial_\xi - a_\xi \partial_x) - \xi a_\xi + ib)v = Ev, \qquad v(0) = v_0 = Tu_0.$$

Apply Gronwall's inequality to the quadratic error term, then use a simple calculus lemma to move to higher derivatives of a and the control we have on the flow to change variables in order to apply the assumptions on a and b.

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Global existence for Gross-Pitaevskii equation on exterior domains of dimension three

RAMONA ANTON

Abstract. We prove global well-posedness in the energy space for the defocusing Gross-Pitaevskii equation on exterior domains of dimension three. The main ingredients are a description of the energy space inspired by that of P.Gérard [11] and a Strichartz estimate with loss of derivatives obtained combining a semi-classical Strichartz estimate [2, 6] with a smoothing effect on exterior domains [9].

The Gross-Pitaevskii equation is a cubic non-linear Schrödinger equation that appears in recent works on superfluidity and Bose-Einstein condensates (e.g. [1]):

$$(1) \qquad \begin{cases} i\partial_t u + \triangle u &= (|u|^2 - 1)u, \ on \ \mathbb{R} \times \Omega \\ u_{|t=0} &= u_0, \ on \ \Omega \\ \frac{\partial u}{\partial \nu}|_{\mathbb{R} \times \partial\Omega} &= 0. \end{cases}$$

It is a Hamiltonian equation, associated with the Hamiltonian

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega (|u|^2 - 1)^2 dx,$$

also called Ginzburg-Landau energy. For smooth enough data the energy is preserved in time, $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$. The natural energy space is

$$E = \{u \in H^1_{loc}(\Omega), \ \nabla u \in L^2(\Omega), \ |u|^2 - 1 \in L^2(\Omega)\}.$$

Notice that this energy space is not a linear space. Previous results where shown by P.E.Zhidkov [17, 18] in Zhidkov space $X^1(\mathbb{R})$, by F.Béthuel – J.-C. Saut [5] in the space of functions $1 + H^1(\mathbb{R}^d)$ for $d = 2, 3$ and recently by P.Gérard [11] in the energy space E, for $\Omega = \mathbb{R}^d$, $d = 2, 3, 4$. Following the work of P.Gérard [11], we show that, for $\Omega \subset \mathbb{R}^3$,

$$(2) \qquad E = \{c + v, c \in \mathbb{C}, \ |c| = 1, \ v \in \dot{H}^1(\Omega), \ |v|^2 + 2\Re(vc^{-1}) \in L^2(\Omega)\}.$$

This is a complete metric space for the distance

$$\delta_E(c+u, \tilde{c}+\tilde{v}) = |c - \tilde{c}| + \|\nabla v - \nabla\tilde{v}\|_{L^2(\Omega)} + \||v|^2 + 2\Re(\bar{c}v) - |\tilde{v}|^2 - 2\Re(\bar{\tilde{c}}\tilde{v})\|_{L^2(\Omega)}.$$

Solving (1) is equivalent with solving the Duhamel integral equation :

$$(3) \qquad u(t) = e^{it\triangle}u_0 - i\int_0^t e^{i(t-\tau)\triangle}F(u(\tau))\mathrm{d}\tau,$$

where $F(u) = (|u|^2 - 1)u$. The classical method of solving this equation consists in using a fixed point method in order to obtain a local existence result in E. Moreover, the fixed point method ensures that the conservation laws are verified. If the existence time depends only on $\mathcal{E}(u_0)$ and not on the profile of $u_0 \in E$, then, as $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$, we can bootstrap the argument and obtain global existence.

In order to apply this program here, since $E \not\subset L^2$, we have to analyze the action of the linear Schrödinger flow $e^{it\triangle}$ on E. We shall use a Strichartz inequality adapted to the exterior domain $\Omega = \complement\Theta$.

A Strichartz inequality [16, 12, 13, 14] is a space-time estimate of the linear flow that reads, on $\mathbb{R}^d$: for all (p, q) such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p \geq 2$, $(p, q, d) \neq (2, \infty, 2)$:

$$(4) \qquad \|e^{it\triangle}u_0\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^d))} \leq c\|u_0\|_{L^2(\mathbb{R}^d)}.$$

On $\mathbb{R}^d$ it follows from $\|e^{it\triangle}u_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c}{|t|^{\frac{d}{2}}}\|u_0\|_{L^1(\mathbb{R}^d)}$, called the dispersive $L^1 \to L^\infty$ estimate. In the Euclidean case this is a consequence of the exact form of the linear flow, known thanks to the Fourier transform.

As opposed to the $\mathbb{R}^d$ case, for bounded domains, the dispersive $L^1 \to L^\infty$ estimate fails for every $t > 0$. Following the approach of Burq-Gérard-Tzvetkov [8] on compact Riemannian manifolds we proved [2] a Strichartz estimate with loss of derivatives on domains ($d = 2, 3$) with compact boundaries using the double manifold. On the double manifold the regularity of the coefficients of the metric is only Lipschitz. To overcome this difficulty we regularize the coefficients of the metric at a frequency depending of the frequency of the initial data (see also [4]) and optimize the supplementary loss in the end. Using a similar method, Blair-Smith-Sogge [6] improved on the loss of derivatives. However, the loss is too important to obtain an existence result for $d = 3$.

In dimension three we restrict ourselves to the case of exterior domains. The non-trapping condition has been assumed by many authors in order to ensure dispersion. On the exterior of a non-trapping obstacle, $\Omega = \complement\Theta$, a light ray that propagates according to the laws of geometric optics quits any compact set in a finite interval of time. As the energy propagates along the geodesics, we have intuitively the dispersion of the flow. Let us mention the work of Staffilani-Tataru [15] : under the assumption $(\mathbb{R}^d, g)$, g non-trapping, they show a Strichartz estimate identical with (4). An important argument is the Kato smoothing effect which holds for such geometries.

Burq-Gérard-Tzvetkov [9] showed that Kato smoothing effect holds on the exterior of non-trapping obstacles. We combine this with the semi-classical Strichartz estimate [2, 6] holding on small intervals of time depending on the frequency of the initial data. From Staffilani-Tataru [15] we know that, away from the obstacle, Strichartz estimate holds without loss of derivatives. Close to the obstacle we have a loss :

$$(5) \qquad \|e^{it\triangle}u_0\|_{L^p([-T,T], L^q(\Omega))} \leq c\|u_0\|_{H^{\frac{1}{3p}+\epsilon}(\Omega)}.$$

For $u_0 \in E$, the decomposition (2) holds : $u_0 = c_0 + v_0$, $c_0 \in \mathbb{C}$, $|c| = 1$, $v_0 \in \dot{H}^1(\Omega)$. We have $e^{it\triangle}u_0 = u_0 + \frac{e^{it\triangle}-1}{\sqrt{-\triangle}}\sqrt{-\triangle}v_0$ and we show that $e^{it\triangle} : E \to E$. Moreover,

$$sup_{|t|<T}\delta_E(e^{it\triangle}u_0, e^{it\triangle}\tilde{u}_0) \leq c\delta_E(u_0, \tilde{u}_0).$$

From (5) we obtain $\|e^{it\triangle}u_0\|_{L^p_T L^\infty(\Omega)} \leq 1 + c\|\nabla v_0\|_{L^2(\Omega)}$. We denote by $u_L(t) = e^{it\triangle}u_0$ the linear flow associated with $u_0 \in E$. We show that $w = u - u_L$, the non-linear part in the Duhamel formula (3), always belongs to $H^1(\Omega)$. We prove

that

$$\Phi(w) = -i \int_0^t e^{i(t-\tau)\triangle} F(u_L + w)(\tau) d\tau$$

has a fixed point in $X_T = C([-T,T], H^1(\Omega)) \cap L^p([-T,T], L^\infty(\Omega))$, for some $T = T(\mathcal{E}(u_0))$. From the conservation of the energy we obtain global existence in the energy space.

Some interesting open problems are the stability of special solutions of the Gross-Pitaevskii equation on exterior domains. For example, imposing Dirichlet boundary conditions we have that the minimizer of the energy is not a constant, as opposed to the $\mathbb{R}^d$ case. The exterior of a cylinder is also an interesting domain for the Gross-Pitaevskii equation, as it describes the motion of a laser in a plasma. Obtaining some Strichartz estimate adapted to this domain would allow to address the Cauchy problem.

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A Centre-Stable Manifold for the Focussing Cubic NLS in $\mathbb{R}^{1+3}$

Marius Beceanu

Consider the focussing cubic nonlinear Schrödinger equation in $\mathbb{R}^3$:

$$(1) \qquad\qquad i\psi_t + \Delta\psi = -|\psi|^2\psi.$$

From a physical point of view, the NLS equation in $\mathbb{R}^3$ with cubic nonlinearity and the focussing sign (1) describes, to a first approximation, the self-focussing of optical beams due to the nonlinear increase of the refraction index. As such, the equation appeared for the first time in the physical literature in 1965, in [Kel].
It admits special solutions of the form $e^{it\alpha}\phi$, where $\phi \in \mathcal{S}(\mathbb{R}^3)$ is a positive ($\phi > 0$) solution of

$$(2) \qquad\qquad -\Delta\phi + \alpha\phi = \phi^3.$$

The space of all such solutions, together with those obtained from them by rescaling and applying phase and Galilean coordinate changes, called standing waves, is the eight-dimensional manifold of functions of the form $e^{i(v\cdot+\Gamma)}\phi(\cdot - y, \alpha)$.
We investigate the stability of standing waves under small perturbations and prove the existence of a codimension-one asymptotically stable manifold.
For a parameter path $\pi = (v_k, D_k, \alpha, \Gamma)$ such that $\|\dot\pi\|_\infty + \|\langle t\rangle\dot\pi(t)\|_1 < \infty$, define the soliton path $W(\pi(t))$ with the respective speed, displacement correction, scale, and complex phase correction at time t. The main result is the following:

Theorem 1. *There exists a local codimension-one Lipschitz manifold $\mathcal{N}$ in $\Sigma = H^1 \cap |x|^{-1}L^2$, containing the eight-dimensional manifold of standing waves, such that equation (1) has a global solution ψ if we start with initial data $\psi(0)$ on the manifold $\mathcal{N}$.*

Furthermore, the solution depends Lipschitz continuously on the initial data and decomposes into a moving soliton and a dispersive term: $\psi = W(\pi(t)) + R(t)$, with

$$(3) \qquad\qquad \|\dot\pi\|_\infty + \|\langle t\rangle\dot\pi(t)\|_1 \leq C\|\psi(0) - W(\pi(0))\|_\Sigma$$

and

$$(4) \qquad\qquad \|R\|_{L^\infty_t L^2_x \cap L^2_t L^6_x \cap \langle t\rangle^{-1/2}L^2_t L^{6+\infty}_x} \leq C\|\psi(0) - W(\pi(0))\|_\Sigma.$$

The dispersive term scatters: $R(t) = e^{it\Delta}f_0 + o_{L^2}(1)$, for some $f_0 \in L^2$.

Moreover, for a solution ψ of initial data $\psi(0) \in \mathcal{N}$, one has that $\psi(t) \in \Sigma$ for all t and $\psi(t) \in \mathcal{N}$ for sufficiently small t.

Finally, $\mathcal{N}$ is a centre-stable manifold for this equation in the sense of Bates, Jones [BatJon].

Outline of the proof The proof is based on the modulation method introduced by Soffer and Weinstein for the L^2-subcritical case and adapted by Schlag to the L^2-supercritical case. An important part of the proof is the Keel-Tao endpoint Strichartz estimate in $\mathbb{R}^3$ for the nonselfadjoint Hamiltonian obtained by linearizing (1) around a standing wave solution.

All results in this paper depend on the standard spectral assumption that the Hamiltonian

$$
(5) \qquad \mathcal{H} = \begin{pmatrix} \Delta + 2\phi(\cdot,\alpha)^2 - \alpha & \phi(\cdot,\alpha)^2 \\ -\phi(\cdot,\alpha)^2 & -\Delta - 2\phi(\cdot,\alpha)^2 + \alpha \end{pmatrix}
$$

has no embedded eigenvalues in the interior of its essential spectrum.

Under this assumption, we completely describe the spectrum of $\mathcal{H}$ following [Sch]. It consists of an absolutely continuous part $(-\infty, -\alpha] \cup [\alpha, \infty)$, a generalized eigenspace at zero with four eigenvectors and four generalized eigenvectors, and two conjugate imaginary eigenvalues.

Linearize the solution to equation (1) around a soliton by writing $\Psi = W + R$, where $W = e^{i\theta}\phi(x - y, \alpha)$ is a moving soliton, determined by the parameter path $\pi = (\Gamma, D, \alpha, v)$, while R is an error that needs to be controlled, satisfying its own equation.

The main difficulty lies in dealing with the unstable mode of the equation, which corresponds to the imaginary eigenvalue $i\sigma$ of $\mathcal{H}$. To address this, [Sch] showed that the solution of the linearized equation does not grow exponentially in time if and only if the initial data $Z(0)$ is on a certain codimension-one manifold, tangent to $Ker(P_+(0))$. This choice eliminates the effect of the unstable eigenvalue.

Another difficulty lies in the presence of the zero eigenvectors. Left unchecked, the generalized zero eigenspace would lead to polynomial growth of the solution. We impose an orthogonality (modulation) condition that bounds the zero eigenspace component of the solution and leads to a system of modulation equations for the parameter path π. However, π must satisfy a condition of the form $\langle t \rangle \dot{\pi}(t) \in L^1$. The modulation equations translate it into

$$
(6) \qquad \int_0^\infty t \|R(t)\|_{6+\infty}^2 \, dt < \infty.
$$

Since this condition goes beyond the decay provided by Strichartz or smoothing estimates, in order to achieve it we need to impose a decay condition on the initial data $R(0)$.

Schlag [Sch] dealt with this problem by imposing an L^1 decay condition on the initial data and using $L^1 \to L^\infty$ dispersive estimates. In this manner, he proved global existence and decay properties for $H^1 \cap W^{1,1}$ initial data on a codimension-one manifold.

A more convenient estimate is

Lemma 2. *Consider the equation*

$$
(7) \qquad i\partial_t U + \mathcal{H}P_c U = RHS(t), \ U(0) \ given.
$$

Then, for $q < 4/3$, $\beta < 2/q - 1$,

$$
(8) \qquad \int_0^\infty \langle t \rangle^{2\beta} \|P_c U\|_{6+\infty}^2 \, dt \le C(\|U(0)\|_{q \cap 2} + \|RHS\|_{\langle t \rangle^{-\beta} L_t^2 L_x^{1 \cap 6/5}})^2.
$$

This leads to the condition that $R(0) \in L^{4/3-\epsilon}$. The equation is $H^{1/2}$-critical. Thus, with the help of endpoint Strichartz estimates we prove global existence for small initial data in $H^{1/2} \cap L^{4/3-\epsilon}$ on the codimension-one submanifold that eliminates the unstable eigenvalue. However, this Banach space is not invariant under the action of the Hamilton flow, so we need to replace it with the weaker space Σ of the main theorem.

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The Interaction Morawetz Inequality on $\mathbb{R}^2$

JAMES COLLIANDER

(joint work with Manoussos Grillakis and Nikolaos Tzirakis)

0. GENERALIZED VIRIAL IDENTITY

This talk describes work[1] with M. Grillakis and N. Tzirakis. We recall discussion from [4] inspired by [7] and [3]. Suppose $\phi : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ nicely solves $i\partial_t \phi + \Delta \Phi = \mathcal{N}$. We define the *Morawetz action with virial weight $a : \mathbb{R}^d \to \mathbb{R}$ of ϕ* to be

$$M_a[\phi](t) = \int_{\mathbb{R}^d} \nabla a \cdot 2\mathrm{Im}(\overline{\phi}\nabla\phi)(t)dx.$$

Local conservation identities produce the *generalized virial identity*

$$(1) \qquad \partial_t M_a[\phi] = \int_{\mathbb{R}^d} (-\Delta\Delta a)|\phi|^2 + 4a_{jk}\Re(\overline{\phi}_j\phi_k) + 2a_j\{\mathcal{N},\phi\}_p^j \; dx$$

where $\{\mathcal{N},\phi\}_p = \Re[\mathcal{N}\nabla\overline{\phi} - \phi\nabla\overline{\mathcal{N}}]$ and a_j denotes $\partial_{x^j}a$, etc. Suppose $i\partial_t u + \Delta u = F'(|u|^2)u$ for $u : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{C}$ with $F' \geq 0$, so defocusing. The tensor product $\phi(t,x_1,x_2) = u(t,x_1)u(t,x_2)$ satisfies[2] a defocusing equation on $\mathbb{R}_t \times \mathbb{R}_x^6$; the choice $a(x_1,x_2) = |x_1 - x_2|$ in (1) and calculations imply the *interaction Morawetz estimate on $\mathbb{R}_x^3$*, valid for all defocusing equations:

$$(2) \qquad \int_0^T \int_{\mathbb{R}_x^3} |u(t,x)|^4 dxdt \lesssim \|u_0\|_{L_x^2}^3 \|\nabla u(t)\|_{L_T^\infty L_x^2}.$$

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[1]At this Oberwolfach workshop, F. Planchon announced [9] similar results obtained independently and simultaneously in work with L. Vega.

[2]This idea is due to Andrew Hassell.

1. Improved $L^4_{t,x}(\mathbb{R}_t \times \mathbb{R}^2_x)$ Estimate with $T^{1/3}$ loss

For an analog of (2) in two space dimensions, consider $a(x_1, x_2) = f(|x_1 - x_2|)$ with f smooth, convex and depending upon a parameter M, such that, for $|x| \lesssim M$, $f(|x|) = (2M)^{-1}|x|^2[1 - \log(|x|M^{-1})]$ and $f(|x|) = C|x|$ for $|x| \gg M$. The calculations leading to (2) produce an error term of unfavorable sign in the region $|x_1 - x_2| \gtrsim M$ which may be crudely estimated using mass conservation. The choice $M = T^{1/3}$ produces the *interaction Morawetz estimate with $T^{1/3}$ loss* valid for all defocusing equations

$$(3) \qquad \int_0^T \int_{\mathbb{R}^2_x} |u(t,x)|^4 dx dt \lesssim T^{1/3}\|u_0\|^3_{L^2_x}\|\nabla u\|_{L^\infty_T L^2_x} + T^{1/3}\|u_0\|^4_{L^2_x}.$$

This estimate was obtained in [5], improving upon [6] which had $T^{1/2}$.

2. Optimal Virial Weight Function

Efforts to absorb the error term into the left side of (3) failed but led to a new estimate. Set $w(s) = s^{-3}$ for $s \geq 1$ and zero otherwise; $r = |x|$. We implicitly define a (up to boundary conditions) by

$$\Delta a(r) = \frac{1}{r_0} \int_{r/r_0}^\infty s \log(\frac{r_0 s}{r}) w(s) ds \geq 0.$$

Explicit calculations reveal a perfect square and the limit as $r_0 \searrow 0$ gives

$$(4) \qquad \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}_2} \frac{\{|u(t,x_1)|^2 - |u(t,x_2)|^2\}^2}{|x_1 - x_2|^3} dx_1 dx_2 dt \lesssim \sup_{t \in [0,T]} M_a[u_1 u_2].$$

The left side of (4) involves [1] the Besov $B_2^{\frac{1}{2},2}$ norm. Thus, *any finite energy defocusing Schrödinger evolution $u : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}$ satisfies*

$$(5) \qquad \|u\|^2_{L^4_t L^8_x} \lesssim \|D^{1/2}|u|^2\|^2_{L^2_{t,x}} \lesssim \|u\|^3_{L^\infty_t L^2_x}\|\nabla u\|_{L^\infty_t L^2_x}.$$

Inequality (5) resembles the bilinear Strichartz estimate from [2].

3. Vector Commutator Proof

A vector commutator proof of (5) generalizes to higher dimensions. On $\mathbb{R}^2$, we have the Fourier transform formula $\mathcal{F}^{-1}(|\xi|^{-1}) = c_2|x|^{-1}$ so write $D^{-1}f(x) = \int_{\mathbb{R}^2} |x - y|^{-1} f(y) dy$. We define the *vector commutator operator $X = [x, D^{-1}]$* so $Xf(x) = \int_{\mathbb{R}^2} \frac{x-y}{|x-y|} f(y) dy$. Let $\rho(t,x) = \frac{1}{2}|u(t,x)|^2$ and $p(t,x) = \Im(\overline{u}\nabla u)(t,x)$. Calculations show that $\partial_j X^j = D^{-1}$ and

$$\partial_j X^k f = (D^{-1}\delta^k_j + [x^k, R_j])f$$

where $R_j = \partial_j D^{-1}$ has symbol $\frac{\xi_j}{|\xi|}$. Finally, we define the action

$$M(t) = \langle X \cdot p(t) | \rho(t) \rangle.$$

Local mass and momentum conservation identities for defocusing Schrödinger equations and rearranging shows that $\dot{M}(t) = S_1 + S_2 + S_3 + S_4$ with $S_i \geq 0, i = 1, \cdots, 4$ and $S_3 = \|D^{1/2}\rho\|_{L_x^2}^2$. Thus, we recover (5),

$$\int_0^t \|D^{\frac{1}{2}}\rho(t)\|_{L_x^2}^2 dt \leq M(t) - M(0) \leq 2\|u\|_{L_t^\infty L_x^2}^3 \|\nabla u\|_{L_t^\infty L_x^2}.$$

On $\mathbb{R}^d$, we have the formula $\mathcal{F}^{-1}(|\xi|^{1-d}) = c_d|x|^{-1}$ and can generalize the preceding discussion with $X = [x, D^{1-d}]$ and maintain $S_1, S_2, S_3, S_4 \geq 0$ but with $S_3 = \|D^{\frac{3-d}{2}}\rho\|_{L_x^2}^2$ to obtain: *any finite energy defocusing Schrödinger evolution* $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ *satisfies*

$$\|D^{\frac{3-d}{2}}|u|^2\|_{L_{t,x}^2}^2 \leq 2\|u\|_{L_t^\infty L_x^2}^3 \|\nabla u\|_{L_t^\infty L_x^2}.$$

Thus, the vector commutator approach also recovers (2).

4. Simplified proof of Nakanishi's Scattering Result

Suppose u solves $i\partial_t u + \Delta u = |u|^{p-1}u$ with initial data $u_0 \in H^1(\mathbb{R}_x^2), p > 3$. Define the Strichartz space S^1 via $\|u\|_{S^1} = \sup_{q,r} \|\langle\nabla\rangle u\|_{L_t^q L_x^r}, \frac{2}{q} + \frac{2}{r} = \frac{2}{2}, 2 \leq q, r \leq \infty, (q,r) \neq (2,\infty)$. Based on (5), we can decompose $\mathbb{R}_t = \cup_{j=1}^J I_j$ into disjoint intervals I_j on which $\|u\|_{L_{I_j}^4 L_x^8} = \delta \ll 1$ with $J = J(u_0) < \infty$. Using Duhamel's formula, finite energy, Strichartz estimates and Hölder's inequality, we have for some $\epsilon > 0$ that for all j

$$\|u\|_{S_{I_j}^1} \lesssim \|u_0\|_{H^1} + \|u\|_{L_{I_j}^4 L_x^8}^\epsilon \|u\|_{S_{I_j}^1}^{p-\epsilon}.$$

Since $J < \infty, u$ is globally bounded in S^1.

This proof gives a better quantification on the spacetime size of u than in [8]. Using the I-method and a Morawetz bootstrap (as in [3]), scattering also holds for defocusing $L^2(\mathbb{R}^2)$-subcritical problems with nonlinearity $|u|^{2k}u, 2 \leq k \in \mathbb{N}$, for initial data $u_0 \in H^s$ provided $1 > s > 1 - \frac{1}{4k-3}$.

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Stability of multipeakons

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(joint work with Khaled El Dika)

Abstract. The Camassa-Holm equation possesses well-known peaked solitary waves that are called peakons. Their orbital stability has been established by Constantin and Strauss in [4]. We present here a result on the stability of ordered trains of peakons as well as on the stability of multipeakons.

The Camassa-Holm equation (C-H)$_\kappa$, $\kappa \geq 0$,

$$(1) \qquad u_t - u_{txx} = -2\kappa u_x - 3uu_x + 2u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}^2,$$

can be derived as a model for the propagation of unidirectional shalow water waves over a flat bottom by writing the Green-Naghdi equations in Lie-Poisson Hamiltonian form and then making an asymptotic expansion which keeps the Hamiltonian structure ([3], [13]). It was also found independently by Dai [7] as a model for nonlinear waves in cylindrical hyperelastic rods.

(C-H) is completely integrable (see [3]). It possesses among others the following invariants

$$(2) \quad E(v) = \int_{\mathbb{R}} v^2(x) + v_x^2(x)\, dx \text{ and } F(v) = \int_{\mathbb{R}} v^3(x) + v(x)v_x^2(x) + 2\kappa v^2(x)\, dx$$

and can be written in Hamiltonian form as

$$(3) \qquad\qquad\qquad \partial_t E'(u) = -\partial_x F'(u) \quad .$$

For $\kappa > 0$ it possesses smooth positive solitary waves $\varphi_{\kappa,c}$ with speed $c > 2\kappa$ whose orbital stability has been proved in [5] by applying the classical spectral method initiated by Benjamin [2] (see also [12]). In [11], following the general method developed in [14] (see also [10]), the authors proved the stability of ordered trains of such solitary waves. It is worth recalling that this general method requires principally two ingredients : A property of almost monotonicity which says that for a solution close to $\varphi_{\kappa,c}$, the part of the energy traveling at the right of $\varphi_{\kappa,c}(\cdot - ct)$ is almost time decreasing; A dynamical proof of the stability of the solitary wave using the spectral approach (as in [2] or [12] for instance).

In this talk we consider the Camassa-Holm equation in the case $\kappa = 0$, that is

$$(4) \qquad\qquad u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}^2.$$

Henceforth, we refer to (4) as the Camassa-Holm equation (C-H). (4) possesses also solitary waves but they are non smooth and are called peakons. They are given by

$$u(t,x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{|x-ct|}, \quad c \in \mathbb{R}.$$

Their stability seems not to enter the general framework mentioned above . However, Constantin and Strauss [4] succeeded in proving their orbital stability by a direct approach. In this work, following the general strategy initiated in [14](note that due to the reasons mentioned above, the general method of [14] is not directly applicable here), we combine the monotonicity result proved in [10] with localized versions of the estimates established in [4] to derive the stability of the trains of peakons.

Before stating the main result we have to introduce the function space where we will define the flow of the equation. For I a finite or infinite interval of $\mathbb{R}$, we denote by $Y(I)$ the function space[1]

$$(5) \quad Y(I) := \left\{ u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})), \ u_x \in L^\infty(I; BV(\mathbb{R})) \right\}.$$

We are now ready to state our main result.

Theorem 1. *Let be given N velocities $c_1, .., c_N$ such that $0 < c_1 < c_2 < .. < c_N$. There exist γ_0, $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $u \in Y([0,T[)$, with $0 < T \leq \infty$, is a solution of (C-H) satisfying*

$$(6) \qquad \|u_0 - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - z_j^0)\|_{H^1} \leq \varepsilon^2$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$, with $L > L_0$, then there exist $x_1(t), .., x_N(t)$ such that

$$(7) \qquad \sup_{[0,T[} \|u(t,\cdot) - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j(t))\|_{H^1} \leq A(\sqrt{\varepsilon} + L^{-1/8})$$

and

$$(8) \qquad x_j(t) - x_{j-1}(t) > L/2, \quad \forall t \in [0,T[\quad .$$

As discovered by Camassa and Holm [3], (C-H) possesses also special solutions called multipeakons given by

$$u(t,x) = \sum_{i=1}^{N} p_j(t) e^{-|x-q_j(t)|},$$

where $(p_j(t), q_j(t))$ satisfy some differential system. In [3] and furthermore in [1] the asymptotic behavior of the multipeakons is studied. In particular, the limits as t tends to $+\infty$ and $-\infty$ of $p_i(t)$ and $\dot{q}_i(t)$ are determined. Combining

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[1]$W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation

these asymptotics with the preceding theorem we get the following result on the stability of the variety Λ of $H^1(\mathbb{R})$ defined by

$$\Lambda := \left\{ v = \sum_{i=1}^N p_j e^{-|\cdot - q_j|}, \ (p_1, .., p_N) \in (\mathbb{R}_+)^N, \ q_1 < q_2 < .. < q_N \right\}.$$

Corollary 2. *Let be given N positive real numbers $p_1, .., p_N$ and N real numbers $q_1 < .. < q_N$. For any $B > 0$ and any $\gamma > 0$ there exists $\alpha > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies $u_0 - u_{0,xx} \in \mathcal{M}_+$ with*

$$(9) \qquad \|u_0 - u_{0,xx}\|_{\mathcal{M}} \le B \quad and \quad \|u_0 - \sum_{j=1}^N p_j \exp(\cdot - q_j)\|_{H^1} \le \alpha$$

then

$$(10) \qquad \forall t \in \mathbb{R}, \quad \inf_{P \in (\mathbb{R}_+)^N, Q \in \mathbb{R}^N} \|u(t, \cdot) - \sum_{j=1}^N p_j \exp(\cdot - q_j)\|_{H^1} \le \gamma.$$

As mentioned above, the proof of the stability of trains of peakons does not enter the general framework ([14], [10], [11]) on orbital stability of ordered trains of solitary waves. However, the strategy of combining the orbital stability of a single solitary wave with a monotonicity result seems to be quite robust.

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On the focusing energy-critical nonlinear Schrödinger equation

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(joint work with Rowan Killip, Monica Visan)

We consider the initial value problem for focusing energy-critical nonlinear Schrödinger equation

$$(1) \qquad \begin{cases} iu_t + \Delta u = -|u|^{\frac{4}{d-2}} u, \\ u(x,0) = u_0(x) \end{cases}$$

in dimension $d \geq 3$. The minus sign in front of the nonlinearity corresponds to the "focusing" case.

The equation (1) is called "energy critical" as the natural scaling transformation

$$(2) \qquad u(t,x) \to u_\lambda(t,x) = \lambda^{-\frac{d-2}{2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$$

leaves both the equation (1) and the energy

$$(3) \qquad E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{d-2}{2d} \|u(t)\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}$$

invariant.

From the classical result of Cazenave and Weissler [1], [2], we know that (1) is locally wellposed; however, due to the critical nature of the problem, the lifespan of the local solution depends on the profile of the initial data, rather than merely on its $\dot{H}^1$ norm. Nevertheless, when the $\dot{H}^1$ norm of u_0 is small enough, the solution is global and scatters.

On the other hand, the equation (1) admits a global soliton solution. That is,

$$W(t,x) = W(x) = \frac{1}{(1 + \frac{|x|^2}{d(d-2)})^{\frac{d-2}{2}}} \in \dot{H}_x^1(\mathbb{R}^d)$$

solves the static nonlinear Schrödinger equation:

$$\Delta W + |W|^{\frac{4}{d-2}} W = 0.$$

It is believed that the ground state W is the minimal kinetic energy blowup solution; here, by blowup we understand that the $L_{t,x}^{\frac{2(d+2)}{d-2}}$-norm is infinite. In this paper, we verify the conjecture in the spherical symmetric case.

Theorem 1. *Let $d \geq 3$ and let u be a spherically symmetric solution to (1) with maximal lifespan I. Assume also that*

$$\sup_{t \in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2.$$

Then $I = (-\infty, \infty)$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt \leq C(E_0).$$

A more effective criterion for wellposedness can be obtained from Theorem 1 via coercive properties of the ground state W.

Corollary 2. *Let $d \geq 3$ and let $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$ be spherically symmetric and such that $\|\nabla u_0\|_2 < \|\nabla W\|_2$ and $E(u_0) < E(W)$. Then the corresponding solution u to (1) is global and moreover*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} dx dt \leq C(E(u_0)).$$

For $d = 3, 4, 5$, Corollary 2 was proved by Kenig-Merle [3]. Their paper also contains Theorem 1 for $d = 3, 4, 5$.

The result in Theorem 1 is sharp in the sense that W is a solution to (1) which blows up at infinity in both time directions.

Moreover, in this paper, we also prove the following concentration result:

Theorem 3. *(Blowup solutions concentrate kinetic energy). Let u be a spherically symmetric solution to (1) that blows up at time $T^* \in [-\infty, +\infty]$. Assume also that*

$$\limsup_{t \to T*} \|\nabla u(t)\|_2 < \infty.$$

If T^* is finite, then there exists a sequence $t_n \to T^*$ such that for any sequence $R_n \in (0, \infty)$ obeying $|T^* - t_n|^{-\frac{1}{2}} R_n \to \infty$,

$$\limsup_{n \to \infty} \int_{|x| \leq R_n} |\nabla u(t_n, x)|^2 dx \geq \|\nabla W\|_2^2.$$

If $|T^*| = \infty$, then there exists a sequence $t_n \to T^*$ such that for any sequence $R_n \in (0, \infty)$ obeying $|t_n|^{-\frac{1}{2}} R_n \to \infty$,

$$\limsup_{n \to \infty} \int_{|x| \leq R_n} |\nabla u(t_n, x)|^2 dx \geq \|\nabla W\|_2^2.$$

In order to prove Theorem 1, we follow the concentration compactness approach of Kenig and Merle [3]. More precisely, using a concentration-compactness technique based on linear profile decomposition [4], we reduce matters to studying a special type of solution.

Definition 1. (Almost periodicity modulo scaling). Let $d \geq 3$. A solution u to (1) with lifespan I is said to be *almost periodic modulo scaling* if there exists a function $N : I \to \mathbb{R}^+$ and a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x| \geq C(\eta)/N(t)} |\nabla u(t,x)|^2 dx \leq \eta \ \text{ and } \int_{|\xi| \geq C(\eta)N(t)} |\xi|^2 |\hat{u}(t,\xi)|^2 d\xi \leq \eta.$$

We prove that any failure of Theorem 1 must be caused by a very special type of solution.

Theorem 4. *(Reduction to almost periodic solutions). Suppose $d \geq 3$ is such that Theorem 1 failed. Then there exists a spherically symmetric maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ such that $\sup_{t \in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2$, which is almost periodic modulo scaling and blows up both forward and backward in time. Moreover, we have the frequency bound*

$$\inf_{t \in I} N(t) > 0.$$

We preclude the existence of the solutions that satisfy the criteria in Theorem 4, thus prove Theorem 1 by the following rigidity result.

Theorem 5. *(Rigidity). There are no spherically symmetric solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1) that are almost periodic modulo scaling and obey*

$$\sup_{t \in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2, \ \|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbb{R}^d)} = \infty, \ \text{and} \ \inf_{t \in I} N(t) > 0.$$

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