Mathematisches Forschungsinstitut Oberwolfach

Report No. 51/2007

Modulformen

Organised by
Siegfried Böcherer (Mannheim)
Tomoyoshi Ibukiyama (Osaka)
Winfried Kohnen (Heidelberg)

October 28th – November 3rd, 2007

Abstract. The meeting brought together 24 mathematicas working on some of the many aspects of modular forms. The main focus was on holomorphic modular forms (mainly in many variables). This theory is very rich in explicit structures. Central themes were explicit liftings and their properties and the detailed study of modular forms for $GSp(4)$, as well as relations to arithmetic and algebraic geometry.

Mathematics Subject Classification (2000): 11xx.

Introduction by the Organisers

The workshop *Modulformen*, organised by Siegfried Böcherer (Mannheim), Tomoyoshi Ibukiyama (Osaka) and Winfried Kohnen (Heidelberg) was held October 28 -November 3, 2007. This meeting was attended by 24 participants with different backgrounds reflecting some of the many aspects of the theory of modular forms. One of the callenging features of this theory is that techniques using Whittaker models do not apply here. On the other hand, the theory is very rich in explicit structures and has immediate connections to arithmetic and geometry.

Some of the main topics (presented in one-hour talks) are

- Maass forms and their role in arithmetic and geometry (Bruinier, Bringman)
- Application to Algebraic Geometry (Dummigan, Yoshida, Gritsenko)
- The fine structure of modular forms for $GSp(4)$ (Poor, Wakatski, Roberts, Schmidt)
- Properties of explicit liftings (Panchishkin, Katsurada, Heim, Schulze-Pillot)

The third and forth topic are not only interesting in their own right but they are also a testing ground for general conjectures. We explictly mention the theory of newforms by Roberts and Schmidt for $GSp(4)$.
Explicit liftings can be viewed as examples for Langlands functoriality. The emphasis is on the explicit description of these liftings in terms of modular forms (not only representations). A prototype is the Duke-Imamoglu-Ikeda-lifting, which appeared in several talks.
Other talks dealt with theta series, Poincare series, generation of spaces of modular forms, converse theorems and spherical functions on p-adic spaces. Beyond the talks, there was much opportunity for scientific interaction among the participants.

Workshop: Modulformen

Table of Contents

Abstracts

Heegner divisors, L-functions and harmonic weak Maass forms

Jan Hendrik Bruinier
(joint work with Ken Ono)

1. Introduction and Statement of Results

Half-integral weight modular forms play important roles in arithmetic geometry and number theory. Thanks to the theory of theta functions, such forms include important generating functions for the representation numbers of integers by quadratic forms. Among weight 3/2 modular forms, one finds Gauss' function ($q := e^{2\pi i\tau}$ throughout)

$$\sum_{x,y,z\in\mathbb{Z}} q^{x^2+y^2+z^2} = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \cdots ,$$

which is essentially the generating function for class numbers of imaginary quadratic fields, as well as Gross's theta functions which enumerate the supersingular reductions of CM elliptic curves.

In the 1980s, Waldspurger [Wa], and Kohnen and Zagier [KZ, K] established that half-integral weight modular forms also serve as generating functions of a different type. Using the Shimura correspondence [Sh], they proved that certain coefficients of half-integral weight cusp forms essentially are square-roots of central values of quadratic twists of modular L-functions. When the weight is 3/2, these results appear prominently in works on the ancient "congruent number problem" [T], as well as the deep works of Gross, Zagier and Kohnen [GZ, GKZ] on the Birch and Swinnerton-Dyer Conjecture.

In analogy with these works, Katok and Sarnak [KS] employed a Shimura correspondence to relate coefficients of weight 1/2 Maass forms to sums of values and sums of line integrals of Maass cusp forms. We investigate the arithmetic properties of the coefficients of a different class of Maass forms, the weight 1/2 harmonic weak Maass forms.

A *harmonic weak Maass form of weight* $k \in \frac{1}{2}\mathbb{Z}$ *on* $\Gamma_0(N)$ (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$) is a smooth function on $\mathbb{H}$, the upper half of the complex plane, which satisfies:

(i) $f\mid_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$;
(ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplacian on $\mathbb{H}$;
(iii) There is a polynomial $P_f = \sum_{n\le 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Throughout, for $\tau \in \mathbb{H}$, we let $\tau = u + iv$, where $u, v \in \mathbb{R}$, and we let $q := e^{2\pi i\tau}$.

The polynomial P_f, the *principal part of f at ∞*, is uniquely determined. If P_f is non-constant, then f has exponential growth at the cusp ∞. Similar remarks apply at all of the cusps.

Spaces of harmonic weak Maass forms include *weakly holomorphic modular forms*, those meromorphic modular forms whose poles (if any) are supported at cusps. We are interested in those harmonic weak Maass forms which do not arise in this way. Such forms have been a source of recent interest due to their connection to Ramanujan's mock theta functions (see [BO1, BO2, Zw1, Zw2]). For example, it turns out that

$$M_f(\tau) := q^{-1} f(q^{24}) + 2i\sqrt{3} \cdot N_f(\tau) \tag{1}$$

is a weight 1/2 harmonic weak Maass form, where

$$N_f(\tau) := \frac{i}{\sqrt{3\pi}} \sum_{n\in\mathbb{Z}} \Gamma(1/2, 4\pi(6n+1)^2 v) q^{-(6n+1)^2}$$

is a period integral of a theta function, $\Gamma(a,x)$ is the incomplete Gamma function, and $f(q)$ is Ramanujan's mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

This example reveals two important features common to all harmonic weak Maass forms on $\Gamma_0(N)$. Firstly, all such f have Fourier expansions of the form

$$f(\tau) = \sum_{n \gg -\infty} c^+(n) q^n + \sum_{n<0} c^-(n) W(2\pi n v) q^n, \tag{2}$$

where $W(x) = W_k(x) := \Gamma(1-k, 2|x|)$. We call $\sum_{n\gg-\infty} c^+(n)q^n$ the *holomorphic part* of f, and we call its complement its *non-holomorphic part*. Secondly, the non-holomorphic parts are period integrals of weight $2-k$ modular forms. Equivalently, $\xi_k(f)$ is a weight $2-k$ modular form on $\Gamma_0(N)$, where ξ_k is a differential operator which is essentially the Maass lowering operator.

Every weight $2-k$ cusp form is the image under ξ_k of a weight k harmonic weak Maass form. The mock theta functions correspond to those forms whose images under $\xi_{1/2}$ are weight 3/2 theta functions. We turn our attention to those weight 1/2 harmonic weak Maass forms whose images under $\xi_{\frac{1}{2}}$ are orthogonal to the elementary theta series. Unlike the mock theta functions, whose holomorphic parts are often generating functions in the theory of partitions (for example, see [BO1, BO2]), we show that these other harmonic weak Maass forms can be "generating functions" simultaneously for both the values and central derivatives of quadratic twists of weight 2 modular L-functions.

Although we treat the general case in [BrOn], to simplify exposition, here we assume that p is prime and that $G(\tau) = \sum_{n=1}^{\infty} B_G(n)q^n \in S_2^{new}(\Gamma_0(p))$ is a normalized Hecke eigenform with the property that the sign of the functional equation of

$$L(G,s)=\sum_{n=1}^{\infty}\frac{B_G(n)}{n^s}$$

is $\epsilon(G)=-1$. Therefore, we have that $L(G,1)=0$.

By Kohnen's theory of plus-spaces [K], there is a half-integral weight newform

$$g(\tau)=\sum_{n=1}^{\infty}b_g(n)q^n\in S^+_{3/2}(\Gamma_0(4p)), \tag{3}$$

unique up to a multiplicative constant, which lifts to G under the Shimura correspondence. For convenience, we choose g so that its coefficients are in F_G, the totally real number field obtained by adjoining the Fourier coefficients of G to $\mathbb{Q}$. It turns out that there is a weight $1/2$ harmonic weak Maass form on $\Gamma_0(4p)$ in the plus space, say

$$f_g(\tau)=\sum_{n\gg-\infty}c_g^+(n)q^n+\sum_{n<0}c_g^-(n)W(2\pi nv)q^n, \tag{4}$$

whose principal part P_{f_g} has coefficients in F_G, which also enjoys the property that $\xi_{\frac{1}{2}}(f_g)=\|g\|^{-2}g$, where $\|g\|$ denotes the usual Petersson norm.

A calculation shows that if $n>0$, then

$$b_g(n)=-4\sqrt{\pi n}\|g\|^2\cdot c_g^-(-n). \tag{5}$$

The coefficients $c_g^+(n)$ are more mysterious. We show that both types of coefficients are related to L-functions. To make this precise, for fundamental discriminants D let χ_D be the Kronecker character for $\mathbb{Q}(\sqrt{D})$, and let $L(G,\chi_D,s)$ be the quadratic twist of $L(G,s)$ by χ_D. These coefficients are related to these L-functions in the following way.

Theorem 1: *Assume that p is prime, and that $G\in S_2^{new}(\Gamma_0(p))$ is a newform. If the sign of the functional equation of $L(G,s)$ is $\epsilon(G)=-1$, then the following are true:*

(1) *If $\Delta<0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then we have*

$$L(G,\chi_\Delta,1)=32\|G\|^2\|g\|^2\pi^2\sqrt{|\Delta|}\cdot c_g^-(\Delta)^2.$$

(2) *If $\Delta>0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right)=1$, then*

$$L'(G,\chi_\Delta,1)=0 \text{ if and only if } c_g^+(\Delta) \text{ is algebraic.}$$

Remark 1: In Theorem 1 (2), we have that $L(G,\chi_\Delta,1)=0$ since the sign of the functional equation of $L(G,\chi_\Delta,s)$ is -1. Therefore it is natural to consider derivatives in these cases.

Remark 2: The f_g are uniquely determined up to the addition of a weight $1/2$ weakly holomorphic modular form with coefficients in F_G. Furthermore, absolute values of the nonvanishing coefficients $c_g^+(n)$ are typically asymptotic to subexponential functions in n. For these reasons, Theorem 1 (2) cannot be simply modified to obtain a formula for $L'(G, \chi_\Delta, 1)$. It would be very interesting to obtain a more precise relationship between these derivatives and the coefficients $c_g^+(\Delta)$.

Theorem 1 (1) follows from Kohnen's theory (see Corollary 1 on page 242 of [K]) of half-integral newforms, the existence of f_g, and (5). The proof of Theorem 1 (2) is more difficult, and it involves a detailed study of Heegner divisors. We establish that the algebraicity of the coefficients $c_g^+(\Delta)$ is dictated by the vanishing of certain twisted Heegner divisors in the Jacobian of $X_0(p)$. This result, when combined with the work of Gross and Zagier [GZ], will imply Theorem 1(2).

Our argument depends on the construction of canonical differentials of the third kind for twisted Heegner divisors. We produce such differentials of the form $\eta_{\Delta,r}(z,f) = -\frac{1}{2}\partial\Phi_{\Delta,r}(z,f)$, where $\Phi_{\Delta,r}(z,f)$ are automorphic Green functions on $X_0(N)$ which are obtained as liftings of weight 1/2 harmonic weak Maass forms f. To define these liftings, we generalize the regularized theta lift due to Borcherds, Harvey, and Moore (for example, see [Bo1], [Br]). We then employ transcendence results of Waldschmidt and Scholl (see [W], [Sch]), for the periods of differentials, to relate the vanishing of twisted Heegner divisors in the Jacobian to the algebraicity of the corresponding canonical differentials of the third kind. By means of the q-expansion principle, we obtain the connection to the coefficients of harmonic weak Maass forms.

References

[Bo1] *R. Borcherds*, Automorphic forms with singularities on Grassmannians, Invent. Math. **132** (1998), 491–562.

[BO1] *K. Bringmann and K. Ono*, The $f(q)$ mock theta function conjecture and partition ranks, Invent. Math. **165** (2006), 243–266.

[BO2] *K. Bringmann and K. Ono*, Dyson's ranks and Maass forms, Ann. of Math., accepted for publication.

[BrOn] *J. Bruinier and K. Ono*, Heegner divisors, L-functions and harmonic weak Maass forms, preprint (2007).

[Br] *J. H. Bruinier*, Borcherds products on $\mathbf{O}(2,l)$ and Chern classes of Heegner divisors, Springer Lecture Notes in Mathematics **1780**, Springer-Verlag (2002).

[GZ] *B.Gross and D.Zagier*, Heegner points and derivatives of L-series, L-series, Invent. Math. **84** (1986), 225–320.

[GKZ] *B. Gross, W. Kohnen, and D. Zagier*, Heegner points and derivatives of L-series. II. Math. Ann. **278** (1987), 497–562.

[KS] *S. Katok and P. Sarnak*, Heegner points, cycles and Maass forms, Israel J. Math. **84** (1993), 193–227.

[K] *W. Kohnen*, Fourier coefficients of modular forms of half-integral weight. Math. Ann. **271** (1985), 237–268.

[KZ] *W. Kohnen and D. Zagier*, Values of L-series of modular forms at the center of the critical strip. Invent. Math. **64** (1981), no. 2, 175–198.

[Sch] *A. J. Scholl*, Fourier coefficients of Eisenstein series on non-congruence subgroups, Math. Proc. Camb. Phil. Soc. **99** (1986), 11–17.

[Sh] *G. Shimura*, On modular forms of half integral weight. Ann. of Math. (2) **97** (1973), 440–481.

[T] *J. Tunnell*, A classical Diophantine problem and modular forms of weight 3/2. Invent. Math. **72** (1983), no. 2, 323–334.

[W] *M. Waldschmidt*, Nombers transcendents et groupes algébraiques, Astérisque **69–70** (1979).

[Wa] *J.-L. Waldspurger*, Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484.

[Zw1] S. P. Zwegers, *Mock ϑ-functions and real analytic modular forms*, q-series with applications to combinatorics, number theory, and physics (Ed. B. C. Berndt and K. Ono), Contemp. Math. **291**, Amer. Math. Soc., (2001), 269–277.

[Zw2] S. P. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, 2002.

Families of Siegel Modular Forms, Their L-Functions, and Lifting Conjectures (A research report)

Alexei Panchishkin

Let p be a prime, and let $\Gamma = \mathrm{Sp}_g(\mathbb{Z})$ be the Siegel modular group of genus g. L-functions of Siegel modular forms are described in terms of motivic L-functions attached to Sp_g, and their analytic properties are given. Rankin's lemma of higher genus is established. A general conjecture on a lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m} (of genus $g = 4m$) is formulated.

We study p-adic families of zeta functions and Siegel modular forms. Critical values for the spinor L-functions and p-adic constructions are discussed. Constructions of p-adic families of Siegel modular forms are given using Ikeda-Miyawaki constructions.

We discuss the following topics:

1) L-functions of Siegel modular forms
2) Motivic L-functions for Sp_g, and their analytic properties
3) Critical values and p-adic constructions for the spinor L-functions
4) Rankin's Lemma of higher genus
5) A lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m} (of genus $g = 4m$)
6) Constructions of p-adic families of Siegel modular forms
7) p-adic versions of Ikeda-Miyawaki constructions

A holomorphic lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m} : a conjecture.

Conjecture 1 (on a lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m})**.** *Let f and g be two Siegel modular forms of genus $2m$ and of weights $k > 2m$ and $l = k - 2m$. Then there exists a Siegel modular form F of genus $4m$ and of weight k with the Satake parameters $\gamma_0 = \alpha_0\beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \cdots, \gamma_{2m} = \alpha_{2m}, \gamma_{2m+1} = \beta_1, \cdots, \gamma_{4m} = \beta_{2m}$ for suitable choices $\alpha_0, \alpha_1, \cdots, \alpha_{2m}$ and $\beta_0, \beta_1, \cdots, \beta_{2m}$ of Satake's parameters of f and g.*

One readily checks that the Hodge types of $M(Sp(f)) \otimes M(Sp(g))$ and $M(Sp(F))$ are the same (of rank 2^{4m}) (it follows from Künneth's-type formulas).

An evidence for this version of the conjecture comes from Ikeda-Miyawaki constructions ([Ike01], [Ike06], [Mur02]): let k be an even positive integer, $h \in S_{2k}(\Gamma_1)$ a normalized Hecke eigenform of weight $2k$, $F_{2n} \in S_{k+n}(\Gamma_{2n})$ the Ikeda lift of h of genus $2n$ (we assume $k \equiv n \bmod 2$, $n \in \mathbb{N}$).

Next let $f \in S_{k+n+r}(\Gamma_r)$ be an arbitrary Siegel cusp eigenform of genus r and weight $k + n + r$, with $n, r \geq 1$. If we take $n = m, r = 2m$, $k := k + m$,

$k+n+r := k+3m$, then an example of the validity of this version of the conjecture is given by

$$(f,g) = (f, F_{2m}(h)) \mapsto \mathcal{F}_{h,f} \in S_{k+3m}(\Gamma_{4m}),$$
$$(f,g) = (f, F_{2m}) \in S_{k+3m}(\Gamma_{2m}) \times S_{k+m}(\Gamma_{2m}).$$

Another evidence comes from the Siegel-Eisenstein series

$$f = E_k^{2m} \text{ and } g = E_{k-2m}^{2m}$$

of even genus $2m$ and weights k and $k-2m$: we have then

$$\alpha_0 = 1, \alpha_1 = p^{k-2m}, \cdots, \alpha_{2m} = p^{k-1},$$
$$\beta_0 = 1, \beta_1 = p^{k-4m}, \cdots, \beta_{2m} = p^{k-2m-1},$$

then we have that

$$\gamma_0 = 1, \gamma_1 = p^{k-4m}, \cdots, \gamma_{2m} = p^{k-1},$$

are the Satake parameters of the Siegel-Eisenstein series $F = E_k^{4m}$.
A similar property can be checked for the Klingen-Eisenstein series: their construction is compatible with the conjecture.

Remark 1. *If we compare the L-function of the conjecture (given by the Satake parameters $\gamma_0 = \alpha_0\beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \cdots, \gamma_{2m} = \alpha_{2m}, \gamma_{2m+1} = \beta_1, \cdots, \gamma_{4m} = \beta_{2m}$ for suitable choices $\alpha_0, \alpha_1, \cdots, \alpha_{2m}$ and $\beta_0, \beta_1, \cdots, \beta_{2m}$ of Satake's parameters of f and g), we see that it corresponds to the tensor product of the spinor L-functions, and this function is not of the same type as that of the Yoshida's lifting* [Yosh81]*, which is a certain product of Hecke's L-functions.*

We would like to mention in this context Langlands's functoriality: The denominators of our L-series belong to local Langlands L-factors (attached to representations of L-groups). If we consider the homomorphisms

$${}^L GSp_{2m} = GSpin(4m+1) \to GL_{2^{2m}}, \quad {}^L GSp_{4m} = GSpin(8m+1) \to GL_{2^{4m}},$$

we see that our conjecture is compatible with the homomorphism of L-groups

$$GL_{2^{2m}} \times GL_{2^{2m}} \to GL_{2^{4m}}, \ (g_1, g_2) \mapsto g_1 \otimes g_2, GL_n(\mathbb{C}) = {}^L GL_n.$$

However, it is unclear to us if Langlands's functoriality predicts a holomorphic Siegel modular form as a lift.

A general program. We plan also to extend our previous p-adic constructions to other situations using the following techniques:

1) Construction of modular distributions Φ_j with values in an infinite dimensional modular tower $\mathcal{M}(\psi)$.
2) Application of a canonical projector of type π_α onto a finite dimensional subspace $\mathcal{M}^\alpha(\psi)$ of $\mathcal{M}^\alpha(\psi)$.
3) General admissibility criterion. The family of distributions $\pi_\alpha(\Phi_j)$ with value in $\mathcal{M}^\alpha(\psi)$ give a h-admissible measure $\tilde{\Phi}$ with value in a module of finite rank.

4) Application of a linear form ℓ of type of a modular symbol produces distributions $\mu_j = \ell(\pi_\alpha(\Phi_j))$, and an admissible measure from congruences between modular forms $\pi_\alpha(\Phi_j)$.
5) One shows that certain integrals $\mu_j(\chi)$ of the distributions μ_j coincide with certain L-values; however, these integrals are not necessary for the construction of measures (already done at stage 4).
6) One shows a result of uniqueness for the constructed h-admissible measures : they are determined by many of their integrals over Dirichlet characters (not all).
7) In most cases we can prove a functional equation for the constructed measure μ (using the uniqueness in 6), and using a functional equation for the L-values (over complex numbers, computed at stage 5).

References

[Boe-Pa2006] Böcherer, S., Panchishkin, A.A., *Admissible p-adic measures attached to triple products of elliptic cusp forms*, Documenta Math. Extra volume : John H.Coates' Sixtieth Birthday (2006), 77-132.

[CourPa] Courtieu,M., Panchishkin ,A.A., *Non-Archimedean L-Functions and Arithmetical Siegel Modular Forms*, Lecture Notes in Mathematics 1471, Springer-Verlag, 2004 (2nd augmented ed.)

[Ike01] Ikeda, T., *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) 154 (2001), 641-681.

[Ike06] Ikeda, T., *Pullback of the lifting of elliptic cusp forms and Miyawaki's Conjecture* Duke Mathematical Journal, **131**, 469-497 (2006)

[Mi92] Miyawaki, I., Numerical examples of Siegel cusp forms of degree 3 and their zeta-functions, *Memoirs of the Faculty of Science, Kyushu University*, Ser. A, Vol. 46, No. 2 (1992), pp. 307–339.

[Mur02] Murokawa, K., *Relations between symmetric power L-functions and spinor L-functions attached to Ikeda lifts*, Kodai Math. J. 25, 61-71 (2002)

[Pa94] Panchishkin, A., *Admissible Non-Archimedean standard zeta functions of Siegel modular forms*, Proceedings of the Joint AMS Summer Conference on Motives, Seattle, July 20–August 2 1991, Seattle, Providence, R.I., 1994, vol.2, 251 – 292

[Pa05] Panchishkin, A., *The Maass-Shimura differential operators and congruences between arithmetical Siegel modular forms.* Moscow Mathematical Journal, v. 5, N 4, 883-918 (2005).

[Yosh81] Yoshida, H., *Siegel's Modular Forms and the Arithmetic of Quadratic Forms*, Inventiones math. 60, 193–248 (1980)

[Yosh01] Yoshida, H., *Motives and Siegel modular forms*, American Journal of Mathematics, 123 (2001), 1171–1197.

Hypergeometric series, automorphic forms and mock theta functions

Kathrin Bringmann

There are famous examples of hypergeometric series that are modular forms. To state one, denote by $p(n)$ the number of partitions of n. By Euler we have

$$P(q) := \sum_{n=1} p(n)\, q^n = \frac{q^{\frac{1}{2}}}{\eta(z)}, \tag{1}$$

where η is Dedekins' η-function. The theory of modular forms can be employed to show many important properties of $p(n)$. For example Rademacher used the circle method to prove an exact formula for $p(n)$. Moreover $p(n)$ satisfies some nice congruence properties, including the Ramanujan congruences:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11}.$$

The function $P(q)$ can also be written as a hypergeometric series, namely

$$P(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}. \tag{2}$$

It is an open problem to show directly the modularity of $P(q)$ using this expansion. The literature on further examples that relate hypergeometric series and modular form is extensive and it is an active area of of research to obtain more of those and to also interpret them because they have applications for example in number theory, Lie theory, combinatorics, and physics. However, there is no comprehensive theory that really describes the interplay between hypergeometric series and automorphic forms. The situation is further complicated by the mock theta functions, a collection of 22 series such as

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} \tag{3}$$

which were defined by Ramanujan in his last letter to Hardy. Even if (2) and (3) look similar, (3) is not just a modifications of the Fourier expansion of modular forms and therefore until recently little was known about the mock theta functions.

The last couple of years things started to become clearer. Through my work with K. Ono combined with work of Zwegers it is now known that the mock theta functions are the holomorphic parts of weak Maass forms, the non-holomorphic parts are certain Mordell-type integrals. Weak Maass forms are generalizations of modular forms, in that they satisfy a transformation law, and (weak) growth conditions at cusps, but instead of being meromorphic, they are annihilated by the weight k hyperbolic Laplacian. I constructed in joint work with K. Ono [7, 8] building on work of Zwegers [12] an infinite family of weak Maass forms arising from Dyson's rank generating functions. Recall that Dyson defined the *rank* of a partition to be its largest part minus the number of its parts. We let $N(m,n)$ the number of partitions of n with rank m and define the generating function

$$R(w;q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq)_n (w^{-1}q)_n}, \tag{4}$$

where $(a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right)$. If we set $w = 1$ in (4), we recover the partition generating function $P(q)$, i.e., a weight $-\frac{1}{2}$ modular form. Moreover if $w = -1$, we obtain the generating function for the number of partitions with even rank minus the number of partitions with odd rank and this equals the mock theta function $f(q)$.

Theorem 1. ([8]) If $w \neq 1$ is a root of unity, then the functions $R(w;q)$ are the holomorphic parts of weight $\frac{1}{2}$ weak Maass forms. The non-holomorphic parts have the form

$$\int_{-\bar{z}}^{i\infty} \frac{\Theta_w(\tau)}{\sqrt{-i(\tau+z)}}\, d\tau,$$

where Θ_w is a cuspidal weight $\frac{3}{2}$ theta function.

Viewing the rank generating functions in the framework of weak Maass forms has found many applications, including an exact formula for the coefficients of $f(q)$ [7], asymptotics for $N(m,n)$ [2], and congruences for $N(s,t;n)$ [8], the number of partitions of n with rank congruent to s modulo t, which give a combinatorial decomposition of congruences for $p(n)$. The next natural question that arises in this context is whether there is a connection to usual modular forms. The answer is yes, this was for example essential for the proof of the above mentioned congruences since the restriction of the Fourier expansion of the occuring Maass forms to certain residue classes turned out to have no non-holomorphic part. Therefore we were able to employ techinques of Serre and Shimura. Moreover there are relations between non-holomorphic parts of different Maass forms, which "explain" interesting identities involving Maass forms. Famous examples are the so-called mock theta conjectures of Ramanujan, a list of ten identities involving mock theta functions. These confirmed to be very difficult to prove since mock theta functions are not modular forms and were only proven by Hickerson in the late 80's [10]. From the new perspective described above the mock theta conjectures arise naturally in the theory of Maass forms from linear relations between non-holomorphic parts. We obtain theorems like the following.

Theorem 2. ([9]) Suppose that $t \geq 5$ is prime, $0 \leq r, s < t$ and $0 \leq d < t$. Then the following are true: Assume either

(1) If $\left(\frac{1-24d}{t}\right) = -1$ or
(2) $\left(\frac{1-24d}{t}\right) = 1$. If $r, s \not\equiv \pm\frac{1}{2}(1+\alpha) \pmod{t}$, for any $0 \leq \alpha < 2t$ satisfying $1-24d \equiv \alpha^2 \pmod{2t}$,

Then

$$\sum_{n=0}^{\infty} \left(N(r,t;tn+d) - N(s,t;tn+d)\right) q^{24(tn+d)-1} \tag{5}$$

is a weight $\frac{1}{2}$ weakly holomorphic modular form.

Theorem 2 is optimal since for all other pairs r and s (apart from trivial cases) we have that (5) is the holomorphic part of a weak Maass form which has a non-vanishing non-holomorphic part. Using Theorem 2 one can prove concrete identites including the mock theta conjectures using the valence formula.

It turns out that the theory of weak Maass forms can also be employed to relate class numbers to interesting combinatorial statistics. To state our results, we let $H(-N)$ be the Hurwitz class number, i.e., the number of classes of quadratic forms of discriminant $-N$, where each class C is counted with multiplicity $\frac{1}{Aut(C)}$. By

[11] one knows that

$$\mathcal{H}(q) := -\frac{1}{12} + \sum_{\substack{n \geq 1 \\ n \equiv 0,3 \pmod 4}} H(-n)q^n$$

is the holomorphic part of a weak Maass forms of weight $\frac{3}{2}$. Next recall that an *overpartition* is a partition, where the first occurence of a number may be overlined. We denote by $\overline{p}(n)$ the number of overpartitions of n. We consider the following mock-theta-type function for overpartitions:

$$\overline{f}(q) := \sum_n \overline{\alpha}(n)q^n = \sum_n \left(\overline{p}_e(n) - \overline{p}_o(n)\ \right) q^n,$$

where $\overline{p}_e(n)$ denotes the number of overpartitions of n with even and $\overline{p}_o(n)$ the number of overpartitions with odd rank. Suprisingly this function has a totally different behavior than Ramanujan's mock theta function $f(q)$. In [4] the author proved that $\overline{f}$ is the holomorphic part of a weak Maass form. Moreover we showed recently that there is a natural connection to Hurwitz class numbers.

Theorem 3. ([5]) We have

$$\overline{f}(-q) = -16\mathcal{H}(q) - \frac{1}{3}\Theta^3(z).$$

In particular we can write $\overline{\alpha}(n)$ in terms of class numbers. For example if $n \equiv 1, 2 \pmod 4$, then

$$\overline{\alpha}(n) = 4(-1)^{n+1}H(-4n).$$

This result is interesting in two ways, namely first of all it relates an algebraic statistic a class number to a combinatorial statistic a rank difference. Secondly it provides a mock theta function with total different behavior than Ramanujan's mock theta functions. For example the coefficients of $f(q)$ grow exponentially whereas the coefficients of $\overline{f}(q)$ only grow polynomial. Also one can conclude other interesting properties of the coefficients of $\overline{f}(q)$ from properties of class number like congruences like the following:

$$\overline{p}_e(1052n + 256) \equiv \overline{p}_o(1052n + 256) \pmod{61}.$$

Recently I found a new class of functions that are related to hypergeometric series. These function "live" between the classical quasimodular forms and weak Maass forms and to describe this I call a function a *quasiweak Maass form* if it is a linear combination of derivatives of weak Maass forms. The forms that I consider are the generating functions for certain 2-marked Durfee symbols. I don't want to give the definition of those combinatorial objects here since it requires setting up a lot of notation and only consider the analytic meaning of these objects. Andrews showed that we have the following generating function of 2-marked Durfee symbols:

$$R_2(q) := \sum_{m_1, m_2 > 0} \frac{q^{(m_1+m_2-1)^2+m_1}}{(q)_{m_1}^2 \left(q^{m_1}\right)_{m_2}^2} = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{Z}\setminus\{0\}} (-1)^{n+1} \frac{q^{\frac{3n}{2}(n+1)}}{(1-q^n)^2}\,. \tag{6}$$

Difficulties in relating (6) to a weak Maass form arise due to double poles and more severely from the fact that the summation only runs over the incomplete lattice $\mathbb{Z} \setminus \{0\}$. The problems mentioned before are responsible for the fact that additional terms arise in the transformation law of $R_2(q)$, which I managed to identify as quasimodular components. Define the function

$$\mathcal{M}(z) := R_2(24z)\, e^{-2\pi i z} - \mathcal{N}(z) - \frac{1}{24\eta(24z)} + \frac{E_2(24z)}{8\eta(24z)},$$

where as usual $E_2(z)$ is the quasimodular weight 2 Eisenstein series. Moreover the non-holomorphic integral is given by

$$\mathcal{N}(z) \sim \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}}\, d\tau.$$

Theorem 4. ([3]) The function $\mathcal{M}(z)$ is a weak Maass form of weight $\frac{3}{2}$.
As an application I solved two conjectures of Andrews. The function $R_2(q)$ is also crutial for understanding k-marked Durfee symbols for $k > 2$ which I am planning to study in joint work with Garvan and Mahlburg [6]. We are planning to construct quasiweak Maass forms of arbitrary high weight.

References

[1] G. E. Andrews, *Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks.* Invent. Math. **169** (2007), 37-73.
[2] K. Bringmann, *Asymptotics for rank partition functions*, Transaction of the AMS, accepted for publication.
[3] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. Journal, accepted for publication.
[4] K. Bringmann and J. Lovejoy, *Dyson's rank, overpartitions, and weak Maass forms*, Int. Math. Res. Not., Article ID rnm063.
[5] K. Bringmann and J. Lovejoy, *Overpartitions and class numbers of binary quadratic forms*, preprint.
[6] K. Bringmann, F. Garvan, and K. Mahlburg, *Partition statistics and quasiweak Maass forms*, in preparation.
[7] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), 243-266.
[8] K. Bringmann and K. Ono, *Dysons ranks and Maass forms*, Ann. Math., accepted for publication.
[9] K. Bringmann, K. Ono, and R. Rhoades, *Eulerian series as modular form*, Journal of the American Mathematical Society, accepted for publication.
[10] D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), 639-660.
[11] F. Hirzebruch and D. Zagier, Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, *Invent. Math.* **36** (1976), 57-113.
[12] S. P. Zwegers, *Mock theta-functions and real analytic modular forms,* q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), 269–277, Contemp. Math., **291**, Amer. Math. Soc., Providence, RI, 2001.

Consequences for standard zeta values of conjectures of Bloch-Kato and Harder

Neil Dummigan

Let f be a classical Hecke eigenform of level one and weight $2k-2$, with k even. Then there is a Saito-Kurokawa lift $\hat{f}$ of weight k and genus 2. Let p be a large prime divisor of the algebraic part of the near-central critical value $L(f,k)$. (For simplicity, imagine that the field of coefficients is $\mathbb{Q}$, but this is not necessary.) Then, assuming that p is not a congruence prime for f, and another weak condition, there exists another Hecke eigenform F (of weight k, level one and genus 2), not a Saito-Kurokawa lift, such that the Hecke eigenvalues of $\hat{f}$ and F are congruent modulo p. Hence, for any prime ℓ,

$$\lambda(\ell) \equiv \ell^{k-1} + \ell^{k-2} + a_\ell \pmod{p},$$

where $\lambda(\ell)$ is a Hecke eigenvalue for F, and $f = \sum a_n q^n$. This was proved by Katsurada [Ka], and a similar theorem independently by Brown [Br].

This congruence implies that a 4-dimensional (mod p) Galois representation $\overline{\rho}_{F,p}$ attached to F is reducible, with a 2-dimensional factor isomorphic to the (mod p) representation $\overline{\rho}_{f,p}$ attached to f (assumed irreducible), and two one-dimensional factors isomorphic to twists of the trivial representation. Brown shows that from this reducible representation (which depends on a good choice of invariant $\mathbb{Z}_p$-lattice inside a p-adic representation $\rho_{F,p}$) may be extracted a non-trivial extension of $\overline{\rho}_{f,p}$, producing an element of order p in an appropriate Bloch-Kato Selmer group, as predicted by the factor of p in $L(f,k)$. That this element satisfies the necessary local conditions follows from the fact that $\rho_{F,p}$ is unramified away from p and crystalline at p. The non-triviality of the extension depends on the irreducibility of $\rho_{F,p}$.

Inside the exterior square of $\overline{\rho}_{F,p}$ is a twist of $\overline{\rho}_{f,p}$, so we can also get an element of order p in a Selmer group for a representation whose L-function is, more-or-less, the standard zeta-function $L(F,s,\mathrm{St})$, and according to the Bloch-Kato conjecture [BK] we should see p appear in certain ratios of critical values of $L(F,s,\mathrm{St})$. Under a further weak assumption, this can be confirmed using $L(\hat{f},s,\mathrm{St}) = \zeta(s)L(f,s+k-1)L(f,s+k-2)$ and the congruence between $\hat{f}$ and F.

According to a conjecture of Harder [Ha], [vdG] there exist vector-valued Siegel modular forms satisfying congruences a bit like those for F, modulo large prime divisors of the algebraic parts of other critical values of $L(f,s)$, further right than $s=k$. Faber and van der Geer [FvdG], [vdG] have produced compelling numerical evidence for this conjecture in special cases. If we accept it then, imitating the above constructions, we expect to see these primes dividing certain ratios of standard zeta values for these vector-valued forms. Here we can no longer use the formula for the standard zeta function of a Saito-Kurokawa lift, but heroic calculations by Ibukiyama and Katsurada, using pullback formulas and differential operators, have confirmed one case numerically (subject to checking some technicality). Some additional similar predictions can be made, using the elements of

Selmer groups, associated to vanishing of $L(f,s)$ at the central point $s = k-1$, constructed by different methods by Skinner-Urban [SU] and Nékovař [N].

References

[BK] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives, The Grothendieck Festschrift Volume I, 333–400, Progress in Mathematics, 86, Birkhäuser, Boston, 1990.

[Br] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, *Compositio Math.,* **143** (2007), 290–322.

[FvdG] C. Faber, G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes, I, II, *C. R. Math. Acad. Sci. Paris* **338** (2004), 381–384 and 467–470.

[Ha] G. Harder, A congruence between a Siegel and an elliptic modular form, manuscript, 2003. http://www.math.uni-bonn.de/people/harder/Manuscripts/Eisenstein/kolloquium.pdf

[Ka] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, preprint, 2005. http://eprints.math.sci.hokudai.ac.jp/archive/00000951/

[N] J. Nekovář, Selmer Complexes, *Astérisque* **310** (2006), 559 pp.

[SU] C. Skinner, E. Urban, Sur les déformations p-adiques de certaines représentations automorphes, *J. Inst. Math. Jussieu* **5** (2006), 629–698.

[vdG] G. van der Geer, Siegel Modular Forms, preprint, 2006. arXiv:math.AG/0605346 v1

[We] R. Weissauer, Four dimensional Galois representations, *Astérisque* **302** (2005), 67–150.

On the computation of modular forms of half-integral weight

Nils-Peter Skoruppa

We indicate a method for a systematic and explicit generation of a basis for a given space of half integral modular forms on $\Gamma_0(4N)$ with arbitrary nebentype.

Preliminary remarks

The computational theory of elliptic modular forms of integral weight is meanwhile quite well understood. The most efficient method for calculating modular forms is based on the theory of modular symbols and goes back directly to [Ma]. It was used (with several improvements by various authors) by Henri Cohen, the author and Don Zagier to produce in the late 80's tables of modular forms [modi] and later by William Stein to produce another extensive database of modular forms [St]. This method is now implemented (mainly by William Stein) into the computer algebra systems MAGMA and SAGE.

Shimura showed [Sh] that modular forms of half-integral weight are intimately connected to forms of integral weight. This connection became even more important by Waldspurger's theorem relating values of the twisted L-series of newforms of integral weight at the critical point to the Fourier coefficients of associated half-integral weight forms. A famous application of these ideas to classical number theory is Tunnell's theorem on congruent numbers.

Despite their importance for the arithmetic theory of modular forms of integral weight there is no good algorithm to compute systematically half-integral weight forms. The main method used so far[1] was described in Basmaji's thesis [Ba,

[1]William Stein implemented this recently in SAGE.

pp. 55]: If N is divisible by 4, if $k = 2l - 1$ denotes an odd integer, and if we set $\theta_2 = \sum_{n\in 2\mathbb{Z}+1} q^{n^2}$ and $\theta_3 = \sum_{n\in\mathbb{Z}} q^{n^2}$ (where q, for z in the upper half plane, is the function $\exp(2\pi iz)$), then the image of the map[2]

$$S_{\frac{k}{2}}(4N,\chi) \to S_l\left(4N,\chi\chi_{-4}^l\right) \times S_l\left(4N,\chi\chi_{-4}^l\right), \quad f \mapsto (f\theta_2, f\theta_3)$$

equals the set of all pairs of modular forms (f_2, f_3) in $S_2(4N,\chi)$ which satisfy $f_2\theta_3 = f_3\theta_2$. This method is for example implemented in recent versions of SAGE. The disadvantages of this method lie at hand: It is based on a prior computation of modular forms of integral weight. It is not compatible in any sense with Hecke theory. Even if one is interested in only a single Hecke eigenform of half-integral weight one needs to compute first of all basis for the spaces $S_{\frac{3}{2}}(4N,\chi)$ and $S_2(4N,\chi)$, where the the assumption that N is divisible by 4 becomes then especially annoying.

We suggest a different approach, which overcomes the listed disadvantages. This approach is based on modular symbols. It allows to produce closed formulas for the Fourier coefficients of modular forms of half-integer weight in a very direct way. In fact, this idea behind this method is not really new. It was developed and used in [Sk1], [Sk2], [Sk3] to produce closed formulas for Jacobi forms (on the full modular group) of arbitrary weight and index. A more detailed account of this method will be published elsewhere [Sk4]. For simplicity we discuss in the following only the case of half-integral modular forms of weight $\frac{3}{2}$.

Statement of Results

The starting point to derive formulas for generators of a space $S_{\frac{3}{2}}(4N,\chi)$ are the following two theorems. Note that we use $S_{\frac{3}{2}}(4N,\chi)$ for the space of *non trivial cusp forms*, i.e. the orthogonal complement with respect to the Petersson scalar product of the space of all cusp forms f of weight $\frac{3}{2}$ on $\Gamma_0(4N)$ and with character χ which are linear combinations of theta series of the form $\sum_n n\psi(n)q^{tn}$ (t an integer and ψ a Dirichlet character).

Theorem 1 ([Sh]). *Let t be a positive squarefree integer. Then the application*

$$f = \sum_{n>0} c_f(n)\, q^n \mapsto \sum_{n>0}\sum_{d|n} (\chi\chi_{-4t})\,(n/d)\, a_f\left(td^2\right)\, q^n,$$

defines a map

$$S_{t,\chi} : S_{\frac{3}{2}}(4N,\chi) \to S_2(2N,\chi^2).$$

This map commutes with all Hecke operators $T(p)$ with $\gcd(p, 2N) = 1$.

(Note that the precise level $2N$ of the image of the maps $S_{t,\chi}$ was only conjectured in [Sh] and later proved in [Ni].)

[2]For a half-integral or integral integer k and a Dirichlet character χ we use $S_k(N,\chi)$ for the space of (non trivial, if $k = \frac{3}{2}$, see below) cusp forms on $\Gamma_0(N)$ of weight k and nebentype χ. If k is half-integral then N is assumed to be divisible by 4. For a discriminant D, we use χ_D for the Dirichlet character modulo D which, for odd primes p, equals the usual Legendre symbol $\left(\frac{D}{p}\right)$.

Theorem 2 ([Sk2]). *For every positive integer m, the application*

$$f \mapsto \lambda_f, \quad \lambda_f(c) := \sum_{s\in\mathbb{P}_1(\mathbb{Q})} c_s \left(\int_s^\infty + \int_{-s}^\infty\right) f(z)\,dz =: \int_{c^+} f$$

($c = \sum_s c_s(s) \in \mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$) defines an isomorphism

$$\pi : S_2(m,\chi) \to \frac{\mathrm{Hom}_{\Gamma_0(\mathrm{m})}\left(\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0, \mathbb{C}(\chi)\right)^{ev.}}{\mathrm{res}\,\mathrm{Hom}_{\Gamma_0(\mathrm{m})}\left(\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})], \mathbb{C}(\chi)\right)^{ev.}}.$$

This isomorphism commutes with all Hecke operators $T(p)$.

Here the notations are as follows. By $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]$ we denote the free abelian group generated by elements (s), where s runs through the points of the rational projective line $\mathbb{P}_1(\mathbb{Q})$, and by $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$ we denote its subgroups of elements $c = \sum_s c_s(s)$ with $\sum_s c_s = 0$. The semigroup of regular integral 2×2 matrices acts on this group by linear extension of its natural action on $\mathbb{P}_1(\mathbb{Q})$. We use $\mathbb{C}(\chi)$ for the $\Gamma_0(m)$-module with underlying vector space $\mathbb{C}$ and the action[3] $([a,b,c,d],z) \mapsto \chi(d)z$. The vectors spaces whose quotient appears on the right of the claimed isomorphism are the spaces of all *even* $\Gamma_0(m)$-equivariant maps from $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$ (resp. $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]$) into $\mathbb{C}(\chi)$. Here a map λ is called even if $\lambda\left(\sum c_s(-s)\right) = \lambda\left(\sum c_s(s)\right)$. The map res restricts a λ on $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]$ to $\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0$. Finally, for a natural number l, the Hecke operator $T(l)$ is defined on each of the two spaces on the right by

$$\left(T(l)\lambda\right)(c) = \sum_{R=[a,b,c,d]} \chi(a)\lambda(Rc),$$

where R runs through a system of representatives for the set of integral 2×2 matrices $R = [a,b,c,d]$ of determinant l with c divisible by m modulo left multiplication by $\Gamma_0(m)$. (Note that $\chi(a) = 0$ unless a is relatively prime to m.)

The quotient on the right of the isomorphism of the last theorem, but with the restriction to even maps dropped, can be naturally identified with the dual of the space

$$C(m,\chi) := \ker\left(\left[\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})]^0 \otimes_Z \mathbb{C}(\chi)\right]_{\Gamma_0(m)} \to \left[\mathbb{Z}[\mathbb{P}_1(\mathbb{Q})] \otimes_Z \mathbb{C}(\chi)\right]_{\Gamma_0(m)}\right),$$

where the subscript $\Gamma_0(m)$ indicates that we consider the respective spaces of $\Gamma_0(m)$-coinvariants.

We now fix a natural number N and a squarefree natural number t and consider the composed map

$$L^*_{t,\chi} = \pi \circ S_{t,\chi} : S_{\frac{3}{2}}(4N,\chi) \to S_2(2N,\chi^2) \to C_{2N}(\chi^2)^*.$$

By dualising this map and identifying the dual space $S_{\frac{3}{2}}(4N,\chi)^*$ with the space $S_{\frac{3}{2}}(4N,\chi\chi_{4N})$ we obtain a map

$$L_{t,\chi} : C(2N,\chi^2) \to S_{\frac{3}{2}}(4N,\chi\chi_{4N}).$$

[3]We use $[a,b,c,d]$ for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The natural map $S_{\frac{3}{2}}(4N, \chi\chi_{4N}) \to S_{\frac{3}{2}}(4N, \chi)^*$ to which we refer is given explicitly by $f \mapsto \langle \cdot, \iota W_{4N} f\rangle$. Here ι and W_{4N} denote the maps $(\iota f)(z) = \overline{f(-\overline{z})}$ and $(W_{4N} f)(z) = f(-1/4Nz)\left(-i\sqrt{4n}\tau\right)^{-3/2}$, respectively, and we use the Petersson scalar product. It can be checked that then $L_{t,\chi}$ is in fact Hecke-equivariant (with respect to the Hecke operators $T(p)$ with p relatively prime to $4N$). Moreover, from the preceding explanations it is not hard to verify that the images of the $L_{t,\chi}$ with t running through all squarefree integers exhaust the space $S_{\frac{3}{2}}(4N, \chi\chi_{4N})$. Hence, for deducing from these considerations an effective algorithm to compute modular forms of weight $\frac{3}{2}$ in closed form we have to answer the question whether we are able to compute the $L_{t,\chi}$ in an explicit way.

By its very definition the map $L_{t,\chi}$ is defined by the identity

$$\langle g, \iota W_{4N} L_{t,\chi}(c)\rangle = \int_{c^+} S_{t,\chi}(g) \qquad (g \in S_{\frac{3}{2}}(4N, \chi).$$

Now, the maps $S_{t,\chi}$ are *theta liftings*, i.e. there exists a so-called theta kernel $\theta_{t,\chi}(z, \tau)$, which transforms in the first variable under $\Gamma_0(4N)$ like an element in $S_{\frac{3}{2}}(4N, \chi)$ and which transforms in the second variable under $\Gamma_0(2N)$ like an element in the space which is obtained from $S_2\left(2N, \chi^2\right)$ by taking the complex conjugates of its forms, and such that

$$S_{t,\chi}(g)(\tau) = \langle g, \theta_{t,\chi}(\cdot, \tau)\rangle$$

Explicit formulas for the theta kernels in question have be obtained in [Ni] and [Ci].

Inserting $\theta_{t,\chi}$ in the defining identity for $L_{t,\chi}$ we obtain after some obvious manipulations the formula

$$L_{t,\chi}(c) = W_{4N}\iota \int_{c^+} \theta_{t,\chi}(\cdot, \tau)\, d\overline{\tau}.$$

It is a priori not clear whether this is a sensible formula since $\theta_{t,\chi}$ is only real analytic and since we need to interchange taking scalar products and integration along hyperbolic lines. However, it can be verified by analyzing explicit expressions for $\theta_{t,\chi}$ that this formula holds in fact true.

It turns out that the right hand side of the last formula can indeed be computed explicitly and is, moreover, given by a simple and appealing combinatorial formula. The method of computation is in essence the same as in [Sk1]. Details have been worked out (for special cases) by Reinhard Steffens [Sts] in his diploma thesis. We state the final result here in a slightly weaker form.

Theorem 3 ([Sts], [Sk4])**.** *For natural numbers D whose squarefree part does not divide $4Nt$, the D-th Fourier coefficient of $L_{t,\chi}\left(\sum_s c_s\,(s)\right)$ is given by*

$$\sum_{Q \in F_N(4NDt)} \chi_t(Q) \sum_s c_s \operatorname{sign}\left(Q(s)\right).$$

Here, for a natural number Δ, we use $F_N(\Delta)$ for the set of all binary integral quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ such that $b^2 - 4ac = \Delta$ and such that $N|a$ and $2N|b$. Moreover, for any such form, we use $\chi_t(Q) = \chi(a/N)\chi_{-4t}(a/N)$

(in particular $\chi_t(Q) = 0$ *if* $\gcd(a, 4N) \neq 1$*) and* $\operatorname{sign} Q(s)$ *for the sign of* $Q(x, y)$ *if* $s = [x : y]$*.*

We leave it to the reader to verify that the inner sum in the given formula is different from 0 for only finitely many Q. The assumption on D can be dropped for the cost of adding certain more complicated terms to the given formula. The given formula can be interpreted in terms of intersection numbers of certain cycles on the modular curve $X_0(2N)$. We finally mention that a similar theorem (with suitable modifications) holds true for arbitrary half integral weight. Details and proofs will be published elsewhere [Sk4].

References

[Ba] J. Basmaji, Ein Algoritmus zur Berechnung von Hecke-Operatoren und Anwendungen auf modulare Kurve, Dissertation, Universität Essen 1996

[Ci] B. Cipra, On the Niwa-Shintani theta kernel lifting of modular forms. Nagoya Math. J. 91 (1983), 49–117

[Ma] J.I. Manin, Parabolic points and zeta functions of modular curves, Izv. Akad. Nauk SSSR 36 (1972), 19–66

[modi] http://modi.countnumber.de

[Ni] S. Niwa, Modular forms of half integral weight and the integral of certain theta functions. Nagoya Math. J. 56 (1974), 147–161

[Sh] G. Shimura, On modular forms of half-integral weight. Ann. of Math. 97 (1973), 440–481

[Sk1] N-P. Skoruppa, Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms. Invent. Math. 102 (1990), 501–520

[Sk2] N-P. Skoruppa, Heegner Cycles, Modular Forms and Jacobi Forms. Sém. Th. des Nombres Bordeaux 3 (1991), 93–116

[Sk3] N-P. Skoruppa, Binary quadratic forms and the Fourier coefficients of elliptic and Jacobi modular forms, J. Reine Angew. Math. 411 (1990), 66–95

[Sk4] N-P. Skoruppa, Explicit Formulas for the Fourier Coefficients of Modular Forms of Half Integral Weight, in preparation

[St] W. Stein, http://modular.fas.harvard.edu/tables/

[Sts] R. Steffens, Explizite Formeln für Fourier-Koeffizienten von Modulformen vom Gewicht $\frac{3}{2}$. Diplomarbeit, Universität Siegen 2006

Even unimodular lattices with a complex structure and their theta series

Aloys Krieg

Let K be an imaginary quadratic number field with class number 1, ring of integers $\mathcal{O}_K$ and discriminant D_K. We consider $\mathbb{C}^k$, $k \in \mathbb{N}$, with the standard Hermitian scalar product. The lattices Λ under consideration are given by $\mathbb{C}$-linearly independent vectors

$$b_1, \dots, b_k \in \mathbb{C}^k \quad \text{with Gram matrix} \quad S = (\langle b_\nu, b_\mu \rangle)$$

satisfying

$$\Lambda = \mathcal{O}_K b_1 + \dots + \mathcal{O}_K b_k, \quad \langle \lambda, \lambda \rangle \in 2\mathbb{Z} \text{ for all } \lambda \in \Lambda, \quad \det S = \left(\frac{2}{\sqrt{|D_K|}} \right)^k .$$

Note that we have

$$\langle \lambda, \mu \rangle \in \frac{2}{\sqrt{D_K}} \mathcal{O}_K \text{ for all } \lambda, \mu \in \Lambda$$

and that these Λ are even unimodular $\mathbb{Z}$-lattices of rank $2k$ (cf. [3]). Such lattices exist if and only if $k \equiv 0 \,(\mathrm{mod}\ 4)$. There are only finitely many isometry classes of such lattices for fixed k.

The associated theta series on the Hermitian half-space

$$\mathcal{H}_n = \left\{ Z \in \mathbb{C}^{n \times n};\ \frac{1}{2i}(Z - \overline{Z}^{tr}) \text{ positive definite} \right\}$$

are given by

$$\Theta_\Lambda^{(n)}(Z) = \sum_{(\lambda_1, \ldots, \lambda_n) \in \Lambda^n} e^{\pi i \cdot \mathrm{trace}((\langle \lambda_\nu, \lambda_\mu \rangle) \cdot Z)}, \quad Z \in \mathcal{H}_n.$$

They are Hermitian modular forms of weight k (cf. [2], [3]) with respect to

$$\Gamma_n(K) := \{ M \in \mathcal{O}_K^{2n \times 2n};\ MJ\overline{M}^{tr} = J \}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Theorem 1 ([6],[4]). *Let $K = \mathbb{Q}(\sqrt{-1})$. The numbers of $\mathcal{O}_K$-isometry classes of even unimodular $\mathbb{Z}$-lattices of rank $2k$ with the structure of $\mathcal{O}_K$-modules are given by*

k	4	8	12
$\sharp$	1	3	28

If $k = 8$ the three Hermitian theta series are linearly independent if and only if $n \geq 4$.

In this case there exist two lattices for $k = 8$, which are non-isometric as $\mathcal{O}_K$-modules, but are both isometric to D_{16}^+ as $\mathbb{Z}$-lattices. In particular there exists a Hermitian analog of the Schottky form over $K = \mathbb{Q}(\sqrt{-1})$. If $k = 12$ all the lattices from the Niemeier list have got the structure of $\mathcal{O}_K$-modules.

Theorem 2 ([5]). *Let $K = \mathbb{Q}(\sqrt{-3})$. The numbers of $\mathcal{O}_K$-isometry classes of even unimodular $\mathbb{Z}$-lattices of rank $2k$ with the structure of $\mathcal{O}_K$-modules are given by*

k	4	8	12
$\sharp$	1	1	5

The dimensions of the spaces of cusp forms spanned by the associated Hermitian theta series for $k = 12$ are given by

n	0	1	2	3	4
dim	1	1	1	1 *or* 2	1 *or* 0

It is conjectured that there is a non-trivial cusp form of degree 4, which is a linear combination of theta series. There is no Hermitian analog of the Schottky form over $K = \mathbb{Q}(\sqrt{-3})$. The root systems of the lattices for $k = 12$ are

$$3E_8,\ 4E_6,\ 6D_4,\ 12A_2,\ \emptyset.$$

Now we can ask the same kind of questions over the Hurwitz quaternions $\mathcal{O}$.

Theorem 3 ([1],[4]). *The numbers of $\mathcal{O}$-isometry classes of even unimodular $\mathbb{Z}$-lattices of rank $4k$ with the structure of $\mathcal{O}$-modules are given by*

k	2	4	6	8
$\sharp$	1	1	3	11

If $k = 6$ the associated theta series on the quaternionic (or even on the Siegel) half-space are linearly independent if and only if $n \geq 2$.

The root systems of the lattices for $k = 6$ are

$$3E_8,\ 6D_4,\ \emptyset.$$

References

[1] Bachoc, C., Nebe, G.: *Classification of two genera of 32-dimensional lattices of rank 8 over the Hurwitz order*, Exp. Math. **6** (1997), 151–162.
[2] Braun, H.: *Hermitian modular functions III*, Ann. Math. **53** (1951), 143–160.
[3] Cohen, D., Resnikoff, H.: *Hermitian quadratic forms and Hermitian modular forms*, Pac. J. Math. **238** (1978), 97–117.
[4] Hentschel, M., Krieg, A.: *A Hermitian Analog of the Schottky Form* In: S. Böcherer, T. Ibukiyama, M. Kaneko, F. Sato (ed.) *Automorphic Forms and Zeta Functions. Proceedings of the Conference in Memory of Tsuneo Arakawa*, World Scientific (2006), 140-149.
[5] Hentschel, M., Krieg, A., Nebe, G.: *Constructing modular forms from Hermitian lattices with respect to* $\mathbb{Q}(\sqrt{-3})$, to appear.
[6] Kitazume, M., Munemasa, A.: *Even unimodular Gaussian lattices of rank* 12, J. Number Theory **95** (2002), 77-94.

Paramodular cusp forms

Cris Poor
(joint work with David Yuen)

1. Abstract

We are gathering evidence for a degree two version of the Shimura-Taniyama conjecture. Our part in this project revolves around computing $S_2^2(K(p))$, the Siegel modular cusp forms of weight 2 and degree 2 for the paramodular group $K(p)$ for primes levels p. For a natural number N, the paramodular group $K(N)$

is defined by:

$$K(N) = \mathrm{Sp}_2(\mathbb{Q}) \cap \begin{pmatrix} * & * & */N & * \\ N* & * & * & * \\ N* & N* & * & N* \\ N* & * & * & * \end{pmatrix}, \quad \text{for } * \in \mathbb{Z}.$$

In 1980, H. Yoshida conjectured that for every abelian surface defined over $\mathbb{Q}$, there exists a group and a degree two Siegel modular form of weight two for that group with the same L-function. He supported this conjecture by constructing lifts and giving specific examples of GL(2)-type, see [3].

The Paramodular Conjecture, given below, posits the paramodular group $K(N)$ as the group corresponding to a simple rational abelian surface of squarefree conductor N. Accordingly, we are studying spaces of Siegel paramodular cusp forms. We believe that the examples given here are the first nonlifts of weight two found. Although we have verified the equality of some Euler factors in our examples, we have not proven the equality of any L-functions.

We have been motivated by the following degree two version of the Shimura-Taniyama Conjecture, as explained to us by A. Brumer. For a more general statement and details, see the future article [2] by A. Brumer and K. Kramer.

1. Paramodular Conjecture. *Let $N \in \mathbb{N}$ be odd and squarefree. Let a totally real number field K of degree d over $\mathbb{Q}$ be given. Let $\mathcal{O}$ be its ring of integers. There is a bijection between Hecke newforms $f \in S_2^2(K(N))^{new}$ whose eigenvalues generate K over $\mathbb{Q}$ and isogeny classes of simple abelian varieties $\mathcal{A}$ defined over $\mathbb{Q}$ of dimension $2d$ and conductor N^d with endomorphisms, also defined over $\mathbb{Q}$, by $\mathcal{O}$.*

In this correspondence, the product of the spinor L-functions of the Galois conjugates of f is equal to the Hasse-Weil L-function of $\mathcal{A}$.

This conjecture may be generalized further but here we wish to focus on the simplest case:

2. Paramodular Conjecture for rational abelian surfaces of prime conductor. *Let p be a prime. There is a bijection between Hecke eigenforms with rational eigenvalues $f \in S_2^2(K(p))$ that are not Gritsenko lifts and isogeny classes of rational abelian surfaces $\mathcal{A}$ of conductor p. In this correspondence, we have*

$$L(\mathcal{A}, s, \textit{Hasse-Weil}) = L(f, s, \textit{spin}).$$

Our computations here cover primes $p < 479$. To the extent that our computations are conclusive, there is a perfect match between these two sets of data. In particular, there are rational nonlift Hecke eigenforms where the Paramodular Conjecture indicates there should be and there are none where it indicates there should be none. We have found examples of nonlift Siegel modular cusp forms for $p = 277$, 349, 353, 389, 461, 523 and (two) for 587. In all of these cases, the rational abelian surfaces are known, see [2], and the Euler 2 and 3 factors of the abelian surface agree with those of the nonlift Hecke eigenform. The case $p = 587$

is particularly interesting because there is one nonlift in the plus space and one nonlift in the minus space.

Theorem 1.1. *For primes* $p < 499$ *and not in* $\{277, 349, 353, 389, 461\}$, $S_2^2(K(p))$ *is spanned by Gritsenko lifts.*

The first entry in the exceptional set is $p = 277$ and the known surfaces of that conductor are all isogenous to the Jacobian $\mathcal{A}_{277}$ of the curve $y^2 + y = x^5 + 5x^4 + 8x^3 + 6x^2 + 2x$.

Theorem 1.2. *The subspace of Gritsenko lifts in* $S_2^2(K(277))$ *has dimension* 10 *whereas* $S_2^2(K(277))$ *has dimension* 11. *There is a rational Hecke eigenform* f *that is not a Gritsenko lift. The Euler factors of* $L(f, s, spin)$ *for* $q = 2,\ 3,\ 5$ *and the linear coefficients of the Euler factors for* $q = 7, 11,\ 13$ *agree with those of* $L(\mathcal{A}_{277}, s, \textit{Hasse-Weil})$.

The abelian surface $\mathcal{A}_{277}$ has rational 15-torsion. We have the following agreement on the modular side.

Theorem 1.3. *Let* f *be as in Thm. 1.2 and be chosen so that* $f \in S_2^2(K(277))(\mathbb{Z})$ *has Fourier coefficients of unit content. Let the first Fourier Jacobi coefficient of* f *be* $\phi \in J_{2,277}$ *and let* $R = \mathrm{Grit}(\phi) \in S_2^2(K(277))(\mathbb{Z})$. *We have* $f \equiv R \mod 15$.

We find nonlift paramodular cusp forms of weight 2 by computing integral closures. This involves using the dimension formulae of Ibukiyama to compute $S_2^k(K(p))$ for $k = 4$ and 8. We find a meromorphic weight 2 form $f = H/g$ that we believe is holomorphic, where $H \in S_2^4(K(p))$ and $g \in S_2^2(K(p))$ is a Gritsenko lift; then we show that $f^2 = F$ for some $F \in S_2^4(K(p))$. That is, we verify the weight 8 identity $H^2 = g^2 F$. This proves that $f \in S_2^2(K(p))$.

The method of integral closure is also useful for weight three. Ash, Gunnells and McConnell [1] asked for Siegel modular forms of weight 3 with certain Euler factors. We have constructed nonlifts in $S_2^3(K(p))$ with these Euler factors for $p = 61$, 73 and 79.

References

[1] A. Ash and P.Gunnells, and M.McConnell, *Cohomology of Congruence Subgroups of* $SL(4, \mathbb{Z})$ *II,* preprint

[2] Brumer,A., Cramer, K., *Abelian surfaces over* $\mathbb{Q}$, preprint

[3] H.Yoshida, *Siegel Modular Forms and the arithmetic o quadratic forms*, Inventiones Mathematica 60(1980), 193-248

On generalizations of the Shimura–Taniyama conjecture

Hiroyuki Yoshida

Let E and F be number fields. Let M be a motive over F with coefficients in E. We are interested in:

Problem: Find a connected reductive algebraic group G defined over F, an automorphic representation π of $G(F_A)$ and a representation r of the L-group ${}^LG$ such that

$$L(M,s) = L(s,\pi,r).$$

The answer (G,π,r) to this question is not unique in general. The main point of this article is to find G as small as possible. Then we could derive other answers by the functoriality principle. (The existence of π on $GL(d)$, $d = \mathrm{rank}(M)$ is a folklore conjecture.)

I thank Professor D. Blasius and Dr. K. Hiraga for useful discussions.

We assume that M is of pure weight w and is polarizable. Let d be the rank of M. Fix an embedding $F \hookrightarrow \mathbf{C}$. On the Betti realization $H_B(M)$, we have an E-rational Hodge structure of weight w. Generalizing the case $E = \mathbf{Q}$, we can define the Hodge group $\mathrm{Hg}(M)$ as follows. It is the smallest algebraic subgroup H of $\mathrm{GL}(H_B(M))$ defined over E such that $H(E \otimes \mathbf{C})$ contains $h(S(\mathbf{R}))$. Here, as usual, $S = R_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)$ and h is a morphism of S into $\mathrm{GL}(H_B(M))$ associated to the Hodge structure. We can show that $\mathrm{Hg}(M)$ is connected reductive.

For a finite place λ of E, we have a λ-adic representation

$$\rho_\lambda : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(H_\lambda(M)) \cong \mathrm{GL}(H_B(M))(E_\lambda).$$

Generalizing the standard conjecture due to Mumford-Tate-Serre ([4]) for the case $E = \mathbf{Q}$, we conjecture

Conjecture A. *There exists an algebraic subgroup H of $\mathrm{GL}(H_B(M))$ defined over E such that: (i) H is independent of λ. (ii) $\mathrm{Im}(\rho_\lambda)$ is Zariski dense in H. (iii) The identity component H^0 of H is equal to $\mathrm{Hg}(M)$.*

Hereafter we assume Conjecture A.

Let v be a finite place of F. Choose λ so that $(v,\lambda) = 1$. By a standard procedure due to Deligne ([1]), we can construct a representation $W'_{F_v}(E_\lambda) \longrightarrow H(E_\lambda)$ with discrete topology on $H(E_\lambda)$. Here W'_{F_v} is the Weil-Deligne group scheme. Taking an embedding $E_\lambda \hookrightarrow \mathbf{C}$, we obtain a representation $\psi_v : W'_{F_v}(\mathbf{C}) \longrightarrow H(\mathbf{C})$.

For an infinite place v of F, we can also define a representation $\psi_v : W_{F_v}(\mathbf{C}) \longrightarrow H(\mathbf{C})$ using the Hodge structure on $H_B(M)$.

Now let K be the finite Galois extension of F such that

$$H^0(\mathbf{C}) \cap \rho_\lambda(\mathrm{Gal}(\overline{F}/F)) = \rho_\lambda(\mathrm{Gal}(\overline{F}/K)), \qquad \mathrm{Gal}(\overline{F}/K) \supset \mathrm{Ker}(\rho_\lambda).$$

Then we have the fundamental commutative diagram
(R)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \rho_\lambda(\mathrm{Gal}(\overline{F}/K)) & \longrightarrow & \rho_\lambda(\mathrm{Gal}(\overline{F}/F)) & \longrightarrow & \mathrm{Gal}(K/F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & H^0(\mathbf{C}) & \longrightarrow & H(\mathbf{C}) & \longrightarrow & \mathrm{Gal}(K/F) & \longrightarrow & 1. \end{array}$$

Here $H^0 = \mathrm{Hg}(M)$ and both rows are exact. The second exact sequence in (R) does not split in general. But it defines a homomorphism

$$\mathrm{Gal}(K/F) \longrightarrow \mathrm{Out}(H^0) = \mathrm{Aut}(H^0)/\mathrm{Inn}(H^0).$$

Choosing a splitting of H^0, we can lift this map to a homomorphism

$$\mu_{H^0} : \mathrm{Gal}(K/F) \longrightarrow \mathrm{Aut}(H^0).$$

We can find a connected reductive algebraic group G defined over F so that (i) G is quasi-split. (ii) ${}^LG^0 = H^0(\mathbf{C})$. (iii) $\mu_G = \mu_{H^0}$. Here μ_G is the homomorphism of $\mathrm{Gal}(K/F)$ into $\mathrm{Aut}({}^LG^0)$ defined by the action on the based root datum.

Now let $\mathfrak{F} = \{a(\sigma), f(\sigma,\tau)\}$, σ, $\tau \in \mathrm{Gal}(K/F)$ be the factor set of $\mathrm{Gal}(K/F)$ taking values in $H^0(\mathbf{C})$ obtained from the exact sequence in the second row of (R). Here $a(\sigma) \in \mathrm{Aut}(H^0(\mathbf{C}))$, $f(\sigma,\tau) \in H^0(\mathbf{C})$.

Theorem 1. *Let $Z(H^0(\mathbf{C}))$ be the center of $H^0(\mathbf{C})$. Assume that $a(\sigma)$ acts on $Z(H^0(\mathbf{C}))$ as the identity and $Z(H^0(\mathbf{C}))$ is connected. Then there exists a finite Galois extension L of F containing K such that the factor set $\mathfrak{F}$ splits after the inflation by the canonical map* $\mathrm{Gal}(L/F) \longrightarrow \mathrm{Gal}(K/F)$.

Call such a field L *a splitting field* for ρ_λ.

Theorem 2. *A splitting field for ρ_λ always exists.*

Let L be a splitting field for ρ_λ. We form the L-group

$${}^LG = {}^LG^0 \rtimes \mathrm{Gal}(L/F).$$

Then take r as the canonical composite map ${}^LG \longrightarrow H(\mathbf{C}) \longrightarrow \mathrm{GL}(H_B(M))(\mathbf{C})$; we can show that ψ_v lifts to the Langlands parameter $\phi_v : W'_{F_v}(\mathbf{C}) \longrightarrow {}^LG$. Now we can formulate a generalized Shimura-Taniyama conjecture.

Main Conjecture. *Take a quasi-split connected reductive algebraic group G defined over F and define r and ϕ_v as above. Then there exists an irreducible automorphic representation $\pi = \otimes_v \pi_v$ of $G(F_A)$ such that $L(s,\pi,r) = L(M,s)$. Moreover*

(i) $\pi_v \in \Pi_{\phi_v}(G/F_v)$ (= the L-packet attached to ϕ_v).
(ii) π is cuspidal if M is absolutely irreducible.
(iii) π is tempered.
(iv) π is essentially unitary

Remark 1. The π corresponding to M is not unique in general. To know which π will appear in the tempered spectrum, we need the (conjectural) multiplicity formula due to Labesse-Langlands ([3] and Kottwitz. To deduce more precise information (holomorphy of automorphic forms, for example), we need to know the structure of the L-packet $\Pi_{\phi_v}(G/F_v)$. Non-tempered π will appear if we consider a mixed motive.

Remark 2. The Tate conjecture implies that M is irreducible if and only if ρ_λ is irreducible.

Remark 3. We can show that the center of $\mathrm{Hg}(M)$ is the 1-dimensional torus if the λ-adic representation restricted to a sufficiently small open subgroup is absolutely irreducible. Is it true that the center of $\mathrm{Hg}(V)$ is $\mathbf{G}_m$ when V is an irreducible Hodge structure? Since no algebraicity conjecture for the Hodge structure is known, such and similar questions are interesting.

N.B. Theorem 2 was stated in my talk at Luminy, June, 2007. As I found a subtle gap in my proof in September, I let Theorem 2 retreat to a hypothesis in my talk. Now I have filled up the gap. The details will appear elsewhere.

I would like to end this article with my personal recollections. My first visit to the institute was the summer of 1980. I gave a talk on the following theorem: Let A be an n-dimensional abelian variety defined over F with sufficiently many complex multiplications. Then there exists a finite Galois extension K of F and a representation ρ of $W_{F,K}$ into $\mathrm{GL}(2n, \mathbf{C})$ such that $L(s, A/F) = L(s, \rho, W_{F,K})$. Here $W_{F,K}$ denotes the relative Weil group. My result was published in the next year ([5]). In my talk, I asked the following question. Does there exist an A for which ρ is not equivalent to a direct sum of monomial representations? It is my pleasure to report that I found such an A during the research sketched in this article.

References

[1] P. Deligne, *Les constantes des equations fonctionnelles des fonctions L*, in modular functions of one variable II, Lecture notes in Math. **349** (1973), 501–597, Springer-Verlag.

[2] P. Deligne, *Hodge cycles on abelian varieties (Notes by J. S. Milne)*, Lecture notes in Math. **900** (1982), 9–100, Springer-Verlag.

[3] J. -P. Labesse and R. P. Langlands, *L-indistingushability for SL(2)*, Can. J. Math. **31** (1979), 726–785.

[4] J. -P. Serre, *Représentations ℓ-adiques*, Kyoto Int. Symposium on Algebraic Number Theory, Japan Soc. for the Promotion of Science (1977), 177–193.

[5] H. Yoshida, *Abelian varieties with complex multiplication and representations of the Weil groups*, Ann. of Math. **114** (1981), 87–102.

Explicit dimension formulas for spaces of vector valued Siegel cusp forms of degree two

Satoshi Wakatsuki

In this talk we gave a trace formula and some explicit dimension formulas for Siegel cusp forms of degree two. We also discussed the surjectivity of the Witt operator as an application of our explicit dimension formulas. In this note we do not write up our trace formula, because the formula is too long. So we describe a detail of the explicit dimension formula for $\Gamma_e(1)$ and the surjectivity of the Witt operator (this part is a joint work with Tomoyoshi Ibukiyama).

Our trace formula is a step towards getting explicit computable formulas and numerical values of traces. In case of cusp forms of one variable, explicit trace formulas were studied by Eichler et al.. We have not gotten explicit trace formula yet. But we obtained explicit dimension formulas for arithmetic subgroups of any levels for each $\mathbb{Q}$-form of $Sp(2;\mathbb{R})$ by our trace formula in [7], though the details are omitted here. For the scalar valued case, the explicit dimension formulas were already known. As for some congruence subgroups of the split $\mathbb{Q}$-form, Tsushima already gave the dimension formulas for the vector valued case in [6]. By our trace formula, we generalized the explicit dimension formulas for the scalar valued case, Christian [2], Morita [5], Arakawa [1] and Hashimoto [3] to the vector valued case via the Selberg trace formula.

We give some definitions shortly. We put

$$Sp(2;\mathbb{R}) = \left\{ g \in GL(4;\mathbb{R})\,;\, g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}.$$

Let $\mathfrak{H}_2 = \{Z \in M(2;\mathbb{C})\,;\, {}^tZ = Z,\ \mathrm{Im}(Z) \text{ is positive definite }\}$. The group $Sp(2;\mathbb{R})$ acts on $\mathfrak{H}_2$ as $g \cdot Z := (AZ+B)(CZ+D)^{-1}$ for $Z \in \mathfrak{H}_2$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\mathbb{R})$. Let $\rho_{k,j} : GL(2;\mathbb{C}) \to GL(j+1;\mathbb{C})$ be the irreducible rational representation of the signature $(j+k,k)$ $(j,k \in \mathbb{Z}_{\geq 0})$, i.e. $\rho_{k,j} = \det^k \otimes Sym_j$ where Sym_j is the symmetric j-tensor representation of $GL(2;\mathbb{C})$. Let χ be a 1-dimensional unitary representation of Γ such that $[\Gamma : \ker(\chi)] < \infty$. Let $S_{k,j}(\Gamma,\chi)$ be the space of Siegel cusp forms of type $(\rho_{k,j},\chi,\Gamma)$, i.e. the space of holomorphic functions $f : \mathfrak{H}_2 \to \mathbb{C}^{j+1}$ satisfying (i) $f(\gamma \cdot Z) = \rho_{k,j}(CZ+D)f(Z)\chi(\gamma)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $Z \in \mathfrak{H}_2$, (ii) $|\rho_{k,j}(\mathrm{Im}(Z)^{1/2})f(Z)|_{\mathbb{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$, where $\mathrm{Im}(Z)^{1/2} \in SM(2;\mathbb{R})$ and $(\mathrm{Im}(Z)^{1/2})^2 = \mathrm{Im}(Z)$. If χ is trivial, we denote $S_{k,j}(\Gamma,\chi)$ simply by $S_{k,j}(\Gamma)$. Let $\Gamma(1) = Sp(2;\mathbb{Z})$ and $\Gamma(2) = \{\gamma \in \Gamma(1)\,;\, \gamma \equiv I_4 \,(\mathrm{mod} 2)\}$. Let $\Gamma_e(1)$ be the normal subgroup of $\Gamma(1)$ such that $[\Gamma(1) : \Gamma_e(1)] = 2$, $\Gamma_e(1)/\Gamma(2) \cong A_6$, and $\Gamma(1)/\Gamma(2) \cong S_6$, where S_6 is the symmetric group of degree six and A_6 is the alternative group of S_6. The unitary character sgn follows from the signature of $S_6 \cong \Gamma(1)/\Gamma(2)$.

From now, we shall discuss the explicit dimension formula for $S_{k,j}(\Gamma_e(1))$, and the images of the Witt operator related to $S_{k,j}(\Gamma(1))$ and $S_{k,j}(\Gamma(1),\mathrm{sgn})$. We note on $\dim_{\mathbb{C}} S_{k,j}(\Gamma(1),\mathrm{sgn}) = \dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1)) - \dim_{\mathbb{C}} S_{k,j}(\Gamma(1))$. For the dimension formula of $S_{k,j}(\Gamma(1))$, we refer to [6] and [7].

The notation $t = [t_0, t_1, \ldots, t_{l-1}; l; m]$ means that $t = t_n$ if $m \equiv n \pmod{l}$. We note that $\dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1)) = 0$ if j is odd. In case of $j = 0$, the dimensions of $S_{k,0}(\Gamma_e(1))$ were calculated by Igusa [4]. In case of $j > 0$, the following result is new.

Theorem *$k \geq 5$. j is even.*

$$\begin{aligned}
&\dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1)) = \\
&\quad 2^{-6}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) - 2^{-6}3^{-2}(j+1)(j+2k-3) \\
&\quad +2^{-4}3^{-1}(j+1) + 2^{-6}3^{-2}(-1)^k(j+k-1)(k-2) \\
&\quad -2^{-4}3^{-1}(-1)^k(j+2k-3) + 3\cdot 2^{-6}(-1)^k \\
&\quad -2^{-3}[(-1)^{j/2},-1,(-1)^{j/2+1},1;4;k] + 2^{-4}[1,(-1)^{j/2},-1,(-1)^{j/2+1};4;k] \\
&\quad +2^{-2}3^{-3}\left([(j+k-1),-(j+k-1),0;3;k] + [(k-2),0,-(k-2);3;j+k]\right) \\
&\quad -2^{-2}3^{-2}\left([1,-1,0;3;k]+[1,0,-1;3;j+k]\right) - 3^{-2}\left([0,-1,-1;3;k]+[1,1,0;3;j+k]\right) \\
&\quad +2^{-2}3^{-2}\left([-(j+k-1),-(j+k-1),0,(j+k-1),(j+k-1),0;6;k]\right. \\
&\qquad\qquad \left.+[(k-2),0,-(k-2),-(k-2),0,(k-2);6;j+k]\right) \\
&\quad -2^{-2}3^{-1}\left([-1,-1,0,1,1,0;6;k]+[1,0,-1,-1,0,1;6;j+k]\right) \\
&\quad +2^{-6}(-1)^{j/2}(j+2k-3) + 2^{-6}(-1)^{j/2+k}(j+1) - 2^{-3}(-1)^{j/2} \\
&\quad +3^{-3}(j+2k-3)[1,-1,0;3;j] + 2^{-1}3^{-3}(j+1)[0,1,-1;3;j+2k] - 2^{-1}3^{-1}[1,-1,0;3;j] \\
&\quad +2^{-1}3^{-2}C_2 + 2\cdot 5^{-1}C_3 + 2^{-3}C_4',
\end{aligned}$$

where $C_2 = [1,0,0,-1,0,0;6;k]$ $(j=6n)$, $[-1,1,0,1,-1,0;6;k]$ $(j=6n+2)$, $[0,-1,0,0,1,0;6;k]$ $(j=6n+4)$, $C_3 = [1,0,0,-1,0;5;k]$ $(j=10n)$, $[-1,1,0,0,0;5;k]$ $(j=10n+2)$, 0 $(j=10n+4)$, $[0,0,0,1,-1;5;k]$ $(j=10n+6)$, $[0,-1,0,0,1;5;k]$ $(j=10n+8)$, $C_4' = [1,1,-1,-1;4;k]$ $(j=8n)$, $[-1,1,1,-1;4;k]$ $(j=8n+2)$, $[-1,-1,1,1;4;k]$ $(j=8n+4)$, $[1,-1,-1,1;4;k]$ $(j=8n+6)$ *for* $n\in\mathbb{Z}_{\geq 0}$.

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1))$**.**

j \ k	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	2	3	4
2	0	0	0	0	0	1	0	1	0	1	1	2	3	3	2	3	4	7
4	0	0	0	0	0	1	1	1	1	2	3	4	4	5	6	7	9	11
6	0	1	0	1	2	2	2	4	4	5	5	8	9	11	12	13	16	19
8	1	1	0	1	2	4	4	4	5	7	9	11	13	15	16	19	23	27

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma(1),\mathrm{sgn}) = \dim_{\mathbb{C}} S_{k,j}(\Gamma_e(1)) - \dim_{\mathbb{C}} S_{k,j}(\Gamma(1))$**.**

j \ k	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	0	1	0	0	0	1	0	1	0	1	0	2	0	3	0	2	0	4
2	0	0	0	0	0	1	0	1	0	1	0	2	1	3	0	3	1	6
4	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	5	3	8
6	0	1	0	1	1	2	1	3	2	4	2	6	4	8	5	9	7	13
8	1	1	0	1	1	3	2	3	2	5	4	7	6	10	7	12	10	17

(∗) Our theorem is not valid for $k=4$. As for $(j,k)=(0,4)$, Igusa calculated it in [4]. As for $k=4$, $j>0$, the values are expected.

We define the Witt operator W as $f(Z) \longrightarrow f\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}$. Put $V_{k,j} = \{f(\tau,\omega) = (f_{j-\nu}(\tau,\omega))_{0\leq\nu\leq j}\ ;\ f_{j-\nu}(\tau,\omega) = (-1)^k f_\nu(\omega,\tau),\ f_{j-\nu}(\tau,\omega) \in S_{k+j-\nu}(SL_2(\mathbb{Z})) \otimes S_{k+\nu}(SL_2(\mathbb{Z}))\}$.

Theorem *$k \geq 10$. j is even.*

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma(1)) - \dim_{\mathbb{C}} S_{k-5,j}(\Gamma(1), \mathrm{sgn}) = \dim_{\mathbb{C}} V_{k,j}.$$

The Witt operator from $S_{k,j}(\Gamma(1))$ to $V_{k,j}$ is surjective for $k \geq 10$.

The surjectivity follows from the equality of the dimension formulas and the fact that $f/\theta_5 \in S_{k-5,j}(\Gamma(1), \mathrm{sgn})$ if $Wf = 0$ for $f \in S_{k,j}(\Gamma(1))$, where $\theta_5 \in S_{5,0}(\Gamma(1), \mathrm{sgn})$. We do not know the explanation for the surjectivity, but this property is interesting.

We also got another equality of the dimensions related to the Witt operator. Put $W_{k,j} = \{f(\tau,\omega) = (f_{j-\nu}(\tau,\omega))_{0\leq\nu\leq j}\,;\, f_{j-\nu}(\tau,\omega) = (-1)^{k+1} f_\nu(\omega,\tau),\ f_{j-\nu}(\tau,\omega) \in S_{k+j-\nu}(SL_2(\mathbb{Z}), \mathrm{sgn}) \otimes S_{k+\nu}(SL_2(\mathbb{Z}), \mathrm{sgn})\}$.

Theorem *$k \geq 10$. j is even.*

$$\begin{aligned}&\dim_{\mathbb{C}} S_{k,j}(\Gamma(1), \mathrm{sgn}) - \dim_{\mathbb{C}} S_{k-5,j}(\Gamma(1)) \\ &= \dim_{\mathbb{C}} W_{k,j} \\ &+ \begin{cases} [1,0,0;3;j] - \dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z}), \mathrm{sgn}) - \dim_{\mathbb{C}} S_j(SL_2(\mathbb{Z}), \mathrm{sgn}) & k \equiv 0 \pmod 2 \\ [1,0,0;3;j] - \dim_{\mathbb{C}} S_j(SL_2(\mathbb{Z}), \mathrm{sgn}) + \dim_{\mathbb{C}} S_{k+j-5}(SL_2(\mathbb{Z})) & k \equiv 1 \pmod 2 \end{cases},\end{aligned}$$

where we put formally $\dim_{\mathbb{C}} S_j(SL_2(\mathbb{Z}), \mathrm{sgn}) = 2^{-2}3^{-1}(j-1) + 3^{-1}[1,0,-1;3;j] - 2^{-2}(-1)^{j/2}$ *for any j.*

References

[1] T. Arakawa, *The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary group*, J. Math. Soc. Japan **33** (1981), 125–145.

[2] U. Christian, *Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q > 2$*, J. Reine Angew. Math. **277** (1975), 130–154; *Zur Berechnung des Ranges der Schar der Spitzenformen zur Modulgruppe zweiten Grades und Stufe $q > 2$*, J. Reine Angew. Math. **296** (1977), 108–118.

[3] K. Hashimoto, *The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two I*, J. Fac. Sci. Univ. Tokyo Sect IA **30** (1983), 403–488; *The dimension of the spaces of cusp forms on Siegel upper half-plane of degree two II, The $\mathbb{Q}$-rank one case*, Math. Ann. **266** (1984), 539–559.

[4] J. Igusa, *On Siegel modular forms of genus two II*, Amer. J. Math. **86** (1964), 392–412.

[5] Y. Morita, *An explicit formula for the dimension of spaces of Siegel modular forms of degree two*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **21** (1974), 167–248.

[6] R. Tsushima, *An explicit dimension formula for the spaces of generalized automorphic forms with respect to $Sp(2,\mathbb{Z})$*, Proc. Japan Acad. Ser A **59** (1983), 139–142.

[7] S. Wakatsuki, *Dimension formula for the spaces of vector valued Siegel cusp forms of degree two*, preprint.

Non-degenerate Siegel vectors in local representations of GSp(4)

Brooks Roberts
(joint work with Ralf Schmidt)

In this report we discuss part of our work on the local theory of Siegel modular forms of degree two with respect to the Siegel congruence subgroups $\Gamma_0(N)$. Suppose that F is a non-archimedean field of characteristic zero with ring of integers $\mathfrak{o}$, prime ideal $\mathfrak{p}$ in $\mathfrak{o}$, and generator ϖ for $\mathfrak{p}$. For n a non-negative integer, let $\Gamma_0(\mathfrak{p}^n)$ be the subgroup of $\mathrm{GSp}(4, \mathfrak{o})$ of elements k such that

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} \end{bmatrix}.$$

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. We let $V_0(n)$ be the subspace of vectors v in V such that $\pi(k)v = v$ for k in $\Gamma_0(\mathfrak{p}^n)$. We refer to the elements of $V_0(n)$ as Siegel vectors of level n.

Our research goal is to understand the Siegel vectors in all irreducible, admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character. In general, there are no conjectures about such fixed point spaces. At the same time, understanding these fixed point spaces is essential for relating Siegel modular forms with level to automorphic representations. In [RS], we determined the structure of all of the spaces of vectors fixed by the paramodular congruence subgroups of $\mathrm{GSp}(4, F)$ of arbitrary level. In [RS] we found that there is always uniqueness at the minimal non-zero paramodular level, that oldforms are obtained from a newform via three level raising operators, and that the minimal paramodular level is the conductor N_π of the L-parameter corresponding to the representation. As concerns Siegel vectors, we are far from having a complete theory, though we can say that the theory of Siegel vectors will be rather different from that governing paramodular vectors. In particular, there is no uniqueness at the minimal non-zero Siegel level, oldforms are not obtained from a newform via three level raising operators even if there is uniqueness at the minimal Siegel level, and the minimal Siegel level is usually lower than the conductor of the L-parameter.

There is a basic decomposition of $V_0(n)$ into a direct sum of four subspaces, and in this report we discuss some of our results about one of the components of this decomposition. For Siegel modular forms, this decomposition appears in [S]. To define the endomorphism of $V_0(n)$ whose eigenspaces form the decomposition assume that $n \geq 2$. If v is in $V_0(n)$, then we consider the vector

$$\sum_{x \in \mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & x\varpi^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v.$$

Typically, this vector is not again in $V_0(n)$. However, we can trace this vector into $V_0(n)$ and thus define an operator called $\mu : V_0(n) \to V_0(n)$. An explicit formula for $\mu(v)$ is:

$$\sum_{x\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&&&x\varpi^{-1}\\&1&&\\&&1&\\&&&1\end{bmatrix})v + \sum_{x,z\in\mathfrak{o}/\mathfrak{p}} \pi(\begin{bmatrix}1&z&&\\&1&&\\&&1&-z\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&x\varpi^{-1}&\\&&1&\\&&&1\end{bmatrix})v.$$

Two facts may be deduced from this formula. First, the second sum can be rewritten to involve only elements of the unipotent radical of the Siegel parabolic subgroup. Second, the formula does not depend on the level n. As a consequence, we find that μ is diagonalizable and that for any eigenvalue c, the space $V_0(n)_c$ is contained in $V_0(n+1)_c$. Some further analysis shows that the possible eigenvalues for μ are $q(q+1)$, q, $2q$ and 0, so that we have the eigenspace decomposition:

$$V_0(n) = V_0(n)_{q(q+1)} \oplus V_0(n)_q \oplus V_0(n)_{2q} \oplus V_0(n)_0.$$

One can prove that the $q(q+1)$-eigenspace is obtained in a simple way from vectors of lower level:

$$V_0(n)_{q(q+1)} = \pi(\begin{bmatrix}1&&&\\&1&&\\&&\varpi&\\&&&\varpi\end{bmatrix})(V_0(n)).$$

The other eigenspaces, however, are more mysterious. In this report we describe some of our results about the $2q$-eigenspace.

We will begin by delineating a useful theoretical tool called the P_3-filtration of V. To define the P_3-filtration we need to define some subgroups of $\mathrm{GSp}(4,F)$. Let

$$\underbrace{Z^J = \begin{bmatrix}1&&&*\\&1&&\\&&1&\\&&&1\end{bmatrix}}_{\text{center of Jacobi group}} \subset \underbrace{G^J = \begin{bmatrix}1&*&*&*\\&*&*&*\\&*&*&*\\&&&1\end{bmatrix}}_{\text{Jacobi group}} \subset \underbrace{Q = \begin{bmatrix}*&*&*&*\\&*&*&*\\&*&*&*\\&&&*\end{bmatrix}}_{\text{Klingen parabolic}} \subset \mathrm{GSp}(4,F).$$

It is easy to verify that Z^J is normal in Q, and one can further check that there is a natural isomorphism

$$Q/Z^J\cdot Z \xrightarrow{\sim} P_3 = \begin{bmatrix}*&*&*\\ *&*&*\\&&1\end{bmatrix} \subset \mathrm{GL}(3,F), \qquad \begin{bmatrix}*&*&*&*\\&a&b&z\\&c&d&x\\&&&1\end{bmatrix} \mapsto \begin{bmatrix}a&b&z\\c&d&x\\&&1\end{bmatrix},$$

where Z is the center of $\mathrm{GSp}(4,F)$. Recalling that our representation (π,V) of $\mathrm{GSp}(4,F)$ has trivial central character, we see that the subspace $V(Z^J)$ of V spanned by the vectors $v-\pi(z)v$ for v in V and z in Z^J is a Q-module, and that $V_{Z^J} = V/V(Z^J)$ is a $Q/Z^J\cdot Z \cong P_3$-module. We call V_{Z^J} the P_3-module of V; it can be thought of as the local version of the 0-th Fourier-Jacobi coefficient. Furthermore, it turns out that V_{Z^J} has finite length as a P_3-module, and that the P_3 filtration can be computed, with irreducible subquotients defined by representations of $\mathrm{GL}(0,F) = 1$, $\mathrm{GL}(1,F) = F^\times$ and $\mathrm{GL}(2,F)$. For more information, see [RS]. The following lemma implies that no information is lost by projecting $2q$-vectors to V_{Z^J}:

Lemma. *Let $v \in V_0(n)_{2q}$. If the image of v under the projection $V \to V_{Z^J}$ is zero, then $v = 0$.*

Using the P_3-filtration we can prove the following results about $2q$-vectors in non-generic representations. In fact, our results in this case are even more precise than stated.

Theorem. *Assume that π is non-generic. Then $\dim V_0(n)_{2q}$ is bounded by the number of irreducible subquotients of V_{Z^J} that are defined by unramified characters of $F^\times$, so that $\dim V_0(n)_{2q} \le 2$ for all $n \ge 2$. Moreover, if π is supercuspidal, then $V_0(n)_{2q} = 0$ for all $n \ge 2$.*

To study $2q$-vectors in generic representations we also use zeta integrals. Assume that π is generic and let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π. For W in $\mathcal{W}(\pi,\psi)$ let $Z(s,W)$ be the zeta integral of W as explained in [RS]. Also, let s_2 be the Weyl group element in $\mathrm{GSp}(4,F)$ as defined in [RS]. The following theorem implies that $2q$-vectors can be studied using zeta integrals.

Theorem. *If $W \in V_0(n)_{q(q+1)} \oplus V_0(n)_q \oplus V_0(n)_0$, then $Z(s,\pi(s_2)W) = 0$. If $W \in V_0(n)_{2q}$, then $W = 0$ if and only if $Z(s,\pi(s_2)W) = 0$.*

The following is our current main theorem about $2q$-vectors in the case of generic representations. In particular, it implies that if π is generic, then π admits non-zero Siegel vectors. In the theorem $\alpha', \beta' : V_0(n) \to V_0(n+1)$ are certain level raising operators. We will not recall the formula for α'; the operator β', however, is just inclusion.

Theorem. *Assume that π is generic. Define*

$$M_\pi = \begin{cases} (N_\pi+1)/2 & \text{if } N_\pi \text{ is odd,} \\ N_\pi/2 & \text{if } N_\pi \text{ is even and } \varepsilon(1/2,\pi) = 1, \\ N_\pi/2+1 & \text{if } N_\pi \text{ is even and } \varepsilon(1/2,\pi) = -1. \end{cases}$$

Then $\dim V_0(n)_{2q} \le n - M_\pi + 1$ for $n \ge 2$. Moreover, $\dim V_0(n)_{2q} = n - M_\pi + 1$ and $V_0(n+1)_{2q} = \alpha' V_0(n)_{2q} + \beta' V_0(n)_{2q}$ for $n \ge N_\pi + 1$.

This work was supported by the National Science Foundation of the United States of America.

References

[RS] Roberts, B., Schmidt, R.: *Local Newforms for* GSp(4). Lecture Notes in Mathematics, vol. **1918**, Springer Verlag, Berlin, Heidelberg, 2007 (307 pp.)

[S] Saito, H.: *A generalization of Gauss sums and its applications to Siegel modular forms and L-functions associated with the vector space of quadratic forms.* J. Reine Angew. Math. **416** (1991), 91–142

Siegel vectors in Saito–Kurokawa representations of GSp(4)

Ralf Schmidt
(joint work with Brooks Roberts)

Let F be a p-adic field, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. Inside the group

$$\mathrm{GSp}(4,F)=\{g\in\mathrm{GL}(4,F):\ {}^tgJg=\lambda(g)J,\ \lambda(g)\in F^\times\},$$

where

$$J=\begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix},$$

we consider the *Siegel congruence subgroup*

$$\Gamma_0(\mathfrak{p}^n)=\mathrm{GSp}(4,\mathfrak{o})\cap\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^n&\mathfrak{p}^n&\mathfrak{o}&\mathfrak{o}\end{bmatrix}$$

of level $\mathfrak{p}^n$. Let (π,V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Motivated by global considerations, our goal is to understand the spaces

$$V_0(n):=\{v\in V:\ \pi(g)v=v\text{ for all }g\in\Gamma_0(\mathfrak{p}^n)\}.$$

These spaces are related by a simple *level raising operator*

$$\beta:\ V_0(n)\to V_0(n+1),$$

given by applying $\pi(\mathrm{diag}(1,1,\varpi,\varpi))$. Let ψ be a fixed character of F with conductor $\mathfrak{o}$. Let

$$Z^J=\{n(z)=\begin{bmatrix}1&&&z\\ &1&&\\ &&1&\\ &&&1\end{bmatrix}:\ z\in F\},$$

and let $V(Z^J,\psi^{-1})$ be the subspace of V spanned by all elements of the form $\pi(n(z))v-\psi^{-1}(z)v$, where $v\in V$ and $z\in F$. It is easy to verify that the quotient $V_{Z^J,\psi^{-1}}:=V/V(Z^J,\psi^{-1})$ carries an action of the *Jacobi group*

$$G^J=\begin{bmatrix}*&*&*&*\\ &*&*&*\\ &*&*&*\\ &&&*\end{bmatrix}\subset\mathrm{GSp}(4,F).$$

We call $V_{Z^J,\psi^{-1}}$ the *Fourier-Jacobi module* associated to (π,V).

Lemma. *Let $p:\ V\to V_{Z^J,\psi^{-1}}$ be the projection. Let $v\in V_0(n)$ for some $n\geq 1$. Then $p(v)=0$ if and only if $v\in\beta(V_0(n-1))$.*

The lemma implies that the spaces $V_0(n)$ can be studied in the Fourier-Jacobi module. Therefore it is desirable to compute these modules more explicitly. It is known (see [BS]) that $V_{Z^J,\psi^{-1}}$, like every representation π^J of G^J for which Z^J acts via the character ψ^{-1}, can be written as $\pi^J = \tilde{\pi} \otimes \pi_{SW}^{-1}$, where $\tilde{\pi}$ is a (genuine) representation of the metaplectic group $\widetilde{\mathrm{SL}}(2,F)$, and where π_{SW}^{-1}, the *Schrödinger–Weil representation*, is a certain representation, independent of π^J, of the double cover of G^J. The metaplectic representation $\tilde{\pi}$ is irreducible if and only if π^J is irreducible. It turns out that if π is generic, then $V_{Z^J,\psi^{-1}}$ has infinite length as a G^J–module. Therefore, the use of the Fourier-Jacobi module in the generic case is limited, and other methods like zeta integrals are more promising. Representations for which Fourier-Jacobi modules are particularly useful are the *Saito–Kurokawa representations.* These are defined as theta liftings $\theta(\tilde{\pi},\psi)$ from a representation $\tilde{\pi}$ (always assumed to be genuine) of $\widetilde{\mathrm{SL}}(2,F)$; here, $\theta(\tilde{\pi},\psi)$ is as defined in [W2]. A global version of the following result is stated in [PS].

Theorem. *Let $\tilde{\pi}$ be an irreducible, admissible representation of $\widetilde{\mathrm{SL}}(2,F)$, and let $\pi = \theta(\tilde{\pi},\psi)$ be its theta lifting to* $\mathrm{GSp}(4,F)$*. Then the Fourier-Jacobi module of π is irreducible as a representation of the Jacobi group. More precisely, $\pi_{Z^J,\psi^{-1}} = \tilde{\pi} \otimes \pi_{SW}^{-1}$.*

At least in the case of odd residue characteristic, this theorem reduces the study of the spaces $V_0(n)$ in a Saito–Kurokawa representation $\pi = \theta(\tilde{\pi},\psi)$ to the study of the spaces of vectors in $\tilde{\pi}$ invariant under $\Gamma_0(\mathfrak{p}^n) := \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathfrak{o}) : c \in \mathfrak{p}^n\}$.

As an example, we consider representations of the form $\pi = \chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \chi^{-1}$, where χ is a character of $F^\times$ such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$; here, ν stands for the normalized absolute value on F, as in [ST]. In the classification of [RS], these are representations of type IIb. The corresponding $\tilde{\pi}$ is a metaplectic principal series representations $\tilde{\pi}_\chi$. The structure of invariant vectors in $\tilde{\pi}_\chi$ can be completely determined, and leads to the following structure of Siegel vectors in π. First, if $\chi(-1) = -1$, then there are no Siegel vectors of any level. Assume that $\chi(-1) = 1$. Then the minimal n for which $V_0(n) \neq 0$ is $n = 2k$, where $\mathfrak{p}^k$ is the conductor of χ. But unless $k = 0$, there is no unique newform: We have $\dim(V_0(2k)) = 2$ for $k \geq 1$. For $n \geq 2k \geq 2$ we have $\dim(V_0(n)) = (n-2k+1)(n-2k+2)$. All the oldforms can be obtained by applying two simple level raising operators to the Siegel vectors at level $2k$ and $2k+1$. In particular, there are *four* primitive vectors, two at level $2k$ and two at level $2k+1$.

This work was supported by the National Science Foundation of the United States of America.

References

[BS] Berndt, R.; Schmidt, R.: *Elements of the Representation Theory of the Jacobi Group.* Progress in Mathematics 163, Birkhäuser, 1998

[PS] Piatetski-Shapiro, I.I.: *On the Saito-Kurokawa Lifting.* Invent. math. **71** (1983), 309–338.

[RS] Roberts, B., Schmidt, R.: *Local Newforms for* GSp(4). Lecture Notes in Mathematics, vol. **1918**, Springer Verlag, Berlin, Heidelberg, 2007 (307 pp.)

[ST] Sally, Jr., P.J.; Tadić, M.: *Induced representations and classifications for* $\mathrm{GSp}(2,F)$ *and* $\mathrm{Sp}(2,F)$. Société Mathématique de France, Mémoire 52, 75–133.

[W1] Waldspurger, J.-L.: *Correspondances de Shimura.* J. Math. pures et appl. **59** (1980), 1–133

[W2] Waldspurger, J.-L.: *Correspondances de Shimura et quaternions.* Forum Math. **3** (1991), 219–307.

Hecke Operators on Siegel Theta Series

Lynne H. Walling

Given a positive definite quadratic form Q on a $\mathbb{Z}$-lattice L, we can construct a Siegel theta series attached to L; the Fourier coefficients of the theta series tell us how many times L represents lower dimensional quadratic forms. The Hecke operators help us study Fourier coefficients of a modular form.

We assume here that the rank of L is $2k$ ($k \in \mathbb{Z}$), and that Q is scaled so that $Q(L) \subseteq 2\mathbb{Z}$. We show the following.

Theorem *Let $\theta(genL)$ be the average Siegel theta series attached to L, p a prime not dividing the level of L. For $j \leq k$,*

$$\theta(genL)|T'_j(p^2) = \lambda_j\ \theta(genL)$$

where $\lambda_j = p^E\beta(n,j)(p^{k-1}+\chi(p))\cdots(p^{k-j}+\chi(p))$ where $T'_j(p^2)$ is a particular linear combination of $T_\ell(p^2)$, $0 \leq \ell \leq j$, χ is the quadratic character associated to L, E is a simple (explicit) expression, and $\beta(m,r)$ is the number of r-dimensional subspaces of an m-dimensional space over $\mathbb{Z}/p\mathbb{Z}$. When $j > k$ ($\chi(p)=1$), $j \geq k$ ($\chi(p)=-1$),

$$\theta(L)|T'_j(p^2) = 0.$$

When $\chi(p) = 1$,

$$\theta(genL)|T(p) = (p^{k-1}+1)\cdots(p^{k-n}+1)\theta(genL),$$

and when $\chi(p) = -1$,

$$\theta(genL)|T(p)^2 = ((p^{k-1}-1)\cdots(p^{k-n}-1))^2\theta(genL).$$

Recall that the Siegel theta series attached to L is

$$\theta(L;\tau) = \sum_C \mathrm{e}\{\,{}^tCAC\tau\}$$

where C varies over $\mathbb{Z}^{2k,n}$, $\tau \in \{X+iY :\ \text{symmetric } X, Y \in \mathbb{R}^{n,n},\ Y > 0\ \}$, and $\mathrm{e}\{*\} = \exp(\pi i \mathrm{Tr}(*))$. (Here $Y > 0$ means that the quadratic form represented by the matrix Y is positive definite.) We set $\theta(\mathrm{gen}L) = \sum_{L'} \frac{1}{o(L')}\theta(L')$ where L' varies over the isometry classes in the genus of L, and $o(L')$ is the order of the orthogonal group of L'. (L' is in the genus of L if, locally everywhere, L' and L are isometric.)

In earlier work with J.L. Hafner, we found explicit matrices giving the action of the Hecke operators $T(p)$, $T_j(p^2)$ ($1 \leq j \leq n$). When analyzing the action of the $T_j(p^2)$ on the Fourier coefficients of a Siegel modular form, we encountered incomplete character sums; to complete these, we replace $T_j(p^2)$ by

$$\widetilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{0\le\ell\le j} \beta(n-\ell, j-\ell) T_\ell(p^2)$$

where $\beta(m,r)$ is the number of r-dimensional subspaces of an m-dimensional space over $\mathbb{Z}/p\mathbb{Z}$.

We write the Fourier series of a Siegel modular form F as

$$F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda) \mathrm{e}^*\{\Lambda\tau\}$$

where clsΛ varies over isometry classes of even integral, positive semi-definite rank n lattices (oriented when k is odd), and

$$\mathrm{e}^*\{\Lambda\tau\} = \sum_G \mathrm{e}\{{}^tGTG\tau\},$$

T a matrix for the quadratic form on Λ, $G \in O(T)\backslash GL_n(\mathbb{Z})$ if k is even, $G \in O^+(T)\backslash SL_n(\mathbb{Z})$ if k is odd. Then with Hafner we showed that the Λth coefficient of $F|\widetilde{T}_j(p^2)$ is

$$\sum_{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda} p^{E(\Lambda,\Omega)} \alpha_j(\Lambda,\Omega) c(\Omega)$$

where $E(\Lambda,\Omega)$ is given by an explicit expression in terms of n, j and the multiplicity of the invariant factors of Ω in Λ, and $\alpha_j(\Lambda,\Omega)$ is the number of totally isotropic, codimension $n-j$ subspaces of $\Lambda\cap\Omega/p(\Lambda+\Omega)$. (We get a similer but simpler formula for the Λth coefficient of $F|T(p)$.)

Applying the coset representatives directly to $\theta(L)$, we obtain

$$\theta(L;\tau)|\widetilde{T}_j(p^2) = \sum_{\Lambda\subseteq L} \left(\sum_{p\Lambda\subseteq\Omega\subseteq\frac{1}{p}\Lambda} p^*\alpha_j(\Lambda,\Omega) \right) \mathrm{e}\{\Omega\tau\}$$

(where $\mathrm{e}\{\Omega\tau\} = \sum_G \mathrm{e}\{{}^tGTG\tau\}$, T a matrix for the quadratic form on Ω, $G \in GL_n(\mathbb{Z})$). Now we interchange the order of summation to get

$$\theta(L;\tau)|\widetilde{T}_j(p^2) = \sum_{\substack{\Omega\subseteq\frac{1}{p}L \\ \Omega \text{ integral}}} \left(\sum_{p\Omega\subseteq\Lambda\subseteq\frac{1}{p}\Omega\cap L} p^*\alpha_j(\Lambda,\Omega) \right) \mathrm{e}\{\Omega\tau\}.$$

Given Ω, we construct the Λ in the inner sum using a two-step modulo p construction, simultaneously computing $\alpha_j(\Lambda,\Omega)$. We find that the sum on Λ can be described by a doubly indexed sum of products of functions of the form

$$\delta(m,r) = (p^m+1)\cdots(p^{m-r+1}+1),\ \mu(m,r) = (p^m-1)\cdots(p^{m-r+1}-1),$$

$\beta(m,r)$ (as defined earlier), and $\varphi_\ell(\overline{\Omega}_1)$; here $\Omega = \frac{1}{p}\Omega_0 \oplus \Omega_1 \oplus p\Omega_2$ where $\Omega_i \subseteq L$ with Ω_0, Ω_1 primitive in L modulo p, and $\varphi_\ell(U)$ is the number of dimension ℓ totally isotropic subspaces of a $\mathbb{Z}/p\mathbb{Z}$-space U.

We compare $\theta(L)|\widetilde{T}_j(p^2)$ to $\sum_{K_j}\theta(K_j)$ where the K_j are sublattices of $\frac{1}{p}L$ so that the invariant factors $\{L : K_j\}$ consist of j factors $\frac{1}{p}$, j factors p, $2(k-j)$ factors 1, and with $K_j \in \mathrm{gen}L$. Notice that we have the constraint that such K_j do not exist unless $j \leq k$ ($\chi(p)=1$), $j<k$ ($\chi(p)=-1$). For j thus constrained, we count which (even) integral, rank n $\Omega \subseteq \frac{1}{p}L$ lie in each K_j; we find that

$$\sum_{K_j}\theta(K_j;\tau) = \sum_{\substack{\Omega\subseteq\frac{1}{p}L \\ \Omega \text{ integral}}} b_j(\Omega)\mathrm{e}\{\Omega\tau\}$$

where $b_j(\Omega)$ is given by a sum of products of the functions $\delta, \mu, \beta, \varphi_\ell(\overline{\Omega}_1)$ (as described above).

We compare $\theta(L)|\widetilde{T}_j(p^2)$ to $\sum_{K_j}\theta(K_j)$; they do not quite match up, although their "leading" terms do. Hence, adjusting our operators again, we get

$$\sum_{0\leq q\leq j} u_j(q)\ \theta(L)|\widetilde{T}_{j-q}(p^2) = \sum_{0\leq q\leq j} v_j(q)\ \sum_{K_{j-q}}\theta(K_{j-q}).$$

Averaging over gen$L$ quickly yields our result for j constrained as described above.

When j does not meet these constraints, we realize $\theta(L)|T'_j(p^2)$ as a linear combination of $\theta(L)|T'_\ell(p^2)$, $\ell \leq k$ when $\chi(p)=1$, $\ell<k$ when $\chi(p)=-1$. We immediately obtain $\theta(L)|T'_j(p^2)=0$ for such j.

The arguments to analyze $\theta(\mathrm{gen}L)|T(p)$ and $\theta(\mathrm{gen}L)|T(p)^2$ are similar but simpler.

References

[1] A.N. Andrianov *Quadratic Forms and Hecke Operators* Grund. Math. Wiss., Vol. 286, Springer-Verlag 1987

[2] E. Freitag *Die Wirkung von Heckeoperatoren auf Thetareihen mit harmonischen Koeffizienten*, Math. Ann. 258 (1982), 419-440

[3] J.L. Hafner, L.H. Walling *Explicit action of Hecke operators on Siegel modular forms*, J. Number Theory 93(2002),34-57

On a set of products of two Eisenstein series which generates the space of cusp newforms

Yves Martin

(joint work with Winfried Kohnen)

1. Products of Eisenstein series and main results

For positive integers s and n let $r_s(n)$ be the number of representations of n as a sum of s integral squares. It is well know that $r_4(n)$ can be expressed in term of powers of divisors of n. One way to explain this formula is that $r_4(n)$ is related to the n-th Fourier coefficient of a modular form of weight 2 over a congruence subgroup, and that such space of modular forms is generated by Eisenstein series. If we want to generalize this fact for larger, even s, one is forced to consider the following issue; how to write the n-th Fourier coefficient of a cusp form using powers of divisors of n in some simple way.

In [3] O. Imamoglu and W. Kohnen solve this problem for all $s \equiv 0 \bmod 8$ and $s \geq 16$ via the following result.

Theorem 1. *(Imamoglu-Kohnen) Let $k \geq 2$. The set of products*

$$\left\{E_r^\infty(\tau)E_{2k-r}^0(\tau) \mid 4 \leq r \leq 2k-4,\ r \text{ even}\right\}$$

spans the vector space of cusp forms of weight $2k$ over $\Gamma_0(2)$.

Here

$$E_r^\infty(\tau) = \frac{1}{2(2^r-1)\zeta(r)}\left\{2^r E_r(2\tau) - E_r(\tau)\right\},$$
$$E_r^0(\tau) = \frac{2^{r/2}}{2(2^r-1)\zeta(r)}\left\{E_r(\tau) - E_r(2\tau)\right\},$$

and $E_r(\tau)$ denotes the Eisenstein series of weight r and level 1.

The purpose of this note is to report on work in progress towards a generalization of this theorem. More precisely, we want to describe a set of Eisenstein series and products of two Eisenstein series which generate the space of modular forms of weight $2k$ over the group $\Gamma_0(q)$ with q prime. To this end, we consider

$$E_r(\tau) = 2\zeta(r) + \frac{2(-2\pi i)^r}{\Gamma(r)}\sum_{n=1}^{\infty}\sigma_{r-1}(n)e(n\tau)$$

for any positive integer $r > 2$, and

$$E_{r,\psi}(\tau) = 2L(\psi, r) + \frac{2\mathcal{G}_\psi(-2\pi i)^r}{q^r\Gamma(r)}\sum_{n=1}^{\infty}\sigma_{r-1,\overline{\psi}}(n)e(n\tau)$$

for a positive integer r and any Dirichlet character ψ mod q with $\psi(-1) = (-1)^r$. Here $\mathcal{G}_\psi$ is equal to the Gauss sum associated to ψ (resp. 1) if ψ is primitive (resp. trivial) and $\sigma_{r-1,\psi}(n) = \sum_{d|n}\psi(d)d^{r-1}$. In this definition the series with $r = 1, 2$ are obtained by analytic continuation and $(r, \psi) \neq (2, \psi_0)$, where ψ_0 denotes the trivial character mod q.

If we denote by $\mathfrak{M}_r(N, \chi)$ (resp. $\mathfrak{S}_r(N, \chi)$) the space of modular forms (resp. cusp forms) of weight r and character χ over the group $\Gamma_0(N)$, we have

$$E_r(\tau) \in \mathfrak{M}_r(1), \qquad E_r(q\tau) \in \mathfrak{M}_r(q) \qquad \text{and} \qquad E_{r,\psi}(\tau) \in \mathfrak{M}_r(q, \psi).$$

Consider next the following modular forms in $\mathfrak{M}_{2k}(q)$:

$$G_r(\tau) = E_r(\tau)E_{2k-r}(\tau) \quad \text{and} \quad F_{t,\psi}(\tau) = E_{t,\psi}(\tau)E_{2k-t,\overline{\psi}}(\tau),$$

where $4 \leq r \leq k$ even, $1 \leq t \leq k$ and ψ is any Dirichlet character mod q such that $\psi(-1) = (-1)^t$, $(t, \psi) \neq (2, \psi_0)$. Then our main results (which may not be in their final form) are

Claim 1 Let q be a prime. The collection

$$\{E_{2k}(\tau),\ E_{2k,\psi_0}(\tau),\ G_r(\tau),\ G_r(q\tau),\ F_{t,\psi}(\tau) \mid \text{for } r,\ t,\ \psi \text{ as above}\}$$

spans the vector space $\mathfrak{M}_{2k}(q)$.

Claim 2 Let q be a prime. There are simple and explicit orthogonal projection operators $pr^{\pm}$ from $\mathfrak{S}_{2k}(q)$ onto the spaces $\mathfrak{S}_{2k}^{\pm}(q)$ of new and old cusp forms so that

$$\left\{pr^{+}\left(F^{*}_{t,\psi}(\tau)\right) \mid \text{for } t,\ \psi \text{ as above}\right\}$$

generates the space of newforms $\mathfrak{S}_{2k}^{+}(q)$. Here $F^{*}_{t,\psi}(\tau)$ denotes the cuspidal part of $F_{t,\psi}(\tau)$.

2. Some remarks about the proofs

It is not difficult to see that the first claim follows from the second one plus the fact that $\mathfrak{M}_{2k}(1)$ is generated by $E_{2k}(\tau)$ and the set of products $G_r(\tau)$ (see for example remark in [5]).

The proof of the second claim depends mainly on two basic ideas that we illustrate in the case of the products $F_{t,\psi}(\tau)$ with ψ a primitive character mod q. Using the Rankin-Selberg method and some algebraic manipulations one gets the following expression for the Petersson inner product of $f(\tau)$ and $F_{r,\psi}(\tau)$ whenever $f(\tau)$ is a Hecke eigenform.

Lemma 1. *Let $f(\tau) \in \mathfrak{S}_{2k}(q)$ be a normalized eigenform for all Hecke operators T_n with* $\gcd(n,q)=1$. *If $1 \le r \le 2k-1$ is an integer and ψ a primitive character mod q such that $\psi(-1)=(-1)^r$, then*

$$\langle f(\tau), F_{r,\psi}(\tau)\rangle = \frac{(-i)^r\Gamma(2k-1)\mathcal{G}_{\psi}}{2^{2k-3}(2\pi)^{2k-r-1}\Gamma(r)q^r} L(f;2k-1)L(f_{\overline{\psi}};2k-r).$$

Here $L(f;s)$ denotes the Hecke L-function of $f(\tau)$ and $\Gamma(s)$ is Euler's Gamma function. This identity was established by Rankin for modular forms of level 1 and generalized by Zagier [5] to arbitrary congruence subgroups and multiplier systems with large enough weights.

The other important ingredient in the proof of claim 2 comes from the Eichler-Shimura theory for congruence subgroups (see [1]). The period polynomial of a cusp form $g(\tau)$ of weight $2k = w+2$ is

$$P_g(X) = \int_0^{i\infty} g(\tau)(X-\tau)^w d\tau = \sum_{j=0}^{w}(-1)^j \binom{w}{j} r_j(g)X^{w-j}.$$

Its coefficient $r_j(g)$ is related to the special value $L(g;j+1)$ of the L-function. If V is the space of polynomials with complex coefficients of degree at most w, there is a function Φ^- from $\mathfrak{S}_{2k}(q)$ into cohomology group $H^1_P(\Gamma_0(q),V)$ given by

$$\Phi^-(g)(\gamma,X) = \frac{1}{2}\int_{\gamma(i\infty)}^{i\infty} g(\tau)\left(X-\tau\right)^w d\tau - \frac{1}{2}\int_{-\gamma(i\infty)}^{i\infty} g(\tau)\left(X+\tau\right)^w d\tau$$

for all $g(\tau)$ in $\mathfrak{S}_{2k}(q)$ and γ in $\Gamma_0(q)$. This is an injective homomorphism from $\mathfrak{S}_{2k}(q)$ into $H^1_P(\Gamma_0(q),V)$. Using a particular set of generators $\gamma_{q,a}$ of $\Gamma_0(q)$ for which we can write $\Phi^-(g)(\gamma_{q,a},X)$ in terms of $P_g(X)$ and $P_{g_\psi}(X)$, we get

Lemma 2. *Let $f(\tau)$ be a Hecke eigenform in $\mathfrak{S}_{2k}^{+}(q)$ with $f|_{2k}[W_q](\tau) = \pm f(\tau)$, where W_q is the Fricke involution. Then the following statements are equivalent.*
i) $f(\tau)$ is the zero function.
ii) $L(f_\psi, r) = 0$ for all integers $1 \leq r \leq 2k-1$ and all Dirichlet characters ψ mod q such that $\psi(-1) = (-1)^r$, $(r, \psi) \neq (2, \psi_0)$.

Claim 2 now follows from Lemma 1, Lemma 2 and a suitable linear algebra argument.

References

[1] J. Antoniadis, *Modulformen auf $\Gamma_0(N)$ mit rationalen Perioden*, Manusc. Math. **74** (1992), 359–384.
[2] W. Kohnen, D. Zagier, *Modular forms with rational periods*, in Modular forms (ed. R. Rankin), Ellis Horwood, Chichester, 1984.
[3] O. Imamoglu, W. Kohnen, *representations of integers as sums of an even number of squares*, Math. Ann. **333** (2005), 815–829.
[4] T. Miyake, *Modular forms*, Springer-Verlag Berlin Heidelberg New York, 1989.
[5] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*, LNM **627** Springer-Verlag Berlin Heidelberg New York, 1977.

Two applications of the spectral theory on the three dimensional hyperbolic space to Hermitian modular forms

Yoshinori Mizuno
(joint work with Roland Matthes)

Imai [2] discovered how one can apply the spectral theory on the upper half-plane to Siegel modular forms of degree two. Take a Fourier series on the Siegel upper half-space whose Fourier coefficients are unimodular invariant and increase reasonably. Let $Z = it^{1/2}W$ be a variable on the Siegel upper half-space, where $t > 0$ and W is a positive definite real symmetric matix of size two whose determinant is one. Identifying W with a variable τ on the upper half-plane, we have the Roelcke-Selberg spectral decomposition of the Fourier series as a function of τ. Then each spectral coefficient with respect to any eigenfunction is the inverse Mellin transform of a certain Dirichlet series now called by a Koecher-Maass series. Thus we can analyze the Fourier series on the Siegel upper half-space by studying each spectral coefficient, in other words by studying each Koecher-Maass series. Using this principle, Imai formulated a converse theorem for Siegel modular forms. In fact Duke-Imamoglu [1] gave a new proof of Saito-Kurokawa lift for Siegel modular forms by Imai's converse theorem combined with Katok-Sarnak's correspondence for Maass forms [3]. In my talk, a three dimensional analogue of Katok-Sarnak's correspondence is given and two applications of the spectral theory on the three dimensional hyperbolic space to Hermitian modular forms are indicated. Let K be an imaginary quadratic field of discriminant $-D$, $\mathcal{O}$ its ring of integers, $\chi_K = (\frac{-D}{*})$ the Kronecker symbol of K and $\mathcal{D}^{-1}$ the inverse different. Let $\mathbf{H} = \{P = z + rj; z \in$

$\mathbf{C}, r > 0\}$ be the three dimensional hyperbolic space. An automorphic function on $\mathbf{H}$ is any function $\mathcal{U}(P)$ on $\mathbf{H}$ satisfying the following three conditions.

(G-i)$\mathcal{U}(\gamma P) = \mathcal{U}(P)$ for all $\gamma \in SL_2(\mathcal{O})$.
(G-ii)$\mathcal{U}(P)$ is a C^2-function on $\mathbf{H}$ with respect to $x, y, r, P = x + yi + rj \in \mathbf{H}$ which verifies a differential equation $-\Delta\mathcal{U} = \lambda\mathcal{U}$ with some $\lambda \in \mathbf{C}$, where $\Delta = r^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}) - r\frac{\partial}{\partial r}$ is the hyperbolic Laplace-Beltrami operator on $\mathbf{H}$.
(G-iii)$\mathcal{U}(P)$ is of polynomial growth at all cusps of $SL_2(\mathcal{O})$.

Let $\mathcal{P}_2$ be the set of all positive definite hermitian matrices of size two and $\mathcal{P}S_2$ the determinant one surface of $\mathcal{P}_2$. We identify $\mathcal{P}S_2$ with the three dimensional hyperbolic space $\mathbf{H}$ by

$$\begin{pmatrix} (|z|^2 + r^2)r^{-1} & zr^{-1} \\ \overline{z}r^{-1} & r^{-1} \end{pmatrix} \to P = z + rj$$

and extend any automorphic function $\mathcal{U}(P)$ on $\mathbf{H}$ to a function on $\mathcal{P}_2$ by setting $\mathcal{U}(T) = \mathcal{U}(P_T)$, where P_T corresponds to $\det T^{-1/2}T$, in other words $T \in \mathcal{P}_2$ is identified with $P_T \in \mathbf{H}$ by

$$T = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \to P_T = \frac{b}{d} + \frac{\sqrt{\det T}}{d}j.$$

Our Katok-Sarnak type result gives a correspondence between the space of automorphic functions on the three dimensional hyperbolic space $\mathbf{H}$ and the space of Maass forms of weight -1 with respect to $\Gamma_0(D)$ on the upper half-plane H_1. For $\mu \in \mathbf{C}$ let T_μ^+ denote the vector space consisting of all functions $f(\tau)$ on the upper half-plane $H_1 = \{\tau = u + iv; v > 0\}$ satisfying the following three conditions.

(M-i) Each $f(\tau)$ is a C^2-function of $u = \Re\tau$ and $v = \Im\tau$ verifying the transformation formula

$$f(\gamma\tau) = \chi_K(d)(c\tau + d)^{-1}f(\tau),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$$

and it is of polynomial growth at all cusps of $\Gamma_0(D)$.
(M-ii)$f(\tau)$ has a Fourier expansion of the form $f(\tau) = \sum_{l\in\mathbf{Z}} B(l, v)e(lu)$, where the Fourier coefficients $B(l, v)$ for $l \neq 0$ are given by

$$B(l, v) = b(l)v^{1/2}W_{-sgn(l)/2,\mu/2}(4\pi|l|v).$$

(M-iii)If $a_D(l) = 0$ then $B(l, v) = 0$, where

$$a_D(l) = \sharp\{b \in \mathcal{D}^{-1}/\mathcal{O}; D\mathcal{N}(b) \equiv -l \pmod{D}\}.$$

The following is a three dimensional analogue of Katok-Sarnak [3] for cusp eigenfunctions and Duke-Imamoglu [1] for non-cusp eigenfunctions.

Theorem 1. *Let $\mathcal{U}(P)$ be an automorphic function on $\mathbf{H}$ which is the Eisenstein series on $SL_2(\mathcal{O})$ or belongs to a complete orthonormal set of eigenfunctions for $-\Delta$ in $L^2(SL_2(\mathcal{O}) \setminus \mathbf{H})$. Assume that $-\Delta\mathcal{U} = (1-\mu^2)\mathcal{U}$ with some $\mu \in \mathbf{C}$. Then there exists $f(\tau) \in T_\mu^+$ which satisfies the relation*

$$b(l) = l^{-1} \sum_{T \in SL_2(\mathcal{O})\setminus L_2^+,\ D \det T = l} \mathcal{U}(T)/\epsilon(T)$$

for any natural number l, where $\epsilon(T)$ is the oder of the unit group of T, L_2^+ is the set of all half integral positive definite hermitian matrices of size two

$$L_2^+ = \left\{ T = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} > O; a, d \in \mathbf{Z}, b \in \mathcal{D}^{-1} \right\},$$

the summation extends over all $T \in L_2^+$ such that $D \det T = l$ modulo the action $T \to [U]T = UT\ {}^t\overline{U}$ of the group $SL_2(\mathcal{O})$.

This is one of our main tools to apply the spectral theory on the three dimensional hyperbolic space to Hermitian modular forms. In my talk, two applications in case of the Gaussian number field are indicated. The first is to give a new proof of Saito-Kurokawa lift for Hermitian modular forms by a converse theorem. The second is to obtain an explicit arithmetic formula for the Fourier coefficients of Hermitian-Eisenstein series of degree two with square-free odd levels.

References

[1] W. Duke, O. Imamoglu, A converse theorem and the Saito-Kurokawa lift. Internat. Math. Res. Notices 1996, no. 7, 347–355.

[2] K. Imai, Generalization of Hecke's correspondence to Siegel modular forms. Amer. J. Math. **102** (1980), no. 5, 903–936.

[3] S. Katok, P. Sarnak, Heegner points, cycles and Maass forms, Israel J. Math. **84** (1993), no. 1-2, 193–227.

Period and congruence of the Ikeda lift

Hidenori Katsurada

In this report, we consider Ikeda's conjecture on the period of the Ikeda lift and its application.

Let k and n be positive even integers. Let $f(z) = \sum_{m=1}^{\infty} a(m)\exp(2\pi i m z)$ be a primitive form in $S_{2k-n}(\Gamma_1)$. Furthermore let $\tilde{f}(z) = \sum_e c(e)\exp(2\pi i e z)$ be the Hecke eigenform in Kohnen's plus subspace $S^+_{k-n/2+1/2}(\Gamma_0(4))$ corresponding to f under the Shimura correspondence. Then Ikeda [Ik1] showed that there exisits a Hecke eigenform $I_n(f)(Z)$ in $S_k(\Gamma_n)$ whose standard L-function is $\zeta(s)\prod_{i=1}^n L(s+k-i, f)$. We call $I_n(f)$ the Ikeda lift of f. We note that $I_2(f)$ is the Saito-Kurokawa lift of f.

To state our first main result, put

$$\Gamma_{\mathbf{C}}(s) = 2(2\pi i)^{-s}\Gamma(s),$$

and

$$\widetilde{\xi}(s) = \Gamma_{\mathbf{C}}(s)\zeta(s).$$

Let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+n/2+1/2}a(p)$. For a Dirichlet character χ we define the L-function $L(s, f, \chi)$ of f twisted by χ as

$$L(s, f, \chi) = \prod_p \{(1 - \alpha_p p^{-s+k-n/2-1/2}\chi(p))(1 - \alpha_p^{-1} p^{-s+k-n/2-1/2}\chi(p))\}^{-1},$$

and put

$$\Lambda(s,\ f, \chi) = \Gamma_{\mathbf{C}}(s)L(s,\ f, \chi).$$

In particular, if χ is the principal character we write $L(s, f, \chi)$ and $\Lambda(s, f, \chi)$ as $L(s, f)$ and $\Lambda(s, f)$, respectively. Furthermore, $L(s,\ f,\ \mathrm{Ad})$ be the adjoint L-function by

$$L(s, f, \mathrm{Ad}) = \prod_p \{(1 - \alpha_p^2 p^{-s})(1 - \alpha_p^{-2} p^{-s})(1 - p^{-s})\}^{-1},$$

and put

$$\widetilde{\Lambda}(s,\ f,\ \mathrm{Ad}) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s + 2k - n - 1)L(s,\ f,\ \mathrm{Ad}).$$

Then our first main result of this report is:

Theorem 1. (Joint with H. Kawamura, [K-K1], [K-K2]) *There exists an integer* $\alpha(n,\, k)$ *such that*

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle} = 2^{\alpha(n,\, k)}\Lambda(k,\ f)\widetilde{\xi}(n) \prod_{i=1}^{n/2-1} \tilde{\Lambda}(2i+1,\ f,\ \mathrm{Ad})\,\widetilde{\xi}(2i)$$

This was conjectured by Ikeda [Ik2]. We note that the above result in case $n = 2$ has been proved by Kohnen and Skoruppa [K-S].

Next, let $\mathbf{L}_n$ denote the Hecke ring over $\mathbf{Z}$ associated with the Hecke pair $(GSp_{2n}(\mathbf{Q})^+ \cap M_{2n}(\mathbf{Z}), \mathit{\Gamma}_n)$. For an element $T \in \mathbf{L}_n$, we define the Hecke operator $f|_k T$ as usual (cf. [A].) If f is an eigenfunction of a Hecke operator $T \in \mathbf{L}_n \otimes \mathbf{C}$, we denote by $\lambda_f(T)$ its eigenvalue. Furthermore, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by eigenvalues of all $T \in \mathbf{L}_n$.

Let K be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. Let f be a Hecke eigenform in $S_k(\mathit{\Gamma}_n)$ and M be a subspace of $S_k(\mathit{\Gamma}_n)$ stable under Hecke operators $T \in \mathbf{L}_n$. Assume that M is contained in $(\mathbf{C}f)^\perp$, where $(\mathbf{C}f)^\perp$ is the orthogonal complement of $\mathbf{C}f$ in $S_k(\mathit{\Gamma}_n)$ with respect to the Petersson product. Let K be an algebraic number field containing $\mathbf{Q}(f)$. A prime ideal $\mathfrak{P}$ of $\mathfrak{O}_K$ is called a congruence prime of f with respect to M if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \bmod \tilde{\mathfrak{P}}$$

for any $T \in \mathbf{L}_n$, where $\tilde{\mathfrak{P}}$ is the prime ideal of $\mathfrak{O}_{K\mathbf{Q}(g)}$ lying above $\mathfrak{P}$. If $M = (\mathbf{C}f)^{\perp}$, we simply call $\mathfrak{P}$ a congruence prime of f.

Now returning to the case of the Ikeda lift, let f be a primitive form in $S_{2k-n}(\Gamma_1)$. Let K be an algebraic number field containing all the Hecke fields of primitive forrms in $S_{2k-n}(\Gamma_1)$. To formulate our conjecture exactly, let $\Omega(f, j; A_{\mathfrak{P}})$ $(j = \pm)$ be canonical periods or the Eichler-Shimura periods arising from the Eichler-Shimura isomorphism (cf. Hida [H].) This $\Omega(f, j; A_{\mathfrak{P}})$ is uniquely determined up to constant multiple of units in $A_{\mathfrak{P}}$. For $j = \pm, 1 \leq l \leq 2k - n - 1$, and a Dirichlet character χ such that $\chi(-1) = j(-1)^{l-1}$, put

$$\mathbf{L}(l, f, \chi) = \mathbf{L}(l, f, \chi; A_{\mathfrak{P}}) = \frac{\Gamma(l)L(l, f, \chi)}{\tau(\chi)(2\pi\sqrt{-1})^l\Omega(f, j; A_{\mathfrak{P}})},$$

where $\tau(\chi)$ is the Gauss sum of χ. In particular, put $\mathbf{L}(l, f; A_{\mathfrak{P}}) = \mathbf{L}(l, f, \chi; \mathfrak{P})$ if χ is the principal character. Furthermore, put

$$\mathbf{L}(l, f, \mathrm{Ad}) = \frac{\widetilde{\Lambda}(l, f, \mathrm{Ad})}{\langle f, f\rangle}.$$

Then it is well-known that $\mathbf{L}(l, f, \chi)$ belongs to the field $K(\chi)$ generated over K by all the values of χ, and $\mathbf{L}(l, f, \mathrm{Ad})$ belongs to $\mathbf{Q}(f)$ (cf. [Z].) Let $S_k(\Gamma_n)^*$ be the subspace of $S_k(\Gamma_n)$ generated by all the Ikeda lifts $I_n(g)$ of primitive forms $g \in S_{2k-n}(\Gamma_1)$. We remark that $S_k(\Gamma_2)^*$ is the Maass subspace of $S_k(\Gamma_2)$. The second main result of this report is as follows:

Theorem 2. *Let K and f be as above. Let $\mathfrak{P}$ be a prime ideal of K. Furthermore assume that*
(1) *$\mathfrak{P}$ divides $\mathbf{L}(k, f)\prod_{i=1}^{n/2-1}\mathbf{L}(2i + 1, f, \mathrm{Ad})$.*
(2) *$\mathfrak{P}$ does not divide*

$$\tilde{\xi}(2m)\prod_{i=1}^{n}\mathbf{L}(2m + k - i, f)\mathbf{L}(k - n/2, f, \chi_D)D(2k - 1)!$$

for some integer $2 \leq m \leq k/2 - n/2$, and for some fundamental discriminant D such that $(-1)^{n/2}D > 0$.
(3) *The q-th Fourier coefficient of f is not divided by $\mathfrak{P}$ if $\mathfrak{P}$ divides q.*
(4) *$\mathfrak{P}$ does not divide*

$$\frac{C_{n.k}\langle f, f\rangle}{\Omega(f, +, A_{\mathfrak{P}})\Omega(f, -, A_{\mathfrak{P}})},$$

where $C_{n,k} = 1$ or $\prod_{p\leq(2k-n)/12}(1+p+\cdots+p^{n-1})$ according as $n = 2$ or not.Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(S_k(\Gamma_n)^*)^{\perp}$.

We note that results similar to above have been proved by Brown [B], and Katsurada [K] independently in case $n = 2$.

References

[A] A. N. Andrianov, Quadratic forms and Hecke operators, Springer, 1987.

[B] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, to appear in Compositio Math.

[H] H. Hida, Modular forms and Galois cohomology, Cambridge Univ. Press, 2000

[Ik1] [Ik1] T. Ikeda, On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$, Ann. of Math. 154(2001), 641-681.

[Ik2] T.Ikeda, Pullback of lifting of elliptic cusp forms and Miyawaki's conjecture, Duke Math. J. 131 (2006), 469-497.

[K] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, to appear in Math. Z.

[K-K1] H. Katsurada and H. Kawamura, A certain Dirichlet series of Rankin-Selberg type associated with the Ikeda lifting, To appear in J.N.T.

[K-K2] H. Katsurada and H. Kawamura, Ikeda's conjecture on the Petersson product of the Ikeda lifting, Preprint 2007.

[K-S] W. Kohnen and N-P. Skoruppa, A certain Dirichlet series attached to Siegel modular forms of degree 2, Invent. Math. 95, 541-558(1989).

[Sh] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. pure appl. Math. 29(1976), 783-804.

[Z] D. Zagier, Modular forms whose Fourier coefficients involve zeta functions of quadratic fields, Lect. Notes in Math. 627(1977), 105-169.

Modular forms of small weights and applications to the algebraic geometry

Valery Gritsenko

The global Torelli theorem for projective K3 surfaces was first proved by Piatetskii-Shapiro and Shafarevich 35 years ago, opening the way to treat moduli problems for K3 surfaces. The moduli space of polarised K3 surfaces of degree $2d$ is a quasi-projective variety of dimension 19. For general d very little has been known about the Kodaira dimension of these varieties. In my joint paper [GHS07] with K. Hulek (Hannover) and G. Sankaran (Bath) we presented an almost complete solution to this problem. Our main result is

Theorem 1. *The moduli space of $2d$-polarised* K3 *surfaces is of general type for $d > 61$ and for $d = 46, 50, 54, 58, 60$.*

In this talk I mainly describe the automorphic part of the solution. A short description of the algebro-geometric ideas of our proof you can find in the Bourbaki talk of C. Voisin (see [V07]) or in my talk given at Arbeitstagung in Bonn (http://www.mpim-bonn.mpg.de/at2005).

The moduli space of the polarised K3 surfaces is a modular variety of orthogonal type. Let L be an integral even lattice of signature $(2, n)$ and $(\, , \,)$ the associated bilinear form. By D_L we denote a connected component of the homogeneous type IV complex domain of dimension n

$$D(L) = \{[v] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (v, v) = 0, \ (v, \bar{v}) > 0\}^+.$$

$O^+(L)$ is the index 2 subgroup of the integral orthogonal group $O(L)$ that leaves D_L invariant. Any subgroup Γ of $O^+(L)$ of finite index determines the modular variety

$$F_L(\Gamma) = \Gamma \setminus D(L).$$

This is a quasi-projective variety of dimension n. The classical problem is to determine the birational type of this variety. We obtain the moduli space of $2d$-polarised K3 surfaces if

$$L = L_{2d} = 2U \oplus 2E_8(-1) \oplus < -2d >, \qquad \Gamma = \widetilde{O}^+(L_{2d}),$$

where U is the unimodular hyperbolic plane and $\widetilde{O}^+(L_{2d})$ denotes the subgroup of the orthogonal group which acts trivially on the discriminant group of the lattice L_{2d}.

I would like to note that the Mumford–Hirzebruch proportionality principle for the orthogonal group gives us a good result about the general type of the modular orthogonal varieties only if its dimension is big enough ($n \geq 35$). See [GHS05] and [GHS06]. If the dimension is smaller than 26 we use the method of cusp forms of small weights. Some applications of this method you can find in [G94], [GH95] and [GS96].

The next theorem follows from the results obtained in [GHS07].

Theorem 2. *Let $n > 8$ and let assume that there exists a non-zero cusp forms F_k of weight $k < n$ vanishing on the branch locus of the modular projection $D(L) \to F_L(\Gamma)$. Then the Kodaira dimension of this modular variety $F_L(\Gamma)$ is maximal, i.e., it is equal to n.*

To construct such a cusp form for $n < 26$ we use the quasi pull-back of the Borcherds function Φ_{12}. Let $L_{2,26} = 2U \oplus 3E_8(-1)$ be the unimodular lattice of signature $(2, 26)$. The Borcherds function $\Phi_{12} \in M_{12}(O^+(L_{2,26}), \det)$ is the unique modular form of weight 12 and character det with respect to $O^+(L_{2,26})$. Φ_{12} is the denominator function of the fake Monster Lie algebra and it has a lot of remarkable properties. In particular, the zeros of $\Phi_{12}(Z)$ lie on rational quadratic divisors defined by (-2)-vectors in $L_{2,26}$, i.e., $\Phi_{12}(Z) = 0$ if and only if there exists $r \in L_{2,26}$ with $r^2 = -2$ such that $(r, Z) = 0$ and the multiplicity of the rational quadratic divisor in the divisor of zeros of Φ_{12} is 1. In the context of the moduli of K3 surfaces this function was firstly used in [BKPS].

Let $l \in E_8(-1)$ satisfy $l^2 = -2d$. The choice of l determines an embedding of L_{2d} into $L_{2,26}$ as well as an embedding of the corresponding homogeneous domain $D(L_{2d})$ into $D(L_{2,26})$. We put $R_l = \{r \in E_8(-1) \mid r^2 = -2,\ (r, l) = 0\}$, and $N_l = \#R_l$. Then the function

$$F_l = \frac{\Phi_{12}(Z)}{\prod_{\{\pm r\} \in R_l} (Z, r)} \Bigg|_{D(L_{2d})} \in M_{12+\frac{N_l}{2}}(\widetilde{O}^+(L_{2d}), \det)$$

is a non-trivial modular form of weight $12 + \frac{N_l}{2}$ vanishing on all (-2)-divisors. In [GHS07] we proved that *this form is a cusp form if the set R_l is not empty and that $F_l(Z)$ is zero along the branch divisor of the modular projection.*

According to Theorem 2 and the construction of the quasi pull-back the main point for us is the following. *We want to know for which* $2d > 0$ *there exists a vector* $l \in E_8$, $l^2 = 2d$, *which is orthogonal to at least* 2 *and at most* 12 *roots.*

Theorem 3. *Such a vector* l *in* E_8 *does exist if*

$$4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d).$$

The numbers of representations $N_{E_6}(2d)$ and $N_{D_6}(2d)$ of $2d$ by the corresponding quadratic form are Fourier coefficients of weight 3 Eisenstein series for $\Gamma_0(3)$ and $\Gamma_0(4)$ respectively. $N_{E_7}(2d)$ is the Fourier coefficient $e_{4,1}(d,0)$ of the Jacobi-Eisenstein series $E_{4,1}(\tau,z)$ of weight 4 and index 1. (A general good organized formula for the singular series of the quadratic forms of odd rank see in [GHS08].) Using the exact formulae for the Fourier coefficients we prove that the last inequality is true for $d > 143$. A more detailed analyze of the these root systems gives us the result of Theorem 1.

References

[BKPS] R.E. Borcherds, L. Katzarkov, T. Pantev, N.I. Shepherd-Barron, *Families of* K3 *surfaces.* J. Algebraic Geom. **7** (1998), 183–193.

[G94] V. Gritsenko, *Modular forms and moduli spaces of abelian and* K3 *surfaces.* Algebra i Analiz **6** (1994), 65–102; English translation in St. Petersburg Math. J. **6** (1995), 1179–1208.

[GH95] V. Gritsenko, K. Hulek, *Appendix to the paper "Irrationality of the moduli spaces of polarized abelian surfaces".* Abelian varieties. Proceedings of the international conference held in Egloffstein, 83–84. Walter de Gruyter Berlin, 1995.

[GHS05] V. Gritsenko, K. Hulek, G.K. Sankaran, *The Hirzebruch-Mumford volume for the orthogonal group and applications.* Documenta Math. 12 (2007), 215–241. (`math.NT/0512595`, 27 pp.)

[GHS06] V. Gritsenko, K. Hulek, G.K. Sankaran, *The orthogonal modular varieties of K3 types.* (`math.AG/0609744`, 19 pp.)

[GHS07] V. Gritsenko, K. Hulek, G.K. Sankaran, *The Kodaira dimension of the moduli of K3 surfaces.* Invent. Math. **169** (2007), 519-567.

[GHS08] V. Gritsenko, K. Hulek, G.K. Sankaran, *Moduli spaces of symplectic fourfolds.* (In preparation).

[GS96] V. Gritsenko, G.K. Sankaran, *Moduli of abelian surfaces with a* $(1,p^2)$ *polarisation.* Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), 19–26; reprinted in Izv. Math. **60** (1996), 893–900.

[V07] C. Voisin, *Géométrie des espaces de modules de courbes et de surfaces* K3. Séminaire Bourbaki (2007) n° 981, 21 pp.

Spherical functions on *p*-adic homogeneous spaces

Yumiko Hironaka

Let $\mathbb{G}$ be a linear algebraic group and $\mathbb{X}$ a $\mathbb{G}$-homogeneous affine algebraic variety both defined over a $\mathfrak{p}$-adic field k, where we assume a minimal k-parabolic subgroup $\mathbb{B}$ of $\mathbb{G}$ has an open orbit $\mathbb{X}^{op}$.

A nonzero K-invariant function Ψ on $X = \mathbb{X}(k)$ is called *a spherical function on* X if it is a common $\mathcal{H}(G,K)$-eigen function, where $\mathcal{H}(G,K)$ is the Hecke algebra of $G = \mathbb{G}(k)$ with respect to a maximal compact open subgroup K. Spherical functions on homogeneous spaces are an interesting object to investigate and basic for the study of harmonic analysis on G-space X.

The explicit formula of spherical functions for group cases are given by I. G. Macdonald [M] and also by W. Casselman [C] by representation theoretical method.

In Theorem 1 we will give *an expression of spherical functions* based on the data of G and their functional equations with respect to (a subgroup of) the Weyl group, which is a refinement of a result in [H1] inspired by a technique used by O. Offen [O]. Then we formulate *functional equations* attached to a simple root (Theorem 2), and explain they are reduced to those of p-adic local zeta functions of small prehomogeneous vector space of limited type (Theorem 3).

We explain some more details. Let $\{f_i(x) \mid 1 \le i \le n\}$ be a set of basic regular *relative* $\mathbb{B}$*-invariant on* $\mathbb{X}$ and $\psi_i \in \mathfrak{X}(\mathbb{B})$ the corresponding rational character. We denote $\mathfrak{X}_1(\mathbb{B}) = < \psi_i \mid 1 \le i \le n >$. We assume that $G = KB = BK$.

For $x \in X$, $s \in \mathbb{C}^n$ and B-open orbit X_u in $\mathbb{X}^{op}(k)$, we define

$$\omega_u(x; s) = \int_K |f(k \cdot x)|_u^s \, dk, \tag{1}$$

where dk is the normalized Haar measure on k, $|\,|$ is the absolute value on k and

$$|f(x)|_u^s = \begin{cases} \prod_{i=1}^n |f_i(x)|^{s_i} & \text{if } x \in X_u \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The right hand side of (1) is absolutely convergent if $\mathrm{Re}(s_i) \ge 0,\ 1 \le i \le n$, analytically continued to a rational function on q^{s_i}, and it gives a spherical function on X.

We assume that

(A) : $|\mathbb{B} \backslash \mathbb{X}| < \infty$ and for each $x \in \mathbb{X},\ x \notin \mathbb{X}^{op}$ there exists ψ in $\mathfrak{X}_1(\mathbb{B})$ such that $\psi \not\equiv 1$ on $\mathbb{B}_x$;

(B) : There exists $x_0 \in X^{op}$ and a complete set $\mathcal{R}$ of representatives of $K \backslash X$ such that $U \cdot x \subset B \cdot x_0$ for every $x \in \mathcal{R}$.

The Weyl group W of $\mathbb{G}$ with respect to $\mathbb{B}$ acts on $\mathfrak{X}(\mathbb{B})$ and $\mathfrak{X}(\mathbb{B})^{\mathbb{C}} = \mathfrak{X}(\mathbb{B}) \otimes \mathbb{C}$. Let δ be the modulus character of B and $H = \mathbb{G}_{x_0}(k)$, and set

$$W_0 = \left\{ \sigma \in W \;\middle|\; \sigma(|\psi|^s \delta^{-\frac{1}{2}}) = \delta^{-\frac{1}{2}} \text{ on } B \cap H \right\}.$$

Theorem 1 *Let $x \in \mathcal{R}$ and $\mathcal{U}$ be the index set of open B-orbits X_ν satisfying $X_\nu \subset G \cdot x$. Then, for generic $s \in \mathbb{C}^n$, we have*

$$(\omega_\nu(x; s))_{\nu \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W_0} \gamma(\sigma(s)) B_\sigma(s) \left(\int_U |f(u \cdot x)|_\nu^{\sigma(s)} du \right)_{\nu \in \mathcal{U}},$$

where the scalar Q and the rational function $\gamma(s)$ of q^{s_i} are explicitly given by the group G, U is the Iwahori subgroup compatible with B and the matrix $B_\sigma(s)$ is given by the functional equation

$$(\omega_\nu(x; s))_{\nu \in \mathcal{U}} = B_\sigma(s) \left(\omega_\nu(x; \sigma(s))\right)_{\nu \in \mathcal{U}}.$$

Next, we consider how we obtain functional equations for $\sigma \in W$ attached to a simple root α. Let $\mathbb{P} = \mathbb{B}_\alpha$ (the standard parabolic subgroup) and consider a k-rational representation

$$\rho : \mathbb{P} \longrightarrow R_{k'/k}(GL_2), \tag{3}$$

where $R_{k'/k}$ is the restriction of scalars for a finite unramified extension k'/k. We assume the following for ρ

(C): $\rho(\mathbb{P}) = R_{k'/k}(GL_2)$ or $R_{k'/k}(SL_2)$, $\rho(\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\rho^{-1}(\mathbb{B}_2) \subset \mathbb{B}$,

and $\rho(K \cap \mathbb{P}) \supset R_{k'/k}(SL_2)(\mathcal{O}_k)$,

where $\mathbb{B}_2$ is the Borel subgroup of $\rho(\mathbb{P})$ consisting of upper triangular matrices. Hereafter we assume (A) and (C).

Theorem 2 *Let $x \in X_u$ and J_u be the index set of open B-orbits X_ν satisfying $P \cdot X_\nu = P \cdot X_u$. Assume that $\sigma \in W_0$ and set $\varepsilon \in \mathbb{Q}^n$ by $\sigma(\delta)\delta^{-1} = |\psi|^{2\varepsilon}$. Then*

$$\omega_u(x; s) = \frac{1 - q^{-2d - \sum_i e_i s_i}}{1 - q^{-2d - \sum_i e_i(\sigma(s)_i - \varepsilon_i)}} \times \sum_{\nu \in J_u} \gamma_{u\nu}(s) \cdot \omega_\nu(x; \sigma(s) - \varepsilon),$$

where $d = [k' : k]$, $\gamma_{u\nu}(s)$'s are rational functions of $q^{\frac{s_i}{e}}$, $e = [\mathfrak{X}(\mathbb{B}) \cap \mathfrak{X}_1(\mathbb{B})^{\mathbb{Q}} : \mathfrak{X}_1(\mathbb{B})]$, $e_i = \deg_v \widetilde{f}_i(x, v)$ (cf. below).

The group $\mathbb{P} \times R_{k'/k}(GL_1)$ acts by $(p, r) \cdot (x, v) = (p \cdot x, \rho(p) v r^{-1})$ on $\mathbb{X} \times \mathbb{V}$ with open orbit, where $\mathbb{V} = R_{k'/k}(M_{21})$. Then there exist regular relative invariants $\widetilde{f}_i(x, v)$ on $\mathbb{X} \times \mathbb{V}$ satisfying $\widetilde{f}_i(x, v_0) = f_i(x)$, $1 \le i \le n$ ($v_0 = {}^t(1, 0)$).

Theorem 3 *Denote by $\mathbb{P}_u$ the stabilizers of $x_u \in X_u$ in $\mathbb{P}$. Then the space $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$ is a prehomogeneous vector space defined over k, open orbits in $\mathbb{V}^{op}(k)$ are parametrized by the same J_u as for $B \backslash P \cdot x_u$ in Theorem 2, and there are functional equations between open orbits*

$$\int_V \mathcal{F}_V(\phi)(v) \left| \widetilde{f}(x_u, v) \right|_u^s dv = \sum_{\nu \in J_u} \gamma_{u\nu}(s) \int_V \left| \widetilde{f}(x_u, v) \right|_\nu^{\sigma(s) - \varepsilon} dv \quad (\phi \in \mathcal{S}(V)),$$

where ε and $\gamma_{u\nu}(s)$ are the same as in Theorem 2, $\left| \widetilde{f}(x, v) \right|_\nu^s$ is defined similarly as in (2) and $\mathcal{F}_V(\phi)$ is the Fourier transform of ϕ.

Further, the identity component of $\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1)$ is isomorphic to $R_{k'/k}(GL_1 \times GL_1)$ over the algebraic closure of k.

If $\mathfrak{X}(\mathbb{B})$ and $\mathfrak{X}_1(\mathbb{B})$ have the same rank, by shifting s for $\delta^{\frac{1}{2}}$, one has functional equations between s and $\sigma(s)$ (cf. [H2]).

References

[C] W. Casselman: The unramified principal series of $\mathfrak{p}$-adic groups I. The spherical functions, *Compositio Math.* **40**(1980), 387–406.

[H1] Y. Hironaka: Spherical functions and local densities on hermitian forms, *J. Math. Soc. Japan* **51**(1999), 553 – 581.

[H2] Y. Hironaka: Functional equations of spherical functions on p-adic homogeneous spaces, *Abh. Math. Sem. Univ. Hamburg* **75**(2005), 285 – 311.

[M] I. G. Macdonald: *Spherical functions on a group of p-adic type*, Univ. Madras, 1971.

[O] O.Offen: Relative spherical functions on p-adic symmetric spaces, *Pacific J. Math.* **215**(2004), 97 – 149.

Poincare series and second order modular forms

Özlem Imamoḡlu

(joint work with Cormac O'Sullivan)

In this talk we report on joint work with Cormac O'Sullivan where, following Petersson, we study the parabolic, hyperbolic and elliptic expansions of holomorphic cusp forms and the associated Poincaré series and show how these ideas extend to the space of second-order cusp forms.

Let $\Gamma \subseteq \mathrm{PSL}$ be a Fuchsian group of the first kind acting on the upper half plane $\mathbb{H}$. We write $x + iy = z \in \mathbb{H}$ and set $d\mu z$ to be the SL_2-invariant hyperbolic volume form $dxdy/y^2$. Assume the volume of the quotient space $\Gamma\backslash\mathbb{H}$ is equal to $V < \infty$.

Let $(f|_k\gamma)(z) := f(\gamma z)/j(\gamma, z)^k$ and $\mathbb{N}$ be the natural numbers $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$, the $\mathbb{C}$-vector space of *n-th order modular forms*, $S_k^n(\Gamma)$, is defined recursively as follows. Let $S_k^0(\Gamma)$ consist only of the function $\mathbb{H} \to 0$. For $n \geqslant 1$, let $S_k^n(\Gamma)$ contain all holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ that satisfy

$$f|_k(\gamma - 1) \in S_k^{n-1}(\Gamma) \quad \text{for all} \quad \gamma \in \Gamma. \tag{1}$$

For all parabolic elements π of Γ we also require

$$f|_k(\pi - 1) = 0. \tag{2}$$

Finally f must decay rapidly in each cusp. Induction shows $S_k^{n_1} \subseteq S_k^{n_2}$ for any two integers $0 \leqslant n_1 \leqslant n_2$. Therefore $S_k \subseteq S_k^n$ and higher-order forms are a generalization of the usual cusp forms.

The identity in Γ is $I = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The remaining elements may be partitioned into three sets: the parabolic, hyperbolic and elliptic elements. As is well-known, the relation $(f|_k\gamma)(z) = f(z)$ for parabolic elements γ leads to a Fourier expansion of f associated to each cusp of Γ. A family of corresponding parabolic Poincaré series can be constructed whose inner products with f produce these Fourier coefficients. Much less well-known are Petersson's hyperbolic and elliptic Fourier expansions, and Poincare series introduced in [1]. The series Petersson constructs are all examples of *relative Poincaré series*.

Theorem 1. *Let Γ_0 be a subgroup of Γ and ϕ a holomorphic function on $\mathbb{H}$ satisfying $\phi|_k\gamma = \phi$ for all γ in Γ_0 and*

$$\int_{\Gamma_0\backslash\mathbb{H}} |\phi(z)|y^{k/2}\, d\mu z < \infty. \tag{3}$$

Then the relative Poincaré series

$$P[\phi](z) := \sum_{\gamma\in\Gamma_0\backslash\Gamma} (\phi|_k\gamma)(z) \tag{4}$$

converges absolutely and uniformly on compact subsets of $\mathbb{H}$ to an element of S_k.

In this work we show how the ideas of Petersson extend naturally to the second-order space S_k^2. Let $\mathrm{Hom}_0(\Gamma,\mathbb{C})$ be the homomorphisms from Γ to $\mathbb{C}$ that are 0 on the parabolic elements of Γ. By well known theorem of Eichler and Shimura there exist unique f^+, f^- in $S_2(\Gamma)$ so that

$$L(\gamma) = \int_z^{\gamma z} f^+(w)\, dw + \overline{\int_z^{\gamma z} f^-(w)\, dw}.$$

The right-side above is independent of z and the path of integration in $\mathbb{H}$. Set

$$\Lambda_L^+(z) := \int_i^z f^+(w)\, dw, \quad \Lambda_L^-(z) := \int_i^z f^-(w)\, dw. \tag{5}$$

Then clearly, for all z in $\mathbb{H}$, $L(\gamma) = \Lambda_L^+(\gamma z) - \Lambda_L^+(z) + \overline{\Lambda_L^-(\gamma z)} - \overline{\Lambda_L^-(z)}$.

Our first theorem show that the following relative Poincaré series, twisted by such a homomorphism, are second-order forms.

Theorem 2. *Let Γ_0 be a subgroup of Γ and ϕ a holomorphic function on $\mathbb{H}$ satisfying $\phi|_k\gamma = \phi$ for all γ in Γ_0. Let $L \in \mathrm{Hom}_0(\Gamma,\mathbb{C})$ with $L(\gamma) = 0$ for all γ in Γ_0. If*

$$\int_{\Gamma_0\backslash\mathbb{H}} \Big(1 + |\Lambda_L^+(z)| + |\Lambda_L^-(z)|\Big)|\phi(z)|y^{k/2}\, d\mu z < \infty$$

then

$$P[\phi, L](z) := \sum_{\gamma\in\Gamma_0\backslash\Gamma} L(\gamma)(\phi|_k\gamma)(z)$$

converges absolutely and uniformly on compact subsets of $\mathbb{H}$ to an element of S_k^2.

Theorem 2 allows us to construct parabolic, hyperbolic and elliptic second-order Poincaré series. We show that, whenever they exist, these Poincaré series of order 1 and 2 always span their respective cusp form spaces.

The final theorem of this talk gives an inner product for second order modular forms. More precisely let

$$\mathbb{S} = \Gamma_\infty\backslash\mathbb{H} = \{z \in \mathbb{H} : -1/2 \leqslant \Re(z) < 1/2\}.$$

For any two functions f, g on $\mathbb{H}$ with period 1 we define the pairing $\langle f, g\rangle^{*n} = \langle f, g\rangle_k^{*n}$ to be the coefficient of $(s-1)^{-n}$ in

$$V \int_{\mathbb{S}} f(z)\overline{g(z)}y^{k+s}\, d\mu z. \tag{6}$$

We prove

Theorem 3. *For $n = 2$ the pairing $\langle\ ,\ \rangle_k^{*2}$ is an inner product for the space S_k^2/S_k^1. For f, g in $S_k = S_k^1$ we have $\langle f, g\rangle_k^{*1} = \langle f, g\rangle/V$ where $\langle f, g\rangle$ is the usual Petersson inner product.*

References

[1] Petersson, H. : Einheitliche Begründung der Vollständigkeitssätze für die Poincaréschen Reihen von reeller Dimension bei beliebigen Grenzkreisgruppen von erster Art, Abh. Math. Sem. Hansischen Univ. **14** (1941), 22–60.

Arithmetic trace formula and Hecke duality

Bernhard Heim

In the first part of my talk I presented a new arithmetic trace formula [7]. This formula relates special values of various kinds of automorphic L-functions. Let $g \in S_k(SL_2(\mathbb{Z}))$ be a newform of integer weight k. Let $(f_i)_i \in S_{2k-2}$ and $(g_j)_j \in S_k$ be primitive eigenbasis. The trace formula compares the weighted average $\sum_i$ of special values of the non-trivial piece of the triple L-function $L(f_i \otimes \mathrm{Sym}^2(g), c_k)$ evaluated at the central value c_k and the average $\sum_j$ of the triple L-function $L(g \otimes g \otimes g_j, 2k-2)$ and an error term expressed by special values $D(g, s)$ related to the Rankin L-function attached to g. This special value $L(f_j \otimes \mathrm{Sym}^2(g), c_k)$ and the related triple L-function recently played a prominent role in the proof of the Gross-Prasad conjecture of Saito-Kurokawa lifts given by Ichino [8]. More generally Ikeda stated in [10] a conjecture on the explicit value of a certain period which involves the central value of L-functions (Conjecture 5.1) of the type studied in this paper. There the non-vanishing of the central value is essential. Recently some progress has been obtained by Katsurada and Kawamura [11].

Theorem 1 [Heim 07] *Let k be an even positive integer. Let $g \in S_k$ be a primitive Hecke eigenform. Then we have*

$$(1) \qquad \sum_{i=1}^{dim\, S_{2k-2}} \frac{\widehat{L}(f_i, 2k-3)\ \widehat{L}\left(f_i \otimes Sym^2(g), 2k-2\right)}{\| f_i \|^2 \| g \|^4}$$

$$= (-1)^{k/2} \cdot 2^{k-2} \sum_{j=1}^{dim\, S_k} \frac{\widehat{L}(g \otimes g \otimes g_j, 2k-2)}{\| g \|^4 \| g_j \|^2}$$

$$+\kappa_1 \left(\frac{\widehat{D}(g, 2k-2)}{\pi^{\frac{k}{2}-1} \| g \|^2} \right)^2 + \kappa_2 \frac{\widehat{D}(g, 2k-2)}{\pi^{\frac{k}{2}-1} \| g \|^2}.$$

Here $(f_i)_i$ and $(g_j)_j$ are primitive Hecke eigenbases of S_{2k-2} and S_k and the constants κ_1 and κ_2 can be explicitly given. We have

$$\kappa_1 = (-1)(-1)^{k/2}2^4\frac{\Gamma(k)^2}{(2k-2)B_{2k-2}\Gamma(k/2)^2}, \tag{2}$$

$$\kappa_2 = (-1)(-1)^{k/2}2^{2k+1}\frac{\Gamma(k+1)}{(2k-2)B_k\Gamma(k/2)}. \tag{3}$$

Here ˆ denotes the completion of the related L-function and B_k the k-th Bernoulli number. We gave a sketch of the proof and indicated several applications.

The concept of Hecke Duality was motivated with an example related to the non-vanishing of the triple L-function evaluated at the center [8], [5] and the characterization of Saito-Kurokawa lifts. The general concept is the following: Let $G_n = Sp_n, GSp_n, U(n,n), Sp_n \ltimes H_n, \ldots$ be any classical group. Let $\mathcal{A}(G_n)$ denote the space of automorphic forms on G_n and $\mathcal{H}_n$ the related Hecke algebra. Further we fix a canonical imbedding $j : j_1 \times j_2 : G_n \times G_n \hookrightarrow G_{2n}$ and define the following subspace of all automorphic forms on G_{2n}:

$$\mathbb{E}(G_{2n}) := \{F \in \mathcal{A}(G_{2n}) \,|\, j_1(T)F = j_2(T)F, T \in \mathcal{H}_n\}. \tag{4}$$

Theorem 2 [Heim 06] *The space of Saito-Kurokawa lifts*

$$M_k^{Sk} = \mathbb{E}(Sp_2) = \mathbb{E}(GSp_2).$$

Let (n,r,m) be related to the positive integral quadratic form $nx^2 + rxy + my^2$ and let $A(n,r,m)$ be the Fourier coefficients of a Siegel cusp form of degree 2. Then F is a Saito-Kurokawa lift if and only $A(n,r,pm) - A(np,r,m)$ is equal to

$$p^{k-1}\left(A\left(\frac{n}{p},\frac{r}{p},m\right) - A\left(n,\frac{r}{p},\frac{m}{p}\right)\right) \tag{5}$$

for all $T = (n,r,m)$ and for all primes p. Pitale and Schmidt[12] improved the result to the statement almost all primes. Recently we obtained a result, which says that if the condition is satisfied for all prime p outside a subset of primes with Dirichlet density smaller then 1/8 then the form has to be a Saito-Kurokawa lift [6].

Theorem 3 [Bringmann, Heim 07] *Let $G = Sp_2 \ltimes H_2$ and $\mathcal{A}(G)$ the space $J_{k,m}^2$ of Jacobi forms of degee 2, weight k and index m. Let $m = 1$ or a prime then we have*

$$\emptyset \subsetneq \mathbb{E}(G) \subsetneq J_{k,m}^2 \tag{6}$$

up to some minor conditions on the weight k.

Finally I reported on a joint project with Paul Garrett.
Theorem 4 [Garrett, Heim 07] *The space of Ikeda lifts satisfies the Hecke duality property.*

Theorem 5 [Garrett, Heim 07] *Let F be a Siegel cuspform of degree 2. Let F be a Hecke eigenform with respect to the even Hecke algebra and let the standard L-function has the same form as a Saito-Kurokawa lift, then F is a Saito-Kurokawa lift.*

This can be seen as a certain weak multiplicity one result for Sp_2, which was only known for GSp_2 until know.

References

[1] K. Bringmann, B. Heim: *Hecke Duality Relations of Jacobi forms submitted* 2007
[2] P. Deligne: *Valeurs de fonctions L et periodes d'integrales.* Proc. Symposia Pure Math. **33** (1979), part 2, 313-346
[3] P. Garrett, B. Heim: *Hecke Duality of Ikeda lifts.submitted* 2007
[4] B. Heim: *Period integrals and the global Gross-Prasad conjecture.* 2006
[5] B. Heim: *On the Spezialschar of Maass. submitted* 2006
[6] B. Heim: *Distribution Theorems for Saito-Kurokawa lifts.* 2007
[7] B. Heim: *A trace formula of special values of automorphic L-functions* 2007
[8] A. Ichino: *Pullbacks of Saito-Kurokawa Lifts.* Invent. Math. **162** (2005), 551-647.
[9] T. Ikeda: *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$. Ann. of Math. **154** no. 3 (2001), 641-681.
[10] T. Ikeda: *Pullback of the lifting of elliptic cusp forms and Miyawakis conjecture.* Duke Math. Journal, **131** no. 3 (2006), 469-497.
[11] H. Katsurada, Hisa-aki Kawamura: *A certain Dirichlet series of Rankin-Selberg type associated with the Ikeda lifting.* Department of Mathemtics Hokkaido University, series 2006, number 806
[12] A. Pitale, R. Schmidt: *Ramanujan-type reuslts for Siegel cusp forms of degree* 2. submitted

Local Theta correspondence and the Lifting of Duke, Imamoglu and Ikeda

Rainer Schulze-Pillot

Let f be an elliptic modular form of weight $2k$ for the full modular group $SL_2(\mathbf{Z})$. By τ we denote the irreducible cuspidal automorphic representation of $GL_2(\mathbf{A})$ associated to f. It was conjectured by Duke and Imamoglu and proven by Ikeda in [1] that for any $n \equiv k \bmod 2$ there exists a nonzero Siegel cusp form $F = F_{2n}(f)$ of weight $n+k$ for the group $Sp_{2n}(\mathbf{Z}) \subseteq SL_{4n}(\mathbf{Z})$ whose standard L-function is equal to

$$\zeta(s)\prod_{i=1}^{2n} L(s+k+n-i, f),$$

where $L(s,f)$ is the usual Hecke L-function of f. We call $F_{2n}(f)$ the Duke-Imamoglu-Ikeda (short: DII) lift of degree $2n$ of f and denote the corresponding automorphic representation of $Sp_{2n}(\mathbf{A})$ by $\pi(2n,\tau)$. Ikeda has announced a representation theoretic version of this lifting which works for slightly more general τ and also for arbitrary totally real grozund field E instead of $\mathbf{Q}$.

In [2] it is proved as a side remark that $F_{2n}(f)$ is not a linear combination of theta series of even unimodular positive definite quadratic forms of rank $m = 2(n+k)$ if n is bigger than k, whereas for $n = k \equiv 0 \bmod 2$ the DII liftings lie in the space generated by theta series subject to a conjecture on L-functions of elliptic cuspidal Hecke eigenforms. The proof uses Böcherer's characterization of the cuspidal Siegel eigenforms that lie in the space of theta series by special values of their standard L-functions.
Using the description of the local theta correspondence between representations of local orthogonal and local symplectic groups in terms of the Bernstein–Zelevinsky data of the representations in [5, 3] we prove the following results:

Theorem *Let V be a vector space over E of even dimension $2r$ with a nondegenerate quadratic form q.*

(1) *If $2n > r - 1$ and there is a finite place v of E for which the completion V_v of the quadratic space (V, q) is not split (i. e. is not an orthogonal sum of hyperbolic planes) the representation $\pi(2n, \tau)$ is not in the image of the theta correspondence with $O_{(V,q)}(\mathbf{A}_E)$.*
(2) *If $2n > r$ the representation $\pi(2n, \tau)$ is not in the image of the theta correspondence with $O_{(V,q)}(\mathbf{A}_E)$.*

Corollary *Let f be an elliptic modular form of weight $2k$ and $\nu \in \mathbf{N_0}$.*
Then for $n > k - \nu$ the DII-lift $F_{2n}(f)$ is not a linear combination of theta series of positive definite quadratic forms with pluriharmonic forms of degrees $\nu' \geq \nu$.

In particular for $n > k$ the DII-lift $F_{2n}(f)$ is not a linear combination of theta series attached to positive definite quadratic forms (with or without pluriharmonic forms).

Proposition *The generalized Duke–Imamoglu–Ikeda lift $\pi(2n, \tau)$ can not be constructed by a series of theta liftings between groups G_i, where for each i the pair G_i, G_{i+1} consists (in either order) of a symplectic or metaplectic group and an orthogonal group, starting with the representation associated to f on SL_2 or the representation on the metaplectic group $\widetilde{SL_2}$ associated to the form g which corresponds to f under the Shimura correspondence.*

On the positive side we have using [4] the (purely local)

Theorem *For $n = k$ the local components π_p of the representation $\pi(F_{2n}(f))$ of the Duke-Imamoglu-Ikeda lift $F_{2n}(f)$ are in the image of the local theta correspondence with the split quadratic space over $\mathbf{Q}_p$ of dimension $4n = 2(k+n)$ (which is the orthogonal sum of $2n$ hyperbolic planes) for all (finite) primes p.*

The component π_∞ at the real place is in the image of the theta correspondence with the orthogonal group of the positive definite quadratic space over $\mathbf{R}$ of dimension $4n$ and also in the image of the theta correspondence with the orthogonal group of the quadratic space of dimension $4n$ and signature $(4n-1, 1)$ over $\mathbf{R}$.

References

[1] T. Ikeda: *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) **154** (2001), 641–681

[2] W. Kohnen, R. Salvati Manni: *Linear relations between theta series*, Osaka J. Math. **41** (2004), 353–356.

[3] S. Kudla: *On the local theta correspondence*, Invent. math. **83** (1986), 229-255

[4] A. Paul: *On the Howe correspondence for symplectic-orthogonal dual pairs*, J. Funct. Anal. **228** (2005), 270–310

[5] S. Rallis: *Langlands functoriality and the Weil representation*, Amer. J. of Math. **104** (1982), 469-515

Reporter: Siegfried Böcherer

Participants

Prof. Dr. Siegfried Böcherer
Fakultät für Mathematik und
Informatik
Universität Mannheim
68131 Mannheim

Prof. Dr. Kathrin Bringmann
Department of Mathematics
University of Wisconsin
Madison , WI 53706
USA

Prof. Dr. Jan Hendrik Bruinier
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Dr. Neil Dummigan
Dept. of Pure Mathematics
Hicks Building
University of Sheffield
GB-Sheffield S3 7RH

Prof. Dr. Masaaki Furusawa
Department of Mathematics
Graduate School of Science
Osaka City University
Sugimoto 3-3-138, Sumiyoshi-ku
Osaka 558-8585
JAPAN

Prof. Dr. A. Valery Gritsenko
U.F.R. de Mathematiques
Universite de Lille I
USTL Bat.M2
F-59655 Villeneuve D'Ascq

Dr. Bernhard Heim
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. Yumiko Hironaka
Dept. of Mathematics
School of Education
Waseda University
Shinjuku-ku
Tokyo 169-8050
JAPAN

Prof. Dr. Tomoyoshi Ibukiyama
Department of Mathematics
Graduate School of Science
Osaka-University Machikaneyama 1-16
Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Özlem Imamoglu
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Hidenori Katsurada
Muroran Institute of Technology
27-1 Mizumoto
Muroran 050-8585
JAPAN

Prof. Dr. Winfried Kohnen
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Aloys Krieg
Lehrstuhl A für Mathematik
RWTH Aachen
Templergraben 55
52062 Aachen

Prof. Dr. Yves Martin
Departamento de Matematicas
Facultad de Ciencias
Universidad de Chile
Casilla 653
Chile

Dr. Yoshinori Mizuno
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223-8522
JAPAN

Prof. Dr. Alexey A. Panchishkin
Institut Fourier
UFR de Mathematiques
UMR 5582
BP 74
F-38402 St Martin d'Heres

Prof. Dr. Cris Poor
Department of Mathematics
Fordham University
Bronx , NY 10458
USA

Prof. Dr. Brooks Roberts
Department of Mathematics
University of Idaho
P.O.Box 441103
Moscow ID 83844-1103
USA

Ralf Schmidt
Dept. of Mathematics
University of Oklahoma
601 Elm Avenue
Norman , OK 73019-0315
USA

Dr. Rainer Schulze-Pillot
Fachrichtung - Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken

Prof. Dr. Nils-Peter Skoruppa
Universität Siegen
Fachbereich 6 Mathematik
Walter-Flex-Str. 3
57068 Siegen

Prof. Dr. Satoshi Wakatsuki
Dept. of Mathematics
Faculty of Science
Kanazawa University
Kakuma-machi
Kanazawa 920-1192
JAPAN

Prof. Dr. Lynne H. Walling
Department of Mathematics
University of Bristol
University Walk
GB-Bristol BS8 1TW

Prof. Dr. Hiroyuki Yoshida
Dept. of Mathematics
Faculty of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN