Mathematisches Forschungsinstitut Oberwolfach

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Tropical Geometry

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Abstract. Tropical Geometry is a new and rapidly developing discipline that touches upon many branches of modern mathematics. It is characterized by the transition of algebro-geometric objects to piecewise-linear ones, thereby retaining much of the algebro-geometric data while unveiling novel sets of techniques for outstanding problems. The workshop provided an invaluable communication platform as the subject develops into a field of its own.

Mathematics Subject Classification (2000): 14Nxx, 14Pxx, 14Qxx, 52Bxx, 52Cxx.

Introduction by the Organisers

This is the first appearance of "Tropical Geometry" as the topic of an Oberwolfach workshop, though the subject figured earlier at the Forschungsinstitut as an Oberwolfach Seminar on "Tropical Algebraic Geometry" in October 2004. The workshop was organized by Eva-Maria Feichtner (U Bremen), Andreas Gathmann (U Kaiserslautern), Ilia Itenberg (U Strasbourg) and Thorsten Theobald (U Frankfurt) and brought together about 50 mathematicians from the many fields Tropical Geometry touches upon. The workshop came timely as the subject develops into a field of its own with numerous connections to diverse branches of mathematics.

Tropical Geometry can be considered as an algebraic geometry over the semifield $(\mathbb{R}, \max, +)$. The name was coined by French computer scientists to honor the pioneering work of their Brazilian colleague Imre Simon on the max-plus semiring. Alternatively, Tropical Geometry can be understood as the geometry resulting from complex geometry by a certain degeneration process: complex toric varieties are replaced by real linear spaces and, more generally, complex algebraic varieties are replaced by polyhedral complexes, i.e., by piecewise-linear objects.

The roots of Tropical Geometry extend at least to Bergman's logarithmic limit sets (in the 70's), Viro's patchworking construction (in the late 70's), Maslov's dequantization of positive real numbers (in the 80's), and to the use of idempotent semirings in applications to optimization, control theory, and max-plus operators (in the 90's). In recent years, the various research directions have been fruitfully merged, generalized and advanced to what is now called Tropical Geometry. Along the way, deep connections to numerous branches of pure and applied mathematics have been unveiled; among them e.g. algebraic geometry, symplectic geometry, complex analysis, dynamical systems, geometric combinatorics, as well as computer algebra, algebraic statistics, and phylogenetics.

Despite of the short time span of its recent development, Tropical Geometry has already been widely recognized as an important discipline and as a unifying viewpoint for the transition of algebro-geometric problems to combinatorial ones. The aim of this Oberwolfach workshop was to furnish this newly emerging field with an outstanding communication platform and to foster the interaction between the various research directions that are involved.

Moreover, Tropical Geometry is still in a phase of forming its foundations: fundamental concepts, such as abstract tropical varieties, and basic definitions, such as a proper notion of tropical intersection multiplicity, are still under construction. It was therefore an additional goal of this workshop to advance the field through discussions among people with different views of the subject.

In both respects, the workshop met all expectations: in 22 fifty-minutes talks the current state of development was outlayed from various viewpoints. An afternoon with half-hour talks by graduate students and an evening session featuring a software presentation completed the picture. This part of the workshop is well accounted for by the following collection of abstracts.

Most importantly - and only forthcoming activities and publications will indirectly report on this - numerous informal discussions evolved between participants with most different mathematical backgrounds. A fruitful atmosphere of exchange developed during the week, notably thanks to the excellent facilities at the Mathematisches Forschungsinstitut.

On behalf of the participants, we wish to express our sincere thanks to the Institute for hosting this workshop.

Eva-Maria Feichtner
Andreas Gathmann
Ilia Itenberg
Thorsten Theobald

January 2008

Workshop: Tropical Geometry

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Abstracts

Phase-tropical curves

Grigory Mikhalkin

Tropical manifolds and in particular tropical curves appear as a result of procedure that disposes of arguments (or *phases*) of complex numbers. However, there is a way to put these phases back to get *phase-tropical* manifolds. As it happens quite often in Mathematics these two operations are not quite opposite to each other: phase-tropical manifolds do not posess an honest complex-analytic structure, but rather a certain degeneration of this structure.

In the talk we considered in details the case of phase-tropical curves and phase-tropical morphisms from them to $\mathbb{R}^n$ and tropical hypersurface and complete intersections in $\mathbb{R}^n$. In the case when $n = 2$, the phase structure is real and the morphism is an embedding we recover the famous patchworking construction of Viro. More generally, we have the following theorem.

Theorem 1. *If a phase-tropical morphism from a curve to a smooth hypersurface or a complete intersection in* $\mathbb{R}^n = (\mathbb{T}^\times)^n$ *is* regular *then it comes as the limit of a family of classical (complex) curves of the corresponding degree and genus under a suitable renormalization.*

Here regularity refers to the underlying tropical morphism (after forgetting the phase) and means that the virtual dimension of the deformation space of tropical morphism (computable from the Riemann-Roch theorem) coincides with the actual dimension. The talk was illustrated by several applications of this theorem in complex and real geometry.

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Amoebas and coamoebas (of discriminants)

Mikael Passare
(joint work with Lisa Nilsson)

Given an algebraic hypersurface $Z = \{ z \in \mathbf{C}_*^n ;\ f(z) = 0 \}$, the *amoeba* and the *coamoeba* of Z are defined as $\mathrm{Log}\,(Z)$, respectively $\mathrm{Arg}\,(Z)$, where the mappings Log and Arg are given by

$$\mathrm{Log}\,(z) = (\log|z_1|, \dots, \log|z_n|) \quad \text{and} \quad \mathrm{Arg}\,(z) = (\arg z_1, \dots, \arg z_n)\,.$$

In this talk we focus on the case where $n = 2$ and Z is an inhomogeneous (or reduced) discriminantal curve in the sense of Gelfand-Kapranov-Zelevinsky. We start from an integer $(N \times 2)$-matrix

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$$

with the property that $b_1 + b_2 + \ldots + b_N = (0,0)$, and such that its 2×2 minors are relatively prime. One then defines the Horn-Kapranov parametrization Ψ_B by the explicit formula

$$\mathbf{C}P_1 \ni [s:t] \mapsto \Big(\prod_{k=1}^{N} (b_{k1}s + b_{k2}t)^{b_{k1}},\ \prod_{k=1}^{N} (b_{k1}s + b_{k2}t)^{b_{k2}} \Big),$$

and considers the image curve $Z_B = \Psi_B(\mathbf{C}P_1) \cap \mathbf{C}_*^2$.

We shall present a result which relates the coamoeba $\Sigma_B = \mathrm{Arg}\,(Z_B)$ to a simple convex polygon Π_B, which in turn determines the domain of convergence for the associated hypergeometric Mellin–Barnes integral.

The convex polygon Π_B is defined to be the Minkowski sum of the line segments $[0, \pi b_1], \ldots, [0, \pi b_N]$. It is thus a zonotope centered at the origin $(0,0)$, and its boundary is obtained by placing the vectors $\pm \pi b_k$ one after another according to the cyclic direction ordering.

The coamoeba Σ_B is also obtained by similarly placing vectors one after another, but now only the vectors $\pi b_1, \ldots, \pi b_N$ and according to the *projective* cyclic ordering; then one adds a mirror image.

Identifying the polygons Π_B and Σ_B with their images (considered as simplicial chains) under the natural projection $\mathbf{R}^2 \to (\mathbf{R}/2\pi\mathbf{Z})^2 = \mathbf{T}^2$, one finds that the chain $\Pi_B + \Sigma_B$ is a 2-cycle, and hence equal to $m_B \mathbf{T}^2$ for some integer m_B.

We then use the area formulas

$$\pi^{-2}\mathrm{Area}\,(\Pi_B) = \sum_{j<k} \Big| \det \begin{pmatrix} b_j \\ b_k \end{pmatrix} \Big| \quad \text{and} \quad \pi^{-2}\mathrm{Area}\,(\Sigma_B) = \sum_{j<k} \det \begin{pmatrix} b_j \\ b_k \end{pmatrix},$$

where in the latter case it is important that the projective ordering go *clockwise*. These formulas allow us to compute the multiplicity.

Theorem 1. *The multiplicity m_B of the chain $\Pi_B + \Sigma_B$ is given by the formula*

$$m_B = \frac{1}{2} \sum_{j<k} \det{}^{+} \begin{pmatrix} b_j \\ b_k \end{pmatrix},$$

where $\det^{+}(\cdot) = \max(0, \det(\cdot))$ and the summation is taken in accordance with the clockwise projective ordering of the vectors b_k.

Theorem 2. *One has $m_B = d_B :=$ the normalized volume of $\mathrm{conv}\,(A)$, where A is the Gale dual of B.*

In order to illustrate the content of the above theorems we now provide two concrete examples.

Example 1. Here we take $N = 3$ and the three vectors $b_1 = (1,0)$, $b_2 = (0,1)$, $b_3 = (-1,-1)$. Notice that this ordering is clockwise projective as required in Theorem 1. The area computations become

$$\pi^{-2}\mathrm{Area}\,(\Pi_B) = \left|\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right| + \left|\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}\right| + \left|\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}\right| = 1+1+1 = 3\,;$$

$$\pi^{-2}\mathrm{Area}\,(\Sigma_B) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1-1+1 = 1\,.$$

We see that the combined chain $\Pi_B + \Sigma_B$ precisely covers the torus $\mathbf{T}^2$, so that in this case $m_B = 1$, which agrees with the formula from Theorem 1:

$$m_B = \frac{1}{2}\sum_{j<k} \det{}^{+}\begin{pmatrix} b_j \\ b_k \end{pmatrix} = \frac{1}{2}\big(1+0+1\big) = 1\,.$$

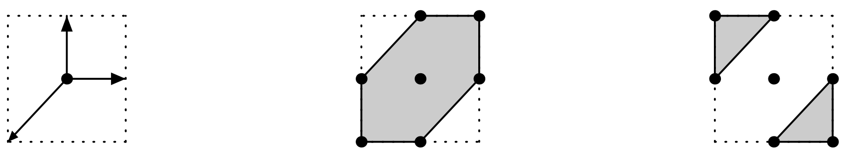

FIGURE 1. *The vectors πb_k (left), the zonotope Π_B (center), and the coamoeba Σ_B (right) from Example 1.*

Example 2. Now we let $N = 5$ and consider the five vectors $b_1 = (1,-1)$, $b_2 = (-1,2)$, $b_3 = (0,-2)$, $b_4 = (1,3)$, and $b_5 = (-1,-2)$. Again we have a clockwise projective ordering.
Computing the areas we get

$$\pi^{-2}\mathrm{Area}\,(\Pi_B) =$$

$$\left|\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}\right| + \left|\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}\right| + \left|\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}\right| + \left|\begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix}\right| + \left|\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix}\right|$$

$$+ \left|\begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix}\right| + \left|\begin{vmatrix} -1 & 2 \\ -1 & -2 \end{vmatrix}\right| + \left|\begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix}\right| + \left|\begin{vmatrix} 0 & -2 \\ -1 & -2 \end{vmatrix}\right| + \left|\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix}\right|$$

$$= 1+2+4+3+2+5+4+2+2+1 = 26\,;$$

$$\pi^{-2}\mathrm{Area}\,(\Sigma_B) = 1-2+4-3+2-5+4+2-2+1 = 2\,.$$

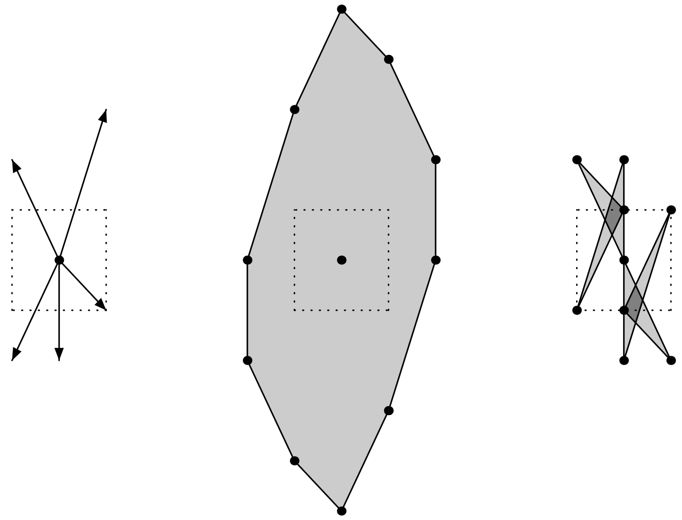

FIGURE 2. *The vectors πb_k (left), the zonotope Π_B (center), and the coamoeba Σ_B (right) from Example 2.*

From Theorem 1 we then obtain $m_B = (1+4+2+4+2+1)/2 = 7$.
The Gale dual A is given by an 3×5 matrix satisfying $AB = 0$, and we can take

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 & 2 \end{pmatrix}.$$

The columns of A are of the form $(1, a_k)$, $k = 1, 2, \ldots, 5$, with the five points $a_k \in \mathbf{N}^2 \subset \mathbf{R}^2$. The meaning of Theorem 2, that is, the identity $m_B = d_B$, is in this example that (twice) the area of the convex hull of the points $a_1, \ldots, a_5$ is equal to 7, see Figure 3.

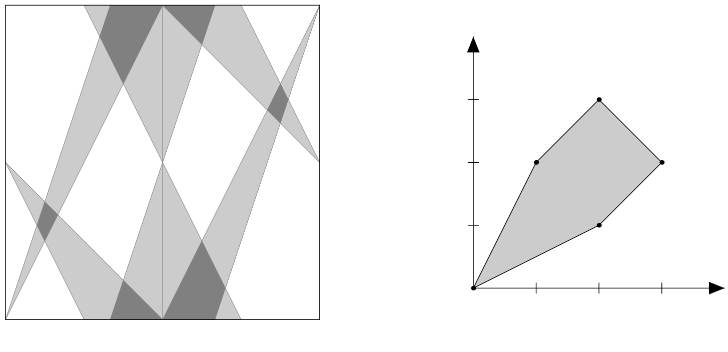

FIGURE 3. *The coamoeba after projection to $\mathbf{T}^2$ (left), and the convex hull of the Gale dual (right).*

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Intersecting Psi-classes on tropical $M_{0,n}$

Hannah Markwig
(joint work with Michael Kerber)

Psi-classes Ψ_i are certain divisor classes on the space of stable curves, $M_{g,n}$, which arise as the first Chern class of the line bundle L_i whose fiver over a point $(C, x_1, \dots, x_n)$ is the cotangent space of C at x_i. If $g = 0$ and $\sum k_i = n-3$, the following equality holds for their intersection: $\Psi_1^{k_1} \cdot \ldots \cdot \Psi_s^{k_s} = \binom{n-3}{k_1 \ldots k_s}$. The aim of this talk is to show that the analogous statement holds tropically.

First, we recall the tropical analogue of $M_{0,n}$ that we denote by $\mathcal{M}_{0,n}$ and introduce tropical Psi-classes as defined by Mikhalkin in [7]. Then we recall some tropical intersection theory as announced by Mikhalkin in [6] and developed in detail by Allermann and Rau in [1]. Finally, we show that Psi-classes are given as the divisor of a rational function and we intersect the divisors.

An n-marked (rational) abstract tropical curve is a metric tree Γ without 2-valent vertices and with n leaves, labeled by numbers $\{1, \dots, n\}$. The space $\mathcal{M}_{0,n}$ of all n-marked tropical curves is a polyhedral fan of dimension $n-3$ obtained by gluing copies of the space $\mathbb{R}^k_{>0}$ for $0 \le k \le n-3$ — one copy for each combinatorial type of a tree with n leaves and exactly k bounded edges. Its face lattice is given by $\tau \prec \sigma$ if and only if the tree corresponding to τ is obtained from the tree corresponding to σ by contracting bounded edges. For details, see [2], section 2, or [4], section 2.

Let the coordinates of $\mathbb{R}^{\binom{n}{2}}$ given by 2-subsets (i, j) of $\{1, \dots, n\}$. We define a map $\varphi_n : \mathcal{M}_{0,n} \to \mathbb{R}^{\binom{n}{2}} : C \mapsto \text{dist}(\{i, j\})_{(i,j)}$ where $\text{dist}(\{i, j\})$ denotes the sum of the lengths of all bounded edges on the path between the leaf marked i and the leaf marked j.

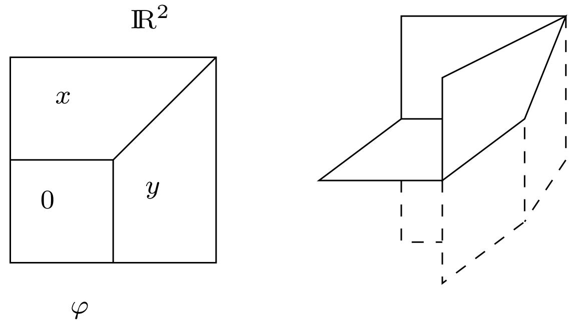
Using this map, we can embed $\mathcal{M}_{0,n}$ as a tropical fan (i.e. a fan satisfying the balancing conditions) into $\mathbb{R}^{\binom{n}{2}-n} = \mathbb{R}^{\binom{n}{2}}/W$, where we divide out the subspace W generated by trees with one bounded edge, a 2-valent vertex on one side and an n-valent vertex on the other side. For a proof of this statement, see theorem 3.4 of [3], section 2 of [7] or theorem 3.4 of [8].

Definition 1 (see [7], definition 3.1.)**.** *For $k \in [n]$, the tropical Psi-class $\Psi_k \subset \mathcal{M}_{0,n}$ is the union of those closed $(n-4)$-dimensional cones that correspond to tropical curves where the leaf marked with k is adjacent to a vertex with valence 4. The weight of each cone is 1.*

Let X be a tropical fan of dimension n and $0 \leq k \leq n$. Then $Z_k(X)$ denotes the group of k-cycles of X, that is the group of k-dimensional subfans satisfying the balancing condition (where negative weights are allowed).

Let φ be a piece-wise linear function on X. We call it a rational function.

Let for example $\varphi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \max\{0, x, y\}$. The following construction shows how to associate a divisor $\mathrm{div}(\varphi)$ (that is, an element in $Z_1(X)$) to a rational function φ. Roughly, this divisor should consist of the corner-locus of the map φ. But we want to attach suitable weights. Consider the graph of φ in $X \times \mathbb{R}$. There is a unique way to make it a balanced fan by attaching cones in the $(0, -1)$-direction. The weight of those cones is the weight that we want to attach to the corresponding cone in the corner-locus of φ. In the example, all the weights will be one.



A general formula to compute the weight for a cone σ in the corner-locus of φ, i.e. a codimension one cone of X, is the following: assume the full-dimensional neighbours of σ are called $\tau_1, \ldots, \tau_n$, and assume they are generated by the vectors $u_{\tau_i/\sigma}$ (modulo σ) and have weight ω_i. By the balancing condition, we have $\sum \omega_i u_{\tau_i/\sigma} = 0$ (modulo σ). The corresponding vectors in the graph are $(u_{\tau_i/\sigma}, \varphi(u_{\tau_i/\sigma}))$. We have to add a multiple of $(0, -1)$ to make it balanced. The weight we need is $(\sum \varphi(\omega_i u_{\tau_i/\sigma}) - \varphi(\sum \omega_i u_{\tau_i/\sigma}))$. Then sum is $(\sum \omega_i u_{\tau_i/\sigma}, \varphi(\sum \omega_i u_{\tau_i/\sigma}))$ which is a vector in the codimension one cone of the graph of φ corresponding to σ.

In general, how can we intersect to cycles Z_1 and Z_2? If we assume that Z_1 is of codimension 1, and we manage to find a rational function φ such that $\mathrm{div}(\varphi) = Z_1$, then we an just compute the divisor of φ on Z_2.

For each subset $I \subset \{0, \ldots, n\}$ of cardinality $1 < |I| < n-1$, define a vector $v_I \in \mathbb{R}^{\binom{n}{2}}/W$ as the image under the "distance map" φ_n of a tree with one bounded edge of length one, the marked ends with labels in I on one side of the bounded edge and the marked ends with labels in $\{0, \ldots, n\} \setminus I$ on the other.

We define $V_k := \{v_S \;:\; k \notin S \text{ and } |S| = 2\}$.

Lemma 1. *The linear span of the set V_k equals $\mathbb{R}^{\binom{n}{2}-n} = \mathbb{R}^{\binom{n}{2}}/W$.*
The sum over all elements $v_S \in V_k$ is 0.

Consequently, we can write any $v \in \mathbb{R}^{\binom{n}{2}-n}$ uniquely as a combination $v = \sum_{v_S \in V_k} \lambda_S v_S$ such that all $\lambda_S > 0$ for all S and at least one $\lambda_S = 0$. We will call such a representation a "positive representation with respect to V_k".

Lemma 2. *Let $k \notin I$. Then a positive representation of the distance vector v_I with respect to V_k is given by $v_I = \sum_{S \subset I, v_S \in V_k} v_S$.*

Now we can define a rational map f_k sending each vector $v_S \in V_k$ to 1. Because we can write each $v \in \mathbb{R}^{\binom{n}{2}-n}$ as a positive representation with respect to V_k, f_k can uniquely be extended to a piece-wise linear map.

Lemma 3. *The divisor associated to the rational function f_k is a multiple of the k-th Psi-class,* $\operatorname{div}(f_k) = \binom{n-1}{2}\Psi_k$.

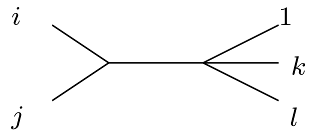
Theorem 1. *The intersection $\Psi_1^{k_1} \cdot \ldots \cdot \Psi_s^{k_s}$ is the subfan of $\mathcal{M}_{0,n}$ consisting of the closure of the cones of dimension $n - 3 - \sum_{i=1}^{s} k_i$ corresponding to tropical curves satisfying: if $i_1, \ldots, i_r \subset \{1, \ldots, s\}$ are adjacent to a vertex V then the valence* $\operatorname{val}(V)$ *of the vertex V is* $\operatorname{val}(V) = k_{i_1} + \cdots + k_{i_r} + 3$.

We define the weight $w(V)$ of such a vertex to be $w(V) = \frac{(k_{i_1}+\cdots+k_{i_r})!}{k_{i_1}! \cdot \ldots \cdot k_{i_r}!}$, and the weight of the cone as the product over the weights of all vertices.

It is an easy corollary that the formula mentioned in the beginning holds for 0-dimensional tropical intersections, too.

The proof of Theorem 1 is an induction on $\sum k_i$. In the induction step, we intersect $\Psi_1^{k_1} \cdot \ldots \cdot \Psi_s^{k_s}$ with $\operatorname{div}(f_1)$. The result is equal to $\binom{n-1}{2} \cdot \Psi_1^{k_1+1} \cdot \ldots \cdot \Psi_s^{k_s}$, and we have to show that it is equal to the subfan as described in the Theorem. To do this, we compute the intersection with $\operatorname{div}(f_1)$. For the general case, this is a quite long computation, and several combinatorial identities have to be shown (see [5]).

Example 1. *Let $n = 5$. Let us compute* $\operatorname{div}(f_1) \cdot \Psi_1$. *We need to check the cones of codimension 1 in Ψ_1 — that is, the cone $\{0\}$. The neighbors of $\{0\}$ in Ψ_1 — that is, in this case, the top-dimensional cones of Ψ_1 — correspond to tropical curves with 1 at a 4-valent vertex:*



There are $\binom{4}{2} = 6$ of these cones. Each such cone is generated by the normal vector $v_{i,j}$. The sum over all normal vectors is 0. Hence the weight of $\{0\}$ is given by $\sum_{i,j \in \{2,3,4,5\}, i \neq j} f_1(v_{i,j}) - f_1(0) = 6$. Thus the weight of $\{0\}$ in $\operatorname{div}(f_1) \cdot \Psi_1$ *is 6, and using Lemma 3, the weight of $\{0\}$ in $\Psi_1 \cdot \Psi_1$ is 1. Analogously, we can show that $\Psi_1 \cdot \Psi_2$ is the cone $\{0\}$ with weight 2.*

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Mixed Fiber Polytopes and Tropical Elimination

Bernd Sturmfels
(joint work with Jenia Tevelev, Josephine Yu)

Let $f_1, \ldots, f_c \in \mathbb{C}[x_1^{\pm 1}, \ldots x_n^{\pm 1}]$ be Laurent polynomials whose Newton polytopes are $P_1, \ldots, P_c \subset \mathbb{R}^n$ and suppose that the coefficients of the f_i are generic. The corresponding variety

$$X = \{u \in (C^*)^n \ : \ f_1(u) = \ldots = f_c(u) = 0\}$$

is a complete intersection of codimension c in the algebraic torus $(C^*)^n$. Fix a matrix $A \in \mathbb{Z}^{(n-c+1)\times n}$, the map $\pi : \mathbb{R}^n \to \operatorname{coker}(A) \cong \mathbb{R}^{c-1}$ and the induced monomial map $\alpha : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-c+1}$.

Theorem 1 ([5],[8]). *The Newton polytope of the hypersurface $Y = \overline{\alpha(X)}$ is affinely isomorphic to the mixed fiber polytope $\Sigma_\pi(P_1, \ldots, P_c)$.*

Question: What is a mixed fiber polytope?

Consider the Minkowski sum $P_\lambda = \lambda_1 P_1 + \cdots + \lambda_c P_c$ with $\lambda = (\lambda_1, \cdots, \lambda_c) \in (\mathbb{R}_{\geq 0})^c$ and form its classical fiber polytope introduced in [2]

$$\Sigma_\pi(P_\lambda) = \int_{q \in \pi(P_\lambda)} (\pi^{-1}(q) \cap P_\lambda)\, dq.$$

Theorem 2 (McMullen 2004 [6]). *The fiber polytope is a homogeneous polynomial of degree c in λ, i.e.*

$$\Sigma_\pi(P_\lambda) = \sum_{i_1 + \cdots + i_c = c} \lambda_1^{i_1} \cdots \lambda_c^{i_c} M_{i_1, \cdots, i_c}$$

where the $M_{i_1,\cdots,i_c}$ are polytopes. The mixed fiber polytope *is defined to be*

$$\Sigma_\pi(P_1, \ldots, P_c) := M_{1,\cdots,1}.$$

Example 2. *Consider the unmixed case, i.e. $P_1 = \cdots = P_c = P$. Then the mixed fiber polytope $\Sigma_\pi(P, \cdots, P)$ equals the fiber polytope $\Sigma_\pi(P)$ scaled by a factor $c!$.*

Example 3. *Assume that all f_i are linear forms with full support. Then the mixed fiber polytope is the secondary polytope of $\pi(\Delta)$.*

Example 4. *Implicitization of surfaces: Suppose we are given three Laurent polynomials* $g_1, g_2, g_3 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. *These data defines a morphism*

$$g : (\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^3,\ (s,t) \longmapsto (g_1(s,t), g_2(s,t), g_3(s,t))\,.$$

Under mild hypotheses, the closure of the image of g *is a hypersurface* $Y \subset (\mathbb{C}^*)^3$ *and we want to compute its Newton polytope. In order to do this, we introduce three new variables* x_1, x_2 *and* x_3 *and consider the Laurent polynomials* $f_i = x_i - g_i(s,t)$ *for* $1 \leq i \leq 3$. *The subvariety defined by* $f_1, \ldots, f_3$ *in* $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^3$ *is a generic complete intersection, namely the graph of the map* g. *Hence the image of* g *is obtained by projecting the variety* $\{f_1 = \ldots = f_n = 0\}$ *onto the last* n *coordinates.*

Question: Given the equations for $X \subset (\mathbb{C}^*)^n$, how can the Newton polytope of $Y = \overline{\alpha(X)} \subset (\mathbb{C}^*)^{n-c+1}$ be computed?

The Newton polytope of Y can be computed using the following three constructions from tropical geometry:

Construction 1 (**Tropical complete intersections** [4]).
The tropical complete intersection is the following subfan of the normal fan of $P_1 + \cdots + P_c$:

$$\begin{aligned}\mathrm{Trop}(X) &= \{v \in \mathbb{R}^n \ : \ \forall\, I \subset \{1,\ldots,c\} : \dim(\mathrm{face}_v(\sum_i P_i)) \geq |I|\} \\ &= \{v \in \mathbb{R}^n \mid MV_c(\mathrm{face}_v(P_1), \cdots, \mathrm{face}_v(P_c)) > 0\}\end{aligned}$$

where the mixed volumes are the weights m_v *which make the fan balanced.*

Construction 2 (**Push-forward of tropical cycles** [1],[7]).
The weights on the fan $\mathrm{Trop}(Y) = A \cdot \mathrm{Trop}(X)$ *are given by*

$$m_w = \frac{1}{deg(\alpha)_X} \sum_v m_v [\mathbb{L}_w \cap \mathbb{Z}^{n-c+1} : A(\mathbb{L}_v \cap \mathbb{Z}^n)],$$

where w *is a regular point on* $\mathrm{Trop}(Y)$ *with link* $\mathbb{L}_w$ *and the sum is over all* $v \in \mathrm{Trop}(X)$ *with* $A \cdot v = w$.

Construction 3 (**Recovering the Newton polytope** [3]).
For generic $u \in \mathbb{R}^{n-c+1}$, *the* i*-th coordinate of the vertex* $\mathrm{face}_u(\mathrm{Newton}(Y))$ *is*

$$\sum_{w \in (Trop(Y) \cap (u + \mathbb{R}_{\geq 0} e_i))} m_w \cdot [\mathbb{Z}^{n-c+1} : (\mathbb{L}_w + \mathbb{Z} e_i)]$$

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Tropical compactifications of subvarieties of tori

Jenia Tevelev

It is well-known that subvarieties of algebraic tori have many unexpected geometric properties that makes them similar to subvarieties of Abelian varieties. For example, they are their own logcanonical models (a conjecture of Miles Reid). I will try to explain this interplay of Mori theory and tropical geometry.

Tropical curves from log geometry

Bernd Siebert
(joint work with Mark Gross)

There are many approaches to tropical geometry, each bringing in new techniques and applications. While this means it is hard to agree on one common framework, I view this a sign of the vitality and reach of the field and do not see a need to restrict to a single point of view. That said, I still want to advertise here one point of view that I find extremely useful in making the connection to algebraic geometry and which in my opinion has not yet obtained the attention it deserves. Of all the approaches I am aware of, it is also the only one not working in some ambient variety such as $(\mathbb{C}^*)^N$, $\mathbb{P}^N$ or some other toric variety. It thus seems the right tool for investigating abstract tropical varieties, a notion which we currently have even more difficulties to agree on.

This approach is implicit in [2]. It is based on abstract log geometry [5],[3], and it refines the transversal toric degeneration approach of [6],[7]. Log geometry enhances the geometry on an algebraic variety by some kind of virtual information, encoded in a monoid sheaf. The central definition runs as follows.

Definition 1. *A* log structure *on a scheme X is a homomorphism*

$$\alpha_X : \mathcal{M}_X \longrightarrow \mathcal{O}_X$$

of sheaves of (multiplicative) monoids such that $\alpha_X : \alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. A log scheme *is a pair $(X, \mathcal{M}_X)$ with the morphism α_X understood.*

To provide enough flexibility the sheaves should be taken in the étale topology. Of course, this definition is far too general to be useful. The relation to toric geometry comes by the following construction. Let P be a toric monoid, that is, given by the integral points of a rational convex cone $C \subset \mathbb{R}^n$. Then a homomorphism of sheaves of monoids

$$\varphi : P_X \longrightarrow \mathcal{O}_X$$

defines an *associated log structure* $\mathcal{M}_X$ on X by

$$\mathcal{M}_X := \big(P_X \oplus \mathcal{O}_X^\times\big)/\big\{(p, \varphi(p)^{-1}) \,\big|\, p \in \varphi^{-1}(\mathcal{O}_X^\times)\big\}, \quad \alpha_X(p,h) := h\cdot\varphi(p). \tag{1}$$

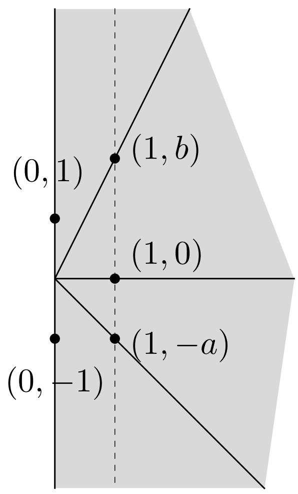
This construction has the effect of attaching $\mathcal{O}_X^\times$ to P in such a way that $\mathcal{O}_X^\times \simeq \alpha_X^{-1}(\mathcal{O}_X^\times)$. If $X = \operatorname{Spec}\mathbb{C}[P]$ is the toric variety defined by P the discrete part

$$\overline{\mathcal{M}}_X := \mathcal{M}_X/\alpha_X^{-1}(\mathcal{O}_X^\times)$$

of $\mathcal{M}_X$ is a submonoid of a constructible sheaf on X, which is constant along any torus orbit. The stalk at a point $x \in X$ equals P/P_x where P_x is generated by the powers of monomials that are invertible at x. Thus a neighbourhood of x is isomorphic to Spec $\big(\mathbb{C}[\overline{\mathcal{M}}_{X,x}]\big) \times O_x$ where O_x is the torus orbit containing x.

As long as we work with log structures arising from some toric situation there is really no need to write down $\mathcal{M}_X$, as one can always work with monomial functions on the ambient space. The point of log geometry is to transfer toric techniques to more general situations.

As an example let us look at a toric degeneration of curves, defined by the fan in $\mathbb{R}^2$ with rays $(0,1), (0,-1), (1,0), (1,-a), (1,b)$, $a, b > 0$ (Figure 1). This fan refines the fan of $\mathbb{A}^1 \times \mathbb{P}^1$. We thus obtain a toric blow up $X \to \mathbb{A}^1 \times \mathbb{P}^1$ contracting two $\mathbb{P}^1$. Moreover, the projection to the second coordinate defines a flat map $\pi : X \to \mathbb{A}^1$ with general fibres $\pi^{-1}(t) = \mathbb{P}^1$, $t \neq 0$, and with $C := \pi^{-1}(0)$ a chain of three $\mathbb{P}^1$. Thus π can be viewed as a toric degeneration of $\mathbb{P}^1$ into a chain of three $\mathbb{P}^1$. For $a, b > 1$ the total space X has two singular points of types A_{a-1} and A_{b-1}, located at the singular points of the central fibre C. These singular points get reflected in the stalks of $\overline{\mathcal{M}}_X$, which at these points are isomorphic to the submonoids of $\mathbb{N}^2$ generated by $(0,1), (1,0), (1,-k)$ for $k = a, b$, respectively.



Now the point is that we can restrict the log structure of X to C, by taking the log structure associated via (1) to $\iota^{-1}\mathcal{M}_X$, $\iota : C \to X$ the inclusion. This process preserves the stalks of $\overline{\mathcal{M}}_X$. Thus $\mathcal{M}_C$ is a log structure on C that somehow remembers the fact that C sits inside X. Moreover, the toric degeneration parameter t defines a section ρ_C of $\mathcal{M}_C$ that even records the "speed" of smoothing of the nodes: If X is described locally by $xy = f(x,y,t) \cdot t^k$ then ρ_C determines $f(0,0,0)$.

In the talk I motivated and explained how to associate a tropical curve to a stable map to the central fibre of a toric degeneration, provided we endow all spaces with a log structure. To state the result, let $\pi : X \to \mathbb{A}^1$ be a flat map of toric varieties. These are obtained from fans with support $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, analogously to the one-dimensional example above. Here $N \simeq \mathbb{Z}^n$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. The intersection of this fan with $N_{\mathbb{R}} \times \{1\}$ defines a polyhedral decomposition $\mathscr{P}$ of $N_{\mathbb{R}}$ with vertices in bijection with the irreducible components of $X_0 := \pi^{-1}(0)$. Let $\mathcal{M}_{X_0}$ be the induced log structure on X_0 and ρ_{X_0} the section of $\mathcal{M}_{X_0}$ defined by t.

Now assume we have a stable map

$$f : C \longrightarrow X_0,$$

so C is a nodal curve without infinitesimal automorphism relative to X_0. If f arises as the limit of embedded curves $C_t \subset \pi^{-1}(t)$, $t \neq 0$, we can lift f to a log morphism making C into a *(pre-) stable log curve*, a notion investigated in [4]. The log structure of a pre-stable log curve (over the so-called standard log point) has a similar shape as the toric example above; let ρ_C be the distinguished section of $\mathcal{M}_C$ thus defined. We obtain a homomorphism of monoid sheaves

$$f^\sharp : f^{-1}\mathcal{M}_{X_0} \longrightarrow \mathcal{M}_C$$

with $f^\sharp(\rho_{X_0}) = d \cdot \rho_C$ for some $d \in \mathbb{N} \setminus \{0\}$.

It is possible to extract the tropical curve in $N_{\mathbb{R}}$ associated to the degeneration C_t just from the limiting stable log map $(f, f^\sharp)$. This works no matter if $(f, f^\sharp)$ arises as a limit or not. The vertices of this tropical curve are defined as follows. Let $\eta \in C$ be the generic point of an irreducible component of C. Then $f(\eta)$ lies in some minimal toric stratum of X_0, say given by the cone $C(\sigma) \subset N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ over a cell $\sigma \in \mathscr{P}$. It is then not hard to see that $\overline{\mathcal{M}}_{X_0, f(\eta)} = C(\sigma)^\vee \cap (N^* \times \mathbb{N})$, the integral points of the dual cone, while $\overline{\mathcal{M}}_{C,\eta} = \mathbb{N}$. Thus $f^\sharp_\eta$ defines an integral point (p, r) of $C(\sigma) \cap (N \times \mathbb{N})$. Moreover, $f^\sharp(\rho_{X_0}) = \rho_C$ implies $r = d$. We define the vertex p_η of the associated tropical curve as $p/d \in \sigma$.

Similarly, one defines an interval or half-line l_x for each nodal point $x \in C$. Here is our result that hopefully will appear with full written details and in much greater generality in the context of Gromov-Witten computations in the context of our mirror symmetry program [2].

Theorem 1. *It is possible to extract weights from the log morphism such that the* p_η *and* l_x *are the vertices and edges of a tropical curve. In the case* $(f, f^\sharp)$ *arises as the limit* $t \to 0$ *of a family of embedded curves* $C_t \subset \pi^{-1}(t)$, *it agrees with the usual tropical curve.*

The case when the image of C intersects only toric strata of codimension at most one is treated in [6]. Some discussion of the situation with contracted components can be found in [1].

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Parameterizing Tropical Curves

David E Speyer

One of the primary applications of tropical geometry is to the study of curves in toric varieties. According to this perspective, one considers curves defined over a field with a non-archimedean valuation. Given a curve X embedded in an (algebraic) torus, one constructs a graph embedded in a real vector space. This graph is known as the tropicalization of the curve. From the tropicalization of X, one tries to read off information about the degree and genus of the original curve X, and its intersections with other subvarieties of the torus. In this introduction, we will write X for a curve embedded in a torus $T \cong (\mathbb{K}^*)^n$ and we will write $\Gamma \subset \mathbb{R}^n$ for the tropicalization of X.

In order to use these tropical methods, we need to know which graphs are tropicalizations of curves. We will refer to a graph which actually is the tropicalization of a curve as a tropical curve. There are certain basic combinatorial conditions which hold for any tropical curve. The first, the zero tension condition, is a description of the possible local structures of a tropical curve around a given vertex. We can assign to X a multiset of lattice vectors, which we will call the degree of X, from which we can determine the homology class represented by the closure of X when this closure is taken in a suitable toric compactification of T. The second combinatorial condition is that the directions of the unbounded rays of Γ are given by the degree of X. Thirdly, we can show that, modulo some technical conditions, the genus of X is greater than or equal to the first Betti number of Γ. We will define a zero-tension curve of genus g and degree δ to be a graph which has first Betti number g and obeys the obvious conditions to be the tropicalization of a degree δ curve.

We attack the inverse problem: Given a zero-tension curve of genus g and degree δ, when does it come from an actual curve of genus g and degree δ? And how can we build such a curve explicitly? When $g = 0$, we can always find such a curve and we can parameterize it by rational functions in an extremely explicit manner. When dealing with curves of higher genus, there are some combinatorial obstructions but, once these obstructions are overcome, we can also give explicit paramaterizations using nonarchimedean elliptic functions and their higher genus analogues.

First, let us state what the tropicalization of a curve is. Let $\boldsymbol{K}$ be the fraction field of a complete dvr, and let $\mathbb{K}$ be the algebraic closure of $\boldsymbol{K}$. Let $v : \mathbb{K} \to \mathbb{Q} \cup \{\infty\}$ be the valuation on $\mathbb{K}$. Then, if T is a torus over $\mathbb{K}$ and X a curve over $\mathbb{K}$ with ϕ a map $X \to T$, the tropicalization $\operatorname{Trop} \phi(X)$ is $v(\phi(X))$.

We also must define the degree of a curve in a torus. Let X be a smooth algebraic curve and let $\phi : X \to T$ be an algebraic map. (Whenever ϕ is nonconstant, the curve X is not complete.) Let $\overline{X}$ be the smooth complete curve compactifying X; we impose the condition that ϕ can not be extended to any point of $\overline{X} \setminus X$. Let x be any point of $\overline{X} \setminus X$. Define σ_x, ρ_x and d_x as before. Let $\rho_1, \ldots, \rho_N$ be the set of distinct values of ρ_x as x ranges over $\overline{X} \setminus X$. For $1 \leq i \leq N$, let $d_i = \sum_{\rho_x = \rho_i} d_x$ and set $\sigma_i = \sum_{\rho_x = \rho_i} \sigma_x = d_i \rho_i$. We define the set $\{\sigma_1, \ldots, \sigma_N\}$ to be the degree of (ϕ, X). Note that this is defined without any choice of toric compactification of T. Note also that $\sum \sigma_i = 0$, because any rational function has equally many zeroes and poles on $\overline{X}$.

We now describe zero tension curves. These are the combinatorial models for tropicalizations of curves. Let Γ be a metric graph with finitely many vertices and edges. We permit edges of infinite length, as long as at least one of their endpoints is of degree one, and we require every vertex of degree one to be at the end of such an edge. Write $\partial\Gamma$ for the set of degree 1 vertices of Γ. Let ι be a continuous map $\Gamma \setminus \partial\Gamma \to \mathbb{R}^n$ such that an edge e of Γ is taken to a translate of the vector $\ell(e)\sigma(e)$, for some $\sigma(e) \in \mathbb{Z}^n$. (So, if $\ell(e)$ is infinite, the image of e is a ray or line segment.) The slope $\sigma(e)$ is only well defined up to sign; we write $\sigma_v(e)$ for the choice of sign pointing away from the endpoint v of e. So, if v_1 and v_2 are both endpoints of e, then $\sigma_{v_1}(e) = -\sigma_{v_2}(e)$. We require that the zero tension condition, $\sum_{e \ni v} \sigma_v(e) = 0$, hold for every vertex v not of degree 1. Such a pair (ι, Γ) is a zero tension curve. The genus of a zero tension curve is its first betti number. We define the degree of a zero tension curve as follows: Let $D \subset \Lambda$ be the (finite) set of values assumed by $-\rho_v(e)$ as v ranges through $\partial\Gamma$. For each $\lambda \in D$, let $m_\lambda = \sum m(e)$ where the sum is over e with an endpoint v in $\partial\Gamma$ such that $-\sigma_v(e)$ is a positive multiple of λ. Then the degree of (ι, Γ, m) is the set $\{m_\lambda \cdot \lambda : \lambda \in D\}$.

The connection between tropicalization and zero tension curves is the following:

Theorem. *Let X be a smooth curve over $\mathbb{K}$ and ϕ a map from X to T. Then there is a zero tension curve (ι, Γ) such that $\iota(\Gamma) = \operatorname{Trop} \phi(X)$, the degree of (ι, Γ) is the same as that of (ϕ, X) and the genus of Γ is less than or equal to that of X.*

For curves of genus zero, there is no difficulty in reversing this result.

Theorem. *Let (ι, Γ) be a zero tension curve of genus zero. Then there is a genus zero curve X and a map $\phi : X \to T$ such that $\iota(\Gamma) = \operatorname{Trop} \phi(X)$.*

For curves of higher genus, there is an obstruction called superabundance, first observed by Mikhalkin. If Γ has genus one, then (ι, Γ) is superabundant if the image under ι of the cycle of Γ lands in a hyperplane. More generally, let (ι, Γ)

be a zero tension graph in $\mathbb{R}^n$. Let $C1(\Gamma)$ be the vector space spanned by the oriented edges of Γ. We write $[e, v]$ for the basis vector of $C1(\Gamma)$ corresponding to the edge e, oriented away from its endpoint v. Then $H1(\Gamma)$ is a quotient of $C1(\Gamma)$ in the obvious way; write $\overline{[e, v]}$ for the image of $[e, v]$ in $H1(\Gamma)$. Let $s : C1(\Gamma) \to \mathbb{R}^n \otimes H1(\Gamma)$ be the map $[e, v] \mapsto \sigma_v(e) \otimes \overline{[e, v]}$. The general definition is that (ι, Γ) is superabundant if s fails to be surjective. It is a pleasant exercise to see that this corresponds to the previous description when Γ has genus one.

Theorem. *Let (ι, Γ) be a superabundant zero tension curve of genus one. Then there is a genus one curve X and a map $\phi : X \to T$ such that $\iota(\Gamma) = \operatorname{Trop} \phi(X)$.*

Theorem. *Let (ι, Γ) be a superabundant zero tension curve of genus $g \geq 2$ and assume that the residue field of $\mathbb{K}$ has characteristic zero. Then there is a genus g curve X and a map $\phi : X \to T$ such that $\iota(\Gamma) = \operatorname{Trop} \phi(X)$.*

Tropical curves, theta functions and Riemann-Roch

Ilia Zharkov
(joint work with Grigory Mikhalkin)

Let Γ be a connected finite graph and $\mathcal{V}_1(\Gamma)$ be the set of its 1-valent vertices. We say Γ is a metric graph if the topological space $\Gamma \setminus \mathcal{V}_1(\Gamma)$ is given a complete metric structure and Γ is its compactification. In particular, all leaves have infinite lengths. Introducing a new two-valent vertex on the interior of an edge is set to give an equivalent metric graph, and a *tropical curve* C is an equivalence class of such graphs. Its genus is $g = b_1(\Gamma)$ for any representative Γ.

The metric allows one to talk about affine and piece-wise linear functions on C with integral slopes. At every vertex v we may define the set of outward primitive tangent vectors ξ_i. Then any PL function f defines a *divisor*

$$(f) = \sum_{p \in C} \left(\sum_{i=1}^{\operatorname{val}(p)} \frac{\partial f(p)}{\partial \xi_i} \right) p.$$

In general, a divisor $D = \sum a_i p_i$ is a formal linear combination of points in C with integral coefficients. We say: $D_1 \sim D_2$ if $D_1 - D_2$ is a principal divisor, $D \geq 0$ if all $a_i \geq 0$, and D has degree $\deg D = \sum a_i$.

Let Aff be the sheaf of $\mathbb{Z}$-affine functions. Define the integral cotangent local system $\mathcal{T}_{\mathbb{Z}}^*$ on C by the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{R} \longrightarrow \operatorname{Aff} \longrightarrow \mathcal{T}_{\mathbb{Z}}^* \longrightarrow 0,$$

The rank g lattice of *1-forms* $\Omega_{\mathbb{Z}}(C)$ (a.k.a. *circuit lattice*) on C is formed by the global sections of $\mathcal{T}_{\mathbb{Z}}^*$. Let $\Omega(C)^*$ be the vector space of $\mathbb{R}$-valued linear functionals on $\Omega_{\mathbb{Z}}(C)$. Then the integral cycles $H_1(C, \mathbb{Z})$ form a lattice Λ in $\Omega(C)^*$ by integrating over them. We define the tropical *Jacobian* (cf. [BHN97]) to be

$$J(C) := \Omega(C)^* / H_1(C, \mathbb{Z}) \cong \mathbb{R}^g / \Lambda.$$

Let us fix a reference point $p_0 \in C$. Given a divisor $D = \sum a_i p_i$ we choose paths from p_0 to p_i. Integration along these paths defines a linear functional on $\Omega_{\mathbb{Z}}(C)$:

$$\hat{\mu}(D)(\omega) = \sum a_i \int_{p_0}^{p_i} \omega.$$

For another choice of paths the value of $\hat{\mu}(D)$ will differ by an element in Λ. Thus, we get a well-defined tropical analog of the *Abel-Jacobi map* $\mu : \mathrm{Div}^d(C) \to J(C)$.

Theorem 1 (Tropical Abel-Jacobi)**.** *For each degree d the map μ factors through $\mathrm{Pic}^d(C)$, the group of divisors modulo linear equivalence:*

$$\begin{array}{ccc} \mathrm{Div}^d(C) & \longrightarrow & \mathrm{Pic}^d(C) \\ & \searrow^{\mu} & \downarrow \phi \\ & & J(C) \end{array}$$

so that ϕ is a bijection.

The metric on C defines a symmetric positive bilinear form Q on $\Omega(C)^*$ by setting $Q(\ell, \ell) := \mathrm{length}(\ell)$ on simple cycles ℓ. That, in turn, defines a convex Λ-quasi-periodic PL function on $\Omega(C)^*$:

$$\Theta(x) := \max_{\lambda \in \Lambda} \{Q(\lambda, x) - \frac{1}{2} Q(\lambda, \lambda)\}, \quad x \in \Omega(C)^*,$$

that can be thought as a section of a polarization line bundle on $J(C)$. Its corner locus defines the *theta divisor* $[\Theta]$ on the Jacobian.

For $\lambda \in J(C)$ let $[\Theta_\lambda] = [\Theta] + \lambda$ be the translated theta divisor on $J(C)$. Let $D_\lambda := \mu^*[\Theta_\lambda]$ denote the pull back divisor of $[\Theta_\lambda]$ to the curve via the Abel-Jacobi map $\mu : C \to J(C)$.

Theorem 2 (Jacobi Inversion)**.** *For any $\lambda \in J(C)$ the divisor $D_\lambda := \mu^*[\Theta_\lambda]$ is effective of degree g. There exists a universal $\kappa \in J(C)$ (depends on p_0) such that $\mu(D_\lambda) + \kappa = \lambda$ for all $\lambda \in J(C)$.*

Corollary 3. *Given a divisor D of degree d there is an effective divisor $D_\lambda = \mu^*[\Theta_{\mu(D)+\kappa}]$ of degree g linearly equivalent to $D + (g-d)p_0$. In particular, any element in* Pic^g *has a* canonical *(i.e. independent of the base point p_0) effective representative in* $\mathrm{Div}^g(C)$*.*

The linear system $|D|$ is the set $\{D' \geq 0 \ : \ D' \sim D\}$. For $|D| \neq \emptyset$ its *dimension* $\dim |D|$ is the maximal integer $r \geq 0$ such that for any divisor $R \geq 0$ of degree r the space $|D - R|$ is non-empty. We set $\dim |D| = -1$ when $|D| = \emptyset$. A point $q \in C$ is in $\mathrm{supp}\, |D|$ if $|D - q| \neq \emptyset$.

The degree $2g-2$ divisor $K := \sum_{p \in C} (\mathrm{val}(p) - 2)p$ is called the *canonical divisor.*

Theorem 4 (Tropical Riemann-Roch, cf. [BN07], [GK06] and [MZ06])**.**

$$\dim |D| - \dim |K - D| = d - g + 1.$$

Theorem 5. *Let $D \in \mathrm{Div}^g(C)$. Then $q \in \mathrm{supp}\, |D| \iff \mu(q) \in [\Theta_{\mu(D)+\kappa}]$.*

Corollary 6 (Riemann's Theorem)**.** *$W_{g-1} + \kappa = [\Theta]$, where $\kappa \in J(C)$ is the constant from Theorem 2.*

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Tropical linear algebra: old and new results

Stéphane Gaubert

Before the recent flourishing of tropical geometry, tropical algebraic structures were considered by several communities under different names: extremal algebra, max-algebra, max-plus algebra, idempotent algebra, incline algebra, . . . We present here some results relating the tropical and pre-tropical civilizations. These concern linear algebra.

The tropical analogues of the notion of vector space or module have been studied under the name of *idempotent spaces* [14] or *semimodules* [4]. We shall rather use the term "cone" to bring to light the analogy with classical convexity: a set C of functions from a set X to $\mathbb{R} \cup \{-\infty\}$ is a *max-plus cone* if for all $u, v \in C$ and for all $\lambda, \mu \in \mathbb{R} \cup \{-\infty\}$, the function $\sup(\lambda + u, \mu + v)$ belongs to C. When $X = \{1, \ldots, n\}$, the elements of C will be seen as vectors of $(\mathbb{R} \cup \{-\infty\})^n$. The tropical polytopes of Develin and Sturmfels [6] yield a remarkable class of max-plus cones.

A classical result of convex analysis, the Minkowski theorem, shows that a finite dimensional pointed closed convex cone is generated by its extreme rays. A similar result holds for max-plus cones. We say that $u \in C$ is an *extreme generator* of C if u cannot be written as the supremum of two elements of C that are both different from it.

Theorem (Tropical Minkowski, [3, 10]) *Any vector of a closed max-plus cone $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is the supremum of at most n extreme generators of C.*

Extreme generators arise in particular, in an infinite dimensional setting, when considering the compactifications of metric spaces. Let (X, d) denote a metric space. Let $\imath$ denote the map sending any point $x \in X$ to the function $y \mapsto -d(x, y) + d(b, y)$, where $b \in X$ is an arbitrary (base) point. When (X, d) is proper, the map $\imath$ is an embedding from X to the space of continuous functions on X, equipped with the topology of uniform convergence on compact sets. A *horofunction* is a map in $\mathrm{clo}(\imath(X)) \setminus \imath(X)$. The horofunctions constitute a boundary which was defined by Gromov and further studied by Rieffel. A function h is a

Busemann point if there exists an almost-geodesic $x_k \in X$ such that $h = \lim_k \imath(x_k)$. By *almost-geodesic*, we mean that $d(x_1, x_2) + \cdots + d(x_{k-1}, x_k) - d(x_1, x_k)$ remains bounded above as $k \to \infty$.

THEOREM (See [2, Th. 8.3]) *A horofunction is a Busemann point if and only if it is an extreme generator of the max-plus cone consisting of* 1*-Lipschitz functions on* X*.*

More generally, the result of [2] characterizes the extreme generators of an eigenspace. This extends the finite dimensional max-plus spectral theorem (see [1] for more background on this result). This is also related to Fathi's characterization of "weak-KAM solutions" by their restrictions to the Aubry set [7].

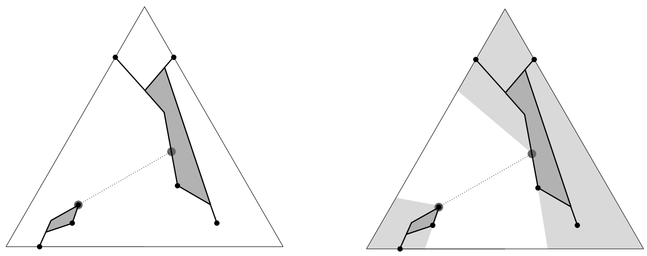
Another topic in classical convexity is separation. A first tropical separation theorem was established by K. Zimmermann [18]. It was simplified and generalised in [16] and in [4, 5]. The latter works rely on projections. The *projection* $P_C(x)$ of a vector x on a closed max-plus cone C is defined as the maximal element of C which is less than x. For all vectors u, let u^- denote the vector obtained by changing the sign of each entry of u. The separation theorem of [4] implies that if C is a closed max-plus cone of $(\mathbb{R} \cup \{-\infty\})^n$, and if $x \notin C$, then, the set

$$H = \{y \mid y \cdot x^- \geq y \cdot (P_C(x))^-\}$$

contains C and not x. Here, "$\cdot$" denotes the max-plus scalar product, so that $y \cdot x^- := \min_i y_i - x_i$, where the convention $-\infty - (-\infty) = -\infty$ is understood. The latter result is reminiscent of the separation theorem between a point and a convex set in a Hermitean space, the map $(y, x) \mapsto y \cdot x^-$ resembling a sesquilinear form. When the entries of x and $P_C(x)$ are finite, the set H is closed in the usual topology and it is called a *Archimedean half-space.* A perturbation argument [5] allows one to separate a point from a closed max-plus cone by a Archimedean half-space. This can be extended to the case of several max-plus cones, with an explicit construction of the half-spaces in terms of cyclic projections.

THEOREM (Separation of several max-plus cones [11]) *If* $V_1, \ldots, V_k$ *are closed max-plus cones of* $(\mathbb{R} \cup \{-\infty\})^n$ *with a zero-intersection, then, for every* $1 \leq i \leq k$*, we can find a Archimedean half-space* H_i *containing* V_i*, in such a way that* $H_1, \ldots, H_k$ *also have a zero intersection.*

By "zero intersection", we mean an intersection reduced to the max-plus "zero" vector, which has $-\infty$ entries. This result is illustrated in the following figure, representing two max-plus cones and a periodic orbit of the cyclic projector (left), together with separating half-spaces supported at the points of this periodic orbit (right). Here, a "non-zero" vector x of $(\mathbb{R} \cup \{-\infty\})^3$ is represented by the center of gravity of the three vertices of the standard simplex, with respective masses $\exp(x_i)$, $i = 1, \ldots, 3$. See [11] for details.



Finally, let us point out another direction in which tropical linear algebra has been developed. It concerns Perron-Frobenius theory. If $A = (A_{ij})$ is a nonnegative matrix, the spectral radius of A is known to be the maximal eigenvalue of A with a nonnegative eigenvector. It is called the Perron root. We denote it by $\rho(A)$. Kingman [13] showed that the logarithm of the Perron root is a convex function of the logarithms of the entries of A. This may be rewritten as

$$\rho(A \circ B) \leq \rho(A^{(p)})^{1/p} \rho(B^{(q)})^{1/q}$$

for all $n \times n$ nonnegative matrices A, B and for all $p, q > 1$ such that $1/p + 1/q = 1$. Here, $A^{(p)}$ is the p-th Hadamard power of A, i.e. the matrix obtained by raising every entry of A to the power p, and $A \circ B$ is the Hadamard product (entrywise product) of A and B. Friedland [8] showed that

$$\rho_\infty(A) := \lim_{p\to\infty} \rho(A^{(p)})^{1/p} = \max_{i_1,\dots,i_k} (A_{i_1 i_2} \cdots A_{i_k i_1})^{1/k} \quad ,$$

where the maximum is taken over all sequences of distinct elements of $\{1, \dots, n\}$. The latter quantity is nothing but the maximal eigenvalue of A in the "max-times" reincarnation of the tropical semiring. By taking $p = \infty$, we get

$$\rho(A \circ B) \leq \rho_\infty(A)\rho(B) \quad .$$

Some bounds for the modulus of the roots of a polynomial follow from this inequality. Indeed, if $P(x) = \sum_{0\leq k\leq n} p_k x^k$ is a polynomial with complex coefficients, define the *tropical roots* of P to be the points $x \geq 0$ at which the maximum $\max_{0\leq k\leq n} |p_k| x^k$ is attained at least twice. The *multiplicity* of a tropical root x is the greatest difference of two indices attaining the maximum in the above expression. Let $\zeta_1 \geq \ldots \geq \zeta_n$ denote the modulus of the roots of P, and let $\alpha_1 \geq \cdots \geq \alpha_n$ denote the tropical roots, all roots being counted with multiplicities. We can bound $\zeta_1 \cdots \zeta_k$ by considering the k-th exterior power of the companion matrix of the polynomial P and applying the inequality above to well chosen matrices. The simplest bound obtained in this way is:

$$\zeta_1 \cdots \zeta_k \leq (k+1)\alpha_1 \cdots \alpha_k \quad .$$

However, Hadamard [12] gave an equivalent bound, which was perhaps not widely known (Fujiwara proved in [9] the $k = 1$ case and Specht [17] proved the weaker inequality $\zeta_1 \cdots \zeta_k \leq (k+1)\alpha_1^k$). Later on, Pólya improved the Hadamard bound by replacing the $k+1$ factor by $\sqrt{(k+1)^{k+1}/k^k}$, see [15]. Some bounds relying on the matrix approach are obtained in a current work with Akian and Brandjesky.

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Tropical and Ordinary Convexity Combined

Michael Joswig

(joint work with Katja Kulas)

A set $S \subseteq \mathbb{R}^{d+1}$ is called *tropically convex* if for all $x, y \in S$ and for all $\lambda, \mu \in \mathbb{R}$ we have $(\lambda \odot x) \oplus (\mu \odot y) \in S$. Here $\oplus$ is the component-wise minimum and $\odot$ is the *tropical scalar multiplication*, that is, $\lambda \odot x = (\lambda + x_0, \ldots, \lambda + x_d)$. Each tropically convex set is invariant under tropical scalar multiplication. Hence we consider the quotient $\mathbb{TA}^d = \mathbb{R}^{d+1}/(\mathbb{R} \odot (0, \ldots, 0))$ which we call *tropical affine space*. Clearly, $\mathbb{TA}^d$ can be identified with $\mathbb{R}^d$ via the map $(x_0, x_1, \ldots, x_d) \mapsto (x_1 - x_0, \ldots, x_d - x_0)$.

Tropical polytopes, that is, tropical convex hulls of finitely many points in $\mathbb{TA}^d$, have been introduced by Develin and Sturmfels [2]. They have shown, in particular, that tropical polytopes, or rather configurations of n points in $\mathbb{TA}^d$ are equivalent to regular subdivisions of the product of simplices $\Delta_{n-1} \times \Delta_d$.

We introduce *polytropes* as tropical polytopes which (via the above identification of $\mathbb{TA}^d$ with $\mathbb{R}^d$) are also convex in the ordinary sense. The Figure 1 shows polytropes in the plane.

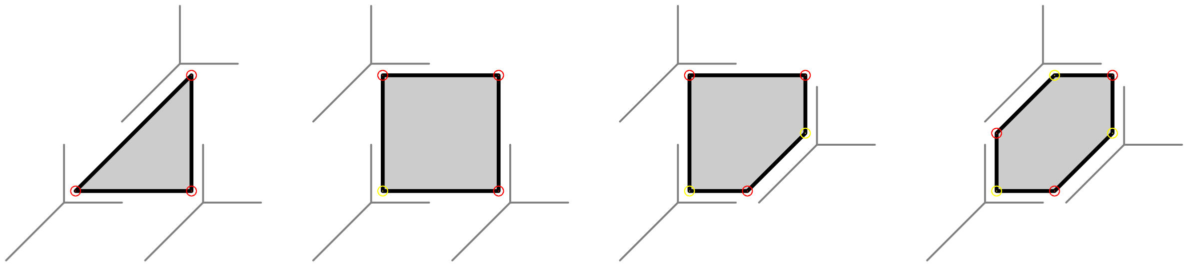

FIGURE 1. Four types of polytropes in $\mathbb{TA}^2$. The tropical vertices are red. The sketches of tropical hyperplanes indicate the facet defining tropical halfspaces.

Polytropes, considered as ordinary polytopes, are precisely the *alcoved polytopes* of Lam and Postnikov [6], or the bounded cells of the *deformations* of the Coxeter hyperplane arrangement of type A_d studied by Postnikov and Stanley [7], or the bounded intersections of apartments in affine buildings of type $\tilde{\mathrm{A}}_d$; see [5].

Our main result is the following.

Theorem 1. *A d-dimensional polytrope is a tropical simplex, that is, it is the tropical convex hull of $d+1$ points.*

The (easy) proof uses the tropical halfspaces from [4] in an essential way. Moreover, via a result of Bruns and Römer [1], our theorem turns out to be equivalent to the known fact that the Segre product of two full polynomial rings (over some field K) has the Gorenstein property if and only if both factors are generated by the same number of indeterminates. The latter statement is a special case of a theorem of Goto and Watanabe [3].

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Donaldson-Thomas invariants for Calabi-Yau categories

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(joint work with Maxim Kontsevich)

Donaldson-Thomas theory, if understood categorically, should produce invariants of the "moduli space of stable objects" in a triangulated category. This point of view goes back to Thomas where (partially motivated by the ideas of Homological Mirror Symmetry he suggested a dictionary between "complex" and "symplectic" geometry of mirror dual Calabi-Yau manifolds . In particular, critical points of the holomorphic Chern-Simons functional on a 3-dimensional Calabi-Yau manifold M equipped with a holomorphic volume form $\Omega^{3,0}$

$$CS_{\mathbb{C}}(A_0+\alpha)=\int_M Tr\left(\frac{1}{2}\overline{\partial}_{A_0}\alpha\wedge\alpha+\frac{1}{3}\alpha\wedge\alpha\wedge\alpha\right)\wedge\Omega^{3,0}$$

correspond to the critical points of the functional

$$f_{\mathbb{C}}(A,L)=\int_{L_0}^{L}(F_A+\omega)^2$$

for the mirror dual Calabi-Yau manifold W equipped with the Kähler form ω. In other words, holomorphic bundles on M correspond to the pairs: a Lagrangian submanifold $L\subset W$ and a flat connection A on a line bundle over L. In both cases the virtual dimension of the moduli space of the objects is zero, hence appropriately defined count of holomorphic vector bundles on M (i.e. holomorphic Casson invariant) corresponds to the appropriately defined count of special Lagrangian submanifolds in W. In order to define the count rigorously one has to impose some sort of stability condition on the objects. For example one can count Hermitian Yang-Mills connections (i.e. stable holomorphic vector bundles) on the "complex" side of the dictionary, and special Lagrangian submanifolds (SLAGs) on the "symplectic" side. It was also observed that the holomorphic Casson invariant does not change when we deform the complex structure on M, but it does change on "walls" of real codimension one as we deform the Kähler form (since the notion of stability of a holomorphic vector bundle depends on it). Clearly one has the "mirror dual" story for W and SLAGs. This wall-crossing phenomenon is well-known in Donaldson theory of 4-dimensional manifolds. It should be compared to the "chamber" structure of the space of stability conditions. Since the functionals defining the notion of stability in the above example are additive on exact sequences, one expects that the theory can be extended to the world of derived categories.

The above considerations suggest that (at least part of) the Donaldson-Thomas theory can be spelled out in the language of derived categories, or, more generally, in a pure algebraic framework of Calabi-Yau categories. We discuss such a theory.

It is concentrated around the idea of wall-crossing formula which plays an important role in Donaldson-Thomas theory. The approach is motivated by motivic integration, cluster transformations and recent results on Calabi-Yau categories.

1. An example of the wall-crossing formula

In the case of holomorphic Casson invariant, the wall-crossing formulas describe the behavior of invariants with respect to the (complexified) Kähler structure. In the general categorical framework the role of the latter is played by a stability condition in the sense of Bridgeland . It turns out that essential part of the story does not depend on the details (e.g. the definition of DT-invariants, precise meaning of the stability condition, etc.). We will see that it leads to a non-trivial wall-crossing formula for DT-invariants. The wall-crossing formula below reproduces correct BPS spectrum for $N = 2, d = 4$ super Yang-Mills theory studied by Seiberg and Witten.

Let Λ be a free abelian group endowed with a skew-symmetric integer-valued bilinear form $\langle \bullet, \bullet \rangle$. Consider a Lie algebra over $\mathbb{Q}$ with the basis $(e_\gamma)_{\gamma \in \Lambda}$ such that

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle \; e_{\gamma_1 + \gamma_2}.$$

This Lie algebra is isomorphic (non-canonically) to the Lie algebra of Laurent polynomials on the algebraic torus $\mathbb{T} := \mathbb{T}_\Lambda = \mathrm{Hom}(\Lambda, \mathbb{G}_m)$, endowed with the translation-invariant Poisson bracket associated with $\langle \bullet, \bullet \rangle$.

Let $Z : \Lambda \to \mathbb{C}$ be an additive map which is generic in the sense that there are no two $\mathbb{Q}$-independent elements of the lattice which are mapped by Z into the same real line. Otherwise we say that Z belongs to the *wall of first kind* (it is called the wall of marginal stability in physics literature). Let us choose an arbitrary norm $\| \bullet \|$ on the real vector space $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$. Finally, assume that we are given an even map $\Omega : \Lambda \setminus \{0\} \to \mathbb{Z}$ supported on the set B of such $\gamma \in \Lambda$ that $\| \gamma \| \leq C|Z(\gamma)|$ for some constant $C > 0$. Let $V \subset \mathbb{R}^2$ be a sector which is less than 180° and has the vertex at the origin, and $C(V)$ be the convex hull of $Z^{-1}(V) \cap B$. We define an element $A_V \in G_V := exp(\prod_{\gamma \in \Lambda \cap C(V)} \mathbb{Q} \cdot e_\gamma)$,

$$A_V := \prod_{\gamma \in C(V) \cap \Lambda}^{\longrightarrow} \exp \left(\Omega(\gamma) \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right),$$

where the product is taken into a clockwise order, and G_V is considered as a pronilpotent group.

A generic path $Z_t, 0 \leq t \leq 1$ in the space of the above additive maps Z intersects the wall of first kind at $t = t_0$ for which there is a lattice $\Lambda_0 \subset \Lambda$ of rank two such that $Z_{t_0}(\Lambda_0)$ belongs to a real line $\mathbb{R} \cdot e^{i\alpha} \subset \mathbb{C}$.

The wall-crossing formula describes change of the values of $\Omega(\gamma)$ for $\gamma \in \Lambda_0$. It depends on the two-dimensional lattice Λ_0 only. Denote by $k \in \mathbb{Z}$ the value of the form $\langle \bullet, \bullet \rangle$ on a fixed basis of $\Lambda_0 \simeq \mathbb{Z}^2$. We assume that $k \neq 0$, otherwise there will be no jump in values of Ω on Λ_0. Let us consider the pronilpotent group generated by products of the following formal symplectomorphisms (automorphisms of $\mathbb{Q}[[x,y]]$ preserving the symplectic form $(xy)^{-1}dx \wedge dy$):

$$T_{a,b} : (x,y) \mapsto \left(x \cdot (1-(-1)^{ab}x^a y^b)^b, y \cdot (1-(-1)^{ab}x^a y^b)^{-a}\right), a,b \geq 0, a+b \geq 1.$$

Any exact symplectomorphism ϕ of $\mathbb{Q}[[x,y]]$ can be decomposed uniquely into a clockwise and an anti-clockwise product which gives a wall-crossing formula:

$$\phi = \overrightarrow{\prod_{a,b}} T_{a,b}^{kc_{a,b}} = \overleftarrow{\prod_{a,b}} T_{a,b}^{kd_{a,b}}$$

with certain exponents $c_{a,b}, d_{a,b} \in \mathbb{Q}$. These exponents play a role of Donaldson-Thomas invariants. The passage from the clockwise order (when the slope $a/b \in [0,+\infty] \cap \mathbb{P}^1(\mathbb{Q})$ decreases) to the anti-clockwise order (the slope increases) gives the change of $\Omega_{|\Lambda_0}$ as we cross the wall

In order to explain all these formulas we define the notion of motivic Hall algebra. It is an algebra over the ring of motivic functions on the space of objects of $3d$ Calabi-Yau category. In order to do this we assume that the space of objects is a countable union of constructible sets (we call such categories *ind-constructible*). The motivic analog of the element A_V (we call them *motivic Donaldson-Thomas invariants*) lives naturally in the motivic Hall algebra. The above elements A_V are obtained in a certain "quasi-classical limit" of the motivic DT-invariants.

Phenomena of tropical flavor in Lagrangian Floer theory

Fukaya Kenji

We consider a symplectic $2n$ manifold M which has a Hamiltonian action of T^n with moment map $\pi : M \to B$. We assume the following for $\pi^{-1}(v) = T^n_v$.

For any nonconstant pseudo-holomorphic disc $u : (D2, \partial D2) \to (M, T^n_v)$, its Maslov index $\eta_{T^n_v}([u])$ is not smaller than 2.

Other case we can study is a Lagrangian submanifold in Calabi-Yau three fold with vanishing Maslov index.

In the first case we have a map

$$\Phi(x_1, \cdots, x_n : v_1, \cdots, v_n)$$

where $(x_1, \cdots, x_n) \in H1(T^n_v; \Lambda_{0,nov})$ and $v = (v_1, \cdots, v_n) \in B$. Φ is called potential function in [2] and is basically the same thing as what is called superpotential by physisists. Here $\Lambda_{0,nov}$ is the universal Novikov ring, which consists of a formal serise $\sum a_i T^{\lambda_i}$, with $\lambda_i \geq 0$ converges to infinity as $i \to \infty$. (We do not use formal parameter e in [2] here.)

In [1], $\Phi(0, \cdots, 0 : v_1, \cdots, v_n)$ is calculated in many cases. (There are earier work by [4].) By a minor modification of their results, we can calculate Φ.

In general, we can show that Φ is of the form

$$\sum_i eT^{f_i(v)} \prod_{j=1}^{n} y_i^{\partial f_i/\partial v_i}. \tag{1}$$

Here f_i are affine functions whose partial derivative $\partial f_i/\partial v_i$ are integers and $y_i = e^{x_i}$.

If

$$\frac{\partial \Phi}{\partial y_i} = 0, \qquad i = 1, \cdots, n \tag{2}$$

has a solution $y = (y_1, \cdots, y_n)$ (for some $v = (v_1, \cdots, v_n)$), then the Floer homology

$$HF((T_v^n, b_y), (T_v^n, b_y))$$

does not vanish. Here b_y is the bounding chain in the sense of [2] that is $b_y = (\log y_1, \cdots, \log y_n) \in H1(T_v; \Lambda_{0,nov})$.

Equation (2) can be studied by an argument with a flavor of tropical geometry. It implies that there exists at least one v where (2) has a solution. Floer theory implies that such T_v^n is not displacable. Namely for any Hamiltonian diffeomorphism $\varphi : M \to M$, the intersection $\varphi(T_v^n) \cap T_v^n$ is nonempty.

This conclusion is closely related to the earlier work by [3].

In the case M is Calabi-Yau and Maslov index of $L \subset M$ vanishes we can study a similar superpotential and a similar structure theorem as (1) can be proved.

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Some application of tropical geometry to mirror symmetry

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(joint work with Denis Auroux, Ludmil Katzarkov)

The best understood geometric procedure for realizing mirror symmetry relies on the SYZ conjecture: given a manifold with a singular special Lagrangian torus fibration over a base B, the mirror manifold is expected to be the “dual” torus fibration over the same base; moreover, the mirror comes equipped with a holomorphic complex-valued superpotential W which counts in an appropriate way the number of holomorphic discs with boundary on every fibre of the original fibration.

Starting with $\mathbb{C} \times \mathbb{C}^*$ together with the fibration

$$\mathbb{C} \times \mathbb{C}^* \to \mathbb{R}^+ \times \mathbb{R} \tag{1}$$

$$(a, b) \mapsto (2\pi|a|^2, \log|b|), \tag{2}$$

it is easy to compute that the mirror is $U \times \mathbb{C}^*$, where $U = \{z | 0 < |z| \leq 1\} \subset \mathbb{C}$, and the superpotential is the coordinate z on U. The torus fibration is simply a product map

$$U \times \mathbb{C} \to \mathbb{R}^+ \times \mathbb{R} \tag{3}$$

$$(z, q) \mapsto (\log|z|, \log|w|). \tag{4}$$

This is one of the most basic toric situations, whose compact case has been studied in considerable detail in [2, 3, 4]. A simple way to extend this construction beyond the toric case is to consider an ϵ-symplectic blow-up of $\mathbb{C} \times \mathbb{C}^*$ at $(0, 1)$; there is no induced toric structure on the blow up since $(0, 1)$ is not a fixed point of the torus action. Nonetheless, this space admits a singular Lagrangian fibration over a base B which can be obtained as follows:

Start with $\mathbb{R}^+ \times \mathbb{R}$, remove the triangle with vertices $(\epsilon, 0)$, $(0, 0)$, and $(0, \epsilon)$, and glue its non-horizontal boundaries together along the affine map

$$(u_1, u_2) \mapsto (u_1 - u_2 + \epsilon, u_2). \tag{5}$$

The resulting space B therefore inherits an integral affine structure away from the points $(0, \epsilon)$, which define a smooth Lagragian torus fibration in this region. It is well known that this fibration can be extended to $(0, \epsilon)$, where the fibre becomes a "focus-focus" singularity. Topologically, the fibre is homeomorphic to a nodal elliptic curve; the result of collapsing an essential curve on the torus to a point.

To study mirror symmetry for this manifold, we consider the two region B_+ and B_- consisting of points in B whose first coordinate is respectively positive and negative. It is easy to see that the mirrors of the regions lying over $B_\pm$ can both be thought of as domains $\Omega_\pm$ in $\mathbb{C}^* \times \mathbb{C}^*$, with coordinates, say (z_1, x_1) and (z_2, x_2). To glue these two regions together, we use the map on Floer complexes of the fibers induced by a path with endpoints in B_- and B_+. By taking into account the behaviour of putative holomorphic discs with boundary on the torus, tropical geometry suggests (and one can readily prove) that the correct identification is

$$z_2 = z_1 \tag{6}$$

$$x_2 = x_1^{-1}(1 + e^{\epsilon} z_1). \tag{7}$$

This is by now a familiar expression in mirror symmetry, see, for example [6, 5].

The above identification yields a smooth mirror manifold (it's a graph over the (x_1, x_2) plane), with superpotential $e^{-\epsilon}(x_1 x_2 - 1)$. In particular, the origin of the (x_1, x_2) plane is the unique critical point, and should be thought of as the mirror to the zero dimensional Lagrangian immersion $S^0 \to \{pt\}$.

Motivated by an attempt to understand the mirrors to higher genus curves, we extend this construction to complex curves

$$C \times \{0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}. \tag{8}$$

The key idea is to exploit the fact that the symplectic topology of such a curve is independent of its complex modulus, by considering a tropical degeneration of the curve C and its image in $\mathbb{R}^2$ which we denote $\mathcal{A}$ (for amoeba). In the limit, the amoeba converges to a graph in $\mathbb{R}^2$, whose edges correspond to pairs of pants in C which are connected, along the edges, by flat cylinders. Since the SYZ construction is completely local in this simple situation (including all "quantum correction"), it is sufficient to construct the mirror of the blow up of $(\mathbb{C}^*)^2 \times \mathbb{C}$ along a curve C which is now assumed to be a generic hyperplane in $(\mathbb{C}^*)^2$. In the tropical limit, the amoeba of such a hyperplane converges to a tree consisting of three edges meeting at the origin. Moreover, away from the origin, and in a neighbourhood of each edge, there is a choice of coordinate on $(\mathbb{C}^*)^2$ such that C is locally symplectomorphic to a product hypersurface $\{pt\} \times \mathbb{C}^*$.

Upon blowing up $(\mathbb{C}^*)^2 \times \mathbb{C}$ along C, we can produce a singular torus fibration on a base B whose singular locus can be identified with

$$\mathcal{A} \times \{\epsilon\} \subset \mathbb{R}^2 \times \{\epsilon\}. \tag{9}$$

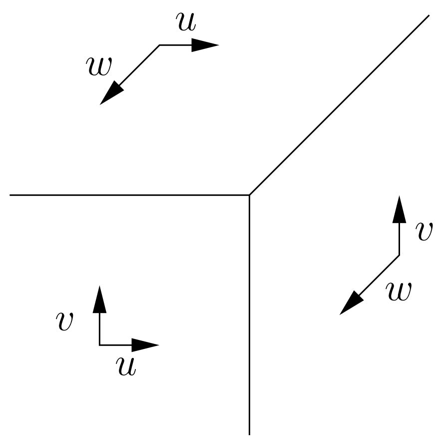

FIGURE 1

The affine structure on the base B is more difficult to describe, but one can still identify regions which correspond to the complement of the trivalent graph, and on whose mirrors one can establish linear coordinates. Further, the fact that we have a product decomposition along each edge implies that the monodromy acts trivially on directions which are tangent to the edge. In particular, we can set up our coordinate system with a global z coordinate, and the other coordinates as in Figure 1.

Using the product decomposition again, it is easy to check that the result of gluing the mirrors of all three regions is the variety

$$\{(z, u, v, w) | uvw = (1 + e^{\epsilon} z)\} \tag{10}$$

equipped with the superpotential z. Observe that the mirror manifold is $\mathbb{C}^3$ and that the critical locus of the superpotential is a union of three lines. One can easily check that $\mathbb{C}^3$ equipped with this superpotential is indeed the complex mirror of the pair of pants by identifying the Fukaya category of the pair of pants (in particular, the version introduced in [1]) with the category of matrix factorizations of the Landau-Ginzburg mirror. In work in progress with Paul Seidel, we extend the machinery developed in [1] to verify that this is in fact true for any curve $C \in (\mathbb{C}^*)^2$.

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Computing tropical varieties

Anders Nedergaard Jensen
(joint work with Tristram Bogart, Komei Fukuda, David Speyer, Bernd Sturmfels, Rekha Thomas)

The title of this talk coincides with the one of [1] where we presented algorithms for computing tropical varieties. In this talk we will discuss these algorithms and demonstrate the software Gfan [4] which contains an implementation of them. The algorithms are no longer new but remain relevant as Gfan is an important research tool for many of the people in the audience.

By a tropical variety we mean the closure in $\mathbb{R}^n$ of the image under coordinatewise valuation of a variety in $(\mathbb{C}\{\{t\}\}^*)^n$ defined by an ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$. Here $\mathbb{C}\{\{t\}\}$ denotes the field of Puiseux series. For simplicity we assume that I is homogeneous. In [6] Speyer and Sturmfels gave an equivalent definition in terms of initial ideals, namely they defined the tropical variety of I as the set of vectors $\omega \in \mathbb{R}^n$ such that the initial ideal $\text{in}_\omega(I)$ does not contain a monomial. This definition is useful for computations as it describes the tropical variety of an ideal in terms of Gröbner cones: We consider two vectors $u, v \in \mathbb{R}^n$ to be equivalent if $\text{in}_u(I) = \text{in}_v(I)$ and call the closure of an equivalence class a Gröbner cone of I. The collection of all Gröbner cones is a polyhedral fan called the Gröbner fan of I. It was first studied in [5]. Since the tropical variety defined by I is a union of Gröbner cones it makes sense to equip the tropical variety with a polyhedral structure. We let $T(I)$ denote the subfan of the Gröbner fan consisting of all cones contained in the tropical variety. Our goal is to compute $T(I)$.

The Gröbner fan of I can be computed by applying the Gröbner walk [2] to traverse its maximal cones; see [3]. Now a naive method for computing $T(I)$ is to check for every Gröbner cone if its initial ideal is monomial-free. In general a better method is needed since the Gröbner fan typically is much more complicated than $T(I)$. In [1] we gave a proof of connectedness of $T(I)$ by ridge paths in case of I being a prime ideal (over $\mathbb{C}$). Given a maximal cone in $T(I)$ as input this property can be exploited to compute $T(I)$ without computing the entire Gröbner fan.

The tropical variety of I can also be defined as the intersection of all tropical hypersurfaces of polynomials in I. A priori, this is an infinite intersection of tropical hypersurfaces. A finite generating set of I whose tropical hypersurfaces cut out the tropical variety of I is called a tropical basis. In [1] it was proved constructively that a tropical basis always exists. We also presented a better algorithm for computing such a basis in the case of a tropical curve. This algorithm serves as an important subroutine in the traversal algorithm for prime ideals described above.

Many ideals possess symmetries which can be exploited in an enumeration of the cones in $T(I)$. Which tropical varieties can be computed in practise depends not only on the number of orbits with respect to the symmetries but also on the complexity of the Gröbner bases of the ideal. We have computed tropical varieties with thousands of maximal cones, but have also found our methods to be infeasible for some small examples.

It is common to consider tropical varieties of ideals in $\mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$. In this case a tropical variety needs not be a polyhedral fan but may consist of polyhedra which are not cones. However, if the ideal is defined by polynomials in $\overline{\mathbb{Q}}(t)[x_1, \ldots, x_n]$ its tropical variety can be gotten by slicing $T(J)$ with an affine hyperplane where J is a suitable ideal in $\mathbb{Q}[x_1, \ldots, x_m]$. Thus, in most cases it is no real restriction to consider only ideals in the polynomial ring over $\mathbb{Q}$.

Our implementation Gfan is a collection of command line programs written in C++ using the libraries GMP and cddlib for arithmetic and polyhedral computations respectively. Gfan runs on UNIX-like systems such as GNU/Linux and MacOS X.

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Tropical duality of cluster varieties

Vladimir Fock
(joint work with Aleksandr Goncharov)

Cluster variety is an algebraic variety (strictly speaking, a scheme) defined by combinatorial data by explicit set of coordinate charts and transition functions. More precisely, for any collection of combinatorial data, called *seed* one associates three varieties $\mathcal{A}_{|\mathbb{I}|}$, $\mathcal{X}_{|\mathbb{I}|}$, and $\mathcal{D}_{|\mathbb{I}|}$. These varieties possess canonical pre-symplectic, Poisson and symplectic structures, respectively. One defines also a discrete group $\mathfrak{D}_{|\mathbb{I}|}$ acting on all the three types of varieties and preserving the respective structures. The manifolds $\mathcal{X}_{|\mathbb{I}|}$ and $\mathcal{D}_{|\mathbb{I}|}$ admit a quantisation (noncommutative deformation of the algebra of functions) which is also $\mathfrak{D}_{|\mathbb{I}|}$-invariant.

Varieties admitting cluster descriptions are simple Lie groups, moduli spaces of Stokes parameters, moduli of flat connections on Riemann surfaces, configuration spaces of flags, Teichmüller spaces and their generalisations, the spaces of measured laminations and some others. One of the important features of cluster varieties is that they are defined not only over a field but also over semifields (semigroups w.r.t. addition and groups w.r.t. the multiplication). For example, one can consider Teichmüller space, space of measured laminations and the space of flat $PSL(2,\mathbb{F})$-connections over a surface Σ as the same cluster manifold but defined over the semifield $\mathbb{R}_{>0}$ of positive real numbers, tropical semifield $\mathbb{R}^t$ (which is ordinary R as a set with maximum for the addition operation and ordinary addition for the multiplication), and a field $\mathbb{F}$, respectively.

Let us give the precise definitions:

A *cluster seed*, or just *seed*, $\mathbf{I}$ is a quadruple (I, I_0, ε, d), where

i) I is a finite set;

ii) $I_0 \subset I$ is its subset;

iii) ε is a matrix ε_{ij}, where $i, j \in I$, such that $\varepsilon_{ij} \in \mathbb{Z}$ unless $i, j \in I_0$.

iv) $d = \{d_i\}$, where $i \in I$, is a set of positive integers, such that the matrix $\widehat{\varepsilon}_{ij} = \varepsilon_{ij} d_j$ is skew-symmetric.

The elements of the set I are called *vertices*, the elements of I_0 are called *frozen vertices*. The matrix ε is called *exchange matrix*, the numbers $\{d_i\}$ are called *multipliers*, and the function d on I whose value at i is d_i is called *multiplier function*. We omit $\{d_i\}$ if all of them are equal to one, and therefore the matrix ε is skew-symmetric, and we omit the set I_0 if it is empty.

An isomorphism σ between two seeds is a map $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ is an isomorphism of finite sets $\sigma : I \to I'$ such that $\sigma(I_0) = I'_0$, $d_{\sigma(i)} = d_i$ and $\varepsilon_{\sigma_i,\sigma_j} = \varepsilon_{ij}$. Observe that the automorphism group of a seed may be nontrivial.

For a seed $\mathbf{I}$ we associate a torus $\mathcal{X}_{\mathbf{I}} = (\mathbb{F}^\times)^I$, called $\mathcal{X}$-*torus*, another torus $\mathcal{A}_{\mathbf{I}} = (\mathbb{F}^\times)^I$, called $\mathcal{X}$-*torus* and the third one $\mathcal{D}_{\mathbf{I}} = (\mathbb{F}^\times)^{I \times I}$ called $\mathcal{D}$-*torus* or a *double torus*. We denote the standard coordinates on these tori by $\{x_i | i \in I\}$, $\{a_i | i \in I\}$ and $\{y_i, b_i | i \in I\}$, respectively.

The $\mathcal{X}$-torus is equipped with the Poisson structure

$$\{x_i, x_j\} = \widehat{\varepsilon}_{ij} x_i x_j \tag{1}$$

The $\mathcal{A}$-torus is equipped with the pre-symplectic structure (closed 2-form ω possibly degenerate)

$$\omega = \frac{1}{2}\sum_{i,j} \widehat{\varepsilon}_{ij} \frac{da_i \wedge da_j}{a_i a_j} \tag{2}$$

The $\mathcal{D}$-torus is equipped with the symplectic form

$$\omega_{\mathcal{D}} = \frac{1}{2}\sum_{i,j} \widehat{\varepsilon}_{ij} \frac{db_i \wedge db_j}{b_i b_j} + \sum_i d_i^{-1} \frac{db_i \wedge dy_i}{b_i y_i} \tag{3}$$

The inverse of this form is a nondegenerate Poisson structure which can be written as

$$\{y_i, y_j\} = \widehat{\varepsilon}_{ij} y_i y_j, \quad \{y_i, b_j\} = \delta^i_j d^i y_i b_j, \quad \{b_i, b_j\} = 0 \tag{4}$$

Observe that these sructures are constant in logarithmic coordinates.

Isomorphism between two $\mathcal{X}$-tori $\mathcal{X}_{\mathbb{I}}$ and $\mathcal{X}_{\mathbb{I}'}$ is a map given in coordinates by $x_{\sigma(i)} = x_i$, where σ is an isomorphism of the seeds. Observe that there are much less isomorphisms of $\mathcal{X}$-tori then just isomorphisms of the corresponding Poisson manifolds. Isomorphisms of $\mathcal{A}$- and $\mathcal{D}$-tori are defined analogously.

There exist the following maps between the tori:

$$\mathcal{A}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = \prod_j a_j^{\varepsilon_{ij}}; \tag{5}$$

$$\mathcal{A}_{\mathbf{I}} \times \mathcal{A}_{\mathbf{I}} \to \mathcal{D}_{\mathbf{I}}, \quad y_i = \prod_j a_j^{\varepsilon_{ij}}, \quad b_i = a_i/\tilde{a}_i, \tag{6}$$

Here $\tilde{a}_i$ are coordinates on the second $\mathcal{A}_{\mathbf{I}}$-factor.

$$\mathcal{D}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = y_i, \tag{7}$$

and

$$\mathcal{D}_{\mathbf{I}} \to \mathcal{X}_{\mathbf{I}}, \quad x_i = y_i \prod_j b_j^{\varepsilon_{ij}}. \tag{8}$$

All the maps are compatible with the respective symplectic, pre-symplectic and Poisson structures. Namely the map (5) is a composition of the quotient by the kernel of the pre-symplectic form and a symplectic map to a symplectic leaf. The map (6) maps the symplectic form to the pre-symplectic one. The map (7) is Poisson, the map (8) is anti-Poisson (Poisson with the opposite Poisson structure on the $\mathcal{X}$-torus). The maps (7) and (8) are dual to each other in the sense on Poisson pairs.

Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, and $k \in I - I_0$. A *mutation in the vertex* k is an isomorphism $\mu_k : I \to I'$ satisfying the following conditions:

(1) $\mu_k(I_0) = I_0'$,
(2) $d'_{\mu_k(i)} = d_i$,
(3) $\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i=k \text{ or } j=k \text{ otherwise} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} < 0 \\ \varepsilon_{ij} + \varepsilon_{ik}|\varepsilon_{kj}| & \text{if } \varepsilon_{ik}\varepsilon_{kj} \geq 0 \end{cases}$

Two seeds related by a sequence of mutations are called equivalent.

Mutations induce rational maps between the corresponding seed tori, which are denoted by the same symbol μ_k and are given by the formulae

$$x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if} \quad i = k \\ x_i(1+x_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geq 0 \\ x_i(1+(x_k)^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \leq 0 \end{cases}.$$

for the $\mathcal{X}$-torus,

$$a_{\mu_k(i)} = \begin{cases} \dfrac{\prod_{j|\varepsilon_{jk}>0} a_j^{\varepsilon_{jk}} + \prod_{j|\varepsilon_{jk}<0} a_j^{-\varepsilon_{jk}}}{a_k} & \text{if } i = k \\ a_i & \text{if } i \neq k \end{cases}$$

for the $\mathcal{A}$-torus and

$$b_{\mu_k(i)} = \begin{cases} \dfrac{(1+x_k)^{-1} \prod_{j|\varepsilon_{jk}>0} b_j^{\varepsilon_{jk}} + (1+(x_k)^{-1})^{-1} \prod_{j|\varepsilon_{jk}<0} b_j^{-\varepsilon_{jk}}}{b_k} & \text{if } i = k \\ b_i & \text{if } i \neq k \end{cases}$$

$$y_{\mu_k(i)} = \begin{cases} y_k^{-1} & \text{if} \quad i = k \\ y_i(1+y_k)^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \geq 0 \\ y_i(1+(y_k)^{-1})^{\varepsilon_{ik}} & \text{if} \quad \varepsilon_{ik} \leq 0 \end{cases}.$$

for the $\mathcal{D}$-torus.

Since in the sequel we shall extensively use compositions of mutations called also *cluster transformations* we would like to introduce a shorthand notation for them. Namely, we denote an expression $\mu_{\mu_i(j)}\mu_i$ by $\mu_j\mu_k$, $\mu_{\mu_{\mu_i(j)}\mu_i(k)}\mu_{\mu_i(j)}\mu_i$ by $\mu_k\mu_j\mu_i$, and so on.

Mutations have the following properties (valid for mutation of seeds as well as for mutations of respective tori):

- Every seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ seed is related to other seeds by exactly $\sharp(I - I_0)$ mutations.

A_1: $\mu_i\mu_i = id$

$A_1 \times A_1$ If $\varepsilon_{ij} = \varepsilon_{ji} = 0$ then $\mu_i\mu_j\mu_j\mu_i = id$.

A_2: If $\varepsilon_{ij} = -\varepsilon_{ji} = -1$ then $\mu_i\mu_j\mu_i\mu_j\mu_i = id$. (This is called the *pentagon relation.*)

B_2: If $\varepsilon_{ij} = -2\varepsilon_{ji} = -2$ then $\mu_i\mu_j\mu_i\mu_j\mu_i\mu_j = id$.

G_2: If $\varepsilon_{ij} = -3\varepsilon_{ji} = -3$ then $\mu_i\mu_j\mu_i\mu_j\mu_i\mu_j\mu_i = id$.

By *id* we mean here an isomorphism of the seeds or tori. Conjecturally all relations between mutation follow from these ones.

Given a seed one can produce a $\sharp(I - I_0)$ seeds by mutations. Continuing this procedure one obtains a $\sharp(I - I_0)$-valent tree whose vertices are seeds (or seed tori) and edges are pairs of mutually inverse mutations. Obviously if we start from any other seed from the tree we obtain the same tree. Every two tori of the tree are related by exactly one composition of mutations. Call two points of two different tori equivalent if they are related by the composition of mutations. The cluster manifold (denoted by $\mathcal{X}_{|\mathbf{I}|}$, $\mathcal{A}_{|\mathbf{I}|}$ or $\mathcal{D}_{|\mathbf{I}|}$ depending on which kind of tori are used) is the affine closure of disjoint union of the tori quotiented by the equivalence relation.

Each particular seed tori can be considered as a coordinate chart of the corresponding cluster manifolds and compositions of mutations can be considered as transition functions between the charts.

Mutations respect the Poisson structure when acting on $\mathcal{X}$ tori, pre-symplectic structure when acting on $\mathcal{A}$-tori and symplectic when acting on $\mathcal{D}$-tori. Thus the cluster manifolds $\mathcal{X}_{|\mathbf{I}|}$, $\mathcal{A}_{|\mathbf{I}|}$ and $\mathcal{D}_{|\mathbf{I}|}$ acquire the respective structures. (In fact the formula for mutation of the matrix ε can be considered as a corollary of this property and the mutation formulae for, say, $\mathcal{X}$-tori).

Mutations commute with the maps (5),(6),(7) and (8) thus these maps are defined between the respective cluster varieties compatible with pre-symplectic, symplectic and Poisson structures thereof.

Mutations are rational maps with positive integral coefficients and thus the cluster manifold can be defined not only over a field but over any semifield as well. For semifields without -1 (like the semifields of positive real numbers or the tropical semifields) the mutations are isomorphisms and thus the whole manifold is isomorphic to every coordinate torus.

The symmetry group $\mathfrak{D}_{|\mathbf{I}|}$ of a cluster manifold permuting the seed tori is called the (generalised) *mapping class group* of the cluster manifold. The name comes from the case of Teichmüller space, when this group is the actual mapping class group. The group depends on the equivalence class of a seed only and is common for cluster manifolds of types $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{D}$. Every sequence of mutations together with an isomorphism of the initial and the final seed gives an element of the mapping class group. Conversely, given a seed, every mapping class group element can be presented by a sequence of mutations starting from the given seed together with the isomorphism between the final seed and the initial one. Two sequences of mutations different by the relations A_1–G_2 correspond to the same mapping class group elements.

Consider the ring of algebraic functions on a cluster manifold in more details. The ring of algebraic functions $\mathcal{O}(\mathcal{X}_\mathbf{I})$ (resp. $\mathcal{O}(\mathcal{D}_\mathbf{I})$, $\mathcal{O}(\mathcal{A}_\mathbf{I})$) on every torus is the ring of Laurent polynomials of cluster variables. This ring contains a subring of Laurent polynomials with integral coefficients $\mathcal{O}^\mathbb{Z}$ and a semiring of Laurent polynomials with positive integral coefficients $\mathcal{O}^\mathbb{Z}_{>0}$ also depending of course of the seed and of the type of the torus. A cluster transformation in general does not presereve the ring $\mathcal{O}$ since it is birational. The ring of algebraic functions on the whole cluster manifold is the intersection of inverse images of the rings $\mathcal{O}$ under all possible cluster transformations of a seed tori. In other words the ring $\mathcal{O}$ consists of Laurent polynomials which stay Laurent under all possible cluster transformations. The celebrated result of Fomin and Zelevinsky called *Laurent phenomenon* claims that for any cluster variety $\mathcal{A}_{|\mathbf{I}|}$ of type $\mathcal{A}$ the coordinate functions belong to the ring $\mathcal{O}^{\mathbb{Z}}$. The ring $\mathcal{O}$ contains a subring $\mathcal{O}^{\mathbb{Z}}$ and a subsemiring $\mathcal{O}^{\mathbb{Z}}_{>0}$. The latter is additively generated by Laurent polynomials from $\mathcal{O}$ with positive integral coefficients indecomposable into a sum of two such polynomials. Such elements of the semiring $\mathcal{O}^{\mathbb{Z}}_{>0}$ are called *irreducibles.* The main conjecture, proven for a sufficiently wide class of cluster manifolds describes the structure of the set of irreducible Laurent polynomials:

(1) The set of irreducible Laurent polynomials is a basis in the ring $\mathcal{O}$.
(2) The set of irreducible Laurent polynomials for the cluster variety $\mathcal{X}_{|\mathbf{J}|}$ (resp. $\mathcal{A}_{|\mathbf{I}|}$, $\mathcal{D}_{|\mathbf{I}|}$) is canonically isomorphic to the set of points of the cluster variety $\mathcal{A}_{|\mathbf{I}|}(\mathbb{Z}^t)$ (resp. $\mathcal{X}_{|\mathbf{I}|}(\mathbb{Z}^t)$, $\mathcal{D}_{|\mathbf{I}|}(\mathbb{Z}^t)$).

One can consider this property as a duality between cluster varieties of type $\mathcal{X}$ (resp. $\mathcal{A}$, $\mathcal{D}$) and the tropical cluster varieties of type $\mathcal{A}$, $\mathcal{X}$ and $\mathcal{D}$, respectively.

The correspondence between irreducible Laurent polynomials and points of the dual tropical variety is especially simple for the variety of type $\mathcal{X}$. In this case the coordinates of the corresponding point of the tropical variety are given by multidegree of the highest term of the corresponding Laurent polynomial.

Example.

Let us consider the simplest nontrivial example: the seed $\mathbf{I} = \{I, \varepsilon\}$ with $I = \{1, 2\}$ and $\varepsilon_{12} = 1$. There are exactly 5 isomorphism classes of seed tori equivalent to a given one, however all the five seeds are isomorphic, thus the mapping class group is $\mathbb{Z}/5\mathbb{Z}$.

The simplest geometric meaning has the space $\mathcal{X}$. It is the space of 5-tuples of points $(p_1, \dots, p_5)$ on the projective line P^1 such that $p_i \neq p_{i+1 \text{ (mod 5)}}$ and modulo the automorphisms of P^1. The 5-tuple of coordinate systems on this space is numerated by triangulations of the pentagon with vertices $1, \dots, 5$. For every internal diagonal one associates the cross-ratio of the four points of the quadrilateral which this diagonal cuts into halves. Mutations correspond to removing a diagonal and replacing it by another one of the quadrilateral. The same variety over $\mathbb{R}_{>0}$ is the configuration space of 5-tuples of points on $\mathbb{R}P^1$ with prescribed cyclic order.

The $\mathcal{A}$-space is the space of collections of 10 nonvanishing vectors $v_1, \dots, v_{10}$ in $\mathbb{F}^2$ equipped with a nonzero bivector Vol. The collections are considered up to the action of the group $SL(2, \mathbb{F})$ of linear transformations preserving Vol and subject to the relations $v_i = -v_{i+5 \text{ (mod 10)}}$ and $v_i \wedge v_{i+1 \text{ (mod 10)}} = Vol$. The map $\mathcal{X}_{|\mathbf{I}|} \to \mathcal{X}_{|\mathbf{I}|}$ is given by the obvious projection of $\mathbb{F}^2 - \{0\} \to P^1$. For the internal diagonal of the pentagon with ends i and j one associates the coordinate $v_i \wedge v_j / Vol$.

The $\mathcal{D}$ variety is the space of flat $SL(2,\mathbb{F})$ connections on a sphere with 5 different points on the equator removed with parabolic monodromy around these points. Consider the associated vector bundle and choose a monodromy invariant section about each singular points. Then trivialise the bundle over the northern hemisphere. The five chosen sections give five vectors $v_1,\dots,v_5$ in $\mathbb{F}^2$. The same procedure over the southern hemisphere gives five vectors $w_1,\dots,w_5$ in another copy of $\mathbb{F}^2$. Given a triangulation of the pentagon we associate to every internal diagonal two coordinates x and b. The coordinate x is just the cross ratio of four points in P^1 defined by the vectors v_i standing at the corners of the quadrilateral cut by the diagonal (just like for the $\mathcal{X}$-space). The coordinate b is given by $b = (v_i \wedge v_j)/(w_i \wedge w_j)$, where i and j are the ends of our diagonal. The two projections to the $\mathcal{X}$ variety are obviously given by projectivising the collections of vectors $\{v_i\}$ and $\{w_i\}$, respectively. The same manifold over $\mathbb{R}_{>0}$ can be identified with the space of complex structures on a sphere with five punctures on the equator.

Given a triangulation of the pentagon one can describe the basis of the ring $\mathcal{O}^{\mathbb{Z}}$ of the corresponding $\mathcal{X}$-variety explicitly as a set of Laurent polynomials $P_{\mathsf{a},\mathsf{b}}(x,y)$ of two variables x,y parameterised by two integers a,b as follows:

$$P_{\mathsf{a},\mathsf{b}}(x,y) = \begin{cases} x^{\mathsf{a}}y^{\mathsf{b}} & \text{if} \quad \mathsf{a}\le 0, \mathsf{b}\le 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+x^{-1})^{-\mathsf{b}} & \text{if} \quad \mathsf{a}\le 0, \mathsf{b}\ge 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+x^{-1})^{-\mathsf{b}}(1+y^{-1}+x^{-1}y^{-1})^{\mathsf{a}} & \text{if} \quad \mathsf{a}\ge 0, \mathsf{b}\le 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+y^{-1})(1+y^{-1}+x^{-1}y^{-1})^{\mathsf{a}-\mathsf{b}} & \text{if} \quad \mathsf{a}\ge \mathsf{b}\ge 0 \\ x^{\mathsf{a}}y^{\mathsf{b}}(1+y^{-1})^{\mathsf{a}} & \text{if} \quad \mathsf{b}\ge \mathsf{a}\ge 0 \end{cases}$$

One can easily check that this set of Laurent polynomials is invariant under simultaneous mutation of the variables x,y and of the variables a,b.

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Applications of tropical geometry to groups and manifolds

Stephan Tillmann

A tropical variety is a set fashioned from valuations. Applying this point of view, the following are tropical varieties: Bergman's logarithmic limit set; the geometric invariants of groups by Bieri, Neumann and Strebel; and the set of ideal points of an affine algebraic set in the sense of Morgan and Shalen. (The geometric invariants of groups are in fact the complements of tropical varieties; should they be termed temperate varieties?)

To motivate the applications of tropical geometry to groups and manifolds, I will describe how valuations arise in the analysis of deformations of a hyperbolic manifold M. All valuations considered are real-valued and of rank one. The key observation is that valuations associated to ideal points of certain affine algebraic sets —arising from linear representations of the fundamental group of M, $\pi_1(M)$— encode information about actions of $\pi_1(M)$ on certain trees. Constructions by Culler, Morgan and Shalen have formalised this relationship and form part of the standard repertoire in the field of low dimensional topology, see [3, 12, 14, 15, 16]. In this setting, a valuation can be thought of as the length function of an action on an $\mathbb{R}$–tree. Dual to such an action is a geometric object in M (such as an essential codimension-one submanifold or a transversely measured essential lamination) and some information about this object can be determined from properties of the valuation. This interplay has been used successfully to prove many results concerning the topology and geometry of 3–manifolds; see, for instance, [1, 9, 11, 17].

Using Bergman's logarithmic limit set and ideal hyperbolic triangulations, a variant of the constructions of Culler, Morgan and Shalen is given in [18] which, combined with work of Bogart, Jensen, Speyer, Sturmfels and Thomas [8], leads to new algorithms in the study of ideally triangulated 3–manifolds.

Equivalent definitions of, and facts about, Bergman's logarithmic limit set from [2, 4] are then recalled. The geometric invariant of groups, $\widehat{\Sigma}$, due to Bieri and Strebel [7, 6] can now be introduced as the *complement* of a generalisation of this tropical variety, since the complement of $\widehat{\Sigma}$ is understood via ring-theoretic valuations in a formally analogous way to one of the definitions of the logarithmic limit set. The exposition then changes gear and the geometric invariant, Σ, of groups due to Bieri, Neumann and Strebel [5] is defined as a generalisation of $\widehat{\Sigma}$ and key applications are given. The complement of Σ is understood via group-theoretic valuations due to work by Brown [10]. Last, computational aspects of the geometric invariants of groups using Gröbner basis methods are discussed.

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Floor decomposition of plane tropical curves and Caporaso-Harris type formulas

Erwan Brugallé
(joint work with Lucia Lopez de Medrano, Grigory Mikhalkin)

Let $d \geq 1$ and ω a generic configuration of $3d-1$ points in $\mathbb{C}P^2$. Consider the set $\mathcal{C}$ of algebraic irreducible rational curves of degree d in $\mathbb{C}P^2$ passing through all points of ω. The set $\mathcal{C}$ is finite, and if d is fixed, then the cardinal of $\mathcal{C}$ does not depend on ω. The numbers $N(d) = \sharp\mathcal{C}$ are known as genus 0 Gromov-Witten invariants of $\mathbb{C}P^2$. These invariants were first computed by Kontsevich, and Caporaso and Harris gave another formula to compute these numbers.

When ω is a real configuration of points (i.e. when $conj(\omega) = \omega$), then it is natural to consider the set

$$\mathbb{R}\mathcal{C} = \{real\ algebraic\ curves\ in\ \mathcal{C}\}$$

Now the cardinal of $\mathbb{R}\mathcal{C}$ depends on ω. However, Welschinger proved that counting curves in $\mathbb{R}\mathcal{C}$ with respect to some sign, one obtain a number which depends only on d and r, where r is the number of pairs of complex conjugate points in ω. We denote by $W(d,r)$ this invariant. The numbers $W(d,0)$ were first computed by Itenberg, Kharlamov and Shustin.

Thanks to Mikhalkin's Correspondence Theorems, and their improvement in the case $r \neq 0$ by Shustin, previous invariants can be computed tropically.

Jointly with Mikhalkin, we developed a technique based on what we called *floor decomposition of tropical curves* which allow the simultaneous computation of all Welschinger invariants $W(d,r)$ and genus 0 Gromov-Witten invariants of $\mathbb{C}P^2$. In particular, this technique allows one to turn the algebraic count into a purely combinatorial problem.

Jointly with Lopez de Medrano, we studied closer the combinatorial problem arising from floor decomposition. In this way, we proved recursion formulas to compute the numbers $W(d,r)$. In particular, we obtain relations between Gromov-Witten invariants and Welschinger invariants.

These formulas generalize the one obtained by Caporaso and Harris for the numbers $N(d)$ (proved by Gathmann and Markwig in the tropical set up) and the one obtained by Itenberg, Kharlamov and Shustin for the numbers $W(d,0)$.

The space of tropically collinear points is shellable

Josephine Yu
(joint work with Hannah Markwig)

Let $(\mathbb{R},\oplus,\odot)$ be the *tropical semiring* where the tropical addition $\oplus$ is taking minimum and the tropical multiplication $\odot$ is the usual addition. We will work in the tropical projective space $\mathbb{TP}^{d-1}=\mathbb{R}^d/(1,\ldots,1)\mathbb{R}$ obtained by quotienting out the tropical scalar multiplication.

The space $T_{d,n}$ is the tropical variety of the determinantal ideal generated by 3×3 minors of a $d\times n$ matrix of indeterminates. Develin conjectured in [1] that $T_{d,n}$ is shellable for all d and n and proved his conjecture for $d=3$ (or $n=3$). Here we prove his conjecture for all d and n.

The space $T_{d,n}$ is the space of $d\times n$ real matrices of tropical or Kapranov rank 2. Thus we can understand an element of $T_{d,n}$ as n points on a tropical line in $\mathbb{TP}^{d-1}$. A *tropical line* in $\mathbb{TP}^{d-1}$ is a one dimensional polyhedral complex in $\mathbb{TP}^{d-1}$ which is combinatorially a tree with unbounded edges in directions $e_1,\ldots,e_d$ and the *balancing condition* at each vertex as follows. At a vertex V, let $u_1,\ldots,u_k$ be the primitive integer vectors pointing from V to its adjacent vertices. The balancing condition holds at V if $u_1+\cdots+u_k=0$ in $\mathbb{TP}^{d-1}$. A configuration of n points in $\mathbb{TP}^{d-1}$ is called *tropically collinear* if there is a tropical line which passes through the n points. $T_{d,n}$ is the space of all such configurations.

The space $T_{d,n}$ is a polyhedral fan in $\mathbb{R}^{d\times n}$. We can derive a simplicial fan structure on it using moduli spaces of tropical curves and the space of phylogenetic trees $\mathcal{T}_{n+d}$. Since $T_{d,n}$ is closed under simultaneous translation of all points and under choosing different representatives for each point, we mod out by these actions and obtain a pointed simplicial fan. We then intersect this fan with the unit sphere centered at the origin to obtain a simplicial complex, which we will also call by $T_{d,n}$ by abuse of notation.

A parametrized tropical line is an abstract tropical curve (a leaf-labeled tree) Γ together with a map

$$h:\Gamma\to\mathbb{TP}^{d-1},$$

such that the image $h(\Gamma)$ is a tropical line as defined above. Our parametrized tropical lines are equipped with certain marked points x_i. The evaluation maps send a tuple (Γ,h,x_i) to $h(x_i)\in\mathbb{TP}^{d-1}$. We show that $T_{d,n}$ is the image of the moduli space of n-marked parametrized tropical lines under the evaluation map.

Moduli spaces of tropical curves can be used to derive results in enumerative tropical geometry. This is why these moduli spaces attracted a lot of attention recently (see e.g. [5], [4] or [3]). Their simplicial fan structure equals the structure of the space of trees, $\mathcal{T}_{n+d}$ (see [3]). In fact, we can identify $T_{d,n}$ with the subcomplex of the space of trees $\mathcal{T}_{n+d}$ on which the evaluation map is injective, a subcomplex induced on the vertices corresponding to "bicolored splits", the splits that contains both colors on both sides.

In [7], Trappmann and Ziegler showed that the space of trees $\mathcal{T}_{n+d}$ is shellable. Since we derive our simplicial complex structure for $T_{d,n}$ using the space of trees $\mathcal{T}_{n+d}$, we use a similar method to show that the space $T_{d,n}$ is shellable. We also compute the homology of $T_{d,n}$ by counting how many times "a loop is closed". Our main results can be summarized as follows:

Theorem. *The simplicial complex $T_{d,n}$ is shellable and has the homotopy type of a wedge of $n+d-4$-dimensional spheres. The number of spheres is equal to the number of simultaneous partitions of an $(n-1)$-set and a $(d-1)$-set into the same number of non-empty ordered parts. This number equals*

$$\sum_{k=1}^{\min(n-1,d-1)} (k!)^2 S(n-1,k) S(d-1,k),$$

where $S(m,k) = \frac{1}{k!}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i} i^m$ *is the Stirling number of the second kind.*

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Geometric properties of logarithmic limit sets over the reals

Daniele Alessandrini

Logarithmic limit sets of complex algebraic sets have been first studied in [4], and they were further studied by many people, for example in [7]. The logarithmic limit set of a complex algebraic set is a polyhedral complex of the same dimension as the algebraic set, it is described by tropical equations and it is the image, under the component-wise valuation map, of an algebraic set over an algebraically closed non-archimedean field. We can extend these properties to the real case, see [1] for details.

Let $V \subset (\mathbb{R}_{>0})^n$ be a real semi-algebraic set. We apply the **Maslov dequantization** to V: for $t \in (0,1)$ the **amoeba** of V is

$$\mathcal{A}_t(V) = \{(\log_{(\frac{1}{t})}(x_1), \ldots, \log_{(\frac{1}{t})}(x_n)) \mid (x_1, \ldots, x_n) \in V\}$$

the **logarithmic limit set** is the limit of the amoebas

$$\mathcal{A}_0(V) = \lim_{t \to 0} \mathcal{A}_t(V)$$

To study the set $\mathcal{A}_0(V)$ we use the following property: the point $(0, \ldots, 0, -1)$ is in $\mathcal{A}_0(V)$ if and only if there exists a sequence $(x_k) \subset V$ and $a_1, \ldots, a_{n-1} > 0$ such that $(x_k) \to (a_1, \ldots, a_{n-1}, 0)$. This proposition shows that the special point $(0, \ldots, 0, -1)$ is particularly easy to control. If we want to study another point $x \in \mathbb{R}^n$, we can act on $\mathbb{R}^n$ by linear maps, moving x to the special point.

Let $B = (b_{ij}) \in GL_n(\mathbb{R})$, then B acts linearly on $\mathbb{R}^n$. By conjugation with the componentwise logarithm map, B acts on $(\mathbb{R}_{>0})^n$:

$$\overline{B}(x) = (x_1^{b_{11}} x_2^{b_{12}} \cdots x_n^{b_{1n}} \ , \ \ldots \ , \ x_1^{b_{n1}} x_2^{b_{n2}} \cdots x_n^{b_{nn}})$$

If $V \subset (\mathbb{R}_{>0})^n$, then $B(\mathcal{A}_0(V))$ is the logarithmic limit set of $\overline{B}(V)$. Anyway, if the entries of B are not rational, $\overline{B}(V)$ is not semi-algebraic. The category of semi-algebraic sets is too small for our methods.

We need to work in a more general setting: sets definable in an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. For example the structure $\mathcal{OS}^{\mathbb{R}}$ of real closed field expanded with all the power functions is o-minimal, polynomially bounded, with field of exponents $\mathbb{R}$ (see [6]).

Theorem 1. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. Then the logarithmic limit set $\mathcal{A}_0(V) \subset \mathbb{R}^n$ is a polyhedral cone, and* $\dim \mathcal{A}_0(V) \leq \dim V$.

In the real case, the behavior of logarithmic limit sets is less regular than the behavior they have in the complex case. It is easy to show examples where $\dim \mathcal{A}_0(V) < \dim V$, and where $\mathcal{A}_0(V)$ is not equidimensional. Also the combinatorics is not well understood.

Let $V \subset (\mathbb{C}^*)^n$ be an algebraic hypersurface with real equation f, and let $V_{>0} = V \cap (\mathbb{R}_{>0})^n$, its positive part. Then $\mathcal{A}_0(V)$ is a polyhedral fan, dual to the Newton polytope of f. The set $\mathcal{A}_0(V_{>0})$ is a subset of $\mathcal{A}_0(V)$, a polyhedral complex, but it is not always a subcomplex. For example, consider the "Cartan umbrella" $V = \{(x,y,z) \in (\mathbb{C}^*)^3 \mid x^2(1-(z-2)^2) = x^4 + (y-1)^2\}$. Then $\mathcal{A}_0(V_{>0})$ is only the ray in the direction $(-1,0,0)$, but this set is in the interior of a face of the dual fan of $f = x^4 + x^2(z^2 - 4z + 3) + y^2 - 2y + 1$.

Let S be an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. The Hardy field can be defined as the set of germs of definable functions of one variable:

$$H(S) = \{(f,\varepsilon) \mid f : (0,\varepsilon) \longrightarrow \mathbb{R} \text{ definable } \}/\sim$$

$$(f,\varepsilon) \sim (g,\varepsilon') \Leftrightarrow \exists \delta > 0 : f_{|(0,\delta)} = g_{|(0,\delta)}$$

The set $H(S)$ inherit an S-structure from $\mathbb{R}$, that is an elementary extension. In particular $H(S)$ is a non-archimedean real closed field, with a surjective real valuation.

Let $W \subset (H(S)_{>0})^n$ be a definable set. We define the Log map as:

$$\text{Log} : (H(S)_{>0})^n \ni (x_1, \dots, x_n) \longrightarrow (-v(x_1), \dots, -v(x_n))$$

Theorem 2. *The set* $\text{Log}(W) \subset \mathbb{R}^n$ *is a polyhedral complex, and* $\dim(\text{Log}(W)) \leq \dim(W)$.

If W is a semi-linear set (a polyhedron) these objects were studied in [5]. These objects are very similar to the Positive Tropical Varieties (see [8]) but there are examples where they differs.

Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set in S, and let $\overline{V} \subset (H(S)_{>0})^n$ be its extension to the Hardy field. We have

Theorem 3.

$$\lim_{t \to 0} \mathcal{A}_t(V) = \mathcal{A}_0(V) = \text{Log}(\overline{V})$$

In the general case, we can construct a family of definable sets $V_t \subset (\mathbb{R}_{>0})^n$ such that

$$\lim_{t \to 0} \mathcal{A}_t(V_t) = \text{Log}(W)$$

this construction is a generalization of the patchworking families, see [9]. If W is defined by a first order formula ϕ:

$$W = \{(x_1, \dots, x_n) \mid \phi(x_1, \dots, x_n, f_1, \dots, f_m)\}$$

where $f_1, \dots, f_m \in H(S)$ are germs of definable functions, then

$$V_t = \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \dots, x_n, f_1(t), \dots, f_m(t))\}$$

Finally, we can describe logarithmic limit sets with tropical equations. A positive formula in the symbols $\mathcal{OS}^{\mathbb{R}}$ is a first order formula containing only the connectives $\vee$ and $\wedge$ and the quantifiers $\forall$, $\exists$. (No $\neg, \Rightarrow, \Leftrightarrow$). Every subset of $(\mathbb{R}_{>0})^n$ that is defined by a quantifier-free positive formula is closed. Every closed semi-algebraic set is defined by a positive quantifier-free formula.

Theorem 4. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable by a positive formula in the symbols $\mathcal{OS}^{\mathbb{R}}$ with parameters in $\mathbb{R}_{>0}$. Note that every closed semi-algebraic set satisfies the hypothesis. Then there exists a positive formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and parameter $a_1, \ldots, a_m \in \mathbb{R}_{>0}$ such that*

$$V = \{x \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

$$\mathcal{A}_0(V) = \{x \mid \phi_{\mathbb{T}}(x_1, \ldots, x_n, 0, \ldots, 0)\}$$

Where $\phi_{\mathbb{T}}$ is the formula ϕ where the operations are interpreted tropically, i.e. "+" becomes "max" and "·" becomes "+".

Our motivations for this work come from low-dimensional topology, see [2] and [3]. Let $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ denote parameter space of convex real projective structures on a closed orientable n-manifold M such that $\pi_1(M)$ is torsion free, virtually centerless and Gromov hyperbolic (for example M can be every hyperbolic manifold whose fundamental group is torsion-free). The space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ can be identified with a closed semi-algebraic subset of the character variety $\text{Char}(\pi_1(M), SL_{n+1}(\mathbb{R}))$. We can construct compactifications of semi-algebraic sets using inverse systems of logarithmic limit sets. The boundary points are tropical images of the extension of the semi-algebraic set to a real closed non-archimedean field $\mathbb{F}$ with a surjective real valuation. In particular the points of $\partial\mathcal{T}^c_{\mathbb{RP}^n}(M)$ are the tropical images of elements of $\text{Char}(\pi_1(M), SL_{n+1}(\mathbb{F}))$, where $\mathbb{F}$ is as above. Using this fact we can give a geometric interpretation of the boundary points of $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ as actions on "tropical projective spaces", constructed using a generalization of the Bruhat-Tits buildings for $SL_{n+1}(\mathbb{F})$.

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A tropical Riemann-Roch theorem

Michael Kerber
(joint work with Andreas Gathmann)

Let Γ be a connected metric graph and let $\text{Div}(\Gamma)$ denote the free abelian group generated by the points of Γ. An element $D = \sum_{i=1}^n a_i P_i \in \text{Div}(\Gamma)$ is called a divisor, and its degree is defined to be the integer $\deg(D) := \sum_{i=1}^n a_i$. If $D = \sum_{i=1}^n a_i P_i$ has the property that $a_i \geq 0$ for all $i \in \{1, \ldots, n\}$, then D is called effective, written $D \geq 0$. For a given graph Γ, we define its canonical divisor by

$$K_\Gamma = \sum_{P \in \Gamma} (\text{val}(P) - 2)P \in \text{Div}(\Gamma),$$

where val denotes the number of edges adjacent to P.

A rational function f on Γ is defined to be a continuous, piecewise-linear real-valued function with integer slope and only a finite number of pieces. To each rational function f, we have an associated divisor $(f) \in \text{Div}(\Gamma)$ defined by

$$(f) = \sum_{P \in \Gamma} \text{ord}_P(f) \in \text{Div}(\Gamma)$$

where the integer $ord_P(f)$ equals the sum of slopes of f on every edge emanating from the point P.

For a given divisor D we denote by $R(D)$ the space of all rational functions f on Γ such that $(f) + D$ is effective. For each divisor D we define its dimension $r(D)$ to be

$$r(D) = \min_{E \geq 0, R(D-E) = \emptyset} \deg(E) - 1.$$

Theorem 1. *Let D be a divisor of degree* $\deg(D)$ *on a graph* Γ *of genus* $g(\Gamma)$*. Then the following equation holds:*

$$r(D) - r(K_\Gamma - D) = \deg(D) + 1 - g(\Gamma).$$

The proof given in [GK06] of this tropical analogue of the Riemann-Roch theorem in algebraic geometry is an extension of an analogous result on combinatorial graphs recently obtained by M. Baker and S. Norine [BN07]. An alternative proof was independently found by G. Mikhalkin and I. Zharkov [MZ06].

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Tropical bases by regular projections

Kerstin Hept
(joint work with Thorsten Theobald)

This talk gives a short introduction to our recent paper about tropical bases [5].

Given a field K with a real valuation $\text{ord} : K \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (i.e. $K = \mathbb{Q}$ with the p-adic valuation, the field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series with the natural valuation) we can extend the valuation map to an algebraic closure $\bar{K}$ and then to $\bar{K}^n$ via

$$\text{ord} : \bar{K}^n \to \bar{\mathbb{R}}^n, \quad (a_1, \ldots, a_n) \mapsto (\text{ord}(a_1), \ldots, \text{ord}(a_n)).$$

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in $K[x_1, \ldots, x_n]$. The *tropicalization of* f is defined as

$$\text{trop}(f) := \bigoplus_\alpha \text{ord}(c_\alpha) \odot x^\alpha = \min_\alpha \{\text{ord}(c_\alpha) + \alpha_1 x_1 + \cdots + \alpha_n x_n\}$$

and the *tropical hypersurface* of f is

$$\mathcal{T}(f) := \{w \in \mathbb{R}^n : \text{the minimum in trop}(f) \text{ is attained at least twice in } w\}$$

For an ideal $I \lhd K[x_1, \ldots, x_n]$, the *tropical variety of* I is $\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(f)$ or equivalently (if the valuation is nontrivial) by the topological closure $\mathcal{T}(I) = \overline{\text{ord}\, \mathcal{V}(I)}$ where $\mathcal{V}(I) \subset (\bar{K}^*)^n$ is the variety of I.

A *tropical basis* of the ideal I is a finite generating set $\mathcal{F}$ of I, such that

$$\mathcal{T}(I) = \bigcap_{f \in \mathcal{F}} \mathcal{T}(f)$$

Bogart, Jensen, Speyer, Sturmfels, and Thomas initiated the systematic computational investigation of tropical bases [2], by providing both Gröbner-related techniques for computing tropical bases as well as by providing lower bounds on the size. They showed that for $1 \leq d \leq n$ there is a linear ideal $I \lhd \mathbb{C}[x_1, \ldots, x_n]$ such that any tropical basis of linear forms of I has size at least $\frac{1}{n-d+1}\binom{n}{d}$.

We showed that there are indeed short bases if we drop the assumption on the degree of the polynomials:

Theorem 1 (Theobald-H.). *Let $I \lhd K[x_1, \ldots, x_n]$ be a prime ideal generated by the polynomials $f_1, \ldots, f_r$. Then there exist $g_0, \ldots, g_{n-\dim I} \in I$ with*

$$\mathcal{T}(I) = \bigcap_{i=0}^{n-\dim I} \mathcal{T}(g_i)$$

and thus $\mathcal{G} := \{f_1, \ldots, f_r, g_0, \ldots, g_{n-\dim I}\}$ is a tropical basis for I of cardinality $r + \text{codim}\, I + 1$.

To prove the result we need the result of Bieri and Groves [1] that there exist $\operatorname{codim} I + 1$ projections $\pi_0, \ldots, \pi_{\operatorname{codim} I} : \mathbb{R}^n \to \mathbb{R}^{\dim(I)+1}$ such that

$$\mathcal{T}(I) \;=\; \bigcap_{i=0}^{\operatorname{codim} I} \pi_i^{-1}\pi_i(\mathcal{T}(I))\,.$$

This leaves to show that for an arbitrary projection $\pi : \mathbb{R}^n \to \mathbb{R}^{\dim(I)+1}$ the preimage $\pi^{-1}\pi(\mathcal{T}(I))$ is a tropical hypersurface. To show this let $m = dim(I)$ and the projection be described by

$$\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}, x \mapsto Ax$$

with a regular rational matrix A whose rows are denoted by $a^{(1)}, \ldots, a^{(m+1)}$ and let $u^{(1)}, \ldots,\, u^{(n-(m+1))} \in \mathbb{Q}^n$ be a basis of the kernel of π. Define the ideal

$$J := \Big\langle g \in K[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_l] \;:$$
$$g = f(x_1 \prod_{j=1}^{n-(m+1)} \lambda_j{}^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{n-(m+1} \lambda_j{}^{u_n^{(j)}}) \text{ for some } f \in I\Big\rangle\,.$$

Then we have the following theorem:

Theorem 2 (Theobald-H.)**.** *Let $I \lhd K[x_1, \ldots, x_n]$ be an m-dimensional prime ideal and $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be a rational projection. Then $\pi^{-1}(\pi(\mathcal{T}(I)))$ is a tropical variety with*

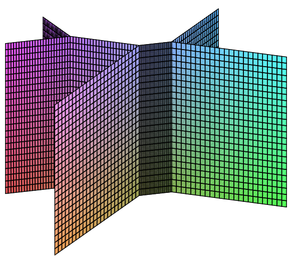
$$\pi^{-1}(\pi(\mathcal{T}(I))) \;=\; \mathcal{T}(J \cap K[x_1, \ldots, x_n])$$

$\mathcal{T}(J \cap K[x_1, \ldots, x_n])$ is a tropical hypersurface if the projection is m-dimensional.

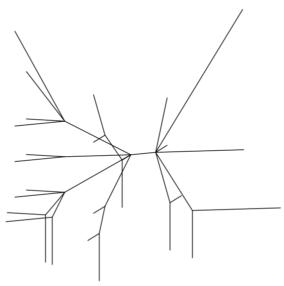
Example 5. *Let I be generated by*

$$f_1 := 2 + y - 4x^2y + x^2y^2 + 2xy^2, \quad f_2 := xyz - 2z + 4xyz^2 - 2 + z^2$$

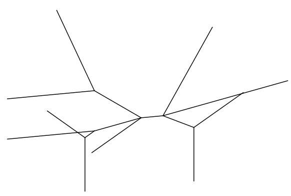
For the first projection π_2 we take the one with kernel $(0,0,1)$, so it is the projection on the plane $z = 0$. Then $J \cap K[x,y,z]$ is generated by f_1. As π_1 we choose the projection with kernel $\langle(1,1,0)\rangle$. This defines the second polynomial f_3 in the tropical basis by $J \cap K[x,y,z] = \langle f_3\rangle$.
The last projection determines the tropical variety, we take $(2,4,1)$ as a kernel for π_0. The polynomial f_4 defining the elimination ideal has 63 terms.



$\mathcal{T}(f_1)$



$\mathcal{T}(f_1) \cap \mathcal{T}(f_3)$



$\mathcal{T}(f_1) \cap \mathcal{T}(f_3) \cap \mathcal{T}(f_4)$
$= \mathcal{T}(I)$

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Some tropical geometry of algebraic groups, minimal orbits, and secant varieties

Jan Draisma

1. Secant varieties and minimal orbits

Given a variety X embedded in a projective space $\mathbb{P}V$, the $(k-1)$-st secant variety of X, denoted kX, is the closure of the union of all $(k-1)$-spaces spanned by k points on X. We usually require that X spans $\mathbb{P}V$, so that $kX = \mathbb{P}V$ for k sufficiently large. We often work with the cone C in V over X rather than with X, and write kC for the cone over kX. Secant varieties appear in applications as diverse as phylogenetics [2, 5, 12], complexity theory [10, 11], and polynomial interpolation [1]. The references in this note are by no means complete, but they themselves contain many further relevant references.

Example 3. *Consider "matrix multiplication of two 2×2-matrices", which can be thought of as a tensor T in $V = \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, and take C equal to the set of pure tensors in this tensor product. Then the ordinary procedure for multiplication, which needs 8 scalar multiplications, shows that T lies in $8C$. Strassen realised that by taking clever linear combinations, T can be written as a sum of 7 pure tensors. This shows that $T \in 7C$, and Strassen used this fact in an algorithm for multiplication of $n \times n$-matrices which needs less than n^3 multiplications* [11]*. Recently Landsberg proved that $T \notin 6C$* [10]*—which means that T cannot be approximated with tensors of rank 6, a much stronger and more difficult statement than that T itself does not have rank 6.*

Example 4. *In phylogenetics, one tries to reconstruct evolution from genetic data of species alive today. One approach runs as follows: given n strings of nucleotides A, C, G, T of DNA of n species and given a hypothetical evolutionary tree leading to those n species, one wants to decide whether the tree matches the data. First, the data leads to an empirical probability distribution on $\{A, C, G, T\}^n$, which can be thought of as an element of $(\mathbb{C}^4)^{\otimes n}$. On the other hand one has a parameterised* *variety, the* General Markov Model*, of probability distributions that match the tree. To test whether the tree matches the data, one tries to find the equations defining the model, which can then be tested on the empirical distribution. Allman and Rhodes reduced the quest for equations defining the model for general trees to the case of* stars*, trees of diameter at most* 2 [2]*. For the star with* 3 *leaves, this model is* $4C$*, where* C *is the set of pure tensors in* $(\mathbb{C}^4)^{\otimes 4}$*—for which we unfortunately do not know equations yet. On a side note, Allman and Rhodes prove only that their procedure would yield set-theoretic equations; we recently showed that they generate the full ideal.*

Theorem 5 ([9])**.** *The Allman-Rhodes equations generate the full ideal of the phylogenetic model.*

Example 6. *The case where* $V = S^d(\mathbb{C}^n)$ *and* C *is the set of pure powers* l^d *with* $l \in \mathbb{C}^n$ *is closely related to polynomial interpolation. The dimensions of the secant varieties* kC *are known from the ground-breaking work of Alexander and Hirschowitz* [1]*.*

This illustrates the omnipresence of secant varieties in mathematics and applications. Two important problems concerning them are: first, to find equations for kC; and second, more modestly, to determine the dimension of kC. We now concentrate on the second problem. Typically one expects $\dim kC$ to be $\min\{k \dim C, \dim V\}$—an obvious upper bound—but one has a hard time proving that this is the case. This is already difficult in the toric case where $V = \mathbb{C}^n$ and C is given as the closure of the image of a monomial map $f : \mathbb{C}^m \to \mathbb{C}^n$—all examples above are of this type. Using tropical geometry we have proved the following lower bound.

Theorem 7 ([8])**.** *Suppose that* $f = (x^\alpha)_{\alpha \in A}$*, where* A *is some subset of* $\mathbb{N}^m$ *of cardinality* n*. Assume that* A *lies on an affine hyperplane, so that* $C := \overline{\operatorname{im} f}$ *is indeed a cone. For any* k*-tuple* $l = (l_1, \ldots, l_k)$ *of affine-linear forms on* $\mathbb{R}^m$ *let* $C_i(l)$ *denote the subset of* A *where* l_i *is strictly smaller than all other* $l_j, j \neq i$*. Then*

$$\dim kC \geq \sum_{i=1}^{k} (1 + \dim \operatorname{Aff}_{\mathbb{R}} C_i),$$

where $\operatorname{Aff}_{\mathbb{R}} C_i$ *is the affine span of* C_i *in* $\mathbb{R}^m$*.*

To find good lower bounds with this theorem, one has to maximise the sum on the right-hand side over all k-tuples l, or, equivalently, over all regular subdivisions of $\mathbb{R}^m$ into k parts. In general this optimisation problem is not easy. Nevertheless, Baur and I have determined the secant dimensions of many embedded varieties in this manner.

Theorem 8 ([3]; see also [6])**.** *The secant varieties of* $(\mathbb{P}^1)^i$ *for* $i = 1, 2, 3$*,* $\mathbb{P}^2$*,* $\mathbb{P}^1 \times \mathbb{P}^2$*, in all equivariant embeddings, are as expected, with an explicit list of exceptions.*

In her Master's thesis [4], Brannetti has reproved the Alexander-Hirschowitz theorem for $S^d(\mathbb{C}^4)$, for all d, with the method of [8]. These result lead to the following intriguing question.

Question 1. *Is the lower bound of Theorem 7, optimised over all k-tuples l, always the exact dimension of kC? I know of no counter-examples.*

Apart from these toric examples, we have also applied this approach to other *minimal orbits* X. Our results include a parameterisation of the cone C over X that when tropicalised hits a full-dimensional subset of the tropicalisation of C. For the smallest interesting case, where X is the collection of all incident point-line pairs in $\mathbb{P}^2$, we computed all secant dimensions of X in all SL_3-equivariant embeddings into projective spaces [3]. Related approaches to secant varieties, which also study their degrees and equations, are [7, 13].

2. Tropical algebraic groups

With Tyrrell McAllister I have initiated the study of tropicalising algebraic groups. There are many issues here: what coordinates to use? Does one expect a tropical multiplication on the result? I report some preliminary observations.

Proposition 9. *The tropicalisation of* SL_n*, with respect to matrix entries, is a monoid with respect to tropical matrix multiplication.*

Also, using Egerváry's theorem on minimal-weight matchings one can describe the maximal cones of this tropicalisation. For a not-so-easy example consider the orthogonal group $\mathrm{O}_n = \{g \mid gg^T = 1\}$. The choice for this non-split form is perhaps justified by the following beautiful observation.

Proposition 10. *The tropicalisation of* O_n *contains the matrices* $(d_{ij})_{ij}$ *satisfying* $d_{ii} = 0, d_{ij} = d_{ji}$*, and* $d_{ij} + d_{jk} \geq d_{ik}$*, as well as the closure of this set of matrices under tropical multiplication.*

(Note that these metric matrices form a cone of dimension $\binom{n}{2} = \dim \mathrm{O}_n$.) This is already rather interesting: combinatorially it is not clear why that closure should still have dimension $\binom{n}{2}$ (and not larger). This ends my preliminary account of tropical geometry of algebraic groups.

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Real Zeuthen numbers for two lines

Benoit Bertrand

In this talk I tackle the following enumerative problem. Given l lines and $k = d(d+3)/2 - l$ points in $\mathbb{C}P^2$, how many nonsingular complex algebraic curves of degree d pass through the k points and are tangent to the l lines? This is a particular instance of the Zeuthen problem. For generic configurations of points and lines there is a finite (and invariant) number of solutions to the problem. Denote this Zeuthen number $N_d(l)$.

Consider the corresponding question for real data: assume that the points and the lines are real, how many degree d real curves pass through the k points and are tangent to the l lines? In other words, what values can take the real Zeuthen number $N_d^{\mathbb{R}}(l, C)$ of real solutions? This number most often depends on the configuration C of points and lines chosen. However the number of complex solutions is clearly always an upper bound for the number of real solutions.

$$N_d^{\mathbb{R}}(l, C) \leq N_d(l)$$

Whether there exists a generic configuration for which all the solutions are real is a natural and classical question in real enumerative geometry. It is said that the problem is maximal if such a configuration exists. In other words a problem is maximal if the number of real solution is equal the the upper bound given by the number of complex solution.

For $l = 1$ it was shown by F. Ronga [Ron00] that the Zeuthen problem is maximal in the above sense (i.e. all the curves can be real). In [Ber07] (on which this extended abstract is based) I prove that the problem for 2 lines is also maximal.

Theorem 1. *For any integer $d \geq 2$ there exists a configuration C of 2 real lines and $d(d+3)/2 - 2$ real points such that all the degree d curves passing through the points and tangent to the lines are real:*

$$N_d^{\mathbb{R}}(2, C) = N_d(2)$$

The techniques I use are those developed by Mikhalkin in [Mik05]. The statement is proved using correspondence theorems to tropicalize the problem. A general correspondence theorem for curves subject to tangency (and incidence) conditions is given in [Mik] but here only tangencies with respect to toric divisors are needed and the required correspondence theorem can be deduced from [Mik05] or [Shu05]. I use a lattice path algorithm to count the number of real tropical curves and find a configuration when this number is maximal.

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Adelic amoebas

Sam Payne

Let K be a field, with $\nu : K^* \to \mathbb{R}$ a group homomorphism. For instance, if K is the field of rational numbers then ν could be a p-adic valuation or $-\log|\ |_\infty$. Let T be a torus with character lattice M. To each point $x \in T(K)$ we associate its "tropicalization" $\mathrm{Trop}(x)$, which is the point in the vector space $\mathrm{Hom}(M, \mathbb{R})$ given by the composition

$$M \xrightarrow{\mathrm{ev}_x} K^* \xrightarrow{\nu} \mathbb{R}.$$

For points over a finite extension $L \subset K$, we can define $\mathrm{Trop}(x)$ similarly, by the composition

$$M \xrightarrow{\mathrm{ev}_x} L^* \xrightarrow{N_{L/K}} K^* \xrightarrow{\nu/[L:K]} \mathbb{R}.$$

Similarly, we can define the tropicalization of points over the completion of K with respect to the norm $\exp(\nu)$ and its algebraic extensions.

If X is a subvariety of T, we define the amoeba of X to be the closure of the image of $X(\overline{K}_\nu)$ in $\mathrm{Hom}(M, \mathbb{R})$, where $\overline{K}_\nu$ is the algebraic closure of the completion of K with respect to $\exp(\nu)$.

We now fix K to be either a number field or the function field of a curve $k(C)$. Let S be the set of places of K, or canonical representatives of the equivalence classes of norms on K. For instance, if K is the field of rational numbers, then S is the set of p-adic norms $|\ |_p$, together with the archimedean norm $|\ |_\infty$. If $K = \mathbb{C}(t)$, then $S = \mathbb{P}^1(\mathbb{C})$. These norms extend uniquely to the algebraic closure of K, and we have the product formula

$$\prod_{p \in S} |a|_p = 1.$$

For example, if K is the field of rational numbers and a is $3/4$, then we have $|a|_2 = 4$, $|a|_3 = 1/3$, $|a|_p = 1$ for all primes greater than three, and $|a|_\infty = 3/4$. Then $4 \cdot (1/3) \cdot (3/4) = 1$.

The adelic amoeba of X is the union of its amoebas with respect to the places p in S. The radial projection of the adelic amoeba to the sphere is of particular interest. Its image is closed and was studied in relation to questions from algebra about finite generation of modules over subsemigroups of abelian groups [1] and questions from dynamical systems about expansiveness along halfspaces [3].

In this talk, I presented several examples of adelic amoebas and proved the following results.

Theorem 1. *Suppose X is a hypersurface and K is a number field. If the radial projection of the adelic amoeba of X is not surjective then X is a translate of a subtorus by a torsion point.*

Theorem 2. *Suppose X is a hypersurface and $K = k(C)$ is the function field of a curve. If the radial projection of the adelic amoeba of X is not surjective then X is defined over k.*

The converse in each case is clear. If X is a translate of a subtorus by a torsion point then the adelic amoeba of X is a hyperplane, and if X is defined over k then the adelic amoeba of X is a fan, so the radial projections are not surjective.

The proof in the function field case is algebraic and essentially elementary. In the number field case, the proof that I gave relies on a deep theorem of Zhang from diophantine geometry that says that the Zariski closure of the set of torsion points in a subvariety of a torus is a finite union of translates of subtori by torsion points [5].

For details, generalizations to higher codimension, relations to a conjecture of Einsiedler, Kapranov, and Lind [2, Conjecture 2.3.5], and further references, see [4].

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The Maslov dequantization and related dequantization procedures for mathematical structures and objects

Grigory L. Litvinov
(joint work with coauthors)

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values, see [1, 3–6, 8]. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to idempotent mathematics over idempotent semirings. For example, the field of real or complex numbers can be treated as a quantum object whereas idempotent semirings can be examined as "classical" or "semiclassical" objects (a semiring is called idempotent if the semiring addition is idempotent, i.e. $x \oplus x = x$).

Tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be treated as a result of the Maslov dequantization applied to traditional algebraic geometry (O. Viro, G. Mikhalkin).)

In the spirit of N.Bohr's correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar results over idempotent semirings. A systematic application of this correspondence principle leads to a variety of theoretical and applied results, see, e.g., [1–10].

In the framework of idempotent mathematics, a new version of functional analysis is developed from idempotent variants of basic theorems (e.g., of the Hahn-Banach type) to the theory of tensor products, nuclear operators and nuclear spaces in the spirit of A. Grothendieck as well as basic concepts and results of the theory of representation of groups in idempotent linear spaces, see, e.g., [3–8].

Last time the Maslov dequantization and related dequantization procedures are applied to different concrete mathematical objects and structures, see, e.g., [6, 8, 10].

Examples:

(1) The Legendre transform can be treated as a result of the Maslov dequantization of the Fourier-Laplace transform (V.P. Maslov).
(2) If f is a polynomial, then a dequantization procedure leads to the Newton polytope of f. Using the so-called dequantization transform it is possible to generalize this result to a wide class of functions and convex sets, see [6].
(3) An application of dequantization procedures to linear operators leads to spectral properties of these operators [10].
(4) An application of a dequantization procedure to metrics leads to the Hausdorff-Besicovich dimension including the fractal dimension [10].

(5) An application of a dequantization procedure to measures and differential forms leads to a notion of dimension at a point [10]. This dimension can be real-valued (e.g. negative).

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Piecewise polynomials in Tropical Geometry

Eric Katz

(joint work with Sam Payne)

It is familiar for tropical geometers to consider piecewise linear functions. Tropical polynomials are convex piecewise linear functions, and one can associated tropical hypersurfaces to them [7, 9]. These tropical hypersurfaces are tropical cycles, that is, integrally-weighted rational polyhedral complexes satisfying the balancing condition. Observe that if we are in the constant coefficient case (where the tropical polynomials are defined as $f(x) = \min_i(x \cdot u_i)$), the tropical hypersurfaces will be fans. Another use of piecewise linear functions is in the Tropical Intersection

Theory [1] as developed by Allermann and Rau. Here, a Cartier divisor ϕ on a tropical cycle C is defined as a piecewise linear section of a tropical line bundle on C. This Cartier divisor has an associated Weil divisor which is a tropical cycle supported on the non-linear locus of ϕ. Intersection theory on $\mathbb{R}^n$ is built up from the operation of intersecting with Cartier divisors.

It is natural to ask what role piecewise polynomial functions play in tropical geometry. We consider the constant coefficient case. Fix a complete rational fan Δ in $\mathbb{R}^n$. Our piecewise polynomial functions will be polynomials on the cones in Δ and they will lead to tropical cycles supported on Δ. The notions of piecewise polynomial functions and tropical cycles can be rephrased in terms of the toric variety $X(\Delta)$. Let T be the open torus in $X(\Delta)$ and $M \cong \mathbb{Z}^n$ its character lattice. Fulton and Sturmfels [3] show that the Chow cohomology groups, $A^k(X(\Delta))$ are isomorphic to the groups of codimension k Minkowski weights. A Minkowski weight is a function $c : \Delta^{(k)} \to \mathbb{Z}$ from the $(n-k)$-dimensional cones of Δ to the integers that satisfies a balancing condition. By may view a Minkowski weight c as the tropical cycle supported on Δ given by

$$\sum_{\sigma \in \Delta^{(k)}} c(\sigma)\sigma.$$

Payne [8] showed that the space of piecewise polynomial functions on Δ is isomorphic to the equivariant Chow cohomology $A^*_T(X(\Delta))$.

There is a canonical homomorphism:

$$\iota^* : A^*_T(X) \to A^*(X)$$

induced by the inclusion of X into the finite dimensional approximation of the Borel mixed space [2]. This homomorphism is well-understood in the smooth case and is described by equivariant localization. In [4], we give a combinatorial description for the general case. We first describe the map $\iota^* : A^n_T(X) \to A^n(X)$. Let $\mathrm{Sym}^{\pm} M$ denote the $\mathbb{Z}$-graded ring obtained by inverting homogeneous elements in $\mathrm{Sym}^* M$. For $\sigma \in \Delta^{(0)}$, we define an *equivariant multiplicty* $e_\sigma \in \mathrm{Sym}^{-n}$. The equivariant multiplicity is characterized by two properties:

(1) If $\sigma_1, \ldots, \sigma_r$ are the maximal cones in a rational polyhedral subdivision of σ, then

$$e_\sigma = e_{\sigma_1} + \cdots + e_{\sigma_r};$$

(2) if σ is unimodular, spanned by a basis $e_1, \ldots, e_n$ for $M^\vee$, then

$$e_\sigma = \frac{1}{e_1^* \cdots e_n^*}\ .$$

It is non-trivial to show that such an equivariant multiplicity exists. The definition of e_σ is the principal part of the Hilbert series counting points in the dual cone σ^*. If f is a degree n piecewise-polynomial on Δ, and f_σ is the restriction of f to the top-dimensional cone σ, then

$$\iota^*(f) = \sum_{\sigma \in \Delta^{(0)}} f_\sigma e_\sigma.$$

Here, $\iota^*(f)$ turns out to be an integer which we view as the value of a Minkowski weight on the unique 0-dimensional cone in Δ. In the smooth case, e_σ is $\frac{1}{e_T(N_{V(\sigma)/X})}$, the reciprocal of the equivariant Euler class of the torus fixed point $V(\sigma)$ in X. In this case, the formula for $\iota^*(f)$ reduces to the standard localization formula. Now, we define ι^* in other degrees. For a cone τ, let Δ_τ

$$\Delta_\tau = \{\gamma + N_\tau/N_\tau | \gamma \in \mathrm{Star}(\tau)\}.$$

Then $X(\Delta_\tau) = V(\tau)$. Given f, a degree k piecewise polynomial, for $\tau \in \Delta^{(k)}$, we may consider the restriction of f to $A_T^k(X(\Delta_\tau))$ and compute $c(\tau) \in \mathbb{Z}$ by applying ι^*. The resulting function c is proven to be a Minkowski weight.

This map ι^* naturally fits into the framework of tropical geometry. The map

$$\iota^* : A_T^1(X) \to A^1(X)$$

is the familiar map taking a tropical polynomial to its tropical hypersurface. Since ι^* is a ring homomorphism, it takes the product of a number of tropical polynomials, viewed as piecewise linear functions to the intersection of their tropical hypersurfaces, viewed as a tropical cycle. From this, in a future paper [5], we will give the proof that the tropical intersection product defined by Allermann and Rau is equivalent to the intersection product defined by Mikhalkin [7] and Fulton-Sturmfels [3]. The proof employs a notion that generalizes tropical Cartier divisors. These are equivariant Minkowski weights (c, f) where c is a Minkowski weight and f is a degree d piecewise polynomial function defined on the support of c. One can associate to (c, f) a tropical cycle which is codimension d in the support of c. These notions have natural extensions to the non-constant coefficient case.

Using these combinatorial localization techniques, we can also give a new proof of Bernstein's theorem on the number of zeroes of a system of polynomials in $(\mathbb{C}^*)^n$ [6]. Given Laurent polynomials $g_1, \dots, g_n \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$, let Δ be the common refinement of the normal fans of the Newton polytopes of $\{g_1, \dots, g_n\}$. Each g_i can be viewed as an equivariant divisor $f_i \in A_T^1(X(\Delta))$. The number of common zeroes of $\{g_1, \dots, g_n\}$ is bounded by the degree of the following intersection

$$\iota^*(f_1) \cdot \dots \cdot \iota^*(f_n) = \iota^*(f_1 \dots f_n) \in A^n(X(\Delta)).$$

By rewriting our combinatorial localization formula for $\iota^*(f_1 \dots f_n)$ as the principal part of a certain Hilbert series and using Brion's formula, we can show that the degree is exactly the mixed volume of the Newton polytopes.

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Gale duality for complete intersections

Frank Sottile
(joint work with Frédéric Bihan)

This talk is based upon the preprint [4]. A complete intersection in $(\mathbb{C}^\times)^{n+m}$ defined by Laurent polynomials,

$$f_1(x_1,\ldots,x_{m+n}) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_{m+n}) \;=\; 0\,, \tag{1}$$

where each polynomial f_i contains the same monomials $\{1, x^{\alpha_1},\ldots,x^{\alpha_{l+m+n}}\}$ may also be viewed as the intersection of a codimension n affine linear space L in $\mathbb{C}^{l+m+n}$ with the image of $(\mathbb{C}^\times)^{m+n}$ under the map

$$\varphi\,:\;(\mathbb{C}^\times)^{m+n}\;\ni\;x\;\longmapsto\;(x^{\alpha_1},\ldots,x^{\alpha_{l+m+n}})\in(\mathbb{C}^\times)^{l+m+n}\subset\mathbb{C}^{l+m+n}\,.$$

When the exponent vectors $\{\alpha_1,\ldots,\alpha_{l+m+n}\}$ span the integer lattice $\mathbb{Z}^{m+n}$, the map φ is injective and the complete intersection (1) in $(\mathbb{C}^\times)^{m+n}$ is scheme-theoretically isomorphic to the intersection $\varphi((\mathbb{C}^\times)^{m+n})\cap L$.

Suppose that $\psi\colon\mathbb{C}^{l+m}\to L$ parameterizes L. Then $\psi^{-1}(\varphi((\mathbb{C}^\times)^{m+n})\cap L)$ is also isomorphic to the original complete intersection (1). In the coordinates for $\mathbb{C}^{l+m+n}$, ψ is given by degree 1 polynomials $p_1(y),\ldots,p_{l+m+n}(y)$, and the inverse image of $(\mathbb{C}^\times)^{l+m+n}$ is the complement M_H of the arrangement H of hyperplanes in $\mathbb{C}^{l+m}$ defined by $\prod_i p_i(y)=0$. If $z_1,\ldots,z_{l+m+n}$ are coordinates for $\mathbb{C}^{l+m+n}$, then $\varphi((\mathbb{C}^\times)^{m+n})$ is defined in $(\mathbb{C}^\times)^{l+m+n}$ by all monomial equations $z^\beta=1$, where $\beta=(b_1,\ldots,b_{l+m+n})\in\mathbb{Z}^{l+m+n}$ is a vector such that

$$b_1\alpha_1+b_2\alpha_2+\cdots+b_{l+m+n}\alpha_{l+m+n}\;=\;0\,.$$

The monomial z^β pulls back to a *master function* on M_H,

$$p(y)^\beta\;:=\;(p_1(y))^{b_1}\cdot(p_2(y))^{b_2}\cdots(p_{l+m+n}(y))^{b_{l+m+n}}\,.$$

Letting $\beta_1,\ldots,\beta_l$ form a basis for the free abelian group of all such linear relations, we see that the pullback $\psi^{-1}(\varphi((\mathbb{C}^\times)^{m+n})\cap L)$ is a complete intersection in M_H defined by the system of master functions,

$$p(y)^{\beta_1}\;=\;p(y)^{\beta_2}\;=\;\cdots\;=\;p(y)^{\beta_l}\;=\;1\,. \tag{2}$$

We say that the system of polynomials (1) in $(\mathbb{C}^\times)^{m+n}$ is *Gale dual* to the system of master functions (2) in M_H.

The isomorphism between schemes defined by Gale dual systems was a key idea behind the new fewnomial bounds in [1, 2, 3]. The number of positive solutions of a system of n polynomials in n variables with $l+n+1$ monomials is at most

$$\tfrac{e^2+3}{4}2^{\binom{l}{2}}n^l\,.$$

This dramatically improves Khovanskii's bound [5], which is $2^{\binom{l+n}{2}}(n+1)^{l+n}$.

We close with an example. Let $n=l=2$ and $m=0$ and consider the system

$$\begin{aligned} x^3y^2 &= x^4y^{-1} - x^4y - \tfrac{1}{2}\,, \\ xy^2 &= x^4y^{-1} + x^4y - 1\,. \end{aligned} \tag{3}$$

in $(\mathbb{C}^\times)^2$. This is isomorphic to $\varphi((\mathbb{C}^\times)^2)\cap L$, where L is defined by

$$z_1 - (z_3 - z_4 - \tfrac{1}{2}) \;=\; z_2 - (z_3 + z_4 - 1) \;=\; 0\,, \quad \text{and}$$

$$\varphi\colon\ (x,y) \longmapsto (x^3y^2, xy^2, x^4y^{-1}, x^4y) \;=\; (z_1,z_2,z_3,z_4)\,.$$

Let s,t be new variables and set

$$\begin{aligned} p_1 &:= s - t - \tfrac{1}{2} \qquad & p_3 &:= s \\ p_2 &:= s + t - 1 \qquad & p_4 &:= t \end{aligned}$$

Then $\psi_p\colon (s,t)\mapsto (p_1,p_2,p_3,p_4)$ parametrizes L.

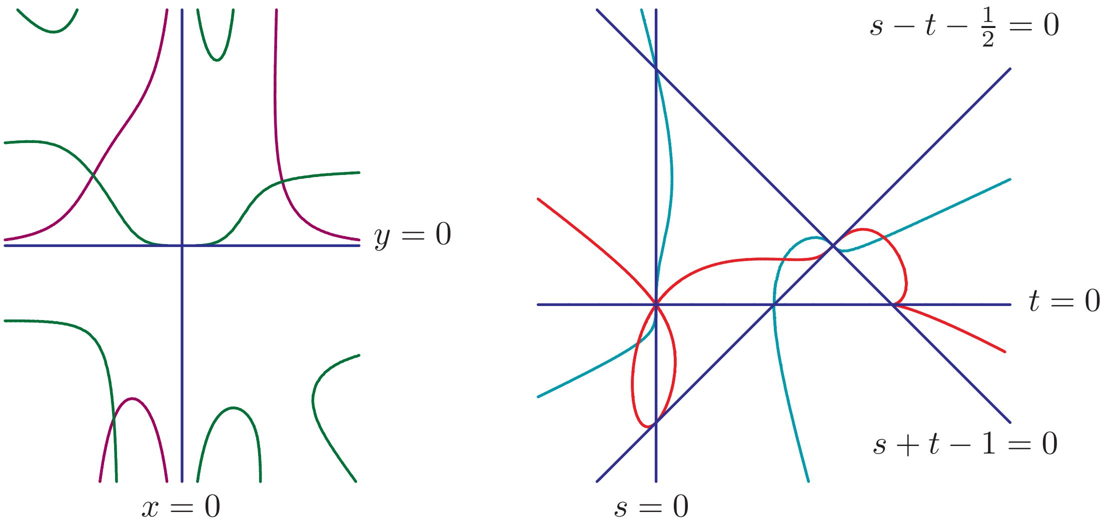
The primitive weights $(-1,3,2,-2)$ and $(3,-1,1,-3)$ annihilate the exponents:

$$(x^3y^2)^{-1}(xy^2)^3(x^4y^{-1})^2(x^4y)^{-2} \;=\; (x^3y^2)^3(xy^2)^{-1}(x^4y^{-1})(x^4y)^{-3} \;=\; 1\,.$$

The polynomial system (3) in $(\mathbb{C}^\times)^2$ is equivalent to the system of master functions

$$\frac{s^2(s+t-1)^3}{t^2(s-t-\frac{1}{2})} \;=\; \frac{s(s-t-\frac{1}{2})^3}{t^3(s+t-1)} \;=\; 1\,. \tag{4}$$

in the complement of the hyperplane arrangement $st(s+t-1)(s-t-\frac{1}{2})=0$.



The polynomial system (3) and the system of master functions (4).

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