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Combinatorics

Organised by
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Abstract. This is the report on the Oberwolfach workshop on Combinatorics, held 6–12 January 2008. Combinatorics is a branch of mathematics studying families of mainly, but not exclusively *discrete* (i.e. finite or countable) structures, often with connections to probability, geometry, computer science, and other areas. Among the structures considered in the workshop were *graphs*, *set systems*, *discrete geometries*, and *matrices*. The programme consisted of 15 invited lectures, 18 contributed talks, and a problem session focusing on recent developments in graph theory, coding theory, discrete geometry, extremal combinatorics, Ramsey theory, theoretical computer science, and probabilistic combinatorics.

Mathematics Subject Classification (2000): 05Cxx, 05Dxx, 52Bxx, 68Rxx.

Introduction by the Organisers

The workshop *Combinatorics* organised by Jeff Kahn (Piscataway), László Lovász (Budapest), and Hans Jürgen Prömel (Berlin) was held January 6st–January 12th, 2008. This meeting was very well attended with 46 participants from many different countries. The programme consisted of 15 plenary lectures, accompanied by 18 shorter contributions and a vivid problem session led by Vera T. Sós.

The conference is a workshop on Combinatorics in a very broad sense, and is in part intended to serve as a framework for all other Oberwolfach meetings that focus on particular areas within combinatorics. This meeting over the years has been extremely successful in achieving the sometimes elusive goal of bringing together and fostering interactions among people with a wide range of interests, and we feel that the program this time was close to ideal in its coverage of a large fraction of the most exciting recent developments across the combinatorial spectrum. Quite a few of the talks — for instance those of Chudnovsky, Kühn, Osthus, Ruciński, Sudakov, Taraz and Vu — reported major progress on well-known problems. Some of the plenary speakers were asked to give overviews of areas somewhat distant from the central interests of many of the participants.

The breadth of the conference makes its contents nearly impossible to summarize. One might say that the central foci of the meeting were extremal and probabilistic aspects of combinatorics, and graph theory; but "extremal combinatorics" is an extremely broad term, and a glance at the titles below shows that the list of topics is not much shorter than the list of talks. Thus, even among the talks that could be considered to fall in the above categories, one finds geometry, Fourier analysis, algebra, physics, connections with social sciences, and multiple connections with computer science and related technology.

Again, we consider this breadth to be not a drawback, but a central feature of the meeting. It has promoted an inspiring, interactive atmosphere, and led to fruitful discussions and collaborations, to new awareness of what's happening in different parts of combinatorics, and to the discovery of some unexpected connections.

More than in past meetings, an emphasis was placed on talks (both plenary and shorter) by younger researchers. This too worked very well, and we hope to make it the pattern for future meetings.

On behalf of all participants, the organisers would like to thank the staff and the director of the *Mathematisches Forschungsinstitut Oberwolfach* for providing a stimulating and inspiring atmosphere.

Workshop: Combinatorics

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Abstracts

The broadcast rate of a hypergraph

Noga Alon
(joint work with Amit Weinstein)

Suppose that a sender S wishes to broadcast a word $x = x_1x_2 \dots x_n$, where $x_i \in \{0,1\}^t$ for all i, to m receivers R_i. Each R_j knows some of the blocks x_i and is interested solely in the block $x_{f(j)}$. Let β_t denote the minimum possible length of a code that enables S to transmit the information so that each receiver R_j will be able to reconstruct $x_{f(j)}$. Thus β_t/t is the average amount of information that has to be transmitted per bit of each x_i when the length of each block is t. Our objective is to study the possible behavior of the numbers β_t/t as t increases. In particular, we show that there are settings in which $\beta_1/1$ is at least $\Omega(\log\log n)$, whereas for sufficiently large t, $\frac{\beta_t}{t} < 2.1$.

The problem above generalizes the setting initiated in [4] and studied in [2], [5] and [1], where the special case considered is $m = n$ and $f(j) = j$ for all j. The motivation is in applications such as Video on Demand, where a network, or a satellite, has to broadcast information to a set of clients, each client is interested in a different part of the data, and each client has a side information, that is, he already knows part of the required data. Note that the assumption that each receiver is interested only in a single block is not a real restriction, as one can simulate a receiver interested in r blocks by r receivers, each interested in one of these blocks, and all having the same side information.

We represent the above setting by a *marked hypergraph* $H = (V, E)$ with a set $V = \{1, 2, \dots, n\}$ of n vertices, corresponding to the input blocks x_i, and a set E of m edges, corresponding to the receivers R_j. For each such receiver R_j, there is an edge $e_j \subset V$ containing a distinguished marked vertex $f(j)$ as well as all other vertices i so that R_j knows x_i. Let $\beta_t(H)$ denote the minimum possible length of a binary code for the scenario represented by H, when each block of x is of size t.

The *confusion graph* $G_t = G_t(H)$ corresponding to H and block-length t is the graph whose vertices are all 2^{tn} binary vectors of length tn corresponding to all possible inputs of S (when the blocks of x are of size t each). Two vertices $x = x_1 \dots x_n$ and $y = y_1 \dots y_n$ are adjacent iff there is at least one edge e of H with a distinguished vertex f so that $x_f \neq y_f$ whereas $x_i = y_i$ for all $i \in e \setminus \{f\}$ (i.e. the receiver which corresponds to the edge e, who must distinguish between these two inputs since $x_f \neq y_f$, can not do so since all the blocks he knows in x and in y are identical). It is not difficult to see that $\beta_t(H)$ is precisely $\lceil \log_2 \chi(G_t) \rceil$, where $\chi(G_t)$ is the chromatic number of G_t. Note that G_t is a Cayley graph of the group Z_2^{nt} since the existence of an edge between two vertices depends only on the Z_2-difference between them that determines which of the blocks are equal and which are not.

Let $t \cdot H$ denote the hypergraph consisting of t pairwise disjoint copies of H. A moment's reflection shows that $G_t(H)$ is a subgraph of $G_1(t \cdot H)$, and hence

$\chi(G_t(H)) \leq \chi(G_1(t \cdot H))$. It is not difficult to check that each of the two functions $g_1(t) = \chi(G_t(H))$ and $g_2(t) = \chi(G_1(t \cdot H))$ is sub-additive, and hence, by Fekete's Lemma, the limits of $g_1(t)/t$ and of $g_2(t)/t$, as t tends to infinity, exist, and are equal to $\inf \frac{g_1(t)}{t}$ and $\inf \frac{g_2(t)}{t}$, respectively.

The graph $G_1(t \cdot H)$ is precisely the OR-product of t copies of $G_1(H)$, and it thus follows, by the main result of [3] and [6], that the limit $\lim_{t \mapsto \infty} \frac{\chi(G_1(t \cdot H))}{t}$ is the fractional chromatic number $\chi^*(G_1(H))$ of $G_1(H)$.

The main result here is an explicit construction of a hypergraph H for which $\chi^*(G_1(H)) = O(1)$ whereas $\chi(G_1(H)) \geq \Omega(\sqrt{\log n})$.

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Higher connectivity: graph-theoretical and topological

Anders Björner

1. Introduction

Several kinds of connectivity are considered: k-connectivity of graphs, k-Cohen-Macaulay connectivity of simplicial complexes, and topological k-connectivity of spaces.

We define a notion of a complex being (k,t)-*connected*, meaning that deletion of any set of at most $k-1$ cells (and all cells containing them) from the complex leaves a topologically t-connected subcomplex of the same dimension. So, for $k=1$ this reduces to the ordinary notion of topological t-connectivity, whereas for $t=0$ we obtain the ususal notion of graph-theoretic k-connectivity of the 1-skeleton. Also, we introduce yet another version of higher connectivity, which, in order to avoid unnecessary confusion, we call by another name: (k,t)-*rigidity*. It applies to finite posets and lattices, and (via the face poset) to regular cell complexes.

One of the cornerstones of the combinatorial theory of convex polytopes is the theorem of Balinski [1], saying that the 1-skeleton of a d-dimensional polytope is graph-theoretically d-connected, meaning that if any subset of $d-1$ vertices is deleted the remaining subgraph is still connected.

Balinski's theorem was extended to 1-skeleta of manifolds by Barnette [2]. In this paper we generalize the Balinski-Barnette theorem in another direction, namely to higher-dimensional skeleta of polytopes and manifolds. This was already done in a homology version by Fløystad [4] using ring theory. Here we work with topological connectivity (vanishing of homotopy groups) leading to stronger results. The techniques rely on methods from poset topological combinatorics, whose generality gives results more widely applicable than to polytopes. For instance, new results on geometric lattices and matroid basis graphs are obtained.

2. RESULTS

The notation is the standard one in topological combinatorics, see e.g. [3] for definitions and explanations. In particular, topology is associated to a poset P via its order complex $\Delta(P)$. Let $\widehat{P} \stackrel{\text{def}}{=} P \cup \{\hat{0}, \hat{1}\}$.

A cell complex (regular CW-complex) is (k,t)-*connected* if removal of any set of $\leq k-1$ cells (and all cells containing them) leaves a topologically t-connected subcomplex of the same dimension. Note that, as distinct from the graph-theoretic concept, we quantify over cells of *all* dimensions. This is because just removing vertices gives a weaker concept in dimensions ≥ 2.

A theorem implying that (k,t)-connected complexes are produced by truncation of certain face posets will be stated later (Theorem 4). First we present some applications to

- Convex polytopes
- Matroid basis graphs

Theorem 1. *The boundary complex of a convex d-polytope is $(d-j,j)$-connected, for $j = 0, 1, \ldots, d-2$.*

Here the $j = 0$ case is equivalent to Balinski's theorem, and the method of proof applies also to Barnette's extension.

Let M be a matroid of rank r on the ground set E. It is said to have the *disjoint basis property* if $\text{rank}(E \setminus B) = \min\{r, |E \setminus B|\}$ for every basis B (i.e., if either there exists a basis C such that $B \cap C = \emptyset$, or else $E \setminus B$ is independent).

The *basis graph* $\Gamma^1(M)$ of M has as vertices the bases of M and as edges the pairs of bases (B_1, B_2) such that $|B_1 \cap B_2| = rank(M) - 1$. For a basis B, an edge (B_1, B_2) is B-*related* if $B_1 \cap B_2 \subset B$.

Theorem 2. *Let M be a matroid of rank r with the disjoint basis property. Then any collection of at most $r-1$ vertices and all related edges can be removed from its basis graph $\Gamma^1(M)$ without losing connectivity.*

In particular, $\Gamma^1(M)$ is (graph-theoretically) r-connected. A paper by Liu [5] contains the result that $\Gamma^1(M)$ is δ-connected (where δ is minimal degree). Since in our theorem more edges are removed (all related edges, not just the incident ones), and in Liu's result more vertices, neither result implies the other.

The *basis complex* $\Gamma^2(M)$ of a matroid M is the polyhedral complex obtained from the basis graph by gluing 2-cells (or "membranes") into all 3- and 4-cycles of the basis graph. Maurer showed [6] that $\Gamma^2(M)$ is 1-connected.

Given a basis B, an 1-cell (edge) or a 2-cell is *B-related* if the intersection of its vertices is a subset of B.

Theorem 3. *Let M be a matroid of rank r with the disjoint basis property. Then, if any collection of at most $r-2$ vertices and all related cells are removed from its basis complex $\Gamma^2(M)$, the remaining cell complex is 1-connected.*

3. Poset rigidity

The results are obtained by a common technique which is best formulated for a general class of posets. Here follows one version of the general statement.

A pure poset P is *(k,t)-rigid* if $P \setminus F$ is topologically t-connected, pure and of the same length as P, for every filter $F \subset P$ generated by at most $k-1$ elements. A poset P is *locally rigid* if for all $x < y \leq z$ in P the order complex of $(x,z)_P \setminus [y,z)_P$ is of length $(\ell_P(x,z)-2)$ and is $(\ell_P(x,z)-3)$-connected. The *truncated poset* $P^{\leq i}$ is obtained by deleting from P all elements of rank $> i$.

Theorem 4. *Let P be a pure poset of length r, and let $0 \leq s \leq t < r$. Assume that*

(i) $\widehat{P}$ *is a lattice,*
(ii) $P \cup \{\hat{0}\}$ *is locally rigid,*
(iii) P *is t-connected,*
(iv) every open interval $(x, \hat{1})$ *in* $\widehat{P}$ *is* $\min\{t, r-2-\mathit{rank}(x)\}$*-connected.*

Then, the truncated poset $P^{\leq(s+1)}$ *is* $(r-s,s)$*-rigid.*

Remark 5. *A concept of k-*HCM-rigidity *is similarly defined in terms of "homotopy Cohen-Macaulay"-ness. The poset* $P^{\leq(s+1)}$ *in Theorem 4 is actually* $(r-s)$*-HCM-rigid.*

For the applicability of Theorem 4 it is of interest to know that the following families of posets are locally rigid:

- posets for which the order complex of each open interval is homotopy-equivalent to a sphere (this is used to prove Theorem 1),
- face posets of matroid complexes $IN(M)$ turned upside-down (this is used to prove Theorems 2 and 3),
- geometric lattices.

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Packing Cubes in a Torus

Tom Bohman
(joint work with Ron Holzman and Venkatesh Natarajan)

Consider the following natural packing problem: How many d-dimensional cubes of side length 2 can we pack into a d-dimensional torus with a fixed, odd side length? This problem can be formulated in terms of graph products as follows. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs then let $G_1 \times G_2$ be the graph with vertex set $V_1 \times V_2$ and an edge between distinct vertices (u_1, u_2) and (v_1, v_2) if and only if $u_i = v_i$ or $\{u_i, v_i\} \in E_i$ for $i = 1, 2$. The graph power G^d is then the product of G with itself d times. A packing of cubes of side length 2 in the d-dimensional torus of side length $2n+1$ corresponds to an independent set in C^d_{2n+1}. (This correspondence between packings of cubes in the torus and independent sets in powers of odd cycles was first noted by Baumert, McEliece, Rodemich, Rumsey, Stanley, and Taylor [1]).

Let $\alpha(G)$ denote the independence number of graph G, i.e., the maximum size of an independent set in G. The independence numbers of the powers of odd cycles are also related to a central open question on the Shannon capacities of graphs. The Shannon capacity of the graph G is defined as

$$c(G) = \sup_d \left(\alpha\left(G^d\right)\right)^{1/d}$$

and gives a measure of optimal zero-error performance of an associated communication channel [7]. The odd cycles on seven or more vertices and their complements are, in a certain sense, the simplest graphs for which the Shannon capacity is not known. This follows from the Strong Perfect Graph Theorem (which was recently proved by Chudnovsky, Robertson, Seymour and Thomas [3]). The Shannon capacity of $C_5 = \overline{C_5}$ was determined in a celebrated paper of Lovász [6]. For a survey of zero-error information theory see [5].

The problem of determining the independence numbers of arbitrary powers of odd cycles remains widely open. The best known upper bounds on these independence numbers are given (in most cases) by the Lovász-theta function $\vartheta(G)$ (which, for the sake of brevity, we do not define here) or the fractional vertex packing number $\alpha^*(G)$ and the simple fact

$$\alpha(G \times H) \leq \alpha(G)\alpha^*(H). \tag{1}$$

The fractional vertex packing number of the graph G is the minimum, over all assignments of non-negative real weights to the vertices of G with the property that the sum of weights over any clique is at most 1, of the sum of weights of the vertices of G. The independence numbers are known in the following cases:

$$\alpha\left(C_5^{2j}\right) = 5^j = \vartheta(C_5)^{2j} \tag{2}$$

$$\begin{aligned} \alpha\left(C_{k2^d+1}^d\right) &= k(k2^d+1)^{d-1} = k2^{d-1}\left(\frac{k2^d+1}{2}\right)^{d-1} \\ &= \alpha(C_{k2^d+1})\alpha^*\left(C_{k2^d+1}^{d-1}\right) \end{aligned} \tag{3}$$

$$\begin{aligned} \alpha\left(C_{k2^d+3}^d\right) = \frac{k(k2^d+3)^d+1}{k2^d+1} &= \left\lfloor \left(\frac{2k(k2^d+3)^{d-1}+1}{k2^d+1}\right)\left(\frac{k2^d+3}{2}\right)\right\rfloor \\ &= \left\lfloor \alpha\left(C_{k2^d+3}^{d-1}\right)\alpha^*(C_{k2^d+3})\right\rfloor \end{aligned} \tag{4}$$

Equation (2) was established in the celebrated paper of Lovász [6]. Hales [4] and Baumert et al [1] independently established (3), and Baumert et al [1] proved (4). Given this state of affairs, the first interesting case is $\alpha(C_{8n+5}^3)$. Setting

$$t_n = (8n+5)\frac{(2n+1)(8n+5)-1}{2},$$

we can summarize the current state of our understanding as follows:

$$t_n \leq \alpha\left(C_{8n+5}^3\right) \leq t_n + 4n + 1.$$

The lower bound is given by construction [1],[2]. The upper bound is one less than the bound given by (1) (as established in [1]). (For small values of n the Lovász theta function gives a slightly better upper bound.) Baumert et al conjectured that $\alpha(C_{8n+5}^3) = t_n$ for all n [1].

Now, we are ready to state our main results.

Theorem 1.

$$\alpha\left(C_{8n+5}^3\right) \leq t_n + 2.$$

Our central interest is the development of new techniques for giving upper bounds on the independence numbers of powers of odd cycles. The proof of Theorem 1 is a combination of structural considerations and a stability Lemma. The structural considerations are based on a classification of all maximum independent sets in C_{4m+1}^2 given by Hales [4] and Baumert et al [1]. The stability Lemma is a classification of all independent sets S in C_{4m+1}^2 such that $|S| = \alpha(C_{4m+1}^2) - 1$. It turns out that all such independent sets are given by simple modifications of maximum independent sets in C_{4m-3}^2, C_{4m+1}^2 or C_{4m+5}^2. Very loosely speaking, we use a stability result for the independent sets in C_{8n+5}^2 to give an upper bound on the independence number of C_{8n+5}^3. For some values of n we determine the independence number of C_{8n+5}^3 exactly.

Theorem 2. *If $8n+5$ is prime then $\alpha\left(C_{8n+5}^3\right) = t_n$.*

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$K_{2,t}$ minors in dense graphs

Maria Chudnovsky
(joint work with Bruce Reed and Paul Seymour)

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

Mader [5] proved that for every graph H there is a constant C_H such that every graph G not containing H as a minor satisfies $|E(G)| \leq C_H|V(G)|$, but determining the best possible constant C_H for a given graph H is a question that has been answered for very few graphs H, and not much research has gone into questions of this type.

A particular case that *has* been intensively studied is when H is a complete graph K_t (this is motivated by Hadwiger's conjecture). One natural way to make a large dense graph with no K_t minor is to take a complete graph of size $t-2$, and add $n-t+2$ more vertices each adjacent to all vertices in the complete graph. This produces an n-vertex graph with no K_t minor and with $(t-2)n - \frac{1}{2}(t-1)(t-2)$ edges, and Mader[5] showed that for $t \leq 7$ this is the maximum possible number of edges in an n-vertex graph with no K_t minor, and it is natural to expect this to extend to all t. But Mader also showed that for $t \geq 8$ this is *not* the correct expression, and Kostochka [2] and Thomason [6] showed that for large t and n the maximum number of edges is $O(t\log(t)^{\frac{1}{2}})n$.

This was something of a disappointment, but what about when H is a complete bipartite graph $K_{s,t}$ say? This case has also attracted a certain amount of attention. Consider the following graph: let H_1 be the union of disjoint copies of the complete graph K_t, and H_2 be a complete graph on $s-1$ (new) vertices; make every vertex of H_1 be adjacent to every vertex of H_2. This graph has no $K_{s,t}$ minor, and therefore $C_{K_{s,t}} > \frac{(t+2s-3)}{2}$.

In [1] Myers proved that this is in fact best possible for $s=2$ and large t:

Theorem 1. *Let $t > 10^{29}$ be a positive integer. Then every graph with n vertices and more than $\frac{t+1}{2}(n-1)$ edges has a $K_{2,t}$-minor.*

Myers also conjectured that the average degree that guarantees a $K_{s,t}$ minor, also guarantees a $K^*_{s,t}$-minor, where $K^*_{s,t}$ is the graph obtained form $K_{s,t}$ by adding all the edges between vertices on the "s-side" of the bipartition:

Conjecture 2. *(Myers): For every positive integer s, there exist a number $C(s)$, such that for every positive integer t a graph with average degree $C(s)t$ has a $K_{s,t}$-minor.*

In [4] Kuhn and Osthus proved a refinement of Myers' conjecture for large t:

Theorem 3. *For every $\epsilon > 0$ and every positive integer s there exists a number $t_0(s,\epsilon)$ such that for all integers $t \geq t_0$ a graph with average degree at least $(1+\epsilon)t$ contains a $K^*_{s,t}$ minor.*

Kostochka and Prince [3] proved a stronger result, but under stronger assumptions:

Theorem 4. *Let s, t be positive integers with $t > (180s\log_2 s)^{1+6s\log_2 s}$. Then every graph with n vertices and at least $\frac{t+3s}{2}(n-s+1)$ edges contains a $K^*_{s,t}$ minor. On the other hand, for arbitrarily large N, there exist graphs with at least N vertices and average degree at least $t+3s-5\sqrt{s}$ that do not have a $K_{s,t}$ minor.*

The theorems above deal with the asymptotics of $C_{K_{s,t}}$ when t is large compared with s. In this paper, we concentrate on the case $s=2$, but we do not impose any restrictions on t. The following is our main theorem:

Theorem 5. *Let $t \geq 2$. Then every graph with n vertices and strictly more than $\frac{1}{2}(t+1)(n-1)$ edges has a $K_{2,t}$ minor.*

Thus we are able to strengthen Theorem 1 to include all values of t. Moreover, Conjecture 2 is an easy corollary of Theorem 5.

We remark, that the graph with no $K_{2,t}$ minor described above is an extremal example for the problem, and therefore the bound in Theorem 5 is best possible when $n-1$ is a multiple of t. However, for other values of n it may not be best possible, and as far as we know, it could be way off. For instance, if $n = \frac{3}{2}t$, Theorem 5 gives an upper bound of about $\frac{1}{2}tn$, but the best lower bound we know is about $\frac{5}{12}tn$.

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$\pi_1(|G|)$, Earrings, and Limits of Free Groups

Reinhard Diestel
(joint work with Philipp Sprüssel)

This talk described our recent combinatorial characterization [2] of the fundamental group of the space $|G|$ formed by a locally finite connected graph G and its ends. We characterize $\pi_1(|G|)$ as a subgroup of a group F_∞ of infinite words of oriented chords of a suitable spanning tree T of G. The group F_∞ in turn embeds as a subgroup in the inverse limit of the free groups F_I on the finite sets $\{\vec{e}_i \mid i \in I\}$ of oriented chords. We thus have subgroup embeddings

$$\pi_1(|G|) \to F_\infty \to \varprojlim F_I,$$

the first of which depends on the structure of G while the second does not.

Although $|G|$, known as the *Freudenthal compactification* of G, is the standard space in which a locally finite connected graph G—such as the Cayley graph of a finitely generated group—is viewed topologically [4], its fundamental group has never been studied explicitly. Implicitly, Higman [5], and later Cannon and Conner [1], characterized it for the first interesting case, that G has exactly one non-trivial end. In this case, $|G|$ is homotopy-equivalent to the Hawaiian Earring (Fig. 1), whose fundamental group is the entire group F_∞.

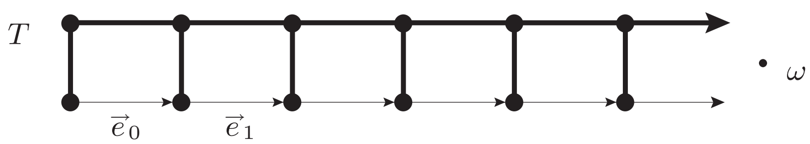

Figure 1. A graph theorist's edition of the Hawaiian Earring

When G is finite, $\pi_1(|G|) = \pi_1(G)$ is the free group on the set of (arbitrarily oriented) chords of any spanning tree T of G. When there are infinitely many chords, $\pi_1(|G|)$ is not a free group [5], and T must be chosen with care. Indeed, an arbitrary spanning tree—such as the subgraph of the ladder obtained by deleting all rungs but the first—may have circles in its closure in $|G|$, in which case there are non-trivial loops in $|G|$ that do not traverse any chords, and hence cannot be represented by a word of chords. However, this is the only obstruction: any spanning tree T whose closure contains no circle, indeed any *topological spanning tree* [4] of G, can be used as the basis for our description of $\pi_1(|G|)$ as a subgroup of F_∞.

As with G finite, we represent loops in $|G|$ by their trace of chords of T, i.e. by a linearly ordered 'word' of oriented chords. These words may now be infinite, and of any countable order type. (However, each letter will occur only finitely often.) For example, the loop α in the ladder of Figure 1 that runs through all the chords along its bottom side from left to right and back, is encoded by the word $w = e_0, e_1, \dots \mid \dots e_1^{-1}, e_0^{-1}$.

In order to capture homotopy of loops, we then have to define reduction of words; for example, the word w above should reduce to the empty word, since α is null-homotopic. When G is finite, one can generate the homotopy class of any loop by local homotopies each retracting just one pass through a chord. Reduction of words, correspondingly, can be achieved by cancelling pairs (e, e^{-1}) of adjacent inverse letters. This fails for our word w above, which contains no such pair. But its desired reduction to the empty word can be achieved by cancelling a 'pair of inverse subsequences', $(e_0, e_1, \dots)$ and $(\dots e_1^{-1}, e_0^{-1})$. So one might hope that cancelling pairs of adjacent inverse subsequences, perhaps of any order type, might capture all homotopies of paths and thereby define a suitable notion of the reduction of words.

However, this is not the case. Indeed, let T_2 be the infinite binary tree with ends, edges arbitrarily oriented. One can construct a loop β through T_2 that traverses every edge exactly once in each direction. The trace of this loop in the edges of T_2 contains no pair of inverse subsequences of length > 1, of any order type. Now turn the edges of T_2 into chords by doubling every edge, subdividing every new edge, and letting T be the tree consisting of all the new edges. Then T is a topological spanning tree of the resulting graph G, the original edges of T_2 are its chords, and β is a null-homotopic loop in $|G|$ whose word of chords cannot be reduced by cancelling any pair of adjacent inverse subsequences of letters. Moreover, the obvious homotopy from β to a constant map obtained by sliding its image down the tree T_2 is not generated by local homotopies that retract β through one chord at a time—at least not if discrete 'time' is expected to be well-ordered.

Our approach to this problem to stick to the idea that reduction of words should happen by cancelling inverse pairs of adjacent letters in some linear order, but to allow arbitrary order types also for this ordering. (In our earlier example, each letter e_i cancels with e_i^{-1}, but this must be preceded by cancellations of all the pairs (e_j, e_j^{-1}) with $j > i$: these have to be deleted first to make e_i and e_i^{-1} adjacent.) We then have to show that from such word reductions we can indeed recover homotopies of loops, and that those homotopies suffice to generate the entire homotopy classes of loops. (Note that covering space theory is not available, since $|G|$ is not semi-locally simply connected at ends.) The first of these tasks accounts for most of the paper. Its difficulty stems from the fact that homotopies between loops have to satisfy continuity requirements at limits of chords that are 'forgotten' in the corresponding words of those chords.

Once reduction of a word w is known to yield a unique reduced word $r(w)$, and the combined map $\alpha \mapsto w_\alpha \mapsto r(w_\alpha)$ is known to be well-defined on homotopy classes, we have a map $\langle\alpha\rangle \mapsto r(w_\alpha)$ from $\pi_1(|G|)$ to F_∞. This map turns out to be an injective homomorphism. And we can determine its image precisely: a sequence $e_0, e_1, \dots$ of chords can clearly occur as a subsequence of the trace of a loop only if the e_n converge to an end of G, and we prove that the image of $\pi_1(|G|)$ in F_∞ consists of precisely those reduced words of chords whose subsequences all converge in $|G|$. Similarly, we can precisely determine the image of F_∞ under its homomorphism to $\varprojlim F_I$.

All in all, our combinatorial characterization of $\pi_1(|G|)$ reads as follows:

Theorem 1 ([2])**.** *Let G be a locally finite connected graph. Let T be a topological spanning tree of G. Let $e_0, e_1, \ldots$ be its chords, arbitrarily oriented.*

(i) *The map $\langle\alpha\rangle \mapsto r(w_\alpha)$ is an injective homomorphism from $\pi_1(|G|)$ to the group F_∞ of reduced finite or infinite words in the letters e_n and their inverses, with image the set of words whose monotonic subwords converge in $|G|$.*

(ii) *The homomorphisms $w \mapsto r(w \restriction I)$ from F_∞ to F_I, for $I \subset \mathbb{N}$ finite, embed F_∞ as a subgroup in $\varprojlim F_I$. It consists of those elements of $\varprojlim F_I$ whose projections $r(w \restriction I)$ use each letter only boundedly often. (The bound may depend on the letter.)*

In [3], we apply Theorem 1 to prove that if G has at least one non-trivial end, then the topological cycle space of G, viewed as a group, is a proper quotient of the first singular homology group of $|G|$.

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Combinatorial Algebraic Topology

Dmitry N. Kozlov

This talk will be introduction to the book [1] recently published by the author. The subject of *Combinatorial Algebraic Topology* is in a certain sense a classical one, since modern algebraic topology derives its roots from dealing with various combinatorially defined complexes and with combinatorial operations on them. Yet the aspects of the theory that we consider here and that we distinguish under the title of this talk are far from classical and have been brought to the attention of the general mathematical public fairly recently.

If one asks oneself the question

What does Combinatorial Algebraic Topology do?

then the answer will be the same of for regular algebraic topology: one computes various algebraic invariants of topological spaces, for example homology groups, or special cohomology elements such as characteristic classes; at times, one is even able to determine the homotopy type. The discriminating feature is provided not by *what* one is computing, but by *how* and for *which classes* of topological spaces it is done.

More precisely, in this talk the focus will be on the algebraic topology of cellular complexes, which are combinatorial both *locally*, meaning that the cell attachments are simple, and *globally*. Being combinatorial locally usually means that we have simplicial complexes, though more and more, further classes of complexes, such as cubical and prodsimplicial ones, find their application in Combinatorial Algebraic Topology. The word "globally" here refers to the fact that the cells themselves are combinatorially enumerated. Of course, the meaning of being combinatorially enumerated is open to interpretation, and probably cannot formally be pinned down without the loss of the desired flexibility. Typically this alludes to the fact that one has a bijection between cells and some objects that are universally perceived as combinatorial, for example graphs, partitions, permutations, and various combinations and enrichments (e.g., by labelings) of these.

Additionally, though the cell attachment maps are easy, the cell inclusions themselves indicate some combinatorial relationship between the objects that are indexing the cells in question. Normally, to obtain the combinatorial objects that are indexing the cells on the boundary of a given cell σ, one would need to perform some combinatorial operation on the object that is indexing σ itself.

Such complexes arise in all sorts of contexts. Sometimes the complexes are simply given directly, though more often they are induced implicitly. For example, frequently one happens to consider a topological space that allows additional structure, such as some kind of stratification. The combinatorial data that can be extracted from such a stratification is the partially ordered set of strata. This is of course a serious trivialization of the space, since only the bare incidence structure is left. There are then standard ways, such as taking the nerve, to associate a simplicial complex to this poset, with the idea that some of the algebro-topological invariants of this complex will reflect something about the initial stratification.

This is an example of a procedure that constitutes the first of perhaps the three major venues of Combinatorial Algebraic Topology: being able to derive new interesting combinatorial objects by building suitable models for topological questions. A classical example of this is the so-called Goresky–MacPherson formula: in short, given a collection of linear subspaces, this formula provides a way to calculate the cohomology groups of the complement of the union of these subspaces, in terms of a certain "combinatorial model," namely homology groups of the so-called order complex of a combinatorial object associated to this family of subspaces, the *intersection lattice*.

The second major venue is that the methods of computation that are established as standard in algebraic topology lead to the unearthing of new discrete structures. For example, spectral sequences are such a tool, and once the filtration on the studied complex is chosen, the calculation, though possibly technically challenging, is nonetheless uniquely determined. The subsequent steps in the computation will unveil new combinatorial objects on a constant basis. As an example, we refer to the computation of the homology groups of certain standard prodsimplicial complexes associated to cycle graphs. In performing the actual calculation along the lines prescribed by the spectral sequence, one uncovers the important `Hom` construction and witnesses the appearance of other classical instances of combinatorial complexes. This can trace its genesis to the original work of Eric Babson and the author on the resolution of the Lovász Conjecture.

Finally, the third major venue is that the combinatorial properties of the indexing objects from discrete mathematics get distinguished by the topology, providing a deeper insight both into the structure theory of these objects and into which part of it is relevant for topology. For example, there are many operations on graphs. However, it is specifically the operation of *fold* that has been singled out in the study of the `Hom` *complexes*, based solely on the fact that it is extremely well behaved from the topological point of view.

We will survey the subject of regular trisps and acyclic categories. If the time will allow we shall also present discrete Morse theory in terms of the new concept *poset maps with small fibers.*

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On disjoint arborescences

András Frank
(joint work with Kristóf Bérczi)

A directed tree is called an *arborescence rooted at* r_0 if each node is reachable from r_0. A *branching with root set* R is a collection of $|R|$ node-disjoint arborescences where R is the set of roots of the arborescences. We call a set-system $\mathcal{F} \subseteq 2^V$ *intersecting* if

$$X, Y \in \mathcal{F},\ X \cap Y \neq \emptyset \ \Rightarrow\ X \cap Y,\ X \cup Y \in \mathcal{F}$$

holds. We say that $D' = (V, F)$ *covers* $\mathcal{F}$ if $\varrho_F(X) \geq 1$ holds for all $X \in \mathcal{F}$ where $\varrho_F(X)$ denotes the number of arcs from F entering X.

In [5] J. Edmonds gave the following characterization of the existence of disjoint spanning arborescences with the same root r_0:

Theorem 1 (Edmonds' theorem (weak form), 1973)**.** *In a digraph* $D = (V, A)$ *there exist* k *disjoint spanning arborescences rooted at* $r_0 \Leftrightarrow \varrho(Z) \geq k$ *for all* $\emptyset \neq Z \subseteq V - r_0$.

A simple proof of the theorem was given by L. Lovász in [1] in 1976. It was observed in [2] that, by using Lovász's proof technique, Edmonds' theorem can be extended as follows:

Theorem 2 (Frank, 1979)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F} \subseteq 2^V$ an intersecting family. Then A can be partitioned into k coverings of $\mathcal{F} \Leftrightarrow \varrho(Z) \geq k$ for all $Z \in \mathcal{F}$.*

By choosing $\mathcal{F} = 2^{V-r_0} - \{\emptyset\}$ one obtains Edmonds' theorem.

Edmonds actually proved a more general result by characterizing the existence of disjoint branchings with prescribed root sets. The following theorem also can be proved using Lovász's approach:

Theorem 3 (Edmonds' theorem (strong form), 1973)**.** *Let $R_1, ..., R_k \subseteq V$ be root sets. There exist k disjoint branchings of root sets $R_1, ..., R_k$, respectively $\Leftrightarrow$ $\varrho(X) \geq p(X)$ for all $\emptyset \neq X \subseteq V$ where $p(X)$ denotes the number of R_i's disjoint from X.*

L. Szegő proved the following common generalization of Theorem 2 and 3:

Theorem 4 (Szegő, 2001)**.** *Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be intersecting families with the following mixed intersecting property:*

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then A can be partitioned into $A_1, ..., A_k$ such that A_i covers $\mathcal{F}_i \Leftrightarrow \varrho(X) \geq p(X)$ for all $X \subseteq V$ where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

It is easy to see that with the choice $\mathcal{F}_1 = ... = \mathcal{F}_k = \mathcal{F}$ we get Theorem 2, while choosing $\mathcal{F}_i = 2^{V-R_i} - \{\emptyset\}$ gives the strong form of Edmonds' theorem. The proof is based on the observation that the mixed intersecting property implies that p is positively intersecting supermodular and hence Lovász's approach works again.

Recently, N. Kamiyama, N. Katoh, and A. Takizawa in [4] proved yet another generalization of Edmonds' strong theorem:

Theorem 5 (Kamiyama-Katoh-Takizawa, 2008)**.** *Let $D = (V, A)$ be a digraph and a $R = \{r_1, ..., r_k\}$ set of roots. Let S_i denote the set of nodes reachable from r_i. There exist disjoint arborescences $(S_1, A_1), ..., (S_k, A_k)$ rooted at $r_1, ..., r_k$, respectively $\Leftrightarrow \varrho(X) \geq p(X)$ for all $X \subseteq V$ where $p(X)$ denotes the number of roots r_i for which $r_i \notin X$ and $S_i \cap X \neq \emptyset$.*

The proof of the theorem also follows Lovász's proof but it is more technical because p is not supermodular in this case. Our main result is an extension of Szegő's theorem to bi-set families which implies the theorem of Kamiyama, Katoh and Takizawa.

We call a pair $X = (X_O, X_I)$ a *bi-set* if $X_I \subseteq X_O \subseteq V$. For $X = (X_O, X_I)$, $Y = (Y_O, Y_I)$ let:

$$X \cap Y = (X_O \cap Y_O, X_I \cap Y_I),$$
$$X \cup Y = (X_O \cup Y_O, X_I \cup Y_I).$$

An arc e *enters* X if e enters both X_O and X_I. Our theorem -that can be considered as the extension of Szegő's theorem to bi-set-systems- is the following:

Theorem 6 (Covering bi-set-systems)**.** *Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be intersecting families of bi-sets on the ground set V, i.e.,*

$$X, Y \in \mathcal{F}_i,\ X_I \cap Y_I \neq \emptyset \ \Rightarrow\ X \cap Y,\ X \cup Y \in \mathcal{F}_i$$

for all $i \in \{1, ..., k\}$. Moreover:

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

The edge set of a digraph $D = (V, A)$ can be partitioned into $A_1, ..., A_k$ such that A_i covers $\mathcal{F}_i \Leftrightarrow \varrho(X) \geq p(X)$ for any bi-set X where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

We call a subset $X \subseteq V$ *separable* if there exists an $i \in \{1, ..., k\}$ such that $X \cap S_i \neq \emptyset$ and $X \setminus S_i \neq \emptyset$. If there is no such i we call X *non-separable*. Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be defined as follows:

$$\mathcal{F}_i = \{(X_O, X_I):\ X_I \subseteq S_i \setminus \{r_i\},\ X_I \text{ is non-separable, } X_O \subseteq V \setminus S_i\}.$$

With this definition we get:

Claim 7. *The bi-set-systems defined above satisfy the mixed intersecting property.*

Claim 8. *If $\varrho(Z) \geq p^{'}(Z)$ for all $Z \subseteq V$, then $\varrho(X) \geq p(X)$ also holds for any bi-set X, where $p^{'}(Z)$ denotes the number of roots r_i for which $r_i \notin Z$, $S_i \cap X \neq \emptyset$ and $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.*

Claim 9. *If $A_i \subseteq A$ covers $\mathcal{F}_i$ then it includes an arborescence F_i rooted at r_i that spans S_i.*

Theorem 6, along with these claims, implies the theorem of Kamiyama, Katoh and Takizawa. Since it also extends Szegő's theorem it can be considered as a generalization of all previous theorems.

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Intersecting families of permutations, an algebraic approach.

Ehud Friedgut
(joint work with Haran Pilpel)

A family $X \subset S_n$ is called *intersecting* if for every $\sigma, \tau \in X$ there exists i such that $\sigma(i) = \tau(i)$. Deza and Frankl [2], in 1977, observed that the maximal size of such a family is $(n-1)!$, and conjectured that the unique extremal examples are the cosets of stabilizers of points. This was confirmed by Cameron and Ku[1], and independently by Larose and Malvenuto [3] in 2003.

We provide a new proof of this using representation theory of S_n. Basically what we do is this:

(1) Define the Cayley graph that describes the problem
(2) Use Hoffman's bound on the independence number in terms of the eigenvalues of the graph.
(3) Characterize the case of equality.

For step (2) we need to calculate the eigenvalues in terms of the different irreducible characters of S_n. We show that the minimal eigenvalue is associated with the $(n-1)$-dimensional representation whose sum with the trivial one gives the permutation representation.

This, in turn, teaches us that in the case of equality the Fourier transform of the intersecting family is concentrated on that representation, the trivial one, and no others. We then recompose the trivial representation and the $(n-1)$ dimensional one to get back the permutation representation, and use the Fourier inversion formula to deduce that for all irreducible ρ

$$\hat{f}(\rho)^2 = \frac{1}{n}\hat{f}(\rho)$$

where f is the characteristic function of our family. This implies that X is a subgroup of index n, and we're done.

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Color critical hypergraphs, a new algebraic proof for Lovász' theorem

Zoltán Füredi
(joint work with Attila Sali)

A k-uniform hypergraph $(V, \mathcal{E})$ is 3-*color critical* if it is not 2-colorable, but for all $E \in \mathcal{E}$ the hypergraph $(V, \mathcal{E} \setminus \{E\})$ is 2-colorable. The only 3-color critical *graphs* are the odd cycles. The investigation of color critical graphs is among the oldest problems of extremal combinatorics (see e.g., Dirac [3] from 1952). Toft [7] proved that there exist 4-chromatic critical graphs G with more than $\frac{1}{16}|V(G)|^2$ edges (conjectured by P. Erdős). He also gave rather sharp upper and lower bounds on the maximum number of edges in c-critical k-chromatic hypergraphs, except in the case 3. For some recent results and generalizations see [4] or [6]. Many problems remain open.

Lovász [5] proved in 1976, that

$$|\mathcal{E}| \le \binom{n}{k-1} \tag{1}$$

for a 3-color critical k-uniform hypergraph. He used a sieve method in a very clever way. The aim of this talk was to present a short proof for (1) using the algebraic method.

The main tool of the proof is a refinement of following result which was proved in [1] and that can be considered as generalization of Lovász' result.

Theorem 1. *Let $\mathcal{E} \subseteq \binom{[m]}{k}$ be a k-uniform set system on an underlying set X of m elements. Let us fix an ordering $E_1, E_2, \dots E_t$ of $\mathcal{E}$ and a prescribed partition $A_i \cup B_i = E_i$ $(A_i \cap B_i = \emptyset)$ for each member of $\mathcal{E}$. Assume that for all $i = 1, 2, \dots, t$ there exists a partition $C_i \cup D_i = X$ $(C_i \cap D_i = \emptyset)$, such that $E_i \cap C_i = A_i$ and $E_i \cap D_i = B_i$, but $E_j \cap C_i \neq A_j$ and $E_j \cap C_i \neq B_j$ for all $j < i$. (That is, the ith partition cuts the ith set as it is prescribed, but does not cut any earlier set properly.) Then*

$$t \le \binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0}. \tag{2}$$

Theorem 1 was applied in estimates concering the size of a matrix with forbidden subconfigurations, more on this, see, e.g. [2].

Proof of Theorem 1 (Sketch): We define a polynomial $p_i(\mathbf{x}) \in \mathbb{R}[x_1, x_2, \dots, x_m]$ for each E_i as follows.

$$p_i(x_1, x_2, \dots, x_m) = \prod_{a \in A_i} (1 - x_a) \prod_{b \in B_i} x_b + (-1)^{k+1} \prod_{a \in A_i} x_a \prod_{b \in B_i} (1 - x_b) \tag{3}$$

Polynomials defined by (3) are multilinear of degree at most $k-1$, since the product $\prod_{e \in E_i} x_e$ cancels by the coefficient $(-1)^{k+1}$. Thus, they are from the space generated by monomials of type $\prod_{j=1}^{r} x_{i_j}$, for $r = 0, 1, \dots k-1$. The dimension of this space over $\mathbb{R}$ is $\binom{m}{k-1} + \binom{m}{k-2} + \dots + \binom{m}{0}$. Finally, one prove that polynomials $p_1(\mathbf{x}), p_2(\mathbf{x}), \dots p_t(\mathbf{x})$ are linearly independent over $\mathbb{R}$, which implies (2). □

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Convex sets in acyclic digraphs

Stefanie Gerke

(joint work with Paul Balister and Gregory Gutin)

Let D be an acyclic digraph of order n. A non-empty set X of vertices in D is *convex* if there is no directed path in D between vertices of X containing a vertex not in X. A non-empty set X of vertices in D is *connected* if the underlying undirected graph of $D[X]$, the subgraph of D induced by X, is connected. The set of all convex sets of D is denoted by $\mathcal{CO}(D)$ and its size by $\mathrm{co}(D)$. The set of all connected convex sets of D is denoted by $\mathcal{CC}(D)$ and its size by $\mathrm{cc}(D)$. Convex sets and connected convex sets in acyclic digraphs are of interest in the field of custom computing in which central processor architectures are parameterized for particular applications, see, e.g., [1, 2].

Gutin, Johnstone, Reddington, Scott, Soleimanfallah, and Yeo [2] introduced an algorithm $\mathcal{A}$ determining all connected convex sets of D in time $O(n \cdot \mathrm{cc}(D))$. where $n = |V(D)|$ is the order of D. They observed that $\mathcal{A}$ can be modified to produce all convex sets in time $O(n \cdot \mathrm{co}(D))$. The authors of [2] conjectured that the sum of the sizes of all convex sets (all connected convex sets, respectively) in D equals $\Theta(n \cdot \mathrm{co}(D))$ ($\Theta(n \cdot \mathrm{cc}(D))$, respectively). If the conjecture were true, then their algorithms would be optimal. The conjecture can be formulated differently. Let $\bar{s}_{\mathrm{co}}(D)$ and $\bar{s}_{\mathrm{cc}}(D)$ be the average size of a convex set and the average size of a connected convex set in D. The conjecture claims that $\bar{s}_{\mathrm{co}}(D) = \Theta(n)$ and $\bar{s}_{\mathrm{cc}}(D) = \Theta(n)$.

We disprove both parts of the conjecture by showing that the following family $\mathcal{F} = \{D_1, D_2, \ldots\}$ of digraphs satisfies $\bar{s}_{\mathrm{co}}(D) = O(\sqrt{n})$ and $\bar{s}_{\mathrm{cc}}(D) = O(\sqrt{n})$. For $t = 1, 2, \ldots$ and $r = \lceil \sqrt{t} \rceil$, the acyclic digraph D_t consists of vertex set $V(D_t) =$

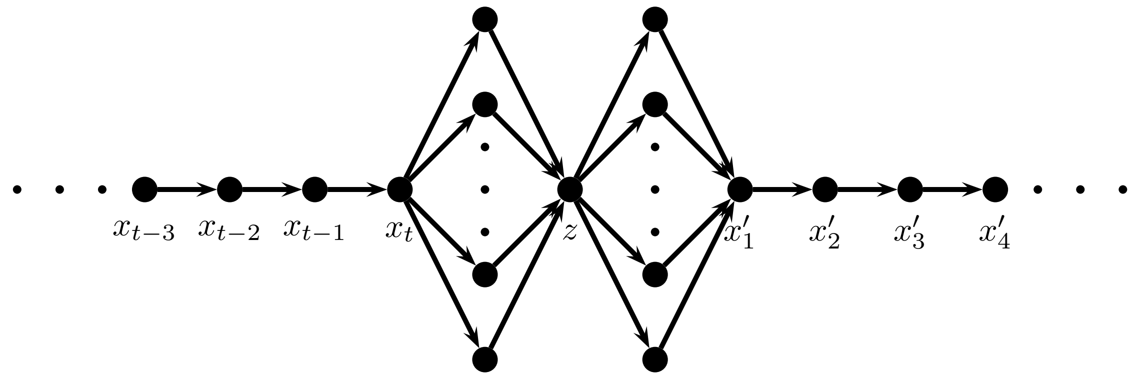

Figure 1. Digraphs from $\mathcal{F}$

$X \cup Y \cup \{z\} \cup Y' \cup X'$, where

$$X = \{x_i : \ i \in [t]\}, \qquad X' = \{x'_i : \ i \in [t]\},$$
$$Y = \{y_j : \ i \in [r]\}, \qquad Y' = \{y'_j : \ i \in [r]\},$$

and arc set

$$A(D_t) = \{x_i x_{i+1}, x'_i x'_{i+1} : \ i \in [t-1]\} \cup \{x_t y_j, y_j z, z y'_j, y'_j x'_1 : j \in [r]\}.$$

For illustration, see Figure 1. We also introduce a simple algorithm that returns all convex sets in time $O(\sum_{C \in \mathcal{CO}(D)} |C|)$ which is clearly asymptotically optimal. The question whether $O(n\mathrm{cc}(D))$ is the asymptotically best running time to compute all connected convex set of a digraph of size n remains open.

We also show that each connected digraph of order n contains at least $n-k+1$ connected convex sets of size k for each $1 \le k \le n$. This extends a result of Gutin and Yeo [3] who showed that each connected acyclic digraph of order n has at least $n(n+1)/2$ connected convex sets.

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Arrow's and Gibbard-Satterthwaite theorems

Gil Kalai

(joint work with Ehud Friedgut and Noam Nisan)

A social choice function aggregates the preferences of all members of society towards a common social choice. Formally, let $[m]$ be a set of m alternatives (candidates), over which n voters have preferences. The preferences of the ith voter are specified as $x_i \in L$, where L denotes the set of full orders over $[m]$ (thus L corresponds to S_m). Using this notation, a social choice function is a function $f : L^n \to [m]$. We will also write the vector x of preferences as (x_i, x_{-i}) when wanting to single out the vote of the i'th voter, or as (x'_i, x_{-i}) after changing the ith coordinate to x'_i.

There is a vast literature on the design of social choice functions, also called voting methods or election rules.

One of the basic desired properties from a social choice function is implied by our thinking of them as "asking the voters about their preferences": voters should not gain from reporting false preferences rather than their true ones. Formally:

Definition 1. *A (profitable) manipulation by voter i of a social choice function f at profile $(x_1, \ldots, x_n)$ is a preference x'_i such that $f(x'_i, x_{-i})$ is preferred by voter i to $f(x_i, x_{-i})$.*

Intuitively, if such a manipulation exists, then voter i would be better off by "voting strategically": reporting x'_i as his preference rather than the true x_i. The Gibbard-Satherwaite theorem [Gib73, Sat75] states that *every* "non-trivial" social choice function is strategically vulnerable, where "nontrivial" means not a dictatorship and whose range contains at least three alternatives. We ask *how often* does this happen: for what fraction of profiles does such a manipulation exist? [1] Can it be tiny? Perhaps exponentially small? Let us define the following quantification of the probability of a random manipulation:

Definition: The *manipulation power* of voter i on a social choice function f, denoted $M_i(f)$, is the probability that x'_i is a profitable manipulation of f by voter i at profile $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ and x'_i are chosen uniformly at random among all full orders on $[m]$.

This definition assumes a uniform distribution over preferences, which while certainly unrealistic, is the natural choice for proving a "lower bound"[2]. In particular, the lower bound, up to a factor δ, applies also to any distribution that gives each preference profile at least a δ fraction of the probability given by the uniform distribution.

To formally state our main theorem, we will require a few standard definitions: A social choice function is neutral if the names of the candidates "do not matter", formally, if f commutes with permutations of $[m]$, i.e. $f(\sigma(x_1), \ldots, \sigma(x_n)) = \sigma(f(x_1, \ldots, x_n))$. A dictatorship is a social choice function that always chooses the top choice of a fixed voter. The distance of f from a dictatorship is simply the minimal fraction of values that need to be changed to turn f into a dictatorship.

[1] Functions that are very close to being a dictatorship may have a very small number of such manipulable profiles so this paper is concerned with social choice functions that are *far* from being trivial.

[2] Note that we cannot hope for an impossibility result for *every* distribution, e.g. since for every social choice function we can take a distribution on its non-manipulable profiles.

Main Theorem: There exists a constant $C > 0$ such that For every $\epsilon > 0$, if f is a neutral social choice function among 3 alternatives for n voters that is ϵ-far from dictatorship, then: $\sum_{i=1}^{n} M_i(f) \geq C\epsilon^2$.

This immediately implies that for fixed ϵ, some voter has non-negligible manipulation power $\max_i M_i(f) \geq \Omega(1/n)$. It is easy to see that one cannot bound $\max_i M_i(f)$ below by a constant independently of n since for the plurality voting method $M_i(f) = \theta(1/\sqrt{n})$ Furthermore, for the plurality voting method only for a $1/\sqrt{n}$ fraction of profiles can manipulated at all by any single player. While it is easy to see that the bound on $\sum_i M_i(f)$ cannot be improved to being more than a constant, the first open problem we leave is whether the bound on $\max_i M_i(f)$ can be improved further. We also do not know how to replace the neutrality condition with the weaker "correct" condition: being far from having a range of size at most 2. We leave this as the second open problem.

Our third open problem concerns the case of more than three alternatives, $m > 3$. While some parts of our proof extend to this case, (and indeed we took the care to state them in the general form), we were not able to extend all required parts of the proof. We do conjecture that the theorem does generalize to $m > 3$, perhaps with the exact form of the bound decreasing polynomially in m. This is our third open problem.

A word is in order regarding our techniques. Our starting point is the recent work of [Ka02] that obtained quantitative versions of Arrow's theorem [Arr51] using methods that involve the Fourier transform on the boolean hypercube. Our proof then has two further components. First, a "quantitative-preserving" reduction from Arrow's theorem to a variant of the Gibbard-Satherwaite theorem that allows multi-voter manipulation, and then a directed isoperimetric inequality that allows us to move to single-voter manipulation. Our proof of the isoperimetric inequality relies on the FKG correlation inequality [FKG71] (or, more precisely, on Harris' inequality, [Ha60]).

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Coloring uniform hypergraphs with few edges

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(joint work with Mohit Kumbhat)

The *girth* of a hypergraph is the length of its shortest cycle. A hypergraph of girth at least three is also called *simple.* Let $m(r,t,g)$ denote the smallest number of edges in an r-uniform hypergraph with girth at least g and chromatic number at least $t+1$. In their seminal paper [1], Erdős and Lovász gave upper bound

$$m(r,t,g) \le 4 \cdot 20^{g-1} r^{3g-5} t^{(g-1)(r+1)} \tag{1}$$

for all g and lower bound

$$m(r,t,3) \ge \frac{t^{2(r-2)}}{16r(r-1)^2} \tag{2}$$

for simple hypergraphs. The ratio of the upper bound to the lower bound for simple hypergraphs is only r^7. The bound (2) was derived from the following famous result.

Theorem 1. [1] *If $t, r \ge 2$, then every r-uniform hypergraph $\mathcal{H}$ with maximum degree at most $\frac{1}{4} t^r r^{-1}$ is t-colorable.*

To derive the bound, they used an interesting trick of *trimming.*

Szabó [2] refined the bound of Theorem 1 for simple hypergraphs as follows.

Theorem 2. *If $t \ge 2$, and $\epsilon > 0$ are fixed and r is sufficiently large, then every r-uniform simple hypergraph $\mathcal{H}$ with maximum degree at most $t^r r^{-\epsilon}$ is t-colorable.*

Actually, he proved the theorem only for $t = 2$, since that was what he needed for his applications, but the technique works for any fixed t. Again, applying trimming and this theorem, one easily gets that for fixed t and ϵ and large r,

$$m(r,t,3) \ge \frac{t^{2r}}{r^{1+\epsilon}}. \tag{3}$$

The main result of this talk says that for fixed $t \ge 2$ and $\epsilon > 0$ and sufficiently large r, if a simple r-uniform hypergraph $\mathcal{H}$ cannot be colored with t colors, then either it has a vertex of degree greater than $r\,t^r$, or there are "many" vertices of degree greater than $t^r r^{-\epsilon}$. This will improve the bound (3) by a factor of r. Furthermore, we extend our bound to b-simple hypergraphs.

A hypergraph $\mathcal{H}$ is *b-simple* if $|e \cap e'| \le b$ for every distinct $e, e' \in E(\mathcal{H})$. A 1-simple hypergraph is a simple hypergraph. Let $f(r,t,b)$ denote the smallest number of edges in an r-uniform b-simple hypergraph that is not t-colorable. From our main result we deduce that for fixed t, b and $\epsilon > 0$ and sufficiently large r,

$$f(r,t,b) \ge \frac{t^{r(1+1/b)}}{r^{\epsilon}}. \tag{4}$$

It turns out that the bound cannot be improved by more than a factor polynomial in r. Using the technique of the proof of (2) in [1], we show that for large r,

$$f(r,t,b) \le 40t^2 \left(t^r r^2\right)^{1+1/b}. \tag{5}$$

We also use our main result and trimming to derive the following lower bounds on $m(r,t,g)$ for arbitrary fixed g (in [1], the bound was only for $g = 3$):

$$m(r,t,2s+1) \ge \frac{t^{r(1+s)}}{r^{\epsilon}}, \tag{6}$$

if r is large in comparison with t, s and $1/\epsilon$.

Remark. It looks that during the meeting, Vojtech Rödl jointly with the speaker managed to somewhat improve the upper bounds (1) and (5).

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On the poset of generalised non-crossing partitions associated to reflection groups

Christian Krattenthaler

In his 2006 thesis [1], Drew Armstrong introduced a new family of posets associated to finite reflection groups, which have many interesting and fascinating properties. We find several of them in [1], but at the same time [1] contains many open problems and conjectures which spurred further work. In this abstract, some results on enumerative aspects of these posets are summarized.

We start by defining the posets. Let Φ be a finite root system of rank n, and let $W = W(\Phi)$ be the corresponding reflection group. For all reflection group terminology, we refer the reader to [8]. We define first the *non-crossing partition lattice* $NC(\Phi)$ (cf. [3, 4]). By definition, any element w of W can be represented as a product $w = t_1 t_2 \cdots t_\ell$, where the t_i's are reflections. We call the minimal number of reflections which is needed for such a product representation the *absolute length* of w, and we denote it by $\ell_T(w)$. We then define the *absolute order* on W, denoted by $\le_T$, via

$$u \le_T w \quad \text{if and only if} \quad \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w).$$

As is well-known and easy to see, this is equivalent to the statement that every shortest representation of u by reflections occurs as an initial segment in some shortest product representation of w by reflections. Let c be a *Coxeter element* in W. Then $NC(\Phi)$ is defined to be the restriction of the partial order $\le_T$ to the set of all elements which are less than or equal to c in this partial order.

This definition makes sense since any two Coxeter elements in W are conjugate to each other; the induced inner automorphism then restricts to an isomorphism of the posets corresponding to the two Coxeter elements. It can be shown that $NC(\Phi)$ is in fact a lattice (see [5] for a uniform proof), and moreover self-dual (this is obvious from the definition). Clearly, the minimal element in $NC(\Phi)$ is the identity element in W, which we denote by ε, and the maximal element in $NC(\Phi)$ is the chosen Coxeter element c. The term "non-crossing partition lattice" is used because $NC(A_n)$ is isomorphic to the lattice of non-crossing partitions of $\{1, 2, \dots, n+1\}$, originally introduced by Kreweras [12] (see also [7]), and since also $NC(B_n)$ and $NC(D_n)$ can be realised as lattices of non-crossing partitions (see [2, 14]).

In addition to a fixed root system, the definition of Armstrong's *generalised non-crossing partitions* require a fixed positive integer m. The poset of *m-divisible non-crossing partitions associated to the root system* Φ has as ground-set the following subset of $(NC(\Phi))^{m+1}$:

$$NC^m(\Phi) = \big\{(w_0; w_1, \dots, w_m) : w_0 w_1 \cdots w_m = c \text{ and } \\ \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c)\big\}.$$

The order relation is defined by

$$(u_0; u_1, \dots, u_m) \le (w_0; w_1, \dots, w_m) \quad \text{if and only if} \quad u_i \ge_T w_i,\ 1 \le i \le m.$$

The poset $NC^m(\Phi)$ is graded by the rank function

$$\operatorname{rk}\big((w_0; w_1, \dots, w_m)\big) = \ell_T(w_0).$$

Thus, there is a unique maximal element, namely $(c; \varepsilon, \dots, \varepsilon)$, where ε stands for the identity element in W, but, for $m > 1$, there are many different minimal elements. In particular, $NC^m(\Phi)$ has no least element if $m > 1$; hence, $NC^m(\Phi)$ is not a lattice for $m > 1$. (It is, however, a graded join-semilattice, see [1, Theorem 3.4.4].) Combinatorial realisations of $NC^m(\Phi)$ for the classical types $\Phi = A_n, B_n, D_n$ are known and are summarised in [11, Sec. 7].

We are interested in *rank selected chain enumeration* in $NC^m(\Phi)$, that is, in the enumeration of all (multi-)chains $x_1 \le x_2 \le \cdots \le x_{l-1}$ in $NC^m(\Phi)$, where $\operatorname{rk}(x_i) = r_i$, $i = 1, 2, \dots, l-1$, for fixed $r_1, r_2, \dots, r_l$. It suffices to solve this problem for the irreducible root systems. For the case of type A_n, the problem had already been solved by Edelman [6, Theorem 4.2].

Theorem 1. *The number of (multi-)chains $x_1 \le x_2 \le \cdots \le x_{l-1}$ in $NC^m(A_n)$ with $\operatorname{rk}(x_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, is given by*

$$\frac{1}{n}\binom{n}{s_1}\binom{mn}{s_2}\cdots\binom{mn}{s_l},$$

where $s_1 + s_2 + \cdots + s_l = n$.

Armstrong solved the corresponding problem in type B_n in [1, Theorem 4.5.7].

Theorem 2. *The number of (multi-)chains $x_1 \le x_2 \le \cdots \le x_{l-1}$ in $NC^m(B_n)$ with $\operatorname{rk}(x_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, is given by*

$$\binom{n}{s_1}\binom{mn}{s_2}\cdots\binom{mn}{s_l},$$

where $s_1 + s_2 + \cdots + s_l = n$.

In joint work with T. Müller, the author proved the corresponding result in type D_n; see [11, Cor. 19].

Theorem 3. *The number of (multi-)chains $x_1 \le x_2 \le \cdots \le x_{l-1}$ in $NC^m(D_n)$ with $\operatorname{rk}(x_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \dots, l-1$, is given by*

$$2\binom{n-1}{s_1}\binom{m(n-1)}{s_2}\cdots\binom{m(n-1)}{s_l}$$
$$+ m\sum_{j=2}^{l}\binom{n-1}{s_1}\binom{m(n-1)}{s_2}\cdots\binom{m(n-1)-1}{s_j-2}\cdots\binom{m(n-1)}{s_l}$$
$$+\binom{n-2}{s_1-2}\binom{m(n-1)}{s_2}\cdots\binom{m(n-1)}{s_l},$$

where $s_1 + s_2 + \cdots + s_l = n$.

While Theorems 1 and 2 are proved in bijective ways, the proof of Theorem 3 in [11] is highly non-bijective. In fact, what is computed in [11] are finer invariants of $W(D_n)$ (and also of $W(A_{n-1})$ and $W(B_n)$): so-called decomposition numbers for finite reflection groups. The result in Theorem 3 then follows upon carrying out certain summations involving the decomposition numbers. Further enumerative results, in which conditions are imposed on the block sizes of elements, can as well be found in [11].

Since the decomposition numbers for the exceptional reflection groups are known as well from [9] and [10], the rank selected chain enumeration can also be carried out in the m-divisible non-crossing partitions associated to the exceptional reflection groups. For example, the number of all chains $x_1 \le x_2 \le x_3$ in $NC^m(E_8)$, where x_1 is of rank 4, x_2 is of rank 6, and x_3 is of rank 7, is given by

$$\frac{75m^3\,(8055m - 1141)}{2}.$$

In closing, we point out that very recently Reading has found a very surprising uniform recurrence relation for rank selected chain enumeration in [13].

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Cycles in directed graphs

Daniela Kühn
(joint work with Peter Keevash, Luke Kelly, Deryk Osthus, and Andrew Treglown)

A central topic in graph theory is that of giving conditions under which a graph is Hamiltonian. One such result is the classical theorem of Dirac [3], which states that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. For an analogue in directed graphs it is natural to consider the *minimum semi-degree* $\delta^0(D)$ of a digraph D, which is the minimum of its minimum outdegree $\delta^+(D)$ and its minimum indegree $\delta^-(D)$. The corresponding result is a theorem of Ghouila-Houri [4], which states that any digraph on n vertices with minimum semi-degree at least $n/2$ contains a Hamilton cycle. (When referring to paths and cycles in directed graphs we always mean that these are directed, without mentioning this explicitly.) Both of these results are best possible.

In 1979 Thomassen [13] raised the natural corresponding question of determining the minimum semi-degree that forces a Hamilton cycle in an *oriented graph* (i.e. in a directed graph that can be obtained from a (simple) undirected graph by orienting its edges). He [15] showed that a minimum semi-degree of $n/2-\sqrt{n/1000}$ suffices (see also [14]). Note that this degree requirement means that the oriented graph is not far from being a regular tournament. Häggkvist [5] constructed an example which gives a lower bound of $(3n-5)/8$. He also improved the upper bound to $(1/2-2^{-15})n$. Later Häggkvist and Thomason [6] improved the upper bound further to $(5/12+o(1))n$. Recently, Kelly, Kühn and Osthus [8] obtained an approximate solution. They proved that an oriented graph on n vertices with minimum semi-degree at least $(3/8 + o(1))n$ has a Hamilton cycle. The following result of Keevash, Kühn and Osthus [7] gives the exact bound for large oriented graphs.

Theorem 1. [7] *There exists a number n_0 so that any oriented graph G on $n \geq n_0$ vertices with minimum semi-degree $\delta^0(G) \geq \lceil \frac{3n-4}{8} \rceil$ contains a Hamilton cycle.*

Note that Theorem 1 implies that every sufficiently large regular tournament on n vertices contains at least $n/8$ edge-disjoint Hamilton cycles. (To verify this, note that in a regular tournament, all in- and outdegrees are equal to $(n-1)/2$. We can then greedily remove Hamilton cycles as long as the degrees satisfy the condition in Theorem 1.) This is the best bound so far towards the classical conjecture of Kelly (see e.g. [1]), which states that every regular tournament on n vertices can be partitioned into $(n-1)/2$ edge-disjoint Hamilton cycles.

Häggkvist [5] also made the following conjecture which is closely related to Theorem 1. Given an oriented graph G, let $\delta(G)$ denote the minimum degree of G (i.e. the minimum number of edges incident to a vertex) and set $\delta^*(G) := \delta(G) + \delta^+(G) + \delta^-(G)$. Häggkvist conjectured that if $\delta^*(G) > (3n-3)/2$, then G has a Hamilton cycle. (Note that this conjecture would not quite imply Theorem 1 as it results in a marginally greater minimum semi-degree condition.) In [8], this conjecture was verified approximately:

Theorem 2. [8] *For every $\alpha > 0$ there exists a number n_0 so that any oriented graph G on $n \geq n_0$ vertices with $\delta^*(G) \geq (3/2 + \alpha)n$ has a Hamilton cycle.*

Using Ghouila-Houri's theorem it is easy to show that every digraph D with minimum semi-degree $> n/2$ is *vertex-2-pancyclic*, i.e. every vertex of D lies on a cycle of length ℓ for every $\ell = 2, \ldots, n$. The following result of Kelly, Kühn and Osthus [9] provides an approximate analogue of this for oriented graphs.

Theorem 3. [9] *For every $\alpha > 0$ there exists a number n_0 so that any oriented graph G on $n \geq n_0$ vertices with minimum semi-degree $\delta^0(G) \geq (3/8 + \alpha)n$ is vertex-4-pancyclic, i.e. every vertex of G lies on a cycle of length ℓ for every $\ell = 4, \ldots, n$.*

Theorem 3 is best possible in the sense that one cannot guarantee vertex-3-pancyclicity: there are infinitely many oriented graphs with minimum semi-degree $2n/5 - 1$ which have a vertex that does not lie on a triangle (see [9]). However, combining Theorem 3 with a result of (e.g.) Shen [12] on the triangle case of the Caccetta-Häggkvist conjecture implies that the minimum semi-degree in Theorem 3 forces G to be pancyclic, i.e. G contains a cycle of length ℓ for every $\ell = 3, \ldots, n$.

Clearly, the bound on the minimum semi-degree in Theorem 3 is best possible up to the term αn. However, to force cycles of fixed length containing any given vertex a much smaller minimum semi-degree suffices:

Theorem 4. [9] *For every $\ell \geq 4$ there exists a number n_0 so that whenever G is an oriented graph on $n \geq n_0$ vertices with minimum semi-degree $\delta^0(G) \geq \lfloor n/3 \rfloor + 1$ then every vertex of G lies on a cycle of length ℓ.*

The bound in Theorem 4 is best possible. Indeed, if ℓ is not divisible by 3 then a 'blow-up' of a triangle has minimum semi-degree $\lfloor n/3 \rfloor$ but does not contain a cycle of length ℓ. If ℓ is divisible by 3 there are infinitely many oriented graphs with minimum semi-degree $\lfloor n/3 \rfloor$ having a vertex that does not lie on a cycle of length ℓ. However, for such ℓ, it is an open problem to determine the minimum semi-degree which forces a cycle of length ℓ (not necessarily containing a given vertex), see [9] for partial results and a discussion of this.

Another way to force Hamilton cycles in graphs is by imposing conditions on the degree sequence. For undirected graphs this question is settled by Chvátal's theorem [2] which characterizes the degree sequences forcing a Hamilton cycle. It states that a graph G is Hamiltonian if its degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

$$d_k \leq k \implies d_{n-k} \geq n-k$$

for all $k < n/2$. This condition on the degree sequence is best possible. Nash-Williams [11] conjectured that for digraphs the following analogue holds. Let $d_1^+ \leq \cdots \leq d_n^+$ and $d_1^- \leq \cdots \leq d_n^-$ be the out- and indegree sequences of a strongly connected digraph D. If

$$d_k^+ \leq k \implies d_{n-k}^- \geq n-k$$

and

$$d_k^- \leq k \implies d_{n-k}^+ \geq n-k$$

for all $k < n/2$ then D contains a Hamilton cycle. The following result of Kühn, Osthus and Treglown [10] provides an approximate version of this conjecture:

Theorem 5. [10] *For every $\alpha > 0$ there exists a number n_0 so that the following holds. Let $d_1^+, \ldots, d_n^+$ and $d_1^-, \ldots, d_n^-$ be the out- and indegree sequences of a digraph D on $n \geq n_0$ vertices. Suppose that*

$$d_k^+ \leq k + \alpha n \implies d_{n-k-\alpha n}^- \geq n-k$$

and

$$d_k^- \leq k + \alpha n \implies d_{n-k-\alpha n}^+ \geq n-k$$

for all $k < n/2$. Then D contains a Hamilton cycle.

See [10] for a related result on pancyclicity.

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Euclidean Ramsey Theory

Imre Leader
(joint work with Paul Russell and Mark Walters)

We say that a finite set S, in some Euclidean space $\mathbb{R}^d$, is *Ramsey* if for any k there is an n such that whenever $\mathbb{R}^n$ is k-coloured there is a monochromatic set isometric to S. The study of which sets are Ramsey is called *Euclidean Ramsey Theory.* Initial progress was made by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, who proved that any subset of a brick (cuboid), in any dimension, is Ramsey, and also that every Ramsey set must be *spherical*, meaning that it lies on the surface of some sphere. The conjecture made there was that every spherical set is Ramsey, and this conjecture has dominated subsequent work in Euclidean Ramsey Theory. Progress towards this has been slow: Frankl and Rödl showed that every triangle is Ramsey, and then showed that every simplex is Ramsey. Kříž showed that every regular polygon is Ramsey, and in fact proved more: that any set on which a cyclic group (or indeed a soluble group) of isometries acts transitively must be Ramsey. We have a new conjecture. Our idea is based on a principle that we have observed in operation in all papers on Euclidean Ramsey Theory so far. In each case, it seems that, to prove a set is Ramsey, one first embeds it in a transitive set (a set whose isometry group acts transitively), and then makes some clever combinatorial arguments to show that this transitive set has the Ramsey property required. The actual machinery for this can differ greatly from paper to paper, and the transitive set can have a *much* higher dimension than the original set, but we have noticed that this transitivity is always present. Based on this, and some other facts, we are led to conjecture that transitivity is the key: that a set is Ramsey if and only if it embeds in a transitive set.

This would have implications for all parts of the theory. On the one hand, all known results that particular sets are Ramsey would be subsumed in, and unified by, this result. On the other hand, the key result on the 'other direction', that Ramsey sets must be spherical, which at the moment is a rather technical and unenlightening argument, would become a natural and automatic deduction (it is a very simple exercise to show that every transitive set must be spherical – by considering its minimal circumscribing sphere).

Until recently, we did not know if our conjecture would *prove* or *disprove* the classical conjecture that all spherical sets are Ramsey. In other words, we did not know if every spherical set embeds in some transitive set. However, Russell has now proved that, for $k \geq 18$, there exist cyclic k-gons that do not embed in any transitive set.

Word maps, graph lifts and spectra

Nati Linial
(joint work with Doron Puder)

Let $w \neq 1$ be a formal word, i.e., an element of the free group $\mathbf{F}_k$ with generators $g_1, \ldots, g_k$. For permutations $s_1, \ldots, s_k \in S_n$, let $w(s_1, \ldots, s_k) \in S_n$ be the permutation obtained by replacing for each i, every occurrence of g_i in w by s_i. In the *word map* associated with w, the permutations s_i are selected uniformly at random, and one considers the resulting probability distribution on S_n. Such distributions are of interest in various fields (see [LSh07] and the references therein for the group-theoretic perspective). In [Nica94], A. Nica made a very interesting discovery: Consider $X_{w,L}^{(n)}$, the random variable on S_n^k that counts the number of cycles of length L in $w(s_1, \ldots, s_k)$. Nica's theorem determines for every fixed w and L the limit distribution (as $n \to \infty$) of $X_{w,L}^{(n)}$. Surprisingly, perhaps, the answer depends only on the largest integer d so that $w = u^d$ for some $u \in \mathbf{F}_k$. Our first result is a significantly shorter and more conceptual proof of this theorem. Our method of proof suggests several conjectures which roughly state that the expectation of $X_{w,L}^{(n)}$ is bounded from below by its limit. We prove several (partial) results in this vein.

In combinatorics, word maps appear in the study of graph spectra. We next consider the second eigenvalue, λ, of random d-regular graphs. Broder and Shamir [BS87] had shown that $\lambda(G) = O(d^{3/4})$ holds almost surely. In a recent tour-de-force Friedman [Fri04] showed that for every $\epsilon > 0$, almost every d-regular graph satisfies $\lambda(G) \leq 2\sqrt{d-1} + \epsilon$. Our method yields a simple proof to a result of intermediate strength. Namely, that $\lambda(G) = O(d^{2/3})$ holds almost surely.

Word maps appear in a natural way in the study of lifts of graphs in that they control the cycle distribution of the lifted graph. In this context we mention another work of Friedman [Fri03] (which is easily seen to include the Broder-Shamir work). Let G be a finite connected graph and let T be (the infinite) universal cover tree of G. Let D be the spectral radius of (the adjacency matrix of) G and let ρ be the spectral radius of T. Friedman showed that asymptotically almost every n-lift of G has spectral radius $\leq O(\sqrt{D\rho})$. He also conjectured that the same holds with spectral radius $\rho + o(1)$. This conjecture, if true would be a far-reaching extension of many theorems and conjectures in this area and in particular of Friedman's own [Fri04]. We are at present able to improve Friedman's result to $O(D^{1/3}\rho^{2/3})$.

Some of our aforementioned conjectures suggest an approach which may lead to a resolution of this problem.

We refer the reader to [HLW06] for a comprehensive coverage of this area.

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How many points can be reconstructed from k projections?

Jiří Matoušek

(joint work with Aleš Přívětivý and Petr Škovroň)

Let A be a finite set of points in the plane. The *discrete X-ray of A in direction u* specifies the number of points of A on every line parallel to u (this terminology is borrowed from the field of geometric tomography). We say that A is *reconstructible for directions $u_1, u_2, \ldots, u_k$* if there is no $B \neq A$ such that for all $i = 1, 2, \ldots, k$, the X-rays of A and B in direction u_i are identical.

As was observed several times (the earliest reference seems to be Rényi [Rén52]), every set A of $k-1$ or fewer points is reconstructible for *any* k distinct directions. (To see this, we suppose that some $A \neq B$ have the same X-rays in directions $u_1, \ldots, u_k$, we fix a point $a \in A \setminus B$, and we note that each line through a parallel to some u_i has to contain a point of B, forcing $|B| \geq k$.) If the directions are chosen by an adversary, then we cannot do any better in general: If $u_1, u_2, \ldots, u_k$ are k equally spaced directions and two k-point sets A and B are obtained by putting the vertices of a suitably rotated regular $2k$-gon alternately into A and into B, then A and B have identical X-rays in each of the directions u_i.

Intuition suggests that the equally spaced directions in this example are "exceptionally bad", and that other sets of directions should allow for reconstruction of much larger sets. For given directions $u_1, u_2, \ldots, u_k$ let us define

$$f(u_1, \ldots, u_k) := \max\{n : \text{every } n\text{-point set is reconstructible for } u_1, \ldots, u_k\}$$

and

$$F(k) := \max_{u_1, u_2, \ldots, u_k} f(u_1, \ldots, u_k).$$

We have just noted that $f(u_1, \dots, u_k) \geq k-1$ for all k-tuples of distinct $u_1, \dots, u_k$. A simple inductive construction (also rediscovered several times) shows that, perhaps counterintuitively, $F(k)$ is finite for every k. More precisely, it yields $F(k) < 2^{k-1}$ for all k.

The only published lower bound on $F(k)$ we could find is roughly $k + \Omega(\sqrt{k})$, due to Bianchi and Longinetti [BL90]. After we started working on the problem, we learned from Attila Pór that Tóth [Tót03] announced an $\Omega(k^{3/2})$ lower bound, which has remained unpublished.

We have the following lower bound:

Theorem 1. *There are constants $c > 0$ and k_0 such that*

$$F(k) > 2^{ck/\log k}$$

for all $k \geq k_0$. Moreover, for every $k \geq k_0$ there exists a finite set $\mathcal{P}_k$ of nonzero polynomials in $2k$ variables and with integer coefficients such that if $u_1 = (x_1, y_1), \dots,\ u_k = (x_k, y_k)$ are directions with $f(u_1, \dots, u_k) \leq 2^{ck/\log k}$, then $(x_1, x_2, \dots, x_k, y_1, \dots, y_k)$ is in the zero set of some polynomial in $\mathcal{P}_k$. Consequently, almost all (in the sense of measure) k-tuples of directions $u_1, \dots, u_k$ satisfy the stated lower bound.

In the proof we establish the following result in extremal graph theory, which may be of independent interest:

Proposition 2. *There exists a constant C such that the following holds. Let $G = (V, E)$ be a graph on $2n \geq 4$ vertices, whose edge set E is a disjoint union $E = E_1 \cup \dots \cup E_k$ of k perfect matchings on V. If $k \geq \lfloor C \log_2 n \log_2 \log_2 n \rfloor$, then there exist disjoint index sets $I_1, I_2 \subset [k]$ and a subset $W \subseteq V$, $|W| \geq 2$, such that the graphs $G[I_1, W]$ and $G[I_2, W]$ are both connected, where $G[I, W]$ denotes the graph with vertex set W and edge set $\{\{u, v\} \in \bigcup_{i \in I} E_i : u, v \in W\}$.*

As for upper bounds, we can improve the bound $F(k) < 2^{k-1}$ mentioned above to $F(k) \leq O(C^k)$ for $C = 6^{1/3} \approx 1.81712$.

Related work. Problems similar to those investigated in the present paper have been studied in a lively area called *geometric tomography*; see, e.g., the book by Gardner [Gar06]. The classical tomography problem deals with reconstructing a set, or more generally a density function, from X-rays in all directions. *Discrete tomography* investigates the reconstruction problem for a finite (discrete) set of X-ray directions. Since reconstructing an arbitrary set is generally impossible, most of the work deals with special sets, say convex ones.

For reconstructing finite sets A, most of the results in the literature concern the case where A is a lattice set, $A \subseteq \mathbb{Z}^2$, and the directions of the X-rays are integer vectors. A seminal paper by Gardner and Gritzmann [GG97] thoroughly examines the case where A is guaranteed to be a convex lattice set (that is, the intersection of $\mathbb{Z}^2$ with a convex set). In this case, they show that any 7 lattice directions suffice for reconstruction of every convex lattice set, while 6 directions need not suffice. Few other papers with somewhat related results are [Hep56], [Gar92], [BDLNP01], [DGP06].

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Hypergraphs with independent neighborhoods

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(joint work with Tom Bohman, Alan Frieze, Zoltan Füredi, and Oleg Pikhurko)

The *neighborhood* $N(S)$ of a $(k-1)$-set S in a k-uniform hypergraph (henceforth a *k-graph*) is the set of vertices v such that $S \cup \{v\}$ is an edge. For $n \geq k \geq 2$, let $f(n,k)$ be the maximum number of edges in a k-graph on n vertices such that all its neighborhoods are *independent* sets (that is, span no edge). Mantel proved in 1907 that $f(n,2) = \lfloor n^2/4 \rfloor$, and this was the first result in extremal graph theory. Thus the problem of computing $f(n,k)$ is a natural generalization of Mantel's result.

A k-graph is *odd* if it has a vertex partition $X \cup Y$ such that all edges have an odd number of points less than k in Y. It is easy to see that all neighborhoods in an odd k-graph are independent sets. Let $B(n,k)$ be an odd k-graph with n vertices and the maximum number of edges. Let $b(n,k) = |B(n,k)|$ be the number of edges in $B(n,k)$. Then the previous observation implies that $f(n,k) \geq b(n,k)$. It was conjectured in [3] that there exists some function $n_0(k)$ such that $n > n_0(k)$ implies

$$f(n,k) = b(n,k). \tag{1}$$

There was some evidence for this, as it reduces to Mantel's theorem for $k = 2$, and it was proved for $k = 3$ by Füredi, Pikhurko, and Simonovits [4, 5], thereby settling a conjecture of Mubayi and Rödl [7]. The next case $k = 4$ was addressed in [3]. Note that the vertex partition of $B(n,4)$ is not into precisely equal parts, but they have sizes $n/2 - t$ and $n/2 + t$, where, as it follows from routine calculations,

$$|\, t - \frac{1}{2}\sqrt{3n-4}\,| < 1.$$

Theorem 1. **(Exact Result** [3]**)** *Let n be sufficiently large, and let G be an n-vertex 4-graph with all neighborhoods being independent sets. Then $|G| \leq b(n,4)$, and if equality holds, then $G = B(n,4)$.*

We also prove an approximate structure theorem, which states that if G has close to $b(n,4)$ edges, then the structure of G is close to $B(n,4)$.

Theorem 2. **(Global Stability** [3]**)** *For every $\delta > 0$, there exists n_0 such that the following holds for all $n > n_0$. Let G be an n-vertex 4-graph with independent neighborhoods, and $|G| > (1/2 - \varepsilon)\binom{n}{4}$, where $\varepsilon = \delta^2/108$. Then G can be made odd by removing at most $\delta\binom{n}{4}$ edges.*

One might suspect that Theorem 2 can be taken further, by showing that if G has minimum degree at least $(1/2 - \gamma)\binom{n}{3}$ for some $\gamma > 0$, then G is already odd. Such phenomena hold for $k = 2$ and 3. For example, when $k = 2$, a special case of the theorem of Andrásfai, Erdős, and Sós [1] states that a triangle-free graph with minimum degree greater than $2n/5$ is bipartite. For $k = 3$, a similar result was proved in [4]. The analogous statement is not true for $k = 4$. Indeed, one can add an edge E to $B(n,4)$ that intersects each part in two vertices, and then delete all edges of $B(n,4)$ that intersect E in three vertices. The resulting 4-graph has independent neighborhoods, and yet its minimum degree is $(1/2)\binom{n}{3} - O(n^{5/2})$. Nevertheless, a slightly weaker statement is true. Let us call a k-graph 2-*colorable* if its vertex set can be partitioned into two independent sets.

Theorem 3. ([3]) *Let G be an n-vertex 4-graph with independent neighborhoods. There exists $\varepsilon > 0$ such that if n is sufficiently large and G has minimum degree greater than $(1/2 - \varepsilon)\binom{n}{3}$, then G is 2-colorable.*

The situation for larger k is more complicated. Since exact results are rare in extremal hypergraph theory, one often studies asymptotics. In this case, we can define $\rho_k = \lim_{n\to\infty} f(n,k)/\binom{n}{k}$ which is easily shown to exist. Now conjecture (1) implies that $\rho_k = 1/2$ for all even k and $\rho_k \uparrow 1/2$ as $k \to \infty$ for odd k. Thus a weaker statement than (1) would be that $\rho_k = \lim_{n\to\infty} b(n,k)/\binom{n}{k}$, and an even weaker statement is that $\rho_k \to 1/2$ as $k \to \infty$.

We prove that conjecture (1) is false for all $k \geq 7$, and in fact $\rho_k \to 1$. This follows from an old construction of Kim and Roush [6] which gives lower bounds for the Turán problem for complete k-graphs. Thus the small cases shed little light on the behavior of ρ_k.

We are able to obtain rather sharp estimates on the rate at which ρ_k converges to 1:

Theorem 4. **(Asymptotic Result** [2]**)** *As $k \to \infty$, we have*

$$1 - \frac{2\log k}{k} + (1 + o(1))\,\frac{\log\log k}{k} \leq \rho_k \leq 1 - \frac{2\log k}{k} + (5 + o(1))\,\frac{\log\log k}{k},$$

where log *denotes the natural logarithm.*

Furthermore, for $k \geq 7$, we have $\rho_k > 1/2$, hence (1) is false for $k \geq 7$.

This leaves open the cases $k = 5$ and 6, where we believe that (1) still holds.

Conjecture 5. ([2]) *$f(n, k) = b(n, k)$ for $k \in \{5, 6\}$ and n sufficiently large.*

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Minors in random and expanding graphs

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(joint work with Nikolaos Fountoulakis and Daniela Kühn)

A graph H is a *minor* of G if for every vertex $h \in H$ there is a connected subset $B_h \subseteq V(G)$ such that all the B_h's are disjoint and G contains an edge between B_h and $B_{h'}$ whenever hh' is an edge of H. The B_h's are called the *branch sets.* We denote by $\mathrm{ccl}(G)$ the order of the largest complete minor in G. The study of the largest complete minor contained in a given graph has its origins in Hadwiger's conjecture which states that if the chromatic number of a graph G is at least k, then G contains a K_k minor. It has been proved for $k \leq 6$.

Bollobás, Catlin and Erdős [1] proved that Hadwiger's conjecture is true for almost all graphs. For this, they estimated the typical order of the largest complete minor in a graph on n vertices and compared it with the typical chromatic number of such a graph. In particular, they proved that for constant p and $\varepsilon > 0$ asymptotically almost surely $\mathrm{ccl}(G_{n,p}) = (1 \pm \varepsilon)n/\sqrt{\log_b n}$, where $b := 1/(1-p)$. Here $G_{n,p}$ is a random graph on n vertices where the edges are present independently and with probability p. We say that an event occurs *asymptotically almost surely* (a.a.s.) if it occurs with probability tending to 1 as n tends to infinity.

Krivelevich and Sudakov [4] considered the order of the largest complete minor in a sparser random graph (and more generally in arbitrary pseudo-random and expanding graphs). They determined the order of magnitude of $\mathrm{ccl}(G_{n,p})$ as long as $p \geq n^{\varepsilon-1}$. Our first result in [2] determines $\mathrm{ccl}(G_{n,p})$ asymptotically as long as $p \geq C/n$ and $p = o(1)$.

Theorem 1. *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that if $pn \geq C$ and $p = o(1)$, then a.a.s.*

$$\mathrm{ccl}(G_{n,p}) = (1 \pm \varepsilon)\sqrt{\frac{n^2 p}{\ln(np)}} = (1 \pm \varepsilon)\frac{n}{\sqrt{\log_b(np)}},$$

where $b := 1/(1-p)$.

In [2], we also estimated $\mathrm{ccl}(G_{n,c/n})$ where $c > 1$ is fixed. Krivelevich and Sudakov [4] observed that there are constants c_1 and c_2 such that $c_1\sqrt{n/\log n} \leq \mathrm{ccl}(G_{n,c/n}) \leq c_2\sqrt{n}$ and asked what the correct order of magnitude is.

Theorem 2. *For every $c > 1$ there exists a constant $\delta = \delta(c)$ such that a.a.s. $\delta\sqrt{n} \leq \mathrm{ccl}(G_{n,c/n}) \leq 2\sqrt{cn}$.*

Note that the upper bound in Theorem 2 is immediate, since for any graph G, the number of edges in any minor of G is at most $e(G)$. The same argument shows that the condition that $p \geq c/n$ for some constant $c > 1$ is necessary to ensure a complete minor of order $\Theta(\sqrt{n})$ in $G_{n,p}$. This follows from the fact that if $pn \to 1$ the number of edges in any component is sublinear in n.

Partial results on $\mathrm{ccl}(G_{n,p})$ during the phase transition (i.e. when $pn \to 1$) were proven by Łuczak, Pittel and Wierman [5].

1. Related results and open questions

While the influence of the chromatic number on the existence of complete minors is far from clear, the corresponding extremal problem for the average degree has been settled for large complete minors: Thomason [8] asymptotically determined the smallest average degree $d(k)$ which guarantees the existence of a K_k minor in any graph of average degree at least $d(k)$. (The order of magnitude $k\sqrt{\log k}$ of $d(k)$ was determined earlier in [3, 6].) The extremal graphs are (disjoint copies of) dense random graphs. Recall that Theorem 2 shows that the behaviour of sparse random graphs is quite different: in that case $\mathrm{ccl}(G_{n,p})$ has the same order of magnitude as $\sqrt{e(G_{n,p})}$, a trivial upper bound which holds for any graph.

There are several results on large complete minors in pseudo-random graphs and expanding graphs. Thomason [7] introduced a notion of pseudo-randomness called jumbledness. Krivelevich and Sudakov [4] gave bounds on $\mathrm{ccl}(G)$ for jumbled graphs G. For $G_{n,p}$ their results only imply the lower bound in Theorem 1 up to a multiplicative constant if $p \geq n^{\varepsilon-1}$. It would be interesting to know whether their bound is best possible or whether (up to a multiplicative constant) the bound in Theorem 1 can be extended to jumbled graphs with appropriate parameters. Krivelevich and Sudakov [4] also considered minors in expanding graphs. Again, their results only imply the lower bound in Theorem 1 up to a multiplicative constant if $p \geq n^{\varepsilon-1}$.

Finally, note that our results do not cover the case where $p \to 1$. Usually, the investigation of $G_{n,p}$ for such p is not particularly interesting. However, any counterexamples to Hadwiger's conjecture are probably rather dense, so in this case it might be worthwile to investigate the values of the chromatic number and the order of the largest complete minor of such a random graph (though it seems rather unlikely that this approach will yield any counterexamples).

2. Strategy of the proofs

As in [1], the upper bound in Theorem 1 is proved by a first moment argument. The main difference between the arguments is that in our case, we need to make use of the fact that the branch sets of a minor have to be connected, whereas this was not necessary in [1].

For the lower bound, let $k := n/\sqrt{\log_b(np)}$ be the function appearing in Theorem 1. The proof in [1] for the case when p is a constant proceeds as follows. One first shows that a.a.s. there are k large pairwise disjoint connected sets B_i in $G_{n,p}$. These are used as candidates for the branch sets. The number U_0 of pairs of B_i which are not connected by an edge is then shown to be $o(k)$. So by discarding a comparatively small number of candidate branch sets, one can obtain the desired minor. For small p, the main problem is that U_0 will be much larger than k. However, we can show that U_0 is at most a small fraction of n. We make use of this as follows. We first find a path P whose length satisfies $U_0 \ll |P| \ll n$ and which is disjoint from the B_i. We will divide this path into disjoint subpaths. Our aim is to join most of those pairs of B_i which are not yet joined by an edge via one of these subpaths. More precisely, we are looking for a matching of size $U_0 - \varepsilon k$ in the auxiliary bipartite random graph G^* whose vertex classes consist of the unjoined pairs of candidate branch sets and of the subpaths and where a subpath is adjacent to such an unjoined pair if it sends an edge to both of the candidate branch sets in this unjoined pair. There are two difficulties to overcome in order to find such a matching. Firstly, some of the B_i are involved in several unjoined pairs, so the edges G^* are not independent. Secondly, if we make the subpaths too short, then the density of G^* is not large enough to guarantee a sufficiently large matching, while if we make the subpaths too long, then there will not be enough of them. We overcome this by using paths of very different lengths together with a greedy matching algorithm which starts off by using short paths to try and join the unjoined pairs. Then in the later stages the algorithm uses successively longer paths to try and join those pairs which were not joined in the previous stages until $U_0 - \varepsilon k$ of the pairs have been joined. To ensure that the dependencies between the existence of edges in G^* are not too large, we also remove some of the unjoined pairs from future consideration after each stage (namely those containing a candidate branch set that is involved in comparatively many pairs which are still unjoined).

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Dirac type results for uniform hypergraphs

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(joint work with Vojtěch Rödl and Endre Szemerédi)

One of the classic theorems of graph theory is the result of Dirac [2] which states that if the minimum degree $\delta(G)$ in a graph G on $n \geq 3$ vertices is at least $n/2$ then G has a hamiltonian cycle, while if $\delta(G) \geq \lfloor n/2 \rfloor$ then G has a hamiltonian path.

For k-uniform hypergraphs (or k-graphs, for short) with $k \geq 3$, a path, and consequently a cycle, may be defined in several ways (see, e.g., [1], [5], [6] and [4]). Here we consider *(tight) paths* which are k-graphs with vertices $v_1, \dots, v_l$ and edges $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$, $i = 1, \dots, l-k+1$. (For $k = 3$, the pairs v_1, v_2 and v_l, v_{l-1} are called *the endpairs* of the path.) A companion notion of *a (tight) cycle* is defined similarly with the additional presence of the edges $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ for $i = l-k+2, \dots, k$, where for $h > l$ we set $v_h = v_{h-l}$. The name *tight* is used only to distinguish this definition from a competitive one (cf. [6] and [4]), where *a loose cycle* is defined as a k-graph with vertices $v_1, \dots, v_{(k-1)l}$ and edges $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$ for $i \equiv 1(\mod k-1)$.

A natural extension of Dirac's theorem to k-graphs, $k \geq 3$, has been conjectured in [5].

Conjecture 1. *Let H be a k-graph with $n \geq k+1 \geq 4$ vertices. If every $(k-1)$-element set of vertices is contained in at least $\lfloor (n-k+3)/2 \rfloor$ edges, then H has a hamiltonian cycle.*

In [9] and [10] we proved the approximate version of the conjecture.

Theorem 2 ([9],[10]). *Let $k \geq 3$, $\gamma > 0$, and let H be a k-graph on n vertices, where n is sufficiently large. If each $(k-1)$-element set of vertices is contained in at least $(1/2 + \gamma)n$ edges, then H has a hamiltonian cycle.*

A similar result holds for loose cycles with $1/2$ replaced by $\frac{1}{4}$ for $k = 3$ (see [6]). Here we announce the exact result for tight hamiltonian cycles and paths when $k = 3$. Let $\delta_2(H)$ be the minimum pair degree in H.

Theorem 3. *Let H be a k-graph on n vertices, where n is sufficiently large.*
(a) If $\delta_2(H) \geq \lfloor n/2 \rfloor$ then H has a hamiltonian cycle.
(b) If $\delta_2(H) \geq \lceil n/2 \rceil - 1$ then H has a hamiltonian path.

Note that for n odd the two thresholds coincide, while for n even they differ by 1. A pivotal role in the proof is played by the minimal extreme k-graph $H_0 = H_0(A, B)$ with vertex set $V = A \cup B$, $|A| = \lceil n/2 \rceil$, $|B| = \lfloor n/2 \rfloor$, and edge set consisting of all triples of vertices with an odd intersection with A. It is easy to see that $\delta_2(H) = \lceil n/2 \rceil - 2$ and H_0 does not have a hamiltonian path. A minor modification of H_0 leads to an extreme k-graph with $\delta_2(H) = \lfloor n/2 \rfloor - 1$ and no hamiltonian cycle.

We sketch the proof of part (b) only which is slightly easier. Below we assume that n is even (equivalently, we will write $2n$ for the number of vertices). The proof is split into two major cases, depending on whether H contains almost entirely a copy of H_0, or not. For this to be precise, we look at the quantity $|H_0 \setminus H|$ defined as $\min |H_0(A, B) \setminus H|$ taken over all partitions of $V(H) = A \cup B$ with $|A| = |B| = n$. Let ε_0 be a carefully chosen absolute constant.

Case I: $|H_0 \setminus H| > \varepsilon_0 n^3$

In this case we actually have a stability result saying that a hamiltonian path is present in H already when $\delta_2(H) \geq (1 - \varepsilon_1)n$. The proof is based on the idea of an absorbing path introduced in [9] and [10]. A path Q in H is called *ε-absorbing* if $|Q| \leq \varepsilon n$ and for every set $U \subset V(H) \setminus V(Q)$, $|U| \leq \varepsilon^2 n$, there exists a path Q_U such that $V(Q_U) = V(Q) \cup U$ and both paths, Q and Q_U, have the same endpairs.

The main proof consists of three steps:

(1) Find an ε_0^2-absorbing path Q in H.
(2) Find a path P which contains Q as a subpath and such that $|V(H) \setminus V(P)| \leq \varepsilon_0^4 n$.
(3) Extend P to a hamiltonian path by applying the absorbing property of Q to the set $U = V(H) \setminus V(P)$.

We comment on steps 1 and 2 only. The absorbing device we use is much more sophisticated that the one in [9]. In addition to the standard *Connecting Lemma*, it is based also on the so called *Comb-Connecting Lemma*, where instead of a tight path we seek a pseudo-path of the form $213, 134, 435, 356, \ldots$. Such a pseudo-path which begins with xy and ends with zw is called an $xy - zw$-comb. The absorbing device is constructed by connecting certain pairs of teeth of an $xy - zw$-comb by short paths, using the Connecting Lemma. We now state both connecting lemmas.

Lemma 4 (Connecting Lemma)**.** *There exists $\varepsilon_1 > 0$ which depends on ε_0 only, such that whenever $\delta_2(H) \geq (1 - \varepsilon_1)n$ then for every two disjoint, ordered pairs of vertices (x, y) and (v, w) there is a path in H of length at most $1/\varepsilon_1$, which connects (x, y) and (v, w).*

Lemma 5 (Comb-Connecting Lemma). *There exists* $\varepsilon_1 > 0$ *which depends on* ε_0 *only, such that whenever* $\delta_2(H) \geq (1-\varepsilon_1)n$ *then for every two disjoint, ordered pairs of vertices* (x,y) *and* (v,w) *there is an* $xy - zw$*-comb in* H *of length at most* $1/\varepsilon_1$.

The almost hamiltonian path found in step (2) has been constructed in [9] and [10] with the help of hypergraph regularity lemmas (strong and weak, resp.). Here, instead, we apply a recursive construction based on a classical result in extremal graph theory due to Kővari, Sós and Turán [7] (cf. [8]). Each new segment of the path is glued to the present one via the Connecting Lemma.

Case II: $|H_0 \setminus H| \leq \varepsilon_0 n^3$

We fix the partition $V(H) = A \cup B$ which minimizes $|H_0(A,B) \setminus H|$. Since H almost contains H_0, most edges of the form AAA and ABB are present in H and so it is relatively easy to draw a long "top"path (of the form $AAA\cdots$), as well as a long "cross" path (of the form $ABBABB\cdots$). To connect them into one hamiltonian path we need *a bridge*, a short path with endpairs of the form AA and AB (or BB). Such a path needs necessarily to contain an atypical edge of the from AAB. Due to the degree condition $\delta_2(H) \geq n-1$, for every pair of vertices $a_1, a_2 \in A$ there does exist a vertex $b \in B$ such that $a_1a_2b \in H$. Still some work has to be done, as b may be atypical in the sense that its link almost entirely is contained in $\binom{A}{2} \cup \binom{B}{2}$ and not in $A \times B$, as it should.

If we wanted to construct a hamiltonian cycle instead (under the stronger assumption that $\delta(H) \geq n$) we would need to build two bridges. which is just an extra technical difficulty. However, with two bridges we have no more freedom to arbitrarily split the vertices into the top path and the cross path which causes some parity problem.

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Generalizations of the removal lemma

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(joint work with Vojtěch Rödl)

Answering a question of Brown, T. Sós, and Erdős [6, 20] Ruzsa and Szemerédi [19] established the *triangle removal lemma*. They proved that every graph which does not contain many triangles can be "made easily" triangle free.

Theorem 1 (Triangle removal lemma)**.** *For every $\eta > 0$ there exists $c > 0$ and n_0 so that every graph G on $n \geq n_0$ vertices, which contains at most cn^3 triangles can be made triangle free by removing at most $\eta\binom{n}{2}$ edges.*

More general statements of that type regarding graphs were successively proved by several authors in [1, 2, 3, 7]. In particular, the result of Alon and Shapira in [2] is a generalization, which extends all the previous results of this type, where the triangle is replaced by a possibly infinite family of graphs and containment is induced. We present an extension of the result of Alon and Shapira from graphs to k-uniform hypergraphs (see Theorem 3).

Before we state Theorem 3 we discuss some of the known extensions of the Ruzsa–Szemerédi theorem for graphs and hypergraphs in more detail.

A k-uniform hypergraph $\mathcal{H}^{(k)}$ on the vertex set V is some family of k-element subsets of V, i.e., $\mathcal{H}^{(k)} \subseteq \binom{V}{k}$. Note that we identify hypergraphs with their edge set and we write $V(\mathcal{H}^{(k)})$ for the vertex set.

A possible generalization of Theorem 1 to hypergraphs was suggested in [7, Problem 6.1]. The first result in this direction was obtained by Frankl and Rödl [8] who extended Theorem 1 to 3-uniform hypergraphs with the triangle replaced by $K_4^{(3)}$ – the complete 3-uniform hypergraph on 4 vertices. The general result, which settles the conjecture from [7] was recently obtained independently by Gowers [11] and Nagle, Skokan and authors [15, 17, 18] and subsequently by Tao in [23].

Theorem 2 (Removal lemma)**.** *For all k-uniform hypergraphs $\mathcal{F}^{(k)}$ on ℓ vertices and and every $\eta > 0$ there exist $c > 0$ and n_0 so that the following holds.*

Suppose $\mathcal{H}^{(k)}$ is a k-uniform hypergraph on $n \geq n_0$ vertices. If $\mathcal{H}^{(k)}$ contains at most cn^ℓ copies of $\mathcal{F}^{(k)}$, then one can delete $\eta\binom{n}{k}$ edges from $\mathcal{H}^{(k)}$ so that the resulting sub-hypergraph contains no copy of $\mathcal{F}^{(k)}$.

Theorem 2 implies Szemerédi's theorem [21] as well as its multidimensional extensions due to Furstenberg and Katznelson [9, 10].

Similarly, as all known proofs of Theorem 1 are based on *Szemerédi's regularity lemma* [22] (see, e.g., [13]), all proofs of Theorem 2 rely on hypergraph generalizations of the regularity lemma (see, e.g., [11, 15, 16, 17, 23]).

A possible generalization of Theorem 2 is to replace the single hypergraph $\mathcal{F}^{(k)}$ by a possibly infinite family $\mathscr{F}$ of k-uniform hypergraphs. Such a result was first proved for graphs by Alon and Shapira [3] in the context of property testing.

The proof of this result relies on a strengthened version of Szemerédi's regularity lemma, which was obtained by Alon, Fischer, Krivelevich, and M. Szegedy [1] by iterating the regularity lemma for graphs.

Recently, the result for monotone properties was extended by Avart and authors in [4] from graphs to hypergraphs. The proof in [4] follows the approach of Alon and Shapira and is based on two successive applications of the hypergraph regularity lemma from [16].

Another natural variant of Theorem 2 would be an *induced* version. For graphs this was first considered by Alon, Fischer, Krivelevich, and M. Szegedy [1]. Note that in this case in order to obtain an induced F-free graph, we may need not only remove, but also add edges. The same result for 3-uniform hypergraphs was obtained by Kohayakawa, Nagle, and Rödl in [12].

In [2] Alon and Shapira proved a common generalization of of the monotone and the induced version of the graph removal lemma, extending the induced version from one forbidden induced graph F to a forbidden family of induced graphs $\mathscr{F}$. Here we present a generalization of their result to k-uniform hypergraphs.

For a family of k-uniform hypergraphs $\mathscr{F}$, let $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ be the family of all hypergraphs $\mathcal{H}^{(k)}$ which contain no induced copy of any member of $\mathscr{F}$. Clearly, $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ is a *hereditary* family (or *hereditary property*) of hypergraphs, i.e., if $\mathcal{H}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ and $\tilde{\mathcal{H}}^{(k)}$ is an induced sub-hypergraph of $\mathcal{H}^{(k)}$, then $\tilde{\mathcal{H}}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$. For a constant $\eta \geq 0$ and a possibly infinite family of k-uniform hypergraphs $\mathscr{P}$ we say a given hypergraph $\mathcal{H}^{(k)}$ **is η-far from $\mathscr{P}$** if every hypergraph $\mathcal{G}^{(k)}$ on the same vertex set $V(\mathcal{H}^{(k)})$ with $|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}| \leq \eta\binom{|V(\mathcal{H}^{(k)})|}{k}$ satisfies $\mathcal{G}^{(k)} \notin \mathscr{P}$, where $\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}$ denotes the symmetric difference of the edge sets of $\mathcal{G}^{(k)}$ and $\mathcal{H}^{(k)}$.

Theorem 3. *For every (possibly infinite) family $\mathscr{F}$ of k-uniform hypergraphs and every $\eta > 0$ there exist constants $c > 0$, $C > 0$, and n_0 such that the following holds.*

Suppose $\mathcal{H}^{(k)}$ is a k-uniform hypergraph on $n \geq n_0$ vertices. If for every $\ell = 1, \ldots, C$ and every $\mathcal{F}^{(k)} \in \mathscr{F}$ on ℓ vertices, $\mathcal{H}^{(k)}$ contains at most cn^ℓ induced copies of $\mathcal{F}^{(k)}$, then $\mathcal{H}^{(k)}$ is not η-far from $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

In other words one can change (add/delete) up to at most $\eta\binom{n}{k}$ k-tuples in $V(\mathcal{H}^{(k)})$ (to/from $\mathcal{H}^{(k)}$) so that the resulting hypergraph $\mathcal{G}^{(k)}$ contains no induced copy of any member of $\mathscr{F}$, i.e., so that $\mathcal{G}^{(k)} \in \mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$.

Moreover, since $\mathrm{Forb}_{\mathrm{ind}}(\mathscr{F})$ is a subset of the family $\bar{\mathscr{F}}$ of all hypergraphs not contained in $\mathscr{F}$, such a hypergraph $\mathcal{H}^{(k)}$ is also not η-far from $\bar{\mathscr{F}}$.

For graphs Theorem 3 was first obtained by Alon and Shapira [2]. The proof in [2] is again based on the strong version of Szemerédi's regularity lemma from [1]. Another proof for graphs was found by Lovász and B. Szegedy [14] (see also [5]).

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On Dissections

Angelika Steger

(joint work with Nicla Bernasconi and Konstantinos Panagiotou)

In the past decades the $G_{n,p}$ model of random graphs, introduced by Erdős and Rényi in the 60's, has led to numerous beautiful and deep theorems. A key feature that is used in basically all proofs is that edges in $G_{n,p}$ appear independently. The independence of the edges allows, for example, to obtain extremely tight bounds on the number of edges of $G_{n,p}$ and its degree sequence by straightforward applications of Chernoff bounds. This situation changes dramatically if one considers graph classes with structural side constraints. For example, in a random planar graph R_n (a graph drawn uniformly at random from the class of all labeled planar graphs on n vertices) the edges are obviously far from being independent. Consequently, so far basically all results about properties of random graphs with structural side constraints rely on completely different methods, mostly from analytic combinatorics.

In this talk we show that recent progress in the construction of so-called Boltzmann samplers by Duchon, Flajolet, Louchard, and Schaeffer [3] and Fusy [4] can be used to reduce the study of degree sequences and subgraph counts to properties of sequences of independent and identically distributed random variables – to which we can then again apply Chernoff bounds to obtain extremely tight results. We exemplify our approach by studying properties of random graphs that are drawn uniformly at random from the class consisting of all dissections of large convex polygons. We obtain very sharp concentration results for the number of vertices of any given degree, and for the number of induced copies of a given fixed graph.

Let $\mathbb{D}_n$ denote the class of dissections of labeled convex n-gons, and let D_n be a graph drawn uniformly at random from $\mathbb{D}_n$, and let $\mathsf{deg}\,(k;\,\mathsf{D}_n)$ denote the number of vertices in D_n with degree k. In our first theorem we determine the asymptotic value of $\mathsf{deg}\,(k;\,\mathsf{D}_n)$ and provide very tight bounds for the tail probabilities. For brevity we write "$(1 \pm \varepsilon)X$" to denote the interval $((1-\varepsilon)X, (1+\varepsilon)X)$.

Theorem 1. *Let $d_k := (k-1)p^2(1-p)^{k-2}$, where $p := 2-\sqrt{2}$, and let $k_0 = k_0(n)$ be the largest integer such that $d_{k_0} n > (\log n)^3$. There is a constant $C > 0$ such that for every $k \le k_0$ and every $\frac{(\log n)^2}{\sqrt{d_k n}} < \varepsilon = \varepsilon(n) < 1$ the following holds for sufficiently large n.*

$$\mathbb{P}\left[\mathsf{deg}\,(k;\,\mathsf{D}_n) \in (1 \pm \varepsilon) \cdot d_k \cdot n\right] \ge 1 - e^{-C\varepsilon^2 \frac{d_k}{k} n}.$$

Furthermore, if $k \in [k_0+1, 10\log n]$, then

$$\mathbb{P}\left[\mathsf{deg}\,(k;\,\mathsf{D}_n) < (\log n)^4\right] \ge 1 - kn^{-\log n}.$$

For all remaining k we have that $\mathbb{P}\left[\mathsf{deg}\,(k;\,\mathsf{D}_n) = 0\right] \to 1$.

From Theorem 1 it is easy to derive information about the *maximum* vertex degree $\Delta(\mathsf{D}_n)$ of a random element from $\mathbb{D}_n$.

Corollary 2. *Let* $p := 2 - \sqrt{2}$*, and set* $b := \frac{1}{1-p}$*. Then*

$$\mathbb{P}\left[\Delta(\mathsf{D}_n) \notin (\log_b n - O(\log\log n), 10\log n)\right] = o(1).$$

With our method it is not directly possible to improve this result, but we believe that the maximum degree for a random element of $\mathbb{D}_n$ is given by the lower bound.

Next we turn to subgraph counts. For an unlabeled dissection H we denote by $\mathsf{copy}\,(H;\ \mathsf{D})$ the number of induced copies of H in D.

Theorem 3. *Let* H *be an unlabeled dissection on* n_H *vertices, such that* $n_H = o(\log n)$*. Denote by* r_H *the number of different ways to root on an edge the external face of* H*. Let* $c_H := \frac{1}{2} r_H \cdot q^{n_H - 3}$*, where* $q := \frac{2-\sqrt{2}}{2}$*. There is a constant* $0 < C < 1$ *such that for every* $0 < \varepsilon < 1$ *and* n *sufficiently large we have*

$$\mathbb{P}\left[\mathsf{copy}\,(H; \mathsf{D}_n) \in (1 \pm \varepsilon) \cdot c_H \cdot n\right] \geq 1 - \exp\{-C^{n_H}\varepsilon^2 n\}.$$

Our method gives similar results for random graphs from the class of triangulations of convex polygons; the class of random triangulations was previously studied by Gao and Wormald [6, 7, 5]; our method provides an alternative approach and gives sharper bounds on the tail distributions.

The results of this talk appeared in [1]. In a follow up paper [2] we extended our approach to the classes of outerplanar and of series-parallel graphs.

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Density theorems for bipartite graphs and related Ramsey-type results

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(joint work with Jacob Fox)

For a graph H, the *Ramsey number* $r(H)$ is the least positive integer n such that every two-coloring of the edges of complete graph K_n on n vertices, contains a monochromatic copy of H. Estimating Ramsey numbers is one of the central (and difficult) problems in modern combinatorics. Among the most interesting questions in this area are the linear bounds for Ramsey numbers of graphs with certain degree constraints. In 1975, Burr and Erdős [2] conjectured that, for each positive integer Δ, there is a constant $c(\Delta)$ such that every graph H with n vertices and maximum degree Δ satisfies $r(H) \leq c(\Delta)n$. This conjecture was proved by Chvatál, Rödl, Szemerédi, and Trotter [3]. Their proof is a beautiful illustration of the power of Szemerédi's regularity lemma. However, the use of this lemma makes an upper bound on $c(\Delta)$ to grow as a tower of 2s with height polynomial in Δ. Since then, the problem of determining the correct order of magnitude of $c(\Delta)$ as a function of Δ has received considerable attention from various researchers. Still using a variant of the regularity lemma, Eaton [4] showed that $c(\Delta) < 2^{2^{c\Delta}}$ for some fixed c. A novel approach of Graham, Rödl, and Rucinski [5] gave the first linear upper bound on Ramsey numbers of bounded degree graphs without using any form of the regularity lemma. Their proof implies that $c(\Delta) < 2^{c\Delta \log^2 \Delta}$.

The case of bipartite graphs with bounded degree was studied by Graham, Rödl, and Rucinski more thoroughly in [6], where they improved their upper bound, showing that $r(H) \leq 2^{\Delta \log \Delta + O(\Delta)}n$ for every bipartite graph H with n vertices and maximum degree Δ. As they point out, their proof does not give a stronger density-type result. In the other direction, they proved that there is a positive constant c such that, for every $\Delta \geq 2$ and $n \geq \Delta + 1$, there is a bipartite graph H with n vertices and maximum degree Δ satisfying $r(H) \geq 2^{c\Delta}n$. Closing the gaps between these two bounds remained a challenging open problem. In this paper, we solve this problem by showing that the correct order of magnitude of the Ramsey number of bounded degree bipartite graphs is essentially given by the lower bound. This follows from the following density-type theorem.

Theorem 1. *Let H be a bipartite graph with n vertices and maximum degree $\Delta \geq 1$. If $\epsilon > 0$ and G is a graph with $N \geq 32\Delta\epsilon^{-\Delta}n$ vertices and at least $\epsilon\binom{N}{2}$ edges, then H is a subgraph of G.*

Taking $\epsilon = 1/2$ together with the majority color in a 2-coloring of the edges of K_N, we obtain a corollary which gives a best possible upper bound up to the constant factor in the exponent on Ramsey numbers of bounded degree bipartite graphs.

Corollary 2. *If H is bipartite, has n vertices and maximum degree $\Delta \geq 1$, then $r(H) \leq \Delta 2^{\Delta+5}n$.*

Moreover, the above theorem also easily gives an upper bound on multicolor Ramsey numbers of bipartite graphs. The k-color Ramsey number $r(H_1, \ldots, H_k)$ is the least positive integer N such that for every k-coloring of the edges of the complete graph K_N, there is a monochromatic copy of H_i in color i for some $1 \leq i \leq k$. Taking $\epsilon = 1/k$ in Theorem 1 and considering the majority color in a k-coloring of the edges of a complete graph shows that for bipartite graphs $H_1, \ldots, H_k$ each with n vertices and maximum degree at most Δ, $r(H_1, \ldots, H_k) \leq 32\Delta k^{\Delta}n$.

One family of bipartite graphs that have received particular attention are the d-cubes. The d-cube Q_d is the d-regular graph with 2^d vertices whose vertex set is $\{0,1\}^d$ and two vertices are adjacent if they differ in exactly one coordinate. Burr and Erdős conjectured that $r(Q_d)$ is linear in the number of vertices of the d-cube. Beck [1] proved that $r(Q_d) \leq 2^{cd^2}$. The bound of Graham et al. [5] gives the improvement $r(Q_d) \leq 8(16d)^d$. Shi [10], using ideas of Kostochka and Rödl [7], proved that $r(Q_d) \leq 2^{(\frac{3+\sqrt{5}}{2})d+o(d)}$, which is a polynomial bound in the number of vertices with exponent $\frac{3+\sqrt{5}}{2} \approx 2.618$. A very special case of Corollary 2, when $H = Q_d$, gives immediately the following improved result.

Corollary 3. *For every positive integer d, $r(Q_d) \leq d2^{2d+5}$.*

A graph is *d-degenerate* if every subgraph of it has a vertex of degree at most d. Notice that graphs with maximum degree d are d-degenerate. This notion nicely captures the concept of sparse graphs as every t-vertex subgraph of a d-degenerate graph has at most td edges. (Indeed, remove from the subgraph a vertex of minimum degree, and repeat this process in the remaining subgraph.) Burr and Erdős [2] conjectured that, for each positive integer d, there is a constant $c(d)$ such that $r(H) \leq c(d)n$ for every d-degenerate graph H on n vertices. This well-known and difficult conjecture is a substantial generalization of the above mentioned results on Ramsey numbers of bounded degree graphs and progress on this problem was made only recently.

Kostochka and Rödl [8] were the first to prove a polynomial upper bound on the Ramsey numbers of d-degenerate graphs. They showed that $r(H) \leq c_d n^2$ for every d-degenerate graph H with n vertices. A nearly linear bound of the form $r(H) \leq c_d n^{1+\epsilon}$ for any fixed $\epsilon > 0$ was obtained in [9]. For bipartite H, Kostochka and Rödl proved that $r(H) \leq d^{d+o(d)}\Delta n$, where Δ is the maximum degree of H. Kostochka and Sudakov [9] proved that $r(H) \leq 2^{O(\log^{2/3} n)}n$ for every d-degenerate bipartite graph H with n vertices and constant d. Here we improve on both of these results.

Theorem 4. *If $d/n \leq \delta \leq 1$, H is a d-degenerate bipartite graph with n vertices and maximum degree $\Delta \geq 1$, G is a graph with N vertices and at least $\epsilon\binom{N}{2}$ edges, and $N \geq 2^{12}\epsilon^{-(1/\delta+3)d-2}\Delta^{\delta}n$, then H is a subgraph of G.*

For δ and H as in the above theorem, taking $\epsilon = 1/2$ and considering the majority color in a 2-coloring of the edges of K_N shows that

$$r(H) \leq 2^{\delta^{-1}d+3d+14}\Delta^{\delta}n.$$

This new upper bound on Ramsey numbers for bipartite graphs is quite versatile. Taking $\delta = 1$, we have $r(H) \leq 2^{4d+14}\Delta n$ for bipartite d-degenerate graphs with n vertices and maximum degree Δ. This improves upon the bound of Kostochka and Rödl. If $\Delta \geq 2^d$, then taking $\delta = (\frac{d}{\log \Delta})^{1/2}$, we have

$$r(H) \leq 2^{2\sqrt{d\log\Delta}+3d+14}n$$

for bipartite d-degenerate graphs H with n vertices and maximum degree Δ. In particular, we have $r(H) \leq 2^{O(\log^{1/2} n)}n$ for constant d. This improves on the bound of Kostochka and Sudakov, and is another step closer to the Burr-Erdős conjecture.

It seems plausible that $r(H) \leq 2^{c\Delta}n$ holds in general for every graph H with n vertices and maximum degree Δ. The following result shows that this is at least true for graphs of bounded chromatic number.

Theorem 5. *If H has n vertices, chromatic number q, and maximum degree Δ, then $r(H) \leq 2^{4q\Delta}n$.*

The proofs of the above results combine probabilistic arguments with some combinatorial ideas. In addition, these techniques can be used to study properties of graphs with a forbidden induced subgraph, edge intersection patterns in topological graphs, and to obtain several other Ramsey-type statements.

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Thresholds for positional games

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(joint work with Heidi Gebauer and Michael Krivelevich)

In this abstract we report about recent progress on three open problems in the theory of positional games.

In the Maker-Breaker *connectivity game* two players, Maker and Breaker alternately claim the edges of the complete graph K_n, until all edges are claimed. The game is won by Maker if he succeeds in occupying a spanning tree, otherwise the game is won by Breaker. In the case when both players claim one edge in each turn, the connectivity game is related to the *Shannon switching game.* That game was fully analyzed by Lehman [10]. He showed that Maker wins "easily" in the sense that he does not even need the full edge set of K_n, it is enough if he restricts his moves to the union of two edge-disjoint spanning trees.

Chvátal and Erdős [4] suggested to even out the advantage of Maker by allowing Breaker to occupy $b > 1$ edges at each of his turns. The natural question then is how large this *bias* b of Breaker can be such that Maker is still able to build a spanning tree. Chvátal and Erdős suggested the following "random graph intuition" to guess the threshold bias. Consider the random game, where the players RandomMaker and RandomBreaker act completely randomly: at each turn RandomMaker occupies 1 free edge selected uniformly at random then RandomBreaker occupies b free edges selected uniformly at random. At the end of this game RandomMaker's graph is a random graph $G(n, M)$ with $M = \left\lceil \frac{\binom{n}{2}}{b+1} \right\rceil$ edges, so $b \approx \frac{n^2}{2M}$. Hence by the classic theorem of Erdős and Rényi, for every $\epsilon > 0$,

- if $b \leq (1-\epsilon)\frac{n}{\ln n}$, then RandomMaker wins the connectivity game almost surely;
- if $b \geq (1+\epsilon)\frac{n}{\ln n}$, then RandomBreaker wins the connectivity game almost surely.

Let $b_{\mathcal{T}}$ be the *threshold bias* of the connectivity game, that is, the the largest integer b such that Maker wins the connectivity game against an arbitrary Breaker who plays with bias b. Surprisingly, Chvátal and Erdős was able to show that the random graph intuition is valid up to a constant factor, that is, for every $\epsilon > 0$ and large enough n, $\left(\frac{1}{4} - \epsilon\right) \frac{n}{\ln n} \leq b_{\mathcal{T}} \leq (1+\epsilon) \frac{n}{\ln n}$. Later Beck [1] improved the lower bound to $(\ln 2 - \epsilon) \frac{n}{\ln n} \approx 0.69 \frac{n}{\ln n}$ using a general Erdős Selfridge-type potential function argument he developed partly for this purpose.

In his recent great treatise of the subject Beck [3] lists numerous open problems on positional games and among them he identifies the "six most humiliating open problems". Here we address three of these.

The first problem concerns the connectivity game.

Problem 1. *Is the random graph intuition valid for the connectivity game asymptotically? That is, is it true that* $b_{\mathcal{T}} = (1 - o(1)) \frac{n}{\ln n}$*?*

Our second problem discusses the hamiltonicity property, which is known to have the same sharp threshold edge number as connectivity.

Problem 2. *Can Maker build a Hamilton cycle while playing against a Breaker of bias* $(1 - o(1)) \frac{n}{\ln n}$*?*

Let $\mathcal{H} \subseteq 2^{E(K_n)}$ be the family of hamiltonian graphs on n vertices. Naturally, this game is harder for Maker to win than connectivity, still the order of the threshold bias turned out to be the same as the one for the connectivity game: Beck [2] used his potential function method coupled with a beautiful ad-hoc argument to show that Maker can create a Hamilton cycle against a bias $(\frac{\ln 2}{27} - o(1))\frac{n}{\ln n}$ of Breaker.

The third problem of Beck we consider here is about the misère version of the hamiltonicity game. In this game the player Avoider wins if he does *not* build a Hamilton cycle, his opponent, Enforcer wins otherwise. The problem was motivated by our inability to adapt the succesful argument of the achievement (Maker-Breaker) version [2] to the misére game, while the random graph intuition suggests that they should have a similar outcome.

Problem 3. *Can player Enforcer, playing with bias* $(1-o(1))\frac{n}{\ln n}$*, play such a way that Avoider, playing with bias* 1*, builds a Hamilton cycle?*

Our Results. Recently the best known bound for Open Problem 2 was improved by Krivelevich and Szabó [9], but the precise asymptotics still eludes us.

Theorem 4. [9] $b_{\mathcal{H}} \geq (\ln 2 - o(1))\frac{n}{\ln n}$

Our proof is more streamlined than the one in [2] as it avoids the ad-hoc reasoning and only applies the potential function criterion of Beck [1]. The key ingredients are a new hamiltonicity criterion of Hefetz, Krivelevich and Szabó [7], and a thinning trick. The method is in fact so flexible that it also allows the resolution of Open Problem 3.

Theorem 5. [9] *Enforcer, playing with bias* $(1-o(1))\frac{n}{\ln n}$*, has a strategy such that Avoider, playing with bias* 1*, does build a Hamilton cycle.*

Most recently, Open Problem 1 was settled by Gebauer and Szabó.

Theorem 6. [5] $b_{\mathcal{T}} = (1-o(1))\frac{n}{\ln n}$

The proof is not based on the potential function technique of Beck, rather on the analysis of a quite natural strategy of Maker, involving a simple yet subtle inductive argument.

Further Open Problems. Several interesting questions arise.

1. Close the remaining gap in Open Problem 2. An easier question is to determine the correct threshold bias for the Maker-Breaker game where Maker's goal is to build a perfect matching. The known upper and lower bounds are the same as the ones for the Hamiltonicity game. The random graph intuition would again suggest $(1-o(1))\frac{n}{\ln n}$ to be the right answer.

2. Another notable graph property having the same random graph threshold edge number is k-connectivity for constant k. It would be interesting to close the existing gap for the Maker–Breaker threshold bias (c.f. [9]) Note that for the related Maker-Breaker game where Maker's goal is to build a graph with minimum degree at least k, the threshold bias was proved to be $(1-o(1))\frac{n}{\ln n}$ by Gebauer and Szabó [5].

3. We do not know much about the threshold bias for the hamiltonicity Avoider-Enforcer game; there is no nontrivial upper bound known. At this time it is even possible that Enforcer can win if his bias is as large as $\Theta(n)$.

4. In fact, before posing the previous problem, one should rather wonder whether there *is* a threshold bias at all!? We do not know the answer. In principal it could happen that Enforcer loses the game with some bias b_1, yet he wins with bias $b_1 + 1$. This might seem impossible at first (superficial) sight, but there exist examples of games where fluctuation in the identity of the winner occurs, see [6]. We remark that with a monotone relaxation of the rules, where in each round both players are allowed to take *at least* as many elements of the board as their respective bias, the threshold bias does exist and is equal to $(1 - o(1))\frac{n}{\ln n}$ (see [8]).

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Testing Properties of Graphs and Functions

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(joint work with László Lovász)

Let $\mathcal{G}$ be the set of finite graphs. A subset $\mathcal{P}$ of $\mathcal{G}$ is called a *graph property.* Our main focus here is *testability* of properties. Roughly speaking, a property is called testable if by sampling a constant size random subgraph from a graph G one can distinguish (with a large probability) between the cases wether G has the property or it is far from having it. In this case "far" means that one has to change at least $\epsilon|V(G)|^2$ edges in G to get a graph in $\mathcal{P}$. For a more precise definition see [5].

The oldest result in the area is the so-called triangle removal lemma by Ruzsa and Szemerédi which, among other things, implies Roth's theorem on 3-term arithmetic progressions in dense subsets of the integers. A far reaching generalization of this result by Alon and Shapira [1] says that every hereditary property is testable. (A property $\mathcal{P}$ is called hereditary if every spanned subgraph of G has $\mathcal{P}$ whenever G has $\mathcal{P}$.) The special case when $\mathcal{P}$ is the set of triangle free graphs is equivalent with the triangle removal lemma.

The converse of the Alon-Shapira theorem is not true. It is easy to create testable properties that are not hereditary. On the other hand it can be shown that if $\mathcal{P}$ is testable and G has $\mathcal{P}$ then its "typical" subgraphs are not far from $\mathcal{P}$. By choosing the definition of "typical" carefully we obtain a characterization of testable graph properties which implies the Alon-Shapira result immediately. To state our result we need some preparation.

Let H and G be two finite graphs. Let $t(H,G)$ denote the probability that a random map $f : V(H) \mapsto V(G)$ is a graph homomorphism (i.e. the image of an edge in H is an edge in G). For two graphs G_1, G_2 let $d(G_1,G_2)$ denote the smallest $\epsilon \geq 0$ such that

$$|t(H,G_1) - t(H,G_2)| \leq \epsilon$$

for every $H \in \mathcal{G}$ with $|V(H)| \leq 1/\epsilon$. If $|V(G_1)| = |V(G_2)|$ we also define the so called edit distance $e(G_1,G_2)$ which is the smallest number of edge deletions or additions which make G_1 isomorphic to G_2.

Our main definition is the following: A property $\mathcal{P}$ is called *weak-hereditary* if for every ϵ there is a number $\delta > 0$ and a natural number n_ϵ such that if $G \in \mathcal{P}$ and G_1 is a spanned subgraph of G with $|V(G_1)| > n_\epsilon$ and $d(G,G_1) \leq \delta$ (i.e. G_1 is "typical") then

$$e(G_1,\mathcal{P}) \leq \epsilon |V(G_1)|^2.$$

We proved the next characterization of testable properties:

Theorem: *A graph property is testable if and only if it is weak-hereditary*

Our main tool is the recently developed graph limit method [6, 7, 4, 3] which is based on a completion of the set $\mathcal{G}$ that we denote by $\bar{\mathcal{G}}$. The set $\bar{\mathcal{G}}$ is a compact topological space with two metrics $\delta_\square$ and δ_1 corresponding to the previous d and e metrics. For every property $\mathcal{P}$ we introduce a certain closure $\bar{\mathcal{P}}$ which is a $\delta_\square$-closed subset of $\bar{\mathcal{G}}$. The closure of a graph property is a geometric object and we show that testability of $\mathcal{P}$ corresponds to a geometric property of $\bar{\mathcal{P}}$ which is roughly speaking the continuity of $\delta_1(x,\bar{\mathcal{P}})$ in $\delta_\square(x,\bar{\mathcal{P}})$ at 0.

Note that our analytic language also enables us to introduce a natural class of testable properties which contains all the hereditary ones. We call these properties *flexible*. Among other results we show that the furthest graph from a flexible property is quasi-random which generalizes a surprising result by Alon and Stav [2].

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Embedding spanning subgraphs into dense graphs

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(joint work with Julia Böttcher and Mathias Schacht)

One of the fundamental results in extremal graph theory is the theorem by Erdős and Stone [9] which implies that any *fixed* graph H of chromatic number r is forced to appear as a subgraph in any sufficiently large graph G if the average degree of G is at least $(\frac{r-2}{r-1} + \gamma)n$, for an arbitrarily small but positive constant γ. In this paper we prove an analogue of this result for *spanning* subgraphs H which was conjectured by Bollobás and Komlós.

When trying to translate the Erdős–Stone theorem into a setting where the graphs H and G have the same number of vertices, then first of all, the average degree condition must be replaced by one involving the minimum degree $\delta(G)$ of G, since we need (to be able to control) every single vertex of G. Also, it is clear that in this regime the lower bound has to be raised at least to $\delta(G) \geq \frac{r-1}{r}n$: simply consider the example where G is the complete r-partite graph with partition classes almost, but not exactly, of the same size (thus G has minimum degree almost $\frac{r-1}{r}n$) and let H be the spanning union of vertex disjoint r-cliques.

There are a number of results where a minimum degree of $\frac{r-1}{r}n$ is indeed sufficient to guarantee the existence of a certain spanning subgraph H. A well known example is Dirac's theorem [7]. It asserts that any graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle. Another classical result of that type by Corrádi and Hajnal [4] states that every graph G with n vertices and $\delta(G) \geq 2n/3$ contains $\lfloor n/3 \rfloor$ vertex disjoint triangles. This was generalised by Hajnal and Szemerédi [10], who proved that every graph G with $\delta(G) \geq \frac{r-1}{r}n$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r.

A further extension of this theorem was suggested by Pósa (see, e.g., [8]) and Seymour [19], who conjectured that, at the same threshold $\delta(G) \geq \frac{r-1}{r}n$, such a graph G must in fact contain a copy of the $(r-1)$-st power of a Hamiltonian cycle (where the $(r-1)$-st power of an arbitrary graph is obtained by inserting an edge between every two vertices of distance at most $r-1$ in the original graph). This was proven in 1998 by Komlós, Sárközy, and Szemerédi [13] for sufficiently large n.

Recently, several other results of a similar flavour have been obtained which deal with a variety of spanning subgraphs H, such as, e.g., trees, F-factors, and planar graphs [1, 2, 3, 5, 6, 12, 14, 15, 16, 17, 18, 20]. Thus, in an attempt to move away from results that concern only graphs H with a special, rigid structure, a naïve conjecture could be that $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$ suffices to guarantee that G contains a spanning copy of any r-chromatic graph H of bounded maximum degree. However, the following simple example shows that this fails in general. Let H be a random bipartite graph with bounded maximum degree and partition classes of size $n/2$ each, and let G be the graph formed by two cliques of size $(1/2 + \gamma)n$ each, which share exactly $2\gamma n$ vertices. It is then easy to see that G cannot contain a copy of H, since in H every set X of vertices of size $(1/2-\gamma)n$ has more than $2\gamma n$ neighbours outside X.

One way to rule out such expansion properties for H is to restrict the *bandwidth* of H. A graph is said to have bandwidth at most b, if there exists a labelling of the vertices by numbers $1,\dots,n$, such that for every edge $\{i,j\}$ of the graph we have $|i-j|\le b$. Bollobás and Komlós [11, Conjecture 16] conjectured that every r-chromatic graph on n vertices of bounded degree and bandwidth limited by $o(n)$, can be embedded into any graph G on n vertices with $\delta(G)\ge(\frac{r-1}{r}+\gamma)n$. Recently, we managed to prove this conjecture.

Theorem 1. *For all $r,\Delta\in\mathbb{N}$ and $\gamma>0$, there exist constants $\beta>0$ and $n_0\in\mathbb{N}$ such that for every $n\ge n_0$ the following holds.*

If H is an r-chromatic graph on n vertices with $\Delta(H)\le\Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G)\ge(\frac{r-1}{r}+\gamma)n$, then G contains a copy of H.

Since planar graphs with bounded maximum degree have sublinear bandwidth (see [21]) our result implies that any sufficiently large graph G with n vertices and $\delta(G)\ge(3/4+\gamma)n$ contains any planar n-vertex graph with bounded degree.

We close by addressing the rôle of the chromatic number in Theorem 1. There are $(r+1)$-chromatic graphs that are forced already as subgraphs when $\delta(G)\ge(\frac{r-1}{r}+\gamma)n$ (simply consider the Hamiltonian cycle on an odd number of vertices). It seems that the crucial question here is whether all $r+1$ colours are needed by *many* vertices, and it was Komlós [12], who introduced the concept of the *critical chromatic number* $\chi_{cr}(H)$ to capture exactly this phenomenon.

Our methods allow an extension of Theorem 1 that goes into a somewhat similar direction. Assume that the vertices of H are labelled $1,\dots,n$. For two positive integers x,y, an $(r+1)$-colouring $\sigma:V(H)\to\{0,\dots,r\}$ of H is said to be (x,y)-*zero free* with respect to such a labelling, if for each $t\in[n]$ there exists a t' with $t\le t'\le t+x$ such that $\sigma(u)\ne 0$ for all $u\in[t',t'+y]$. We also say that the interval $[t',t'+y]$ is *zero free.*

Theorem 2. *For all $r,\Delta\in\mathbb{N}$ and $\gamma>0$, there exist constants $\beta>0$ and $n_0\in\mathbb{N}$ such that for every $n\ge n_0$ the following holds.*

Let H be a graph with $\Delta(H)\le\Delta$ whose vertices are labelled $1,\dots,n$ such that, with respect to this labelling, H has bandwidth at most βn, an $(r+1)$-colouring that is $(8r\beta n,4r\beta n)$-zero free, and uses colour 0 for at most βn vertices in total.

If G is a graph on n vertices with minimum degree $\delta(G)\ge(\frac{r-1}{r}+\gamma)n$, then G contains a copy of H.

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Fingerprinting capacity

Gábor Tardos
(joint work with Ehsan Amiri)

Digital fingerprinting is a basically combinatorial question motivated by security and cryptography, namely the need for including a unique identifier code in each copy of a digital document so that unauthorized copies could be traced back to their source. Fingerprint codes resistant against *collusion attacks* was first introduced by Boneh and Shaw [2]. See definition below. The present talk surveyed recent constructions and upper bounds on the rate. The construction combines features of the earlier papers [3, 1]. The upper bounds use novel combinatorial and probability theory tools.

The model A fingerprint code of length n for a set of N users consists of a probabilistic strategy to assign a codeword from $\{0,1\}^n$ to each user (the *code generation*) and the *accusation algorithm*, that on input of another word in $\{0,1\}^n$ (the *forged code*) and the codewords of the users outputs one accused user. It is *ϵ-secure against t pirates* if in case a t-subset of users (*the pirates*) use a valid strategy to generate the forged copy, then the probability that the accused user is not one of the pirates is at most ϵ.

Here a *valid pirate strategy* is any algorithm that takes the codewords of the pirates as input and outputs the forged copy and obeys the *marking condition*: if for some index i the ith bit of all the pirates' codewords agree then the corresponding bit of the forged copy must agree with them too.

The object of digital fingerprinting is to design secure fingerprint codes of high rate $R = \log N/n$. In particular one wants to find the *t-fingerprinting capacity*, the largest rate for which ϵ-secure codes against t pirates exist with ϵ tending to 0.

The paper [3] gives constructions achieving rates $1/(100t^2)$. The paper [1] gives another construction that achieves higher rates for small values of t but the rates of that construction deteriorates exponentially with t. We combine the techniques from the two papers. Namely our code generation is a modification of that of [3] while our accusation algorithm resembles that of [1]. This combination lets us achieve better rates for all values of t and we conjecture that these rates are optimal for all t.

While we cannot prove this conjecture we prove that the rates are asymptotically optimal, both the rates achieved and our upper bounds are $\frac{1+o(1)}{2\ln 2\cdot t^2}$. For the upper bound we use two new tools. The simpler is tighter form of Pinsker inequality that depends on the distributions. The other tool is a partial solution to following simple question:

Let D_1 and D_2 be two distributions on the product set $S_1 \times S_2 \times \cdots \times S_k$. There *box-distance* is defined as the maximum of the differences $|D_1(A) - D_2(A)|$ for boxes $A = A_1 \times \cdots \times A_k$, where $A_i \subseteq S_i$. We call a distribution D on k-tuples of integers *monotone*, if $D(x_1, \ldots, x_k) = 0$ unless $x_1 < x_2 < \ldots < x_k$. We call such a distribution *symmetric* if $D(x_1, \ldots, x_k) = D(y_1, \ldots, y_k)$ whenever $(y_1, \ldots, y_k)$ is a permutation of $(x_1, \ldots, x_k)$. What is the minimum box-distance between a symmetric and a monotone distribution on k-tuples of integers from $\{1, 2, \ldots, n\}$?

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Extremal problems for cycles in graphs

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(joint work with Benny Sudakov)

We survey recent progress on extremal problems for cycles in graphs, and some new applications. The first type of problem we consider, given a set $\mathcal{C}$ of cycles, is to estimate $\mathrm{ex}(n,\mathcal{C})$ – the maximum number of edges in an n-vertex graph which does not contain any of the cycles in $\mathcal{C}$. In particular, the well-known Moore Bound deals with the case that $\mathcal{C}$ consists of all cycles of length at most g. In general, the following holds [8]:
If the longest even cycle in $\mathcal{C}$ has length $2k$, then it holds that

$$ex(n,\mathcal{C}) \leq 4(k-1)n^{1+1/k}$$

It is known that this result is tight up to constant factors when $k \in \{2,3,5\}$. Our result improves earlier bounds by Erdős and by Bondy and Simonovits [1], and the technique we use also shows that a graph with no cycle of length zero modulo k has average degree at most $8(k-1)$. The problem of constructing graphs without short cycles can be directly translated into estimating the largest size of subgroups of non-abelian groups in which no word of length $2k$ of a particular form is the identity. For $k = 2$, the existence of dense Sidon sets – also called perfect difference sets in projective geometry – gives rise to projective planes which lead to the Erdős-Rényi extremal graphs with no 4-cycles. The best bounds known for the case of 6-cycles are given by Füredi, Naor and the author [4] using a mix of extremal and probabilistic methods. No Cayley graph construction of close-to-extremal graphs for 6-cycles is known, and in fact only one infinite family of generalized quadrangles is known. It would be interesting to find new constructions, as it can be deduced from a recent paper of Cohn and Umans [2] using group algebras that if a Cayley graph construction of close-to-extremal graphs with no 6-cycles exists, then the exponent of matrix multiplication is 2. Odlyzko and Smith [6] did construct close-to-optimal subsets of non-abelian groups in which no product of $2k$ letters is zero, but for Cayley graphs one would need closure under inverses. We leave, therefore, the following open question:
Do there exist infinitely many groups G_n of order n containing a subset S_n such that the equation $ab^{-1}cd^{-1}ef^{-1} = 1$ with $a,b,c,d,e,f \in S_n$ has only trivial solutions and such that $|S_n| \gg n^{1/3}$?

Finally, we use the methods of the proof of the statement $\mathrm{ex}(n,\mathcal{C}) \ll n^{1+1/k}$ when $2k \in \mathcal{C}$ to prove some results on unavoidable sequences of cycle lengths. A sequence σ of positive integers is unavoidable if every graph of large enough but constant average degree contains a cycle of length in σ. Erdős [3] conjectured the existence of an unavoidable sequence of density zero, and this was proved by

the author. It was also conjectured by Erdős and Gyárfás [3] that the sequence of powers of two is unavoidable. While this conjecture is open, Sudakov and the author [7] recently proved the following positive result:
Let σ be an infinite sequence of positive even integers and suppose that there exists a constant $K > 1$ such that $\sigma_k \leq K^{\sigma_{k-1}}$ for all $k \in \mathbb{N}$ – we could say σ has at most tower growth. Then every n-vertex graph of average degree at least $\exp(\log^* n)$ contains a cycle of length in σ. Furthermore, there are infinitely many n for which there exists an n-vertex graphs containing no cycle of length in the sequence sequence $\sigma_k = 2k^{\sigma_{k-1}}$ where $\sigma_1 = 4$.

Erdős and Hajnal [3] (see also [5]) conjectured that a graph of infinite chromatic number has infinite sum of reciprocals of odd cycle lengths. Many other conjectures of Erdős [3] on unavoidable odd cycles in graphs of large chromatic number remain, and it is hopeful that some of the techniques used to prove the above-mentioned result can be fruitful for solving these problems.

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Perfect factors in random (hyper)graphs

Van H. Vu
(joint work with Anders Johansson and Jeff Kahn)

Let H be a fixed graph on v vertices. For an n-vertex graph G with n divisible by v, an *H-factor* of G is a collection of n/v copies of H whose vertex sets partition $V(G)$. The definition generalizes in the straightforward manner to r-uniform hyeprgraphs.

The problem of estimating the threshold $th_H(n)$ of the property that an Erdős-Rényi random (hyper) graph (on n points) contains an H-factor, for a general H has been, for quite some time, a central problem in probabilistic combinatorics. The only case when the order of magnitude of $th_H(n)$ has been determined (prior to the main result of this abstract) is when H is the graph K_2 (in other words, H a graph edge and a perfect factor is a perfect matching). A classical result of

Erdős and Rényi from the 1960s showed that in this case $th_H(n) = \Theta(\log n/n)$. Even the analogue of Erdős-Rényi result for hypergraphs (Shamir's problem on the threshold of perfect matching in random hypergraphs) has been open for a long time. The monograph [1] contains a good summary on partial results on these problems and related questions.

In a recent paper with A. Johansson and J. Kahn [2], we obtained a considerable progress, which provides the right order of magnitude of $th_H(n)$ for a large class of (hyper) graphs H (which contain Shamir's problem as a special case).

Define $d(H) = \frac{m}{v-1}$, where m is the number of edges of H. We say that H is *strictly balanced* if $d(H) > d(H')$ for any H' proper subgraph (sub-hypergraph) of H.

Theorem 1. *Let H be an r-uniform strictly balanced hypergraph with m edges. Then*

$$th_H(n) = \Theta(n^{-1/d(H)}(\log n)^{1/m}).$$

Corollary 2. *The threshold for containing a perfect matching in a random r-uniform hypergraph is $\Theta(\log n/n^{r-1})$.*

For general (hyper)graphs which may not be strictly balanced, we obtained an asymptotic result, which is tight up an $n^{o(1)}$ factor. This establishes a conjecture of Alon-Yuster.

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Conjectures on f-vectors

Volkmar Welker

1. Basic Definitions

We consider specific conjectures on f-vectors of simplicial complexes and their generating polynomials. Recall, that an abstract simplicial complex Δ is a subset $\Delta \subseteq 2^\Omega$ of the power set 2^Ω of the ground set Ω which is closed under taking subsets. Throughout this text the ground set Ω will always be a finite set. We call $B \in \Delta$ a face of Δ and denote by $\dim B := |B| - 1$ its dimension. The dimension $\dim(\Delta)$ of Δ is the maximum dimension of one of its faces. Now the f-vector $\mathfrak{f}^\Delta := (f_{-1}^\Delta, \ldots, f_{d-1}^\Delta)$ of a $(d-1)$-dimensional simplicial complex Δ is the vector with components f_i^Δ counting the number of i-dimensional faces of Δ. By $\mathfrak{f}^\Delta(t) := \sum_{i=0}^d f_{i-1}^\Delta t^{d-i}$ we denote the f-polynomial of Δ. Then $\mathfrak{h}^\Delta(t) := \mathfrak{f}^\Delta(t-1) := \sum_{i=0}^d h_i^\Delta t^{d-i}$ is the h-polynomial of Δ and $\mathfrak{h}^\Delta = (h_0^\Delta, \ldots, h_d^\Delta)$ the h-vector of Δ. There is a rich history of results, conjectures and counterexamples on f-vectors in combinatorics and discrete geometry (see for example [1]). In this abstract we will concentrate on some conjectures that are on the one hand side motivated from commutative algebra and on the other side from the theory of real rooted polynomials.

The first conjecture is concerned with the entry f_{d-1}^{Δ} of $\mathfrak{f}^{\Delta}$ for a $(d-1)$-dimensional simplicial complex Δ.

2. The Combinatorial Multiplicity Conjecture

The Multiplicity Conjecture by Huneke and Herzog & Srinivasan (see [8]) bounds the multiplicity of a standard graded k-algebra by the shifts in the graded Betti numbers.

More precisely the conjecture states:

Conjecture 1 (Multiplicity Conjecture)**.** *Let I be a homogeneous ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$ and $A = S/I$. Let h be the codimension of I and β_{ij} the graded Betti numbers of the minimal free resolution of A as an S-module. Set $M_i := \max\{j \mid \beta_{ij} \neq 0\}$. Then the multiplicity $e(A)$ of A is bounded by:*

$$e(A) \leq \frac{1}{h!} \prod_{i=1}^{h} M_i.$$

This conjecture has raised great interest in the commutative algebra community. The most far reaching result known is the proof of the conjecture for Cohen-Macaulay rings by Eisenbud & Schreyer [5] (see [6] for an overview of known cases before the Eisenbud & Schreyer proof). On the first sight this conjecture has little to do with combinatorics. But indeed it has a strong combinatorial consequence in the case $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ. Let Δ be a simplicial complex on ground set $[n] := \{1, \ldots, n\}$. A simplicial complex is completely specified by the set of its maximal faces and by the set $MN(\Delta)$ of its minimal non-faces. Now I_Δ is the ideal in $k[x_1, \ldots, x_N]$ generated by the monomials $\prod_{i \in B} x_i$ for all $B \in MN(\Delta)$. If $A = k[\Delta] := S/I_\Delta$ is the Stanley-Reisner ring of Δ then by Hochster's [9] formula:

$$\beta_{ij} = \sum_{\substack{W \subseteq [n] \\ |W| = j}} \dim_k \widetilde{H}_{|W|-i-1}(\Delta_W; k).$$

Here Δ_W denotes the induced subcomplex $\{B \in \Delta \mid B \subseteq W\}$ and $\widetilde{H}_\bullet(\bullet; k)$ reduced simplicial homology with coefficients in k. Since it is also well known that $e(k[\Delta]) = f_{d-1}$ and $codim(I_\Delta) = n - d$ we get the following form of the Multiplicity Conjecture for Stanley-Reisner rings.

Conjecture 2 (Semi-Combinatorial Multiplicity Conjecture)**.** *Let Δ be a simplicial complex on ground set $[n]$ of dimension $d-1$. Set*

$$M_i := \max\{j \mid \sum_{\substack{W \subseteq [n] \\ |W| = j}} \dim_k \widetilde{H}_{|W|-i-1}(\Delta_W; k) \neq 0\}.$$

Then

$$f_{d-1} \leq \frac{1}{(n-d)!} \prod_{i=1}^{n-d} M_i.$$

Note that if $A = k[\Delta]$ is a Stanley-Reisner ring then Conjectures 1 and 2 are equivalent. But there is no known argument that shows that proving Conjecture 2 implies Conjecture 1 for all standard graded k-algebras. For Stanley-Reisner rings Conjecture 2 was verified for matroid complexes, 2-CM complexes etc. by Novik & Swartz [13], for barycentrically subdivided simplicial complexes by Kubitzke & Welker [11] and 3-dimensional simplicial complex etc. by Goff [7]. Clearly, the work of Eisenbud & Schreyer implies the conjecture of all Cohen-Macaulay simplicial complexes.

Even though Conjecture 2 may appeal to geometric and topological combinatorialists, it still fails to be formulated in a purely combinatorial setting. For that one has to resort for a last time to commutative algebra. The well known Taylor resolution [14] implies that for Stanley-Reisner rings of simplicial complexes Δ the Betti number β_{ij} vanishes for any j such that there are is no subset $\mathcal{C} \subseteq MN(\Delta)$ of cardinality i with $\left|\bigcup_{B\in\mathcal{C}} B\right| = j$. Thus if true Conjecture 2 would imply:

Conjecture 3 (Combinatorial Multiplicity Conjecture)**.** *Let Δ be a simplicial complex on ground set $[n]$ of dimension $d-1$. Set*

$$M'_i := \max\{j \mid \exists \mathcal{C} \subseteq MN(\Delta) : |C| = i \text{ and } \big|\bigcup_{B\in C} B\big| = j \ \}.$$

Then

$$f_{d-1} \leq \frac{1}{(n-d)!} \prod_{i=1}^{n-d} M'_i.$$

In this form the conjecture was proved by Kummini [12] for flag simplicial complexes (i.e. simplicial complexes Δ for which all elements of $MN(\Delta)$ are of size 2).

Clearly, even if at some point there is a combinatorial proof for the upper bound, it seems desirable to have a combinatorial proof. On the other hand classical methods from extremal set theory (e.g. shifting) do not seem to be applicable.

3. Real rooted polynomials and g-theorems

Next we are interested in cases when the f-polynomial $\mathfrak{f}^\Delta(t)$ has only real roots and numerical consequences of the fact that the f-polynomial has only real roots.

The first results in this direction arise in connection with barycentric subdivision. Recall, that for a simplicial complex Δ its barycentric subdivision $\mathrm{s}d(\Delta)$ is the simplicial complex on ground set $\Delta \setminus \{\emptyset\}$ whose simplices are the chains $F_0 \subset \cdots \subset F_i$ of faces of $\Delta \setminus \{\emptyset\}$. For any $n \geq 1$ we write $\mathrm{s}d(\Delta)^n$ for the nth barycentric subdivision of Δ.

Theorem 4 ([2]). *For any simplicial complex Δ there is a number $N > 0$ such that for any $n \geq N$ the h-polynomial $\mathfrak{h}^{\mathrm{sd}^n(\Delta)}(t)$ has only real roots. If $h_i^\Delta \geq 0$ for $0 \leq i \leq d$ then $\mathfrak{h}^{\mathrm{sd}^n(\Delta)}(t)$ has only real roots for all $n \geq 1$.*

A second example which is geometrically more involved arises from r-edgewise subdivisions, an operation that for any $r \geq 2$ appears naturally in computational geometry and algebraic topology (see references in [3]). For a simplicial complex Δ denote by $\Delta(r)$ its r-edgewise subdivison.

Theorem 5 ([3]). *For any simplicial complex Δ there is an $R > 0$ such that for any $r \geq R$ the h-polynomial $\mathfrak{h}^{\Delta(r)}(t)$ has only real roots.*

Thus it appears that subdivision operations increase the number of real roots of f- or equivalently h-polynomials of simplicial complexes. Indeed there is the following conjecture :

Conjecture 6 ([4]). *If ∂P is the boundary complex of a (not necessarily simplicial) d-polytope P then the h-polynomial of $\mathrm{sd}(\partial P)$ has only real roots.*

Note that if P is a simplicial polytope then Conjecture 6 is implied by Theorem 4. The concept of real rooted f- or h-polynomials is absent in the classical theory of f- or h-vectors. Here, motivated by the classification of certain combinatorial or geometric classes of simplicial complexes, theorems often refer to the binomial expansion of the entries in the f-vector. Recall, that for numbers $n \geq i \geq 1$ there is a unique expansion $n = \binom{a_i}{i} + \cdots + \binom{a_j}{j}$ with numbers $a_i > \cdots > a_j \geq j \geq 1$. In this situation we write $n^{\langle i \rangle}$ for $\binom{a_i - 1}{i-1} + \cdots + \binom{a_j - 1}{j-1}$.

We say that an integer vector $(h_0, \ldots, h_d) \in \mathbb{N}^{d+1}$ satisfies a g-theorem if $1 = h_0 < \cdots < h_r \geq h_{r+1} \geq \cdots \geq h_d$ and for $g_0 = h_0 = 1$, $g_i = h_i - h_{i-1}$, $1 \leq i \leq r$, we have $g_{i+1} \leq g_i^{\langle i \rangle}$ for $1 \leq i \leq r - 1$.

The classical g-theorem states:

Theorem 7 (g-Theorem by Billera & Lee, Stanley). *A vector $(h_0, \ldots, , h_d) \in \mathbb{N}^{d+1}$ is the h-vector of the boundary complex of a simplicial polytope if and only if $h_i = h_{d-i}$, $0 \leq i \leq d$, and $(h_0, \ldots, , h_d)$ satisfies a g-theorem.*

The conjecture claiming that the same classification holds for simplicial spheres is known as the g-conjecture in discrete geometry. Now the following conjecture relates real rootedness and g-theorems.

Conjecture 8 ([4]). *Let $h(t) = h_0 + \cdots h_d t^d \in \mathbb{N}[t]$ be a polynomial of degree d with coefficients $h_0 = 1$ and $h_i \geq 1$ for $1 \leq i \leq d$. If $h(t)$ has only real roots then $(h_0, \ldots, h_d)$ satisfies a g-theorem.*

Besides calculations for small d, supportive evidence comes from the following results which shows that Conjecture 8 holds if $h(t) = h^{\mathrm{sd}(\Delta)}(t)$ for a Cohen-Macaulay simplicial complex of dimension $d - 1$. Note that in this situation by Theorem 4 the polynomial $h^{\mathrm{sd}(\Delta)}(t)$ has only real roots.

Theorem 9 ([10]). *Let Δ be a Cohen-Macaulay simplicial complex of dimension $d - 1$. Then $\mathfrak{h}^{\mathrm{sd}(\Delta)}$ satisfies a g-theorem.*

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Face numbers of centrally-symmetric polytopes: Conjectures, examples, counterexamples

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(joint work with Raman Sanyal and Axel Werner)

The f-vectors of centrally-symmetric convex polytopes are the subject of three conjectures A, B, C of increasing strength by Kalai [4] from 1989. Such basic open questions illustrate that our understanding of the f-vectors of centrally-symmetric polytopes is dramatically incomplete. (Our understanding of f-vectors of general convex polytopes is also quite limited outside the range of simple/simplicial polytopes; compare [7], [8].)

In our lecture, based on [6], we described the three conjectures, and reported that Conjectures A and B hold for $d \leq 4$, while Conjecture C fails for $d \geq 4$ and Conjecture B fails for $d \geq 5$.

1. The case $d = 3$

The case of $d = 3$ is easy, but it serves as a model for a complete answer: The set of f-vectors of centrally-symmetric 3-polytopes is

$$\begin{aligned}\mathcal{F}_3^{cs} \;=\; \{(f_0, f_1, f_2) \in (2\mathbb{Z})^3 : \;& f_0 - f_1 + f_2 = 2,\\ & f_0 \leq 2f_2 - 4,\\ & f_2 \leq 2f_0 - 4,\\ & f_0 + f_2 \geq 14 \;\}.\end{aligned}$$

2. Three conjectures

Kalai [4] proposed the following three conjectures of increasing strength about the f-vectors of d-dimensional centrally-symmetric polytopes.

The first one, **Conjecture A**, claims that every such polytope has at least 3^d non-empty faces,

$$\sum_{i=0}^{d} f_i \;\geq\; 3^d.$$

This became known as the 3^d-*conjecture*. In its strong form, it would claim that equality occurs only for *Hanner polytopes*, which arise from centrally-symmetric 1-polytopes (intervals $[-1, 1]$) by repeated application of "taking products" and dualization.

The second one, **Conjecture B**, claimed that the f-vector of every centrally symmetric d-polytope P should componentwise dominate the f-vector of one of the Hanner polytopes, $f(P) \geq f(H)$.

The third one, **Conjecture C**, claimed that the flag-vector *flag*(P) of every centrally symmetric d-polytope P should dominate in flag-vector space some flag vector *flag*(H), not only componentwise, but with respect to all linear flag-vector functionals that are nonnegative on all flag-vectors of general d-polytopes.

Kalai noted that quite obviously Conjecture C implies Conjecture B, which in turn implies the "3^d-conjecture", Conjecture A.

3. The cases 4A and 4B

While all three conjectures clearly hold for $d \leq 3$, we report that Conjectures A and B also hold for $d = 4$. The proof involves simple f-vector combinatorics, known elementary inequalities, some case distinctions, and one crucial non-trivial inequality, $g_2^{tor} \geq 2$. In its more general form for centrally-symmetric d-polytopes,

$$g_2^{tor}(P) \;=\; f_1 + f_{02} - 3f_2 - df_0 + \tbinom{d+1}{2} \;\geq\; \tbinom{d}{2} - d.$$

This inequality was derived by a'Campo-Neuen [1] via toric geometry. Following a suggestion by Kalai, we also derive an elementary proof via rigidity theory in [6].

4. Examples

As noted by Kalai, the Hanner polytopes (introduced by Hanner [2] in 1956, described above) provide a first, very interesting class of examples. A second class was described by Hansen [3] in 1977: The antiprisms over the independence polytopes of self-dual perfect graphs yield self-dual centrally-symmetric polytopes with interesting f-vectors. None of the two classes includes the other one: For examples take the sum of two 3-cubes, resp. the Hansen polytope of the path on 4 vertices. Both classes are examples of *weak Hanner polytopes* as introduced by Hansen, which have the property that any pair of opposite facets includes all the vertices. The hypersimplex $\Delta(k, 2k)$ of dimension $2k-1$ is an example of a weak Hanner polytope that is neither Hanner nor Hansen in general.

5. The case 4C fails

Consider the flag vector functional for 4-dimensional polytopes

$$\alpha(P) \ := (f_{02} - 3f_2) + (f_{13} - 3f_1),$$

which is non-negative, and vanishes exactly if P is 2-simplicial (first term) and 2-simple (second term).

This functional takes the values 9 and 12 on the 4-dimensional Hanner polytopes. Examples of centrally-symmetric 2-simplicial 2-simple 4-polytopes include Schläfli's 24-cell. Infinite families of 2-simple 2-simplicial 4-polytopes, with $\alpha = 0$, which may also be obtained to be centrally symmetric, are described in [5].

Thus for $d = 4$ Conjecture C fails strongly, in the sense that there are infinitely many polytopes whose flag-vectors are separated from *all* flag-vectors of Hanner polytopes by a common nonnegative linear functional.

6. The cases 5B and 5C fail

For $d = 5$, we consider the linear f-vector functional

$$\beta(P) \ := \ f_0 + f_4.$$

This functional satisfies $\beta \geq 34$ on all Hanner polytopes, while $\beta = 32$ both for the Hansen polytope associated with the path on 4 vertices, with f-vector $(16, 64, 98, 64, 16)$, and on the central hypersimplex $\Delta(3, 6)$, whose f-vector is $(20, 90, 120, 60, 12)$.

Thus for $d = 5$ Conjecture B fails strongly, in the sense that there are are polytopes whose f-vectors are separated from *all* f-vectors of Hanner polytopes by a common nonnegative linear functional.

One can derive from this that indeed Conjecture C fails for all $d \geq 4$ and that Conjecture B fails for all $d \geq 5$. Conjecture A remains open for $d \geq 5$. We refer to [6] for details.

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