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Nonlinear Evolution Equations

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Abstract. In this workshop three types of nonlinear evolution problems - geometric evolution equations (essentially all of parabolic type), nonlinear hyperbolic equations and dispersive equations - were the subject of 21 talks.

Mathematics Subject Classification (2000): 47J35, 74J30, 35L70, 53C44, 35Q55.

Introduction by the Organisers

Following the successful pattern of the meeting in 2005, this year's workshop on 'Nonlinear Evolution Problems' focussed on a small number of currently very active areas in this field. By far the dominant theme, however, were geometric evolution equations of parabolic type, followed by the topic of wave equations and water waves/Navier-Stokes equations both in a classical and relativistic framework.

Among the geometric evolution equations, curvature flows of hypersurfaces were considered with priority. In several talks, applications of these to isoperimetric problems and to formulation of appropriate local mass concepts were presented. One talk dealt with mean curvature flow in certain degenerate spaces. Several talks looked into harmonic and biharmonic map flow as well as Ricci flow.

Among the nonlinear hyperbolic equations, the water wave equation, Einstein's equations, semilinear wave equation and, for the first time, the d-brane equation from string theory were featured. For the latter, an ϵ-regularization method was used to establish an existence proof. This is another example of a method which has now been applied to different types of nonlinear equations. It had previously been applied to mean curvature flow and to minimal surfaces.

The last theme concerned dispersive equations. This was represented only in a talk by Markus Keel, albeit on seminal work on resonant decompositions applicable to a wide class of example.

All together, 21 talks were presented by international specialists from Australia, Canada, Germany, Great Britain, Italy, Sweden, Switzerland and the United States. Many of the speakers were only a few years past their Ph.D., some even still working towards their Ph.D.; 6 out of 42 participants and 4 out of 21 speakers were women.

As a rule, three lectures were delivered in the morning session; two lectures were given in the late afternoon, which left ample time for individual discussions.

Workshop: Nonlinear Evolution Equations

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Abstracts

A sub-Riemannian analogue of the mean curvature flow

Luca Capogna

In a joint project with Giovanna Citti (Bologna), we study a weak form of sub-Riemannian mean curvature flow in the setting of Carnot groups (in particular, in the model space of the Heisenberg group). Our motivation for this work is twofold: First, it is plausible that such mean curvature flow may prove useful in the solution of the isoperimetric problem in the Heisenberg group [2]; Second, a related sub-Riemannian mean curvature flow arises in the recent work [3] in conjunction with a model of the first layer of the visual cortex $V1$ in mammals.

Let G be an analytic and simply connected Lie group with topological dimension n and such that its Lie algebra $\mathcal{G}$ admits a stratification $\mathcal{G} = V^1 \oplus V^2 \oplus ... \oplus V^r$, where $[V^1, V^j] = V^{j+1}$, if $j = 1, ..., r-1$, and $[V^k, V^r] = 0$, $k = 1, ..., r$. Such groups are called *stratified nilpotent Lie groups*. Fix $X_1, ..., X_m$ a basis of V^1, called the horizontal frame, and complete it to a basis $(X_1, ..., X_n)$ of $\mathcal{G}$ by choosing for every $k = 2, \cdots r$ a basis of V_k. If X_i belongs to V_k, then we will set $d(i) = k$. We will denote by $xX = \sum_{i=1}^n x_i X_i$ a generic element of $\mathcal{G}$. Since the exponential map $\exp : \mathcal{G} \to G$ is a global diffeomorphism we use exponential coordinates in G, and denote $x = (x_1, \cdots, x_n)$ the point $\exp\big(xX\big)$. We also set $x_H = (x_1, \cdots, x_m)$ and $x_V = (x_{m+1}, ..., X_n)$ so that $x = (x_H, x_V)$. Define non-isotropic dilations as $\delta_s(x) = (s^{d(i)} x_i)$, for $s > 0$.

We denote by $(X_1, ..., X_n)$ (resp. $(\tilde{X}_1, ..., \tilde{X}_n)$) the left invariant (resp. right invariant) translation of the frame $(X_1, ..., X_n)$ of $\mathcal{G}$. Set $H(0) = V^1$, and for any $x \in G$ we let $H(x) = xH(0) = \text{span}[X_1, ..., X_m](x)$. The distribution $x \to H(x)$ is called *the horizontal sub-bundle* H. On H we define a left invariant positive definite bilinear form g_0, so that $X_1 \cdots, X_m$ is an orthonormal frame. We let $\nabla = (X_1, \cdots, X_m)$ denote the *horizontal gradient* operator. The vectors $X_1 .... X_m$ and their commutators span all the Lie algebra $\mathcal{G}$, and consequently one can define a control distance $d_C(x, y)$ associated to the distribution $X_1 .... X_m$, which is called *the Carnot-Carathéodory metric* (denote by $\tilde{d}_C$ the corresponding right invariant distance). We call the couple (G, d_C) a *Carnot Group*. We define a family of left invariant Riemannian metrics g_ε, $\varepsilon > 0$ in $\mathcal{G}$ by requesting that $\{X_1, \cdots, X_m . \varepsilon X_{m+1}, \cdots, \varepsilon X_n\}$ is an orthonormal frame. We will denote by d_ε the corresponding distance functions. Correspondingly we use ∇_ε, (resp. $\tilde{\nabla}_\varepsilon$) to denote the left (resp. right) invariant gradients. It is well known[1] that (G, d_ε) converges in the Gromov-Hausdorff sense as $\varepsilon \to 0$ to the sub-Riemannian space (G, d_C).

[1] See for instance [4]

1. Horizontal mean curvature flow of hypersurfaces

Let $M \subset G$ be a C^2 smooth hypersurface, denote by $\boldsymbol{n^\varepsilon}$ the unit normal in the metric g_ε and by $\boldsymbol{n^0} = \sum_{d(i)=1}(\boldsymbol{n^0})_i X_i$ its normalized projection in the g_ε norm onto the horizontal plane. Note that this is not dependent on ε and is well defined only outside the characteristic set $\Sigma(M) = \{x \in M|\ H(x) \subset T_xM\}$. The vector $\boldsymbol{n^0}$ is called *horizontal normal* and its (horizontal) divergence

$$K_0 = \sum_{d(i)=1} X_i \boldsymbol{n^0}_i \tag{1}$$

is known as the horizontal mean curvature of M at x.

The horizontal mean curvature flow is defined as an evolution M_t of an initial manifold M such that the velocity is horizontal and proportional to $-K_0$, i.e.

$$\partial_t F(x,t) = -K_0 \boldsymbol{n^0}, \tag{2}$$

where $F : M \times [0,T) \to G$ is a family of embeddings and K_0 is the horizontal mean curvature of $M_t = F(M,t)$. This is the horizontal flow along which the sub-Riemannian perimeter of M_t decreases faster. In the study of this flow we are then faced with two novel features: (a) Generically there are singularities (characteristic points) at all times, even with smooth initial data; (b) The PDE is only defined outside $\Sigma(M_t)$. The level set approach consists in studying a PDE describing the evolution of a function $u(x,t)$ such that[2] $M_t = \{x \in G|\ u(x,t) = 0\}$. In this setting one has $\boldsymbol{n^\varepsilon} = \nabla_\varepsilon u/|\nabla_\varepsilon u|$ and $\boldsymbol{n^0} = \nabla_0 u/|\nabla_0 u|$. Consequently, on a formal level, the relevant PDE is $\partial_t u = K_0|\nabla_0 u|$. This equation is "well approximated" by the Riemannian mean curvature flows $\partial_t x = -K_\varepsilon \boldsymbol{n^\varepsilon}$, where $K_\varepsilon = \sum_{i=1}^n X_i^\epsilon \boldsymbol{n^\varepsilon}_i$ is the g_ε mean curvature of M. The corresponding evolution PDE for the level sets is $\partial_t u^\varepsilon = K_\varepsilon |\nabla_\varepsilon u|$. In fact, we observe that for a given hypersuface, $\boldsymbol{n^\varepsilon} \to \boldsymbol{n^0}$ and $K_e \to K_0$ as $\varepsilon \to 0$. Away from characteristic points, the sub-Riemannian flow can be rewritten more explicitly as

$$u_t = \sum_{i,j=1}^m \left(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_0 u|^2}\right) X_i X_j u, \text{ for } x \in G, t > 0. \tag{3}$$

If the Carnot group is a product $G \times \mathbb{R}$ and we use coordinates $(x,e) \in G \times \mathbb{R}$, then a special class of evolutions is given by graphs over G of the form $M_t = \{(x, u(x,t)) \mid x \in G,\ t > 0\}$ where $u : G \to \mathbb{R}$ is a solution of

$$u_t = \sum_{i,j=1}^m \left(\delta_{ij} - \frac{X_i u X_j u}{1 + |\nabla_0 u|^2}\right) X_i X_j u, \text{ for } x \in G, t > 0. \tag{4}$$

Note that such graphs are always non-characteristic. Since the PDE (3) is not defined at characteristic points we will interpret the flow in a generalized form, as flow of singular surfaces, and use the viscosity solutions approach

[2]When a manifold is defined as a level set, we tacitly assume that the gradient of the defining function does not vanish in a neighborhood of the manifold.

Definition 1. A function $u \in C(G \times [0,\infty)$ is a weak subsolution of (3) in $G \times (0,\infty)$ if for any $(x,t) \in G \times (0,\infty)$ and any function $\phi \in C^2(G) \times (0,\infty)$ such that $u - \phi$ has a local maximum at (x,t) then

$$(5)\quad \partial_t \phi \le \begin{cases} \sum_{i,j=1}^m \left(\delta_{ij} - \frac{X_i\phi X_j\phi}{|\nabla_0\phi|^2}\right) X_i X_j \phi & \text{if } |\nabla_0 \phi| \neq 0 \\ \sum_{i,j=1}^m (\delta_{ij} - p_i p_j) X_i X_j \phi & \text{for some } p \in \mathbb{R}^m, |p| \le 1, \text{ if } |\nabla_0\phi| = 0. \end{cases}$$

Weak supersolutions and solutions are defined accordingly.

Our results are: (1) Comparison principles, (2) Existence of weak solutions, (3) Constructions of explicit bounded barriers.

Theorem 2. *Assume that u is a bounded weak subsolution and v is a bounded weak supersolution of* (3). *Suppose further (i) For all $(x_H, x_V), (x_H, y_V) \in G$ $u(x_H, x_V, 0) \le v(x_H, y_V, 0)$. (ii) Either u or v is uniformly continuous when restricted to $G \times \{t = 0\}$. Then $u(x,t) \le v(x,t)$ for all $x \in G$ and $t \ge 0$.*

Theorem 3. *Assume that u is a bounded weak subsolution and v is a bounded weak supersolution of* (4). *Suppose further (i) For all $x \in G$ $u(x,0) \le v(x,0)$. (ii) Either u or v is uniformly continuous when restricted to $G \times \{t = 0\}$. Then $u(x,t) \le v(x,t)$ for all $x \in G$ and $t \ge 0$. In particular, bounded weak solutions of* (4) *are unique.*

Remark 4. For bounded domains and in the special case of the Heisenberg group this theorem follows from the results of Bieske [1]. See also the comparison principle for the Gauss curvature flow established in [5].

In proving the existence of weak solutions to the initial value problem for (3), such solution will arise as limit of solutions of regularized parabolic equations: For $\delta, \sigma > 0$, for all $\xi \in G$ and $1 \le i,j \le n$ we define the coefficients of the approximating equations $A_{ij}^{\varepsilon,\delta}(\xi) = \left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2 + \delta}\right)$, and $A_{ij}^{\varepsilon,\delta,\sigma}(\xi) = A_{ij}^{\varepsilon,\delta}(\xi) + \sigma\delta_{ij}$.

Proposition 5. *For any $f \in C^\infty(G)$ there exists a unique solution $u^{\varepsilon,\delta} \in C^\infty(G) \times (0,\infty))$ of the initial value problem*

$$(6)\quad \frac{\partial}{\partial t} u^{\varepsilon,\delta} = \sum_{i,j=1}^n A_{ij}^{\varepsilon,\delta}(\nabla_\varepsilon u^{\varepsilon,\delta}) X_i^\varepsilon X_j^\varepsilon u^{\varepsilon,\delta} \text{ in } x \in G, t > 0,$$

$$\text{and } u^{\varepsilon,\delta}(x,0) = f(x) \text{ for all } x \in G.$$

Moreover, for all $t > 0$ one has $||u^{\varepsilon,\delta}(\cdot,t)||_{L^\infty(G)} \le ||f||_{L^\infty(G)}$ and $||\tilde{\nabla}_\varepsilon u^{\varepsilon,\delta}(\cdot,t)||_{L^\infty(G)} \le ||\tilde{\nabla}_\varepsilon f||_{L^\infty(G)}$. For any compact set $K \subset G$ there exists $C = C(K,G) > 0$ such that if $0 \le \varepsilon < 1$, $||\nabla_\varepsilon u^{\varepsilon,\delta}(\cdot,t)||_{L^\infty(K)} \le C||\nabla_E f||_{L^\infty(G)}$.

Theorem 6. *For any bounded $f \in C(G)$ there exists a viscosity solution $u \in Lip_{g_1}(G \times (0,\infty))$ of*

$$\partial_t u = \sum_{i,j=1}^{m} A_{ij}^{0,0}(\nabla_0 u) X_i X_j u \quad \text{in } G \times (0,\infty) \quad \text{and } u(x,0) = f(x). \tag{7}$$

Theorem 7. *Let G be a Carnot group of step two. If we assume that the function $f \in C(G)$ is constant in a neighborhood $G \setminus K$ of infinity then any weak solution u of the initial value problem* (7) *constructed as in Theorem 6 is constant in a set of the form $\{|x| + t \geq R\}$, with R depending on K.*

We show two basic geometric properties for the flow, namely (i) separation property and (ii) show that the right invariant distance between level sets is not increasing with time. We say that a level set $M = \{u(x) = 0\}$ is *cylindric* if $u(x_H, x_V)$ is constant in the x_V variables.

Proposition 8. *Let $M_0, \hat{M}_0$ be subset of G and denote by M_t and $\hat{M}_t$ the corresponding generalized flows. We have (i) If $M_0 \subset \hat{M}_0$ and $\hat{M}_t$, $t \geq 0$ is cylindric, then $M_t \subset \hat{M}_t$, for all $t > 0$. (ii) For every compact initial data M_0, the corresponding evolution M_t has a finite extinction time. (iii) For this part we consider the flows M_t, $\hat{M}_t$ arising as level set of the solutions constructed in Theorem 6. If we denote by $\tilde{d}(\cdot,\cdot)$ the right invariant CC distance, then $\tilde{d}(M_0, \hat{M}_0) \leq \tilde{d}(M_t, \hat{M}_t)$ for all $t > 0$.*

In a joint project with Mario Bonk (Michigan) we study smooth solutions of the flow $F^\perp(x,t) := \langle F(x,t), \boldsymbol{n^1} \rangle_1 = -K_0 \langle \boldsymbol{n^1}, \boldsymbol{n^0} \rangle_1$ in the setting of the Heisenberg group $\mathbb{H}^n = (z, x_{2n+1}) = (x_1, ..., x_{2n+1}) \in R^{2n+1}$. We interpret the PDE to hold at a characteristic point x_0 if the limit of both sides exist and coincide for some sequence of non-characteristic points converging to x_0. This is always the case for the class of cylindrically symmetric solutions, i.e. level sets of functions of the form $u(x,t) = w(x_1^2 + ... + x_{2n}^2, x_{2n+1}, t)$. In the special case when M_t is the graph of a radial function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, that is $u(x,t) = x_{2n+1} - f(|z|, t)$, the PDE becomes $\partial_t f = \frac{4r^2 \partial_r^2 f + (\partial_r f)^3/r}{(\partial_r f)^2 + 4r^2}$, here we have let $r = |z|$. We show:

1) A flow $\{M_t\}$ evolving by horizontal mean curvature is *self-similar* with respect to homogeneous group dilations if there exists a real valued function $\lambda(t) > 0$ such that for all times $M_t = \delta_{\lambda(t)} M_0$.

Theorem 9. *A smooth flow $\{M_t\}$ is self-similar with respect to dilations if there exists a constant α and a time $t_0 > 0$ such that $-\frac{\alpha}{2}\mathcal{D}^\perp(y) = K_0 \langle \vec{n}, \nu_h \rangle_0$ for all $y \in M_{t_0} \setminus \Sigma(M_{t_0})$. Here $\mathcal{D}(y) = (y_1, \ldots, y_{2n}, 2y_{2n+1})$ is the vector field generating the dilations group in $\mathbb{H}^n$. In other words, if the surface M_t is self-similar then the flow produces the same normal velocity as if the evolution M_t were driven by the vector $-\alpha \mathcal{D}(y)/2$.*

Lemma 10. *The only self-similar analytic solutions of horizontal mean curvature flow that have cylindrical symmetry and correspond to initial data that satisfy the conditions $f(0) = c_0$, $f'(0) = 0$, are the paraboloids level sets of $u(x,t) = x_{2n+1} - c_0[1 - \alpha t - \alpha(x_1^2 + ... + x_{2n}^2)/2]$.*

2) Two C^2 solutions corresponding to disjoint initial closed hypersurfaces cannot meet for the first time at non-characteristic points.

3) Two C^2 cylindrically symmetric solutions, corresponding to disjoint initial closed hypersurfaces cannot intersect for all $t > 0$ for which the flows are defined.

4) A closed, C^2 cylindrically symmetric solution, corresponding to a strictly convex initial data has a legendrian foliation composed of curves with strictly positive curvature for all $t > 0$ for which the flow is defined.

References

[1] Bieske, T. Comparison principle for parabolic equations in the Heisenberg group. *Electron. J. Differential Equations* (2005), No. 95, 11 pp. (electronic).

[2] Capogna, L., Danielli, D., Pauls, S., and Tyson, J. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, vol. 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[3] Citti, G., and Sarti, A. A cortical based model of perceptual completion in the roto-translation space. *J. Math. Imaging Vision 24*, 3 (2006), 307–326.

[4] Gromov, M. *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser Boston Inc., Boston, MA, 1999. Edited by J. LaFontaine and P. Pansu. Based on the 1981 French original, Translated from the French by Sean Michael Bates.

[5] Haller, E. Generalized gauss curvature flow of graphs over carnot groups. Preprint (2008).

Highly degenerate harmonic mean curvature flow

Maria-Cristina Caputo

(joint work with Panagiota Daskalopoulos)

We study the evolution of a weakly convex surface Σ_0 in $\mathbb{R}^3$ with flat sides by the Harmonic Mean Curvature flow. We establish the short time existence as well as the optimal regularity of the surface and we show that the boundaries of the flat sides evolve by the curve shortening flow. It follows from our results that a weakly convex surface with flat sides of class $C^{k,\gamma}$, for some $k \in \mathbb{N}$ and $0 < \gamma \leq 1$, remains in the same class under the flow. This distinguishes this flow from other, previously studied, degenerate parabolic equations, including the porous medium equation and the Gauss curvature flow with flat sides, where the regularity of the solution for $t > 0$ does not depend on the regularity of the initial data. We consider the motion of a compact, weakly convex two-dimensional surface Σ_0 in space $\mathbb{R}^3$ under the *harmonic mean curvature flow* (HMCF)

$$\text{(HMCF)} \qquad \frac{\partial P}{\partial t} = \frac{K}{H} N$$

where each point P of Σ_0 moves in the inward normal direction N with velocity equal to the *harmonic mean curvature* of the surface, namely the harmonic mean

$$\frac{K}{H} = \frac{\lambda_1 \, \lambda_2}{\lambda_1 + \lambda_2}$$

of the two principal curvatures λ_1, λ_2 of the surface.

The existence of solutions to the HMCF with strictly convex smooth initial data was first shown by Andrews in [1]. Andrews also showed that, under the HMCF, strictly convex, smooth surfaces converge to round points in finite time. In [2], Dieter established the short time existence of solutions to the HMCF with weakly convex smooth initial data and mean curvature $H > 0$. More precisely, Dieter showed that if at time $t = 0$ the surface Σ_0 satisfies $K \geq 0$ and $H > 0$, then there exists a unique strictly convex smooth solution Σ_t of the HMCF defined on $0 < t < \tau$, for some $\tau > 0$. By the results of Andrews, this solution exists up to the time where its enclosed volume becomes zero. However, the highly degenerate case where the initial data is weakly convex and both K and H vanish in a region is not studied in [2] .

We will consider in this work the evolution of a surface Σ_0 with flat sides by the HMCF. The parabolic equation describing the motion of the surface becomes degenerate at points where both curvatures K and H become zero. Our main objective is to study the solvability and optimal regularity of the evolving surface for $t > 0$, by viewing the flow as a *free-boundary* problem. It will be shown that a surface Σ_0 of class $C^{k,\gamma}$ with $k \in \mathbb{N}$ and $0 < \gamma \leq 1$ at $t = 0$, will remain in the same class for $t > 0$. In addition, we will show that the strictly convex parts of the surface become instantly C^∞ smooth up to the flat sides and the boundaries of the flat sides evolve by the curve shortening flow.

For simplicity we will assume that the surface Σ_0 has only one flat side, namely $\Sigma = \Sigma_1 \cup \Sigma_2$, with Σ_1 flat and Σ_2 strictly convex (both principal curvatures are strictly positive). We may also assume that Σ_1 lies on the $z = 0$ plane and that Σ_2 lies above this plane since the equation is invariant under rotation and translation. Therefore, the lower part of the surface Σ_0 can be written as the graph of a function

$$z = h(x, y)$$

over a compact domain $\Omega \subset \mathbb{R}^2$ containing the initial flat side Σ_1. Let Γ denote the boundary of the flat side Σ_1. We define $g = h^p$, for some $0 < p < 1$. Our main assumption on the initial surface Σ_0 is that it satisfies the following *non-degeneracy condition* ($\star$):

$$(\star) \qquad |Dg(P)| \geq \lambda \qquad \text{and} \qquad g_{\tau\tau}(P) \geq \lambda, \qquad \text{for all } \; P \in \Gamma$$

for some number $\lambda > 0$. Here τ denotes the tangential direction to the level sets of g and $g_{\tau\tau}$ denotes the second order derivative in this direction.
Under the above conditions, our main results show that for $t \in (0, T)$:

(1) The HMCF admits a solution $\Sigma_t = (\Sigma_1)_t \cup (\Sigma_2)_t$ of class $C^{k,\gamma}$, for some $k \in \mathbb{N}$ and $0 < \gamma \leq 1$ depending on p, which is smooth up to $\Gamma_t = \partial(\Sigma_1)_t$.
(2) $(\Sigma_1)_t$ is flat and its boundary Γ_t evolves by the curve shortening flow.

The fact that the solution Σ_t remains in the class $C^{k,\gamma}$ distinguishes this flow from other, previously studied, degenerate free-boundary problems (such as the Gauss curvature flow with flat sides, the porous medium equation and the evolution p-laplacian equation) in which the regularity of the solution for $t > 0$ does not depend on the regularity of the initial data. For more details, the reader is invited to read her PhD thesis [3].

References

[1] B. Andrews. Motion of hypersurfaces by Gauss curvature *Pacific J. Math.* 195, no. 1, 1-34, 2000

[2] S. Dieter. Nonlinear degenerate Curvature flow for weakly convex hypersurfaces *Calculus of Variations and Partial Differential Equations*, 22, 2: 229 - 251, 2005

[3] M.C. Caputo. Highly Degenerate Harmonic Mean Curvature Flow *PhD Thesis, Columbia University*, May 2006

Shock Reflection and Free Boundary Problems

Mikhail Feldman

(joint work with Gui-Qiang Chen)

One of important problems in mathematical fluid dynamics is reflection of shock by a wedge. It arises in many physical application, and in the study of mutidimentional conservation laws since its solutions are building blocks and asymptotic attractors for the gemeral solutions of Euler equations for compressible fluids. The reflection picture was first described by Ernst Mach in 1878. In later works, experimental, computational, and asymptotic analysis have shown that various patterns of reflected shocks may occur, including regular and Mach reflection [2, 5, 6, 7, 8, 9]. However, there has been no rigorous mathematical results on the global existence and structural stability of shock reflection, especially for potential flow equation, which has been used in aerodynamics. Such problems involve several difficulties in the analysis of nonlinear partial differential equations including equations of elliptic-hyperbolic mixed type, free boundary problems, degenerate ellipticity along the sonic line.

In the talk I describe recent results on regular shock reflection for potential flow equation in dimension two. For potential flow, velocity $\mathbf{u}$ is $D_{\mathbf{x}}\Phi$, where Φ is the potential.

A plane shock in the $(\mathbf{x}, t)$–coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with left state $(\rho, D_{\mathbf{x}}\Phi) = (\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0), u_1 > 0, \rho_0 < \rho_1$, hits a symmetric wedge $W := \{(x_1, x_2) : |x_2| < x_1 \tan\theta_w, x_1 > 0\}$ at time zero. We can consider only upper half-plane $\mathbb{R}^2_+ = \{x_2 > 0\}$. We are looking for a solution in $\Lambda = \mathbb{R}^2_+ \setminus W$ of the time-dependent potential flow system satisfying initial data

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \text{for } |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (\rho_1, u_1 x_1) & \text{for } x_1 < 0, \end{cases} \tag{1}$$

and boundary condition

$$\nabla\Phi \cdot \nu|_{\partial\Lambda} = 0. \tag{2}$$

Since self-similar solutions are expected, we rewrite this as a quasi-static problem in self-similar plane.

Potential flow equation for self-similar solutions, in self-similar variables $(\xi,\eta) = (\frac{x}{t},\frac{y}{t})$, is

$$\operatorname{div}(\rho(|D\varphi|^2,\varphi)D\varphi) + 2\rho(|D\varphi|^2,\varphi) = 0, \tag{3}$$

with $\rho(|D\varphi|^2,\varphi) = \left(\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2)\right)^{\frac{1}{\gamma-1}}$, where $\varphi(\xi,\eta)$ is the pseudo-velocity potential, ρ is density, and $\gamma > 1, \rho_0 > 0$ are constants. Equation is elliptic-hyperbolic mixed, which is elliptic (resp. hyperbolic) if and only if

$$|D\varphi| < c(|D\varphi|^2,\varphi), \quad \left(\text{resp. } |D\varphi| > c(|D\varphi|^2,\varphi)\right),$$

where $c(|D\varphi|^2,\varphi)$ is the sonic speed defined by $c^2 = \rho^{\gamma-1}$. Solution is called subsonic (resp. supersonic) in elliptic (resp. hyperbolic) regions. Shocks are discontinuities in the pseudo-velocity $D\varphi$. That is, if Ω^+ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are two nonempty open subsets of $\Omega \subset \mathbb{R}^2$ and $S := \partial\Omega^+ \cap \Omega$ is a C^1–curve where $D\varphi$ has a jump, then $\varphi \in W^{1,1}_{loc}(\Omega) \cap C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution of (3) in Ω if and only if φ satisfies equation (3) in $\Omega^\pm$ and the Rankine-Hugoniot conditions on S:

$$[\varphi]_S = 0, \qquad \left[\rho(|D\varphi|^2,\varphi)D\varphi\cdot\nu\right]_S = 0. \tag{4}$$

The plane incident shock solution in the $(\mathbf{x},t)$–coordinates with states $(\rho,\nabla_{\mathbf{x}}\Psi) = (\rho_0,0,0)$ and $(\rho_1,u_1,0)$ corresponds to a weak solution φ of (3) of the form:

$$\varphi_0(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) \qquad \text{for } \xi > \xi_0, \tag{5}$$

$$\varphi_1(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_1(\xi-\xi_0) \qquad \text{for } \xi < \xi_0, \tag{6}$$

respectively, where $S_0 = \{\xi = \xi_0\}$ is the incident shock. Here ξ_0 is uniquely determined by (ρ_0,ρ_1,γ) through (4). Denote by P_0 the point of intersection of S_0 with the wedge boundary, that is, $P_0 = (\xi_0, \xi_0\tan\theta_w)$. Shock reflection problem is now reduced to the following problem in self-similar plane:

Problem 1. *Seek a solution φ of equation (3) in the self-similar domain Λ with the slip boundary condition (2) and the asymptotic boundary condition at infinity:*

$$\varphi \to \bar\varphi := \begin{cases} \varphi_0 & \text{for } \xi > \xi_0, \eta > \xi\tan\theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0, \eta > 0, \end{cases} \qquad \text{when } \xi^2+\eta^2 \to \infty,$$

where the convergence holds in the sense that $\lim_{R\to\infty} \|\varphi - \overline{\varphi}\|_{C(\Lambda\setminus B_R(0))} = 0$.

Since φ_1 does not satisfy the slip boundary condition (2), the solution must differ from φ_1 in $\{\xi < \xi_0\} \cap \Lambda$ and thus a shock diffraction by the wedge occurs.

Denote by $P_0 = (\xi_0, \xi_0\tan\theta_w)$ the point of intersection of the incident shock S_0 with the wedge boundary. There exists an angle $\theta_{sonic} \in (0,\pi/2)$ determined by ρ_0, ρ_1, γ such that for the wedge angles $\theta_w \in (\theta_{sonic}, \pi/2)$ there exists uniform state

$$\varphi_2(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) + u_2(\xi - \xi_0) + (\eta - \xi_0 \tan\theta_w)u_2 \tan\theta_w, \tag{7}$$

which satisfies (2) on the wedge boundary $\{\eta = \xi \tan\theta_w\}$, and satisfies Rankine-Hugoniot conditions (4) with φ_1 at P_0 and thus along the line $S_1 = \{\varphi_1 = \varphi_2\}$. Constant velocity $(u_2, u_2 \tan\theta_w)$ and density ρ_2 are determined by $(\theta_w, \rho_0, \rho_1, \gamma)$ from the two algebraic equations expressing the conditions above. Moreover $\rho_2 > \rho_1$, and φ_2 is supersonic(hyperbolic) at the point P_0. For such wedge angles $\theta_w \in (\theta_{sonic}, \pi/2)$ the structure of global solution φ to Problem 1 is expected to be regular reflection which described as following:

Let B be the sonic circle for state (2) with center $(u_2, u_2 \tan\theta_w)$ and radius $c_2 = \rho_2^{(\gamma-1)/2} > 0$ (the sonic speed of φ_2). Denote by P_1 (resp P_4) the point of intersection of ∂B with S_1 (resp. with the wedge boundary $\{\eta = \xi \tan\theta_w\}$). It is expected that the solutions φ and φ_1 differ within $\{\xi < \xi_0\}$ only in the domain $P_0P_1P_2P_3P_4$, where $P_2 \in \{\xi < 0,\ \eta = 0\}$ and $P_3 = (0, 0)$. The curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1. Then, within $P_0P_1P_2P_3P_4$, solution φ differs from φ_2 in the domain $\Omega = P_1P_2P_3P_4$, where the equation (3) is elliptic. Boundary of Ω consists of the sonic arc P_1P_4, line segments P_2P_3 and P_3P_4 and the curved part of the reflected shock P_1P_2, which is apriory unknown (the free boundary).

Theorem 1 ([3])**.** *For any $\gamma > 1$ and $\rho_1 > \rho_0 > 0$ there exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1)$ such that, when $\theta_w \in [\theta_c, \frac{\pi}{2})$, there exists a weak solution of Problem 1, which satisfies the following:*

(i)

$$\varphi \in C^{0,1}(\Lambda), \qquad \varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega}),$$

$$\varphi = \begin{cases} \varphi_0 & \text{for } \xi > \xi_0 \text{ and } \eta > \xi \tan\theta_w, \\ \varphi_1 & \text{for } \xi < \xi_0 \text{ and above the reflection shock } P_0P_1P_2, \\ \varphi_2 & \text{in } P_0P_1P_4. \end{cases}$$

(ii) *equation* (3) *is elliptic in Ω;*
(iii) $\varphi \geq \varphi_2$ *in* Ω*;*
(iv) *the reflected shock $P_0P_1P_2$ is C^2 at P_1 and C^∞ elsewhere;*
(v) *φ is $C^{1,1}$ across the part $\Gamma_{sonic} = P_1P_4$ of the sonic circle.*

Theorem 2 ([4])**.** *Let $\gamma > 1$ and $\rho_1 > \rho_0$ satisfy the condition $u_1 < c_1$, where $c_1^2 = \rho^{\gamma-1}$. Then solution of Problem 1 satisfying properties (i)-(v) of Theorem 1 exists for all $\theta_w \in (\theta_{sonic}, \pi/2)$.*

The condition in Theorem 2 is an explicit algebraic condition in terms of γ, ρ_0, ρ_1.

Next we show that $C^{1,1}$ regularity near and accross sonic arc $\Gamma_{sonic} = P_1P_4$ where ellipticity degenerates is optimal:

Theorem 3 ([1]). *Let φ be a solution of Problem 1 satisfying properties (i)-(v) of Theorem 1. Then:*

(i) *φ is $C^{2,\alpha}$ in Ω up to Γ_{sonic} away from the point P_1 for any $\alpha \in (0,1)$,*
(ii) *φ is $C^{1,1}$ but not C^2 across Γ_{sonic}, specifically $D^2\varphi$ has a jump across Γ_{sonic},*
(iii) *The limit $\lim_{\substack{(\xi,\eta)\to P_1\\(\xi,\eta)\in\Omega}} D^2\varphi$ does not exist.*

For the proofs, we reformulate Problem 1 as a free boundary problem for the free boundary Γ_{sonic} and φ in the elliptic region Ω. Free boundary conditions are Rankine-Hugoniot conditions on Γ_{sonic}. We solve this problem by method of continuity, which involves deriving some regularity estimates for degenerate elliptic equations, and controlling geometry of free boundary using maximum principle.

References

[1] M. Bae, G.-Q. Chen and M. Feldman, Regularity of Solutions to Regular Shock Reflection for Potential Flow, submitted. Also, preprint arXiv:0804.2500 on arXiv.
[2] G. Ben-Dor, *Shock Wave Reflection Phenomena*, Springer-Verlag: New York, 1991.
[3] G.-Q. Chen and M. Feldman, Global Solutions to Shock Reflection by Large-Angle Wedges for Potential Flow, to appear in *Annals of Math.*
[4] G.-Q. Chen and M. Feldman, Global Solutions to Regular Shock Reflection by Wedges up to the Sonic Angle for Potential Flow, preprint.
[5] J. Glimm and A. Majda, *Multidimensional Hyperbolic Problems and Computations*, Springer-Verlag: New York, 1991.
[6] J. K. Hunter, Transverse diffraction of nonlinear waves and singular rays, *SIAM J. Appl. Math.* **48** (1988), 1–37.
[7] J. Hunter and J. B. Keller, Weak shock diffraction, *Wave Motion,* **6** (1984), 79–89.
[8] J. B. Keller and A. A. Blank, Diffraction and reflection of pulses by wedges and corners, *Comm. Pure Appl. Math.* **4** (1951), 75–94.
[9] C. S. Morawetz, Potential theory for regular and Mach reflection of a shock at a wedge. *Comm. Pure Appl. Math.*, **47** (1994), 593–624.

Isoperimetric inequalities in Riemannian and Lorentzian manifolds

Gerhard Huisken

Let $L^4(h)$ be a cosmological space-time, ie a globally hyperbolic Lorentzian manifold diffeomorphic to $\Sigma^3 \times (0,T)$, where Σ^3 is a compact 3-manifold. We assume that the Lorentzian metric h satisfies the timelike convergence condition $Ric_h(X,X) \geq 0$ for all timelike vectors X and admits a crushing initial singularity. Cosmological space-times of this type admit foliations by hypersurfaces satisfying geometric partial differential equations related to their mean curvature: Constant mean curvature foliations have been found by Gerhardt [2], solutions of mean curvature flow were constructed by Ecker and Huisken [1], and solutions to inverse mean curvature flow were constructed by Holder [4]and, in greater generality, Gerhardt [3].

In the lecture it is demonstrated that certain integrals of mean curvature are monotone under the mean curvature flow and the inverse mean curvature flow respectively: Solutions $F : \Sigma^3 \times (0,T) \to L^4(h)$ of mean curvature flow and inverse mean curvature flow satisfy the equations

$$\frac{d}{dt}F = H\,\nu \ \text{ and } \ \frac{d}{dt}F = -\frac{1}{H}\,\nu$$

respectively, where ν is the timelike past-directed normal to the evolving hypersurfaces and H is the mean curvature. Using these monotonicity formulae it is possible to derive reverse isoperimetric inequalities of the type

$$|\Sigma^3|^{\frac{4}{3}} \le C_0 \mathrm{Vol}(\Sigma^3),$$

where $\mathrm{Vol}(\Sigma^3)$ denotes the 4-volume of the space-time between Σ^3 and the crushing singularity. In this Lorentzian setting the mean curvature flow moves in the expanding direction whereas inverse mean curvature flow decreases the area of the hyperurfaces exponentially and approaches the crushing singularity. It turns out that the constant C_0 is related to the behavior of the curvature integrals mentioned above both in the expanding and the crushing directions; in a constant mean curvature foliation it is determined by the asymptotic behavior of the scaling invariant quantity $H|\Sigma^3|^{\frac{1}{3}}$.

The lecture also explains that the above relation between the isoperimetric inequality and geometric evolution equations holds in Riemannian manifolds of nonnegative Ricci-curvature. In this case mean curvature flow is used to sweep out the interior of a bounded region while inverse mean curvature flow can relate a bounded region to the behavior of the manifold near infinity. The constant C_1 in the isoperimetric inequality is then determined by the asymptotic behavior of the integral of H^n near infinity, a quantity that turns out to be monotonically decreasing under inverse mean curvature flow in manifolds of non-negative Ricci curvature. We get

$$|\Sigma^n|^{\frac{n+1}{n}} \ge C_1 \mathrm{Vol}(\Sigma^n)$$

with the constant determined by the infimum of the integral of H^n on outward minimising boundaries in the manifold. The method relies in this case on inverse mean curvature flow in Riemannian manifolds developed by Huisken and Ilmanen [5] and on the regularity theory for mean curvature flow by White [6].

In the special case of Riemannian 3-manifolds the above results can be extended to the case of non-negative scalar curvature, leading to a concept of isoperimetric mass in asymptotically flat 3-manifolds related to the supremum of the Hawking mass on outward minimising boundaries.

References

[1] K. Ecker and G. Huisken, *Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes*, Comm. Math. Phys. **135** (1991), no. 3, 595–613.

[2] C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Commun. Math. Phys. **89** (1983), 523–553.

[3] C. Gerhardt, *The Inverse Mean Curvature Flow in Robertson-Walker Spaces and its Application*, Methods Appl. Anal. **13**, Number 1 (2006), 19–28.

[4] M. Holder, *Contracting spacelike hypersurfaces by their inverse mean curvature*, J. Austral. Math. Soc. Ser. A **68** (2000), no. 3, 285–300.

[5] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Diff. Geom. **59** (2001), 353–437.
[6] B. White, *Partial regularity of mean-convex hypersurfaces flowing by mean curvature*, Int. Math.Res.Not. **4** (1994), 185–192.

Dynamics of topological defects in nonlinear field theories

Robert L. Jerrard

A vigorous line of research, dating back at least 30 years, establishes various ways in which solutions of semilinear elliptic equations such as

$$(1) \qquad -\Delta u + \frac{1}{\varepsilon^2}(1-|u|^2)u = 0 \qquad u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^k, \quad k = 1 \text{ or } 2$$

are connected to minimal surfaces when $N > k$ and $0 < \varepsilon \ll 1$. In the scalar case $k = 1$, these results (see for example [9, 11, 5, 6, 10]) demonstrate with varying degrees of precision that a if solution u satisfies conditions such as suitable boundary conditions together with bounds on an associated energy functional, then roughly speaking u has the form

$$(2) \qquad u \;\approx\; q(\frac{d}{\varepsilon})$$

where

$$(3) \qquad q : \mathbb{R} \to \mathbb{R} \text{ solves } \; -q'' + (q^2-1)q = 0, \quad q(\pm\infty) = \pm 1, \quad \text{and } q(0) = 0,$$

and $d : \Omega \to \mathbb{R}$ is the signed distance function to some minimal hypersurface Γ of Ω. That is, Γ is a hypersurface whose mean curvature vanishes identically, and d is characterized in a neighborhood of Γ by the properties

$$(4) \qquad d = 0 \text{ on } \Gamma, \qquad |\nabla d|^2 = 1 \text{ near } \Gamma.$$

Other results of the same general character assert for example that an energy density associated with a solution u of (1) concentrates around a minimal submanifold Γ.

In the case of a vector-valued solution $u : \Omega \to \mathbb{R}^2$ of (1), no descriptions exactly analogous to (2) are known, due to the difficulty in pinning down rotational degrees of freedom, but there are numerous results ([8, 2, 1], among many others) showing that for suitable solutions, energy concentrates around a codimension 2 submanifold Γ with mean curvature identically equal to zero.

Analogous results are also known for the parabolic equation

$$(5) \qquad u_t - \Delta u + \frac{1}{\varepsilon^2}(1-|u|^2)u = 0 \quad \text{in } u : (0,T) \times \mathbb{R}^N \to \mathbb{R}^k, \quad k = 1 \text{ or } 2$$

when $N > k$ and $0 < \varepsilon \ll 1$. For example, in the scalar case, solutions u are again known have roughly the form (2), where for every t, $d(t,\cdot)$ is the signed distance function from a hypersurface Γ_t, so that $d(t,\cdot) = 0$ on Γ_t and $|\nabla d(t,\cdot)|^2 = 1$ near Γ_t; and the hypersurfaces $\{\Gamma_t\}_{0\le t\le T}$ evolve by mean curvature flow.

In the case $k = 2$, results establishing a relationship between (5) and codimension 2 mean curvature flow are known and are typically stated, roughly speaking, in terms of concentration of energy densities.

We prove results of a similar character for the semilinear wave equation

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2} g(|u|^2)u = 0 \quad \text{in } u : (0,T) \times \mathbb{R}^N \to \mathbb{R}^k, \quad k = 1 \text{ or } 2 \tag{6}$$

when $N > k$ and $0 < \varepsilon \ll 1$. Here $g(|u|^2) = 1 - |u|^2$ for $N \leq 4$, and for general N, $g = \frac{1}{2}G'$ for some smooth $G : [0,\infty) \to [0,\infty)$ such that $G(1) = 0, G''(1) > 0, G(s) > 0$ for $s \neq 1$; and with G satisfying growth conditions that guarantee global well-posedness of (5).

There are very few prior results on this problem. Some work [4, 7] has analyzed dynamics of vortices in the case when $N = k = 2$. This is easier in that the topological defect in question are points rather than submanifolds, and in addition they move at subrelativistic speeds (in the situations considered by [4, 7].) A recent preprint [3] establishes the asymptotic stability of a flat kink when $N = 3$, with respect to very smooth, compactly supported perturbations.

One of our main results is

Theorem 1. *Let $N \geq 2$, and let $u : R^{1+N} \to \mathbb{R}$ solve* (6) *with initial data*

$$u(0,x) = q(\frac{d_0(x)}{\varepsilon}), \qquad u_t(0,x) = 0. \tag{7}$$

where

$$q : \mathbb{R} \to \mathbb{R} \quad \text{solves} \quad -q'' + g(q^2)q = 0, \quad q(\pm\infty) = \pm 1, \quad \text{and } q(0) = 0, \tag{8}$$

$d_0 : \mathbb{R}^N \to \mathbb{R}$ is the signed distance function from a smooth hypersurface $\Gamma_0 \subset \mathbb{R}^N$. In particular d_0 satisfies (4) *(with Γ replaced by Γ_0).*

Let $\Gamma \subset [0,T) \times \mathbb{R}^N$ be a timelike Minkowski minimal surface such that $\Gamma \cap \{t = 0\} = \Gamma_0$, and with zero velocity at $t = 0$.

Then for any compact subset $K \subset [0,T) \times \mathbb{R}^N$, there exists a constant $C(K)$, independent of ε, such that

$$\|u - U_\varepsilon\|_{L^2(K)} \leq C\sqrt{\varepsilon}$$

where U_ε is an explicitly constructed function that has the form

$$U_\varepsilon = q(\frac{d}{\varepsilon})$$

near Γ, with d defined in a neighborhood of Γ by

$$d = 0 \text{ on } \Gamma, \qquad -d_t^2 + |\nabla d|^2 = 1 \text{ near } \Gamma. \tag{9}$$

and satisfying $d(0,x) = d_0(x)$ near Γ_0. In other words, d is the signed Minkowski *distance to Γ,*

The condition $d(0,x) = d_0(x)$ near Γ_0 can be arranged to hold due to the assumption that Γ has velocity 0 at $t = 0$. It is needed only to fix a sign.

A timelike Minkowski minimal surface is a critical point of the (Minkowski) area functional. For example, if Γ_0 in the statement of the theorem is the graph of some smooth, compactly supported function h_0, ie if

$$\Gamma_0 = \{y, h_0(y)) : y \in \mathbb{R}^{N-1}\}$$

then a minimal surface Γ with zero initial velocity and $\Gamma \cap \{t = 0\} = \Gamma_0$ is given by

$$\Gamma = \{(t, y, h(t, y)) : (t, y) \in \mathbb{R} \times \mathbb{R}^{N-1}\}$$

where h solves

$$\partial_t \left(\frac{h_t}{\sqrt{1 - h_t^2 + |\nabla h|^2}} \right) - \nabla \left(\frac{\nabla h}{\sqrt{1 - h_t^2 + |\nabla h|^2}} \right) = 0 \tag{10}$$

with initial data

$$h(0, y) = h_0(y), \quad h_t(0, y) = 0.$$

A smooth solution of the above equation is known to exist locally in t. Note that the left-hand side of (10) is exactly the mean curvature of Γ with respect to the Minkowski (pseudo) metric.

The most important step in the proof is to perform a change of variables that reduces the problem under consideration to the studying behavior of an equation of roughly the form

$$\Box_{g,\tau} v - \partial^2_{X_N} v + \frac{1}{\varepsilon^2} g(|v|^2) v = b \cdot Dv$$

where $\Box_{g,\tau}$ denotes a wave-like operator in the "tangential" variables $t, x_1, .., x_{N-1}$ and $|b_N| \leq C|x_N|$. This equation is studied for initial data that is a small perturbation of

$$v(0, x) = q(\frac{x_N}{\varepsilon}), \quad v_t(0, x) = 0.$$

In a second main result, we prove that if $u : \mathbb{R}^{1+N} \to \mathbb{R}^2$ solves (6) for suitable initial data, then u exhibits energy concentration around a codimension 2 timelike submanifold that is a critical point of the Minkowski area functional, at least as long as this submanifold remains smooth. Again, a main point is a change of variables similar to that used in the scalar case.

References

[1] G. Alberti, S. Baldo, G. Orlandi, "Variational Convergence for Functionals of Ginzburg-Landau Type," *Indiana University Math. J.*, **54**, no. 5 (2005), 1411-1472.

[2] F. Bethuel, H. Brezis and G. Orlandi, " Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions", *J. Func. Anal.*, **186**, 432-520, (2001).

[3] S. Cuccagna," On asymptotic stability in 3D of kinks for the ϕ^4 model " arXiv:0801.2678v1.

[4] R. Jerrard, "Vortex dynamics for the Ginzburg-Landau wave equation", *Calculus Variations Partial Differential Equations*, **9** (8) (1999) 683-688.

[5] R.V. Kohn and P. Sternberg, "Local Minimizers and Singular Perturbations," *Proc. Roy. Soc. of Edin. Sect. A*, **111**, no. 1-2, (1989), 69-84.

[6] M. Kowalczyk, "On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions," *Annali di Matematica,*, **184**, (2005), 17-52.
[7] F.-H. Lin, "Vortex dynamics for the nonlinear wave equation," *Comm. Pure Appl. Math.* **52** (6) (1999) 737-761.
[8] F.H. Lin and T. Rivière, "Complex Ginzburg-Landau equations in higher dimensions and codimension two area minimizing currents", *J. Eur. Math. Soc.*, **1**, 237-322, (1999).
[9] L. Modica and S. Mortola, "Un Essempio di $\Gamma-$convergenza,", *Boll. U.M.I.*, **14-B**, (1977), 285 - 299.
[10] F. Pacard and M. Ritore, "From constant mean curvature hypersurfaces to the gradient theory of phase transitions," *J. Diff. Geom.*, **64**, no. 3, (2003),359-423.
[11] P. Sternberg, "The Effect of a Singular Perturbation on Nonconvex Variational Problems," *Arch. Rational Mech. Anal.*, **101**, no. 3, (1988), 209-260.

Resonant decompositions and almost conservation laws for dispersive PDE in higher dimensions

Markus Keel
(joint work with James Colliander, Gigliola Staffilani, Hideo Takaoka, Terence Tao)

1. Introduction

We consider the Cauchy problem for the cubic defocusing nonlinear Schrödinger (NLS) equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u, \\ u(0,x) = u_0(x) \in H^s_x(\mathbb{R}^2), \end{cases} \tag{1}$$

in a Sobolev space $H^s_x(\mathbb{R}^2)$, where the unknown function $u : J \times \mathbb{R}^2 \longmapsto \mathbb{C}$ is a strong solution to (1) on a time interval $J \subset \mathbb{R}$ in the sense that $u \in C^0_{t,\mathrm{loc}} H^s_x(\mathbb{R}^2)$ and u obeys the integral equation

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-t')\Delta}[|u|^2 u(t')]\; dt'$$

for $t \in J$. Here of course the propagators $e^{it\Delta}$ are defined via the Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\; dx$$

by the formula

$$\widehat{e^{it\Delta}f}(\xi) := e^{-it|\xi|^2}\hat{f}(\xi)$$

and the Sobolev space $H^s_x(\mathbb{R}^2)$ is similarly defined via the Fourier transform using the norm

$$\|f\|_{H^s_x(\mathbb{R}^2)} := \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^2)}$$

where $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$. For later use we shall also need the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s_x(\mathbb{R}^2)} := \||\xi|^s \hat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^2)}.$$

We are interested primarily in the global-in-time problem, in which we allow J to be the whole real line $\mathbb{R}$.

Both the local and global-in-time Cauchy problems for this NLS equation (1) have attracted a substantial literature [23], [8], [18], [16],[3], [4], [14], [2], [9]. One has local well-posedness in $H^s_x(\mathbb{R}^2)$ for all $s \geq 0$, and if s is strictly positive then a solution can be continued unless the $H^s_x(\mathbb{R}^2)$ norm of the solution goes to infinity at the blowup time (see e.g. [7], [21]). Also, due to the smooth nature of the nonlinearity, any local $H^s(\mathbb{R}^2)$ solution can be expressed as the limit (in $C^0_{t,\mathrm{loc}} H^s_x$) of smooth solutions. The space $L^2_x(\mathbb{R}^2)$ is the critical space for this equation, as it is invariant under the scaling symmetry

$$u(t,x) \mapsto \frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \tag{2}$$

of (1).

Now we turn our attention to the global-in-time well-posedness problem. Based on the local well-posedness theory, standard limiting arguments, and the time reversal symmetry $u(t,x) \mapsto \overline{u(-t,x)}$, global well-posedness of (1) for arbitrarily large data[1] in $H^s_x(\mathbb{R}^2)$ for some $s > 0$ follows if an *a priori bound* of the form

$$\|u(T)\|_{H^s_x(\mathbb{R}^2)} \leq C(s, \|u_0\|_{H^s_x(\mathbb{R}^2)}, T) \tag{3}$$

can be established for all times $0 < T < \infty$ and all smooth-in-time, Schwartz-in-space solutions $u : [0,T] \times \mathbb{R}^2 \to \mathbb{C}$, where the right-hand side is some finite quantity depending only upon s, $\|u_0\|_{H^s_x(\mathbb{R}^2)}$, and T. Thus we shall henceforth restrict our attention to such smooth solutions, which will in particular allow us to justify all formal computations, such as verification of conservation laws.

As is well known, the equation (1) enjoys two useful conservation laws, the *energy conservation law*

$$E(u(t)) := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(t,x)|^4 \, dx = E(u_0). \tag{4}$$

and the *mass conservation law*

$$\|u(t)\|_{L^2_x(\mathbb{R}^2)} = \|u_0\|_{L^2_x(\mathbb{R}^2)}. \tag{5}$$

¿From these laws one easily establishes (3) for $s = 1$ (with bounds uniform in T). The mass conservation law (5) also gives (3) for $s = 0$, but unfortunately this does not immediately imply any result for $s > 0$ except in the small mass case.

It is conjectured that the equation (1) is globally well-posed in $H^s_x(\mathbb{R}^2)$ for all $s \geq 0$, and in particular (3) holds for all $s > 0$. This conjecture remains open (though in the radial case, this conjecture has recently been settled in [22] and [19]). However, there has been some progress in improving the $s \geq 1$ results mentioned earlier. The first breakthrough was by Bourgain [3], [4], who established (3) (and

[1] Global well-posedness and even scattering is known when the mass $\|u_0\|_{L^2_x(\mathbb{R}^2)}$ is sufficiently small (see e.g. [7], [21]), or if suitable decay conditions (e.g. $xu_0 \in L^2_x(\mathbb{R}^2)$ are also imposed on the initial data [23]). Our interest here however is in the large data case with no further decay conditions beyond the requirement that u_0 lies in $H^s_x(\mathbb{R}^2)$.

hence global well-posedness in $H^s_x(\mathbb{R}^2)$) for all $s > 3/5$, using what is now referred to as the *Fourier truncation method*.

In [14] the bound (3) was established for all $s > 4/7$, using the "I-method" developed by the authors in [11], [12] (see also [20]). The main result of this abstract is the following improvement:

Theorem 1 (Main theorem)**.** *The bound* (3) *holds for all* $s > 1/2$*. In particular, the Cauchy problem* (1) *is globally well-posed in* $H^s_x(\mathbb{R}^2)$ *for all* $s > 1/2$*.*

Our arguments refine our previous analysis in [14] by adding a "correction term" to a certain modified energy functional $E(Iu)$, as in [12] or [13], in order to damp out some oscillations in that functional; also, we establish some more refined estimates on the multilinear symbols appearing in those integrals. The main new difficulty is that, due to the multidimensional setting of this equation, the direct analogue of the correction terms used in [12], [13] contains a singular symbol and is thus intractable to estimate. We get around this new difficulty by truncating the correction term to non-resonant interactions, and dealing with the resonant interactions separately by some advanced estimates of $X^{s,b}$ type. This method seems quite general and should lead to improvements in global well-posedness results for other non-integrable evolution equations which are currently obtained by the "first-generation" I-method (i.e. without correction terms). A resonant decomposition similar to that employed here appeared previously in the work [5], and more recently in [1].

Fang and Grillakis [17] have obtained a stronger version of Theorem 1 - their result holds for $s \geq 1/2$, by a different method based upon a new type of Morawetz inequality. The Fang-Grillakis interaction Morawetz estimate has recently [10] been improved and combined with the I-method (following the general scheme from [15]) to prove that (1) is globally well-posed in H^s for $s > 2/5$. The techniques leading to the improved energy increment control obtained in the above theorem which is $N^{-1/2}$ better than what was obtained in [14] and used in [15], [10]) may also improve the "almost Morawetz" increment in [10] by $N^{-1/2}$. Such an improvement would improve the global well-posedness result to $s > 4/13$. The arguments in [17], [10] are based on Morawetz inequalities and are thus restricted to the defocusing case. Provided the mass of the initial data is less than the mass of the ground state, Theorem 1 also holds true, using [24], for the focusing analog of (1). The focusing problem is expected to be globally well-posed and scatter for L^2 initial data with mass less than the ground state mass.

One of the main Lemmas used to handle the resonant terms which arise (specifically, that part of the correction term which contains a singularity in the symbol - now truncated so that it is supported on a set where the various frequencies involved obey a certain orthogonality condition) is the following modification of the bilinear Strichartz estimate from [3].

Lemma 2 (Angularly refined bilinear Strichartz estimate)**.** *Let* $0 < N_1 \leq N_2$ *and* $0 < \theta \ll 1$*. Suppose that* v_1, v_2 *are solutions of the linear Schrödinger equation on* $\mathbb{R}^2$ *with spatial frequency supports* $|\xi| \sim N_1, N_2$ *respectively. Assume in addition* *that these supports also satisfy* $|\cos\angle(\xi_1, \xi_2)| \leq \theta$ *for any* $\xi_1 \in supp\,(\hat{v}_1(t,\xi)), \xi_2 \in supp\,(\hat{v}_2(t,\xi))$.

We conclude,

$$\|v_1 \cdot v_2\|_{L^2_{t,x}} \lesssim \theta^{1/2} \|v_1(0,\cdot)\|_{L^2(\mathbb{R}^2)} \|v_2(0,\cdot)\|_{L^2(\mathbb{R}^2)}. \tag{6}$$

References

[1] D. Bambusi, J.-M. Delort, B. Grebert, J. Szeftel, *Almost global existence for Hamiltonian semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, preprint, 2007. (`arXiv:math.DS/0510292`)

[2] B. Birnir, C. Kenig, G. Ponce, N. Svanstedt, L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*, J. London Math. Soc. **53**:3 (1996), 551–559.

[3] J. Bourgain, *Refinement of Strichartz inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices, **5** (1998), 253–283.

[4] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, AMS Colloquium Publications, **46** (1999).

[5] J. Bourgain, *A remark on normal forms and the "I-method" for periodic NLS*, J. Anal. Math. **94** (2004), 125–157.

[6] J. Bourgain, *On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE*, Internat. Math. Res. Notices **1996**, no. 6, 277–304.

[7] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, **10**. New York University, Courant Institute of Mathematical Sciences, AMS, 2003.

[8] T. Cazenave, F.B. Weissler, *The Cauchy problem for the critical nonlinear Schrodinger equation*, Non. Anal. TMA, **14** (1990), 807-836.

[9] M. Christ, J. Colliander, T. Tao, *Ill-posedness for nonlinear Schrodinger and wave equations*, preprint, 2003. (`arXiv:math.AP/0311048`)

[10] J. Colliander, M. Grillakis, N. Tzirakis, *Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on* $\mathbb{R}^2$, preprint, 2007. (`arXiv:math.AP/0703606`)

[11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global well-posedness for KdV in Sobolev spaces of negative index*, Electron. J. Diff. Eq. **2001** (2001) No 26, 1–7.

[12] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp Global Well-Posedness of KdV and Modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc., **16** (2003), 705–749.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T.Tao, *A refined global well-posedness result for Schrödinger equations with derivative*, SIAM J. Math. Anal., **34** (2002), 64–86.

[14] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Almost conservation laws and global rough solutions to a Nonlinear Schrödinger Equation*, Math. Res. Letters, **9**, (2002), 1 - 24.

[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka. T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on* $\mathbb{R}^3$, Comm. Pure Appl. Math. **57**:8, (2004), 987–1014.

[16] J. Ginibre, G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **144**:1 (1992), 163–188.

[17] M. Grillakis, Y. Fang, *On the global existence of rough solutions of the cubic defocusing Schrödinger equation in* $\mathbb{R}^{2+1}$, to appear J.H.D.E., 2007.

[18] T. Kato,*On nonlinear Schrödinger equations. II.* H^s*-solutions and unconditional well-posedness*, J. Anal. Math. **67** (1995), 281–306.

[19] R. Killip, T. Tao, M. Visan, *The cubic nonlinear SchrÃ¶dinger equation in two dimensions with radial data*, preprint, 2007. `arXiv:0707.3188`

[20] M. Keel, T. Tao, *Local and global well-posedness of wave maps on* $\mathbb{R}^{1+1}$ *for rough data*, Internat. Math. Res. Not. **21** (1998), 1117–1156.

[21] T. Tao, Nonlinear dispersive equations: local and global analysis, CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.
[22] T. Tao, M. Visan, X. Zhang, *Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation for radial data in higher dimensions*, preprint, 2006. (arXiv:math.AP/0609692)
[23] Y. Tsutsumi, *Scattering problem for nonlinear Schrodinger equations*, Ann. Inst. H. Poincare, Phys. Theor. **43** (1985), 321–347.
[24] M.I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Commun. Math. Phys. **87** (1983), 567–576.

Asymptotic stability for the Kadomtsev-Petviashili II equation

Herbert Koch
(joint work with Martin Hadac, Sebastian Herr)

In [3] we study the Kadomtsev-Petviashvili-II (KP-II) equation

$$\begin{aligned} &\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) + \partial_y^2 u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^2 \\ &u(0,x,y) = u_0(x,y) \quad (x,y)\in\mathbb{R}^2 \end{aligned} \tag{1}$$

which has been introduced by B.B. Kadomtsev and V.I. Petviashvili to describe weakly transverse water waves in the long wave regime with small surface tension. It generalizes the Korteweg - de Vries equation, which is spatially one dimensional and thus neglects transversal effects. The KP-II equation has a remarkably rich structure. Let us begin with its symmetries and assume that u is a solution of (1).

- *Translation:* Translates of u in x, y and t are solutions.
- *Scaling:* If $\lambda > 0$ then also

$$u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y) \tag{2}$$

  is a solution.
- *Galilean invariance:* For all $c \in \mathbb{R}$ the function

$$u_c(t,x,y) = u(t, x - cy - c^2 t, y + 2ct) \tag{3}$$

  satisfies equation (1).

The KP-II equation is integrable in the sense that there exists a Lax pair. Formally, there exists an infinite sequence of conserved quantities [7], the two most important beeing the L^2 norm and the energy. The conserved quantities besides the L^2 norm seem to be useless for proofs of well-posedness, because of the difficulty to define ∂_x^{-1} and because the quadratic term is indefinite. The line solitons are among the simplest solutions. An analysis of the spectrum of the linearization and inverse scattering indicate that the line soliton is stable [5, 6]. A satisfactory nonlinear stability result for the line soliton is an outstanding problem.

I report on a modest step towards this challenging question: Well-posedness and scattering in a critical space.

We study the Cauchy problem (1) for initial data u_0 in the non-isotropic Sobolev space $H^{-\frac{1}{2},0}(\mathbb{R}^2)$ and in the homogeneous variant $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$, respectively, which are defined as spaces of distributions with $-\frac{1}{2}$ generalized x-derivatives in $L^2(\mathbb{R}^2)$.

The well-posedness of (1) has been thoroughly studied in the last two decades. After a first well-posedness result by S. Ukai [10] in more regular spaces, J. Bourgain established global well-posedness in $L^2(\mathbb{T}^2;\mathbb{R})$ and $L^2(\mathbb{R}^2;\mathbb{R})$ in his seminal paper [1] by combining the Fourier restriction norm method with the L^2 conservation law. N. Tzvetkov [9] improved the local theory within the scale of non-isotropic Sobolev spaces. Local well-posedness in the full sub-critical range $s > -\frac{1}{2}$ was obtained by H. Takaoka [8] in the homogeneous spaces and by the first author [2] in the inhomogeneous spaces. Global well-posedness for large, real valued data in $H^{s,0}(\mathbb{R}^2)$ has been pushed down to $s > -\frac{1}{14}$ by P. Isaza - J. Mejía [4].

The first main result is concerned with small data global well-posedness in $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$. For $\delta > 0$ we define

$$\dot{B}_\delta := \{u_0 \in \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2) \mid \|u_0\|_{\dot{H}^{-\frac{1}{2},0}} < \delta\},$$

and obtain the following:

Theorem 1. *There exists $\delta > 0$, such that for all initial data $u_0 \in \dot{B}_\delta$ there exists a solution*

$$u \in \dot{Z}^{-\frac{1}{2}}([0,\infty)) \subset C([0,\infty); \dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2))$$

of the KP-II equation (1) on $(0,\infty)$. If for some $T > 0$ a solution $v \in Z^{-\frac{1}{2}}([0,T])$ on $(0,T)$ satisfies $v(0) = u(0)$, then $v = u|_{[0,T]}$. Moreover, the flow map

$$F_+ : \dot{B}_\delta \to \dot{Z}^{-\frac{1}{2}}([0,\infty)),\ u_0 \mapsto u$$

is analytic.

The definition of the spaces $\dot{Z}^{-\frac{1}{2}}(I)$ and $Z^{-\frac{1}{2}}(I)$ is of central importance. A consequence of Theorem 1 is scattering for small data in $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ and hence asymptotic stability for small data.

The proof relies on bilinear estimates, a strategy, which are by now standard in the context of the $X^{s,b}$ spaces of Bourgain. To access critical problems by $X^{s,b}$ spaces one is forced to use $b = \pm 1/2$. But then crucial embeddings fail. A remedy is the use of a Besov type modification $X^{s,1/2,1}$ resp. $X^{s,-1/2,\infty}$. In our case this seems to be insufficient and we use functions spaces V^p and U^p with

$$X^{0,1/2,1} \subset U^2 \subset V^2 \subset X^{0,1/2,\infty},$$

which are based on entirely different ideas.

Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_K = \infty$ and let $\mathcal{Z}_0$ be the set of finite partitions $-\infty < t_0 < t_1 < \ldots < t_K < \infty$. In the following, we consider functions taking values in $L^2 := L^2(\mathbb{R}^d;\mathbb{C})$, but in the general part of this section L^2 may be replaced by an arbitrary Hilbert space. The following spaces were introduced by N. Wiener [11].

Definition 2. Let $1 \leq p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L^2$ such that $v(\infty) := \lim_{t\to\infty} v(t) = 0$ and $v(-\infty)$ exists and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^{K} \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} \tag{4}$$

is finite. Likewise, let V_-^p denote the normed space of all functions $v : \mathbb{R} \to L^2$ such that $v(-\infty) = 0$, $v(\infty)$ exists, and $\|v\|_{V^p} < \infty$, endowed with the norm (4).

Let $S(t)$ be the unitary group defined by the linear equation,

$$\|u\|_{V_S^p} = \|S(-t)u(t,.)\|_{V^p}$$

and V_S^p the space of right continuous functions for which this norm is finite. Then V_S^2 is a suitable replacement for $X^{0,1/2}$.

References

[1] J. Bourgain. On the Cauchy problem for the Kadomtsev-Petviashvili equation. *Geom. Funct. Anal.*, 3(4):315–341, 1993.

[2] M. Hadac. Well-posedness for the Kadomtsev-Petviashvili II equation and generalisations. *Trans. Amer. Math. Soc., to appear*, 2006.

[3] M. Hadac, S. Herr, and H. Koch. Well-posedness and scattering for the KP-II equation in a critical space. *Annales de l'Institut Henri Poincare / Analyse non lineaire*, 2008.

[4] P. Isaza J. and J. Mejía. Global solution for the Kadomtsev-Petviashvili equation (KPII) in anisotropic Sobolev spaces of negative indices. *Electron. J. Differential Equations*, pages No. 68, 12 pp. (electronic), 2003.

[5] B. B. Kadomtsev and V. I. Petviashvili. On the stability of solitary waves in weakly dispersing media. *Sov. Phys., Dokl.*, 15:539–541, 1970.

[6] F. Rousset and N. Tzvetkov. Transverse nonlinear instability for two-dimensional dispersive models, 2007.

[7] E. I. Schul'man and V. E. Zakharov. Degenerative dispersion laws, motion invariants and kinetic equations. *Phys. D*, 1(2):192–202, 1980.

[8] H. Takaoka. Well-posedness for the Kadomtsev-Petviashvili II equation. *Adv. Differential Equations*, 5(10-12):1421–1443, 2000.

[9] N. Tzvetkov. On the Cauchy problem for Kadomtsev-Petviashvili equation. *Comm. Partial Differential Equations*, 24(7-8):1367–1397, 1999.

[10] S. Ukai. Local solutions of the Kadomtsev-Petviashvili equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(2):193–209, 1989.

[11] N. Wiener. The quadratic variation of a function and its Fourier coefficients. In Pesi Rustom Masani, editor, *Collected works with commentaries. Volume II: Generalized harmonic analysis and Tauberian theory; classical harmonic and complex analysis*, volume 15 of *Mathematicians of Our Time.* Cambridge, Mass. - London: The MIT Press. XIII, 969 p. , 1979 (1924).

A construction of weak solutions of a biharmonic map heat flow

Roger Moser

For $n \geq 4$, let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary. We consider a compact, smooth Riemannian manifold N that is embedded isometrically in a Euclidean space $\mathbb{R}^m$. A sufficiently smooth map $u : \Omega \to N$ induces a pull-back vector bundle $u^{-1}TN$ with a covariant derivative ∇^u coming from the Levi-Civita connection on N. The functional

$$E_2(u) = \frac{1}{2} \int_\Omega |\nabla^u du|^2 \, dx$$

can be thought of as a second order counterpart to the Dirichlet energy. Therefore, the L^2-gradient flow for E_2 is a higher order analogue of the harmonic map heat flow

$$\frac{\partial u}{\partial t} = \tau(u) := \operatorname{trace} \nabla^u du. \tag{1}$$

It gives rise to a fourth order parabolic problem given by

$$\frac{\partial u}{\partial t} + \nabla^u_\alpha \Delta^u \frac{\partial u}{\partial x^\alpha} + R(u) \left(\nabla^u_\alpha \frac{\partial u}{\partial x^\beta}, \frac{\partial u}{\partial x^\alpha} \right) \frac{\partial u}{\partial x^\beta} = 0,$$

where we use a standard summation convention, $\Delta^u = \operatorname{trace}(\nabla^u)^2$ is the Laplacian belonging to ∇^u, and R denotes the Riemann curvature tensor on N. The equation can also be written in the form

$$\frac{\partial u}{\partial t} + \Delta^2 u + \operatorname{div} a(u, du, \nabla^u du) + b(u, du, \nabla^u du) = 0, \tag{2}$$

where a and b are smooth functions that satisfy

$$|a(y, \xi, \zeta)| \leq C_0 |\xi| (|\xi|^2 + |\zeta|)$$

and

$$|b(y, \xi, \zeta)| \leq C_0 |\zeta| (|\xi|^2 + |\zeta|)$$

for a constant C_0 that depends only on n and N.

In the form (2), the equation has a weak interpretation if u belongs to the space $L^\infty((0, \infty), H^1(\Omega, N))$, where

$$H^1(\Omega, N) = \left\{ v \in H^1(\Omega, \mathbb{R}^m) : \; v(x) \in N \text{ for almost every } x \in \Omega \right\},$$

and if in addition the second derivative $\nabla^2 u$ exists in the weak sense with $|\nabla^2 u| \in L^2_{\mathrm{loc}}((0, \infty) \times \Omega)$. We consider initial and boundary conditions of the form

$$u(t, x) = u_0(x) \quad \text{for } t = 0 \text{ or } x \in \partial\Omega, \tag{3}$$

$$du(t, x) = du_0(x) \quad \text{for } x \in \partial\Omega, \tag{4}$$

for a given map $u_0 \in H^2(\Omega, N)$. The conditions (3) and (4) can be understood in the sense of traces, provided that $u \in L^\infty((0, \infty), H^1(\Omega, N))$ with $|\frac{\partial u}{\partial t}| \in L^2((0, \infty) \times \Omega)$ and $|\nabla^u du(t, \cdot)| \in L^2(\Omega)$ for almost every $t \in (0, \infty)$.

Weak solutions of the harmonic map heat flow (1) have first been constructed by Chen and Struwe [1] for all dimensions n, and their arguments also give partial regularity of the solutions. But despite some formal similarities of the problems, these methods seem inappropriate for the L^2-gradient flow of E_2. On the other hand, there exists a different approach to the harmonic map heat flow, due to Haga, Hoshino, and Kikuchi [2], which is more suitable for the higher order equation. It is based on a time discretization method, and its implementation for E_2 involves the construction of a sequence of maps $u_0^h, u_1^h, \dots$ for a fixed $h > 0$ as follows. The map u_0^h coincides with the initial map u_0, and $u_1^h, u_2^h, \dots$ are chosen recursively such that u_{k+1}^h minimizes the functional

$$E_2(u) + \frac{1}{2h} \int_\Omega |u - u_k^h|^2 \, dx$$

for $k = 1, 2, \dots$ under the boundary conditions $u_{k+1}^h = u_0$ and $du_{k+1}^h = du_0$ on $\partial\Omega$. Formally, these minimization problems give rise to the equation

$$\frac{u_{k+1}^h - u_k^h}{h} + \Delta^2 u_{k+1}^h + \operatorname{div} a\left(u_{k+1}^h, du_{k+1}^h, \nabla^{u_{k+1}^h} du_{k+1}^h\right) \\ + b\left(u_{k+1}^h, du_{k+1}^h, \nabla^{u_{k+1}^h} du_{k+1}^h\right) = 0.$$

This is a time discretized version of the biharmonic map heat flow. For $h \searrow 0$, one hopes to obtain a limit that solves equation (2).

In order to carry out such a scheme successfully, we obviously have to be able to minimize a functional as above. The minimization problem for E_2 has been studied in a recent paper [3], and the additional term is not difficult to handle in this context. The same paper provides some tools to study the regularity of the minimizers, most importantly a monotonicity formula that permits estimates of $|\nabla^u du|$ in the appropriate Morrey spaces. For the functional that we study here, however, we have to assume that $n \leq 8$ in order to use these arguments. Using also an idea of Scheven [5], we obtain some initial regularity results, and other arguments of Wang [6, 7, 8] give higher regularity. We can then prove a uniform estimate for the H^4-norm of the minimizers under small energy assumptions. This is crucial when we pass to the limit $h \searrow 0$, because such an estimate implies that there is a limit map (for a certain subsequence) that solves (2). With this method, we can prove the following result.

Theorem 1. *Let $n \leq 8$ and suppose that $u_0 \in H^2(\Omega, N)$. Then there exists a map $u \in L^\infty((0,\infty), H^1(\Omega, N))$ with $|\nabla^2 u| \in L^2_{\mathrm{loc}}((0,\infty) \times \Omega)$, $|\frac{\partial u}{\partial t}| \in L^2((0,\infty) \times \Omega)$, and $|\nabla^u du| \in L^\infty((0,\infty), L^2(\Omega))$, such that (2) holds weakly and (3) and (4) are satisfied in the sense of traces.*

The details of the proof are given in another paper [4].

References

[1] Y. M. Chen and M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201** (1989), 83–103.

[2] J. Haga, K. Hoshino, and N. Kikuchi, *Construction of harmonic map flows through the method of discrete Morse flows*, Comput. Vis. Sci. **7** (2004), 53–59.

[3] R. Moser, *A variational problem pertaining to biharmonic maps*, to appear in Comm. Partial Differential Equations.
[4] R. Moser, *Weak solutions of a biharmonic map heat flow*, preprint, 2008.
[5] C. Scheven, *An optimal partial regularity result for minimizers of an intrinsically defined second-order functional*, preprint, 2008.
[6] C. Wang, *Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold*, Math. Z. **247** (2004), 65–87.
[7] C. Wang, *Remarks on biharmonic maps into spheres*, Calc. Var. Partial Differential Equations **21** (2004), 221–242.
[8] C. Wang, *Stationary biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold*, Comm. Pure Appl. Math. **57** (2004), 419–444.

Monotone volume formulas for geometric flows

Reto Müller

Perelman's result from [6] that the Ricci flow $\partial_t\, g_{ij} = -2R_{ij}$ may be interpreted up to a pull-back with a family of diffeomorphisms as the gradient flow of the $\mathcal{F}$-energy

$$\mathcal{F}(g,f) = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV$$

was successfully adopted to related flows, e.g. certain renormalization group flows of worldsheet nonlinear sigma models arising in quantum field theory, cf. [5], or List's extended Ricci flow system [1] which is motivated by a problem from general relativity. Moreover, it also led to the creation of new geometric flows like the Ricci Yang-Mills flow introduced by Streets in [7] or the Ricci flow coupled with harmonic map heat flow

$$\partial_t\, g_{ij} = -2R_{ij} + 2\alpha \nabla_i \phi^\kappa \nabla_j \phi^\kappa, \qquad \partial_t\, \phi = \tau_g \phi, \tag{1}$$

which we introduce in [3]. Here, $\phi : M \to N \hookrightarrow \mathbb{R}^k$ is a map between closed manifolds (M,g) and (N,γ) with tension field $\tau_g\phi$ and $\alpha = \alpha(t)$ is a positive and non-increasing coupling function. This is the gradient flow of

$$\mathcal{F}_\alpha(g,\phi,f) = \int_M \left(R + |\nabla f|^2 - \alpha\, |\nabla \phi|^2\right) e^{-f} dV.$$

Note that setting $\alpha \equiv 2$ and $N = \mathbb{R}$, this flow reduces to List's system from [1]. One motivation to study (1) is the fact that for large α we can bound the energy density of ϕ along the flow without any restriction on the curvature of the target manifold N.

A natural question is to what extent the other monotone functionals of Perelman can be adopted to these geometric flows, or more generally to any flow of the form

$$\partial_t\, g_{ij}(t) = -2S_{ij}(t), \tag{2}$$

for a symmetric tensor S_{ij}. Of particular interest to us was the monotonicity of his reduced volume, since it implies useful non-collapsing results for the flow – a crucial step in Perelman's proof of the Poincaré conjecture. Given any symmetric tensor S_{ij} and a point $p \in M$, we define the (forwards) reduced distance between $(p, 0)$ and (q, t_1) by

$$\ell(q,t_1) := \inf_{\gamma\in\Gamma} \left\{ \frac{1}{2\sqrt{t_1}} \int_0^{t_1} \sqrt{t}\left(S + |\partial_t \gamma|^2\right) dt \right\},$$

where $\Gamma = \{\gamma : [0, t_1] \to M \mid \gamma(0) = p, \gamma(t_1) = q\}$ and $S = g^{ij} S_{ij}$ is the trace of S_{ij}. The (forwards) reduced volume is then given by

$$V(t) := \int_M (4\pi t)^{-n/2} e^{\ell(q,t)} dV(q).$$

While studying Perelman's work in [4], we already noticed that for a static manifold, i.e. $S_{ij} = 0$, the monotonicity of $V(t)$ only holds under the additional assumption that the Ricci curvature of M is nonnegative. After solving the corresponding problem for List's flow in joint work with Valentina Vulcanov, we found the following general condition.

Theorem 1 ([2]). *Along the flow (2), if the symmetric tensor S_{ij} satisfies*

(3) $\partial_t S - \triangle S - 2\,|S_{ij}|^2 + 4(\nabla_i S_{ij})X_j - 2(\nabla_j S)X_j + 2R_{ij}X_iX_j - 2S_{ij}X_iX_j \geq 0,$

then the (forwards) reduced volume $V(t)$ is non-increasing.

A similar statement holds true for a backwards reduced volume quantity which is non-decreasing along the flow. Condition (3) is obviously satisfied for the Ricci flow or for a static manifold with nonnegative Ricci curvature. Moreover, it also holds for the Ricci flow coupled with harmonic map flow – and thus also for List's flow. A further example, pointed out by Mu-Tao Wang, is the mean curvature flow of spacelike hypersurfaces in a Lorentzian manifold, where (3) yields a curvature condition on the ambient Lorentzian manifold.

Towards the proof: The quantity on the left hand side of (3) can be written as the difference of two Harnack type quantities for the flow (2), which also appear in the first and second variation formulas for the reduced length functional. The proof then proceeds similar to Perelman's proof in the Ricci flow case by comparing with a carefully chosen variation. It is then easy to see that at the points where $\ell(q, t)$ is smooth, the integrand $v(q, t) = (4\pi t)^{-n/2} e^{\ell(q,t)}$ of the reduced volume is a subsolution to the adjoint heat equation under the flow (2). By constructing barriers at the null set of points in space-time where $\ell(q, t)$ fails to be smooth, we then conclude that the inequality for $v(q, t)$ still holds in the distributional sense, which is good enough for the monotonicity of $V(t)$.

References

[1] B. List. *Evolution of an extended Ricci flow system.* PhD thesis, AEI Potsdam, 2005.
[2] R. Müller. Monotone volume formulas for geometric flows. preprint.
[3] R. Müller. The Ricci flow coupled with harmonic map heat flow. preprint.
[4] R. Müller. *Differential Harnack inequalities and the Ricci flow.* Series of Lectures in Mathematics. EMS, 2006.

[5] T. Oliynyk, V. Suneeta, and E. Woolgar. A gradient flow for worldsheet nonlinear sigma models. *Nuclear Physics*, B739:441–458, 2006.
[6] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *ArXiv:math/0211159*, 2002.
[7] J. Streets. *Ricci Yang-Mills flow.* PhD thesis, Duke University, 2007

Translating solutions of Lagrangian mean curvature flow

André Neves

1. Introduction

The idea of Lagrangian mean curvature flow is to deform a given initial Lagrangian into a minimal Lagrangian (also called Special Lagrangian) using mean curvature flow. One case where this procedure has been successful is when we restrict ourselves to simple curves on the plane. It is a theorem of Grayson that if we apply curve shortening flow to an embedded noncompact curve, then the flow will exist for all time and if it converges, it will converge to a straight line.

Optimistically, one could expect that some sort of long-time existence result should also hold for Lagrangian mean curvature flow in $\mathbb{C}^2$. Unfortunately, it is known from [1] that there are "very good" initial conditions for which the flow, nonetheless, develops a finite time singularity.

Thus, if we want to use the flow to produce Special Lagrangians, we need to be able to understand how singularities for. Before we proceed, I need to introduce some definitions.

Let J and ω denote, respectively, the standard complex structure on $\mathbb{C}^2$ and the standard symplectic form on $\mathbb{C}^2$. We consider also the closed complex-valued 2-form given by

$$\Omega \equiv dz_1 \wedge dz_2$$

where $z_j = x_j + iy_j$ are complex coordinates of $\mathbb{C}^2$, and the Liouville form

$$\lambda = \sum_{j=1}^{2} x_j dy_j - y_j dx_j.$$

IA smooth 2-dimensional submanifold L in $\mathbb{C}^2$ is said to be *Lagrangian* if $\omega_L = 0$ and this implies that

$$\Omega_L = e^{i\theta}\mathrm{vol}_L,$$

where vol_L denotes the volume form of L and θ is a multivalued function called the *Lagrangian angle*. When the Lagrangian angle is a single valued function the Lagrangian is called *zero-Maslov class* and if

$$\cos\theta \geq \varepsilon_0$$

for some positive ε_0, then L is said to be *almost-calibrated*. The Lagrangian L is said to be *exact* if the Liouville form is an exact form on L. Finally, the relation between the Lagrangian angle and the mean curvature is given by

$$H = J\nabla\theta.$$

The expectation is that if L is an almost-calibrated Lagrangian in $\mathbb{C}^2$, then the singularities are isolated for the flow. The almost calibrated condition is necessary because otherwise there are know counterexamples (see [2]).

The way to approach this problem is to understand rescales of finite time singularities. Assume that $(L_t)_{0<t<T}$ is a solution to Lagrangian mean curvature flow that becomes singular at x_0 at time T. Consider sequences $(\lambda_i)_{i\in\mathbb{N}}$, $(t_i)_{i\in\mathbb{N}}$, and $(x_i)_{i\in\mathbb{N}}$ that converge to infinity, T, and x_0 respectively. Set

$$L^i_s := \lambda_i(L_{t_i+s/\lambda_i^2} - x_i),$$

which is still a solution to Lagrangian mean curvature flow. If $t_i = T$ and $x_i = x_0$ for all $i \in \mathbb{N}$, then the optimal compactness theorem regarding the sequence L^i_s was proven in [1]. If we allow the points (x_i, t_i) move in space-time, then we can always carefully chose them so that L^i_s converges smoothly to an eternal solutions $(L_t)_{-\infty<t<\infty}$ of Lagrangian mean curvature flow that is almost calibrated.

If singularities are indeed isolated, the expectation is that $(L_t)_{-\infty<t<\infty}$ will have zero mean curvature. We remark that without the almost calibrated condition one can construct examples of finite time singularities for which a sequence os rescales converges to an eternal solution which is not minimal. For this reason it is important to study eternal solutions to Lagrangian mean curvature flow in their own right.

Next I describe all the know examples of eternal solutions in to Lagrangian mean curvature flow in $\mathbb{C}^2$. First is the case where L_0 is a Special Lagrangian. Second is the case where $(\gamma_t)_{-\infty<t<\infty}$ denotes the grim reaper in $\mathbb{C}$ and

$$L_t := \gamma_t \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}.$$

The final set of examples was discovered by Dominic Joyce, Yng-ing Lee, and Mao-Pei Tsui. They are described as follows. Let w be a curve in $\mathbb{C}$ such that

$$w_t := \sqrt{2t}w \quad \text{for} \quad t > 0$$

is a solution to curve shortening flow in $\mathbb{C}$. This curve can be chosen in a way that the angle θ that the tangent vector makes with the x-axis has arbitrarily small oscillation. Set

$$L := \left\{ \left(\frac{|w|^2(y) - x^2}{2} - i\theta(y), xw(y) \right), |\, x, y \in \mathbb{R} \right\} \subset \mathbb{C} \times \mathbb{C}. \tag{1}$$

Using the fact that the curvature of w satisfies

$$\vec{k} = w^{\perp},$$

it is a straightforward computation to check that L is Lagrangian and that

$$L_t = L + t(1, 0, 0, 0)$$

is a solution to Lagrangian mean curvature flow. Moreover, the Lagrangian angle of L coincides with θ and hence its oscillation can be made arbitrarily small.

In this talk I presented two theorems (joint work with Tian) that look at the structure of eternal solutions. In order to state the first theorem we need one more definition. Given an eternal solution $(L_t)_{-\infty<t<\infty}$ and a sequence $(\lambda_i)_{i\in\mathbb{N}}$ converging to zero, we define the sequence of blow-downs to be

$$L^i_s := \lambda_i L_{s/\lambda_i^2}.$$

Theorem 1. *If $(L_t)_{-\infty<t<\infty}$ is an eternal solutions which is almost calibrated and exact for all t, then any sequence of blow-downs converges, after passing to a subsequence, to weak solution L^∞_s which is a union of planes with multiplicities for all $s \leq 0$ and a self expander $\sqrt{s}L^\infty_1$ for all $s > 0$.*

The almost calibrated condition is necessary because otherwise the grim reaper is a counterexample. We should point out that the fact that we can say something for L^∞_s when s is positive is a very unique property of Lagrangian mean curvature flow.

The second theorem gives conditions that assure when a translating solution is trivial, i.e., a plane.

Theorem 2. *Let $(L_t)_{-\infty<t<\infty}$ be a translating solution to Lagrangian mean curvature flow that satisfies*

i) *L_0 is almost calibrated.*
ii) *There is a sequence of blow-downs that converges to a union of planes for* all *s.*

Then L_0 is a plane.

Condition i) is important to exclude the grim reaper. Condition ii) is important to exclude the non-trivial solutions found by Joyce, Lee and Tsui.

References

[1] A. Neves, Singularities of Lagrangian Mean Curvature Flow: Zero-Maslov class case. **Invent. Math. 168** (2007), 449–484.
[2] A. Neves, Translating solutions to Lagrangian mean curvature flow, preprint.

Invariant curvature cones and the Ricci flow

Huy Nguyen

My research centres on the construction of invariant curvature cones and the Ricci flow. In my thesis and together in a paper with my supervisor, Dr. Ben Andrews, we develop a new technique to construct sets of curvature operators that are preserved by the Ricci flow. This technique is based on the maximum principle for geometric evolution equations. The idea is as follows, we consider a set of curvature operators defined by an inequality of a curvature function of the orthonormal frame bundle, $F(R_{iijk}) \geq 0.$. Examples of such functions are linear combinations of sectional curvature. To show that such sets are preserved, by the advanced maximum principle for tensors, it suffices to show that the ODE associated to the nonlinearity of the Ricci flow,

$$\frac{d}{dt}F(R) = F(R)^2 + F(R)^\#,$$

preserves the set.We note here that the nonlinearity is quadratic in the curvature. furthermore, to show that curvature cone is preserved, we need only show that the ODE preserves the set at the boundary, that is where $F(R_{ijkl}) = 0$. However,F is a function of the orthonormal frame bundle, and as it takes a minimum at boundary, we may differentiate the equation with respect to derivatives in $O(n)$. Consequently, the first order derivatives are zero and the matrix of second order derivatives is non-negative. Using the differential equality, we simplify the curvature evolution equation. To show that the evolution equations preserves the curvature cone, it remains to use the matrix of second derivatives and control the remaining terms in the nonlinearity. This part of the proof has additional subtleties, the matrix of second derivatives has entries whose terms are linear in curvature, whereas the nonlinearity is quadratic. Using generalized determinants we are able to overcome this problem. We carry out the computation in two cases, that of positive isotropic curvature in dimensions $n \geq 4$ and for quarter pinched flag curvature for $n = 4$ and prove the following two theorems. Firstly we will need a definition,

Definition 1 (Non-negative Isotropic Curvature (PIC))**.** Let $(\mathcal{M}, g_{ij})$ be a Riemannian manifold, then $(\mathcal{M}, g_{ij})$ has non-negative isotropic curvature if for any set of four orthonormal vectors, $\{e_1, \ldots, e_4\} \subset T_x\mathcal{M}^n$, we have

$$R_{1313} + R_{2424} + R_{1414} + R_{2323} \geq \pm R_{1234}.$$

The curvature condition above was first introduced in [MM88], where it was used to study the space of minimal two spheres in a manifold using harmonic maps. Then we have to following theorem,

Theorem 2 ([Ngu07])**.** *Let $(M, g_{ij}(t))$ be a solution to Ricci flow equation such that the initial metric has nonnegative isotropic curvature. Then $g_{ij}(t)$ has non-negative isotropic curvature.*

This was first shown in by Hamilton in dimension four [Ham97], where a partial classification of manifold with PIC was proven . We note that the second theorem was also proved by Brendle and Schoen,[BS07].

Theorem 3 ([AN07])**.** *Let M be a compact four-manifold, and g_0 a Riemannian metric on M which has λ-pinched flag curvatures, with $\lambda > 1/4$. Then M is diffeomorphic to a space form.*

The condition quarter-pinched flag curvature is explained as follows, let (M, g) be a compact Riemannian 4-manifold, with curvature tensor R. We suppose that M has positive sectional curvatures and that for every $x \in M$ and every orthonormal basis $\{e_1, \ldots, e_4\}$ for T_xM, we have

$$R(e_2, e_1, e_2, e_1) \geq \lambda R(e_3, e_1, e_3, e_1). \tag{1}$$

To put this in a more geometric way, for each e_1 in T_xM there is an associated bilinear form R_{e_1} on the orthogonal subspace, the flag curvature in direction e_1, defined by $R_{e_1}(v, v) = R(e_1, v, e_1, v)$. The condition (1) says precisely that the ratio of any two eigenvalues of R_{e_1} is bounded below by λ. That is, each of the flag curvatures of M is λ-pinched.

References

[AN07] Ben Andrews and Huy Nguyen, *Four-manifolds with* 1/4*-pinched flag curvatures*, Preprint (2007).

[BS07] S. Brendle and R. M. Schoen, *Manifolds with 1/4-pinched curvature are space forms*, arXiv (2007).

[Ham97] Richard S. Hamilton, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), no. 1, 1–92. MR MR1456308 (99e:53049)

[MM88] Mario J. Micallef and John Douglas Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. of Math. (2) **127** (1988), no. 1, 199–227. MR MR924677 (89e:53088)

[Ngu07] H. Nguyen, *Isotropic curvature and the ricci flow*, Preprint (2007).

Regularity of solutions to the Navier-Stokes equations evolving from small initial data in a critical space

Nataša Pavlović

(joint work with Pierre Germain and Gigliola Staffilani)

1. Introduction

In this note we present an overview of our results [2] concerning regularity, decay and analyticity of solutions to the Navier-Stokes equations in $\mathbb{R}^d$.

The Navier-Stokes equations for the incompressible fluid in $\mathbb{R}^d$ are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \Delta u + f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

and the initial condition

$$u(x, 0) = u_0(x), \tag{3}$$

for the unknown velocity vector field $u = u(x,t) \in \mathbb{R}^d$ and the pressure $p = p(x,t) \in \mathbb{R}$, where $x \in \mathbb{R}^d$ and $t \in [0, \infty)$.

Existence of global in time solutions to (1)-(3) when $d = 3$, their uniqueness and regularity are long standing open problems of fluid dynamics. One approach in addressing these problems is to construct solutions to the corresponding integral equation via a fixed point theorem, so called "mild" solutions. However the existence of mild solutions to the Navier-Stokes equations (1) - (3) in $\mathbb{R}^d$ for $d \geq 3$ has been established only locally in time and globally for small initial data. Before we address the types of initial data for which the existence of solutions has been established, we recall the scaling invariance of the Navier-Stokes equations. If the pair $(u(x,t), p(x,t))$ solves (1) in $\mathbb{R}^d$ then $(u_\lambda(x,t), p_\lambda(x,t))$ with

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

is a solution to the system (1) for the initial data $\lambda u_0(\lambda x)$. The spaces which are invariant under such a scaling are called critical spaces for the Navier-Stokes equations. Examples of critical spaces for the Navier-Stokes in $\mathbb{R}^d$ are:

$$\dot{H}^{\frac{d}{2}-1} \hookrightarrow L^d \hookrightarrow \dot{B}^{-1+\frac{d}{p}}_{p|p<\infty,\infty} \hookrightarrow BMO^{-1}. \tag{4}$$

Kato [4] initiated the study of the Navier-Stokes equations in critical spaces, which was then continued by many authors (see [2] for references). In 2001 Koch and Tataru [5] established the existence of global solutions to (1) - (3) in $\mathbb{R}^d$ corresponding to initial data small enough in BMO^{-1}. The space BMO^{-1} has a special role since it is the largest critical space among the spaces listed in (4) where such existence results are available.

Motivated by the work [5] of Koch and Tataru, in [2] we analyze regularity properties of the solution constructed in [5] and show that under certain smallness condition of the initial data in BMO^{-1}, the solution u to the Navier-Stokes equations (1) - (3) satisfies the following regularity property:

$$t^{\frac{k}{2}}\nabla^k u \in X^0, \text{ for all } k \in \mathbb{N}\cup\{0\}\,, \tag{5}$$

where X^0 denotes the space where the solution constructed by Koch and Tataru belongs (for a precise definition of X^0, see Section 2). As a corollary we obtain:

(a) A decay estimate in time for any space derivative.
(b) Space analiticity of the solution.
(c) A regularity result for self-similar solutions.

Similar regularity properties of solutions to the Navier-Stokes equations in the Lebesgue space L^d were obtained in [3] and [1], and in the homogeneous Sobolev space $\dot{H}^{d/2-1}$ in [8].

2. Statements of the results

2.1. Preliminaries. First, let us recall the definition of BMO^{-1}:

$$\|f(\cdot)\|_{BMO^{-1}} = \sup_{x_0,R}\left(\frac{1}{|B(x_0,\sqrt{R})|}\int_0^R\int_{B(x_0,\sqrt{R})}|e^{t\Delta}f(y)|^2\,dy\,dt\right)^{\frac{1}{2}}. \tag{6}$$

In [5] Koch and Tataru proved the following existence theorem for the solutions to the Navier-Stokes equations:

Theorem 1. *The Navier-Stokes equations* (1) - (3) *with* $f = 0$ *have a unique global solution in* X^0

$$\|u\|_{X^0} = \|u\|_{N^0_\infty} + \|u\|_{N^0_C}, \tag{7}$$

where

$$\|u(\cdot,\cdot)\|_{N^0_\infty} = \sup_t t^{\frac{1}{2}}\|u(\cdot,t)\|_{L^\infty}$$

$$\|u(\cdot,\cdot)\|_{N^0_C} = \sup_{x_0,R}\left(\frac{1}{|B(x_0,\sqrt{R})|}\int_0^R\int_{B(x_0,\sqrt{R})}|u(y,t)|^2\,dy\,dt\right)^{\frac{1}{2}},$$

for all initial data u_0 *with* $\nabla\cdot u_0 = 0$ *which are small in* BMO^{-1}.

We shall call such a solution the Koch-Tataru solution to the Navier-Stokes equations.

2.2. **Formulation of results.** Now we are ready to formulate our main result:

Theorem 2. *There exists* $\epsilon = \epsilon(d)$ *such that if* $\|u_0\|_{BMO^{-1}} < \epsilon$*, the Koch-Tataru solution* u *associated to the initial value problem* (1) - (3) *with* $f = 0$ *verifies*

$$t^{\frac{k}{2}} \nabla^k u \in X^0$$

for any $k \geq 0$*.*

Theorem 2 implies the following decay in time of the space derivatives:

Corollary 3. *If* $\|u_0\|_{BMO^{-1}} < \epsilon(d)$*, the Koch-Tataru solution* u *satisfies*

$$\|\nabla^k u\|_{BMO^{-1}} \leq \frac{C}{t^{k/2}}, \tag{8}$$

for any $t \geq 0$ *and any* $k \geq 0$*.*

Also the proof of Theorem 2 implies the following result:

Theorem 4. *If* $\|u_0\|_{BMO^{-1}} < \epsilon(d)$*, then the Koch-Tataru global solution* u *is space analytic.*

We prove Theorem 2 via a fixed point algorithm. Our arguments are based on the following three results of harmonic analysis:

(a) A Carleson-type estimate (a bound on the space-time L^2 norm of

$$\beta_k(x,t) = t^{\frac{k}{2}} (-\Delta)^{\frac{k+1}{2}} e^{t\Delta} \int_0^t N(x,s)\, ds$$

in terms of L^1 norms, one of which is over parabolic cylinders)

(b) Generalized maximal regularity of the heat kernel

(c) Estimates of the Oseen kernel

In [2] we prove (a) by applying the TT^* argument followed by a sequence of integration by parts, while we prove (b) by applying the Fourier transform in space and time. A version of (c) can be found in [6].

Remark: We note that regularity of solutions to the Navier-Stokes equations in BMO^{-1} was considered by Miura and Sawada [7] too. More precisely, in [7] Miura and Sawada prove that the global solution to the system (1) - (3) evolving from small initial data in BMO^{-1} satisfies the following regularity property:

$$t^{\frac{k}{2}} \nabla^k u \in N^0_\infty, \text{ for all } k \in \mathbb{N} \cup \{0\}\,. \tag{9}$$

Recalling (7), we see that our regularity result can be understood as an extension of the result of Miura and Sawada. Indeed, a major part of our paper [2] concentrates on obtaining the regularity result for the Carleson part of the norm[1] given by N^0_C.

[1] We use this to obtain a regularity result for self-similar solutions too.

References

[1] H. Dong and D. Du, *On the local smoothness of solutions of the Navier-Stokes equations*, J. Math. Fluid Mech. **9**, No. 2 (2007), 139–152.

[2] P. Germain, N. Pavlović and G. Staffilani, *Regularity of solutions to the Navier-Stokes equations evolving from small data in BMO^{-1}*, Int. Math. Res. Notices, 2007 Volume: article ID rnm087, (2007).

[3] Y. Giga and O. Sawada, *On regularizing-decay rate estimates for solutions to the Navier-Stokes initial value problem*, Nonlinear analysis and applications **1**, **2** (2003), 549–562.

[4] T. Kato, *Strong L^p-solutions of the Navier-Stokes equations in R^m with applications to weak solutions*, Math. Zeit. **187** (1984), 471–480.

[5] H. Koch and D. Tataru, *Well Posedness for the Navier Stokes equations*, Adv. Math. **157** (2001), 22–35.

[6] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapman & Hall/CRC (2002).

[7] H. Miura, O. Sawada, *On the regularizing rate estimates of Koch-Tataru's solution to the Navier-Stokes equations*, Asymptotic Analysis **49**, No 1-2 (2006), 1–15.

[8] O. Sawada, *On analyticity rate estimates of the solution to the Navier-Stokes equations in Bessel-potential spaces*, J. Math. Anal. Appl. **312**, No. 1 (2005), 1–13.

Future stability of the Einstein non-linear scalar field system, power law expansion

Hans Ringström

Let us define what we mean by the Einstein-non-linear scalar field system. To begin with, we are interested in Einstein's equations:

$$G = T. \tag{1}$$

Here G is the *Einstein tensor*, i.e.

$$G = \mathrm{Ric} - \frac{1}{2}Sg, \tag{2}$$

where Ric is the Ricci tensor and S is the scalar curvature of a Lorentz manifold (M, g). Furthermore, T is the stress energy tensor, the exact form of which depends on the choice of the matter model. We shall only consider the case

$$T = d\phi \otimes d\phi - \left[\frac{1}{2}\langle \mathrm{grad}\phi, \mathrm{grad}\phi\rangle + V(\phi)\right] g, \tag{3}$$

where $\langle\cdot,\cdot\rangle := g$, $V \in C^\infty(\mathbb{R})$ is referred to as the *potential* (specifying V corresponds to specifying the matter model) and $\phi \in C^\infty(M)$ is referred to as the *scalar field*. If the stress energy tensor is of the form (3), we shall say that the matter model is of *non-linear scalar field* type. The scalar field should satisfy a matter equation, given by

$$\Box_g \phi - V'(\phi) = 0. \tag{4}$$

Note that (4) ensures that the stress energy tensor is divergence free, and thereby that the choice of matter model is consistent with (1) (since the Bianchi identities imply that the divergence of the Einstein tensor is zero). We shall refer to the system (1)-(4) as the *Einstein non-linear scalar field system.*

One reason why it is of interest to study the above system is that the spacetimes used to model the universe are ones with accelerated expansion nowadays. One way to induce accelerated expansion is by means of a positive cosmological constant. Another way is to couple Einstein's equations to a non-linear scalar field. Since it is unclear what potential to use, it is of interest to study different cases.

In an earlier paper, [4], we studied the case where V has a positive non-degenerate minimum at the origin, i.e.

$$V(0) > 0, \quad V'(0) = 0, \quad V''(0) > 0.$$

This matter model includes Einstein's vacuum equations with a positive cosmological constant as a special case, and the model solutions exhibit exponential expansion. In [4], we developed a rather general framework for considering the question of future stability in the Einstein non-linear scalar field setting. As a test of this framework, it is of interest to use it to prove future stability for some other potential. We here consider potentials of the form

$$V(\phi) = V_0 e^{-\lambda\phi}, \tag{5}$$

where V_0 and λ are positive constants. We shall restrict the values of λ later. In this case, the model solutions exhibit power law expansion. To the best of our knowledge, the first person to consider this case was Halliwell, cf. [2].

The question we wish to discuss here is that of future stability of certain spatially locally homogeneous solutions. In order to be able to give a precise definition of what future stability means, it is necessary to formulate the inital value problem in the Einstein non-linear scalar field setting.

Definition 1. *Initial data* for (1)-(4) are given by $(\Sigma, h, k, \phi_a, \phi_b)$, where Σ is an n dimensional manifold, h is a Riemannian metric, k is a symmetric covariant 2-tensor and ϕ_a and ϕ_b are two functions on Σ, all assumed to be smooth and to satisfy

$$r - k_{ij}k^{ij} + (\mathrm{tr}_h k)^2 = \phi_b^2 + D^i\phi_a D_i\phi_a + 2V(\phi_a), \tag{6}$$

$$D^j k_{ji} - D_i(\mathrm{tr}_h k) = \phi_b D_i \phi_a, \tag{7}$$

where D is the Levi-Civita connection of h, r is the associated scalar curvature and indices are raised and lowered by h.

Definition 2. Let $(\Sigma, h, k, \phi_a, \phi_b)$ be initial data for (1)-(4). A *development* of the initial data is given by (M, g, ϕ), where M is an $n+1$ dimensional manifold, g is a Lorentz metric on M and $\phi \in C^\infty(M)$. Furthermore, (M, g, ϕ) should satisfy (1)-(4). Finally, there should be an embedding $i : \Sigma \to M$ such that $i(\Sigma)$ is a spacelike hypersurface in (M, g), $i^*g = h$, $\phi \circ i = \phi_a$, and if N is the future directed unit normal and κ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = k$ and $(N\phi) \circ i = \phi_b$. If $i(\Sigma)$ is a Cauchy hypersurface, we shall say that (M, g, ϕ) is a *globally hyperbolic development*.

Remark. A Cauchy hypersurface in a Lorentz manifold is a set which is intersected exactly once by every inextendible timelike curve, cf. [4]. Not all Lorentz manifolds admit Cauchy hypersurfaces. Those that do are called globally hyperbolic.

The fundamental theorem concerning developments is due to Yvonne Choquet-Bruhat and Robert Geroch, cf. [1]. However, in order to formulate it, we need to introduce some more terminology.

Definition 3. Given initial data $(\Sigma, h, k, \phi_a, \phi_b)$ for (1)-(4), a *maximal globally hyperbolic development* of the data is a globally hyperbolic development (M, g, ϕ), with embedding $i : \Sigma \to M$, such that if (M', g', ϕ') is any other globally hyperbolic development of the same data, with embedding $i' : \Sigma \to M'$, then there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image such that $\psi^* g = g'$, $\psi^* \phi = \phi'$ and $\psi \circ i' = i$.

Theorem 4. *Given initial data for (1)-(4), there is a maximal globally hyperbolic development (MGHD) of the data which is unique up to isometry.*

Finally, we are in a position to state the question of **future stability**: Given a globally hyperbolic and future causally geodesically complete solution to the equations, do small perturbations of the corresponding initial data also yield future causally geodesically complete MGHD's?

Recall that causal geodesics are curves along which freely falling test particles and light travel. Future causal geodesic completeness thus means that neither freely falling test particles nor light exit the spacetime to the future after a finite parameter "time" (proper time in the case of freely falling test particles). To demand that the MGHD's corresponding to the perturbed initial data are future causally geodesically complete is a minimal requirement if one wishes to claim that the solution is global to the future. It is of course also of interest to calculate asymptotic expansions of the solutions. One can do so, see [5], but we do not wish to state the results here, due to lack of space.

Let us state our main result in the 4-dimensional spatially locally homogeneous case (for previous results, see [3]).

Theorem 5. *Let V be given by (5), where V_0 is a positive number and $\lambda \in (0, \sqrt{2})$. Let M be a connected and simply connected 3-dimensional manifold and let $(M, h, k, \phi_a, \phi_b)$ be initial data for (1)-(4). Assume, furthermore, that one of the following conditions is satisfied:*

- *M is a unimodular Lie group different from $\mathrm{SU}(2)$ and the isometry group of the initial data contains the left translations.*
- *$M = \mathbb{H}^3$, where $\mathbb{H}^n$ is the n-dimensional hyperbolic space, and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^3$.*
- *$M = \mathbb{H}^2 \times \mathbb{R}$ and the initial data are invariant under the full isometry group of the standard metric on $\mathbb{H}^2 \times \mathbb{R}$.*

Assume finally that $\mathrm{tr}_h k > 0$. Let Γ be a cocompact subgroup of M in the case that M is a unimodular Lie group and a cocompact subgroup of the isometry group otherwise. Let Σ be the compact quotient. Then $(\Sigma, h, k, \phi_a, \phi_b)$ are initial data.

Make a choice of Sobolev norms $\|\cdot\|_{H^l}$ *on tensorfields on* Σ. *Then there is an* $\varepsilon > 0$ *such that if* $(\Sigma, \rho, \kappa, \varphi_a, \varphi_b)$ *are initial data for (1)-(4) satisfying*

$$\|\rho - h\|_{H^4} + \|\kappa - k\|_{H^3} + \|\varphi_a - \phi_a\|_{H^4} + \|\varphi_b - \phi_b\|_{H^3} \leq \varepsilon,$$

then the maximal globally hyperbolic development corresponding to $(\Sigma, \rho, \kappa, \varphi_a, \varphi_b)$ *is future causally geodesically complete.*

References

[1] Y. Choquet-Bruhat, R. Geroch, *Global aspects of the Cauchy problem in General Relativity*, Commun. Math. Phys. **14** (1969), 329–335.
[2] J.J. Halliwell, *Scalar fields in cosmology with an exponential potential*, Phys. Lett. B **185** (1987), 341–344.
[3] J. M. Heinzle, A. D. Rendall, *Power-law Inflation in Spacetimes without Symmetry*, Commun.Math.Phys. **269** (2007), 1–15.
[4] H. Ringström, *Future stability of the Einstein-non-linear scalar field system*, Invent. math. **173** (2008), 123–208.
[5] H. Ringström, *Power law inflation*, *preprint*, http://www.math.kth.se/∼hansr.

Improved Uniqueness for the harmonic map heat flow in two dimensions

Melanie Rupflin

Let M^m and $N \subset \mathbb{R}^k$ be compact Riemannian manifolds.
We consider the harmonic map heat flow

$$\partial_t u - \Delta u = A(u)(\nabla u, \nabla u), \tag{1}$$

where A is the second fundamental form of N.
This is the negative gradient flow of the Dirichlet energy

$$E(u) = \frac{1}{2} \int_M |Du|^2 \, dvol_M$$

of maps $u : M \to N$.
For smooth solutions of the flow, uniqueness to a given initial condition is a consequence of the structure of the equation. On the other hand, when considering weak solutions, i.e. functions $u \in H^1$ satisfying (1) in the sense of distributions, the question of uniqueness is much more complex.
We consider this problem in the critical dimension ($m = 2$) and assume that M is closed.
Then given any initial condition $u_0 \in H^1$, there is a unique global weak solution constructed by Struwe [6], which is smooth away from finitely many points in space-time and has non-increasing energy.
It was shown by Freire [2] that every weak solution with non-increasing energy is identical to the corresponding Struwe solution. Thus, uniqueness holds for weak solutions with non-increasing energy.
As a generalization, we show

Theorem 1 ([5]). *For M a closed Riemannian surface and N compact, there exists $\varepsilon_1 > 0$ such that for weak solutions $u \in H^1([0,T] \times M)$ of (1), the condition*

$$\overline{\lim_{s \searrow t}} E(u(s)) < E(u(t)) + \varepsilon_1 \quad \textit{for all } t \in [0,T) \tag{2}$$

is sufficient for uniqueness, i.e. any such u is identical to the corresponding Struwe solution.

Examples of non-uniqueness were constructed by Topping [7] and Bertsch et al. [1] based on backwards bubbling. This causes a positive energy jump of at least $\varepsilon^\star$ to occur, where

$$\varepsilon^\star = \min\left\{ \frac{1}{2}\int_{S^2} |\nabla u|^2\, dx, u : S^2 \to N \text{ is a non-constant harmonic map} \right\}.$$

Topping conjectured that non-uniqueness in the critical dimension was always caused this way and therefore that assuming condition (2) with $\varepsilon^\star$ instead of ε_1 was enough to ensure uniqueness.
Assuming weak a priori regularity of the energy functional, we can indeed prove this conjecture. More precisely,

Theorem 2 ([5]). *Let M, N be as above. Then for weak solutions $u \in H^1([0,T] \times M)$, the conditions*

- $TV(E(u(\cdot))) < \infty$
- $\overline{\lim_{s \searrow t}} E(u(s)) < E(u(t)) + \varepsilon^\star \quad$ *for all $t \in [0,T)$*

are sufficient for uniqueness.

A few words on the proof:
The proof is built on techniques introduced by Moser in [4] by stopping time and considering (1) as a perturbation of the harmonic map equation. This allows to apply techniques from the stationary case.
Studying such perturbed harmonic map equations in general, we can derive local estimates for $|\nabla u|^4 + |\nabla^2 u|^2$ for fixed times, if the local energy is small enough. It is important to note that this estimate is based on an interpolation inequality holding only in two dimensional domains and it "separates" the influence of the energy distribution and the perturbation term. This separation allows to pass to space-time estimates, assuming that the local energy can be controlled uniformly for small time intervals.
Finally, the assumptions about the global energy allow to obtain the necessary control of the local energy, which then leads to uniqueness.

Outlook:
While the technique is restricted to the critical dimension, it may be extended to higher order equations. More precisely, for the extrinsic biharmonic flow on a four dimensional closed domain manifold with flat metric, the analogous result may be shown as a generalization of [3].

References

[1] Bertsch, Michael; Dal Passo, Roberta; Van der Hout, Rein, *Nonuniqueness for the Heat Flow of Harmonic Maps on the Disk*, Arch. Rat. Mech. An. 161 (2002), 93-112
[2] Freire, Alexandre; *Uniqueness of the harmonic map flow in two dimensions* Cal. Var. 3 (1995), 95-105
[3] Lamm, Tobias; Rivière, Tristan; *Conservation laws for fourth order systems in four dimensions*, Comm.P.D.E., 33 (2008), no. 2, 245-262
[4] Moser, Roger; *Regularity for the approximated harmonic map equation and application to the heat flow for harmonic maps*, Math. Z. 243 (2003), 263-289
[5] Rupflin, Melanie; *An improved uniqueness result for the harmonic map heat flow in two dimensions*, to appear in Calc. Var.
[6] Struwe, Michael; *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. 60 (1985), 558-581.
[7] Topping, Peter; *Reverse Bubbling and Nonuniqueness in the Harmonic Map Flow*, Int. Math. Research Notices 10 (2002)

Nonlinear evolution by mean curvature and isoperimetric inequalities

Felix Schulze

Let $F_0 : M^n \to N^{n+1}$ be an immersion of a hypersurface $M_0^n = F_0(M^n)$ into a Riemannian manifold (N^{n+1}, h). We study smooth one-parameter families $F : M^n \times [0,T] \to N^{n+1}$ of hypersurfaces $M_t^n = F(M^n, t)$ satisfying the following initial value problem:

$$(\star) \quad \begin{cases} F(\cdot, 0) = F_0(\cdot) \\ \dfrac{dF}{dt}(\cdot, t) = -H^k(\cdot, t)\,\nu(\cdot, t)\ , \end{cases}$$

where $\nu(p,t)$ is a choice of unit normal at $F(p,t)$, $k \geq 1$, and the mean curvature H of the hypersurface is given by the sum $\lambda_1 + \ldots + \lambda_n$ of the principal curvatures at a each point on the surface.

Let $M_0 \subset \mathbb{R}^{n+1}$ be a closed hypersurface with positive mean curvature. The positivity of the mean curvature ensures that there exists a smooth solution $\big(M_t\big)_{0 \leq t < T}$ to the initial value problem $(\star)$ on a maximal, finite time interval $[0,T)$. Let us denote by $A(t)$ its surface area and by $V(t)$ the enclosed volume at time t. The central observation is that the 'isoperimetric difference'

$$A(t)^{\frac{n+1}{n}} - c_{n+1} V(t) \tag{1}$$

is decreasing under the flow for $k \geq n-1$. Here c_{n+1} denotes the Euclidean isoperimetric constant. If the flow contracts smoothly to a point, as it is the case for convex surfaces (see [2]), this proves the Euclidean isoperimetric inequality for the initial configuration. But it is to be expected, that in general, as for the mean curvature flow, this flow develops singularities before the enclosed volume goes to zero. To overcome this obstacle we develop a weak level-set formulation for such a flow. More precisely, let $M_0 = \partial\Omega$ with positive mean curvature and $\Omega \subset \mathbb{R}^{n+1}$

be open and bounded. A weak level-set solution $u : \Omega \to \mathbb{R}$ of the H^k-flow is then formally a solution to the equation

$$\operatorname{div}\Big(\frac{Du}{|Du|}\Big) = -\frac{1}{|Du|^{\frac{1}{k}}} \ ,$$

with $u = 0$ on $\partial\Omega$. Using elliptic regularisation we show the existence of approximative solutions to this equation, as well as the existence of appropriate weak solutions. We furthermore show that the isoperimetric difference, appropriately defined for the weak flow, is still decreasing. Since this flow is defined past singularities, we apply it to prove the isoperimetric inequality for all initial configurations Ω, where $\partial\Omega$ has positive mean curvature. By a direct replacement argument one can then show that for $n \leq 7$ this suffices to prove the isoperimetric inequality for any open and bounded set $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary.
It is furthermore possible to show that such a weak flow also exists on a 3-dimensional, simply connected manifold with nonpositive sectional curvatures and that again the isoperimetric difference is decreasing along the flow. This yields a new, alternative proof of the result of Kleiner [1] that on such a manifold the Euclidean isoperimetric inequality is satisfied. If the the sectional curvatures are bounded from above by $-\kappa, \kappa \geq 0$, it also possible to use this flow approach to give as well an alternative proof of the fact that the isoperimetric profile of such a manifold is comparable with the isoperimetric profile of the model space with sectional curvatures equal to $-\kappa$.

References

[1] Bruce Kleiner, *An isoperimetric comparison theorem*, Invent. Math. **108** (1992), no. 1, 37–47.

[2] Felix Schulze, *Evolution of convex hypersurfaces by powers of the mean curvature*, Mathematische Zeitschrift **251** (2005), no. 4, 721–733.

Phase transitions in elastic atomistic chains

Hartmut R. Schwetlick

(joint work with Johannes Zimmer)

1. Summary

We study the existence of travelling wave solution for a Fermi-Pasta-Ulam chain. Motivated by martensitic phase transitions the elastic interaction energy is assumed to be a multi-well potential. We focus on the special case where the potential is piecewise quadratic, with two wells representing two stable phases. In the physically interesting regime of subsonic speeds we prove rigorously the existence of 'heteroclinic' travelling waves, that is, the asymptotic strains are contained in different wells of the potential. The existence proof is able to provide a very detailed insight into the structure and regularity of the solution. Thus, we are able to deduce important information on the macroscopic dissipation, namely, the kinetic relations governing the dependence of the configurational force on the speed of the moving interface.

2. Introduction

This article is concerned with travelling waves and the pertaining kinetic relations for the Fermi-Pasta-Ulam chain with a piecewise quadratic interaction potential. The precise setting is described below. The aims of this article are threefold. First, the existence of a family of travelling heteroclinic waves is established. Here, heteroclinic is understood in the sense that the asymptotic states are in different wells of the on-site potential. The existence result is an extension of earlier work [3], where the existence of one travelling wave is shown. Here, we prove that this solution is in fact embedded in a one-parameter family of solutions. Second, the chosen parametrisation of solutions gives give to a parametrisation of the so-called kinetic relation (relating the wave speed to the applied configuratinal force). This significantly extends the previous result [3], where only the trivial force-free kinetic relation was found. Third, we demonstrate that in the framework employed here, it is easy to give a good approximation of the solutions in the sense that it is proven that the plots differ from the real solution by at most $\frac{1}{2}$ in the L^∞-norm. This should be contrasted with the traditional representation of the solution as an infinite sum of Fourier-like components, where error bounds on the solution do not seem to exist in the literature.

The precise setting is as follows. The Fermi-Pasta-Ulam chain is defined by the Equation of motion

$$\ddot{u}_j(t) = V'(u_{j+1}(t) - u_j(t)) - V'(u_j(t) - u_{j-1}(t)) \tag{1}$$

for every $j \in \mathbb{Z}$; it describes the motion of a one-dimensional chain of atoms $\{q_j\}_{j\in\mathbb{Z}}$ on the real line by the deformation of atom $j \in \mathbb{N}$ by $u_j : \mathbb{R} \to \mathbb{R}$. Equation (1) describes the evolution governed by Newton's law, with neighbouring atoms being linked by springs.

The argument of the elastic potential is the discrete strain, which is given by the difference of the deformations $u_{j+1}(t) - u_j(t)$. We consider phase transitions and thus face the challenge that $V : \mathbb{R} \to \mathbb{R}$ is nonconvex. As in several previous studies [2, 4, 5, 3], we consider the simplest possible elastic potential V, namely a piecewise quadratic function. Specifically, we define

$$V(\varepsilon) := \frac{1}{2}\min\{(\varepsilon+1)^2, (\varepsilon-1)^2\}. \tag{2}$$

For the strain, this implies

$$\sigma(\varepsilon) := \varepsilon + 1 - 2H(\varepsilon) = \varepsilon + H(-\varepsilon) - H(\varepsilon) \tag{3}$$

equals $V'(\varepsilon)$ wherever V is differentiable, that is, for every $\varepsilon \neq 0$. Here, H is the symmetrised Heaviside function,

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}.$$

With the travelling wave ansatz $u_j(t) = u(j-ct)$ for $j \in \mathbb{Z}$, Equation (1) reduces to

$$c^2\ddot{u}(x) = V'\left(u(x+1)-u(x)\right) - V'\left(u(x)-u(x-1)\right).$$

In terms of the discrete strain $\varepsilon(x) := u(x)-u(x-1)$, the travelling wave equation is

$$c^2\varepsilon''(x) = \Delta_1 V'\left(\varepsilon(x)\right), \tag{4}$$

where

$$\Delta_1 f(x) := f(x+1) - 2f(x) + f(x-1)$$

is the *discrete Laplacian.* Specialising the potential to the choice made in (2), Equation (4) becomes

$$c^2\varepsilon''(x) = \Delta_1\left[\varepsilon(x) + H\left(-\varepsilon(x)\right) - H\left(\varepsilon(x)\right)\right] = \Delta_1\varepsilon(x) - 2\Delta_1 H\left(\varepsilon(x)\right). \tag{5}$$

For the sake of clarity, we order into linear and nonlinear part and rewrite (5) as

$$c^2\varepsilon'' - \Delta_1\varepsilon = -2\Delta_1 H(\varepsilon). \tag{6}$$

We want to study the existence of heteroclinic travelling wave solutions for this nonlinear advance-delay equation. To keep the technicalities to a minimum we assume that the dispersion relation

$$D(\kappa) := -c^2\kappa^2 + 4\sin^2\left(\frac{\kappa}{2}\right), \tag{7}$$

associated to the linear operator above, has only one positive real zero κ_0, which says that κ_0 is not too large, or c^2 is not too far away from the sonic speed 1. In fact, we show that there is a family of solutions, rather than one solution.

Theorem 1. *Suppose the dispersion relation* (7) *has one positive zero* κ_0 *with* $\kappa_0^2 < \frac{1}{2}$*. Then there exists a family of heteroclinic solution to Equation* (6)*. The solutions all have odd symmetry and satisfy the one-transition property*

$$\varepsilon > 0 \text{ for } x > 0 \text{ and } \varepsilon < 0 \text{ for } x < 0. \tag{8}$$

3. The Rankine-Hugoniot condition

We now show that the family of waves of Theorem 1 satisfy the Rankine-Hugoniot condition. Our ansatz to describe the solution $\varepsilon(x) = \varepsilon_{\mathrm{pr}}(x) - \varepsilon_{\mathrm{cor}}(x)$ with an approximate profile $\varepsilon_{\mathrm{pr}}$ and a remainder $\varepsilon_{\mathrm{cor}} \in L^2(\mathbb{R})$ is here very helpful since it allows that macroscopic quantities such as the Rankine-Hugoniot condition can be directly read off from the profile function $\varepsilon_{\mathrm{pr}}$, which is known explicitly.

Let us assume that the position of the interface is $s(t)$, and let us introduce the notation $[\![f]\!]$ for $f(s(t)+,t) - f(s(t)-,t)$, (here, $f(x_0\pm)$ is the shorthand notation for the limits from the left and the right at x_0). For an interface moving with velocity c, either the strain u_x or the velocity $\dot{u}$ may be discontinuous at the interface. However, the moving interface must satisfy the *Rankine-Hugoniot* conditions [1, Equations (2.6) and (2.7)]

$$[\![\sigma(u_x)]\!] = -\rho c\,[\![\dot{u}]\!],$$
$$c\,[\![u_x]\!] = -\,[\![\dot{u}]\!],$$

which we combine by writing for $\varepsilon = u_x$

$$\rho c^2 \llbracket \varepsilon \rrbracket = \llbracket \sigma(\varepsilon) \rrbracket . \tag{9}$$

Here, one has $\rho \equiv 1$ and, thanks to (3), $\llbracket \sigma(\varepsilon) \rrbracket = \llbracket \varepsilon \rrbracket - 2$, so (9) is equivalent to

$$\llbracket \varepsilon \rrbracket = \frac{2}{1-c^2}. \tag{10}$$

Although the strain is continuous, it oscillates at $\pm\infty$. Thus, the jump in ε in (10) needs to be understood in the sense

$$\llbracket \varepsilon \rrbracket = \bar{\varepsilon}_+ - \bar{\varepsilon}_-, \tag{11}$$

where $\bar{\varepsilon}_\pm$ are the limits of the *averaged strains*

$$\bar{\varepsilon}_+ := \lim_{x\to\infty} \lim_{s\to\infty} \frac{1}{s} \int_x^{x+s} \varepsilon(\xi)\, \mathrm{d}\xi,$$

and

$$\bar{\varepsilon}_- := \lim_{x\to-\infty} \lim_{s\to\infty} \frac{1}{s} \int_{x-s}^{x} \varepsilon(\xi)\, \mathrm{d}\xi.$$

As mentioned earlier, only $\varepsilon_{\mathrm{pr}}$ contributes to the asymptotic strains $\bar{\varepsilon}_\pm$. A direct calculation shows that

$$\bar{\varepsilon}_+ = \alpha\left(\frac{1+\xi}{\kappa_0^2} + \frac{1}{\beta^2}\right) + \frac{-2}{c^2}\frac{1}{2} = \frac{\alpha}{\kappa_0^2}\gamma^{-2} - \frac{1}{c^2} = \frac{1}{1-c^2} + \frac{\alpha\xi}{\kappa_0^2}.$$

Analogously

$$\bar{\varepsilon}_+ = \alpha\left(\frac{-1+\xi}{\kappa_0^2} + \frac{1}{\beta^2}\right) - \frac{-2}{c^2}\frac{1}{2} = \frac{\alpha}{\kappa_0^2}\gamma^{-2} - \frac{1}{c^2} = -\frac{1}{1-c^2} + \frac{\alpha\xi}{\kappa_0^2}.$$

Thus, as in the symmetric case,

$$\bar{\varepsilon}_+ - \bar{\varepsilon}_- = 2\frac{1}{1-c^2}, \tag{12}$$

and, via (11), we have verified the Rankine-Hugoniot condition (10).

4. The kinetic relation

A central observation of this paper is that the kinetic relation is essentially determined by function in the kernel of the linear operator L from Equation (**??**), in the following sense: if the profile $\varepsilon_{\mathrm{pr}}$ contains no functions in $\ker(L)$ other than the zero function, then the kinetic relation is trivial, namely the zero function. This is the case for a symmetric profile [3]. In the asymmetric case, the profile $\varepsilon_{\mathrm{pr}}$ contains non-zero functions from $\ker(L)$, and it is those kernel functions that render the kinetic relation non-trivial.

Before discussing this in detail, let us recall the definition of a kinetic relation, after introducing the notation. $\{\sigma\} := \frac{1}{2}\left(\sigma(s(t)+,t) + \sigma(s(t)-,t)\right)$ for the average stress across the discontinuity. A moving interface can dissipate energy, and the amount of dissipation is measured by the *configurational force* (or *driving force*). Furthermore, if the strain on both sides of the interface is constant, say ε_l for the strain on the left and ε_r for the strain on the right, then, the *configurational force* acting on an interface is

$$f := \int_{\varepsilon_l}^{\varepsilon_r} \sigma(\varepsilon)\,\mathrm{d}\varepsilon - \{\sigma\}\,[\![\varepsilon]\!] \tag{13}$$

(see, for example, [1, Equation (2.11)]). Since the configurational force depends on the speed c of the interface, we write $f = f(c)$. Furthermore,

$$R(c) := cf(c) \tag{14}$$

is the (macroscopic) *rate of the energy dissipation* or *energy flux* [1, Equation (2.10)]. The entropy inequality requires that $fc \geq 0$.

Here, the waves can oscillate, possibly widely, on both sides of the interface. We thus have to interpret Equation (13) in an averaged sense by setting $\varepsilon_l := \bar{\varepsilon}_-$ and analogously $\varepsilon_r := \bar{\varepsilon}_+$. We find

$$\{\sigma\} = \{\varepsilon\} = \frac{\bar{e}_l + \bar{e}_r}{2} = \frac{\alpha\xi}{\kappa_0^2}. \tag{15}$$

Furthermore, by (12),

$$[\![\varepsilon]\!] = \bar{\varepsilon}_+ - \bar{\varepsilon}_- = 2\frac{1}{1-c^2}. \tag{16}$$

Finally,

$$\begin{aligned} \int_{\varepsilon_l}^{\varepsilon_r} \sigma(\varepsilon)\,\mathrm{d}\varepsilon &= \int_{\frac{1}{1-c^2}}^{\frac{1}{1-c^2}+\frac{\alpha\xi}{\kappa_0^2}} \sigma(\varepsilon)\,\mathrm{d}\varepsilon - \int_{-\frac{1}{1-c^2}}^{-\frac{1}{1-c^2}+\frac{\alpha\xi}{\kappa_0^2}} \sigma(\varepsilon)\,\mathrm{d}\varepsilon \\ &= 2\cdot\frac{1}{1-c^2}\cdot\frac{\alpha\xi}{\kappa_0^2} - 2\cdot\frac{\alpha\xi}{\kappa_0^2} \end{aligned} \tag{17}$$

In summary, the kinetic relation (13) becomes with (15), (16) and (17)

$$f = -2\cdot\frac{\alpha\xi}{\kappa_0^2}. \tag{18}$$

Thus, the kinetic relation is non-zero unless $\xi = 0$. In particular, the force on the interface is exactly determined by functions in the kernel of the linear operator L in (6). We emphasise that is is this local asymmetry which imposes dissipation on the macroscopic scale.

References

[1] Rohan Abeyaratne and James K. Knowles. Kinetic relations and the propagation of phase boundaries in solids. *Arch. Rational Mech. Anal.*, 114(2):119–154, 1991.

[2] Alexander M. Balk, Andrej V. Cherkaev, and Leonid I. Slepyan. Dynamics of chains with non-monotone stress-strain relations. I. Model and numerical experiments. *J. Mech. Phys. Solids*, 49(1):131–148, 2001.

[3] Hartmut Schwetlick and Johannes Zimmer. Existence of dynamic phase transitions in a one-dimensional lattice model with piecewise quadratic interaction potential. Preprint.

[4] Lev Truskinovsky and Anna Vainchtein. Kinetics of martensitic phase transitions: lattice model. *SIAM J. Appl. Math.*, 66(2):533–553 (electronic), 2005.

[5] Lev Truskinovsky and Anna Vainchtein. Quasicontinuum models of dynamic phase transitions. *Contin. Mech. Thermodyn.*, 18(1-2):1–21, 2006.

Some Results concerning the Parabolic Scalar Curvature Equation

Brian Smith

This talk focuses on the *parabolic scalar curvature equation* (pse), which is as follows:

$$\bar{H} r \frac{\partial u}{\partial r} = u^2 \Delta_\gamma u + \left(r \frac{\partial \bar{H}}{\partial r} - \bar{H} + \frac{1}{2}\left(\bar{H}^2 + |\bar{\chi}|^2\right) \right) u - \left(\kappa(\gamma) - \frac{r^2 R}{2} \right) u^3$$

Here, $(\gamma(r), \Sigma)$ is a smooth family of 2-manifolds defined on an interval I, with κ the Gauss curvature, $\bar{\chi} = \gamma + \frac{1}{2} r \frac{\partial \gamma}{\partial r}$, and $\bar{H} = \bar{\chi}_{AB}\gamma^{AB}$. A positive solution of the pse yields a metric g on $M = I \times \mathbb{S}^2$ of prescribed scalar curvature R in the form

$$g = u^2 dr^2 + r^2 \gamma.$$

The first version of this equation was derived by Robert Bartnik [1] for quasi-spherical metrics. It was later fully generalized by Smith and Weinstein [7], and Shi and Tam [3].

In order to obtain short time existence, one assumes $\bar{H} \geq \delta > 0$, which is equivalent to the strict positivity of the mean curvature of the foliation surfaces $\Sigma_r = \{r\} \times \Sigma$. When this assumption is made and Σ remains smooth, standard parabolic theory implies "short time" existence; i.e. given inial data at r_0, one has existence on some interval $[r_0, r_0 + \varepsilon)$. Note that when $R \equiv 0$ and $\kappa > 0$ we get sufficient bounds for global existence by the maximum principle.

All of the results discussed today will be related to global existence in more general cases. Before summarizing these, we should consider briefly what one hopes to accomplish. The following problem, which is strongly motivated by general relativity, is certainly broad enough: one would like to be able to use the pse to construct as many asymptotically flat manifolds of non-negative scalar curvature as possible—outside of any apparent horizons. By results of Huisken and Sinestrari [2], it might be sufficient to assume that the foliation at each r consists of a union of topological spheres. However, κ cannot be assumed to be positive in general, and we must also allow for foliations that have singularities. Dealing with singular foliations seems to be very difficult. Also, it essentially requires a solution to the problem of obtaining global existence in certain situations in which $\kappa - r^2 R/2$ must change signs, and so for the time being we focus on the latter.

Related to this problem, this talk addresses the following results: (1) Blow-up occurs in many cases, but if the foliation is homothetic, $r^2 R/2 - \kappa$ is non-decreasing, and the blow-up occurs *at least* as fast as $(r_1 - r)^{-1/2}$, then the blow-up rate is uniform so that the blow-up corresponds to a maximal area totally geodesic outer boundary [4]. Moreover, spherically symmetric blow-up is stable. (2) In the case of null prescribed scalar curvature and a foliation by spheres, let A denote the area of the foliation spheres, and V the enclosed volume. Then as long as $\partial A / \partial V$ remains bounded away from 0, one has a supremum bound, and hence global existence [5].

(3) Solving an "initial time" blow-up problem in the case of prescribed null scalar curvature leads to the following result: If $\kappa(\gamma) > 0$, the surface $(\mathbb{S}^2, \gamma)$ can be realized as a stably embedded minimal surface in a null scalar curvature manifold. That is, one may construct black hole initial data with a horizon of prescribed geometry $(\mathbb{S}^2, \gamma)$ [6].

Finally, I would like to point out new avenues of investigation in which γ is assumed to flow by a geometric flow such as, for instance, the area preserving Ricci flow. To obtain a parabolic system one can introduce a cross term in the metric to obtain $\bar{H} \equiv 2$, which implies that the foliation flows by inverse mean curvature as well.

References

[1] R. Bartnik, *Quasi-spherical metrics and prescribed scalar curvature*, J. Differential Geom. **37** No. 1 (1993) 31–71.

[2] Huisken and Sinestrari, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Inventiones Mathematicae - to appear

[3] Y. Shi and L. Tam, *Positive mass thoerem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, Journal of Differential Geometry **62** (2002) 79–125.

[4] B. Smith, *Blow-up in the parabolic scalar curvature equation*, ArXiv: 0705.3774

[5] B. Smith, *The parabolic scalar curvature equation*, preprint

[6] B. Smith, *Black hole initial data with a horizon of prescribed intrinsic geometry*, ArXiv: 0710.1005

[7] B. Smith and G. Weinstein, *On the connectedness of the space of initial data for the Einstein equations*, Electron. Res. Announc. Amer. Math. Soc. **6** (2000) 52–63.

An Analysis of the Euler-Nordström System via the Method of Energy Currents

Jared Speck

1. Overview

My research focuses on the analysis of nonlinear partial differential equations (PDEs) used to model physical phenomena. In particular, I have studied the motion of self-gravitating relativistic fluids and relativistic electrodynamics, which are examples of evolutionary processes that are typically modelled by quasilinear hyperbolic systems of PDEs. The most fundamental questions concerning a system of hyperbolic PDEs are

(1) Given initial data belonging to the function/distribution space X, does the system posses a unique solution that exists locally in time, remains in X, and depends continuously on the initial data? If these conditions are satisfied, the system is said to be *well-posed* in X. Often times, the search for a space X in which the system is well-posed is a challenging question in itself.

(2) Does a given solution exist for all time, or does a singularity form in finite time? If a singularity forms, what is its nature?

In my research, I strive to answer (1), (2), and many related questions for systems that are only partially understood.

2. Techniques and Recent Results

2.1. **The Method of Energy Currents.** It is well-known that for first-order *symmetric* hyperbolic (FOSH) systems of PDEs, an energy principle is available that implies well-posedness in the Sobolev space H^N, if the integer N is large enough. The proof is based on the fact the symmetry of the equations allows one to estimate the time derivative of the L^2 norm of the solution in terms of the L^2 norm of the solution itself, which allows one to make a Gronwall estimate. There is also a generalization of this energy principle available that applies to strictly-hyperbolic (in the sense of Leray) systems.

However, not all first order hyperbolic systems are treatable via these two methods. For example, the Euler-Nordström system (see Section 2.2 below) is neither manifestly symmetric hyperbolic or strictly hyperbolic. Fortunately, Christodoulou has constructed a framework of alternate techniques ([Chr00], [Chr07]) that provide energy currents for any hyperbolic system that is derivable from a Lagrangian. The energy currents, which are vectorfields on the domain, enable one to make the same energy estimates that are available in the theory of FOSH systems; in particular, for a hyperbolic system derivable from a Lagrangian, well-posedness in the Sobolev space H^N for N large enough follows.

While Christodoulou's methods are not the only techniques available for proving the well-posedness of a hyperbolic system in a Sobolev space, they are powerful and natural in the sense that they exploit the inherent geometry of the equations. In contrast, one may proceed by seeking a change of state-space variables that renders the system FOSH. For example, Makino applies this symmetrizing technique to the Euler-Poisson equations in [Mak86], and Makino and Ukai apply it to the relativistic Euler equations without gravitational interaction in [MU95a] and [MU95b]. Yet the symmetrizing method is not without disadvantages: one must solve a formally over-determined system of equations to find the symmetrizing variables, and the resulting state-space variables, if they exist, may place un-physical and/or mathematically unappealing restrictions on the function spaces with which one would like to work. However, it should be noted that Makino's symmetrization of the Euler-Poisson system is currently capable of dealing with a restricted class of compactly supported data, while the technique of energy currents as applied to my study of the Euler-Nordström system (which is discussed in Section 2.2) cannot yet handle such data due to singularities in the energy current when the proper energy density ρ of the fluid vanishes.

2.2. **The Euler-Nordström System.** The Euler-Nordström (EN) system is a Lorentz covariant scalar caricature of the general covariant Euler-Einstein system describing a gravitationally self-interacting fluid. Mathematically speaking, the EN system is a quasilinear hyperbolic system of PDEs. In [Spe08b], we introduced a positive cosmological constant κ^2 into the EN system (and designated the resulting system EN_κ) in order to ensure the existence of non-zero constant solutions. Accordingly, we studied the initial value problem for an H^N perturbation of an infinitely extended uniform quiet fluid. Although the EN_κ system is neither symmetric hyperbolic nor strictly hyperbolic, Christodoulou's constructive results on the existence of energy currents for hyperbolic systems derivable from a Lagrangian can be adapted to provide energy currents that can be used in place of the standard energy principle available for first-order symmetric hyperbolic systems. After providing such energy currents, we proved the following theorem:

Theorem (Well Posedness for $\mathbf{EN}_\kappa$). *Let $N \geq 3$ be an integer. Assume that the initial data $\mathring{\mathbf{V}}$ for the EN_κ system are an $H^N(\mathbb{R}^3)$ perturbation of a constant background solution $\bar{\mathbf{V}}$. Then this data launch a unique solution $\mathbf{V}$ existing on a spacetime slab $[0,T] \times \mathbb{R}^3$ and which possesses the regularity property $\mathbf{V} - \bar{\mathbf{V}} \in C^0([0,T], H^N(\mathbb{R}^3)) \cap C^1([0,T], H^{N-1}(\mathbb{R}^3))$. Furthermore, the map from the initial perturbation $\mathring{\mathbf{V}} - \bar{\mathbf{V}}$ to $\mathbf{V} - \bar{\mathbf{V}}$ is a continuous map from an open subset of $H^N(\mathbb{R}^3)$ into $C^0([0,T], H^N(\mathbb{R}^3))$.*

2.3. **The Non-relativistic limit of the $\mathbf{EN}_\kappa$ System.** We have also studied the non-relativistic (also known as the "Newtonian") limit (i.e. $c \to \infty$) of the family of Euler-Nordström systems indexed by the parameters κ and c (EN_κ^c), where κ^2 is the cosmological constant and c is the speed of light. The limit $c \to \infty$ is singular because the EN_κ^c system is hyperbolic for all finite c, while the limiting system, namely the Euler-Poisson system with a cosmological constant (EP_κ), is not hyperbolic. Using Christodoulou's techniques to generate energy currents, together with harmonic analysis, we developed Sobolev estimates and used them to prove [Spe08a] the following theorem:

Theorem (The Non-relativistic Limit of $\mathbf{EN}_\kappa^c$). *For initial data belonging to an appropriate Sobolev space, the corresponding solutions to the EN_κ^c system converge uniformly on a spacetime slab $[0,T] \times \mathbb{R}^3$ to the solution of the EP_κ system as the speed of light c tends to infinity.*

As mentioned above and discussed in detail in [Spe08b], we consider the EN_κ^c system to be a mathematical scalar caricature of the Euler-Einstein (EE) system. We now provide some justification for this point of view. The above theorem shows that for large c, the EN_κ^c system well-approximates the EP_κ system. Furthermore, in [Oli07], Oliynyk shows the existence of a class of non-stationary solutions to the Euler-Einstein equations which converge to solutions of the EP_0 system in the Newtonian limit. Hence, both the EN_κ^c system and the EE system have the same Newtonian limit, and we therefore expect[1] that achieving an understanding of the

[1] We temper this expectation by noting that our proof does not work in the case $\kappa = 0$ and that in contrast to the initial value problem we studied in [Spe08b], Oliynyk considered compactly supported data under an adiabatic equation of state. This special class of equations of state allows Oliynyk to make a "Makino" change of variable which regularizes the equations and overcomes the singularities that typically occur in the equations in regions where the proper energy density vanishes. See [Mak86] and [Ren92] for additional examples of this change of variables in the context of various fluid models.

evolution of the EN^c_κ system will provide insight into understanding the behavior of the vastly more complicated EE system.

References

[Chr00] Demetrios Christodoulou, *The action principle and partial differential equations*, Princeton University Press, Princeton, NJ, 2000.

[Chr07] ———, *The formation of shocks in 3-dimensional fluids*, European Mathematical Society, Zürich, Switzerland, 2007.

[Mak86] Tetu Makino, *On a local existence theorem for the evolution equation of gaseous stars*, Patterns and Waves - Qualitative Analysis of Nonlinear Differential Equations (1986), 459–479.

[MU95a] Tetu Makino and Seiji Ukai, *Local smooth solutions of the Relativistic Euler Equation*, Journal of Mathematics of Kyoto University **35** (1995), 105–114.

[MU95b] ———, *Local smooth solutions of the Relativistic Euler Equation, ii*, Kodai Mathematical Journal **18** (1995), no. 2, 365–375.

[Oli07] Todd A. Oliynyk, *The Newtonian limit for perfect fluids*, Communications in Mathematical Physics **276** (2007), no. 1, 131–188.

[Ren92] Alan D. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, Journal of Mathematical Physics **33** (1992), no. 3, 1047–1053.

[Spe08a] Jared Speck, *The non-relativistic limit the Euler-Nordström system with cosmological constant*, (preprint) (2008).

[Spe08b] ———, *Well-posedness for the Euler-Nordström system with cosmological constant*, (preprint) (2008).

Blow-up results for energy critical semilinear wave equations

Daniel Tataru

The aim of the talk is to survey some recent results concerning the existence of blow-up solutions for energy critical semilinear wave equations. This is joint work with Joachim Krieger and Wilhelm Schlag. There are two models we have considered so far. The first is the $2+1$ dimensional wave-map equation into the sphere,

$$\Box U = U(\partial_\alpha U \partial_\alpha U), \qquad U : \mathbb{R}^{2+1} \to \mathbb{S}^2$$

Here the energy is preserved,

$$E(U) = \int_{\mathbb{R}^2} \langle DU(\cdot,t), DU(\cdot,t)\rangle \, dx = const$$

and it is invariant with respect to the scaling of the equation, $u(x,t) \to u(\lambda x, \lambda t)$.

We note that a fairly satisfactory understanding has been achieved for small-energy wave maps from $\mathbb{R}^{2+1}$ to general targets, see Tao [9], Tataru [10]–[11], and Krieger [4], as well as for rotationally invariant wave maps and general initial data by Christodoulou, Tahvildar-Zadeh [2], and Struwe [8]. In particular, the latter never develop singularities, see [8].

Instead, we consider equivariant wave maps of co-rotation index 1,

$$U(\omega x, t) = \omega U(x,t), \qquad \omega \in SO^2$$

where SO^2 acts in standard fashion on $\mathbb{R}^2$, and the action on S^2 is induced from that on $\mathbb{R}^2$ via stereographic projection. A recent paper by Rodnianski, Sterbenz [7] considers equivariant wave maps of higher co-rotation index.

In polar coordinates one obtains an equation for the longitudinal angle u,

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2} \tag{1}$$

This equation admits a soliton-like solution, namely the stereographic projection and its rescaled versions,

$$Q(r) = 2\arctan r, \qquad Q_\lambda(r) = Q(\lambda r), \quad \lambda > 0$$

Numerical evidence in Bizon, Tabor [1], and Isenberg, Liebling [3] suggests singularity development following a rescaled soliton profile,

$$u(t,r) \approx Q(\lambda(t)r) + o(1)$$

A second model we consider is the focusing quintic semilinear wave equation in $3+1$ dimensions,

$$\partial_{tt}u - \Delta u - u^5 = 0 \tag{2}$$

This has a conserved energy

$$E(u) = \int_{\mathbb{R}^3} \left[\frac{1}{2}(u_t^2 + |\nabla u|^2) - \frac{|u|^6}{6}\right] dx$$

which is invariant with respect to the scaling associated to the equation, $u(x,t) \to \lambda^{\frac{1}{2}} u(\lambda x, \lambda t)$.

For this equation we seek spherically symmetric blow-up solutions. It admits a soliton-like solution and its rescaled versions

$$Q(r) = (1 + r^2/3)^{-\frac{1}{2}}, \qquad Q_\lambda(r) = \lambda^{\frac{1}{2}} u(\lambda r), \quad \lambda > 0.$$

Our main result asserts that for both models there are blow-up solutions with soliton profiles corresponding to a large range of polynomial blow-up rates:

Theorem 1. *Let $\nu > \frac{1}{2}$ and $\delta > 0$. Then there exists an energy solution u of* (1) *respectively* (2) *which blows up precisely at $r = t = 0$ and which has the following property: in the cone $|x| = r \leq t$ and for small times t the solution has the form, with $\lambda(t) = t^{-1-\nu}$,*

$$u(x,t) = Q_\lambda(t)(r) + \eta(x,t)$$

where $E(\eta)(t) \to 0$ as $t \to 0$ and outside the cone $u(x,t)$ satisfies

$$E(u)_{[|x| \geq t]} < \delta$$

for all sufficiently small $t > 0$. In particular, the energy of these blow-up solutions can be chosen arbitrarily close to $E(Q)$, i.e., the energy of the stationary solution.

The restriction $\nu > \frac{1}{2}$ is technical, and should probably be replaced by $\nu > 0$. These blow-up solitions are also unstable.

A key role in the analysis is played by the linearized evolution around the soliton. After a suitable renormalization this has the form

$$(\partial_t^2 + \mathcal{L})v = 0$$

where L is given by

$$\mathcal{L} = -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$$

in the case of wave-maps, respectively

$$\mathcal{L} = -\partial_R^2 - 5(1 + R^2/3)^{-2}$$

A key common feature of these operators is that they have a resonance at 0, which is given by

$$\phi_0 = \frac{d}{d\lambda} Q_\lambda(R)_{|\lambda=1}$$

This resonance appears to be the main driving force behind the above blow-up dynamics.

References

[1] Bizon, P., Tabor, Z. *Formation of singularities for equivariant* $2+1$*-dimensional wave maps into the* 2*-sphere*, Nonlinearity 14 (2001), no. 5, 1041–1053.

[2] Christodoulou, D., Tahvildar-Zadeh, A. S. *On the regularity of spherically symetric wave maps*, Comm. Pure Appl. Math. 46 (1993), 1041–1091.

[3] Isenberg, J., Liebling, S. *Singularity formation for* $2+1$ *wave maps*, J. Math. Phys. 43 (2002), no. 1, 678–683.

[4] Krieger, J. *Global regularity of wave maps from* $\mathbf{R}^{2+1}$ *to* H^2*. Small energy.* Comm. Math. Phys. 250 (2004), no. 3, 507–580.

[5] Krieger, J., Schlag, W., Tataru, D. *Slow blow-up solutions for the* H^1 *critical focusing semi-linear wave equation in* $\mathbb{R}^3$, To appear, Duke Math J.

[6] Krieger, J., Schlag, W., Tataru, D. *Renormalization and blow up for charge one equivariant critical wave maps.*, to appear, Inventiones.

[7] Rodnianski, I., Sterbenz, J. *On the formation of singularities in the critical* $O(3)$ σ*-model*, preprint 2006.

[8] Struwe, M. *Radially symmetric wave maps from* $(1+2)$*-dimensional Minkowski space to general targets*, Calc. Var. Partial Differential Equations 16 (2003), no. 4, 431–437.

[9] Tao, T. *Global regularity of wave maps. II. Small energy in two dimensions.* Comm. Math. Phys. 224 (2001), no. 2, 443–544.

[10] Tataru, D. *On global existence and scattering for the wave maps equation.* Amer. J. Math. 123 (2001), no. 1, 37–77.

[11] Tataru, D. *Rough solutions for the wave maps equation*, Amer. J. Math 127 (2005), no. 2, 293–377.

On the notion of quasi-local mass (energy-momentum) in general relativity

Mu-Tao Wang
(joint work with Shing-Tung Yau)

As is well known, by the equivalence principle there is no well-defined concept of energy density in general relativity. On the other hand, when there is asymptotic symmetry, concepts of total energy and momentum can be defined. This is called the ADM energy-momentum and the Bondi energy-momentum when the system is viewed from spatial infinity and null infinity, respectively. These concepts are fundamental in general relativity and have been proven to be natural and to satisfy the important positivity condition in the work of Schoen-Yau, Witten, etc. However, there are limitations to such definitions if the physical system is not isolated and cannot quite be viewed from infinity where asymptotic symmetry exists. It was proposed more than 40 years ago to measure the energy of a system by enclosing it with a membrane, namely a closed spacelike 2-surface, and then attach to it an energy-momentum four-vector. It is natural to expect that the four-vector will depend only on the induced metric, the second fundamental form, and the connection on the normal bundle of the surface embedded in spacetime. This is the idea behind the definition of quasilocal mass of this surface. Obviously there are a few conditions the quasilocal mass has to satisfy: Firstly, the ADM or Bondi mass should be recovered as spatial or null infinity is approached. Secondly, the correct limits need be obtained when the surface converges to a point. Thirdly and most importantly, quasilocal mass must be nonnegative in general and zero when the ambient spacetime of the surface is the flat Minkowski spacetime. It should also behave well when the spacetime is spherically symmetric. Many proposals were made by Hawking, Penrose, etc. The most promising one was proposed by Brown-York where they motivated their definition by using the Hamiltonian formulation of general relativity (see also Hawking-Horowitz). They found interesting local quantities from which the definition of quasilocal mass was extracted. Their definition depends on the choice of gauge along the three dimensional spacelike slice which the surface bounds. It has the right asymptotic behavior but is not positive in general. Shi and Tam proved that it is positive when the three dimensional slice is time symmetric. Motivated by geometric consideration, Liu and Yau introduced a mass which is gauge independent, and proved that it is always positive. However, it was pointed out by Ó Murchadha, Szabados and Tod that the Liu-Yau mass can be strictly positive even when the surface is in a flat spacetime. In [1] and [2], we explore more in the direction of the Hamilton-Jacobi analysis of Brown-York. Combining some ideas from Liu-Yau, we define a quasilocal mass which is gauge independent and nonnegative. Moreover, it is zero whenever the surface is in the flat Minkowski spacetime. We believe that the present definition satisfies all the requirements necessary for a valid definition of quasilocal mass, and it is likely to be the unique definition that satisfies all the desired properties. A variational characterization of the quasilocal mass and its evolution equation in the spherical symmetric case are also discussed in this talk.

References

[1] M.-T. Wang and S.-T. Yau, *Local mass in general relativity*, arXiv:0804.1174v2

[2] M.-T. Wang and S.-T. Yau, *Isometric embeddings into the Minkowski space and new quasi-local mass*, arXiv:0805.1370v2.

Almost global wellposedness of the 2-D full water wave problem

Sijue Wu

We consider the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density in the two dimensional space. We assume the surface tension on the interface is zero. Assume that the density of the fluid is one, the gravitational field is $(0, -1)$, and at the time $t \geq 0$, the free interface is $\Gamma(t)$, and the fluid occupies the region $\Omega(t)$ below the interface $\Gamma(t)$. The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = (0, -1) - \nabla P \qquad \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, \qquad \text{on } \Omega(t),\ t \geq 0, \\ P = 0, \qquad \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \end{cases} \tag{1}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. In [15] we showed that the initial value problem of the full nonlinear water wave system (1) is uniquely solvable **locally** in time in Sobolev spaces, for any initially nonself-intersecting interfaces and incompressible irrotational velocities. Earlier Nalimov [10], and Yosihara [18] obtained the existence and uniqueness of solutions **locally** in time in Sobolev classes for small initial data, for 2-D water wave of infinite and finite depths. Other related results concerning local in time wellposedness of the water wave motion in higher dimensions and with additional effects such as a bottom, a non-zero surface tension and non-zero vorticity can be found in [1, 4, 5, 6, 7, 8, 9, 11, 12, 13, 16, 19]. However the question concerning the solution behaviors globally in time remains open.

In this paper we study the global in time behavior of solutions of the infinite depth water wave system (1), assuming that the initial data is small.

In what follows, we regard the 2-D space as a complex plane and use the same notation for a complex form $z = x + i\,y$ and a point $z = (x, y)$. So $\bar{z} = x - i\,y$.

Let $z = z(\alpha, t) = x(\alpha, t) + i\,y(\alpha, t)$, $\alpha \in R$, be the equation of the free interface $\Sigma(t)$ at time t in Lagrangian coordinate α, that is $z_t(\alpha, t) = \mathbf{v}(z(\alpha, t), t)$. Let

$$\mathfrak{H} f(\alpha, t) = \frac{1}{\pi i} p.v. \int \frac{f(\beta, t) z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)}\, d\beta \tag{2}$$

be the Hilbert transform associated with $z = z(\cdot, t)$. We know system (1) is equivalent to the following nonlinear system on the interface:

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{3}$$

where $\mathfrak{a}$ is a real valued function determined by system (3). (In fact, $\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}}\frac{1}{|z_\alpha|}$.) Taking a derivative to t to the first equation in (3), we obtain

$$z_{ttt} - i\mathfrak{a} z_{t\alpha} = i\mathfrak{a}_t z_\alpha \tag{4}$$

We showed in [15, 16] that the left hand side of (4) consists of the higher order terms, and constructed energy using these higher order terms and proved local in time well-posedness of the system (3).

The study of the long time behavior of solutions of (3) requires us to understand better the nature of the nonlinearity of the water wave system (3).

For $f = f(\alpha, t)$, $g = g(\alpha, t)$, we use the notation $U_g f(\alpha, t) = f \circ g(\alpha, t) = f(g(\alpha, t), t)$. Let $\Phi(\cdot, t) : \Omega(t) \to P_-$ be the Riemann mapping from the fluid domain $\Omega(t)$ to the lower half plane P_-, satisfying $\lim_{z\to\infty} \Phi_z(z, t) = 1$. Let $h(\alpha, t) = \Phi(z(\alpha, t), t)$ and

$$k(\alpha, t) = 2x(\alpha, t) - h(\alpha, t). \tag{5}$$

Let k^{-1} be such that $k \circ k^{-1}(\alpha, t) = \alpha$. Let $\psi(\alpha, t) = \phi(z(\alpha, t), t)$, where ϕ is the velocity potential.

In this paper, we find that the quantities

$$\chi = U_{k^{-1}}(I - \mathfrak{H})y, \qquad v = U_{k^{-1}}\partial_t(I - \mathfrak{H})y, \qquad \lambda = U_{k^{-1}}(I - \mathfrak{H})\psi, \tag{6}$$

we name $\chi,\ v,\ \lambda$ by Θ, satisfy equations of type

$$(\partial_t^2 - i\partial_\alpha)\Theta = G$$

where G consists of nonlinear terms of only cubic and higher orders. Using these equations for $\chi,\ v,\ \lambda$ and the method of invariant vector fields, we prove the following almost global well-posedness result for the 2-D water wave.

Let the initial interface be a graph $z(\alpha, 0) = \alpha + i\,\epsilon f(\alpha)$, the initial velocity $z_t(\alpha, 0) = \epsilon g(\alpha)$, $\alpha \in R$, f and g are smooth and decay fast at infinity, and $\bar{g} = \mathfrak{H}^0\bar{g}$, here $\mathfrak{H}^0$ is the Hilbert transform associated with the initial interface $z(\cdot, 0)$.

Theorem 1. *There exist $\epsilon_0 > 0$, $T > 0$, depending only on f and g, such that for $0 < \epsilon < \epsilon_0$, the initial value problem of the 2-D water wave system (3) (equivalently (1)) has a unique classical solution for a time period $[0, e^{T/\epsilon}]$. During this time period, the solution has the same regularity as the initial data and remains small, and the interface is a graph.*

The details of this work is contained in [17].

References

[1] D. Ambrose, N. Masmoudi *The zero surface tension limit of two-dimensional water waves.* Comm. Pure Appl. Math. 58 (2005), no. 10, 1287-1315

[2] T. Beale, T. Hou & J. Lowengrub *Growth rates for the linearized motion of fluid interfaces away from equilibrium* Comm. Pure Appl. Math. V.46 no.9, 1993 1269-1301.

[3] G. Birkhoff *Helmholtz and Taylor instability* Proc. Symp. in Appl. Math. Vol. XIII, pp.55-76.

[4] D. Christodoulou, H. Lindblad *On the motion of the free surface of a liquid* Comm. Pure Appl. Math. 53 (2000) no. 12, 1536-1602

[5] D. Coutand, S. Shkoller *Wellposedness of the free-surface incompressible Euler equations with or without surface tension* J. AMS. 20 (2007), no. 3, 829-930.

[6] W. Craig *An existence theory for water waves and the Boussinesq and Korteweg-devries scaling limits* Comm. in P. D. E. 10(8), 1985 pp.787-1003

[7] T. Iguchi *Well-posedness of the initial value problem for capillary-gravity waves* Funkcial. Ekvac. 44 (2001) no. 2, 219-241.

[8] D. Lannes *Well-posedness of the water-wave equations* J. Amer. Math. Soc. 18 (2005), 605-654

[9] H. Lindblad *Well-posedness for the motion of an incompressible liquid with free surface boundary* Ann. of Math. 162 (2005), no. 1, 109-194.

[10] V. I. Nalimov *The Cauchy-Poisson Problem* (in Russian), Dynamika Splosh. Sredy 18, 1974, pp. 104-210.

[11] M. Ogawa, A. Tani *Free boundary problem for an incompressible ideal fluid with surface tension* Math. Models Methods Appl. Sci. 12, (2002), no.12, 1725-1740.

[12] G. Schneider and E. Wayne *The long wave limit for the water wave problem I. The case of zero surface tension* Comm. Pure. Appl. Math. 53, 2000, no.12, 1475-1535.

[13] J. Shatah, C. Zeng *Geometry and a priori estimates for free boundary problems of the Euler's equation* Comm. Pure Appl. Math. V. 61. no.5 (2008) pp.698-744

[14] G. I. Taylor *The instability of liquid surfaces when accelerated in a direction perpendicular to their planes I.* Proc. Roy. Soc. London A 201, 1950 192-196

[15] S. Wu *Well-posedness in Sobolev spaces of the full water wave problem in 2-D* Inventiones Mathematicae 130, 1997, pp. 39-72

[16] S. Wu *Well-posedness in Sobolev spaces of the full water wave problem in 3-D* Journal of the AMS. 12. no.2 (1999), pp. 445-495.

[17] S. Wu *Almost global well-posedness of the 2-D full water wave problem* submitted.

[18] H. Yosihara *Gravity waves on the free surface of an incompressible perfect fluid of finite depth,* RIMS Kyoto 18, 1982, pp. 49-96

[19] P. Zhang, Z. Zhang *On the free boundary problem of 3-D incompressible Euler equations.* Comm. Pure. Appl. Math. V. 61. no.7 (2008), pp. 877-940

[20] S. Klainerman *The null condition and global existence to nonlinear wave equations*, Lectures in Appl. Math. vol. 23 (1986), 293-325.

[21] J. Shatah *Normal forms and quadratic nonlinear Klein-Gordon equations* Comm. Pure Appl. Math. 38 (1985), 685-696.

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