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Applied Dynamics and Geometric Mechanics

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Abstract. The workshop was organized around core topics in applied dynamics and geometric mechanics. Each speaker and invitee presented a new mathematical or computational tool. They also showed how their tool or technique is useful in applications, or how it bridges the gap between these areas and some interesting aspect of the pure mathematical side of dynamical systems or geometric mechanics. The meeting was not restricted to any particular application area, but rather focused on cross fertilizing a number of topics in Engineering and the Sciences.

The meeting was well attended with about 45 participants with broad geographic representation from all continents. There was an excellent blend of senior researchers, students, postdocs and junior faculty.

Mathematics Subject Classification (2000): 37-xx, 58-xx, 76-xx.

Introduction by the Organisers

The meeting was inspired by the fact that the number of interesting new applicable basic tools and techniques in applied dynamical systems and geometric mechanics continues to grow and be developed in interesting ways, both from the point of view of fundamental theory as well as from their applicability. Some examples of specific areas are basic advances in geometric mechanics, techniques for dealing with complex interconnected systems, new optimization methods, new geometric structures such as Dirac structures, discrete mechanics and its related geometric integrators, particle methods, variational principles, etc. Applications included fluid mechanics, stochastic mechanics, nonholonomic systems, plasticity, atmospheric dynamics, impact mechanics, nonlinear geometric control, and classical field theory. The specific themes of the workshop were as follows:

Core Dynamical Systems. The basic theory of dynamical systems continues to develop with numerous new ideas that have importance in applications. For

example, the concept of Finite Time Lyapunov Exponents and their derivative, Lagrangian Coherent Structures are tools that extend invariant manifold theory in a nice way to the time dependent case. These ideas have found application in, for example, the detection of recirculation zones in the heart and atmospheric barriers to transport. Another example is the extension of classical concepts from bifurcation and stability theory to time dependent systems with applications to phase transitions and materials with memory.

Complex Interconnected Systems. The computational limitations of dealing with complex interconnected systems, such as fuel cells, aircraft, etc have already been reached. New methods are being developed that move away from the monolithic approach (that is, thinking of systems of thousands of coupled equations, be they ode's or pde's) to the idea of many computations running in parallel with message passing and information exchange. This is an area in which dynamical systems theory is playing a key role. Set oriented methods and multiobjective optimization is an example of a classical area that has seen significant applications of dynamical systems.

Structured Model Reduction. Related to the previous topic is that of forming dimensionally reduced models for computational feasibility. This is important in, for example, complex fluid flows, where resolvability of the Navier-Stokes equations is simply not possible and where reduced order models have been developed. POD (proper orthogonal decomposition) methods are also undergoing continued development and their limitations being better understood. In carrying out such reductions, it is of special interest to do so in a way that preserves structure (such as symmetry structures, mechanical structures, etc).

Uncertainty and Stochastic Methods. Another area in which dynamical systems can play a key role is how to deal with systems whose very models are uncertain; for example, think of modeling an asteroid moving mainly in the field of the sun and Jupiter. How much error is introduced by neglecting the effects of Saturn? Is this more important than errors in initial conditions? How do error balls propagate under the dynamics? Of course in many systems, such as laboratory based mechanical systems, there are sources of uncertainty due to noise as well and one needs techniques deal with this and to distinguish this from the other sources of uncertainty and from numerical uncertainties.

Core Geometric Mechanics. Despite its maturity, especially over the last few decades, the basic theory of geometric mechanics continues to thrive and develop. For example, the reduction of mechanical systems with symmetry continues to grow and find applications in, for instance, computation and control. Areas that are currently undergoing particularly interesting growth that have links with applications are reduction by stages (for example, applied to fluid–solid interactions), singular cotangent bundle reduction (relevant for instance, to the dynamics of multiple pendula), and the development of Dirac structures. Also, the theory of integrable systems continues to be a valuable link between geometric mechanics

and pure mathematics. Another area that is of great interest is the application of ideas from geometric mechanics to classical field theories such as electromagnetism.

Geometric Integrators. The area of structured integrators for mechanical systems continues to undergo strong growth for problems in which preserving structure is important, such as symplectic structures for backward error analysis, in fluid systems in which it is important to conserve circulation, and in the development of asynchronous integrators. New insight into the development of methods that are robust to uncertainty are also quite promising. Both variational integrators and particle methods were represented at the meeting.

Optimization and Control of Mechanical Systems. New methods for optimization for mechanical systems such as DMOC (discrete mechanics and optimal control) are showing promise for the optimization of, for example, complex systems of vehicles—for instance a swarm of micro-air vehicles that is sent to investigate a biohazard. In these methods, the use of techniques that are successful already in internet congestion control as well as the use of parallel computation are quite attractive. Other types of control, such as stabilization, continue to benefit from basic advances in the theory and to make strong links with, for example, geometric integrators and discrete mechanics.

Structure of the Meeting. The meeting had a balance of senior researchers, postdoctoral fellows and graduate students. Consistent with the general approach advocated by Oberwolfach, there were only about 20 main lectures at the meeting. These senior people suggested students, postdoctoral fellows and junior faculty all of whom participated through two poster sessions.

Posters. The poster session was one of the most interesting aspects of the meeting. The organizers decided to award prizes for the best poster and an ad hoc committee (consisting of Reich, Ratiu, Marsden and Scheurle) was formed to choose the four best posters. This number matched the number of gifts that were available. The winners were (in alphabetical order):

- Ueli Aeberhard (ETH Zürich), *Perfect multi-contact collisions*
- Philip Du Toit (Caltech), *Hurricanes, Horseshoes, and Homoclinic Tangles*
- Andreas Johann (TU München), *Spiral Waves and Spiral Solitons in Lattice Differential Equations*
- Sigrid Leyendecker and Sina Ober-Blöbaum (Caltech), *Dynamic Optimization of a Three-Dimensional Walker*.

Impressions, Connections, Insights. The mixture of participants, from those who were very applied and those who came from more of a dynamical systems or

geometric mechanics background, but all with a strong mathematical dedication, was very fruitful.

Dynamical systems and geometric mechanics methods are now used in a spectacular array of very diverse application areas from complex fluids and plasticity to impact mechanics and spacecraft control. There is a wide range of research opportunities, including the possibility for new theoretical and methodological progress.

Amongst the specific new connections and insights that were gained, the power of dynamical systems ideas, such as invariant manifolds or Lyapunov exponents, seemed to be quite impressive as well as useful in the context of atmospheric science as well as oceanography. The use of dynamical systems ideas in control of mechanical systems was also quite interesting for a number of the participants.

Workshop: Applied Dynamics and Geometric Mechanics

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Abstracts

Degenerate relative equilibria and the concept of criticality in fluid mechanics

Thomas J. Bridges

Criticality is a central issue in fluid mechanics. Typically, a textbook on open channel hydraulics will devote several chapters to the concept of criticality and its implications [1]. For one-dimensional uniform flow in a channel, with depth h and velocity u, criticality corresponds to Froude number unity, $u^2 = gh$, where g is the gravitational constant.

There has been very little work on the generalization of criticality to more general states (see the introduction of [5] for a review). Like many generalizations, the key is to choose a definition which is general enough. It turns out that the natural definition for generalization is to consider uniform flows as relative equilibria (RE), and associate criticality with degeneracy of the RE characterization. This definition works, and generalizes to any fluid flow which can be characterized as a RE.

But why is criticality interesting? In classical hydraulics it is important for determining the maximum flow rate for a given energy, and is used in design of river controls [10]. When nonlinearity is included criticality signals a bifurcation of solitary waves. In the classical case of a one-dimensional channel, criticality generates the KdV solitary wave. It is this mechanism for creating solitary waves that is of interest here.

One of the main implications of degeneracy of RE is the generation of a homoclinic orbit, which in the spatial setting represents a solitary wave. Once this connection between degenerate RE and criticality is established our problem is reduced to a question in geometric mechanics.

Consider the standard setup of a Hamiltonian system which is equivariant with respect to an Abelian Lie group acting symplectically and generating a momentum map. Such systems have a natural definition of RE [11]. Given an $n-$parameter family of RE of a finite-dimensional Hamiltonian system, what is a degenerate RE and what is the implication of degeneracy of RE? There are a number of ways that an RE can become degenerate: singularity of the momentum map, failure of the G-Morse hypothesis, and degeneracy of the reduced momentum map $\mathbf{P} : \mathfrak{g} \to \mathfrak{g}^*$. It is the latter degeneracy that corresponds to criticality in fluid mechanics. Surprisingly the nonlinear implications of this degeneracy had not previously been studied in the Hamiltonian system literature. A new theory is presented in [3]. One of the interesting features of this theory is that the coefficient of the nonlinear term, in the normal form near a degenerate RE, is determined by the curvature of the momentum map. This observation has both theoretical and practical implications. Theoretically it means the all the properties of the leading order normal form near a degenerate RE are determined by the geometry of the RE. Practically, it means that the properties of the bifurcating homoclinic

can be determined by properties of the RE from which it bifurcates. For example, when computing the homoclinics bifurcating from periodic orbits [4] (in the spatial setting, computing the solitary waves bifurcating from periodic travelling waves) a lengthy computation is reduced to an elementary one [5].

In summary, any fluid flow that can be characterized as a RE has a concept of criticality. The more exotic the RE the more exotic the homoclinic orbit that is created at degeneracy.

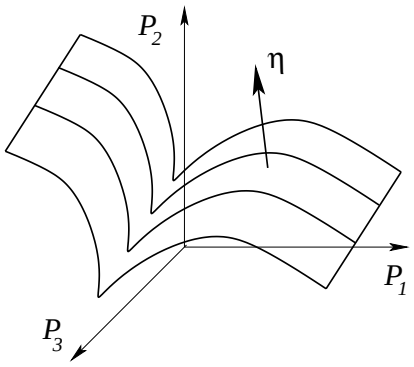


FIGURE 1. Schematic of surface of degeneracy in $\mathfrak{g}^*$.

For an $n-$parameter family of RE, the surface of degeneracy is typically a smooth hypersurface in the Lie algebra $\mathfrak{g}$, and its image in $\mathfrak{g}^*$ is also a hypersurface but can have singularities. Figure 1 shows an example with $n = 3$ of a surface of degenerate RE in $\mathfrak{g}^*$.

Examples presented in the talk were (a) the bifurcation of solitary waves at the interface between two fluids [7, 8] and (b) the steady dark solitary waves which are created by criticality of periodic travelling waves in shallow water hydrodynamics. These latter waves are localized and asymptotic as $x \to \pm\infty$ to the Stokes periodic travelling wave (with a phase shift between $-\infty$ and $+\infty$) [5]. These latter waves were discovered using the theory of degenerate RE, and are a pervasive new family of waves in coastal hydrodynamics.

The degeneracy surfaces in $\mathfrak{g}$ and $\mathfrak{g}^*$ found in [7] are shown in Figures 2(a) and 2(b).

Criticality can be generalized to time dependent states, and this leads to a connection between criticality and the Benjamin-Feir instability [6], and explains some of the properties of breaking waves ("micro-breakers") due to superharmonic instability [2].

There are connections between degenerate RE and degenerate conservation laws. Consider a family of conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0\,, \quad \mathbf{U} \in \mathbb{R}^n\,,$$

where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is the flux vector. Such a conservation law is said to be degenerate, when evaluated on a uniform state $\mathbf{U}_0 \in \mathbb{R}^n$, when the Jacobian of

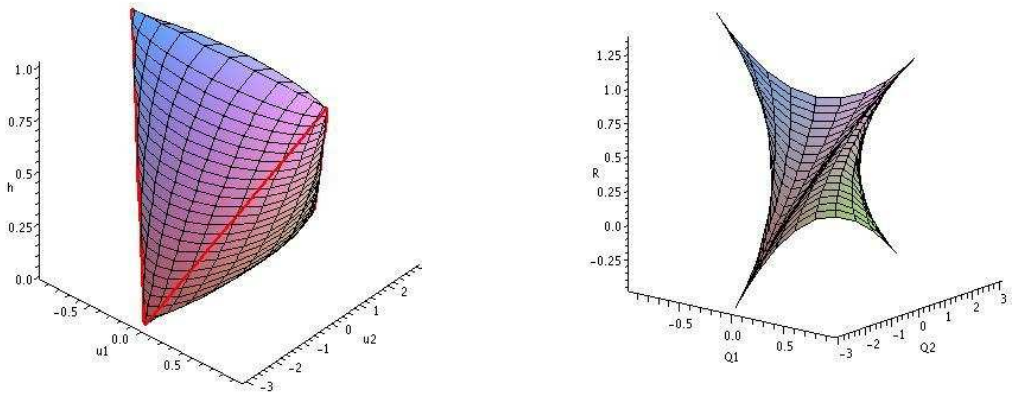
the flux vector is degenerate

$$\det[D\mathbf{F}(\mathbf{U}_0)] = 0\,.$$

It is shown in the talk how this degeneracy can be connected with degeneracy of RE. When regularization is added (dispersive or dissipative) the degeneracy of the flux vector generates a solitary wave or front. In recent work [9] the validity of reduced models near degeneracy of the flux vector is considered for conservation laws with dissipation

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{D}\mathbf{U}_{xx}\,, \quad \mathbf{U} \in \mathbb{R}^n\,,$$

where $\mathbf{D}$ is a symmetric positive definite matrix.



(a) The Lie algebra $\mathfrak{g}$ (b) The dual of the Lie algebra $\mathfrak{g}^*$

FIGURE 2. Surface of degeneracy for the case of stratified flows found in [7].

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Variational and Dissipative Aspects of Nonholonomic Systems

Anthony M. Bloch

In this work I consider variational and dissipative aspects of nonholonomic systems. The dissipation encompasses external dissipation, including that induced from coupling to an external field, and natural internal dissipation where system energy is preserved but there is a contraction in the phase space. I discuss work with Zenkov (see [9] and [1]) where the latter kind of dissipation is observed in systems defined on groups. Examples of such systems are the Chaplygin sleigh and the Suslov system. The latter system is a rigid body system with a constraint on the body angular velocities.

The equations of the Chaplygin sleigh for example are

$$\begin{aligned} \dot{v} &= a\omega^2 \\ \dot{\omega} &= -\frac{ma}{I+ma^2}v\omega \end{aligned}$$

These equations have a family of relative equilibria given by $(v,\omega)|v = \text{const}$, $\omega = 0$. Linearizing about any of these equilibria one finds one zero eigenvalue and one negative eigenvalue.

In fact the solution curves are ellipses in $v-\omega$ plane with the positive v-axis attracting all solutions. Thus one sees immediately that the system cannot be Hamiltonian. This is a simple but key example of nonholonomic systems which do not preserve volume in the phase space.

Our work in this area builds on earlier work of Kozlov, Jovanovic and others (see [1] and references therein). I also discuss generalizations to systems where there is additional internal dynamics as described in [9].

I then discuss work with Hagerty and Weinstein (see [8]) on mechanical systems with radiation damping, which involves dissipation arising from the coupling of the mechanical system with a wave field. Related earlier work on dissipation induced instabilities with Krishnaprasad, Marsden and Ratiu may be found in [3].

The key in this analysis is the original Lamb model (see [8]) of an oscillator physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence the oscillator loses energy and is effectively damped by the string.

Let q denote the position of the oscillator and let $w(x,t)$ denote the displacement of the string. with mass density ρ, tension T. Assuming a singular mass density at $x = 0$, we couple dynamics of an oscillator of mass M to obtain:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$
$$M\ddot{q} + Vq = T[w_x]_{x=0}$$
$$q(t) = w(0,t).$$

$[w_x]_{x=0} = w_x(0+,t) - w_x(0-,t)$ is the jump discontinuity of the slope of the string. Note that this is a Hamiltonian system.

We can solve for w and reduce to obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0.$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ in the dynamics of the oscillator which loses energy and is effectively damped by the string.

In recent work with Rojo (see [6]) I show how one can realize the nonholonomic constraint that arises in the Chaplygin sleigh problem by taking the limit of a suitable coupling to a wave field. This enables one to study the dynamics within the class of (infinite) Hamiltonian systems. This builds on earlier work by Kozlov and others which show how to represent the nonholonomic constraint as the limit of a certain kind of nonlinear Rayleigh dissipation. Once we have the system in Hamiltonian form we can then quantize the system using quantum field theory. We discuss aspects of this process. We have carried out related work on quantum subRiemmanian systems in [2].

We show that the sleigh equations can be obtained from a variational principle as reduced equations of motion after the system is coupled to an environment described by an $U(1)$ infinite field of the form $\mathbf{a}(\mathbf{z},t) \equiv [\cos\alpha(\mathbf{z},t), \sin\alpha(\mathbf{z},t)]$. For the Lagrangian of the free field we choose

$$L_{\mathrm{F}} = \frac{K}{2}\int d^2\mathbf{z}\ \dot{\mathbf{a}}^2, \tag{1}$$

and we couple the sleigh and the field with a term of the form

$$L_1 = \int d^2\mathbf{z}\ \delta(\mathbf{z}-\mathbf{x})\ \left[\gamma\dot{\mathbf{x}}\cdot\mathbf{a} + \mu\cos\left(\alpha(\mathbf{z},t)-\theta\right)\right]. \tag{2}$$

The first term in square brackets corresponds to a minimal coupling that favors $\dot{\mathbf{x}}$ in the direction of $\mathbf{a}$; the second has the form of a potential coupling that favors an alignment of the internal variable θ with the local direction of $\mathbf{a}$.

I also discuss work with Fernandez and Mestdag (see [4] and [7]) where we use the inverse problem in the calculus of variations to define Lagrangian systems which yield the dynamics of certain nonholonomic systems on the constraint manifold. Again this yields a Hamiltonian version of the dynamics which is of interest

for the purpose of quantization. The relationship of nonholonomic systems to the inverse problem of the calculus of variations is of independent interest however.

Finally I discuss work with Marsden and Zenkov (see [5]) which describes the rich dynamics of the Chaplyin sleigh coupled to a harmonic oscillator.This analysis uses the theory of quasivelocities. This work is supported in part by the NSF.

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Reduction of Dirac Structures and Dirac Systems

Hernán Cendra

(joint work with Jerrold E. Marsden, Tudor S. Ratiu)

Dirac Systems The geometric approach to Dirac structures was introduced in [Courant(1990)], and has found many developments and applications afterwards. Dirac structures satisfying the so-called ***integrability condition (or closedness condition)*** include presymplectic and Poisson structures as particular cases. In recent times the significance of Dirac structures (integrable or not) in representing geometrically the fundamental equations in several fields, such as Lagrangian or Hamiltonian mechanics, nonholonomic mechanics, several theories of circuits and interconnected systems, control systems, has become clarified, from the Lagrangian side, thanks to the work of many researchers, see [Yoshimura and Marsden(2006a), Yoshimura and Marsden(2006b)] and references therein. See also the important work of Van der Schaft and Blankenstein [Blankenstein and van der Schaft(2001)]. One must remark that those equations are in general ***implicit differential equations*** (IDE), see [Cendra and Etchechoury(2006)] for general references on IDE.

One of the interesting feature of our approach is the representation of those equations in a unified form as a ***Dirac system***, by definition, is written as follows

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in D_x, \tag{1}$$

where D is a given (not necessarily closed) Dirac structure on a manifold M and $\mathcal{E} : M \to \mathbb{R}$ is a given function which often has the meaning of ***energy***.

There are two important questions that can be addressed in this general framework, namely, the ***Theory of Constraints*** and ***Reduction***.

Theory of Constraints The geometrically inspired Gotay-Nester constraint algorithm for presymplectic manifolds [Gotay et al.(1978)Gotay, Hinds, and Nester] was originally motivated by the Dirac theory of constraints, [Dirac(1950)]. One can generalize the Gotay-Nester algorithm to solve a given Dirac system, for the case of a general Dirac structures (not necessarily integrable) rather than presymplectic forms. We can also generalize the Dirac algorithm, obtaining explicit equations of motion in terms of brackets. [Cendra, Etchechoury and Marsden,(2008)] (preprint). Both algorithms are closely related and, in a sense, equivalent.

Reduction If M is a principal bundle with structure group G and the energy function and the Dirac structure are both invariant one would like to reduce the Dirac system (1). This leads in a natural way to define the category of ***anchored vector bundles*** and the category of ***Dirac anchored vector bundles***, which is ***stable under reduction***. Examples of anchored vector bundles should be reduced tangent bundles $TM/G \to M/G$, where M is a principal bundle with structure group G. Let us explain those concepts more precisely.

An ***anchored vector bundle*** is a pair $(\pi_{(E,M)}, \rho_E)$ where $\pi_{(E,M)} : E \to M$ is a given vector bundle and $\rho_E : E \to TM$ is a given vector bundle map over the identity (that is, $\rho_E(e) \in T_{\pi_{(E,M)}(e)}M$ for $e \in E$) called the anchor. For $x \in M$, we denote by $\rho_x : E_x \to T_xM$ the restriction of ρ to the fiber E_x.

Let $\pi_{(E,M)} : E \to M$ be a given vector bundle and let $\pi_{(E^*,M)} : E^* \to M$ be the dual vector bundle of E. A fiberwise Dirac structure on $\pi_{(E,M)}$, or simply a Dirac structure on $\pi_{(E,M)}$, is a vector subbundle $D_E \subseteq E \oplus E^*$ such that, for each $x \in M$, $(D_E)_x \subseteq E_x \oplus E_x^*$ is a linear Dirac structure on the vector space E_x. A ***Dirac anchored vector bundle*** is a triple $(\pi_{(E,M)}, \rho_E, D_E)$ where $(\pi_{(E,M)}, \rho_E)$ is an anchored vector bundle and D_E is a Dirac structure on $\pi_{(E,M)}$.

Let $(\pi_{(E,M)}, \rho_E)$ be an anchored vector bundle. A curve $e(t)$, $t \in (a, b)$ on E is called ***admissible*** if the following condition is satisfied

$$\rho_E\left(e(t)\right) = \frac{d}{dt}\pi_{(E,M)}(e(t)), \tag{2}$$

for all $t \in (a, b)$.

Let $(\pi_{(E,M)}, \rho_E, D_E)$ be a given Dirac anchored vector bundle.

Let $\varphi \in \Gamma(E^*)$ be a given section of the dual bundle of E, called the ***energy form***. In many cases $\varphi = d\mathcal{E}$, where $\mathcal{E}$ represents energy.

By definition, the associated ***Dirac system*** is defined as follows

$$e \oplus \varphi(\pi_{(E,M)}(e)) \in (D_E)_{\pi_{(E,M)}(e)}. \tag{3}$$

A solution to the Dirac system (3) is an admissible curve $e = e(t) \in E$, $t \in (a, b)$, such that (3) is satisfied for each $t \in (a, b)$. By definition, a ***Dirac dynamical system*** is a pair $(\varphi, \mathcal{D})$ where $\mathcal{D} = \left(\pi_{(E,M)}, \rho_E, D_E\right)$ is an Dirac anchored vector bundle and φ is an energy form.

The definition of ***morphism*** in the category of Dirac anchored vector bundles is a natural one, but it is delicate and we shall not give more details here.

Reduced Anchored Vector Bundles Let $\mathcal{D} = (\pi_{(E,M)}, \rho_E, D_E)$ be a given Dirac anchored vector bundle. Assume that M is a principal bundle with group G acting on M on the left. Assume, in addition, that that G acts on $E \oplus E^*$ by isomorphisms $f_g \oplus \tilde{f}_g : E \oplus E^* \to E \oplus E^*$ of Dirac anchored vector bundles, covering the action of G on M and satisfies the condition $(f_g)^* = (\tilde{f}_g)^{-1}$, for each $g \in G$. One can show that everything passes to the quotient and the following assertion holds.

Under the above assumptions and constructions, the triple

$$\mathcal{D}/G = \left(\pi_{(E/G,M/G)}, \rho_{E/G}, D_E/G\right), \tag{4}$$

is a Dirac anchored vector bundle, called the reduced Dirac anchored vector bundle.

Moreover, there is a natural morphism of Dirac anchored vector bundles $\mathcal{P}_G : \mathcal{D} \to \mathcal{D}/G$ covering the projection $\pi_{M,M/G} : M \to M/G$. Restricted to each fiber of $E \oplus E^*$, the associated map of $E \oplus E^*$ to $(E \oplus E^*)/G$ is an isomorphism.

Let $(\varphi, \mathcal{D}) = \left(\varphi, (\pi_{(E,M)}, \rho_E, D_E)\right)$ be a given Dirac dynamical system and assume that the group G is a symmetry of $(\varphi, \mathcal{D})$. If a curve $e(t)$, $t \in (a, b)$ is a solution of the Dirac system, that is,

$$e \oplus \varphi(m) \in (D_E)_m \tag{5}$$

where $m = \pi_{(E,M)}(e)$, then the reduced curve $[e]_G(t)$, $t \in (a, b)$ is a solution of the reduced Dirac system, that is,

$$[e]_G \oplus [\varphi]_G([m]_G) \in ([D_E]_G)_{[m]_G}. \tag{6}$$

General Reduction. The reduction by the symmetry is only a part of the general notion of reduction in the category of Dirac anchored vector bundles. The general notion, explained in [Cendra, Marsden, Ratiu and Yoshimura, (2008)], includes as particular cases, Poisson reduction, symplectic reduction, Hamilton-Poincare reduction, and many other examples of reduction.

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Set Oriented Numerics for Transport Phenomena and Optimization

Michael Dellnitz

Over the last years so-called *set oriented* numerical methods have been developed in the context of the numerical treatment of dynamical systems, see [1, 2]. The basic idea is to cover the objects of interest – for instance *invariant manifolds* or *invariant measures* – by outer approximations which are created via adaptive multilevel subdivision techniques. These schemes allow for an extremely memory and time efficient discretization of the phase space and have the flexibility to be applied to several problem types. An overview about these set oriented methods can be found in [3].

In this talk we show that set oriented techniques can particularly be useful for the approximation of *transport processes* which play an important role in many real world phenomena. We mainly focus on two related applications: first we analyze the transport of asteroids in the solar system – this work is particularly motivated by the explanation of the existence of the asteroid belt between Mars and Jupiter. Secondly we show how to analyze transport phenomena in ocean dynamics. Here the related mathematical models depend explicitly on time and this makes the numerical treatment inherently more difficult. However, following [5] we demonstrate the strength of an appropriate set oriented approach by a study of transport in Monterey Bay which is based on real data, see figure 1.

In addition we illustrate how to make use of these set oriented numerical techniques for the solution of multiobjective optimization problems. In these problems several objective functions have to be optimized at the same time. For instance, for a perfect economical production plan one wants to simultaneously minimize cost and maximize quality. As indicated by this example the different objectives typically contradict each other and therefore certainly do not have identical optima. Thus, the question arises how to approximate the "optimal compromises" which, in mathematical terms, define the so-called Pareto set. In order to make

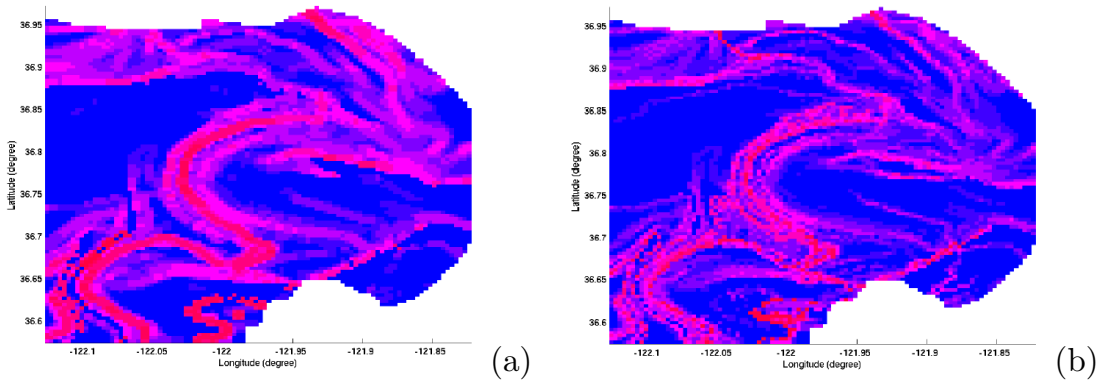

FIGURE 1. Approximation of Lagrangian coherent structures using direct transport processes, cf. [5]. (a) "Fast" method only using the center points; (b) standard direct transport process approach.

our set oriented numerical methods applicable we first construct a dynamical system which possesses the Pareto set as an attractor. In a second step we develop appropriate step size strategies. The corresponding techniques are applied to the optimization of an active suspension system for cars, see [4].

Our approach can briefly be illustrated as follows, see figure 2: We consider the three objective functions $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$,

$$\begin{aligned} f_1(x_1, x_2, x_3) &= (x_1 - 1)^4 + (x_2 - 1)^2 + (x_3 - 1)^2 , \\ f_2(x_1, x_2, x_3) &= (x_1 + 1)^2 + (x_2 + 1)^4 + (x_3 + 1)^2 , \\ f_1(x_1, x_2, x_3) &= (x_1 - 1)^2 + (x_2 + 1)^2 + (x_3 - 1)^4 . \end{aligned}$$

The basic domain is chosen as $Q = [-5, 5]^3$. The resulting box collections are shown in figure 2 (a)-(c). Here, we have taken a $3 \times 3 \times 3$ grid as test points for every box.

Alternatively, we have used a combination of three different algorithms from [4] to achieve a better performance. The result shown in figure 2 (d)-(f) was obtained by the following steps: first, the subdivision algorithm was applied for 21 steps using only the center point of every box as the test point for the dynamical system (figure 2 (d)). Using only these few test points, the computed box collection $\mathcal{B}_{21}$ reveals already the shape of the set of Pareto points, but it contains also many holes. These holes could be filled by an application of the recovering algorithm on $\mathcal{B}_{21}$ (figure 2 (e)). Finally, the covering was tightened using the sampling algorithm (figure 2 (f)).

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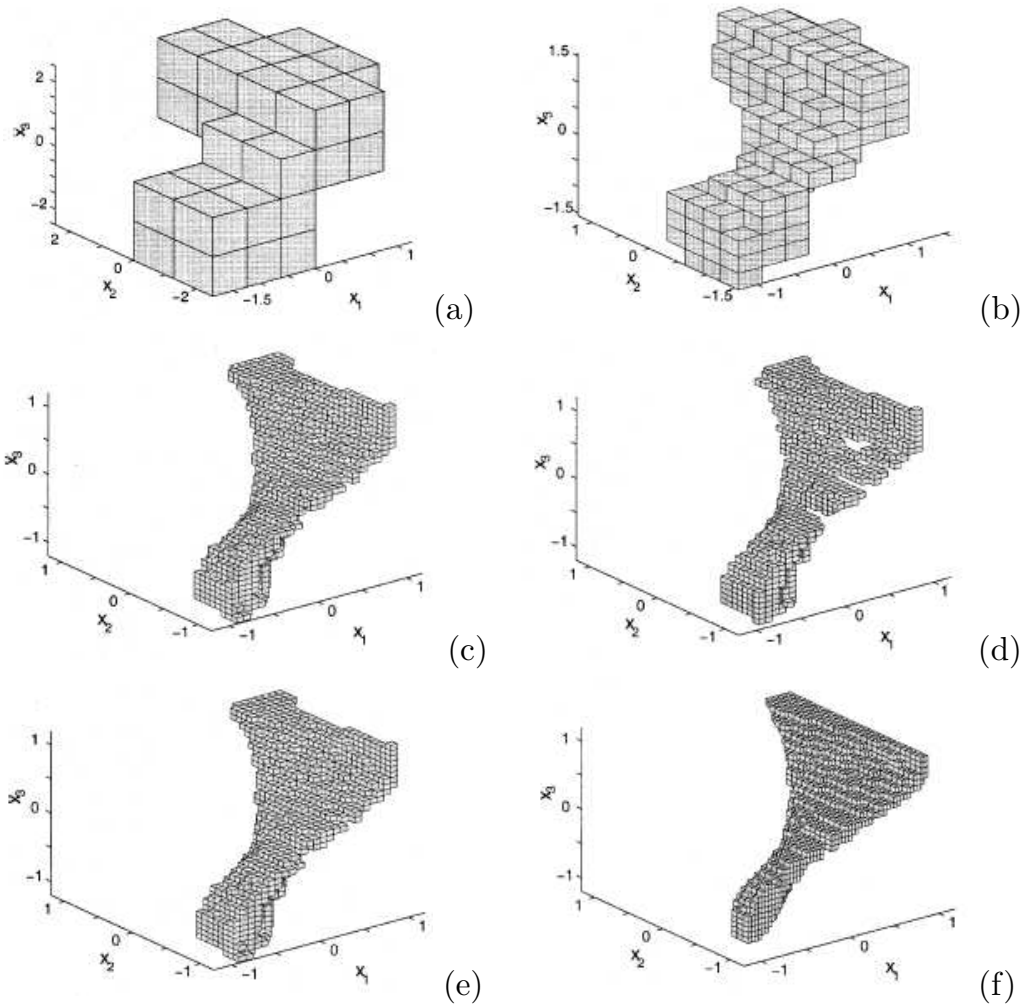

FIGURE 2. Resulting box collections using 27 test points per box after (a) 10, (b) 15, and (c) 25 steps. (d)-(f) Combination of the three algorithms from [4].

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A discussion of the standard inequality impact laws of Newton type

CHRISTOPH GLOCKER

(joint work with Ueli Aeberhard, Remco Leine)

In dynamics, collisions are normally met when bodies come into contact with each other. If the contacts are modeled without compliance, collisions have to

be treated within impact theory. As an impact we understand a velocity jump which occurs at a discrete point in time, and which is associated with impulsive forces as a consequence on finite, non-disappearing masses in the system. Processes with rapidly changing but continuous velocities will not be understood as impacts. Only impacts with constant generalized force directions will be investigated, which excludes discontinuities from crossing sharp bends in the configuration manifold. Impulsive forces applied from outside that can be regarded as external impact excitation, e.g. the impact from the queue on the ball when playing billiard, will also not be examined, and all discussions will be limited to scleronomic systems. Only the impact itself will be examined. Pre- and post-impact motions will not be discussed.

The most established and widespread impact law is the one of Newton, which changes for a collision at time t the relative normal velocity $\gamma(t)$ according to the rule $\gamma^+ = -\varepsilon\gamma^-$. The indices $(\,)^+$ and $(\,)^-$ denote the right and left limit of $\gamma(t)$, corresponding to the post- and pre-impact normal relative velocity. The parameter ε is called the Newtonian coefficient of restitution and is normally taken from the interval $[0, 1]$. For $\varepsilon = 0$, one classically speaks about a perfect inelastic impact, whereas collisions with $\varepsilon = 1$ are called perfectly elastic as they invert the relative velocity to $\gamma^+ = -\gamma^-$.

In the above form, Newton's impact law can be applied to model collisions at only a single contact point. One of the first approaches to extend it to multi-contact problems within the setting of analytical dynamics is found in [6], in which it is still stated as an equality and thus not yet respecting the unilateral nature of the contacts. The latter is taken into account in [5] and extended to frictional behavior by a Coulomb type impact law for the tangential directions of the contact. It turns out that both, the normal and the tangential portion of the impact law can be written in the form

$$(\boldsymbol{\gamma}^{i+} + \varepsilon^i \boldsymbol{\gamma}^{i-}) \in \mathcal{N}_{\mathcal{C}_i}(-\boldsymbol{\Lambda}_i)\,, \tag{1}$$

where $\mathcal{N}_{\mathcal{C}_i}$ denotes the normal cone of convex analysis to the convex set $\mathcal{C}_i$ of negative impulsive forces $\boldsymbol{\Gamma}_i = -\boldsymbol{\Lambda}_i$, defined as

$$\mathcal{N}_{\mathcal{C}_i}(\boldsymbol{\Gamma}_i) = \{\, \boldsymbol{\xi}^i \mid \boldsymbol{\xi}^{i\mathrm{T}}(\boldsymbol{\Gamma}_i^* - \boldsymbol{\Gamma}_i) \le 0 \quad \text{for}\, \boldsymbol{\Gamma}_i \in \mathcal{C}_i,\ \forall\, \boldsymbol{\Gamma}_i^* \in \mathcal{C}_i \,\}\,. \tag{2}$$

Apparently, the impact laws (1) are kinematic laws, because they relate the pre- and post-impact relative velocities $\boldsymbol{\gamma}^{i-}$ and $\boldsymbol{\gamma}^{i+}$ directly to each other as in the original law of Newton. In addition, they take care of restrictions on the impulsive forces by the inequalities in (2). We therefore call (1) the *standard inequality impact laws of Newton type* and propose to accept them with an even more general meaning as in [5], namely as a class of constitutive laws for impacts with the structure (1) that are applied on various types of constraints after the sets $-\mathcal{C}_i$ of transferable impulsive forces have been determined by integration from impact-free motion. In a succeeding step, those laws have to be analyzed on kinetic, kinematic and energetic consistency, as well as on the physical effects that they are able to reproduce adequately. The latter defines then the class of mechanical impact problems on which they can be applied successfully.

Important examples on which the impact law (1) may be applied are geometric and kinematic inequality constraints, representing hard unilateral contacts and one-way clutches, as well as planar and spatial Coulomb type friction laws. The impact laws (1) differ for the particular constraints only in the reservoirs of impulsive forces $-\mathcal{C}_i$, which are $\mathcal{C}_i = \mathbb{R}_0^-$ for the geometric and kinematic inequality constraints, $\mathcal{C}_i = [-1,1]\,\mu\,\Lambda_N$ for planar and $\mathcal{C}_i = \mathcal{D}_2\,\mu\,\Lambda_N$ for spatial Coulomb type friction. For the latter, μ denotes the friction coefficient, Λ_N is the associated impulsive force in normal contact direction, and $\mathcal{D}_2$ the two-dimensional unit disk.

In order to obtain a complete set of equations for resolving the impact, the Newton-Euler equations on impulsive level as well as the relative velocities $\boldsymbol{\gamma}^i$ need still to be introduced. For the former, one starts with the virtual work of the system by including the inertia terms and all the external forces, interprets the resulting balance law as an equality of measures, and integrates it over one singleton $\{t\}$ in time. This yields

$$\mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \sum_{i=1}^{n} \mathbf{W}^i \boldsymbol{\Lambda}_i \tag{3}$$

with $\mathbf{M}(\mathbf{q}_0)$ being the symmetric and positive definite mass matrix of the system evaluated at the location $\mathbf{q}_0$ of the impact, $\mathbf{u} = \dot{\mathbf{q}}$ a.e. the generalized velocities, and $\mathbf{F} = \sum \mathbf{W}^i \boldsymbol{\Lambda}_i$ the generalized impulsive force of all force elements i that may have survived integration over $\{t\}$. Each of the matrices $\mathbf{W}^i = (\mathbf{w}^{i_1}, \ldots, \mathbf{w}^{i_k})$ consists of i_k generalized force directions $\mathbf{w}^{i_j}(\mathbf{q}_0)$ that are necessary to characterize the force element i, and $\boldsymbol{\Lambda}_i = (\Lambda_{i_1}, \ldots, \Lambda_{i_k})^{\mathrm{T}}$ is the tuple of associated Lagrangian multipliers with the physical meaning of scalar impulsive forces. In addition, the relative velocities $\boldsymbol{\gamma}^i$ in each force element can be described as

$$\boldsymbol{\gamma}^i = \mathbf{W}^{i\mathrm{T}}\,\mathbf{u}\,. \tag{4}$$

The whole system (1)–(4) has now to be checked for kinetic, kinematic and energetic consistency. We call a system kinetically consistent if the impact equations (3) and the force restrictions $-\boldsymbol{\Lambda}_i \in \mathcal{C}_i$ are met, which is obviously the case for any post-impact velocity $\mathbf{u}^+$ determined from (1)–(4).

For ensuring kinematic consistency, each of the impact elements (1) has to be analyzed separately. For a unilateral geometric constraint, the values of the pre- and post-impact relative velocities have to have different signs. To ensure this in the impact law (1), the restitution coefficient has to be restricted to non-negative values, $\varepsilon^i \geq 0$. In contrast, the relative velocities at a unilateral kinematic constraints do not change sign at the impact. Therefore, $\varepsilon^i \leq 0$ has to be demanded. There are no kinematical restrictions for the friction elements. For them, the restitution coefficients can freely be chosen according to physical needs.

In a last step, energetic consistency has to be ensured which leads to additional restrictions on the restitution coefficients. The difference in kinetic energy for the post- and pre-impact configuration,

$$T^+ - T^- = \tfrac{1}{2}\,\mathbf{u}^{+\mathrm{T}}\mathbf{M}\mathbf{u}^+ - \tfrac{1}{2}\,\mathbf{u}^{-\mathrm{T}}\mathbf{M}\mathbf{u}^-\,, \tag{5}$$

may be stated with the help of (3) and (4) in two useful forms. The first one is

$$T^+ - T^- = \tfrac{1}{2}\sum_{i=1}^{n} \boldsymbol{\Lambda}_i^{\mathrm{T}}\boldsymbol{\xi}^i + \tfrac{1}{2}\sum_{i=1}^{n}(1-\varepsilon^i)\boldsymbol{\Lambda}_i^{\mathrm{T}}\boldsymbol{\gamma}^{i-}\,, \tag{6}$$

where $\boldsymbol{\xi}^i = \boldsymbol{\gamma}^{i+} + \varepsilon^i\boldsymbol{\gamma}^{i-}$. It holds that $\boldsymbol{\Lambda}_i^{\mathrm{T}}\boldsymbol{\xi}^i \leq 0$. This can be seen from (2) together with the fact that $0 \in \mathcal{C}_i$ for all impact elements considered here. Further, it can be shown [5, 1] that $\boldsymbol{\Lambda}_i^{\mathrm{T}}\boldsymbol{\gamma}^{i-} \leq 0$ for geometric unilateral constraints, which physically means that the impulsive force Λ_i always acts against the approaching velocity γ^{i-}. In other words, frictionless collisions are always energetically consistent as soon as the restitution coefficients are restricted to $\varepsilon^i \leq 1$. Together with the restrictions from kinematics, one ends up with $\varepsilon^i \in [0,1]$ which is the well-known interval from which the restitution coefficients are normally taken. One further sees from (6) that energetic inconsistency might appear when miscellaneous impact elements are contained in the system and that particular instances like slip reversal in the Coulomb type elements could cause energetic problems.

To further elaborate on sufficient conditions for energetic consistency, another form of the energy difference has proven to be useful,

$$T^+ - T^- = \boldsymbol{\Lambda}^{\mathrm{T}}(\mathbf{I}+\boldsymbol{\epsilon})^{-1}\boldsymbol{\xi} - \tfrac{1}{2}\,\boldsymbol{\Lambda}^{\mathrm{T}}(\mathbf{I}+\boldsymbol{\epsilon})^{-1}(\mathbf{I}-\boldsymbol{\epsilon})\mathbf{G}\boldsymbol{\Lambda}\,. \tag{7}$$

Here, $\boldsymbol{\Lambda}^{\mathrm{T}} = (\boldsymbol{\Lambda}_1^{\mathrm{T}}, \ldots, \boldsymbol{\Lambda}_n^{\mathrm{T}})$, $\boldsymbol{\xi}^{\mathrm{T}} = (\boldsymbol{\xi}^{1\mathrm{T}}, \ldots, \boldsymbol{\xi}^{n\mathrm{T}})$, $\boldsymbol{\epsilon} = \mathrm{diag}\,(\varepsilon^i)$, $\mathbf{I}$ denotes identity, and $\mathbf{G}$ is the symmetric and at least positive semi-definite Delassus operator defined by $\mathbf{G} = \mathbf{W}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{W}$ with $\mathbf{W} = (\mathbf{W}^1, \ldots, \mathbf{W}^n)$. As before, the first term $\boldsymbol{\Lambda}^{\mathrm{T}}(\mathbf{I}+\boldsymbol{\epsilon})^{-1}\boldsymbol{\xi}$ in (7) is non-positive as soon as the restitution coefficients are restricted to $\varepsilon^i > -1$. To ensure energetic consistency under all cases, one has to work on the second term which yields as sufficient conditions [4]

$$\frac{2\,\varepsilon_{\max}}{1+\varepsilon_{\max}} \leq \frac{1}{\mathrm{cond}\,\mathbf{G}} \qquad \text{or} \qquad \frac{\varepsilon_{\max}-\varepsilon_{\min}}{1-\varepsilon_{\min}\,\varepsilon_{\max}} \leq \frac{1}{\mathrm{cond}\,\mathbf{G}}\,, \tag{8}$$

where $\mathrm{cond}\,\mathbf{G} = \lambda_{\max}(\mathbf{G})/\lambda_{\min}(\mathbf{G})$ denotes the condition of $\mathbf{G}$. The first inequality in (8) guarantees energetic consistency for small, the second one for similar restitution coefficients. They contain as special cases the one of all restitution coefficients being equal to zero, and the one of all restitution coefficients being equal to each other, which can directly be verified from (7).

Although there are obvious restrictions on the validity and consistency of the standard inequality impact laws presented here, they have successfully been used in a great number of application problems. However, even frictionless collisions which are represented by geometric unilateral constraints can not completely be captured, because the local restitution coefficients in (1) do not fully parameterize the set of consistent post-impact velocities. The most prominent counter-example is Newton's cradle which is discussed within this context in [1].

In [3], energy increase has been reported for the frictional impact of a double pendulum against a rigid wall. For a particular configuration, the authors have been used normal restitution coefficients of $\varepsilon_N = 0.7$ and $\varepsilon_N = 0.5$ together with a friction coefficient of $\mu = 0.5$ to observe energy increase for slip and stick at the end of the impact, respectively. Nothing is said about the tangential restitution

coefficient, because the authors have implicitly assumed $\varepsilon_T = 0$. Apparently, the restitution coefficients here are not similar enough to ensure energetic consistency.

Even Coulomb type friction is not needed to cause energetic inconsistency. The most simple counter-example the authors have found is a system on $\mathbb{R} \times \mathbb{R}$ consisting of two masses with a unilateral geometric constraint between them and another unilateral kinematic constraint between one of the masses and the environment. The source for the inconsistency is the cone of admissible pre-impact generalized velocities, which is much too big in the sense that it is no longer the polar to the cone of transferable impulsive forces, as it is for systems with only unilateral geometric constraints. In the author's opinion, the same reasoning applies to the frictional impact at the double pendulum: It has been shown in [2] that Coulomb friction can be build up by a serial arrangement of unilateral kinematic constraints, which means that this most critical element is also always present when frictional impacts take place.

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General covariance and parametrization in classical field theory

Mark J. Gotay

(joint work with Marco Castrillón López and Jerrold E. Marsden)

To efficiently investigate the structure of classical field theories ("CFTs") it is useful to require that

(I) they be generally covariant, and

(II) all constituent fields be dynamic.

Examples of such CFTs include Einstein's theory of gravity, topological field theories and a nucleon moving in a dynamic Klein–Gordon field. But not all CFTs have these attributes. For instance, Nordstrøm's theory of gravity (see §17.6 of [7]), has a dynamic metric which propagates on a background Minkowskian spacetime, and thus contains an "absolute object" in the sense of [1]. This theory is not generally covariant, since (arbitrary) spacetime diffeomorphisms are not symmetries of the

Minkowski metric. A nonrelativistic free particle is not generally covariant either (in this case, general covariance means time-reparametrization covariance), even though the system contains no absolute objects.

If a given CFT does not possess attributes (I) and (II), we wish to modify it so as to produce a *physically equivalent* system in which every field is variational and which is spacetime diffeomorphism covariant.

As luck would have it, this is not always a straightforward task. Consider again Nordstrøm's theory; we could make it generally covariant by letting the Minkowski metric h "flap in the breeze," that is, letting spacetime diffeomorphisms move it. Then (I) will be satisfied, but (II) is more difficult to arrange. If one simply demands that h be variational then, as it does not derivatively couple in the Lagrangian L, we would have

$$0 = \frac{\delta L}{\delta h} = \frac{\partial L}{\partial h}\,!$$

To avoid this contradiction we must modify L, say by adding in a free field term for h. But it is absolutely unclear as to what that should be. In any case, one will end up with a modified CFT which cannot be expected to be physically equivalent to that with which we started. So we must be more subtle.

Here we restrict consideration to CFTs whose Lagrangians are built from dynamic matter or other fields and a non-dynamic background metric. Such theories often have at most the isometry group of the metric as a spacetime symmetry group. We will indicate, following [6] (see also [5]) and [2], how CFTs of this type can be **parametrized**, that is, made generally covariant, if one introduces spacetime diffeomorphisms themselves as dynamic fields. Many of the ideas herein are applicable to a wider range of field theories, as [6] already indicates, but in this paper we confine ourselves to this important class. The general case is presented in [4].

Our setup is the following. We start with a configuration bundle $Y \to X$ over the $(n+1)$-dimensional spacetime (X, g). Sections of this bundle, denoted ϕ, are the dynamic fields under consideration. We assume for simplicity that the theory is first order, and that the metric g does not derivatively couple to the ϕ (although our results below remain valid in the derivatively coupled case). Let

$$\mathcal{L} : J^1Y \times_X \mathrm{Met}(X) \to \Lambda^{n+1}X$$

be a Lagrangian density, where J^1Y is the first jet bundle of Y, $\mathrm{Met}(X)$ is the bundle whose sections are metrics on X, and $\Lambda^{n+1}X$ is the bundle of top forms on X. We suppose that a means of lifting a spacetime diffeomorphism $\sigma \in \mathrm{Diff}(X)$ to $\sigma_Y \in \mathrm{Aut}(Y)$ has been chosen, and set

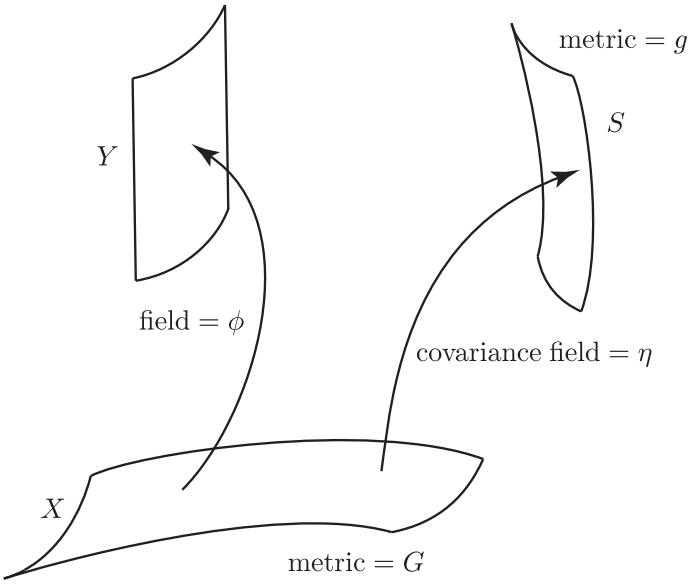
$$\sigma_Y \cdot \phi = \sigma_Y \circ \phi \circ \sigma^{-1}.$$

We further assume that $\mathcal{L}$ transforms as a scalar density under $\mathrm{Diff}(X)$. (This does *not* mean that $\mathrm{Diff}(X)$ is a symmetry group of the CFT since, as emphasized above, it will not preserve the metric g.)

We sidestep both the issues of making g variable, and then dynamic, in one fell swoop as follows. Introduce a copy (S, g) of spacetime into the fiber of the configuration bundle, and view diffeomorphisms $\eta : X \to S$ as sections of $S \times X \to X$. We regard the diffeomorphisms η as new fields—**covariance fields**—and correspondingly replace the configuration bundle by $\widetilde{Y} = Y \times_X (S \times X) \to X$. Next, replace the Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}X$ by $\tilde{\mathcal{L}} : J^1\widetilde{Y} \to \Lambda^{n+1}X$ defined according to

$$\widetilde{\mathcal{L}}(j^1\phi, j^1\eta) = \mathcal{L}(j^1\phi, \eta^* g). \tag{1}$$

Thus, we obtain a modified field theory with the underlying bundle $\widetilde{Y}$ and first order Lagrangian $\tilde{\mathcal{L}}$. The general set up is shown in the figure following.



The general set up for the introduction of covariance fields.

We pause to point out the salient features of this construction. First, the fixed metric g on spacetime is no longer regarded as living on X, but rather on the copy S of X in the fiber of the configuration bundle $\widetilde{Y}$. Consequently g is no longer considered a field—it has been demoted to a mere geometric object on the fiber S. Second, the role formerly played by g on X is now played by $G = \eta^* g$, which acquires its variability from that of η. Finally, we gain a field η which we allow to be dynamic; we will soon see that this imposes no restrictions on the theory at all.

We now show that with this construction we have attained goals (I) and (II). To this end define the lift σ_S of $\sigma \in \mathrm{Diff}(X)$ to the trivial bundle $S \times X$ by $\sigma_S(u, x) = (u, \sigma(x))$. Then on sections $\eta : X \to S \times X$ (i.e., diffeomorphisms $\eta\colon X \to S$) we have the action

$$\sigma_S \cdot \eta = \eta \circ \sigma^{-1}. \tag{2}$$

As our construction has removed the only absolute object (viz., g) from the ranks of fields (so it no longer necessary that it transforms under diffeomorphisms),

to establish (I) we need only verify that the new Lagrangian transforms as a scalar density. This is the content of the following result.

Theorem 1. *The Lagrangian density* $\widetilde{\mathfrak{L}}\colon J^1(Y\times_X(S\times X))\to\Lambda^{n+1}X$ *is* $\mathrm{Diff}(X)$*-equivariant, that is,*

$$\widetilde{\mathfrak{L}}\big(j^1(\sigma_Y\cdot\phi),j^1(\sigma_S\cdot\eta)\big)=\sigma_*\Big(\widetilde{\mathfrak{L}}\big(j^1\phi,j^1\eta\big)\Big).$$

for all $\sigma\in\mathrm{Diff}(X)$.

Proof. By construction and from (2), (1) yields

$$\begin{aligned}\widetilde{\mathfrak{L}}\left(j^1(\sigma_Y\cdot\phi),j^1(\sigma_S\cdot\eta)\right)&=\mathfrak{L}\left(j^1(\sigma_Y\cdot\phi),(\eta\circ\sigma^{-1})^*g\right)\\&=\mathfrak{L}\left(j^1(\sigma_Y\cdot\phi),(\sigma^{-1})^*(\eta^*g)\right)\\&=\sigma_*\big(\mathfrak{L}(j^1\phi,\eta^*g)\big)\\&=\sigma_*\Big(\widetilde{\mathfrak{L}}(j^1\phi,j^1\eta)\Big)\,.\end{aligned}$$

as was to be shown, where the third equality is a consequence of the fact that $\mathfrak{L}$ transforms as a scalar density. □

This property is the reason we call η the "covariance field." Thus $\mathrm{Diff}(X)$ acts on the modified CFT by symmetries and so we have attained goal (I).

Next we show something remarkable:

Theorem 2. *The Euler–Lagrange equation for the covariance field* η *is vacuous.*

This is why we can introduce η as a dynamic field with impunity, namely, its Euler–Lagrange equation does not add any new information to, or impose any restrictions upon, the system. Since, as is readily verified, the Euler–Lagrange equations for the fields ϕ remain unaltered, we see that *the parametrized system is physically equivalent to the original system.*

Proof. From (1) and the chain rule we find [2] that

$$\frac{\delta\tilde{L}}{\delta\eta^\nu}=0\Longleftrightarrow\nabla_\mu\mathfrak{T}^\mu{}_\nu=0$$

where $\mathfrak{T}$ is the SEM tensor density of the original field theory with the Lagrangian $\mathfrak{L}(j^1\phi,G)$:

$$\mathfrak{T}^{\mu\nu}=2\frac{\delta L}{\delta G_{\mu\nu}},\tag{3}$$

$G=\eta^*g$, and ∇ is the G-covariant derivative.

It is known from Proposition 5 in [3] that the SEM tensor is covariantly conserved when the metric G is the *only* nondynamic field. Thus, in our context, the equation $\nabla_\mu\mathfrak{T}^\mu{}_\nu=0$ is an identity, whence the result follows. □

Consequently we are free to suppose η is dynamic, and so we have accomplished goal (II): we have constructed a new field theory in which *all* fields are dynamic.

As an aside, it is interesting to compare the SEM tensors $\mathfrak{T}$ and $\widetilde{\mathfrak{T}}$ for the original and parametrized systems, respectively. Using [3] we compute that

$$\widetilde{\mathfrak{T}}^{\mu}{}_{\nu} = \mathfrak{T}^{\mu}{}_{\nu} - 2\frac{\partial L}{\partial G_{\mu\rho}} G_{\rho\nu}.$$

But then $\widetilde{\mathfrak{T}}^{\mu}{}_{\nu} = 0$ by the Hilbert formula (3). It follows that the SEM tensor density for the fully covariant, fully dynamic modified theory vanishes. One can also obtain this result directly by applying the generalized Hilbert formula (3.13) in [3] to the parametrized theory, since it is fully dynamic.

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Euler-Poincaré Flows on the Space of Tensor Densities and Integrable Systems

Partha Guha

The one–parameter family of shallow water equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{1}$$

where b is a real parameter, has recently drawn some attention. This equation is known as the *b-field equation*. It was introduced by Degasperis, Holm and Hone [3, 4], who showed the existence of multi-peakon solutions for any value of b, although only the special cases $b = 2, 3$ are integrable, having bihamiltonian formulations. The $b = 2$ case is the well-known Camassa-Holm (CH) equation [1] and $b = 3$ is the integrable system discovered by Degasperis and Procesi [5].

Using the Helmholz field $m := u - u_{xx}$, the DHH equation (1) allows reformulation in the compact form

$$m_t + um_x + bu_xm = 0 \ , \tag{2}$$

where the three terms correspond respectively to evolution, convection and stretching of the one-dimensional flow.

In a more recent paper Fokas et al. [6] proposed an algorithmic construction of (2+1) dimensional integrable systems which yield peakon/dromion type solutions.

$$(3)\quad q_{xt} - \nu q_{xxxt} + aq_{xy} + bq_{xxxy} + c\Big(q_{xx}q_y + 2q_xq_{xy}\Big) - c\nu\Big(q_{xxxx}q_y + 2q_{xxx}q_{xy}\Big).$$

This equation can be identified with the potential form of the Camassa-Holm analogue of the Calogero-Bogoyavlenskii-Schiff equation. For $\nu = 0$, this reduces to Camassa-Holm equation in potential form $u = q_x$. In this talk we study an Euler-Poincaré formulation of $(2+1)$-dimensional b-field equation.

Recently an Euler-Poincaré formalism has been studied for the Degasperis and Procesi (DP) equation. It turns out that the DP equation is the Euler-Poincaré flow on the combined space of Hill's (second order) and first order differential operators on circle. It has been generalized to the two component generalization of the DP equation. It has been shown also that the Hamiltonian structure obtained from the EP framework exactly coincides with the Hamiltonian structues of the DP equation obtained by Degasperis, Holm and Hone. In this paper we give a short derivation of the DP and the b-field equation using the deformation of vector field structure on S^1. We study a new kind of deformation of loop Virasoro algebra.

1. Euler-Poincaré framework of $1+1$-dimensional b-field equation

Denote $\mathcal{F}_\mu(S^1)$ the space of tensor-densities of degree μ on S^1

$$\mathcal{F}_\mu = \{a(x)dx^\mu \mid a(x) \in C^\infty(S^1)\},$$

where μ is the degree, x is a local coordinate on S^1. As a vector space, $\mathcal{F}_\mu(S^1)$ is isomorphic to $C^\infty(S^1)$.

Geometrically we say

$$\mathcal{F}_\lambda \in \Gamma(\Omega^{\otimes\lambda}), \qquad \text{where} \quad \Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda},$$

$\Omega = T^*S^1$ is the cotangent bundle of S^1. Here $\mathcal{F}_0(M) = C^\infty(M)$, the space $\mathcal{F}_1(M)$ and $\mathcal{F}_{-1}(M)$ coincide with the spaces of differential forms and vector fields respectively.

A vector field $f(x)\frac{d}{dx}$ acts on the space of tensor densities $\mathcal{F}_\mu$ by the Lie derivative

$$(4)\qquad L^\mu_{f(x)\frac{d}{dx}}(a(x)(dx)^\mu) = \Big(f(x)a'(x) + \mu f'(x)a(x)\Big)(dx)^\mu.$$

Definition 1.1. *The b- bracket between $v(x)\frac{d}{dx}$ and $w(x)\frac{d}{dx}$ is defined as*

$$(5)\qquad [v,w]_b = vw_x - (b-1)v_xw$$

This b-bracket can also be expressed as

$$(6)\qquad [v,w]_b = \frac{b}{2}[v,w] - \frac{b-2}{2}[v,w]^{sym},$$

where $[v,w] = vw_x - v_xw$ and $[v,w]^{sym} = vw_x + v_xw \equiv \partial_x(vw)$.

Remark. The b-bracket can be interpretred as an action of $Vect(S^1)$ on $\mathcal{F}_{-(b-1)}(S^1)$, a tensor densities on S^1 of degree $-(b-1)$. For $b=2$ this is just a vector field action corresponding to a Lie algebra. Moreover because of $[v,w]^{sym}$ term b-bracket is not a skew-symmetric bracket, it is a deformation of the bracket of vector fields.

There is a pairing

$$\langle , \rangle : \ \mathcal{F}_b \otimes \mathcal{F}_{1-b} \to \mathbb{R}$$

given by

$$\langle a(x)(dx)^b, b(x)(dx)^{1-b} \rangle = \int_{S^1} a(x)b(x)\,dx \tag{7}$$

which is $Diff(S^1)$-invariant. We can define a H^1 Sobolev metric on b-algebra according to

$$\langle a(x)(dx)^{-(b-1)}, b(x)(dx)^b \rangle_{H^1} = \int_{S^1} a(x)b(x)\,dx + \int_{S^1} a_x(x)b_x(x)\,dx.$$

Let us compute the coadjoint action with respect to the b–field equation.

Lemma 1.2.

$$(ad^{H^1})^*_f(u) = (1-\nu\partial^2)^{-1}[f(1-\nu\partial^2)u_x + bf_x(1-k\partial^2)u. \tag{8}$$

Proof: We know

$$\begin{aligned} &< ad^*_f(u), g >_{H^1} = - \ < u, [f,g]_b >_{H^1} \\ \equiv & - < udx^b, (fg' - (b-1)f'g)(dx)^{1-b} >_{H^1}, \end{aligned}$$

hence the pairing is well-defined. Let us compute

$$\begin{aligned} \text{R.H.S.} \ &= \int_{S^1} (ufg' - (b-1)uf'g)dx + \nu \int_{S^1} u'(fg' - (b-1)f'g)'dx \\ &= \int_{S^1} [f(1-\nu\partial^2)u' + bf'(1-\nu\partial^2)u \\ L.H.S. &= \int_{S^1} (ad^{H^1})^*_f u)gdx + \nu \int_{S^1} (ad^{H^1})^*_f u'g'dx \\ &= \int_{S^1} [(1-\nu\partial^2)ad^{H^1})^*_f u]gdx. \end{aligned}$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.
□

Using the Helholtz operator we express $m = (1-\nu\partial^2)u$. Thus, we express the Hamiltonian operator corresponding to (8) as

$$\mathcal{O}_1 = -(1-\nu\partial^2)^{-1}(m_x + bm\partial). \tag{9}$$

The Euler-Poincaré equation

$$u_t = \mathcal{O}_1 \frac{\delta H}{\delta u} \qquad \text{for} \ \ H = \int_{S^1} u^2\,dx,$$

can be rewritten as $m_t = \mathcal{O}\frac{\delta H}{\delta u}$, where $\mathcal{O} = -(m_x + bm\partial)$. Using the EP equation we construct b-field equation.

Proposition 1.3. *The Euler-Poincaré flow on the dual space of b-algebra yields the b-field equation*

$$m_t + m_x u + bmu_x = 0.$$

This is a new derivation of the b-field equation is given in [2]. Earlier [7] we gave a different derivation of the DP equation and its two component generalization.

1.1. **DHH Hamiltonian structure and EP framework.** Degasperis et al studied Hamiltonian structures for $b = 3$ case of DHH equation, in other words, they exhibits bihamiltonian features of the Degasperis-Procesi system. They expressed the Degasperis-Procesi equation as

$$m_t = B_i \frac{\delta H_i}{\delta m} \qquad i = 0, 1, \tag{10}$$

where $m = u - u_{xx}$. Thus they studied the flow of Helmholtz function. They showed that there is only one local Hamiltonian structure

$$B_0 = \partial_x(1 - \partial_x^2)(4 - \partial_x^2), \tag{11}$$

and the second Hamiltonian structure is given by

$$B_1 = m^{2/3}\partial_x m^{1/3}(\partial_x - \partial_x^3)^{-1} m^{1/3}\partial_x m^{2/3}, \tag{12}$$

which can be simplified to

$$B_1 \equiv \hat{B} = \frac{2}{9}(3m\partial + m_x)(\partial - \partial^3)^{-1}(3m\partial + 2m_x).$$

Proposition 1.4. *The Degasperis-Procesi equation*

$$m_t = \hat{B}\frac{\delta H_1}{\delta m}, \qquad \hat{B} = (3m\partial + m_x)(\partial - \partial^3)^{-1}(3m\partial + 2m_x) \tag{13}$$

is equivalent to $m_t = \mathcal{O}\frac{\delta H}{\delta u}$ *for* $H = \int_{S^1} u^2\, dx$, *where* $\mathcal{O} = (m_x + bm\partial)$.

Proof: Our goal is to show

$$\frac{2}{9}(\partial - \partial^3)^{-1}(3m\partial + 2m_x)\frac{\delta H_1}{\delta m} = \frac{\delta H}{\delta u},$$

where $H_1 = \frac{9}{4}\int_{S^1} m\, dx$. If we insert $\frac{\delta H_1}{\delta m} = \frac{9}{4}$ to left hand side of above equation we obtain $(\partial - \partial^3)^{-1} m_x = u$, where we use $u = (1 - \partial^2)^{-1} m$. Thus we obtain

$$m_t = (3m\partial + m_x)\frac{\delta H}{\delta u},$$

where $H = \frac{1}{2}\int_{S^1} u^2\, dx$.

Therefore the Degasperis-Holm-Hone form of Hamiltonian structure coincides with our Hamiltonian structure. □

Incidentally the proof of this section is in the same spirit of [7].

2. EP formalism for $2+1$-dimensional b-field equation

We wish to extend the Virasoro algebra to the case of two space variables. A natural way to do this is to consider the loops on it. One defines the loop group on $Diff(S^1)$ as follows

$$L\left(Diff(S^1)\right) = \left\{\phi : S^1 \to Diff(S^1) \,|\quad \phi \quad \text{is differentiable}\right\},$$

the group law being given by

$$(\phi \circ \psi)(y) = \phi(y) \circ \psi(y), \qquad y \in S^1.$$

In the similar way, we construct the Lie algebra $L(Vect(S^1))$ consisting of vector fields on S^1 depending on one more independent variable $y \in S^1$. The loop variable is thus denoted by y and the variable on the "target" copy of S^1 by x. The elements of $L(Vect(S^1))$ are of the form: $f(x,y)\frac{\partial}{\partial x}$ where $f \in C^\infty(S^1 \times S^1)$ and the Lie bracket reads as follows [8]

$$\left[f(x,y)\frac{\partial}{\partial x}, g(x,y)\frac{\partial}{\partial x}\right] = \left(f(x,y)\, g_x(x,y) - f_x(x,y)\, g(x,y)\right)\frac{\partial}{\partial x}. \tag{14}$$

It is easy to convince oneself that $L\left(Vect(S^1)\right)$ is the Lie algebra of $L\left(Diff(S^1)\right)$ in the usual weak sense for the infinite-dimensional case; a one-parameter group argumentation gives an identification between the tangent space to $L\left(Diff(S^1)\right)$ at the identity and $L\left(Vect(S^1)\right)$, equipped with its Lie bracket. The natural pairing between the loop Virasoro algebra and its dual is given by

$$\left\langle f(x,y)\,\frac{\partial}{\partial x}, v(x,y)\,dx^2\right\rangle = \int_{S^1\times S^1} f v\, dxdy. \tag{15}$$

2.1. Loop tensor density algebra and $2+1$-dimensional b-field equation.

Consider $\tilde{G}_1 = LG_1$ be the associated loop group corresponding to G_1 whose Lie algebra is given by

$$\tilde{\mathfrak{g}}_1 = L(\mathcal{F}_{-(b-1)}).$$

Consider an action of $L(Vect(S^1))$ on $L(\mathcal{F}_{-(b-1)})$

$$L_{f(x,y)\frac{\partial}{\partial x}}\left(g(x,y)\,(dx)^{-(b-1)}\right) = (f\, g_x - (b-1) f_x\, g)(dx)^{-(b-1)}, \tag{16}$$

this yields the loop b- bracket (14). Let us introduce H^1 norm on the loop tensor density algebra.

Definition 2.1. *The H^1- Sobolev norm on the loop tensor density algebra is defined as*

$$< f(x,y)(dx)^{-(b-1)}, v(x,y)(dx)^b >_{H^1} = \int_{S^1} f v dx + \nu \int_{S^1} \mathrm{d}_x f \mathrm{d}_x v dx. \tag{17}$$

We perform same type of calculation to compute the (deformed) coadjoint action $ad^*_{f(x,y)\frac{\partial}{\partial x}} v(x,y)(dx)^b|_{H^1}$. The Hamiltonian operator corresponding to this action is given by

$$\widehat{\mathcal{O}} \;=\; -(1-\nu\partial_x^2)^{-1}\left(\partial_x \tilde{v} + (b-1)\tilde{v}\partial_x\right) \tag{18}$$

where $\tilde{v} = (1 - \nu\,\partial^2)v$.

Proposition 2.2. *The Euler-Poincaré flow on the dual space of loop tensor algebra yields the* $2+1$*-dimensional b-field equation*

$$v_t - \nu v_{xxt} + (v_x - \nu v_{xxx})\partial_x^{-1} v_y + b(v - \nu v_{xx})v_y = 0 \tag{19}$$

where the Hamiltonian is given by $H = \frac{1}{2}\int_{S^1\times S^1} v\partial_x^{-1} v_y \, dxdy$*. The potential form* $v = q_x$ *yields the Fokas-Olver-Rosenau type systems.*

In our forthcoming paper we will give Euler-Poincaré formalism of various other $2+1$-dimensional integrable systems.

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Pursuit and Cohesion

P.S. Krishnaprasad

1. Talk Summary: Pursuit is a familiar mechanical activity that humans and animals engage in, – athletes chasing balls, predators seeking prey, insects maneuvering in aerial territorial battles. In this talk, we discussed strategies for pursuit, the occurrence in nature of a strategy known as *motion camouflage*, and some evolutionary arguments to support claims of prevalence of *this* strategy, as opposed to alternatives. We discussed feedback laws for a pursuer to realize motion camouflage, as well as two alternative strategies. We stated a dynamics in the probability simplex in three dimensions that captures an evolutionary game model of competition between these strategies. We sketched the analysis of this dynamics as an ascent equation solving a linear programming problem. Convergence to the

maximum is consistent with observed behavior in Monte-Carlo experiments.

In the final component of the talk, we discussed how pursuit strategies may be useful as building blocks for *coherent* structures, – flocks, swarms, schools, etc. We explored this aspect through the example of a two-body problem involving mutual motion camouflage.

The talk was confined to the setting of a pair of interacting particles in two dimensions.

2. Model: Consider a system of two interacting particles of unit mass in the plane, denoted as p (for pursuer) and e (for evader/pursuee). The speeds of the particles are constant, respectively 1 for p, and ν for e, where $0 < \nu < 1$. The particles interact through forces that leave invariant their speeds, and hence the forces are necessarily perpendicular to their respective velocities. Letting $\mathbf{r}_p$ and $\mathbf{r}_e$ denote the position vectors of p and e, and $(\mathbf{x}_p, \mathbf{y}_p)$ and $(\mathbf{x}_e, \mathbf{y}_e)$ respective *natural moving frames* attached to the particles, the forces on the particles are expressed as

$$\mathbf{f}_p = u_p \mathbf{y}_p \ ; \ \ \mathbf{f}_e = \nu^2 u_e \mathbf{y}_e$$

where u_p and u_e are respective signed curvatures of the particle trajectories. The equations of motion are:

$$\dot{\mathbf{r}}_p = \mathbf{x}_p; \ \ \dot{\mathbf{x}}_p = u_p \mathbf{y}_p; \ \ \dot{\mathbf{y}}_p = -u_p \mathbf{x}_p$$

and

$$\dot{\mathbf{r}}_e = \nu \mathbf{x}_e; \ \dot{\mathbf{x}}_e = \nu u_e \mathbf{y}_e; \ \dot{\mathbf{y}}_e = -\nu u_e \mathbf{x}_e.$$

The state space of the system consisting of all possible pairs $((\mathbf{r}_p, \mathbf{x}_p, \mathbf{y}_p), (\mathbf{r}_e, \mathbf{x}_e, \mathbf{y}_e))$ is $SE(2) \times SE(2)$, the cartesian product of two copies of the rigid motion group in the plane. Interactions of the particles arise from state-dependent curvatures (controls). By a *strategy* we mean an $SE(2)$-invariant submanifold of the state space. The pursuit problem is to design an $SE(2)$-invariant feedback law for u_p that makes a strategy (approximately) accessible, irrespective of the actions of the evader/pursuee.

3. Strategies: Particular strategies may be discernible in nature as fulfilling certain ecological imperatives. Three strategies are identified by suitable anti-parallelism conditions in state space (minus collision set). These are:

Classical Pursuit (CP) defined by $\Lambda = \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|} \cdot \mathbf{x}_p = -1$;

Constant Bearing (CB$_\theta$) defined by $\Lambda_\theta = \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|} \cdot Rot(\theta)\mathbf{x}_p = -1$, where $Rot(\theta)$ denotes rotation by a specific angle θ;

and Motion Camouflage (MC) defined by $\Gamma = \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|} \cdot \frac{\mathbf{x}_p - \nu \mathbf{x}_e}{|\mathbf{x}_p - \nu \mathbf{x}_e|} = -1$.

Definition: Motion Camouflage (MC) is accessible in finite time if for any $\epsilon > 0$, there is a $t_1 > 0$ and a feedback law for u_p such that $\Gamma(t_1) \leq -1 + \epsilon$.

(Analogous definitions apply for the other strategies.)

4. Feedback Laws: Let $\mathbf{r} = \mathbf{r}_p - \mathbf{r}_e$ and let $\mathbf{a}^\perp$ denote the vector obtained by rotating $\mathbf{a}$ through $\pi/2$ in the counter-clockwise direction. In their work (see **Notes on References**), Justh and Krishnaprasad showed:

Theorem For given system and feedback law

$$u_p = -\mu \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^\perp \right),$$

with hypotheses

(i) $0 < \nu < 1$
(ii) u_e is continuous and $|u_e|$ is bounded
(iii) $\Gamma_0 = \Gamma(t)\Big|_{t=0} < 1$
(iv) $|\mathbf{r}(0)| > 0$,

MC is accessible in finite time, for choice of gain μ high enough.

Feedback laws for other strategies and settings have been derived. See **Notes on References**.

5. Prevalence and a Game: In empirical studies of dragonflies engaged in territorial battles and echolocating bats chasing evasive insect prey, the motion camouflage strategy MC has been observed in a statistically significant way. The prevalence of MC over other strategies was explored by Wei, Justh and Krishnaprasad using Monte-Carlo trials and an evolutionary game model, viewed as repeated bat (p) and insect (e) encounters (see **Notes on References**). Let p_i $i = 1, 2, 3$ denote the proportion of trials (out of N trials in each generation) employing strategy i, where $i = 1$ corresponds to CB_θ, $i = 2$ corresponds to CP, and $i = 3$ corresponds to MC. The *fitness* of the i^{th} strategy in each generation is $W_i = \frac{1}{N} \sum_{k=1}^{N} (\tau_k^i)^{-1}$, where τ_k^i = time-to-prey-capture in trial k using strategy i. The probability vector $\mathbf{p} = (p_1, p_2, p_3)$ is updated from generation-to-generation by the rule $p_i \longrightarrow p_i \frac{W_i}{\bar{W}}$ where $\bar{W} = \sum_{i=1}^{3} p_i W_i$. It is observed that $\mathbf{p} \longrightarrow (0, 0, 1)$, that is, MC prevails. A deterministic ordinary differential equation limit of this update rule supports the Monte-Carlo experimental results. This equation is an ascent equation in the simplex for a linear programming problem. The underlying Riemannian geometry (associated with Fisher, Rao and Shahshahani) is of interest to information theorists, game theorists and researchers in mathematical population genetics.

6. Mutual Pursuit and Cohesion: Letting the particles cooperate by using the (essentially same) mutual pursuit law of the MC type:

$$u_p = u = -\mu \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^{\perp} \right)$$
$$u_e = \frac{u}{\nu} = -\frac{\mu}{\nu} \left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \dot{\mathbf{r}}^{\perp} \right)$$

we observe that the resulting dynamics is reducible to a two dimensional system with an interesting hamiltonian structure. There are parallels to earlier examples investigated by Yavuz Nutku(1990). We consider this (oscillator) system to be a building block for further work in cohesion laws.

7. Notes on References: In joint work of Eric Justh and P.S. Krishnaprasad (2006), the mathematical framework for investigation of motion camouflage was introduced and a basic accessibility theorem was proved in two dimensions.

In the joint work of Reddy-Justh-Krishnaprasad (2006,2007) feedback laws for three dimensions and sensorimotor delay in motion camouflage were derived.

In joint work of Galloway-Justh-Krishnaprasad (2007), the case of stochastic inputs was analyzed.

In the work of Wei-Justh-Krishnaprasad (2008), strategy selection is explored using an evolutionary game approach which goes back to the work of Maynard Smith and Price (1973) and Taylor and Jonker (1978).

The empirical data for study of motion camouflage in dragonflies originates in the paper of Mizutani, Chahl and Sreenivasan (2003). The empirical data for prey capture behavior in echolocating bats is first presented in Ghose, Horiuchi, Krishnaprasad and Moss (2006).

8. Collaborators: Work on motion camouflage was begun with Eric W. Justh. Other collaborators who have since joined the project are P. Viswanadha Reddy, Kevin Galloway and Matteo Mischiati. The work on Mutual Pursuit and Cohesion discussed here is with Mischiati. Empirical work involving bat echolocation and prey capture behavior is joint with Kaushik Ghose, Timothy Horiuchi and Cynthia Moss. Monte-Carlo simulation and evolutionary game approach to strategy selection is joint with Ermin Wei and Justh.

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Lagrangian Coherent Structures, Homoclinic Tangles, and Hurricanes

Jerrold E. Marsden
(joint work with Philip Du Toit)

This talk concerned a basic technique in dynamical systems, namely Lagrangian Coherent Structures (LCS). This is a technique that is used for determining mixing properties of fluid flows as well as transport barriers. The basic idea is to use Finite Time Liapunov Exponents (FTLE) to extract the time dependent analog of Invariant Manifolds. This technique is applied to wind field data for several tropical cyclones (hurricanes). The resulting structures reveal sharply defined boundaries that take the form of homoclinic tangles with lobe dynamics from classical geometric dynamics, which are ideas going back to Poincaré and further analyzed by Smale. The analysis indicates that the dominant transport mechanism for large scale (exterior to the eye) flow into and out of typhoons is via lobe dynamics.

Introduction. Recently, Lagrangian methods using Finite Time Liapunov Exponents (FTLE) have been developed to uncover the underlying skeletal structure that dictates how transport occurs in aperiodic flows; see [3], [9] and [6]. Interestingly, these Lagrangian methods reveal well-defined surfaces in the flow that act as barriers to transport and separate regions of different dynamical behavior. This method has been applied to many situations, such as transport of pollutants (see [4]) and for detailed studies of the structure of vortices in turbulent flows (see [2].

We apply this method to extract LCS to the manifestly turbulent wind field data for hurricanes. A main result is the discovery of sharply defined surfaces in the flow surrounding the hurricane that govern the transport of air both into and out of the storm. Furthermore, the evolution of these surfaces indicate very plainly that transport in the large-scale flow occurs via the mechanism of *lobe dynamics* associated with a *homoclinic tangle*, a process well-understood in classical geometric dynamics (see [7] and [10]). The LCS method reveals that transport in hurricanes is a low-dimensional process whose salient features are adequately described by a simple two-dimensional chaotic tangle.

We begin with a brief review of the mathematical definitions and practical implementation of the LCS method as first proposed by [3] and later developed by [9]. We then illustrate the LCS method by applying it to an actual hurricane data set. The transport structures uncovered clearly show the low-dimensionality of the transport process in the hurricane flow and its similarity with lobe dynamics one sees in simple planar time dependent systems. Here we focus on the 2D case, although LCS ideas and computations extend to the 3D case as well (see [5]). These extensions are important when studying, for instance, the eye wall structure of hurricanes.

Finite Time Liapunov Exponents. In many time dependent flows, inspection of the velocity field or streamlines yields little insight into Lagrangian transport in the flow and can lead to erroneous conclusions about flow structure. Furthermore, an attempt to uncover Lagrangian information about the flow by simply integrating particle trajectories at different locations and times very quickly leads to 'spaghetti' plots that are also not helpful. However, the LCS method provides a systematic approach for analyzing aperiodic flows and extracting the coherent structures that govern transport. The method has been successfully applied to a wide range of flows including HF radar data for oceanic flows [4], numerically modeled data for flow over airfoils, laboratory flows for vortex rings, flows surrounding jelly-fish [8], and flows in the phase space of the dynamical system defined, for instance, by the three body problem, in which the resulting coherent structures dictate transport of trajectories in phase space [1].

Let the open set $D \subset \mathbb{R}^2$ be the domain of interest in the flow, and let $\mathbf{v} : D \times \mathbb{R} \to TD$ be a smooth time-dependent vector field on D. Consider a trajectory $\xi_{(\mathbf{x}_0,t_0)} : \mathbb{R} \to D$ that passes through $\mathbf{x}_0 \in D$ at time $t_0 \in \mathbb{R}$; that is, the unique time parametrized curve in D whose tangent vectors satisfy:

$$\frac{d\xi_{(\mathbf{x}_0,t_0)}(t)}{dt} = \mathbf{v}(\xi_{(\mathbf{x}_0,t_0)}(t), t)$$

and the initial condition

$$\xi_{(\mathbf{x}_0,t_0)}(t_0) = \mathbf{x}_0 \,.$$

Let $\Phi_{t_0}^{t_0+T} : D \to D$ be the the associated flow map, so that

$$\Phi_{t_0}^{t_0+T}(\mathbf{x}_0) := \xi_{(\mathbf{x}_0,t_0)}(t_0 + T) \,.$$

The FTLE, denoted $\sigma_T : D \times \mathbb{R} \to \mathbb{R}$, is a time-dependent scalar field on D defined by

$$\sigma_T(\mathbf{x}, t) := \frac{1}{|T|} \ln \left\| \frac{d\Phi_t^{t+T}(\mathbf{x})}{d\mathbf{x}} \right\|_2 .$$

The FTLE defined in this way is a measure of the separation of trajectories induced by the flow over the interval of time $[\, t \,,\, t + T \,]$. Following Shadden [9], we define the LCS to be locally maximizing surfaces, or 'ridges', in the scalar field σ_T. Hence, LCS can be thought of as surfaces of greatest separation. Review of the LCS computed for the wide range of flows in the previously-mentioned examples indicates that LCS defined in this way are remarkably sharp ridges that act as invariant separatrices in the flow.

The definition of FTLE admits both positive and negative flow times T. For positive values of T, the FTLE measures separation forward in time and yields LCS that act as *repelling* surfaces; while for negative values of T, the FTLE measures separation backward in time and hence yields LCS that act as *attracting* surfaces in forward time.

In order to extract the LCS for a given flow, we require as an input the velocity field defined by an analytical expression, or the velocity data recorded discretely

in space and time. Given a velocity field in either of these forms, we begin by numerically approximating the flow map at each desired output time by numerically integrating a uniform grid of trajectories. The requisite derivatives of the flow map are then computed using finite differencing. Once the numerical approximation for $\frac{d\Phi_t^{t+T}(\mathbf{x})}{d\mathbf{x}}$ is obtained, calculation of the FTLE follows in a straightforward manner from the definition. The most computational intensive aspect of the FTLE calculation is the integration of trajectories for each time frame and for each location on the spatial grid. Since these integrations are all independent, they may be computed in parallel, allowing for dramatic speed-up of the computation. For example, the computation of the FTLE for a typical velocity field on a 512 x 512 spatial grid at 64 time frames requires approximately 1 minute on 64 processors.

Transport in Typhoon Nabi. We have applied the method of LCS to extract coherent structures in the velocity field data for a Pacific typhoon. The data is obtained from the NCAR-NCEP repository at `http://www.cdc.noaa.gov/cdc/data.ncep.reanalysis.html`. Specifically, we use the two-dimensional velocity field at the 850mb pressure level over the Western Pacific for late summer of 2005. The particular typhoon of interest during this period is Typhoon Nabi, a category 5 tropical storm, that made landfall in Japan on September 6, 2005. A snapshot of the velocity field provided in the data set is shown in figure 1.

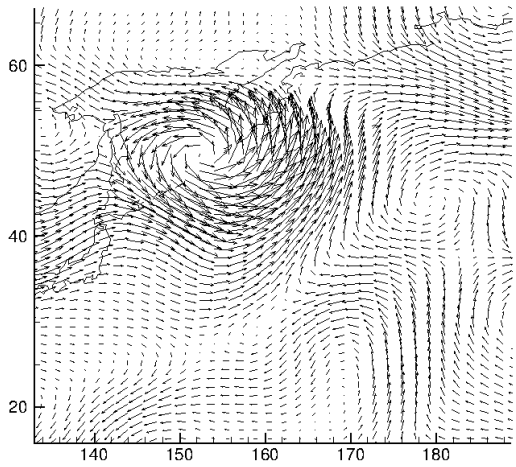


FIGURE 1. Snapshot of the velocity field above the Western Pacific at the 850mb pressure level provided by the NCAR-NCEP reanalysis data set. The prominent vortical flow is associated with Typhoon Nabi (2005).

Computation of the FTLE was performed with an integration time $T = 120$ hours. Snapshots depicting both the repelling and attracting LCS are shown in Figure 2.

Despite the complex flow surrounding the typhoon, we observe that the repelling and attracting LCS accurately capture the boundary of the storm vortex and has the shape of the homoclinic connection. Furthermore, the evolution of the LCS reveals that the transport mechanism that governs entrainment into and detrainment out of the typhoon across this boundary is lobe dynamics. For example,

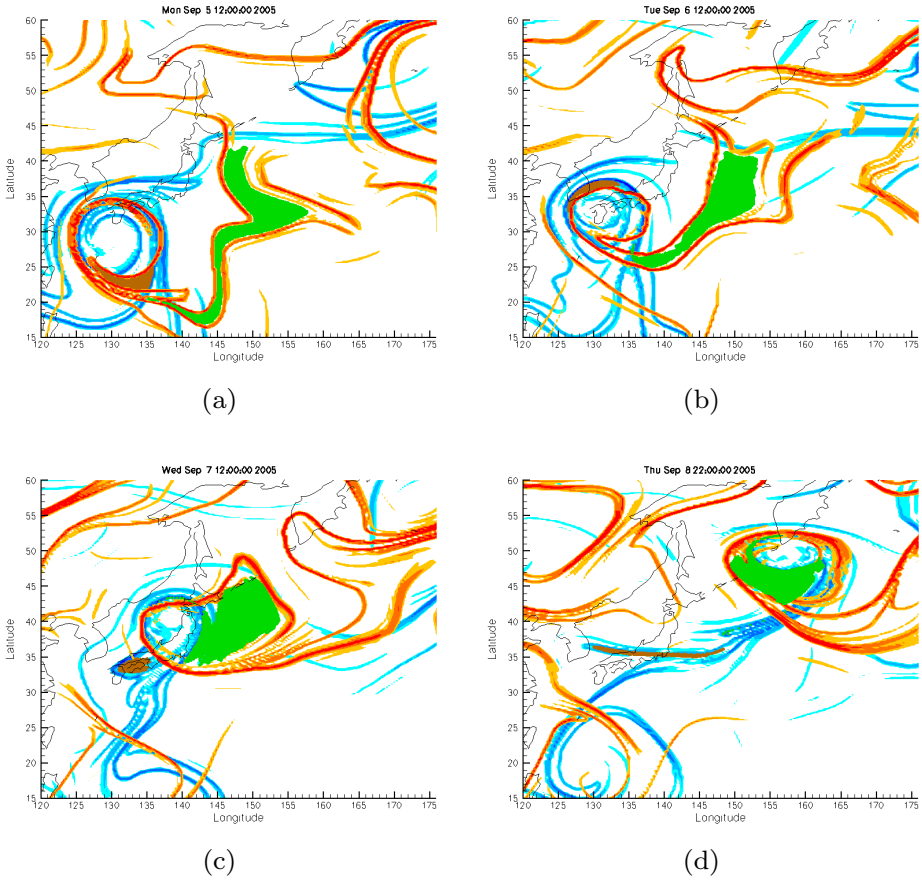
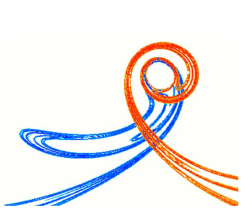


FIGURE 2. LCS for typhoon Nabi. The intersections of the repelling (red) and attracting (blue) LCS define lobes that enclose regions of fluid that will be either entrained into or detrained out of the cyclone. For clarity, only two lobes have been colored although many more are evident during the animation. The LCS reveals that transport into and out of the cyclone is well-described by lobe dynamics. Indeed, the LCS forms a boundary to the cyclone that is a homoclinic tangle. Over the three day period shown, the green lobe is entrained, while the brown lobe is detrained.

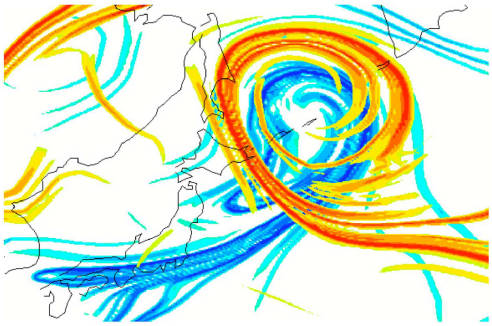
the region of fluid colored brown and enclosed by the intersection of the repelling and attracting LCS is a lobe that will be detrained out of the hurricane, while the region colored green is a lobe that will be entrained into the storm. Indeed, without computing the LCS, the location of the boundary to the storm is not clear, and hence the concepts of detrainment and entrainment are not well-defined. In the literature, plots of vorticity and humidity are typically provided to indicate the size and location of the storm. These quantities are indeed correlated with the motion of the vortex, but do not provide sharp boundaries so that the transport mechanisms can be inferred.

Computations of the LCS for several typhoons has revealed that mixing via lobes in a homoclinic tangle is a generically dominant feature in tropical storms.

Figure 3 shows the homoclinic tangles revealed via LCS computations for Pacific Typhoons Tip (1979), Banyan, and Nabi (2005) with the homoclinic tangle computed in the flow of a simple explicit 2d time dependent ode model provided for comparison.

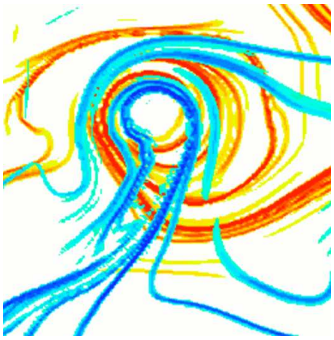


(a) LCS in a simple 2D ode model



(b) LCS in Typhoon Nabi.

(c) LCS in Typhoon Banyan



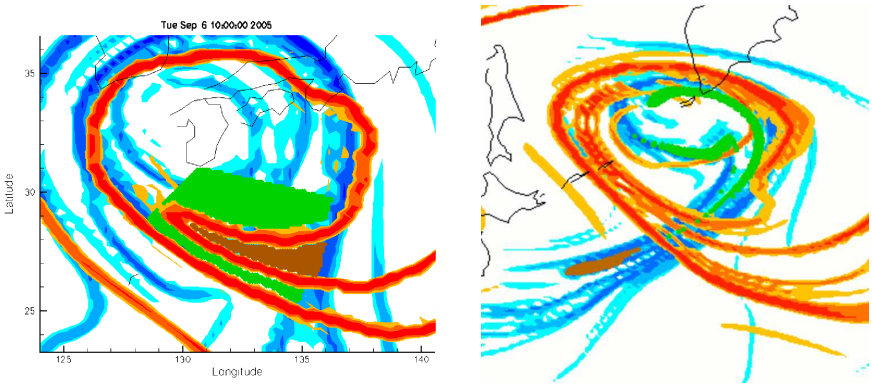
(d) LCS in Typhoon Tip

FIGURE 3. The LCS computed for a simple planar ode model is shown in (a) and compared with the LCS computed for the wind fields in three other Pacific typhoons (b)-(d). The homoclinic tangle structure is evident in all three storms and indicates that lobe dynamics is a dominant transport structure in tropical storms.

The similarity between the LCS computed for Typhoon Nabi wind field data and the LCS computed for the simple kinematic model is quite striking. In figure 2(a), we see that the LCS defines a boundary to the vortex, as well as lobes both inside and outside this boundary. The LCS dictate how the lobes will be transported by the flow, and how the processes of entrainment and detrainment from the vortex will occur. More revealing movies of this process may be viewed online at `http://www.cds.caltech.edu/ pdutoit/Hurricanes`. In the movies, one sees that the structure of the flows is remarkably similar in the real flow and in the simple 2D ode model, despite the complexity of the atmospheric flow compared with the low-dimensionalty of the simple model. To this point, we assert that the computations of the LCS not only allow us to identify and characterize the dominant transport mechanism in hurricane flows, but also to observe that

the essential structure of that transport mechanism can be faithfully reproduced by a very simple low-dimensional model.

Careful study of the LCS computed for hurricanes flows at fine detail reveals all the intricacies of the homoclinic tangle. For instance, the action of the Smale horseshoe map and the first iteration in the formation of a Cantor set can be discerned in the evolution of the LCS. These concepts are typically studied with regard to the Poincaré maps of periodically perturbed systems and are presented in abstraction, whereas here we see quite surprisingly that the FTLE method for extracting LCS uncovers these very same notions in the seemingly unrelated turbulent and aperiodic flows of hurricane data and presents them in a way that their evolution can be observed naturally through animation. Figure 4 illustrates how the central "third" region shaded brown is advected out of the storm, while the outer two "thirds" shaded green remain inside the storm, just as is prescribed by a single iteration of the Smale horseshoe map in the formation of the Cantor set, as in [10].



(a) (b)

FIGURE 4. The action of lobe dynamics in the homoclinic tangle induces a map that, as has been shown by Smale, leads to the formation of a Cantor set when iterated. Here we see the first iteration of this 'horseshoe' map in the flow of the typhoon: the middle 'third', colored brown, is removed from the storm, while the two outer 'thirds' remain.

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Lie groups and plasticity at finite strain

Alexander Mielke

Introduction. The theory of polyconvex materials introduced in [Bal76] provided a basis for a general theory of elastostatics that allows us to treat the geometric nonlinearities arising in physically correct models. A stored-energy density $W : \mathbb{R}^{d\times d} \to \mathbb{R}\cup\{\infty\}$ has to satisfy frame indifference $W(RF) = W(F)$ for $R \in \mathrm{SO}(d)$ and $F \in \mathbb{R}^{d\times d}$, blow up for volume compression $W(F) \to \infty$ for $\det F \to 0^+$, and local non-self-interpenetration $W(F) = \infty$ for $\det F \le 0$. Thus the proper domain for W is $\mathrm{GL}(d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}$ rather than $\mathbb{R}^{d\times d}$.

In plasticity the strain tensor F or its linearized version $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ are decomposed into an elastic and a plastic part, where the latter evolves under time-dependent loadings according to a suitable flow law. While in the case of small strains the decomposition $e(u) = e_{\mathrm{el}} + e_{\mathrm{pl}}$ is additive, elastoplasticity at finite strain is based on the multiplicative decomposition $\nabla\varphi = F = F_{\mathrm{el}}F_{\mathrm{pl}}$, introduced in [Lee69]. Here $P \stackrel{\mathrm{def}}{=} F_{\mathrm{pl}}$ lies in the plastic Lie group $\mathbf{P}$, which is usually chosen to be $\mathrm{SL}(d) = \{\, F \in \mathbb{R}^{d\times d} \mid \det F = 1 \,\}$. This decomposition is often used in engineering and is quite successful in predicting macroscopic deformation processes like deep drawing and other forming processes [SiO85, MiS92, NeW03]. A major advance was the observation in [OrS99] that the time-incremental problems in rate-independent and in the viscoplastic case can be written as minimization problems for the sum of the increments in the stored energy and in the dissipated energy to obtain the state at the next time level [OrR99, CHM02, Mie03, GM*06].

The issue of this note is to address the question how one can use the methods of the calculus of variations to combine the tools from linear functional analysis (like

weak convergence) with the strong geometrical nonlinearities inherent to elastoplasticity at finite strain. In fact, so far this is only possible in the case that the flow law is rate independent, which is a common assumption in engineering.

The mechanical model. Consider the body $\Omega \subset \mathbb{R}^d$ in the reference configuration and let $\varphi : \Omega \to \mathbb{R}^d$ denote the deformation, $P : \Omega \to \mathrm{SL}(d) = \{ P \in \mathbb{R}^{d\times d} \mid \det P = 1 \}$ the plastic tensor, and $p : \Omega \to \mathbb{R}^m$ some hardening variables. Then, we assume that the stored-energy functional takes the form

$$\mathcal{E}(t, \varphi, P, p) = \int_\Omega W_{\mathrm{el}}(x, \nabla\varphi, P) + W_{\mathrm{h,r}}(P, p, \nabla P, \nabla p) \ \mathrm{d}x - \langle \ell(t), \varphi \rangle.$$

Here the gradients $(\nabla P, \nabla p)$ introduce a length scale for the materials, which is essential to provide compactness and prevent formation of microstructures. Elastic equilibrium is obtained via

$$\varphi \text{ minimizes } \mathcal{E}(t, \cdot, P, p) \text{ subject to } \varphi = g_{\mathrm{Dir}} \text{ on } \Gamma_{\mathrm{Dir}}. \tag{1}$$

To formulate the plastic flow law we combine the plastic variables into one inner variable $z = (P, p) \in Z \stackrel{\mathrm{def}}{=} \mathbf{P}\times\mathbb{R}^m$. Using a dissipation metric $R : \mathrm{T}Z \to [0, \infty]$ the flow law can be written as an internal force balance (Biot's law) that is equivalent to the "principle of maximal dissipation":

$$0 \in \partial_{\dot z} R(z, \dot z) + \begin{pmatrix} \partial_P W_{\mathrm{el}}(F,P) \\ 0 \end{pmatrix} + \partial_z W_{\mathrm{h,r}}(z, \nabla z) - \mathrm{div}\, \partial_{\nabla z} W_{\mathrm{h,r}}(z, \nabla z). \tag{2}$$

Rate independence is enforced by the assumption that $R(z, \cdot) : \mathrm{T}_z Z \to [0, \infty]$ is homogeneous of degree 1, which implies that the subdifferential $\partial_{\dot z} R(z, \cdot)$ is homogeneous of degree 0, and hence insensitive to the modulus of the argument.

The invariances we have to satisfy are *frame indifference* and *plastic invariance*:

$$W_{\mathrm{el}}(RFQ, PQ) = W_{\mathrm{el}}(F, P) \quad \text{and} \quad R(PQ, p, \dot P Q, \dot p) = R(P, p, \dot P, \dot p) \tag{3}$$

for all $F \in \mathrm{GL}(d)$, $(P, p) \in \mathbf{P}\times\mathbb{R}^m$, $R \in \mathrm{SO}(d)$, and $Q \in \mathbf{P}$. Here plastic invariance means that previous distortions Q of the crystallographic lattice are forgotten. Note that this invariance does not apply to the hardening and regularizing energy contribution $W_{\mathrm{h,r}}$. In particular, (3) implies the multiplicative decomposition $F_{\mathrm{el}} = FP^{-1}$ via the existence of $\widehat{W}$ and $\widehat{R}$:

$$W_{\mathrm{el}}(F, P) = \widehat{W}(FP^{-1}) \quad \text{and} \quad R(P, p, \dot P, \dot p) = \widehat{R}(p, \dot P P^{-1}, \dot p). \tag{4}$$

Energetic formulation. A proper solution theory for the system (1) and (2) is so far not available. Hence we concentrate on the so-called energetic solutions that are especially adapted to rate-independent processes with strong nonlinearities, see [MaM05, Mie05, FrM06].

Since solutions may develop jumps in time we replace the (infinitesimal) dissipation metric R by the dissipation distance $D : Z\times Z \to [0, \infty]$ such that $D(z_0, z_1)$ is the infimum of $\int_0^1 R(z(s), \dot z(s)) \ \mathrm{d}s$ over all smooth curves z starting in z_0 and ending in z_1. With this we define the functional $\mathcal{D}(z_0, z_1) = \int_\Omega D(x, z_0(x), z_1(x)) \ \mathrm{d}x$, and emphasize that $\mathcal{D}$ satisfies a triangle inequality.

We choose now a suitable state space $\mathcal{Q}$ for the triple $\mathbf{q} = (\varphi, P, p)$ which satisfies $\varphi = g_{\mathrm{Dir}}$ on Γ_{Dir} and $(P(x), p(x)) \in Z$ in Ω. Then, a function $\mathbf{q} : [0, T] \to \mathcal{Q}$ is

called an *energetic solution* for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ if for all $t \in [0,T]$ the *stability condition* (S) and the *energy balance* (E) hold:

$$(5)\qquad \begin{array}{ll} \text{(S)} & \mathcal{E}(t,\mathbf{q}(t)) \le \mathcal{E}(t,\widehat{\mathbf{q}}) + \mathcal{D}(\mathbf{q}(t),\widehat{\mathbf{q}}) \text{ for all } \widehat{\mathbf{q}} \in \mathcal{Q}, \\ \text{(E)} & \mathcal{E}(t,\mathbf{q}(t)) + \mathrm{Diss}_{\mathcal{D}}(\mathbf{q};[0,t]) = \mathcal{E}(0,\mathbf{q}(0)) - \int_0^t \langle \dot{\ell}(t), \varphi(t)\rangle \ \mathrm{d}s, \end{array}$$

where $\mathrm{Diss}_{\mathcal{D}}(\mathbf{q};[r,s]) = \sup \sum_1^N \mathcal{D}(P(\tau_{j-1}),p(\tau_{j-1}),P(\tau_j),p(\tau_j))$ taken over all partitions of $[r,s]$.

The advantage of the energetic formulation via (S) and (E) is that it is totally derivative-free and purely geometric, i.e., independent of choices of coordinates. Thus it is especially adapted to treat strongly nonlinear problems. However, to obtain existence results one has to combine it with some functional analysis.

Existence result. We choose $\mathcal{Q}$ as a weakly closed subset of the Sobolev space

$$\mathrm{W}^{1,q_{\mathrm{Y}}}(\Omega;\mathbb{R}^d) \times \big(\mathrm{W}^{1,r}(\Omega;\mathbb{R}^{d\times d}) \cap \mathrm{L}^{q_{\mathrm{P}}}(\Omega;\mathbb{R}^{d\times d})\big) \times \mathrm{W}^{1,r}(\Omega;\mathbb{R}^m)$$

by asking the pointwise restrictions given in the previous sections. The essential conditions are the *polyconvexity* of $\widehat{W}$ in (4) and the *coercivity*

$$\widehat{W}(F_{\mathrm{el}}) + W_{\mathrm{h,r}}(P,p,\nabla P,\nabla p) \ge c\big(|F_{\mathrm{el}}|^{q_{\mathrm{E}}} + |P|^{q_{\mathrm{P}}} + |p|^r + |(\nabla P,\nabla p)|^r\big) - C.$$

Under some more technical conditions and the crucial assumptions

$$\frac{1}{q_{\mathrm{E}}} + \frac{1}{q_{\mathrm{P}}} = \frac{1}{q_{\mathrm{Y}}}, \quad q_{\mathrm{Y}} > d, \quad r > 1,$$

it is shown in [MaM08] that energetic solutions in the sense of (5) exist. The proof uses the abstract ideas developed in [FrM06] to complete the ideas developed in [CHM02, Mie04, MiM06], where only finitely many time-incremental minimization steps were considered.

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Bifurcations of relative equilibria at zero momentum

James Montaldi

We consider Hamiltonian systems with $G = \mathbf{SO}(3)$ symmetry, and in particular where $\mathbf{SO}(3)$ is acting freely on the phase space P. Zero momentum is special as it is the only momentum value which is fixed by the whole group of rotations. Indeed the symplectic reduced spaces P_μ have dimension $\dim P - 3 - \dim G_\mu$ ($3 = \dim G$) so that for $\mu \neq 0$, $\dim P_\mu = \dim P_0 + 2$. This difference in the geometry of reduction has implications for the dynamics; here we discuss the implications for the family of relative equilibria in a neighbourhood of one with zero momentum. In particular, we discuss what typical sets of relative equilibria look like, and how this set depends on the choice of Hamiltonian H and how it changes when the Hamiltonian is varied.

To analyze such questions, the first step is to quotient by the group action and to consider the reduced dynamics on P/G. Let $p \in P$ be a point with zero momentum: $J(p) = 0$, and write $\bar{p}$ for the corresponding point in $P_0 \subset P/G$. In a neighbourhood of $\bar{p}$, one can write

$$P/G \simeq P_0 \times \mathfrak{g}^*,$$

where $\mathfrak{g} = \mathfrak{so}(3)$ is the Lie algebra of $\mathbf{SO}(3)$, and $\mathfrak{g}^*$ its dual [4]. The reduced space P_0 can often be thought of as the phase space associated to shape dynamics, while $\mathfrak{g}^*$ corresponds to rotational motion, as for a rigid body. An application of the implicit function theorem reduces the search for relative equilibria to finding the critical points of the restriction of a function $h : \mathfrak{g}^* \to \mathbf{R}$ to the spheres $|\mu|^2$=const, see [4, 5] for details. If we write $\phi(\mu) = |\mu|^2$ (the Casimir on $\mathfrak{g}^*$), then this is equivalent to finding the singular points of the reduced energy-momentum map (or energy-Casimir map) $(h, \phi) : \mathfrak{g}^* \to \mathbf{R}^2$. We use Singularity Theory to study these singular points.

Instead of working directly with (h, ϕ), we use an equivalence relation on the differential $F = \mathsf{d}(h, \phi)$. This F is a map from $\mathfrak{g}^*$ to $\mathrm{Mat}(2, 3)$, the set of 2×3 matrices. Let $V \subset \mathrm{Mat}(2, 3)$ be the set of matrices of rank at most 1. Then μ is a singular point of (h, ϕ) if and only if $F(\mu) \in V$. The set $\mathcal{R}$ of relative equilibria is therefore equal to $F^{-1}(V)$. The equivalence relation we use is called $\mathcal{K}_V$ equivalence, developed by J. Damon [1, 2]. This equivalence relation is defined by a diffeomorphism of (a neighbourhood of the origin in) $\mathfrak{g}^* \times \mathrm{Mat}(2, 3)$ of the form

$$\Psi(\mu, A) = (\sigma(\mu), \tau(\mu, A)),$$

where σ is a diffeomorphism of $\mathfrak{g}^* = \mathbf{R}^3$ and τ preserves V in the sense that if $A \in V$ then $\tau(\mu, A) \in V$ $(\forall \mu)$. Two maps $F, G : \mathfrak{g}^* \to \mathrm{Mat}(2, 3)$ are then $\mathcal{K}_V$-equivalent if there is such a diffeomorphism Ψ which maps the graph of F to that of G. This implies in particular that $F^{-1}(V)$ and $G^{-1}(V)$ are diffeomorphic (via σ).

Choose a basis on $\mathfrak{g}^* \simeq \mathbf{R}^3$ so that $\phi(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$. Then with $h_0 = \frac{1}{2}(ax^2 + by^2 + cz^2)$ the map F becomes $F_0(x, y, z) = \begin{pmatrix} ax & by & cz \\ x & y & z \end{pmatrix}$. If a, b, c are distinct then the set $\mathcal{R} = F^{-1}(V)$ of relative equilibria consists of the three axes (as should be familiar for Euler's equations for the rigid body, where P_0 is just a single point). One shows using singularity theory that this map F_0 is 1-determined with respect to $\mathcal{K}_V$-equivalence, which implies that if $h = h_0 + O(3)$ then the set of relative equilibria for h will be diffeomorphic to the three axes, always providing a, b, c are distinct, as in Fig. 1(a).

Now define the family of functions

$$h_u(x, y, z) = h_0(x, y, z) + \alpha x + \beta y + \gamma z,$$

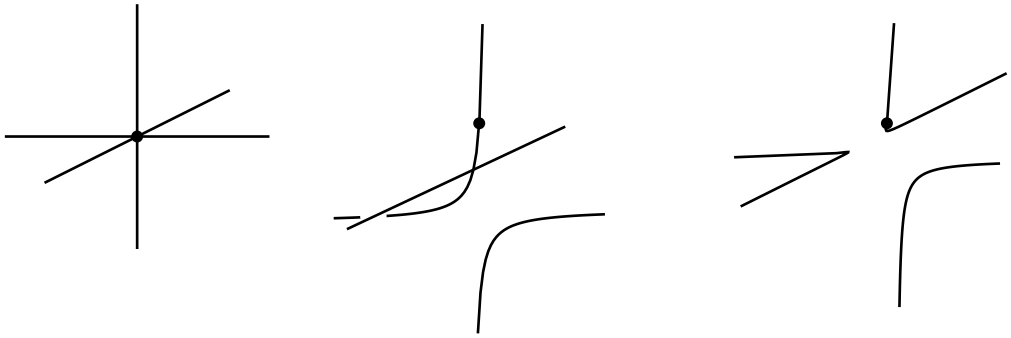
where $u = (\alpha, \beta, \gamma) \in U$ (a neighbourhood of 0 in $\mathbf{R}^3$). The resulting maps F_u are given by

$$F_u(x, y, z) = \begin{pmatrix} ax + \alpha & by + \beta & cz + \gamma \\ x & y & z \end{pmatrix}.$$

Using singularity theory methods, one shows that F_u is a $\mathcal{K}_V$ versal deformation of F_0, implying that if h'_ϵ is any parametrized deformation of h_0 then there is a smooth map $k : \epsilon \mapsto u = k(\epsilon) \in U$ for which F'_ϵ is $\mathcal{K}_V$-equivalent to $F_{k(\epsilon)}$, and so in particular the sets of relative equilibria for h'_ϵ and $h_{k(\epsilon)}$ are diffeomorphic.

To understand all possible deformations of the set $\mathcal{R}_0$ consisting of the three axes, one need only study the family F_u, which is straightforward. In U, the deformation $\mathcal{R}_u$ is singular whenever $u = (\alpha, \beta, \gamma)$ lies on one of the coordinate planes, otherwise it is a smooth curve (or union of curves), and hence a smooth manifold in the original phase space P, as in Fig. 1(c). In Fig. 1(b) we show a typical deformation of $\mathcal{R}_0$ when it is singular. The large dot in each of the figures represents the point $\mu = 0$.

Stability of relative equilibria. One can proceed further with this analysis and determine the stabilities of the relative equilibria. Returning to the splitting $P/G \simeq P_0 \times \mathfrak{g}^*$, suppose the central relative equilibrium at p is Lyapounov stable



(a) $\alpha = \beta = \gamma = 0$ (b) $\alpha = 0,\ \beta, \gamma \neq 0$ (c) $\alpha, \beta, \gamma \neq 0$

FIGURE 1. Deformations of the set of relative equilibria

on P_0 and satisfies Dirichlet's criterion there: the hessian of H_0 on P_0 is positive definite (this is essentially the energy-Casimir method for determining stability). We wish to know which nearby relative equilibria are stable.

Suppose $\mathsf{d}h_u(0) \neq 0$. Then in a neighbourhood of $\mu = 0$, of the two relative equilibria in P_μ one is stable (by Dirichlet's criterion) while the other is elliptic (all eigenvalues are imaginary). As $|\mu|$ is increased, there are two saddle-centre bifurcations (for nonzero α, β, γ): one produces a linearly unstable RE and an elliptic one, and the other produces a linearly unstable RE and a Lyapounov stable one. If on the other hand, say $\alpha = 0$, as in Fig. 1(b), then one of the saddle-centre bifurcations becomes a pitchfork bifurcation, while if say $\alpha = \beta = 0$ then both become pitchforks.

Note that $\mathsf{d}h(0) \neq 0$ is the condition for the relative equilibrium at $\mu = 0$ to be a "transverse relative equilibrium", in the sense of Patrick and Roberts [6].

I would like to thank J.E. Marsden and P.S. Krishnaprasad for suggesting during the conference that these results may be applicable to coupled rotor systems, and J.E. Marsden for asking about reconstruction of the dynamics from P/G to P in the presence of these bifurcations. This is still to be elaborated.

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Stochastic Hamiltonian dynamical systems

Juan-Pablo Ortega
(joint work with Joan-Andreu Lázaro-Camí)

The generalization of classical mechanics to the context of stochastic dynamics has been an active research subject ever since K. Itô introduced the theory of stochastic differential equations in the 1950s (see for instance [14, 2, 18, 19, 20, 16, 17, 1, 3, 4], and references therein). The motivations behind some pieces of work related to this field lay in the hope that a suitable stochastic generalization of classical mechanics should provide an explanation of the intrinsically random effects exhibited by quantum mechanics within the context of the theory of diffusions . In other instances the goal is establishing a framework adapted to the handling of mechanical systems subjected to random perturbations or whose parameters are not precisely determined and are hence modeled as realizations of a random variable.

The approach followed in our work (see [10, 11]) is closer to the one introduced in [2] in which stochastic Hamiltonian systems are those that satisfy a stochastic differential equation that generalizes the standard Hamilton's equations. We then show that this SDE is characterized by a critical action principle where the action has its image in the space of real valued processes and the variations are taken in the space of processes with values in the phase space of the system that we are modeling.

1. The Hamiltonian stochastic differential equation

Let $(M, \{\cdot,\cdot\})$ be a finite dimensional Poisson manifold, $X : \mathbb{R}_+ \times \Omega \to V$ a continuous semimartingale that takes values on the vector space V with $X_0 = 0$, and let $h : M \to V^*$ be a smooth function with values in V^*, the dual of V. Let $\{\epsilon^1, \ldots, \epsilon^r\}$ be a basis of V^* and let $h_1, \ldots, h_r \in C^\infty(M)$ be such that $h = \sum_{i=1}^r h_i \epsilon^i$. The **stochastic Hamiltonian system** associated to h with **stochastic component** X is the stochastic differential equation

$$\delta\Gamma^h = H(X, \Gamma)\delta X \tag{1}$$

defined by the Stratonovich operator $H(v, z) : T_vV \to T_zM$ defined by

$$H(v, z)\,(u) := \sum_{i=1}^{r} \left\langle \epsilon^i, u \right\rangle X_{h_i}\,(z)\,, \tag{2}$$

where X_{h_i} is the Hamiltonian vector field associated to $h_i \in C^\infty\,(M)$. The dual Stratonovich operator $H\,(v, z) : T_z^*M \to T_v^*V$ of $H(v, z)$ is given by $H^*(v, z)(\alpha_z) = -\mathbf{d}h(z) \cdot B^\sharp(z)(\alpha_z)$, where $B^\sharp : T^*M \to TM$ is the vector bundle map naturally associated to the Poisson tensor $B \in \Lambda^2(M)$ of $\{\cdot,\cdot\}$ and $\mathbf{d}h = \sum_{i=1}^r \mathbf{d}h_i \otimes \epsilon^i$. We will summarize this construction by saying that $(M, \{\cdot,\cdot\}, h, X)$ is a ***stochastic Hamiltonian system***.

Evolution of the observables. Let $f \in C^\infty(M)$ and ζ^h maximal stopping time of the Hamiltonian system $(M, \{\cdot,\cdot\}, h, X)$. Then, for any stopping time $\tau < \zeta^h$,

the solution semimartingales Γ^h satisfy

$$f(\Gamma_\tau^h) - f(\Gamma_0^h) = \sum_{j=1}^{r} \int_0^\tau \{f, h_j\}(\Gamma^h) \delta X^j. \tag{3}$$

The Itô representation of this expression is given by

$$f\left(\Gamma_\tau^h\right) - f\left(\Gamma_0^h\right) = \sum_{j=1}^{r} \int_0^\tau \{f, h_j\}\left(\Gamma^h\right) dX^j + \frac{1}{2} \sum_{j,i=1}^{r} \int_0^\tau \{\{f, h_j\}, h_i\}\left(\Gamma^h\right) d\left[X^j, X^i\right].$$

Stochastic Hamiltonian systems locally preserve the symplectic leaves of the Poisson manifolds they live in and satisfy a stochastic version of the Liouville Theorem. More specifically:

Theorem 1.1. *Let (M, ω) be a symplectic manifold, $X : \mathbb{R}_+ \times \Omega \to V^*$ a semimartingale, and $h : M \to V^*$ a Hamiltonian function. Let F be the associated Hamiltonian flow. Then, for any $z \in M$ and any $(t, \eta) \in [0, \zeta(z)]$,*

$$F_t^*(z, \eta)\, \omega = \omega.$$

2. Critical action principle for the stochastic Hamilton equations

We will show that the stochastic Hamilton equations can be characterized by a variational principle that generalizes the one used in the classical deterministic situation. In this section, our phase space will be an exact symplectic manifold (M, ω), that is, there exist a one-form $\theta \in \Omega(M)$ such that $\omega = -\mathbf{d}\theta$. The archetypical example of an exact symplectic manifold is the cotangent bundle T^*Q of any manifold Q, with θ the Liouville one-form.

Definition 2.1. *Let $(M, \omega = -\mathbf{d}\theta)$ be an exact symplectic manifold, $X : \mathbb{R}_+ \times \Omega \to V$ a semimartingale taking values on the vector space V, and $h : M \to V^*$ a Hamiltonian function. We denote by $\mathcal{S}(M)$ and $\mathcal{S}(\mathbb{R})$ the sets of M and real-valued semimartingales, respectively. We define the* **stochastic action** *associated to h as the map $S : \mathcal{S}(M) \to \mathcal{S}(\mathbb{R})$ given by*

$$S(\Gamma) = \int \langle \theta, \delta\Gamma \rangle - \int \left\langle \widehat{h}(\Gamma), \delta X \right\rangle,$$

where in the previous expression, $\widehat{h}(\Gamma) : \mathbb{R}_+ \times \Omega \to V \times V^*$ is given by $\widehat{h}(\Gamma)(t, \omega) := (X_t(\omega), h(\Gamma_t(\omega)))$.

We now specify the kind of variations that we will use in the critical action principle. Let M be a manifold and Γ a M-valued semimartingale. Let $s_0 > 0$; we say that the map $\Sigma : (-s_0, s_0) \times \mathbb{R}_+ \times \Omega \to M$ is a **pathwise variation** of Γ whenever $\Sigma_t^0 = \Gamma_t$ for any $t \in \mathbb{R}_+$ a.s.. We say that the pathwise variation Σ of Γ **converges uniformly** to Γ whenever the following properties are satisfied:

(1) For any $f \in C^\infty(M)$, $f(\Sigma^s) \to f(\Gamma)$ in *ucp* as $s \to 0$.
(2) There exists a process $Y : \mathbb{R}_+ \times \Omega \to TM$ over Γ such that, for any $f \in C^\infty(M)$, the Stratonovich integral $\int Y[f]\, \delta X$ exists for any continuous real semimartingale X (this is for instance guaranteed if Y is a

semimartingale) and, additionally, the increments $(f(\Sigma^s) - f(\Gamma))/s$ converge in *ucp* to $Y[f]$ as $s \to 0$. We will call such a Y the **infinitesimal generator** of Σ.

We will say that Σ (respectively, Y) is **bounded** when its image lies in a compact set of M (respectively, TM).

Theorem 2.2 (Critical Action Principle)**.** *Let $(M, \omega = -d\theta)$ be an exact symplectic manifold, $X : \mathbb{R}_+ \times \Omega \to V$ a semimartingale that takes values in the vector space V, and $h : M \to V^*$ a Hamiltonian function. Let m_0 be a point in M and $\Gamma : \mathbb{R}_+ \times \Omega \to M$ a continuous adapted semimartingale defined on $[0, \zeta_\Gamma)$ such that $\Gamma_0 = m_0$. Let $K \subseteq M$ be a compact set that contains m_0 and τ_K the first exit time of Γ from K. Suppose that $\tau_K < \infty$ a.s.. Then,*

(1) *For any bounded pathwise variation Σ with bounded infinitesimal generator Y which converges uniformly to Γ^{τ_K} uniformly, the action has a directional derivative that equals*

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} S(\Sigma^s) &:= \lim_{s\to 0}\frac{1}{s}\left[S(\Sigma^s) - S(\Gamma^{\tau_K})\right] \\ &= \int \langle \mathbf{i}_Y d\theta, \delta\Gamma^{\tau_K}\rangle - \int \left\langle \widehat{Y[h]}(\Gamma^{\tau_K}), \delta X\right\rangle \\ &\qquad + \langle \theta(\Gamma^{\tau_K}), Y\rangle - \langle \theta(\Gamma^{\tau_K}), Y\rangle_{t=0},\end{aligned}$$

where the symbol $\widehat{Y[h]}(\Gamma^{\tau_K})$ is consistent with the notation introduced in Definition 2.1

(2) *The semimartingale Γ satisfies the stochastic Hamilton equations with initial condition $\Gamma_0 = m_0$ up to time τ_K if and only if, for any bounded pathwise variation $\Sigma : (-s_0, s_0) \times \mathbb{R}_+ \times \Omega \to M$ with bounded infinitesimal generator which converges uniformly to Γ^{τ_K} and such that $\Sigma_0^s = m_0$ and $\Sigma_{\tau_K}^s = \Gamma_{\tau_K}$ a.s. for any $s \in (-s_0, s_0)$,*

$$\left[\left.\frac{d}{ds}\right|_{s=0} S(\Sigma^s)\right]_{\tau_K} = 0 \quad a.s..$$

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Lagrangian and Hamiltonian structure of complex fluids

Tudor S. Ratiu
(joint work with François Gay-Balmaz)

This talk is based on some results in [2], where the geometric structure of many complex fluids models is studied in detail.

The equations of motion of an *adiabatic compressible fluid* on an oriented Riemannian manifold $\mathcal{D}$ are given by

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}}\mathbf{u} = \dfrac{1}{\rho}\operatorname{grad} p, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \quad \dfrac{\partial S}{\partial t} + \operatorname{div}(S\mathbf{u}) = 0, \end{cases}$$

where ∇ is the Levi-Civita connection of the metric, $\mathbf{u}$ is the spatial velocity of the fluid, ρ is the mass density, S is the entropy density, and p is the pressure. The boundary conditions are $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the outside pointing unit normal vector field to the boundary. It was shown in [13] that this system, as well as its magnetohydrodynamic extension, admit a non-canonical Hamiltonian formulation, that is, these equations can be written as $\dot{f} = \{f, h\}$, where h is a Hamiltonian function and $\{\cdot, \cdot\}$ is a Poisson bracket. In [12], this non-canonical Poisson bracket was obtained via Lie-Poisson reduction for the semidirect product group of diffeomorphisms with the vector space that is the product of functions and densities. The Lagrangian formulation of these equations and the associated variational principle were given in [10].

In the same spirit, the non-canonical Hamiltonian structure for *adiabatic Yang-Mills charged fluids* discovered in [3] was obtained by reduction from a canonical formulation in [1], by using a Kaluza-Klein point of view involving the automorphism group of the principal bundle of the theory. The Euler-Poincaré formulation of these equations is also found in this paper.

There is a long list of non-canonical Hamiltonian structures for a wide class of non-dissipative fluid models; see, for example, [7], [4], [8], [9], and [5]. These models include *Yang-Mills magnetohydrodynamics*, *spin glasses*, and several models of *superfluids*. Many of these models have their Hamiltonian structure given by Lie-Poisson brackets *with cocycles*; see [6] where the corresponding variational principles are also studied and the theory is applied to liquid crystals.

The goal of this talk is to present a unified mathematical approach that explains the geometric structure of the equations of motion for all these models and to show that they are all derived from one and the same reduction theorem generalizing Lie-Poisson reduction to systems that have built-in cocycles. The physical systems to which this theory is applicable are the *complex fluids* which are fluids whose particles in material representation have an *order parameter* attached to them.

1. Prerequisites

In this short section we summarize the necessary concepts, definitions, and notations needed later.

Let G be a Lie group, $\rho : G \to \mathrm{Aut}(V)$ a *right* representation, and $S := G \circledS V$ the semidirect product of G with V. Recall that the multiplication is given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)),$$

where $g_1, g_2 \in G$ and $v_1, v_2 \in V$. The Lie algebra $\mathfrak{s} := \mathfrak{g} \circledS V$ of S has bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $\xi_1, \xi_2 \in \mathfrak{g}$ and $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{d}{dt}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

If $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$ we have

$$\mathrm{ad}^*_{(\xi,v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are defined by

$$a\xi := \left.\frac{d}{dt}\right|_{t=0} \rho^*_{\exp(-t\xi)}(a) \quad \text{and} \quad \langle v \diamond a, \xi\rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V,$$

and $\langle\cdot,\cdot\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

Let $\mathcal{F}(G, V^*)$ denote the vector space of smooth V^*-valued functions on G. Then $c \in \mathcal{F}(G, V^*)$ is a *right one-cocycle* if it verifies the identity $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$ for all $f, g \in G$. This implies that $c(e) = 0$ and $c(g^{-1}) = -\rho^*_g(c(g))$. Instead of the contragredient representation $\rho^*_{g^{-1}}$ of G on V^* form the *affine right representation* of G on V given by

$$\theta_g(a) = \rho^*_{g^{-1}}(a) + c(g).$$

Note that

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi).$$

and

$$\langle a\xi + \mathbf{d}c(\xi), v\rangle_V = \langle \mathbf{d}c^T(v) - v \diamond a, \xi\rangle_{\mathfrak{g}},$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ and $\mathbf{d}c^T : V \to \mathfrak{g}^*$ are defined by $\mathbf{d}c(\xi) := T_e c(\xi)$ and $\langle \mathbf{d}c^T(v), \xi\rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v\rangle_V$, respectively.

2. Affine Lagrangian semidirect product theory

Let $L : TG \times V^* \to \mathbb{R}$ be a right G-invariant Lagrangian function under the affine G-action $(v_h, a) \in TG \times V^* \mapsto (T_h R_g(v_h), \theta_g(a)) = (T_h R_g(v_h), \rho^*_{g^{-1}}(a) + c(g)) \in TG \times V^*$. Thus, if $a_0 \in V^*$, we define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of $G^c_{a_0}$ on G, where $G^c_{a_0} := \{g \in G \mid \theta_g(a_0) = a_0\}$. Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by $l := L|_{\mathfrak{g}\times V^*}$ and hence we have

$$l(T_g R_{g^{-1}}(v_g), \theta_{g^{-1}}(a_0)) = L(v_g, a_0) \quad \text{for all} \quad v_g \in T_g G, \quad a_0 \in V^*.$$

For a curve $g(t) \in G$, let $\xi(t) := TR_{g(t)^{-1}}(\dot{g}(t))$ and define the curve $a(t) \in V^*$ as the unique solution of the following affine differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)), \quad a(0) = a_0.$$

The solution of this equation is $a(t) = \theta_{g(t)^{-1}}(a_0)$.

Theorem. *The following statements are equivalent:*

(i) *With $a_0 \in V^*$ fixed, Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) *The curve $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.*

(iii) *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$*, upon using variations of the form*

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta : [t_1, t_2] \to \mathfrak{g}$ *is an arbitrary smooth curve with* $\eta(t_1) = \eta(t_2) = 0$.

(iv) *The affine Euler-Poincaré equations hold on* $\mathfrak{g} \times V^*$*:*

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta l}{\delta a}\right).$$

3. Affine Hamiltonian semidirect product theory

The Hamiltonian analogue of the previous theorem necessitates considerably more background material related to the reduction by stages process (see [11]). We summarize here only the final result.

Let $H : T^*G \times V^* \to \mathbb{R}$ be a right-invariant Hamiltonian under the G-action

$$(\alpha_h, a) \in T^*G \times V^* \mapsto (R_g^{T^*}(\alpha_h), \theta_g(a)) := (R_g^{T^*}(\alpha_h), \rho^*_{g^{-1}}(a) + c(g)) \in T^*G \times V^*,$$

where $R_g^{T^*}(\alpha_h) := T^*_{hg}R_{g^{-1}}\alpha_h$ is the cotangent lift of right translation. In particular, if $a_0 \in V^*$ is fixed, then $H_{a_0} := H|_{T^*G\times\{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the a_0-isotropy subgroup $G^c_{a_0}$ relative to the affine action θ.

Theorem. *For* $\alpha(t) \in T^*_{g(t)}G$ *and* $\mu(t) := T^*_e R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$*, the following are equivalent:*

(i) $\alpha(t)$ *satisfies Hamilton's equations for* H_{a_0} *on* T^*G*.*

(ii) *The following affine Lie-Poisson equation holds on* $\mathfrak{s}^*$*:*

$$\frac{\partial}{\partial t}(\mu, a) = \left(-\operatorname{ad}^*_{\frac{\delta h}{\delta \mu}}\mu - \frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -a\frac{\delta h}{\delta \mu} - \mathbf{d}c\left(\frac{\delta h}{\delta \mu}\right)\right), \ a(0) = a_0.$$

The evolution of the advected quantity $a(t)$ *is given by* $a(t) = \theta_{g(t)^{-1}}(a_0)$*.*

4. Perfect complex fluids

The passage from usual fluids to complex fluids is based on two key observations. First, one needs to enlarge the configuration manifold $\operatorname{Diff}(\mathcal{D})$ to a bigger group G that contains variables in the Lie group $\mathcal{O}$ of order parameters. Second, the usual advection equations (for the mass density, the entropy, the magnetic field, etc) need to be augmented by a new advected quantity on which the group G acts by an *affine representation.*

The general setup. We shall denote in what follows by $\mathcal{O}$ the *order parameter Lie group* and let $\mathcal{F}(\mathcal{D}, \mathcal{O}) := \{\chi : \mathcal{D} \to \mathcal{O} \text{ smooth}\}$ be the infinite dimensional group of $\mathcal{O}$-valued functions on $\mathcal{D}$ relative to pointwise multiplication. The Lie algebra of $\mathcal{O}$ is denoted by $\mathfrak{o}$ and its dual by $\mathfrak{o}^*$. The basic idea in the geometric

theory of complex fluids is to enlarge the "particle relabeling group" $\mathrm{Diff}(\mathcal{D})$ to the semidirect product $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ whose multiplication is given by $(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi)$ for any $(\eta, \chi), (\varphi, \psi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$.

The Lie algebra of G is the semidirect product $\mathfrak{g} = \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})$ whose Lie bracket is

$$\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v}, \zeta) = (\mathrm{ad}_{\mathbf{u}}\, \mathbf{v}, \mathrm{ad}_\nu\, \zeta + \mathbf{d}\nu \cdot \mathbf{v} - \mathbf{d}\zeta \cdot \mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}}\, \mathbf{v} = -[\mathbf{u}, \mathbf{v}]$, $\mathrm{ad}_\nu\, \zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathrm{ad}_\nu\, \zeta(x) := \mathrm{ad}_{\nu(x)}\, \zeta(x)$, and $\mathbf{d}\nu \cdot \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathbf{d}\nu \cdot \mathbf{v}(x) := \mathbf{d}\nu(x)(\mathbf{v}(x))$ for all $x \in \mathcal{D}$.

The dual of $\mathfrak{g}$ is $\mathfrak{g}^* = \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ through the pairing

$$\langle (\mathbf{m}, \kappa), (\mathbf{u}, \nu) \rangle = \int_{\mathcal{D}} (\mathbf{m} \cdot \mathbf{u} + \kappa \cdot \nu)\, \mu,$$

where μ is a volume form on $\mathcal{D}$, $(\mathbf{m}, \kappa) \in \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$, and $(\mathbf{u}, \nu) \in \mathfrak{X}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o})$. The dual map to $\mathrm{ad}_{(\mathbf{u},\nu)}$ is

$$\mathrm{ad}^*_{(\mathbf{u},\nu)}(\mathbf{m}, \kappa) = \big(\boldsymbol{\pounds}_{\mathbf{u}}\mathbf{m} + (\mathrm{div}\, \mathbf{u})\mathbf{m} + \kappa \cdot \mathbf{d}\nu, \mathrm{ad}^*_\nu\, \kappa + \mathrm{div}(\mathbf{u}\kappa)\big).$$

Explanation of the symbols:

• $\kappa \cdot \mathbf{d}\nu \in \Omega^1(\mathcal{D})$ denotes the one-form defined by

$$\kappa \cdot \mathbf{d}\nu(v_x) := \kappa(x)(\mathbf{d}\nu(v_x))$$

• $\mathrm{ad}^*_\nu\, \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ denotes the $\mathfrak{o}^*$-valued mapping defined by

$$\mathrm{ad}^*_\nu\, \kappa(x) := \mathrm{ad}^*_{\nu(x)}(\kappa(x)).$$

• $\mathbf{u}\kappa$ is the 1-contravariant tensor field with values in $\mathfrak{o}^*$ defined by

$$\mathbf{u}\kappa(\alpha_x) := \alpha_x(\mathbf{u}(x))\kappa(x) \in \mathfrak{o}^* \quad \text{for all} \quad \alpha_x \in T^*\mathcal{D}.$$

So $\mathbf{u}\kappa$ is a generalization of the notion of a vector field. $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ denotes the space of all $\mathfrak{o}^*$-valued 1-contravariant tensor fields.

• $\mathrm{div}(\mathbf{u})$ denotes the divergence of the vector field $\mathbf{u}$ with respect to the fixed volume form μ. Recall that it is defined by the relation $(\mathrm{div}\, \mathbf{u})\mu = \boldsymbol{\pounds}_{\mathbf{u}}\mu$. This operator can be naturally extended to the space $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ as follows. For $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ we write $w = w_a\varepsilon^a$ where (ε^a) is a basis of $\mathfrak{o}^*$ and $w_a \in \mathfrak{X}(\mathcal{D})$. We define $\mathrm{div} : \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*) \to \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ by

$$\mathrm{div}\, w := (\mathrm{div}\, w_a)\varepsilon^a.$$

Note that if $w = \mathbf{u}\kappa$ we have

$$\mathrm{div}(\mathbf{u}\kappa) = \mathbf{d}\kappa \cdot \mathbf{u} + (\mathrm{div}\, \mathbf{u})\kappa.$$

Now, that the particle relabeling group has been replaced by the semidirect product $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$, we construct the space of advected quantities. These are of two kinds: usual ones (as in the theory of adiabatic fluids, for example) and new ones, intimately connected to the order parameter group, that involve affine actions and cocycles.

We take as the affine representation space $V_1^* \oplus V_2^*$, where V_i^* are subspaces of the space of all tensor fields on $\mathcal{D}$, possibly with values in a vector space. In addition, we assume that V_1^* is only acted upon by the component $\mathrm{Diff}(\mathcal{D})$ of G and

that the action of G on V_2^* is affine, with the restriction that the affine term only depends on the second component $\mathcal{F}(\mathcal{D},\mathcal{O})$ of G. The right affine representation of $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D},\mathcal{O})$ on $V_1^* \oplus V_2^*$ is given by

$$(a,\gamma) \in V_1^* \oplus V_2^* \mapsto (a\eta, \gamma(\eta,\chi) + C(\chi)) \in V_1^* \oplus V_2^*,$$

where $\gamma(\eta,\chi)$ denotes the action of $(\eta,\chi) \in G$ on $\gamma \in V_2^*$, and $C \in \mathcal{F}(\mathcal{F}(\mathcal{D},\mathcal{O}), V_2^*)$ satisfies the cocycle identity

$$C((\chi\circ\varphi)\psi) = C(\chi)(\varphi,\psi) + C(\psi) \quad \text{for all} \quad \chi,\psi \in \mathcal{F}(\mathcal{D},\mathcal{O}), \quad \varphi \in \mathrm{Diff}(\mathcal{D}).$$

Thus, the representation ρ and the affine term c in the general theory are

$$\rho^*_{(\eta,\chi)^{-1}}(a,\gamma) = (a\eta, \gamma(\eta,\chi)) \quad \text{and} \quad c(\eta,\chi) = (0, C(\chi)).$$

The infinitesimal action of $(\mathbf{u},\nu) \in \mathfrak{g}$ on $\gamma \in V_2^*$ is given by $\gamma(\mathbf{u},\nu) := \gamma\mathbf{u} + \gamma\nu$. Therefore, the diamond operation has the expression

$$(v,w)\diamond(a,\gamma) = (v\diamond a + w\diamond_1\gamma, w\diamond_2\gamma), \quad (v,w) \in V_1\oplus V_2, \quad (a,\gamma)\in V_1^*\oplus V_2^*,$$

where $\diamond_1$ and $\diamond_2$ are associated to the induced representations of the first and second component of G on V_2^*. On the right hand side, $\diamond$ is associated to the representation of $\mathrm{Diff}(\mathcal{D})$ on V_1^*. Usually, V_1^* is naturally the dual of some space V_1 of tensor fields on $\mathcal{D}$. For example the (p,q) tensor fields are naturally in duality with the (q,p) tensor fields. For $a \in V_1^*$ and $v \in V_1$, the duality pairing is

$$\langle a, v\rangle = \int_{\mathcal{D}} (a\cdot v)\mu,$$

where $\cdot$ denotes the contraction of tensor fields.

The affine cocycle is $c(\eta,\chi) = (0, C(\chi))$. Hence

$$\mathbf{d}c^T(v,w) = (0, \mathbf{d}C^T(w)).$$

The Lagrangian formulation. For a Lagrangian

$$l = l(\mathbf{u},\nu,a,\gamma) : [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D},\mathfrak{o})] \circledS [V_1^*\oplus V_2^*] \to \mathbb{R}\,,$$

the *affine Euler-Poincaré equations* become

$$\begin{cases} \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \mathbf{u}} = -\boldsymbol{\pounds}_{\mathbf{u}}\dfrac{\delta l}{\delta \mathbf{u}} - (\operatorname{div}\mathbf{u})\dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu}\cdot\mathbf{d}\nu + \dfrac{\delta l}{\delta a}\diamond a + \dfrac{\delta l}{\delta\gamma}\diamond_1\gamma \\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta \nu} = -\operatorname{ad}^*_\nu\dfrac{\delta l}{\delta\nu} - \operatorname{div}\left(\mathbf{u}\dfrac{\delta l}{\delta\nu}\right) + \dfrac{\delta l}{\delta\gamma}\diamond_2\gamma - \mathbf{d}C^T\left(\dfrac{\delta l}{\delta\gamma}\right), \end{cases}$$

and the advection equations are

$$\begin{cases} \dot a + a\mathbf{u} = 0 \\ \dot\gamma + \gamma\mathbf{u} + \gamma\nu + \mathbf{d}C(\nu) = 0. \end{cases}$$

In the concrete case of complex fluids, one chooses $V_2 = \mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$ and so $V_2^* := \Omega^1(\mathcal{D},\mathfrak{o})$. We still let V_1 be arbitrary. In concrete examples, its dual V_1^* is formed by the classical convected quantities such as mass density, entropy density, or magnetic field, for example. The affine representation is given by

$$(a,\gamma) \mapsto (a\eta, \mathrm{Ad}_{\chi^{-1}}\,\eta^*\gamma + \chi^{-1}T\chi),$$

where $(\eta,\chi)\in \mathrm{Diff}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathcal{O})$, $(a,\gamma)\in V_1^*\times\Omega^1(\mathcal{D},\mathfrak{o})$, and $\mathrm{Ad}_{\chi^{-1}}\,\eta^*\gamma+\chi^{-1}T\chi$ is the $\mathfrak{o}$-valued one-form

$$\left(\mathrm{Ad}_{\chi^{-1}}\,\eta^*\gamma+\chi^{-1}T\chi\right)(v_x):=\mathrm{Ad}_{\chi(x)^{-1}}(\eta^*\gamma(v_x))+\chi(x)^{-1}T_x\chi(v_x),$$

for $v_x\in T_x\mathcal{D}$. One can check that $\gamma(\eta,\chi):=\mathrm{Ad}_{\chi^{-1}}\,\eta^*\gamma$ is a right representation of G on V_2^* and that $C(\chi)=\chi^{-1}T\chi$ verifies the cocycle identity. The formula of the affine representation is identical to the action of the automorphism group of the trivial principal bundle $\mathcal{O}\times\mathcal{D}$ on the space connections.

For this example we have

$$\gamma\mathbf{u}=\boldsymbol{\pounds}_{\mathbf{u}}\gamma,\quad \gamma\nu=-\,\mathrm{ad}_\nu\,\gamma\quad\text{and}\quad \mathbf{d}C(\nu)=\mathbf{d}\nu,$$

where $\mathrm{ad}_\nu\,\gamma\in\Omega^1(\mathcal{D},\mathfrak{o})$ and $\mathbf{d}\nu\in\Omega^1(\mathcal{D},\mathfrak{o})$ are the one-forms defined by

$$(\mathrm{ad}_\nu\,\gamma)(v_x):=\mathrm{ad}_{\nu(x)}(\gamma(v_x))=[\nu(x),\gamma(v_x)],\quad \mathbf{d}\nu(v_x):=T_x\nu(v_x)\in\mathfrak{o}$$

for all $v_x\in T_x\mathcal{D}$. A direct computation shows that

$$\begin{aligned} w\diamond_1\gamma&=(\mathrm{div}\,w)\cdot\gamma-w\cdot\mathbf{i}_{\_}\,\mathbf{d}\gamma\in\Omega^1(\mathcal{D}),\\ w\diamond_2\gamma&=-\,\mathrm{Tr}(\mathrm{ad}^*_\gamma\,w)\in\mathcal{F}(\mathcal{D},\mathfrak{o}^*),\\ \mathbf{d}C^T(w)&=-\,\mathrm{div}\,w\in\mathcal{F}(\mathcal{D},\mathfrak{o}^*),\end{aligned}$$

where Tr denotes the trace of the $\mathfrak{o}^*$-valued $(1,1)$ tensor

$$\mathrm{ad}^*_\gamma\,w:T^*\mathcal{D}\times T\mathcal{D}\to\mathfrak{o}^*,\quad(\alpha_x,v_x)\mapsto\mathrm{ad}^*_{\gamma(v_x)}(w(\alpha_x)).$$

In coordinates we have $\mathrm{Tr}(\mathrm{ad}^*_\gamma\,w)=\mathrm{ad}^*_{\gamma_i}\,w^i$.

The affine Euler-Poincaré equations become in this case

$$\begin{cases}\dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta\mathbf{u}}=-\boldsymbol{\pounds}_{\mathbf{u}}\dfrac{\delta l}{\delta\mathbf{u}}-(\mathrm{div}\,\mathbf{u})\dfrac{\delta l}{\delta\mathbf{u}}-\dfrac{\delta l}{\delta\nu}\cdot\mathbf{d}\nu+\dfrac{\delta l}{\delta a}\diamond a+\left(\mathrm{div}\,\dfrac{\delta l}{\delta\gamma}\right)\cdot\gamma-\dfrac{\delta l}{\delta\gamma}\cdot\mathbf{i}_{\_}\,\mathbf{d}\gamma\\ \dfrac{\partial}{\partial t}\dfrac{\delta l}{\delta\nu}=-\,\mathrm{ad}^*_\nu\,\dfrac{\delta l}{\delta\nu}+\mathrm{div}\left(\dfrac{\delta l}{\delta\gamma}-\mathbf{u}\dfrac{\delta l}{\delta\nu}\right)-\mathrm{Tr}\left(\mathrm{ad}^*_\gamma\,\dfrac{\delta l}{\delta\gamma}\right),\end{cases}$$

and the advection equations are

$$\begin{cases}\dot a+a\mathbf{u}=0\\ \dot\gamma+\boldsymbol{\pounds}_{\mathbf{u}}\gamma-\mathrm{ad}_\nu\,\gamma+\mathbf{d}\nu=0.\end{cases}$$

These are, up to sign conventions, the equations for complex fluids given by Holm[2002].

We close with some comments regarding the geometry of these equations. The first observation is that γ defines a connection, namely,

$$(v_x,\xi_{\mathsf{h}})\in T_x\mathcal{D}\times T_{\mathsf{h}}\mathcal{O}\mapsto\mathrm{Ad}_{\mathsf{h}^{-1}}(\gamma(x)(v_x)+TR_{\mathsf{h}^{-1}}(\xi_{\mathsf{h}}))\in\mathfrak{o}.$$

The covariant differential of this connection is denoted by $\mathbf{d}^\gamma$. For example, if $\nu\in\mathcal{F}(\mathcal{D},\mathfrak{o})$, then

$$\mathbf{d}^\gamma\nu(\mathbf{v}):=\mathbf{d}\nu(\mathbf{v})+[\gamma(\mathbf{v}),\nu].$$

The *covariant divergence* of $w\in\mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$ is the function

$$\mathrm{div}^\gamma\,w:=\mathrm{div}\,w-\mathrm{Tr}(\mathrm{ad}^*_\gamma\,w)\in\mathcal{F}(\mathcal{D},\mathfrak{o}^*),$$

defined as minus the adjoint of the covariant differential, that is,

$$\int_{\mathcal{D}} (\mathbf{d}^\gamma \nu \cdot w)\, \mu = -\int_{\mathcal{D}} (\nu \cdot \operatorname{div}^\gamma w)\, \mu$$

for all $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$.

Note that the Lie derivative of $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{o})$ can be written as

$$\begin{aligned} \pounds_{\mathbf{u}} \gamma(\mathbf{v}) &= \mathbf{d}(\gamma(\mathbf{u}))(\mathbf{v}) + \mathbf{i}_{\mathbf{u}} \mathbf{d}\gamma(\mathbf{v}) \\ &= \mathbf{d}^\gamma(\gamma(\mathbf{u}))(\mathbf{v}) - [\gamma(\mathbf{v}), \gamma(\mathbf{u})] + \mathbf{d}\gamma^\gamma(\mathbf{u}, \mathbf{v}) - [\gamma(\mathbf{u}), \gamma(\mathbf{v})] \\ &= \mathbf{d}^\gamma(\gamma(\mathbf{u}))(\mathbf{v}) + \mathbf{i}_{\mathbf{u}} B(\mathbf{v}), \end{aligned}$$

where

$$B := \mathbf{d}^\gamma \gamma = \mathbf{d}\gamma + [\gamma, \gamma],$$

is the *curvature* of the connection induced by γ.

Note also that, using covariant differentiation, we have

$$w \diamond_1 \gamma = (\operatorname{div} w) \cdot \gamma - w \cdot \mathbf{i}_{\_}\, \mathbf{d}\gamma = (\operatorname{div}^\gamma w) \cdot \gamma - w \cdot \mathbf{i}_{\_}\, B.$$

Therefore, in terms of $\mathbf{d}^\gamma, \operatorname{div}^\gamma$, and $B = \mathbf{d}^\gamma \gamma$, the equations of motion of a complex fluid read

$$\begin{cases} \dfrac{\partial}{\partial t} \dfrac{\delta l}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}} \dfrac{\delta l}{\delta \mathbf{u}} - (\operatorname{div} \mathbf{u}) \dfrac{\delta l}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \nu} \cdot \mathbf{d}\nu + \dfrac{\delta l}{\delta a} \diamond a + \left(\operatorname{div}^\gamma \dfrac{\delta l}{\delta \gamma} \right) \cdot \gamma - \dfrac{\delta l}{\delta \gamma} \cdot \mathbf{i}_{\_}\, B \\ \dfrac{\partial}{\partial t} \dfrac{\delta l}{\delta \nu} = -\operatorname{ad}^*_\nu \dfrac{\delta l}{\delta \nu} - \operatorname{div}\left(\mathbf{u} \dfrac{\delta l}{\delta \nu} \right) + \operatorname{div}^\gamma \dfrac{\delta l}{\delta \gamma}, \end{cases}$$

and

$$\begin{cases} \dot{a} + a\mathbf{u} = 0 \\ \dot{\gamma} + \mathbf{d}^\gamma(\gamma(\mathbf{u})) + \mathbf{i}_{\mathbf{u}} B + \mathbf{d}^\gamma \nu = 0. \end{cases}$$

The Hamiltonian formulation. The Lie-Poisson space is

$$\Big([\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})] \circledS [V_1 \oplus V_2] \Big)^* \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^*$$

with *affine Lie-Poisson bracket* given by

$$\begin{aligned} \{f, g\}(\mathbf{m}, \kappa, a, \gamma) = &\int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}} \right] \mu \\ &+ \int_{\mathcal{D}} \kappa \cdot \left(\operatorname{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d} \frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d} \frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}} \right) \mu \\ &+ \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a} \frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a} \frac{\delta f}{\delta \mathbf{m}} \right) \\ &+ \int_{\mathcal{D}} \gamma \cdot \left(\frac{\delta f}{\delta \gamma} \frac{\delta g}{\delta \mathbf{m}} + \frac{\delta f}{\delta \gamma} \frac{\delta g}{\delta \kappa} - \frac{\delta g}{\delta \gamma} \frac{\delta f}{\delta \mathbf{m}} - \frac{\delta g}{\delta \gamma} \frac{\delta f}{\delta \kappa} \right) \mu \\ &+ \int_{\mathcal{D}} \left(\mathbf{d}C \left(\frac{\delta f}{\delta \kappa} \right) \cdot \frac{\delta g}{\delta \gamma} - \mathbf{d}C \left(\frac{\delta g}{\delta \kappa} \right) \cdot \frac{\delta f}{\delta \gamma} \right) \mu. \end{aligned}$$

For a Hamiltonian $h = h(\mathbf{m}, \kappa, a, \gamma) : \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^* \to \mathbb{R}$, the *affine Lie-Poisson equations* are therefore

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\boldsymbol{\pounds}_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\dfrac{\delta h}{\delta \kappa} - \dfrac{\delta h}{\delta a} \diamond a - \dfrac{\delta h}{\delta \gamma} \diamond_1 \gamma \\ \dfrac{\partial}{\partial t}\kappa = -\operatorname{ad}^*_{\frac{\delta h}{\delta \kappa}}\kappa - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) - \dfrac{\delta h}{\delta \gamma} \diamond_2 \gamma + \mathbf{d}C^T\left(\dfrac{\delta h}{\delta \gamma}\right) \\ \dfrac{\partial}{\partial t}a = -a\dfrac{\delta h}{\delta \mathbf{m}} \\ \dfrac{\partial}{\partial t}\gamma = -\gamma\dfrac{\delta h}{\delta \mathbf{m}} - \gamma\dfrac{\delta h}{\delta \kappa} - \mathbf{d}C\left(\dfrac{\delta h}{\delta \kappa}\right). \end{cases}$$

Recall that in the concrete case of complex fluids one takes $V_2 = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$. Thus $V_2^* := \Omega^1(\mathcal{D}, \mathfrak{o})$. We let V_1^* be arbitrary; this is the space of the classical convected quantities such as mass, entropy, or magnetic field. The *affine Lie-Poisson equations* become in this concrete case

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\boldsymbol{\pounds}_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\dfrac{\delta h}{\delta \kappa} - \dfrac{\delta h}{\delta a} \diamond a \\ \qquad\qquad - \left(\operatorname{div}^\gamma \dfrac{\delta h}{\delta \gamma}\right)\gamma + \dfrac{\delta h}{\delta \gamma} \cdot \mathbf{i}_{\_}\, \mathbf{d}^\gamma \gamma \\ \dfrac{\partial}{\partial t}\kappa = -\operatorname{ad}^*_{\frac{\delta h}{\delta \kappa}}\kappa - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) - \operatorname{div}^\gamma \dfrac{\delta h}{\delta \gamma} \\ \dfrac{\partial}{\partial t}a = -a\dfrac{\delta h}{\delta \mathbf{m}} \\ \dfrac{\partial}{\partial t}\gamma = -\mathbf{d}^\gamma\left(\gamma\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\right) - \mathbf{i}_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{d}^\gamma\gamma - \mathbf{d}^\gamma\dfrac{\delta h}{\delta \kappa} \end{cases}$$

and the associated *affine Lie-Poisson bracket* is

$$\begin{aligned} \{f, g\}(\mathbf{m}, \kappa, a, \gamma) = &\int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu \\ &+ \int_{\mathcal{D}} \kappa \cdot \left(\operatorname{ad}_{\frac{\delta f}{\delta \kappa}}\frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ &+ \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ &+ \int_{\mathcal{D}} \left[\left(\mathbf{d}^\gamma\frac{\delta f}{\delta \kappa} + \boldsymbol{\pounds}_{\frac{\delta f}{\delta \mathbf{m}}}\gamma\right) \cdot \frac{\delta g}{\delta \gamma} - \left(\mathbf{d}^\gamma\frac{\delta g}{\delta \kappa} + \boldsymbol{\pounds}_{\frac{\delta g}{\delta \mathbf{m}}}\gamma\right) \cdot \frac{\delta f}{\delta \gamma}\right]\mu. \end{aligned}$$

One can also formulate all of this in terms of the curvature. For details see [2].

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Large-scale atmospheric circulation, semi-geostrophic motion and Lagrangian particle methods

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(joint work with Colin Cotter)

The compressible non-viscous Euler equations provide the starting point for modeling atmospheric and ocean dynamics [5, 6]. Given typical length and time-scales for global circulation patterns, approximations are often employed which filter non-significant flow patterns from the equations of motion. Among the most popular and useful approximations are the hydrostatic and the semi-geostrophic approximations, which reads [5]

$$\begin{aligned} \frac{D\mathbf{u}_g}{Dt} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} &= 0, \\ \rho_t + \nabla\cdot(\rho\mathbf{u}) &= 0, \\ \theta_t + \mathbf{u}\cdot\nabla\theta &= 0, \end{aligned}$$

with the geostrophic wind approximation

$$\mathbf{u}_g = \left[\begin{array}{ccc} u_g & v_g & 0 \end{array}\right]^T$$

and

$$fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv_g = +\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

A practical implication in the northern hemisphere is that pressure increases to the right if you stand with our back to the wind.

The semi-geostrophic equations make use of the geostrophic wind approximation in a particularly clever way giving rise to many interesting underlying geometric features including links to optimal transportation, variational mechanics and constraint dynamics. One can explain these ideas by going first to the shallow water equations and then further on to a single fluid parcel approximation

$$\begin{aligned} \dot{\mathbf{p}} &= J_2\mathbf{p} - \varepsilon\,\nabla\mu(\tau,\mathbf{q}), \qquad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \dot{\mathbf{q}} &= \mathbf{p}, \end{aligned}$$

with state variable $\mathbf{z} = (\mathbf{q}^T, \mathbf{p}^T)^T \in \mathbb{R}^4$, μ a given (time-dependent) potential, and the small parameter $\varepsilon > 0$. The associated "semi-geostrophic" equations are given by

$$\dot{\mathbf{p}}_g = J_2\mathbf{p} - \varepsilon\nabla\mu(\tau,\mathbf{q}), \tag{1}$$

$$\dot{\mathbf{q}} = \mathbf{p}, \tag{2}$$

with geostrophic "wind" $\mathbf{p}_g = -\varepsilon J_2\nabla\mu(\tau,\mathbf{q})$. For time-independent μ (which we assume from now on), the energy

$$E = \frac{1}{2}\|\mathbf{p}_g\|^2 + \varepsilon\mu(\mathbf{q})$$

is preserved.

Much insight into the semi-geostrophic approximation has been gained by the Hoskins' transformation [5]

$$\mathbf{q}_\varepsilon = \mathbf{q} + \varepsilon\nabla\mu(\mathbf{q}) = \mathbf{q} + J_2\mathbf{p}_g,$$

which leads to the following equation in the transformed variable $\mathbf{q}_\varepsilon$:

$$\dot{\mathbf{q}}_\varepsilon = -\varepsilon J_2\nabla\mu(\mathbf{q}). \tag{3}$$

It turns out that the Hoskin's transform is linked to an optimal transportation problem. See [5] for the fascinating details.

While the semi-geostrophic equations are well studied much less is known about its range of validity in terms of the small parameter ε. Improved semi-geostrophic models can be found in [6]. More recently, asymptotic expansions have been considered within the Lagrangian variational framework in [8, 9].

A different approach has been taken in [3], which applies Hamiltonian normal form theory to the gyroscopic particle problem (1)-(2), i.e., one finds a canonical near-identity change of coordinates $\Psi_n : \mathbf{z}_\varepsilon \to \mathbf{z}$ so that

$$H_n = H_0 \circ \Psi_n = K + \varepsilon G_n + \varepsilon^{n+1} R_n, \tag{4}$$

where

$$\{G_n, K\} = 0, \qquad K = \frac{1}{2}\|\mathbf{p}_\varepsilon\|^2, \tag{5}$$

with $\{\cdot,\cdot\}$ being the Poisson bracket for (1)-(2). Optimal truncation in the index n yields the desired exponential dependence on ε and the preservation of "geostrophic/gyroscopic" balance over exponentially long periods of time.

As a consequence of (4) and (5), we may consider the reduced equations

$$\begin{aligned} 0 = \dot{\mathbf{p}}_\varepsilon &= J_2 \nabla_{\mathbf{p}} G_n(\mathbf{q}_\varepsilon, 0) - \varepsilon G_n(\mathbf{q}_\varepsilon, 0), \\ \dot{\mathbf{q}}_\varepsilon &= \nabla_{\mathbf{p}} G_n(\mathbf{q}_\varepsilon, 0), \end{aligned}$$

for initial conditions satisfying $\mathbf{p}_\varepsilon(0) = 0$. These equations are equivalent to

$$\dot{\mathbf{q}}_\varepsilon = -\varepsilon J_2 \nabla_{\mathbf{q}} G_n(\mathbf{q}_\varepsilon, 0)$$

and the leading order term coincide with Hoskin's transformed equation (3).

Furthermore, the normal form estimates remain valid for many particle systems of type (1)-(2), which couple through a multi-particle potential $\mu(\mathbf{q}_1, \ldots \mathbf{q}_N)$. This observation allows one to go back to the continuum limit by first considering finite dimensional particle approximations of the shallow-water equations (see, e.g., [1, 2]). The continuum limit gives rise to a set of regularized fluid equations which can be interpreted as Euler's equations subject to a regularized pressure field [7]. Similar pressure regularizations arise from semi-implicit time-stepping methods, which are widely used in numerical weather prediction. See [4] and references therein.

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Symmetry reduction for low-dimensional models of self-similar fluid flows

Clarence W. Rowley
(joint work with Mingjun Wei, Miloš Ilak)

Fluid flows are, of course, described by nonlinear partial differential equations that are typically difficult to solve. For all practical situations, numerical simulations are necessary, and the resulting numerical models are too complex for many useful analysis tools to apply. This talk concerns techniques for developing reduced-order models that approximate the full dynamics of fluid flows, in the form of ordinary differential equations that are useful for analysis and, ultimately, for designing model-based feedback control laws to achieve a desired behavior in the fluid.

Reduced-order models. First, we review some existing methods for developing reduced-order models. A typical technique is to use *Galerkin projection*, in which one projects the governing equations onto a particular subspace. For instance, if the full-order dynamics evolve on an inner product space V (possibly infinite-dimensional), then we write the full-order equations as

$$\dot{x}(t) = f(x(t)), \qquad x(t) \in V, \tag{1}$$

where f is a vector field on V. Given an orthonormal basis $\{\varphi_1, \ldots, \varphi_n\}$ for a subspace $S \subset V$, Galerkin projection specifies dynamics on this subspace by writing

$$r(t) = \sum_{j=1}^{n} a_j(t)\varphi_j, \qquad r(t) \in S \subset V \tag{2}$$

and prescribing the reduced-order dynamics by projecting $f(r(t))$ onto the subspace S, to obtain

$$\dot{a}_j(t) = \langle \varphi_j, f(r(t)) \rangle, \qquad j = 1, \ldots, n, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on V. Galerkin projection thus involves two distinct choices: the choice of the subspace S, and the choice of the inner product on V. Both of these choices affect the quality of the approximate models obtained.

A popular method for determining a suitable subspace S, of fixed dimension n, is to use *Proper Orthogonal Decomposition* (POD), in which one starts with a particular dataset $x(t) \in V$ and finds a subspace S of dimension n such that the orthogonal projection of the data onto S best approximates the original data. For more details of this modeling approach, see [1].

Unfortunately, models developed by projection onto POD modes often behave unpredictably. One reason for this is that low-energy modes are often important for the dynamics, but because of their low energy content do not appear as dominant modes in POD. For instance, we show an example of a linearized channel flow in which the first 5 POD modes capture over 99.7% of the energy in a particular dataset that exhibits large transient growth, but a 5-dimensional POD-Galerkin

model captures none of the transient growth. In contrast, a 5-dimensional model containing POD modes 1–3, 10, and 17 captures the transient growth nearly perfectly. That is, replacing the more energetically dominant modes 4 and 5 with the less energetic 10 and 17 produces vastly superior models [2].

For linear problems, an alternative to POD-Galerkin is to use *balanced truncation* [3]. Balanced truncation of an input-output system involves balancing the properties of controllability (loosely equivalent to the energy in a dataset consisting of impulse responses) and observability (related to "dynamical importance," or the sensitivity of the output to perturbations in a given direction in state space). Recently, an approximate procedure was developed that makes computation of balanced-truncation model reduction tractable even for systems of very large dimension, such as arise in fluids simulations [4]. For the linearized channel flow example presented earlier, balanced truncation produces dramatically superior models to POD-Galerkin: a 3rd-order model using balanced truncation performs better than a 16th-order POD-Galerkin model, and the models always improve as more modes are included, unlike POD.

Symmetry reduction. Many problems in fluid mechanics exhibit some type of continuous symmetry, which manifests itself in the form of traveling wave solutions or self-similar solutions. It is usually desirable for reduced-order models to respect the same continuous symmetries as the full problem, and the typical approach is to place a requirement on the subspace S, in particular that it is invariant to actions of the symmetry group. For instance, for translation invariance, this requirement implies that the optimal POD modes in the direction of translation are Fourier modes [1].

An alternative approach is to *factor out* the symmetry, using a type of symmetry reduction, and write reduced-order models in a setting in which the symmetry no longer appears. This approach has been shown to give superior models, since in the reduced setting, solutions such as traveling waves or self-similar solutions appear simply as fixed points [5], whose dynamics are trivial to model.

In this talk, we carry out a version of symmetry reduction for the temporal development of a two-dimensional free shear layer [6]. The domain is periodic in the streamwise (x) direction, and infinite in the transverse (y) direction. In such a domain, an exact solution of the Navier-Stokes equations exists, in the form of a self-similar solution

$$u(x,y,t) = \frac{U_\infty}{2} \operatorname{erfc}\left(-y\left(\frac{Re}{4(t-t_0)}\right)^{1/2}\right), \tag{4}$$

where Re is the Reynolds number, U_∞ is the fluid velocity as $y \to \infty$, and t_0 is the singularity time.

This self-similar solution is always unstable for a certain range of wavenumbers in the x-direction: this is called Kelvin-Helmholtz instability. High-wavenumber perturbations grow exponentially at first, and then saturate due to nonlinearities. As the shear layer spreads in the y-direction, lower wavenumbers become more unstable, and the energy from high-wavenumber perturbations is transferred to the

lower wavenumbers, also through nonlinearities, through a process known as *pairing*. We seek to develop models of this phenomenon, that capture the exponential growth, nonlinear saturation, and pairing.

Our approach is to write the equations in a scaled reference frame, in which the spreading of the shear layer in the y-direction is removed. We do this by scaling the velocity as

$$u(x,y,t) = \tilde{u}(x, g(t)y, t), \qquad v(x,y,t) = \frac{1}{g(t)}\tilde{v}(x, g(t)y, t), \tag{5}$$

where $g(t) > 0$ is a scaling factor, and obtaining reduced-order models for the dynamics of the scaled velocities $(\tilde{u}, \tilde{v})$. We choose the scaling factor $g(t)$ such that the solution best matches a given reference function, called a *template function*. For instance, for the self-similar solution (4), $g(t)$ would be the coefficient multiplying y, and the scaled velocities $(\tilde{u}, \tilde{v})$ would be constant in time. For more complicated solutions, we still write dynamics in this scaled reference frame, along with dynamics of the scaling $g(t)$, and then obtain reduced-order models in the scaled frame. In this setting, we obtain a model using 4 complex POD modes plus the scaling factor $g(t)$ (related to the shear layer thickness), that capture all of the desired effects: growth, saturation, and pairing. For further details and preliminary results, see [6].

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Reachability Analysis for Hybrid Dynamic Systems

Olaf Stursberg

1. Introduction

Hybrid Dynamic Systems (HDS), which combine discrete event dynamics with ordinary differential equations, have evolved as a suitable means to represent a wide range of applications, including not only simple mechanic systems but also manufacturing and processing systems as well as robotic and automotive applications (see, e.g., [2] for a survey). The analysis, design, and control of HDS is often an intricate task as the interaction of the two types of dynamics requires the consideration of techniques for smooth systems as well as search methods for discrete models. This contribution addresses techniques for analysis and synthesis for hybrid models based on reachability analysis (RA). The latter is the task of computing the subset of the state space of a HDS, which can be attained starting from an initial state set (and, if the HDS is not autonomous, for a given set of input trajectories).

Different analysis and design tasks can be formulated such that RA is the key step of computation. In order to make the computations efficient for real-world computations, much effort has been invested in recent years to develop RA for different classes of HDS and for the use in analysis and control synthesis – this contribution presents a survey on different approaches which, in particular, introduce the use of model abstraction and refinement into RA-based techniques for HDS.

2. Hybrid Dynamic Models and Reachability Analysis

Among a large set of formulations for HDS, the class of *hybrid automata* (HA) can be regarded as the most important one. While [4] presents a very general class of HA, this contribution refers to the class specified in [8]. The automaton $HA = (X, U, V, Z, inv, \Theta, g, r, f)$ consists of: the continuous state space $X \subseteq \mathbb{R}^{n_x}$ (on which the *state vector* x is defined), the continuous input space $U \subseteq \mathbb{R}^{n_u}$ (with input vector $u \in U$), the discrete input space $V \subset \mathbb{N}$, the finite set of *locations* $Z = \{z_1, \ldots, z_{n_z}\}$, an invariant mapping $inv : Z \to 2^X$ (assigning a permitted subset of X to each $z \in Z$), the set of *transitions* $\Theta \subseteq Z \times Z$, a mapping $g : \Theta \to 2^X$ that associates a *guard* $g((z_1, z_2)) \subseteq X$ with each $(z_1, z_2) \in \Theta$, a *reset function* $r : \Theta \times X \to X$ which assigns an updated state $x' \in X$ to each $(z_1, z_2) \in \Theta$, and a *vector field* $f : Z \times X \times U \times V \to \mathbb{R}^{n_x}$ that defines an ordinary differential equation (ODE) $\dot{x} = f(z, x, u, v)$ for each location $z \in Z$.

A feasible execution of HA is an alternating sequence of continuous evolutions and discrete transitions, where the following applies: (a) a continuous evolution is the solution of the ODE constrained by the invariant function, initialized by the preceding reset, and subject to given input trajectories for u and v, (b) a transition is enabled if the continuous state is inside of the guard $g((z_1, z_2))$, and

(c) the continuous state is updated according to the reset function r (see [8] for details).

The computation of reachable sets involves to determine and represent (or conservatively approximate) the subset of hybrid states of the HA that are attainable for a given initialization and sets of input trajectories (as imposed by a controller). Let $s = (z, x)$ denote a hybrid state, S the hybrid state set, Φ_u the set of continuous input trajectories, Φ_v the set of discrete input trajectories, and Φ_s the set of hybrid state trajectories according to the sketched model dynamics. For a given set of initializations $S_0 = (z_0, X_0) \subset S$ with $X_0 \subseteq inv(z_0)$ and given Φ_u and Φ_v, the reachable set of HA is defined as $R := \{s \in S \mid \exists s_0 \in S_0, \phi_u \in \Phi_u, \phi_v \in \Phi_v : s \in \phi_s \text{ for any feasible } \phi_s \in \Phi_s\}$.

Assuming that the inputs trajectories ϕ_u and ϕ_v of HA are fixed by a given controller function (i.e. the resulting system can be seen as autonomous), the following simple algorithm for computing R can be formulated:

$S_0 := (z_0, X_0),\ k := 0,\ D := S_0,\ R := \emptyset$
`WHILE` $D := \emptyset$
    $k := k + 1,\ R := R \cup D,\ S_k := Reach(D),\ D := S_k \setminus R$
`END`

k denotes an iteration counter, R the accumulated set of reachable hybrid states, D the set of newly reached states in the current iteration, and *Reach* an operator that computes one-step successors of the argument set. Critical issues of executing this algorithm are the representation of the sets S_k, D, and R as well as the operator *Reach*. Further details can be found in [6].

Typical procedures for analyzing and synthesizing HDS based on such reachability computations are sketched in the sequel.

3. Verification based on Reachability Analysis

Given a HDS (possibly representing the composition M of a plant P and a controller C for fixing the input trajectories, i.e. $M = C||P$) and a specification γ, the task of *verification* is to show that $M \models \gamma$. A specification considered in many approaches is the one of *safety*: given HA, S_0, and an unsafe set S_u, show that $R \cup S_u = \emptyset$. Since the algorithmic proof of this property is computationally expensive for HDS in general [6], this contribution describes the approach of *abstraction-based and counterexample-guided verification* [3]. It maps the model M into an abstract model A given as finite state automaton, which can be obtained, e.g., simply by omitting the continuous dynamics and for which γ can be shown (or refuted) with relatively low effort. In case of refutation, the violating execution (the *counterexample*) is mapped back onto the composition M and is (in-)validated for it by using local reachability computation.

If the counterexample is invalid for the hybrid model M, the abstract model is adapted according to the reachability information by eliminating transitions and/or splitting states. By iterating between the hybrid model and the abstract

representation, the verification of the satisfaction / violation of γ for the model M can be completed with - in average - much lower effort as with a full computation of the reachable set.

The task of verifying safety using reachability computations can be extended to HDS with uncertain parameters as described in [1].

4. Controller Synthesis using Reachable Sets

With a slightly modified structure, the abstraction-based iterative computation can also be used to generate a controller C for a hybrid model HA representing a given plant P. More precisely, given P and a specification γ comprising a safety specification as a above, S_0, and a specification for a control goal (i.e. a subset of hybrid states $S_G \subset S$ into which P has to be driven), the task is to generate C such that $P||C \models \gamma$. The task can be solved by the following steps [9, 7]: It is assumed that C is a controller that only encodes discrete input trajectories where v is defined on a finite set and can be changed only at points of time when P executes a transition. Then, a finite state automaton C' is generated that models all possible input trajectories. For the composition $C'||P$ an abstraction A is generated which is searched for a candidate path to drive P from the initial into the goal set. The path is (in-)validated by reachability analysis for the model P. If it is invalid, A is refined similarly as in the verification procedure (i.e. C' is reduced by the invalid path).

The iteration continues until a feasible control strategy is found, leading to the controller C which limits the possible executions of P to a trajectory bundle that transfers any initial state into a goal region while avoiding unsafe state sets.

5. Optimization based on Abstractions

The synthesis procedure can be extended by performance measures such that it selects from a set of feasible controllers the one with lowest cost for transferring the HDS into a goal region. Formally this means that for a hybrid plant model P, a specification γ defining an initial set S_0, a goal set S_G, and an unsafe set S_u, as well as a performance measure ψ, the task is to compute a controller C such that it minimizes ψ subject to $P||C \models \gamma$.

The proposed solution is to first generate an abstract model A with local transition costs by using reachability computations. A cost-optimal candidate path is then computed for A and the path (i.e. the sequence of discrete states z and inputs v) is projected back onto P. Re-optimization evaluates the achievable performance over variations of u for the hybrid model P. Iterations between A and P, which lead to refinement of A with respect to transition costs and feasible discrete inputs, result in convergence to a (sub-)optimal solution of the control problem.

6. Conclusions

This contribution presents three algorithmic procedures for HDS that employ reachability computations as the basic step combined with the principle of model

abstraction and refinement. Set-valued reachability computations are indispensable if either uncertainties of the initial state or the parametrization of HA exists, or if the continuous space is partitioned in order to achieve a finite number of steps of computation. The use of abstractions is favorable, if the inherently high computational load of algorithmic techniques for HDS has to be lowered to enable applicability to real world systems. It has been found for applications from the areas of automobiles, robotics, and processing systems that the proposed techniques can significantly decrease the computation time compared to techniques that analyze or optimize directly the HDS.

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Bilinear discretization of quadratic vector fields

Yuri B. Suris
(joint work with Matteo Petrera, Andreas Pfadler)

This talk deals with some aspects of the *problem of integrable discretization*, as defined in [1]: consider a completely integrable flow

$$\dot{x} = f(x) = \{H, x\} \tag{1}$$

with a Hamilton function H on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot,\cdot\}$. Thus, the flow (1) possesses sufficiently many functionally independent integrals $I_k(x)$ in involution. The problem consists in finding a family of diffeomorphisms $\mathcal{P} \to \mathcal{P}$,

$$\widetilde{x} = \Phi(x;\epsilon), \tag{2}$$

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:

- The maps (2) *approximate* the flow (1): $\Phi(x;\epsilon) = x + \epsilon f(x) + O(\epsilon^2)$.
- The maps (2) are *Poisson* w. r. t. the bracket $\{\cdot,\cdot\}$ or some its deformation $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
- The maps (2) are *integrable*, i.e. possess the necessary number of independent integrals in involution, $I_k(x;\epsilon) = I_k(x) + O(\epsilon)$.

The talk is devoted to discretizations of the type introduced in [2, 3] and missing from the book [1], despite its encyclopedic nature. Reasons for this omission: discretization of the Euler top [2] seemed to be an isolated curiosity; discretization of the Lagrange top [3] seemed to be incomprehensible, if not even wrong.

It turns out that the discretizations of Hirota-Kimura are instances of a general method for discretizing differential equations with quadratic vector fields, proposed by W. Kahan in [4]. According to this method, a differential equation

$$\dot{x} = Q(x) + Bx,$$

where $B \in \mathbb{R}^{n\times n}$ and $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *quadratic* function, is discretized as

$$(\widetilde{x} - x)/\epsilon = Q(x, \widetilde{x}) + B(x + \widetilde{x}),$$

where $Q(x, \widetilde{x}) = Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})$ is the corresponding symmetric *bilinear* function. General features of this discretization:

- discrete equations are linear w.r.t. $\widetilde{x}$ and define therefore an *explicit* (rational) map $\widetilde{x} = f(x, \epsilon)$;
- moreover, this map is *reversible* (therefore birational):

$$f^{-1}(x, \epsilon) = f(x, -\epsilon). \tag{3}$$

Kahan illustrated his method with an application to the famous Lotka-Volterra system, where it produces non-spiralling orbits, unlike the majority of conventional integrators. A sort of an explanation of this favorable behavior was given by J. Sanz-Serna in [5], where it was shown that Kahan's integrator for the Lotka-Volterra system has the Poisson property. A bi-Hamiltonian structure of the Hirota-Kimura discretizations of the Euler top was established in [6].

The following definition is a formalization and an extension of the method proposed in [3].

Definition (Hirota-Kimura basis). *For a given birational map $f : \mathbb{R}^n \to \mathbb{R}^n$, a set of functions $\Phi = (\varphi_1, \ldots, \varphi_l)$, linearly independent over $\mathbb{R}$, is called a HK-basis, if for every $x \in \mathbb{R}^n$ there exists a vector $c = (c_1, \ldots, c_l) \neq 0$ such that*

$$c_1\varphi_1(f^i(x)) + \ldots + c_l\varphi_l(f^i(x)) = 0 \quad \forall i \in \mathbb{Z}.$$

For a given $x \in \mathbb{R}^n$, the set of all vectors $c \in \mathbb{R}^l$ with this property will be denoted by $K_\Phi(x)$ and called the null-space of the basis Φ (at the point x). This set clearly is a vector space.

Note: we cannot claim that $h = c_1\varphi_1 + ... + c_l\varphi_l$ is an integral of motion, since vectors $c \in K_\Phi(x)$ vary from one initial point x to another. However, the existence of a HK-basis Φ with $\dim K_\Phi(x) = d$ confines the orbits of f to $(n-d)$-dimensional invariant sets.

Proposition (from HK-bases to integrals). *If Φ is a HK-basis for a map f, then $K_\Phi(f(x)) = K_\Phi(x)$. In particular, if $\dim K_\Phi(x) = 1$ for all $x \in \mathbb{R}^n$, and $K_\Phi(x) = [c_1(x) : \ldots : c_l(x)] \in \mathbb{RP}^{l-1}$, then the functions c_j/c_k are integrals of motion for f.*

A sufficient condition for a given set of functions Φ to be a HK-basis, as well as a theoretical basis for a numerical algorithm for finding HK-bases, is given by the following result.

Theorem (finding HK-bases). *Let, for all $x \in \mathbb{R}^n$, the dimension of the solution space of the homogeneous system for $c_1, \ldots, c_l$,*

$$c_1\varphi_1(f^i(x)) + \ldots + c_l\varphi_l(f^i(x)) = 0, \quad i = 0, \ldots, s-1,$$

be equal to $l-s$ for $1 \leq s \leq l-d$ and to d for $s = l-d+1$. Then $K_\Phi(x)$ coincides with the solution space for $s = l-d$, and, in particular, $\dim K_\Phi(x) = d$.

We applied these notions and results for studying the Hirota-Kimura type discretization of the *Clebsch system*, which describes the motion of a rigid body in an ideal fluid:

$$\dot{m} = p \times \Omega p, \quad \dot{p} = p \times m,$$

where $m, p \in \mathbb{R}^3$, $\Omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$. Its Hirota-Kimura type discretization was proposed by T. Ratiu on Oberwolfach workshop "Geometric Integration" (March 2006):

$$\widetilde{m} - m = \epsilon(\widetilde{p} \times \Omega p + p \times \Omega\widetilde{p}), \quad \widetilde{p} - p = \epsilon(p \times \widetilde{m} + \widetilde{p} \times m).$$

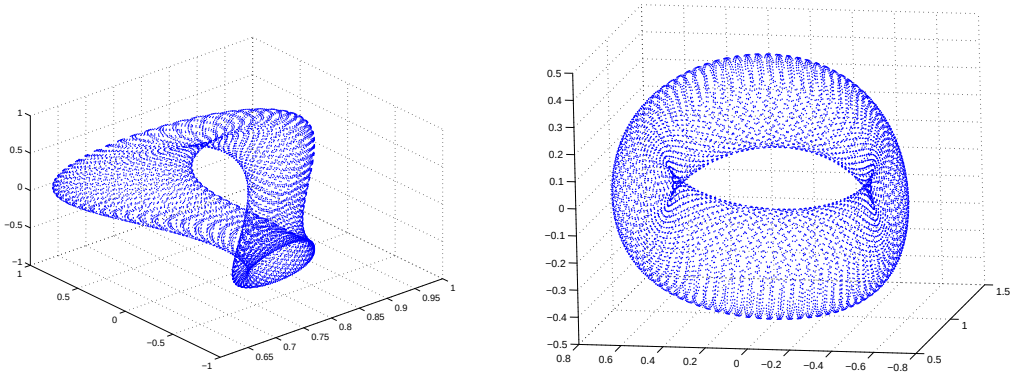
This defines a birational map $(\widetilde{m}, \widetilde{p}) = f(m, p, \epsilon)$, reversible as in (3).

The Clebsch system is Hamiltonian w.r.t. Lie-Poisson bracket of $e(3)$, and has four functionally independent integrals in involution:

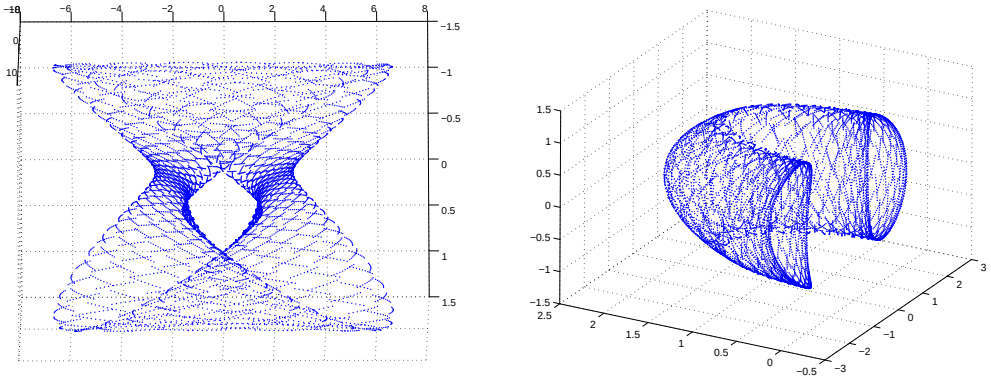
$$I_i = p_i^2 + \frac{m_j^2}{\omega_k - \omega_i} + \frac{m_k^2}{\omega_j - \omega_i}, \ (i,j,k) = c.p.(1,2,3), \quad H_4 = m_1p_1 + m_2p_2 + m_3p_3.$$

We are interested in establishing the integrability of the map f. The integrability is suggested by considering the phase portraits. The first pair of plots below shows the projections to (m_1, m_2, m_3) and to (p_1, p_2, p_3) of one orbit of the discrete

Clebsch system with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$ and $\epsilon = 1$; ; initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.



On the second pair of plots the parameters have been changed to $\omega_1 = 1$, $\omega_2 = 0.2$, $\omega_3 = 30$.



These (and similar) pictures suggest that the orbits are confined to two-dimensional submanifolds in the six-dimensional phase space. These observations are confirmed by the following results.

Theorem (integrability of the discrete Clebsch system). *a) The set of functions*

$$\Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1p_1, m_2p_2, m_3p_3, 1)$$

is a HK-basis for f, with $\dim K_\Phi(m, p) = 4$. Thus, any orbit of f lies on an intersection of four quadrics in $\mathbb{R}^6$.

b) The following four sets of functions are HK-bases for f with one-dimensional null-spaces:

$$\begin{aligned}
\Phi_0 &= (p_1^2, p_2^2, p_3^2, 1),\\
\Phi_1 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1p_1),\\
\Phi_2 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2p_2),\\
\Phi_3 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3p_3).
\end{aligned}$$

There holds: $K_\Phi = K_{\Phi_0} \oplus K_{\Phi_1} \oplus K_{\Phi_2} \oplus K_{\Phi_3}$.

Our proof of this theorem is computer assisted. A general structure which would provide us with less computational proofs and with more insight remains to be found. In particular, no Lax representation has been found. Nothing is known about the existence of an invariant Poisson structure for the map f. In order to appreciate the difficulty of symbolic computations necessary to establish our results, one should be aware of the *complexity* of the map f. Numerators and denominators of components of $(\widetilde{m},\widetilde{p}) = f(m,p,\epsilon)$ are polynomials of degree 6, the numerators of $\widetilde{p}_i$ consist of 31 monomials, the numerators of $\widetilde{m}_i$ consist of 41 monomials, the common denominator consists of 28 monomials. The claims in part b) of the above theorem refer to the solutions of the following systems:

$$(c_1p_1^2 + c_2p_2^2 + c_3p_3^2) \circ f^i = 1,$$
$$(\alpha_1p_1^2 + \alpha_2p_2^2 + \alpha_3p_3^2 + \alpha_4m_1^2 + \alpha_5m_2^2 + \alpha_6m_3^2) \circ f^i = m_1p_1 \circ f^i,$$
$$(\beta_1p_1^2 + \beta_2p_2^2 + \beta_3p_3^2 + \beta_4m_1^2 + \beta_5m_2^2 + \beta_6m_3^2) \circ f^i = m_2p_2 \circ f^i,$$
$$(\gamma_1p_1^2 + \gamma_2p_2^2 + \gamma_3p_3^2 + \gamma_4m_1^2 + \gamma_5m_2^2 + \gamma_6m_3^2) \circ f^i = m_3p_3 \circ f^i.$$

The first one has to be solved for $i = 0, 1, 2$, each of the last three systems has to be solved for $i = -2, \ldots, 3$ (say). This can be done numerically without any difficulties, but becomes (nearly) impossible for a symbolic computation, due to complexity of f^2. The following table lists degrees of numerators and denominators of f^2:

	deg	$\deg_{p_1}$	$\deg_{p_2}$	$\deg_{p_3}$	$\deg_{m_1}$	$\deg_{m_2}$	$\deg_{m_3}$
Denom. of f^2	27	24	24	24	12	12	12
Num. of $p_1 \circ f^2$	27	25	24	24	12	12	12
Num. of $p_2 \circ f^2$	27	24	25	24	12	12	12
Num. of $p_3 \circ f^2$	27	24	24	25	12	12	12
Num. of $m_1 \circ f^2$	33	28	28	28	15	14	14
Num. of $m_2 \circ f^2$	33	28	28	28	14	15	14
Num. of $m_3 \circ f^2$	33	28	28	28	14	14	15

The numerator of the p_1-component of $f^2(m,p)$, as a polynomial of m_k, p_k, contains 64 056 monomials; as a polynomial of m_k, p_k, and ω_k, it contains 1 647 595 terms. Symbolic manipulations with polynomials of this complexity are virtually impossible. In order to prove the above theorem, one needs new ideas! The main one: find (observe numerically) linear relations between the components of $K_\Phi(x_0)$, and then use them to replace the dynamical relations. On this way, one arrives at the following, more detailed, results.

Theorem (HK-basis Φ_0). *At each point $(m,p) \in \mathbb{R}^6$ there holds:*

$$K_{\Phi_0}(m,p) = [c_1 : c_2 : c_3 : -1],$$

where $c_i = 1/J_0 + \epsilon^2\omega_i$ with

$$J_0(m,p,\epsilon) = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1p_1^2 + \omega_2p_2^2 + \omega_3p_3^2)}\,.$$

This function is an integral of motion of the map f.

This is the only "simple" integral of f!

Theorem (additional HK-basis Ψ). *Set $\Psi = (p_1^2, p_2^2, p_3^2, m_1p_1, m_2p_2, m_3p_3)$. At each point $(m,p) \in \mathbb{R}^6$ there holds:*

$$K_\Psi(m,p) = [-1 : -1 : -1 : d_7 : d_8 : d_9],$$

with

$$d_k = \frac{(p_1^2+p_2^2+p_3^2)(1+\epsilon^2 d_k^{(2)} + \epsilon^4 d_k^{(4)} + \epsilon^6 d_k^{(6)})}{\Delta}, \quad k = 7,8,9,$$

$$\Delta = m_1p_1 + m_2p_2 + m_3p_3 + \epsilon^2\Delta^{(4)} + \epsilon^4\Delta^{(6)} + \epsilon^6\Delta^{(8)},$$

where $d_k^{(2q)}$ and $\Delta^{(2q)}$ are homogeneous polynomials of degree $2q$ in phase variables. The functions d_7, d_8, d_9 are integrals of motion of the map f. They satisfy

$$(\omega_2-\omega_3)d_7 + (\omega_3-\omega_1)d_8 + (\omega_1-\omega_2)d_9 = 0.$$

Any two of them together with J_0 are functionally independent.

Each of the functions d_k takes about three pages of MAPLE output!

Theorem (HK-bases Φ_1, Φ_2, Φ_3). *At each point $(m,p) \in \mathbb{R}^6$ there holds:*

$$\begin{aligned} K_{\Phi_1}(m,p) &= [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1], \\ K_{\Phi_2}(m,p) &= [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1], \\ K_{\Phi_3}(m,p) &= [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1], \end{aligned}$$

where α_j,β_j, and γ_j are rational functions of (m,p) of the form

$$h = \frac{h^{(2)} + \epsilon^2 h^{(4)} + \epsilon^4 h^{(6)} + \epsilon^6 h^{(8)} + \epsilon^8 h^{(10)} + \epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2+p_2^2+p_3^2)\Delta}.$$

Here h stands for any of the functions $\alpha_j, \beta_j, \gamma_j$, $j = 1,2,3$, and the corresponding $h^{(2q)}$ are homogeneous polynomials of degree $2q$ in phase variables. For instance,

$$\begin{array}{lll} \alpha_1^{(2)} = H_3 - I_1, & \alpha_2^{(2)} = -I_1, & \alpha_3^{(2)} = -I_1, \\ \beta_1^{(2)} = -I_2, & \beta_2^{(2)} = H_3 - I_2, & \beta_3^{(2)} = -I_2, \\ \gamma_1^{(2)} = -I_3, & \gamma_2^{(2)} = -I_3, & \gamma_3^{(2)} = H_3 - I_3, \end{array}$$

where $H_3 = p_1^2 + p_2^2 + p_3^2$. The functions $\alpha_j, \beta_j, \gamma_j$ are integrals of motion of the map f. The four integrals J_0, α_1, β_1 and γ_1 are functionally independent.

Thus, we established the integrability of the Hirota-Kimura discretization of the Clebsch system, in the sense of

- existence, for every initial point $(m,p) \in \mathbb{R}^6$, of a four-dimensional pencil of quadrics containing the orbit of this point;
- existence of four functionally independent integrals of motion (conserved quantities).

Analogous theorems hold also for an arbitrary flow of the Clebsch system (with one "simple" and three very big integrals).

Conjecture. *For any algebraically completely integrable system with a quadratic vector field, its Hirota-Kimura discretization remains algebraically completely integrable.*

This conjecture is supported by the results of [2, 3, 6] and of the present work, as well as our preliminary results on the following systems: Zhukovsky-Volterra

gyrostat; $so(4)$ Euler top and its commuting flows; Volterra lattice; Toda lattice; classical Gaudin magnet. See also [7] on the Suslov system. If true, this statement could be related to addition theorems for multi-dimensional theta-functions.

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Multiscale time integration for aerosol dynamics

Matthew West

(joint work with Jim Barnard, Dick Easter, Adrian Lew, Jerry Marsden, Michael Ortiz, Nicole Riemer, Minyong Shin, Ronny Votel and Rahul Zaveri)

Atmospheric aerosol modeling involves complex physics over a wide ranges of timescales, including coagulation, evaporation/condensation, and chemistry processes. Both the evaporation/condensation and coagulation processes do not have a clear timescale separation, making it difficult to homogenize or average out the fast scales. We consider box models of aerosols with deterministic chemistry and stochastic coagulation models and present two new multiscale algorithms that enable efficient simulation of all scales without requiring scale separation.

First, we consider asynchronous multistep ODE integrators for deterministic systems that enable adaptive time resolution. This is an extension of work in [4, 5] where Asynchronous Variational Integrators (AVIs) were developed and generalized to Asynchronous Splitting Methods (ASMs). These are methods to numerically integrate systems of the form

$$\dot{x}(t) = f(x(t)) = \sum_{i=1}^{M} f_i(x(t)) \tag{1}$$

that integrate each component f_i asynchronously in time. Multistep Asynchronous Splitting Methods (MASMs) generalize ASMs to integrators of the form

$$x_{k+1} + \sum_{\ell=p-n+1}^{p} \sum_{i=1}^{M} \alpha_{i,\ell} x_{i,\ell} = \sum_{\ell=p-n+1}^{p} \sum_{i=1}^{M} \beta_{i,\ell} f_i(x_{i,\ell}) \tag{2}$$

where x_{k+1} is the next solution value to be computed, $x_{i,\ell}$ is the ℓ-th previous solution value computed via an f_i update, and $\alpha_{i,\ell}$ and $\beta_{i,\ell}$ are coefficients that define the step. These coefficients must be determined to satisfy both order and stability conditions and for a fully adaptive and asynchronous scheme they will be time-varying and i-dependent.

The second new algorithm we present is a multiscale sampling method for stochastic coagulation. This is described in more detail in [6]. We are concerned with coagulation for a multivariate number density $n(\mu, t)$ where μ_i is the mass of species i in an aerosol particle, for $\mu \in \mathbb{R}^A$. The continuum limit coagulation process is described by the Smolukowski equation

$$\begin{aligned} \frac{\partial n(\mu,t)}{\partial t} &= \frac{1}{2}\int_{\mathbb{R}^{A+}} K(\mu', \mu-\mu') n(\mu', t)\, n(\mu-\mu', t)\, d\mu' \\ &\quad - \int_{\mathbb{R}^{A+}} K(\mu, mu') n(\mu, t)\, n(\mu', t)\, d\mu' \end{aligned} \tag{3}$$

where $K(\mu, \mu')$ is the probability rate that a particle with masses μ coagulates with one with masses μ'.

We use a particle-based solution method, where we store N_{MC} particles with masses $\mu_1, \ldots, \mu_{N_{\mathrm{MC}}} \in \mathbb{R}^A$ in a volume $V \in \mathbb{R}$. The coagulation model is a Markov process where each particle coagulates with every other with probability rate $K(\mu_i, \mu_j)$. The standard exact simulation method for such systems is Gillespie's SSA method [3], but this is very inefficient for systems with a wide range of scales in the rates K. Accelerated multiscale variants of SSA have been developed in recent years, such as [1, 2], but these require scale separation which is not present in atmospheric aerosol coagulation. We use a new solution method, where we group the particles into bins by total mass and use the fact that the primary dependence of K on μ is via the total mass $\|\mu\|_1$ of the particle. Coagulation events are then generated for each bin pair, using an approximate sampling method. This gives the algorithm:

divide total mass axis into bins
$N_{\mathrm{MC}}(b)$ is the number of particles in bin b
$\mu(b, i)$ is the mass vector of the i-th particle in bin b
$K_{\max}(b_1, b_2)$ is a precomputed upper bound on the kernel for any particles from bins b_1 and b_2
Δt is the timestep
for all bin pairs (b_1, b_2) **do:**
    $N_{\mathrm{event}} = N_{\mathrm{MC}}(b_1) N_{\mathrm{MC}}(b_2)/2$
    $N_{\mathrm{test}} = \left\lceil K_{\max}(b_1, b_2)\, \Delta t\, N_{\mathrm{event}}/V \right\rceil$
    for N_{test} repetitions **do:**
        randomly choose particles i_1 and i_2 uniformly in bins b_1 and b_2
        $K_{12} = K\big(\mu(b_1, i_1), \mu(b_2, i_2)\big)$
        randomly choose r uniformly in $[0, 1]$

if $r < K_{12}\,\Delta t\,N_{\text{event}}/(N_{\text{test}}V)$ **then:**
coagulate the two particles, updating the arrays $N(b)$ and $\mu(b,i)$
end if
end for
end for

Both of these algorithms mean that the computational work performed is locally adapted to be proportional to the complexity of the evolution, measured either by curvature of a vector field or by the event rate in a Markov process.

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Dirac cotangent bundle reduction

Hiroaki Yoshimura

(joint work with Jerrold E. Marsden)

In this talk, we show a reduction theory for the canonical Dirac structures D on the cotangent bundle T^*Q of a configuration manifold Q and we carry out the reduction under the assumption that a Lie group G acts freely and properly on Q, in which case there is an associated principal bundle $\pi : Q \to Q/G$. This procedure induces, in particular, a reduction theory for standard implicit Lagrangian systems, which includes the case of degenerate Lagrangians.

The notion of Dirac structures is a generalized notion of unifying pre-symplectic and almost Poisson structures, which was introduced by Courant and Weinstein [2] in conjunction with Hamiltonian mechanics, inspired from Dirac's theory of constraints. Interestingly, an idea of *interconnections*, which we now know can be modeled by Dirac structures, has been often employed in engineering (see, for instance, [3, 6]), and it was shown by van der Schaft and Maschke [5] and Bloch and Crouch [1] that dynamics of L-C circuits, which is a typical interconnected system, can be formulated by using Dirac structures in the context of Hamiltonian mechanics, namely, implicit Hamiltonian systems. On the other hand, the link with Lagrangian mechanics had not been sufficiently studied though Dirac's theory of constraints was originally started from degenerated Lagrangians, until the notion

of implicit Lagrangian systems was established by Yoshimura and Marsden [7, 8] in the context of Dirac structures D_{Δ_Q} on T^*Q, which is induced from given constraint distributions Δ_Q on Q, where D_{Δ_Q} can be defined by using the canonical symplectic structure Ω on T^*Q.

Given a Lagrangian L (possibly degenerate) on the tangent bundle TQ, an implicit Lagrangian system is a triple (L, D, X), which satisfies, for each $(q, v, p) \in TQ \oplus T^*Q$,

$$(X(q, v, p), \mathbf{d}E(q, v, p)|_{TP}) \in D_{\Delta_Q}(q, p)$$

where $X : TQ \oplus T^*Q \to TT^*Q$ is a partial vector field, $P = \mathbb{F}L(\Delta_Q) \subset T^*Q$ and E is the generalized energy defined by $E(q, v, p) = \langle p, v\rangle - L(q, v)$. For the case in which $\Delta_Q = TQ$, one can obtain a standard implicit Euler-Lagrange equations

$$\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v},$$

which can be also obtained from the Hamilton-Pontryagin principle

$$\delta \int_a^b \{L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle\}\, dt = 0$$

and with fixed endpoints $\delta q(a) = \delta q(b) = 0$.

Next, we consider the case in which a Lie group act on Q freely and properly, where there is a principal bundle $Q \to Q/G$ and let $L : TQ \to \mathbb{R}$ be a left-invariant Lagrangian, possibly degenerate. Let $A : TQ \to \mathfrak{g}$ be a chosen principal connection on the principal bundle $\pi : Q \to Q/G$ and we employ the isomorphism $(TQ \oplus T^*Q)/G \cong_A T(Q/G) \oplus T^*(Q/G) \oplus \tilde{V}$, where $\tilde{V} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}} = (Q \times \mathfrak{g})/G$ is the associated bundle to $\mathfrak{g}$, regarded as a bundle over Q/G. We explore geometry of variations of reduced curves in $(TQ \oplus T^*Q)/G$ and we develop a reduced Hamilton-Pontryagin principle as

$$\delta \int_{t_0}^{t_1} \left\{ l(x(t), u(t), \bar{\eta}(t)) + \langle y(t), \dot{x}(t) - u(t)\rangle + \left\langle \bar{\mu}(t), \bar{\xi}(t) - \bar{\eta}(t)\right\rangle \right\} dt = 0,$$

under appropriate variations of curves as well as boundary conditions, where $l = L|\mathfrak{g}$ is the reduced Lagrangian. Then, it follows that one can obtain **horizontal implicit Lagrange-Poincaré equations**

$$\frac{Dy}{Dt} = \frac{\partial l}{\partial x} - \left\langle \bar{\mu}, \widetilde{B}(\dot{x}, \cdot)\right\rangle, \quad \dot{x} = u, \quad y = \frac{\partial l}{\partial u}$$

as well as **vertical implicit Lagrange-Poincaré equations**

$$\frac{D\bar{\mu}}{Dt} = \mathrm{ad}^*_{\bar{\xi}}\bar{\mu}, \quad \bar{\mu} = \frac{\partial l}{\partial \bar{\eta}}, \quad \bar{\xi} = \bar{\eta}.$$

We establish **Dirac cotangent bundle reduction**, namely, a reduction procedure for the canonical Dirac structure on the cotangent bundle T^*Q by employing a connection dependent isomorphism $T^*Q \cong \widetilde{Q}^* \times \mathfrak{g}^*$, (an unreduced version of the Sternberg space, where $\tilde{Q}^*$ is a pull-back bundle) via which we develop a G-invariant Dirac structure $\bar{D}$ on $\widetilde{Q}^* \times \mathfrak{g}^*$. By using an isomorphism $(\widetilde{Q}^* \times \mathfrak{g}^*)/G \cong T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ between the Sternberg and Weinstein spaces and

by taking the quotient of $\bar{D}$ by G, one is lead to a **gauged Dirac structure** $[\bar{D}]_G = [\bar{D}]_G^{\mathrm{Hor}} \oplus [\bar{D}]_G^{\mathrm{Ver}}$ on the bundle $(TT^*Q)/G \cong \tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})$ over $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$, where $[\bar{D}]_G^{\mathrm{Hor}}$ is a *horizontal Dirac structure* on the bundle $\tilde{\mathfrak{g}}^* \times TT^*(Q/G)$ over $T^*(Q/G)$ and $[\bar{D}]_G^{\mathrm{Ver}}$ is a *vertical Dirac structure* on the bundle $\tilde{\mathfrak{g}}^* \times \widetilde{V}$ over $\tilde{\mathfrak{g}}^*$.

Finally, it is shown that making use of the gauged Dirac structure, one gets a reduction procedure for a standard implicit Lagrangian system (L, D, X), which we shall call **Lagrange-Poincaré-Dirac reduction**. Using the trivialized vector field $\bar{X}$ of $X : TQ \oplus T^*Q \to TT^*Q$, this procedure naturally yields the reduced implicit Lagrangian system $(l, [\bar{D}]_G, [\bar{X}]_G) = (l, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}}) \oplus (l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$ such that it satisfies

$$([\bar{X}]_G, [\mathbf{d}\bar{E}]_G|_{\tilde{\mathfrak{g}}^* \times (TT^*(Q/G) \oplus \widetilde{V})}) \in [\bar{D}]_G,$$

where $[\bar{X}]_G = [\bar{X}]_G^{\mathrm{Hor}} \oplus [\bar{X}]_G^{\mathrm{Ver}}$ is the reduced partial vector field and $[\mathbf{d}\bar{E}]_G = [\mathbf{d}\bar{E}]_G^{\mathrm{Hor}} \oplus [\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}$ is the reduction of the differential of the trivialized generalized energy $\bar{E}$. Then, it follows that the *horizontal implicit Lagrangian system* $(l, [\bar{D}]_G^{\mathrm{Hor}}, [\bar{X}]_G^{\mathrm{Hor}})$ that satisfies $([\bar{X}]_G^{\mathrm{Hor}}, [\mathbf{d}\bar{E}]_G^{\mathrm{Hor}}|_{\tilde{\mathfrak{g}}^* \times TT^*(Q/G)}) \in [\bar{D}]_G^{\mathrm{Hor}}$ induces *horizontal implicit Lagrange-Poincaré equations*, while the *verticall implicit Lagrangian system* $(l, [\bar{D}]_G^{\mathrm{Ver}}, [\bar{X}]_G^{\mathrm{Ver}})$ that satisfies $([\bar{X}]_G^{\mathrm{Ver}}, [\mathbf{d}\bar{E}]_G^{\mathrm{Ver}}|_{\tilde{\mathfrak{g}}^* \times \widetilde{V}}) \in [\bar{D}]_G^{\mathrm{Ver}}$ induces *vertical implicit Lagrange-Poincaré equations*. We can also develop the case in which a Hamiltonian is given (perhaps coming from a regular Lagrangian); namely, it is shown that *Hamilton-Poincaré-Dirac reduction* yields *horizontal implicit Hamilton-Poincaré equations* as well as *vertical implicit Hamilton-Poincaré equations*. Finally, illustrative examples of artificial spacecraft with rotors and space robots are shown.

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