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Control Theory: On the Way to New Application Fields

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Abstract. Control theory is an interdisciplinary field that is located at the crossroads of pure and applied mathematics with systems engineering and the sciences. Recently, deep interactions are emerging with new application areas, such as systems biology, quantum control and information technology. In order to address the new challenges posed by the new application disciplines, a special focus of this workshop has been on the interaction between control theory and mathematical systems biology. To complement these more biology oriented focus, a series of lectures in this workshop was devoted to the control of networks of systems, fundamentals of nonlinear control systems, model reduction and identification, algorithmic aspects in control, as well as open problems in control.

Mathematics Subject Classification (2000): 93xx.

Introduction by the Organisers

Control theory is an interdisciplinary field that is located at the crossroads of pure and applied mathematics with systems engineering and the sciences. Traditionally, the interaction with systems engineering and signal processing has been particularly strong. More recently, deep interactions are emerging with new application areas, such as systems biology, quantum control and information technology. The field therefore covers a wide variety of topics, ranging from fundamental mathematical aspects and new control paradigms in the sciences to real world engineering applications of industrial relevance, In particular, it has deep connections to different branches of pure and applied mathematics, including e.g. ordinary and

partial differential equations, operator theory, real and complex analysis, probability theory, numerical analysis, discrete mathematics as well as algebraic and differential geometry.

The Oberwolfach workshop "Control Theory: On the Way to New Application Disciplines" brought together about 45 internationally active researchers from Austria, Australia, France, Germany, Israel, Italy, The Netherlands, Sweden, and the United States, with both a mathematical and systems engineering background. In order to address the new challenges posed by the new application disciplines, a special focus of this workshop has been on the interaction between control theory and mathematical systems biology. This was enhanced by two special sessions with 8 lectures on systems biology and complemented by an open discussion on systems biology, to explore future perspectives in the interaction to control theory. Topics of these lectures included persistency in chemical networks, modelling and robustness issues of biochemical reaction networks, monotone systems, the dynamics of gene regulatory networks and mathematical models for metabolism of bacteria. To complement these more biology oriented talks by challenging systems engineering topics, a series of lectures was devoted to the control of networks of systems, another hot topic that currently very actively explored by the systems engineering community. In all these talks, the interaction of mathematical methods from nonlinear dynamics and control with those from discrete mathematics (esp. graph and information theory) played a crucial role. Although many fundamental mathematical questions in systems biology are apparently still unanswered or even unasked, it became evident through the discussions that the appropriate combination of such mathematical tools will be instrumental for further success in this area.

The program comprised over 30 stimulating talks on the theory and applications of control theory. Each talk had a length of thirty five minutes with at least 5 minutes discussion time. The lectures were organized into rather coherent sessions on the topics:

- Systems Biology
- Fundamentals of Nonlinear Control Systems
- Model reduction and Identification
- Networks and Control
- Algorithmic Aspects in Control
- Fundamental Control Problems

In addition to these lectures and the very active discussions throughout the workshop there was a 45 Minutes open discussion on systems biology and an Thursday evening informal open problem session, in which 10 participants presented open mathematical problems in control. Another highlight of the conference was the after dinner surprise lecture on Monday evening by Prof. R. E. Kalman (Zürich) on the classical problem of electrical network synthesis. He gave a beautiful talk on the history of the subject, spanning the period from the classical work Cauer, Forster, Brune and Bott-Duffin till the still unsolved aspects of the problem.

Weather conditions prevented the otherwise traditional Wednesday afternoon walk to St. Roman. Thus, in addition to the excellent scientific program, most participants enjoyed instead the excursion to Wolfach, to attend the very famous Nasenparade.

Workshop: Control Theory: On the Way to New Application Fields

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Abstracts

On Petri Net invariants for the study of persistence of chemical reaction networks

David Angeli
(joint work with P. De Leenheer and E.D. Sontag)

For differential equations defined on the closed positive orthant of $\mathbb{R}^n$ persistence is the property that solutions starting in the interior should not asymptotically approach the boundary of the orthant. For models describing evolutions of interacting species this physically amounts to a "non-extinction" property and is a very well studied subject in the field of mathematical ecology. Similarly, for chemical reaction networks, persistence amounts to availability of all chemical compounds at all times. While this is an interesting property both from a practical and theoretical point of view, a systematic investigation of how network's structure can affect persistence of its solutions has only recently be undertaken in [1, 2]. From the purely algebraic point of view a chemical reaction network is a list of chemical reactions $\mathcal{R} = \{R_1, R_2, \dots R_m\}$:

$$R_i : \sum_{j=1}^{n} \alpha_{ij} S_j \to \sum_{j=1}^{n} \beta_{ij} S_j \tag{1}$$

where $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ is the set of chemical species and α_{ij}s, β_{ij}s are non-negative integers called the stoichiometry coefficients. In particular, $S_R := \{S_j : \alpha_{ij} > 0\}$ are the so called reactants of the i-th reaction, whereas, $S_P = \{S_j : \beta_{ij} > 0\}$ are its products. In order to derive a dynamical system from the list in (1) several options are possible. In the following we adopt a continuous deterministic framework, in which case a system of differential equations of the following form can be associated to the network:

$$\dot{s} = \Gamma R(s) \tag{2}$$

with $s = [s_1 \dots s_n]'$ the vector of species concentrations, Γ the so called stoichiometry matrix, defined by

$$\Gamma_{ij} = \beta_{ji} - \alpha_{ji}$$

and $R(s) : [0, +\infty)^n \to [0, +\infty)^m$ the vector of reaction rates. One of the most common choices for $R(s)$ are the so called mass-action kinetics, corresponding to the following polynomial expression for $R(s)$

$$R_i(s) = k_i \prod_j s_j^{\alpha_{ij}}$$

with the convention that $0^0 = 1$. It turns out that the most natural way to study persistence (as well as many other qualitative properties of the network) is to investigate the topology of the bipartite graph which may be associated to each network. This is done by considering a node for each chemical species and a node for each reaction; directed edges indicate which species are the reactants of a given

reaction (for incoming edges) and which ones are its products (for outgoing edges). This graphical structure is, from the algebraic point of view, totally isomorphic to a Petri Net, and indeed, if modeling of the network were to be carried out by taking into account discrete numbers of molecules (rather than continuous concentrations), the corresponding discrete-event system would turn out to be a Markov Chain with a state-space and transition diagram isomorphic to the reachable set of the Petri-Net described by the bipartite graph previously defined. Several topological invariants have been defined in the study of discrete-event systems which can be represented as Petri Nets. Some of them turned out to be extremely useful also in deriving results for persistence of chemical reaction networks described by means of ODEs. Additional definitions needed in order to state our main results are provided below.

Definition A P-semiflow is any row vector $v \in \mathbb{N}^n$ such that $v\Gamma = 0$. Its support, usually denoted $\sigma(v) := \{S_j \in \mathcal{S} : v_j > 0\}$.

Definition The set of input reactions to a species S_j, is defined as $\rightarrow S_j := \{R_i : \beta_{ij} > 0\}$; similarly the set of output reactions from a species S_j, is defined to be $S_j \rightarrow := \{R_i : \alpha_{ij} > 0\}$. For a set $\Sigma \subset \mathcal{S}$, the set of input reactions is $\rightarrow \Sigma := \bigcup_{S\in\Sigma} \rightarrow S$, and similarly, the set of output reactions is $\Sigma \rightarrow := \bigcup_{S\in\Sigma} S \rightarrow$. A siphon is a subset of $\mathcal{S}$ with the property that $\rightarrow \Sigma \subset \Sigma \rightarrow$.

Our main result is a necessary as well as a sufficient condition for persistence of a chemical reaction network.

Theorem 1 A necessary condition for (2) to be persistent is that Ker[Γ] intersects the interior of the positive orthant.

Theorem 2 A sufficient condition for (2) to be persistent is that all solutions be bounded and each siphon of the network contain the support of at least one P-semiflow.

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An overview of model reduction methods and some new results

Athanasios C. Antoulas

1. Abstract of the talk

In many applications one is faced with the task of simulating or controlling complex dynamical systems. Such applications include for instance, weather prediction, air quality management, VLSI chip design, molecular dynamics, control of high rise buildings, etc. In all these cases complexity manifests itself as the number of first order differential equations which arise. For the above examples, depending on the level of modeling detail required, complexity may range anywhere from a few thousand to a few million first order equations, and above. Simulating (controlling) systems of such complexity becomes a challenging problem, irrespective of the computational resources available. In this talk we first briefly described some motivating examples, then defined the problem in mathematical terms and sketched several methodologies for its solution. The talk concluded with open problems and directions for future research.

1.1. **Some basic details.** Model reduction methods can be classified in two broad categories. The methods in the first are labeled *SVD-based* and those in the second *Krylov-based*. The former category derives its name from the fact that the corresponding reduction methods are related to the SVD (Singular Value Decomposition) and the associated 2-norm. It includes:

- Balanced truncation,
- Hankel norm approximation,
- POD (Proper Orthogonal Decomposition), and the
- Empirical gramian method.

These methods are based on the computation of controllability and observability *gramians*, which leads to the determination of states which are difficult/easy to reach and observe. The bottleneck in applying these methods consists in the fact that the computation of the gramians involves the solution of matrix Lyapunov (or Riccati) equations, which are expensive.

The reduction methods in the latter category, are based on *moment matching*, that is, matching of the coefficients of the power series expansion of the transfer function at points in the complex plane. They include

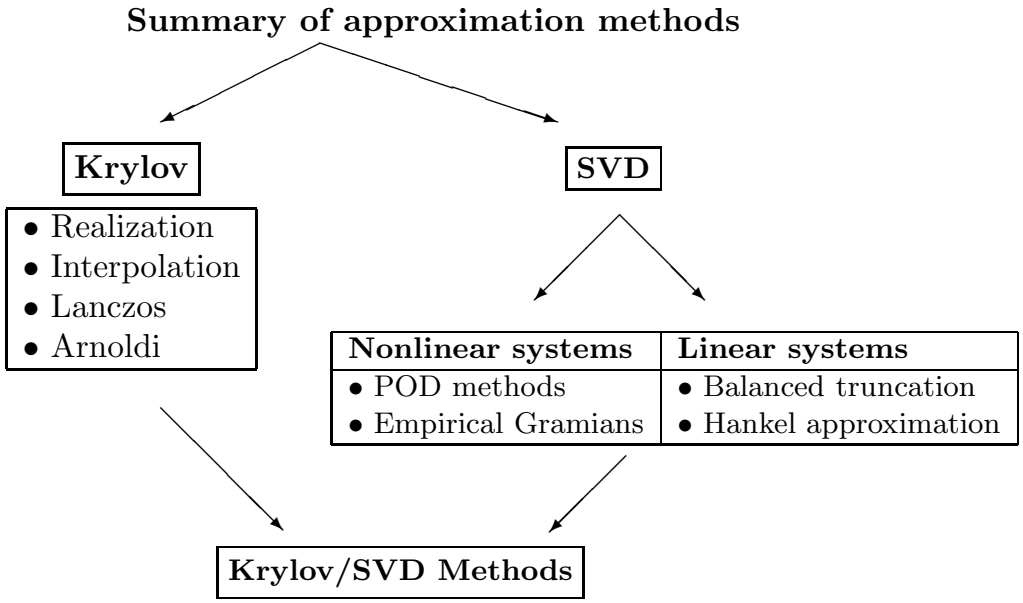
- Realization, and
- Rational interpolation.

They are closely related to the so-called Krylov iteration, encountered in numerical linear algebra, and are implemented using the

- Arnoldi procedure, or the
- Lanczos procedure.

The advantages of the BT reducrion methods include preservation of stability and an apriori computable error bound. The latter methods are numerically efficient and have lower computational cost, but in general there is no guarantee of preservation of any other properties. The third reduction method, sometimes referred to as Krylov-SVD, aims at combining the best attributes of these two families and includes methods like approximate balanced truncation by iterative methods. A

pictorial representation of reduction methods and their relationships is given next. For details we refer to the book [1].



Subsequently, the talk was dedicated to a tutorial exposition of the model reduction methods outlined above. We concluded by discussing the results of [2, 3, 4] as well as open problems and future research directions.

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Topological Methods in Nonlinear Oscillations

Christopher I. Byrnes

Periodic phenomena are pervasive in nature and in engineered systems. They are exhibited, for example, in idealized models of the solar system and in observed circadian rhythms that regulate basic biological functions. Electronic devices producing stable periodic signals underlie the electrification of the world and wireless communications. In this talk, we present conditions that guarantee that a periodic motion exists for a dynamical system.

We will begin by analyzing a nonlinear three-dimensional model of an AC motor that uses oscillations in its magnetic field to produce stable mechanical rotations. These features can be expressed very simply in terms of the vector field defining the dynamics and an "angular variable", a concept with roots in earlier work of G. D. Birkhoff.

A Liapunov function computation, together with Morse-Bott theory and the Palias-Cerf Theorem, shows that this system evolves on a solid torus in three dimensions. Forty years ago, Smale asked whether every nonvanishing smooth vector field X on the solid 3-torus had a periodic orbit. In 1996, G. and K. Kuperberg answered this in the negative. Nonetheless, under the conditions we identified, periodic orbits exist for any vector field on an n-dimensional solid torus. Indeed, the language of fields and forms allows one to use global topological methods, for example the fruitful combination of homotopy and cobordism, to understand the existence of nonlinear oscillations. Moreover, using these methods in higher dimensions and the proofs of Poincare Conjecture in dimensions three and four, we prove the existence of an invariant solid torus and an angular variable are necessary for the existence of an asymptotically stable periodic orbit.

Genetic regulatory networks: structure and qualitative dynamics

Madalena Chaves

(joint work with F. Allgöwer, T. Eissing, L. Tournier)

Genetic networks typically consist of a group of messenger RNAs and proteins, which interact and influence each other, with the goal of regulating biological events such as cell growth or division, programed cell death, or even metabolic changes. The mathematical modelling and analysis of genetic networks leads to decoding and understanding (some of) the mechanisms that regulate and control biological processes. Several different frameworks and many different mathematical techniques have been used to study biological networks. Discrete and, in particular, Boolean models have been playing an increasingly important role in the study and analysis of complex biological systems [4, 1, 2]. Boolean models describe the network structure of a system without involving any kinetic details [5], by representing the states of each component, X_i $(i = 1, \ldots, N)$, simply as "expressed" or "not expressed" (respectively, $X_i = 1$ or 0). The interactions among the various components are usually classified as "inhibition" or "activation" links (these can generally be deduced from gene/protein expression data), and represented by logical (eg., compositions of "AND", "OR") functions ($F_i(X)$, $F_i : \{0,1\}^N \to \{0,1\}$). The time evolution of the system can be written:

$$X_i[k+1] = F_i(X_1[k], X_2[k], \ldots, X_N[k]), \quad i = 1, \ldots, N.$$

In contrast to the continuous approach, discrete models will not provide quantitative details on the time evolution or variables' concentrations, but also require little (if any) knowledge of the systems' parameters. One of the advantages of discrete and Boolean models is that they characterize the possible *qualitative dynamics* of

the system based only on the structure of the network's interactions. Thus, this is a very attractive framework for the systematic study of large systems, which may include hundreds of variables and their interactions. Even if most parameters are unknown, qualitative properties such as generation of specific gene expression patterns, stability or multistability, and oscillatory modes of the network can easily be studied.

In particular, we have studied an apoptosis (or programmed cell death) network [2], composed of two modules (joint work with T. Eissing and F. Allgöwer): an anti-apoptotic signalling pathway (activation of the NFκB transcription factor and resulting events), and a pro-apoptotic pathway (activation of a cascade of caspases). The network of interactions among the anti- and pro-apoptotic pathways roughly shapes the dynamics of the system and the decision between cell survival or initiation of programmed cell death. Combining a Boolean model of the apoptosis network with continuous degradation rates for each components, a hybrid system can be created, following the idea first proposed by Glass and co-authors [3]:

$$\frac{d\,x_i}{dt} = -a_i x_i + b_i F_i(X_1, X_2, \ldots, X_N), \quad i = 1, \ldots, N.$$

where x_i and X_i represent, respectively, the continuous and Boolean variables associated with component i of the network. The parameters a_i and b_i ($\in (0, \infty)$) represent, respectively, the degradation and synthesis rates of i. At each instant t, the discrete and continuous variables are related as:

$$X_i(t) = \begin{cases} 0, & x_i(t) \leq \frac{1}{2}\frac{b_i}{a_i} \\ 1, & x_i(t) > \frac{1}{2}\frac{b_i}{a_i} \end{cases}.$$

The apoptosis system admits two steady states, representing cell survival or apoptosis. This hybrid system was used to explore different variants for the network structure, and predict effects on the probabilities of convergence to either steady state.

A more detailed analysis of the apoptosis Boolean network has also been carried out, using a general asynchronous updating approach [6] (joint work with L. Tournier). Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denote the diagram of interactions (where $\mathcal{V}$ denotes the set of variables or nodes and $\mathcal{E}$ the interactions). Assuming that only one variable is updated at each time, an asynchronous transition graph is constructed ($G(V, E)$, where V is the set of Boolean states and E a set of directed edges). The graph $G(V, E)$ describes the state space of the system and qualitatively characterizes its dynamical trajectories. In this context, we have developed a method for identifying the active or *operational interactions* responsible for a given dynamical behaviour. Given a set of states $c \subset V$, let $G(c)$ be the subgraph of $G(V, E)$ generated by the reachable set from c. An edge $e \in \mathcal{E}$ is defined to be a non-operational interaction associated with c, if the asynchronouos subgraph $\hat{G}(c)$, generated by $\hat{\mathcal{G}}(\mathcal{V}, \mathcal{E} \setminus \{e\})$, satisfies $\hat{G}(c) = G(c)$. That is, an interaction of the original Boolean model is non-operational if the asynchronous transition graph is unchanged when that interaction is removed. The set of states c may be, for

instance, a strongly connected component of the transition graph $G(V, E)$. Using an identification algorithm (such as REVEAL), a minimal family of logical rules that yields the subgraph $G(c)$ can be obtained. This family represents, in fact, the set of operational interactions associated with the region c of the state space. For the apoptosis network, two core groups of variables and interactions are identified: it is shown that these correspond to two different mechanisms responsible for the decision between programmed cell death or cell survival.

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Invariance entropy for control systems

Fritz Colonius

(joint work with Christoph Kawan)

This talk concerned with the amount of information necessary to keep a continuous time control system in a given subset Q of the state space. We introduce 'invariance entropy' that measures, how often open loop control functions must be adjusted in order to avoid exit from a subset Q of the state space. Due to the analysis of the open loop problem this information measure does not depend on a specific class of feedback strategies and hence is intrinsic.

The increasing relevance of control systems with restricted digital communication channels has spurred interest in the information necessary for accomplishing control tasks. The paper by Nair, Evans, Mareels, and Moran [2] develops a method to describe data rates necessary to render subsets Q of the state space invariant. Motivated by the classical notion of topological entropy for dynamical systems, they study for discrete time systems, how many feedbacks defined on open covers of Q are necessary in order to make Q invariant (or asymptotically stable) up to time N; then they let N tend to infinity and take the infimum over all open covers and obtain what they call feedback entropy. In particular, they show that this number is equal to the minimum data rate for a symbolic controller rendering Q invariant.

We introduce various versions of open loop entropies and discusses their relations by adapting several constructions traditionally used for topological entropy of dynamical systems to control systems.

A preliminary definition of our information measure is the following: For systems with compact control range let Q be a compact subset of the state space. Then, for $T > 0$, we let $r_{\text{inv}}(T; Q)$ be the minimal number of controls $u \in \mathcal{U}$ such that for every initial value $x \in Q$ there is u with corresponding trajectory $\varphi(t, x, u) \in Q$ for all $[0, T]$. Then we consider the exponential growth rate of these numbers as T tends to infinity,

$$\limsup_{T\to\infty} \frac{1}{T} \log r_{\text{inv}}(T; Q).$$

A characteristic feature of this information measure is that no information on the present state of the system is involved. Since no conditions are available which guarantee that this number is finite, we relax the requirement that the trajectories remain in Q by requiring that for fixed $\varepsilon > 0$ the trajectories remain in the ε-neighborhood of Q, and consider the corresponding limit superior for $T \to \infty$. Then we let $\varepsilon \to 0$.

Our main results provide upper and lower bounds for the invariance entropy; in particular, it is shown that the invariance entropy is finite. For linear control systems (with compact control range) the invariance entropy is given by the sum of the real parts of the unstable eigenvalues. Furthermore, we can also give a characterization of invariance entropy in terms of covers and a feedback construction akin to the contribution in [2].

Finally, extensions to systems with output are discussed, where one is interested in characterizing the invariance entropy of compact subsets of the output space.

This talk is partially based on the paper [1] and on the forthcoming thesis of Christoph Kawan (Universität Augsburg, 2009).

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A Dynamical System that Solves Algebraic Equations

Christian Ebenbauer

(joint work with Alessandro Arsie)

The search for continuous-time dynamical systems to compute eigenvalues has a long history. Over the last decades, several dynamical systems has been derived and investigated to compute eigenvalues. In the late 1980's, Roger Brockett introduced the so-called double-bracket equation

$$\dot{H} = [[N, H], H], \tag{1}$$

where H is an $n \times n$ real symmetric matrix, N is a constant $n \times n$ real symmetric matrix, and $[A, B] = AB - BA$. This dynamical system has several remarkable properties. For example (1) can be used to sort numbers or to diagonalize symmetric matrices. Moreover, as shown by Antony Bloch, the well-known Toda lattice is a special case of the double-bracket equation.

In this talk, a dynamical system of the form

$$\dot{A} = [[N, A^T + A], A] + \nu[[A^T, A], A], \tag{2}$$

where A is an $n \times n$ real matrix, N is a constant $n \times n$ real symmetric matrix, and ν is a positive constant, is considered. The dynamical system (2) has similar properties as (1), but it also diagonalizes and computes eigenvalues of nonsymmetric matrices. This ability of solving the eigenvalue problem for *generically* any matrix A has many new potential applications. Moreover, in the case of A being symmetric, the self-commutator $[A^T, A]$ vanishes and (2) reduces to (1). Thus, the flow in the space of nonsymmetric matrices described by (2) can be considered as a generalization of the flow in the space of symmetric matrices described by (1).

The motivation to design and study dynamical systems like (1) and (2) has several roots. For example, solving computational problems with the help of continuous-time dynamical systems can be seen as a way to design and implement analog algorithms. Analog algorithms and analog computation have been investigated in various fields or research, including neuroscience, biology, informatics, mathematics. Another reason stems from the wish to obtain a new way to design numerical algorithms and to analyze their behavior and the underlying geometry of these algorithms.

The purpose of the presentation was to explain the basic ideas and the underlying geometry behind the strongly nonlinear but highly structured equation (2) as well as to analyze and generalize equation (2) in various ways. In particular, the following points have been addressed: (a) the convergence behavior and the ability to compute all eigenvalues under generic conditions, (b) the sorting behavior as a consequence of a continuous solution of a certain underlying optimization problem, (c) a Lie-algebraic generalization based on the Cartan decomposition, and finally, (d) structure preserving properties including a new equation for upper Hessenberg matrices: $\dot{A} = [[N, A_l^T + A_l], A] + \nu[[A^T, A]_{du}, A]$, where A_l, A_{du} denotes the strictly lower triangular and upper triangular (including the diagonal) of a matrix and $N = \mathrm{diag}(1, \ldots, n)$.

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I/O Monotone Dynamical Systems

Germán A. Enciso
(joint work with Eduardo D. Sontag)

(compare with the very similar context of the talk by Nicole Radde.)

1. I/O Monotone Systems

Understanding the mechanisms underlying biochemical reactions inside the cell remains a challenging long term research goal. These systems present considerable challenges, not least because of the large number of variables involved (e.g. protein concentrations), the strongly nonlinear interactions, and the nondeterministic nature of the reactions. Another important problem is that when these systems are modeled using systems of differential equations, most of the parameter values in the system are unknown or known only within a large margin of error.

Fortunately, some features about these systems suggest a certain underlying structure. For instance, when they are simulated on the computer, their dynamics is generally 'boring': the solutions tend to converge towards a globally attractive equilibrium, or else there is a small number of equilibria which attract almost all solutions (a behavior known as *multistability*). Sometimes the dynamics may also consist of a single periodic orbit which is globally attractive.

Also, the effect of a variable, say x_i, on another, say x_j, tends to be either consistently promoting or consistently inhibitory. If the system is written in the form $x' = f(x)$, this can be stated by saying that $\partial f_j(x)/\partial x_i$ has a constant sign ($\geq$ or $\leq$), regardless of the state x.

2. Main Result

Definition 2.1. *A dynamical system $x' = f(x)$ is said to be* monotone *if its underlying signed digraph has the property that every (undirected) closed loop has an even number of '$-$' signs.*

The concept of monotonicity has been extensively studied, especially during the 1980s by Morris Hirsch [6], and also by Hal Smith [7]. Perhaps the most important result for these systems is that the generic solution of a (strongly) monotone system converges towards an equilibrium (see the references for details).

Definition 2.2. *An* I/O monotone system *is a controlled system $x' = f(x, u)$ such that every (undirected) closed loop in its extended, I/O signed digraph has an even number of '$-$' signs.*

The extended digraph of a controlled system is created by using the state, input and output variables as nodes, and by writing signed directed edges according to the partial derivatives of the regulatory and output functions with respect to every state and input.

We will also assume that the I/O system has a well defined steady state response (or DC gain) function $S(u)$. For a fixed value of the input u, $S(u)$ is the limit of

the output of the system using the constant input u. Finally, we define the system to be under *I/O positive (negative) feedback* if every path from an input to an output in the associated digraph has an even (odd) number of '$-$' signs.

Theorem 2.3. *Consider a single input, single output I/O monotone control system with steady state response function $S(u)$, and under either I/O positive or I/O negative feedback. Suppose that the associated discrete system*

$$u_{n+1} = S(u_n)$$

is such that every solution converges towards an equilibrium (which may depend on the initial condition). Then every solution of the closed-loop system

$$x' = f(x, h(x))$$

converges towards an equilibrium. Moreover, there is a bijective correspondence between the stable equilibria of the two systems.

This result is a restatement of theorems first introduced by Sontag and Angeli [1, 2]. The I/O positive and negative feedback cases are originally separate results, and each of them has been separately generalized to the case of multiple inputs and outputs, diffusion and delay equations with arbitrary time delays; see for details [3, 4].

3. Example

We provide here a very simple example in the I/O negative feedback case to illustrate the use of this result. For details, see [5] – this model is especially interesting because it was erroneously proposed in the Murray's book *Mathematical Biology* as a possible source of periodic oscillations. Consider the delay system

$$R' = \frac{A}{K+T} - b_1 R, \quad L' = g_1 R - b - 2L, \quad T' = g_2 L(t-\tau) - b_3 T,$$

which is a simple model of testosterone (T) regulation through the hormones L and R. Notice that this system is under negative feedback and therefore it is not itself a monotone system. However, we decompose it as the closed loop of the control system

$$R' = u - b_1 R, \quad L' = g_1 R - b - 2L, \quad T' = g_2 L(t-\tau) - b_3 T,$$

$$y = \frac{A}{K+T}.$$

It is easy to see that this system is an I/O monotone system (in this case the digraph has no closed loops whatsoever), and that it is under I/O negative feedback. Moreover, the response function $S(u)$ is well defined and has the form $S(u) = \frac{p}{q-u}$ for certain constant parameters p, q. Its iterations can be easily shown to converge towards an equilibrium. The I/O negative feedback version of the main result, stated for time-delay systems, implies that the original testosterone model is actually globally attractive for arbitrary delays.

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New approaches for Identification and Model (Inv-)alidation in Systems Biology

Rolf Findeisen

(joint work with Eric Bullinger, Steffen Borchers, Sandro Bosio, Dirk Fey, Philipp Rumschinski, Robert Weismantel)

Current approaches to parameter estimation and model invalidation are often inappropriate for the identification challenges appearing in systems biology. For example in the modeling of biochemical reaction networks typically different reaction schemes (i.e., model hypotheses) are in principle possible and the reaction rates depend strongly on the frequently unknown parameters [9, 4]. The task of model validation and parameter estimation is further complicated by the sparse and noisy measurements; experiments might be time intensive, expensive, and give large measurement uncertainties even under optimal conditions. Even if significant progress has been achieved in the field of identification over the past decades (see e.g. [10]), parameter estimation and model validation for nonlinear systems is still a very challenging and difficult problem.

In this talk we outline two new methods for parameter identification and model invalidation that are specifically suited to biochemical reaction networks. Both approaches allow to take the special structure appearing in biochemical reaction networks into account.

The basic idea of the first approach is to estimate the reaction parameters using an observer based strategy. The method is particularly suited for biochemical reaction networks in which the reaction kinetics are described by polynomial or rational functions. The parameter estimation is performed in three steps[7, 6]. First, the system is transformed into coordinates, in which the system is parameters free. This facilitates the design of a standard observer in the second step. Finally, the parameter estimates are obtained in a straight-forward way from the

observer states, transforming them back to the original coordinates. A major advantage of the proposed parameter estimation approach is that it performs a global parameter search and ensures a guaranteed convergence. An important limitation is the reliance on estimates of derivatives of the output. This can be overcome using other obervers that do not relay explicte on the derivative estimates [5].

The second approach is based on a discretization of the ODEs describing the biochemical reaction network under consideration into a set of implicit polynomial difference equations subject to uncertain, but (set-)bounded measurements. The parameter estimation and model invalidation technique is based on outer-bounding the set of parameters/states that lead to output trajectories being consistent with the measurements. The core of the method is a polynomial feasibility problem [11] that provides an efficient and conclusive test to certify whole parameter/state regions, instead of single points, as inconsistent with the measurements. First results with respect to this approach can be found in [1, 2, 3]. The infeasibility certificate allows to prove model invalidity, and are also used to assess the feasible parameter and state space regions by outer-bounding. This is done by means of a bisection algorithm, which allows to systematically discard parameter and state space regions that are not consistent with the experimental data, while guaranteeing that no feasible solution is lost. In comparison to other approaches, instead of checking (possibly) many separate points which might lead to non-conclusive answers, our approach allows to check whole parameter and state regions for feasibility.

The feasibility approach is a reliable and computationally manageable method for dynamic model invalidation, as well as parameter and state estimation. The method can be extended to general nonlinear systems and systems containing discrete variables considering the same methods as outlined in [8].

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Model-free control and "intelligent" PID controllers

Michel Fliess
(joint work with Cédric Join)

We are introducing a model-free control and a control with a restricted model for finite-dimensional complex systems. This control design may be viewed as a contribution to "intelligent" PID controllers, the tuning of which becomes quite easy, even with highly nonlinear and/or time-varying systems. Our main tool is a newly developed numerical differentiator. Differential algebra provides the theoretical framework. Our approach is validated by several numerical experiments.

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Recent progresses on the metabolism modelling of Bacteria: from a static to a dynamic model

Vincent Fromion
(joint work with A. Goelzer, F. Bekkal Brikci, C. Tanous, and G. Scorletti)

Recently, the reconstruction and analysis of genome-scale genetic and metabolic regulatory networks has become an area of active research. It requires the integration of existing knowledge as a first step towards systems biology, which we define as the study of the interactions between the components of biological systems to

understand how these interactions give rise to the function and behavior of the system.

Genetic and metabolic networks have been constructed for various organisms as for example *Escherichia coli* [8], and *Homo sapiens* [1], by incorporating the description of chemical reactions into a stoichiometric matrix. Transcriptional regulation by regulatory proteins has been included in the last update of the model of *Saccharomyces cerevisiae* [4]. However, most of these models do not explicitly include regulation mediated by metabolites, with the exception of the most recent version of the EcoCyc database [9]. Metabolites can modulate the activity of transcription factors and enzymes and are therefore key regulators.

Reconstruction of the metabolic network and its associated regulatory network contributes to resolving various problems such as unravelling how the bacterium coordinates its genetic and metabolic networks to adapt to environmental changes and elucidating the global organization of the regulatory network Such work will also help develop tools and concepts to handle and to analyze the inherent complexity of biological functions.

Here, we present the recent reconstruction and mathematical analysis of the genetic and metabolic regulatory network for the Gram-positive bacterium *Bacillus subtilis* presented in [2]. *B. subtilis* has been studied for over 40 years and is one of the best-characterized bacteria, easily amenable to genetic and physiological studies. The publication of the *B. subtilis* genome sequence [6] and subsequent international programmes of systematic gene disruption, functional analysis and regulatory network studies [5], makes this microorganism a prime candidate to develop systems biology.

Our model includes the biochemical reactions of the metabolic network and all the known levels of regulation involved in metabolic pathways: transcriptional, translational, post-translational and modulation of enzymatic activities. To obtain the most complete view of the interplay between the metabolic network and genetic regulation, the description of each regulatory mechanism includes known roles of metabolite concentrations, ions, and any other quantities related to the state of the metabolic network. The entire model (reactions, enzymes, genes and regulations) has been curated manually, using published data and expert knowledge [2].

Using this model, we were able to examine various aspects of the general organization of metabolic regulation. We find that metabolite pools are strongly involved in regulation of the central metabolism of *Bacillus subtilis*, in agreement with the findings from an analysis of the genetic regulatory network of *Escherichia coli* [7]. Moreover, by introducing the notion of local and global regulation, we reveal that the complex regulatory network can be broken down into sets of locally regulated modules, which are coordinated by global regulators. Local regulations ensure that the control of elementary pathways through genetic and/or enzymatic regulation depends upon the level of key metabolites. In contrast, global regulations ensure the coordination between these elementary pathways in response to

environmental changes. The integration of these local/global levels and the use of the classification of sensing signals in [7] lead to recover the main physiological aspects of metabolism of *Bacillus subtilis*. Finally, we then have shown that the metabolic network regulation is highly structured through the definition of a rigorous and well defined notion of modules.

The existence of such a strong structure in the regulation network led to look for the constraints acting on bacteria and allowing its emergence. So, we then present another recent result [3] about the problem of resource management in bacteria [3]. Actually, for a fixed growth rate, we formalised the problem of resource management into a non-differentiable convex constraint-based feasibility problem through the integration of three structural constraints. This feasibility problem can be easily transformed into an equivalent Linear Programming (LP) feasibility problem for which many classical polynomial-time solvers are available. The resolution of the LP feasibility problem leads to predict not only the flux distribution and the maximal growth rate, but also the concentrations of ribosomes, and of the proteins involved in the metabolic network and thus the composition of the cell for different growth rates. Moreover, the modular structure of the metabolic network can also be predicted with respect to the medium composition.

Another major conclusion of the talk is the successful use of tools and methods based on convex optimisation in biology. The formalisation of the cell behavior is suitable for convex optimisation and strong structural properties have been obtained allowing to explain the emergence of functional modules in the metabolic network. The links between these two fields (biology and optimization) have to be strengthened in order to investigate fundamental questions such as the evolution of regulatory networks of organisms with respect to the ecological niche.

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Tensored models, Bezoutians and intertwining maps

Paul A. Fuhrmann
(joint work with Uwe Helmke)

The talk focuses on the study of tensor products of polynomial models, both over he underlying field $\mathbb{F}$ as well as over the polynomial ring $\mathbb{F}[z]$. The results are used to derive a natural definition of the Anderson-Jury generalized Bezoutians. In turn, this is used to give a new derivation of the characterization of maps intertwining two polynomial models. We indicate some of the results. For the full details, and more, see Fuhrmann and Helmke [2009].

Tensor products of vector spaces are usually taken over the underlying field $\mathbb{F}$. In that case we have the isomorphism

$$\boxed{Y \otimes_{\mathbb{F}} X^* \simeq \mathrm{Hom}_{\mathbb{F}}(X, Y) = L(X, Y)}$$

However, given linear transformations $A : X \longrightarrow X$ and $B : Y \longrightarrow Y$, the vecor sopaces can be endowed with an $\mathbb{F}[z]$-module structure by defining, for $p(z) \in \mathbb{F}[z]$ and $x \in A$, $p \cdot x = p(A)x$ and similarly for Y. We denote by X_A and Y_B the vector spaces with the induced module structures. This allows us to take the tensor product over the ring $\mathbb{F}[z]$ and yields the isomorphism

$$\boxed{Y_B \otimes_{\mathbb{F}[z]} X^*_{A^*} \simeq \mathrm{Hom}_{\mathbb{F}[z]}(X_A, Y_B)}$$

We note that $Z \in \mathrm{Hom}_{\mathbb{F}[z]}(X_A, Y_B)$ if and only if $ZA = BZ$.

Given a nonsingular polynomial matrix $D(z) \in \mathbb{F}[z]^{p\times p}$, the corresponding polynomial model X_D and rational model X^D were introduced, see Fuhrmann [1976], as functional models suitable for realization theory. We have $X_D \simeq \mathbb{F}[z]^p/D(z)\mathbb{F}[z]^p$ and $X^D = \mathrm{Ker}\, D(\sigma)$, where $D(\sigma) : z^{-1}\mathbb{F}[[z^{-1}]]^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m$ is the Toeplitz operator defined by $D(\sigma)h = \pi_- Dh$. Under a natural induced duality, coming from the identification

$$\boxed{(\mathbb{F}[z]^m)^* \simeq z^{-1}\mathbb{F}[[z^{-1}]]^m}$$

we have the isomorphism $X_D \simeq X^D$ as well as the identification $X^*_D = X_{\tilde{D}}$.

Specializing to the $\mathbb{F}$-tensor product of two polynomial models, given $D_2(z) \in \mathbb{F}[z]^{p\times p}$, $D_1(z) \in \mathbb{F}[z]^{m\times m}$ nonsingular, we have

$$\boxed{X_{D_2} \otimes_{\mathbb{F}} X_{\tilde{D}_1} \simeq X_{D_2(z)\otimes \tilde{D}_1(w)} \simeq \mathrm{Hom}_{\mathbb{F}}(X_{D_1}, X_{D_2})}$$

Here $X_{D_2(z)\otimes\tilde{D}_1(w)} \subset \mathbb{F}[z,w]^{p\times m}$ is a concrete representation of the tensor product, with $Q(z,w) \in X_{D_2(z)\otimes\tilde{D}_1(w)}$ if and only if $D_2(z)^{-1}Q(z,w)D_1(w)^{-1}$ strictly proper in both variables. For such a polynomial, we always have a, nonunique, representation of the form $Q(z,w) = L_2(z)\tilde{L}_1(w)$, with $D_2^{-1}L_2$ and $D_1^{-1}L_1$ strictly proper. The isomorphism $\Psi : X_{D_2(z)\otimes\tilde{D}_1(w)} \longrightarrow \mathrm{Hom}_{\mathbb{F}}(X_{D_1}, X_{D_2})$ is given by

$$\boxed{\Psi(Q)(g) =< g, \widetilde{Q(z,\cdot)} > .}$$

This tensor product has many applications to the study of Sylvester, Stein and Lyapunov equations, the study of stability of high order systems, dissipative systems, invariant factors of Sylvester map, Clebsch-Gordan decomposition, as well as to model reduction.

Let $D_2(z) \in \mathbb{F}[z]^{p\times p}, D_1(w) \in \mathbb{F}[w]^{m\times m}$ nonsingular. There is a natural $\mathbb{F}[z,w]$-module structure on $X_{D_2(z)\otimes\tilde{D}_1(w)}$ given by

$$\boxed{p \cdot Q(z,w) = \pi_{D_1(z)\otimes_{\mathbb{F}}D_2(w)}p(z,w)Q(z,w)}$$

A special case is the Sylvester map S defined by

$$\boxed{(SQ)(z,w) = \pi_{D_2(z)\otimes\tilde{D}_1(w)}(z-w)Q(z,w)}$$

Then the following statements are equivalent:

(1) $Q(z,w)$ is a solution of the homogeneous polynomial Sylvester equation.
(2) $Q(z,w)$ is a generalized Anderson-Jury Bezoutian, i.e. it has the representation
$$\boxed{Q(z,w) = \frac{D_2(z)N_1(w) - N_2(z)D_1(w)}{z-w}}$$
with $D_2(z)N_1(z) = N_2(z)D_1(z)$ satisfied.
(3) The map $Z : X_{D_1} \longrightarrow X_{D_2}$ defined by
$Zg =< g, \widetilde{Q(z,\cdot)} >$ satisfies $S_{D_2}Z = ZS_{D_1}$ i.e. it is intertwining or, equivalently, is an $\mathbb{F}[z]$-homomorphism

In this case the map Z has the representation

$$\boxed{Zg = \pi_{D_2}N_2g, \qquad g \in X_{D_1}}$$

with

$$\boxed{N_2(z)D_1(z) = D_2(z)N_1(z)}$$

holding for some $N_1(z), N_2(z) \in \mathbb{F}[z]^{p\times m}$. This characterization of intertwining maps is the algebraic version of the commutant lifting theorem in operator theory. It is of fundamental importance in the derivation of isomorphism results for various representations of linear systems.

Next, we are after a concrete representation of the tensor product $X_{D_2} \otimes_{\mathbb{F}[z]} X_{\tilde{D}_1}$. Using duality theory, we have the isomorphism $X_{D_2} \otimes_{\mathbb{F}} X_{\tilde{D}_1} \simeq X^{D_2 \otimes I} \cap X^{I \otimes \tilde{D}_1}$. $H(z) \in X^{D_2 \otimes I} \cap X^{I \otimes \tilde{D}_1}$ if and only if it is rational and strictly proper and has representations of the form $H(z) = D_2(z)^{-1} N_2(z) = N_1(z) D_1(z)^{-1}$. The isomorphism $\psi : X^{D_2 \otimes I} \cap X^{I \otimes \tilde{D}_1} \longrightarrow \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2})$ is given by $\psi H(g) = \pi_{D_2} N_2 g$, for $g \in X_{D_1}$. Now of course, there is a natural embedding $i : \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2}) \longrightarrow \mathrm{Hom}_{\mathbb{F}}(X_{D_1}, X_{D_2})$ and hence there exists a uniquely determined map $\beta : X^{D_2 \otimes I} \cap X^{I \otimes \tilde{D}_1} \longrightarrow X_{D_2(z) \otimes \tilde{D}_1(w)}$. We call this the **Bezoutian map**, as it can be show that it is given, for $H \in X^{D_2 \otimes I} \cap X^{I \otimes \tilde{D}_1}$, by

$$\boxed{\beta(H) = Q(z,w) = \frac{D_2(z)N_1(w) - N_2(z)D_1(w)}{z - w}}$$

With the maps defined as above, we have the commutativity of the following diagram.

$$\begin{array}{lcccc}
X_{D_2} \otimes_{\mathbb{F}} X_{\tilde{D}_1} \simeq & X_{D_2(z) \otimes \tilde{D}_1(w)} & \xrightarrow{\Psi} & \mathrm{Hom}_{\mathbb{F}}(X_{D_1}, X_{D_2}) \\
& \uparrow \beta & & \uparrow i \\
X_{D_2} \otimes_{\mathbb{F}[z]} X_{\tilde{D}_1} \simeq & X^{D_2 \otimes I} \cap X^{I \otimes \tilde{D}_1} & \xrightarrow{\psi} & \mathrm{Hom}_{\mathbb{F}[z]}(X_{D_1}, X_{D_2})
\end{array}$$

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Digital redesign of nonlinear multi-input systems

Lars Grüne

(joint work with Dragan Nešić, Karl Worthmann)

1. Introduction

At the Oberwolfach Control Theory Meeting 2005 I presented the following open problem:

Consider a single input control affine closed loop system

$$\dot{x}(t) = g_0(x(t)) + g_1(x(t))u(x(t)) \tag{1}$$

with $x \in \mathbb{R}^n$ and a smooth feedback controller $u : \mathbb{R}^n \to \mathbb{R}$ and the corresponding sampled–data system

$$\dot{x}_T(t) = g_0(x_T(t)) + g_1(x_T(t))u_T(x_T(iT))), \quad t \in [iT, (i+1)T),\ i = 0, 1, \ldots \tag{2}$$

with a family of sampled-data controllers $u_T : \mathbb{R}^n \to \mathbb{R}$ parameterized with the (sufficiently small) sampling rate $T > 0$ which are locally bounded uniformly in T but not necessarily continuous. We consider the mismatch after one time step given by

$$\Delta_T(x_0) := \|x(T, x_0, u) - x_T(T, x_0, u_T)\|,$$

with $x(t, x_0, u)$ and $x_T(t, x_0, u_T)$ denoting the solutions of (1) and (2), respectively, with initial value x_0 at time $t = 0$.

It is easy to prove that for $u_T \equiv u$ we obtain $\Delta_T = O(T^2)$[1] while for

$$u_T(x) = u(x) + \frac{T}{2}\frac{\partial u(x)}{\partial x}[g_0(x) + g_1(x)u(x)] \tag{3}$$

we obtain $\Delta_T = O(T^3)$ (this follows from [5, Theorem 4.11] setting $V(x) = x_i$ observing that positive definiteness of V is not needed). Remark 4.12 in [5] suggests that higher order cannot be obtained in general.

Problem: Find conditions on g_0, g_1, u under which $\Delta_T \leq O(T^4)$ can be achieved.

In this report a solution to the problem and an extension to multi-input systems will be presented.

2. Single-Input systems

We use the following notation: for two vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ we define the usual Lie bracket by $[f, g] = \frac{d}{dx}g \cdot f - \frac{d}{dx}f \cdot g$. Furthermore, for $k \in \mathbb{N}$ we define

$$u^k(x_0) := \left.\frac{d^k}{dt^k}\right|_{t=0} u(x(t, x_0, u)). \tag{4}$$

Note that with this notation (3) can be written as

$$u_T(x) = u(x) + \frac{T}{2}u^1(x).$$

Theorem 2.1: A feedback law u_T with $\Delta_T = O(T^4)$ exists if and only if there exists a bounded function $\alpha : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$[g_0, g_1](x)u^1(x) = \alpha(x)g_1(x). \tag{5}$$

If this condition holds, then the feedback laws u_T are given by

$$u_T(x) = u(x) + \frac{T}{2}u^1(x) + \frac{T^2}{6}u^2(x) + \frac{T^2}{12}\alpha(x)$$

and these u_T are uniquely determined up to terms of order $O(T^3)$ for all x with $g_1(x) \neq 0$.

The proof of this theorem relies on comparing the Taylor expansion of $x(T, x_0, u)$ with the Fliess expansion of $x_T(T, x_0, u_T)$ in $T = 0$, see [2, Theorem 3.6] for details.

[1] $\Delta_T = O(T^m)$ means: for each compact $K \subset \mathbb{R}^n$ there is $C > 0$ with $\sup_{x \in K} \Delta_T(x) \leq CT^m$

Remark 2.2: (i) Conditions for higher order $\Delta_T \leq O(T^5)$ can be stated similarly but become more and more involved. However, computer mathematics systems like, e.g., MAPLE can be used to check the conditions recursively and compute the corresponding u_T.

(ii) The condition (5) is rather restrictive. Hence, Theorem 2.1 shows that a mismatch $\Delta_T \leq O(T^4)$ can hardly be expected in general, regardless of how u_T is chosen. In particular, the seemingly "natural" Taylor-like choice

$$u_T(x) = u(x) + \frac{T}{2}u^1(x) + \frac{T^2}{6}u^2(x)$$

only works if $\alpha \equiv 0$. A sufficient condition for $\alpha \equiv 0$ is $[g_0, g_1] \equiv 0$, i.e., the vector fields commute.

(iii) A sufficient condition for (5) is $[g_0, g_1] \in \operatorname{span}\langle g_1 \rangle$. In [4] it was shown that this condition is necessary and sufficient for the fact that for each smooth controller $u : \mathbb{R}^n \to \mathbb{R}$ there exists u_T satisfying $\Delta_T \leq O(T^k)$ for arbitrary $k \in \mathbb{N}$.

3. Multi-Input systems

We now extend our result to multi-input control affine systems of the form

$$\dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(x(t)) \tag{6}$$

with vector fields $g_i = (g_{i,1}, \ldots, g_{i,n})^T$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, $m \leq n$, and controller $u = (u_1, \ldots, u_m)^T$. We write the right hand side of the system briefly as

$$g_0(x) + G(x)u(x) \quad \text{with} \quad G(x) = \begin{pmatrix} g_{1,1}(x) & \cdots & g_{m,1}(x) \\ \vdots & \ddots & \vdots \\ g_{1,n}(x) & \cdots & g_{m,n}(x) \end{pmatrix}. \tag{7}$$

and use definition (4) also for these vector valued feedback laws.

As in the single input case for $u_T \equiv u$ we get $\Delta_T = O(T^2)$ sets while for $u_T(x) = u(x) + \frac{T}{2}u^1(x)]$ we obtain $\Delta_T = O(T^3)$, cf. [3, Theorem 4.1 (i)-(ii)]. For $\Delta_T \leq O(T^4)$, Theorem 2.1 generalizes as follows, see [3, Theorem 4.1 (iii)]. Again, the proof relies on Taylor and Fliess expansions of the solution.

Theorem 3.1: For the multi-input system (6), a feedback law u_T with $\Delta_T \leq O(T^4)$ exists if there exists a bounded function $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\sum_{i=1}^{m} \left[[g_0, g_i](x) + \sum_{\substack{j=1 \\ j \neq i}}^{m} [g_j, g_i](x)u_{0,j}(x) \right] u_i^1(x) = \sum_{i=1}^{m} \alpha_i(x)g_i(x). \tag{8}$$

If this condition holds, then the feedback laws u_T are given by

$$u_T(x) = u(x) + \frac{T}{2}u^1(x) + \frac{T^2}{6}u^2(x) + \frac{T^2}{12}\alpha(x)$$

and these u_T are uniquely determined up to terms of order $O(T^3)$ for all x for which $G(x)$ has full column rank. For these x condition (8) is also necessary.

As in the case of Theorem 2.1, the results can be extended to higher orders which is most conveniently done recursively using a computer mathematics system such as MAPLE. This recursive design procedure leads to a feedback of the form

$$u_T(x) = u(x) + \frac{T}{2}\tilde{u}^1(x) + \frac{T^2}{6}\tilde{u}^2(x) + \dots$$

in which each $\tilde{u}^k$ is the solution of a least squares problem of the form $G(x)\tilde{u}^k(x) = b^k(x)$. If this problem is solvable with residual 0 for $k = 1, \dots, m$, then u_T is a sampled-data feedback yielding $\Delta_T \le O(T^{m+2})$. In particular, this shows that

(i) the problem is solvable for arbitrary order $O(T^k)$, $k \in \mathbb{N}$, if $G(x)$ is square and invertible for all $x \in \mathbb{R}^n$
(ii) the problem is in general not solvable for $\Delta_T \le O(T^4)$ if $G(x)$ is not square, i.e., when $m < n$.

In the latter case, optimization based approaches can often provide very good solutions even if the sampling period is rather large. For details on this approach we refer to [1] and the references therein.

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Funnel control for positive systems

Achim Ilchmann

(joint work with Norman Hopfe, Pham Huu Anh Ngoc)

We study linear Volterra-Stieltjes m-input $u(\cdot)$, m-output $y(\cdot)$ systems

$$(1) \quad \begin{cases} \dot{x}(t) &= A(t)\,x(t) + \int_0^t \mathrm{d}[\eta(\theta)]\,x(t-\theta) + B\,u(t), & x|_{[0,\sigma]} = \phi \\ y(t) &= C\,x(t) & \text{for a.a. } t \ge \sigma, \end{cases}$$

where

$$(A(\cdot), B, C, \eta(\cdot)) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0}, \mathbb{R}^{n\times n}) \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\ge 0}, \mathbb{R}^{n\times n})$$

initial data $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\ge 0})$, $\sigma \ge 0$, and input function $u(\cdot) \in \mathcal{L}^1_{\mathrm{loc}}([\sigma, \infty), \mathbb{R}^m)$. Due to the limitation of space, we present only some results of [1].

Definition 1: The homogeneous part of system (1) is said to be *positive* if, and only if, for every non-negative initial data $\sigma \geq 0$, $\phi(\cdot) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0})$, the unique solution to the initial value problem (1), $B = 0$, $x|_{[0,\sigma]} = \phi$ is also non-negative: $x(t;\sigma,\phi,0) \in \mathbb{R}^n_{\geq 0}$ for all $t \geq \sigma$.

Theorem 2: Let $(A(\cdot), \eta(\cdot)) \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n}) \times \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$, $B = 0$. Then (1) is positive if, and only if, $A(t)$ is a Metzler matrix for all $t \geq 0$ and $\eta(\cdot)$ is a non-decreasing matrix function. The implication "$\Leftarrow$" only requires that $A(\cdot) \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n\times n})$.

Theorem 3: Suppose (1) has constant $A(\cdot) = A$, its homogeneous part is positive and η is of finite total variation, i.e. $\int_0^\infty |\mathrm{d}\eta(\theta)| < \infty$. Then (1) is exponentially asymptotically stable if, and only if, $\max_i \Re\lambda_i \left(A + \int_0^\infty \mathrm{d}[\eta(\theta)]\right) < 0$ and $\| \int_0^\infty \mathrm{d}[\eta(\theta)] e^{\alpha\theta} \| < \infty$ for some $\alpha > 0$.

It can be shown that in presence of the Volterra term, uniform asymptotical stability of (1) does not necessarily imply exponential stability.

Definition 4: The *zero dynamics* of (1) are defined as the real vector space of trajectories

$$\mathcal{ZD}(A,B,C,\eta(\cdot)) := \{(x,u,y)\,|(x,u,y) \text{ solves (1) with } y \equiv 0\}\,.$$

The system (1) is said to have *stable zero dynamics* if, and only if, for any $(x(\cdot),u(\cdot),y(\cdot)) \in \mathcal{ZD}(A,B,C,\eta(\cdot))$ it holds that $\lim_{t\to\infty} x(t) = 0$ and $x(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.

Proposition 5: Suppose the system (1) has constant A and $\det CB \neq 0$. Writing $V \in \mathbb{R}^{n\times(n-m)}$ such that $\ker C = \operatorname{im} V$ and $N := (V^TV)^{-1}V^T[I_n - B(CB)^{-1}C] \in \mathbb{R}^{(n-m)\times n}$, the following statements are equivalent:

(i) (1) has stable zero dynamics.

(ii) $\forall s \in \mathbb{C}_0 \;:\; \det \begin{pmatrix} sI_n - A - \int_0^\infty \mathrm{d}[\eta(\theta)]\, e^{-s\theta} & B \\ C & 0 \end{pmatrix} \neq 0$

(iii) The subsystem

$$\dot{z}(t) = NAVz(t) + \int_0^t \mathrm{d}[\mathrm{N}\eta(\theta)\mathrm{V}]\,\mathrm{z}(\mathrm{t}-\theta), \qquad \mathrm{t} \geq 0,$$

is uniformly asymptotically stable.

Moreover, if $(x(\cdot),u(\cdot),y(\cdot)) \in \mathcal{ZD}(A,B,C,\eta(\cdot))$, then $\lim_{t\to\infty} u(t) = 0$ and $u(\cdot) \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$.

We are now ready to introduce the concept of 'funnel control': A one dimensional funnel is specified by a funnel boundary $\psi : [\sigma,\infty) \to [\lambda,\infty)$, $\sigma \geq 0$, assumed bounded with global Lipschitz constant $\ell \geq 0$ on any interval $[\sigma+\epsilon,\infty)$, $\epsilon > 0$ and bounded away from zero by $\lambda > 0$ arbitrarily small, and associated with

$$\mathcal{F}_{(\sigma,\psi)} := \{(t,e) \in [\sigma,\infty) \times \mathbb{R} \,|\, |e| < \psi(t)\}\,.$$

The *control objective* is to design a simple output feedback controller such that, for any prespecified funnel $\mathcal{F}_{(\sigma,\psi)}$, every reference signal $r(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$,

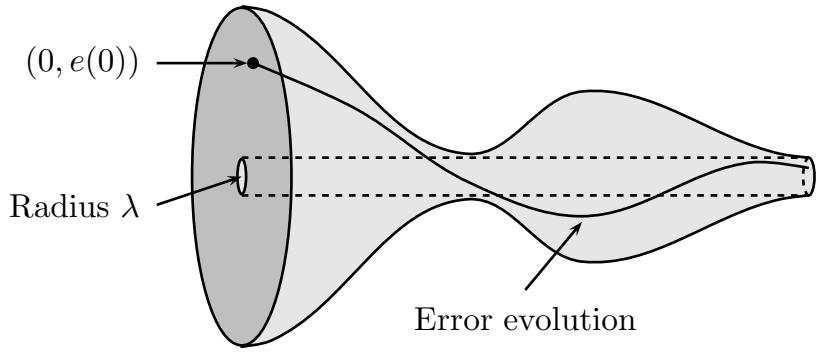

FIGURE 1. Prescribed performance funnel $\mathcal{F}_{(\sigma,\psi)}$.

i.e. bounded with essentially bounded derivative, is tracked within this funnel and all variables of the closed-loop system are bounded.

This is achieved, under suitable assumptions including relative degree one and stable zero dynamins, by the simple proportional time-varying feedback (2). The idea behind the funnel controller is that the feedback structure essentially exploits an intrinsic high-gain property of the system to ensure that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact. Note that the controller does not depend on the actual system data but on structural data only, and the transient behavior can be "shaped" in a predetermined manner through choice of the function $\psi(\cdot)$.

For example, if in the single-input single-output case with $\sigma = 0$ and $\psi(\cdot) = 0$ we choose, for $\ell > 0$ and $\lambda \in (0, 1)$, the funnel boundary $t \mapsto \psi(t) = \max\{1 - \ell\, t, \lambda\}$, then attainment of the tracking objective implies that a prescribed time $t^* = \frac{1-\lambda}{\ell}$ guarantees: $|y(t) - r(t)| < \lambda$ for all $t \geq t^*$.

Theorem 6: Let a funnel $\mathcal{F}_{(\sigma,\psi)}$ be determined by its boundary $\psi(\cdot)$ with Lipschitz constant $\ell \geq 0$ and lower bound $\lambda > 0$, $\sigma \geq 0$ be prespecified. Suppose the system (1) has constant matrix A, stable zero dynamics, its homogeneous part is positive and the input-output matrices have structure

$$B = (b, 0, \ldots, 0)^T,\ C = (c, 0, \ldots, 0), \quad b,\, c > 0.$$

Then for any initial data $\phi(\cdot) \in \mathcal{C}([0, \sigma], \mathbb{R}^n)$ and any reference signal $r(\cdot)$ belonging to $\mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and satisfying $|c\phi(\sigma) - r(\sigma)| < \psi(\sigma)$, the application of the funnel controller

$$\boxed{u(t) = -k(t)\, e(t), \quad k(t) = \tfrac{1}{\psi(t) - |e(t)|}, \quad e(t) = y(t) - r(t)} \tag{2}$$

to (1) yields a closed-loop initial value problem with the properties

(i) there exists a maximal solution $x : [0, \omega) \to \mathbb{R}^n$, $\omega \in (\sigma, \infty]$;
(ii) the solution $x(\cdot)$ is unique and has no finite escape time, i.e. $\omega = \infty$;
(iii) all signals $x(\cdot)$, $u(\cdot)$, $k(\cdot)$ are bounded;
(iv) the tracking error is bounded away from the funnel boundary:

$$\exists\, \varepsilon > 0\ \forall\, t \geq \sigma\ :\ |e(t)| \leq \psi(t) - \varepsilon;$$

(v) if the homogenous part of (1) is positive and $\phi(\cdot)$ and $r(\cdot)$ are non-negative, i.e. $(\phi(\cdot), r(\cdot)) \in \mathcal{C}([0,\sigma], \mathbb{R}^n_{\geq 0}) \times \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, then the signals $x(\cdot;\sigma,\phi)$ and $y(\cdot)$ are non-negative, i.e.

$$\forall\, t \geq 0 \;:\; x(t;\sigma,\phi) \geq 0 \qquad \text{and} \qquad y(t) \geq 0.$$

Remark 7: Theorem 6 can be modified and generalized in various ways:

(i) Multivariable systems are allowed if B and C are diagonal matrices.
(ii) We can capture funnel control in the presence of input constraints if a feasibility assumptions in terms of the systems data, the funnel data, the initial conditions and the reference trajectory is satisfied.
(iii) The funnel can be "infinity" at 0 to capture all possible initial conditions.
(iv) The funnel can be non-symmetric.
(v) We may choose different funnels in each channel.

We have applied the funnel controller to a classical positive example of general anesthesia.

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A unified approach to output regulation for nonlinear systems

Alberto Isidori
(joint work with Lorenzo Marconi)

We consider a problem of output regulation for nonlinear system in normal form

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w,z,\zeta) \\ \dot{\zeta} &=& q(w,z,\zeta) + u \end{array} \tag{1}$$

in which $z \in I\!R^n$ and $\zeta = e \in I\!R$ is the regulated variable. Assume the existence of a smooth map $\pi : W \;\to\; I\!R^n$ satisfying

$$\frac{\partial \pi}{\partial w} s(w) = f_0(w, \pi(w), 0, \ldots, 0) \qquad \forall w \in W\,. \tag{2}$$

The graph of $\pi(\cdot)$ is rendered invariant by the control $u^*(w) = -q(w, \pi(w), 0)$. Since the controller is only driven by e, this "feed-forward" control has to be generated by an *internal model*. To this end, we assume the existence of $d \in \mathbb{N}$, a map $F : I\!R^d \to I\!R^d$, a $d \times 1$ vector G_0, a map $\gamma : I\!R^d \to I\!R$ and a map $\tau : W \;\to\; I\!R^d$ satisfying

$$\begin{array}{rcll} \dfrac{\partial \tau}{\partial w} s(w) &=& F(\tau(w)) + G_0 \gamma(\tau(w)) & \forall w \in W \\ u^*(w) &=& \gamma(\tau(w)) & \forall w \in W\,. \end{array} \tag{3}$$

The control to (1) is provided by

$$
\begin{aligned}
u &= \dot{N}(\varphi) + \gamma(\eta) + v \\
v &= -k[\zeta - N(\varphi)] \\
\dot{\eta} &= F(\eta - G_0[\zeta - N(\varphi)]) + G_0[\gamma(\eta) + v] \\
\dot{\varphi} &= L(\varphi + M[\zeta - N(\varphi)]) - Mv
\end{aligned} \tag{4}
$$

which is a dynamic system, with internal state (η, φ), "driven" only by the measured variable ζ. Change variables as

$$\theta = \zeta - N(\varphi)\,, \qquad \chi = \varphi + M\theta\,, x \qquad = \eta - G_0\theta$$

to obtain a system

$$
\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, \theta + N(\chi - M\theta)) \\
\dot{\chi} &= L(\chi) + M[q(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta)] \\
\dot{x} &= F(x + G_0\theta) - G_0 q(w, z, \theta + N(\chi - M\theta)) \\
\dot{\theta} &= h(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta) - k\theta\,.
\end{aligned} \tag{5}
$$

This system can be seen as feedback interconnection of a system with input θ and state (w, z, χ, x) and of a system with input (w, z, χ, x) and state θ, namely

$$\dot{\theta} = h(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta) - k\theta\,.$$

It is well known that the latter subsystem – by choosing a large k - can be made input-to-state stable (with arbitrarily large restrictions) with a linear gain function, whose gain coefficient can be rendered arbitrarily small by increasing k. Set now $\theta = 0$ in the former subsystem, to obtain

$$
\begin{aligned}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, N(\chi)) \\
\dot{\chi} &= L(\chi) + M[q(w, z, N(\chi)) + \gamma(x)] \\
\dot{x} &= F(x) - G_0 q(w, z, N(\chi))\,.
\end{aligned} \tag{6}
$$

Suppose that the latter possesses a compact invariant set $\mathcal{A}$ which is locally exponentially stable, with a domain of attraction that contains the set of all admissible initial conditions, and suppose that the map $h(w, z, N(\chi)) + \gamma(x)$ vanishes on this invariant set. Then, by the small-gain theorem it can be concluded that there is a number k^* such that, for all $k \geq k^*$, all trajectories of the composite system (5) remain bounded and are such that (w, z, χ, x) converges to $\mathcal{A}$ while θ converges to 0. If $N(\chi)$ vanishes on $\mathcal{A}$, then also e converges to 0 and the problem of output regulation is solved.

We have in this way identified an auxiliary problem which, if solved, makes the controller (4) solving the problem of output regulation for the original plant: find, if possible, a triplet $\{L(\varphi), M, N(\varphi)\}$ such that system (6) possesses a compact invariant set $\mathcal{A}$ which is locally exponentially stable and attracts all admissible initial conditions, and such that $N(\chi)$ and $h(w, z, N(\chi)) + \gamma(x)$ vanish on this set.

Recall that, by assumption, there exists $\pi(w)$ and $\tau(w)$ satisfying (2) and (3). Hence, it is readily seen that if $L(0) = 0$ and $N(0) = 0$, the set

$$\mathcal{A} = \{(w, z, \chi, x) : w \in W, z = \pi(w), \chi = 0, x = \tau(w)\}$$

is a compact invariant set of (6). Moreover, $N(\chi)$ and $q(w, z, N(\chi)) + \gamma(x)$ vanish on this set. Thus, this is the set for which local exponential stability will be sought (with a domain of attraction that contains the compact set of all admissible initial conditions).

To determine whether this is achievable, it is convenient to change z into $z_{\rm a} = z - \pi(w)$, which changes $f(w, z, \zeta)$ and $h(w, z, \zeta)$ into

$$f_{\rm a}(w, z_{\rm a}, \zeta) = f(w, z_{\rm a} + \pi(w), \zeta) - \frac{\partial \pi}{\partial w} s(w) = f(w, z_{\rm a} + \pi(w), \zeta) - f(w, \pi(w), 0)$$

and to set

$$h_{\rm a}(w, z_{\rm a}, \zeta) = q(w, z_{\rm a} + \pi(w), \zeta) - q(w, \pi(w), 0)\,.$$

This being done, system (6) can be interpreted as interconnection of three subsystems:

- a system which we call the *auxiliary plant*, modelled by equations of the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z}_{\rm a} & = & f_{\rm a}(w, z_{\rm a}, u_{\rm a}) \\ y_{\rm a} & = & h_{\rm a}(w, z_{\rm a}, u_{\rm a})\,, \end{array} \tag{7}$$

- a system which we call a *weighting filter*, modelled by equations of the form

$$\begin{array}{rcl} \dot{\tilde{x}} & = & F(\tilde{x} + \tau(w)) - F(\tau(w)) - G_0 u_{\rm f} \\ y_{\rm f} & = & \gamma(\tilde{x} + \tau(w)) - \gamma(\tau(w))\,, \end{array} \tag{8}$$

- a *controller* modelled by equations of the form

$$\begin{array}{rcl} \dot{\chi} & = & L(\chi) + M u_{\rm c} \\ y_{\rm c} & = & N(\chi)\,. \end{array} \tag{9}$$

as described in the figure

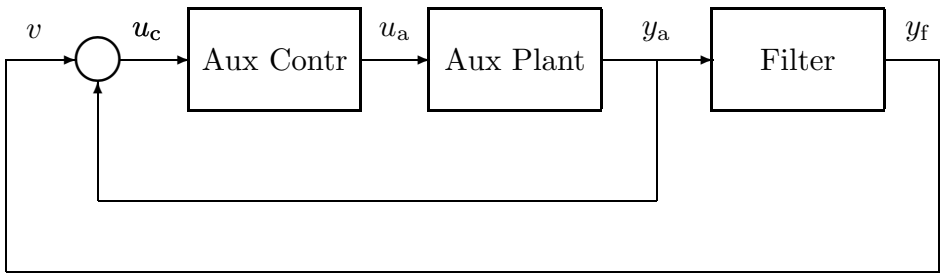

FIGURE 1. System (6).

By the Small Gain theorem, we know that if the system inside the *outer loop* is input-to-state stable, with a linear gain function whose gain coefficient is strictly

less than 1, the reduced system has the desired asymptotic properties and the problem is solved. Thus, we concentrate on that system which, as we see, consists of an *inner loop* obtained by controlling the auxiliary plant (7) by means of controller (9), *cascaded* with the auxiliary filter (8).

The filter can always be made asymptotically stable. Thus, the entire problem is simply reduced to the problem of stabilizing the auxiliary plant, with the constraint – though – that the gain function of the system consisting of the inner loop cascaded with the filter is a contraction. Note that in the stabilization problem thus identified, the choice of internal model has no role. It does have a role, though, in the constraint on the gain.

The proposed construction covers all previously known cases. In this sense, it can be seen as " universal". In fact, if the original plant was minimum-phase, the auxiliary plant is a stable system and hence any arbitrarily small gain coefficient can be imposed on the inner loop. Thus, the required gain constraint on the system inside the outer loop can always be met. If not, the idea is to try to address the stabilization problem of the inner loop by keeping the least possible gain, and – then – to use the degrees of freedom in the choice of the internal model, if any, to meet the gain constraints. Against all odds, this program works in a number of relevant (non-minimum phase) cases.

On the Hautus test for exponentially stable C_0-groups

Birgit Jacob
(joint work with Hans Zwart)

We consider the abstract system

$$\dot{x}(t) = Ax(t),\ x(0) = x_0, \quad t \geq 0 \tag{1}$$

$$y(t) = Cx(t), \quad t \geq 0, \tag{2}$$

on a Hilbert space H. Here A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and by the solution of (1) we mean $x(t) = T(t)x_0$, the weak solution. If C is a bounded linear operator from H to a second Hilbert space Y, then it is straightforward to see that $y(\cdot)$ in (2) is well-defined, and continuous. However, in many PDE's, rewritten in the form (1)–(2), C is only a bounded operator from $D(A)$, the domain of A, to Y, although the output y is a well-defined (locally) square integrable function. In the following C will always be a bounded operator from $D(A)$, equipped with the graph norm, to Y. If the output is square integrable on the time interval $(0, \infty)$, then C is called an *infinite-time admissible observation operator for* $(T(t))_{t\geq 0}$, see Weiss [9] and Jacob and Partington [2]. Using the uniform boundedness theorem, we see that the observation operator C is infinite-time admissible if and only if there exists a constant $L > 0$ such that

$$\int_0^\infty \|CT(t)x_0\|^2\, dt \leq L\|x_0\|^2, \qquad x_0 \in D(A). \tag{3}$$

Note that the first norm is in Y, whereas the second norm is in H. In the following we will always assume that C is an infinite-time admissible observation operator for $(T(t))_{t\geq 0}$. We introduce the following observability concepts.

Definition. The pair (A, C) is called *exactly observable in time* $t_0 > 0$ if there exists a constant $\kappa_{t_0} > 0$ such that

$$\kappa_{t_0} \int_0^{t_0} \|CT(t)x_0\|^2 \, dt \geq \|x_0\|^2, \qquad x_0 \in D(A).$$

The pair (A, C) is called *exactly observable* if there exists a constant $\kappa > 0$ such that

$$\kappa \int_0^{\infty} \|CT(t)x_0\|^2 \, dt \geq \|x_0\|^2, \qquad x_0 \in D(A).$$

The pair (A, C) is called *final state observable* if there exist constants $\kappa, t_0 > 0$ such that

$$\kappa \int_0^{\infty} \|CT(t)x_0\|^2 \, dt \geq \|T(t_0)x_0\|^2, \qquad x_0 \in D(A).$$

The pair (A, C) is called *approximately observable* if

$$\int_0^{\infty} \|CT(t)x_0\|^2 \, dt > 0, \qquad x_0 \in D(A)\backslash\{0\}.$$

Clearly, approximate observability and final state observability are weaker concepts than exact observability, whereas exact controllability in time t_0 is a stronger concept. For C_0-groups the concept of exact observability and final state observability are equivalent notions. In Russell and Weiss [8] it is shown that a necessary condition for exact observability of exponentially stable systems is the following version of the Hautus test:

> There exists a constant $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in D(A)$:
>
> $$\|(sI - A)x\|^2 + |\operatorname{Re} s| \, \|Cx\|^2 \geq m|\operatorname{Re} s|^2 \|x\|^2, \qquad \text{(HT)}$$

Here $\mathbb{C}_-$ denotes the open left half plane. The Hautus test (HT) is sufficient for approximate observability of exponentially stable systems [8] and for polynomially stable systems [3]. Further, the Hautus test (HT) is sufficient for exact observability of strongly stable Riesz-spectral systems with finite-dimensional output spaces [4], for exponentially stable systems with A is bounded on H [8], and for exponentially stable systems if the constant m in (HT) equals one [1], a short proof of this last result can be found Section 4. However, in general the Hautus test (HT) is not sufficient for exponentially stable systems [6]. We refer the reader to Russell and Weiss [8], and Jacob and Zwart [4, 5] for more information on this Hautus test. Related to the Hautus test (HT) is an equivalent condition for exact observability of groups of unitary operators, see Section 2, [10], [7].

We show in particular that the Hautus test (HT) is sufficient for exponentially stable systems generated by a normal C_0-group, and we prove that the Hautus test (HT) is in general not sufficient for strongly stable systems even if the operator

C is bounded and A generates a contraction semigroup. More precisely, the main results are as follows.

Theorem. Let A be the generator of a C_0-group $(T(t))_{t\in\mathbb{R}}$ satisfying

$$M_1 e^{\alpha_1 t}\|x_0\| \le \|T(t)x_0\| \le M_2 e^{\alpha_2 t}\|x_0\|,$$

for every $t \ge 0$ and $x_0 \in H$ and some constants $M_1, M_2 > 0$ and $\alpha_1 < \alpha_2 < 0$. Further we assume that the Hautus test (HT) is satisfied for $s = \alpha_2 + i\omega$, $\omega \in \mathbb{R}$.

If

$$\frac{\alpha_2 - \alpha_1}{|\alpha_2|} < \frac{\sqrt{m}M_1}{4eM_2},$$

then the pair (A, C) is exactly observable in time $t_0 = (\alpha_2 - \alpha_1)^{-1}$.

Theorem. Let A be the generator of an exponentially stable normal C_0-semigroup $(T(t))_{t\ge 0}$. Then the Hautus test (HT) is sufficient for final state observability.

Corollary. Let A be the generator of an exponentially stable normal C_0-group $(T(t))_{t\in\mathbb{R}}$. Then the Hautus test (HT) is equivalent to exact observability.

We recall that the C_0-semigroup $(T(t))_{t\ge 0}$ is said to be strongly stable if $\|T(t)x_0\| \to 0$ as $t \to 0$ for every $x_0 \in H$.

Theorem. There exists a strongly stable contraction semigroup on a Hilbert space with generator A such that

$$\|(sI - A)x\| \ge m|\mathrm{Re}\, s|\,\|x\|, \quad \mathrm{Re}\, s < 0, x \in D(A). \tag{4}$$

In particular, the pair $(A, 0)$ satisfies the Hautus test (HT). Clearly the zero operator is infinite-time admissible for the semigroup, but the pair $(A, 0)$ is not approximately observable.

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Set oriented numerics in control theory – on the construction of optimal feedbacks

Oliver Junge
(joint work with Lars Grüne)

An elegant way to construct a globally stabilizing controller for a (nonlinear) control system is via Bellman's *optimality principle*. This is a fixed point equation which – together with a suitable boundary condition – characterizes the *value function* of the system.[1] From the value function, the associated optimally stabilizing controller can be computed by repeatedly solving a finite-dimensional optimization problem. The value function will then act as a Lyapunov function for the closed loop system.

In the case of linear dynamics and a quadratic (instantaneous) cost function, an explicit formula for V as well as the controller can be derived by solving an associated *Ricatti equation*. In the nonlinear case, one typically has to resort to a numerical approximation of V. In this case, one needs to project the Bellman equation onto a finite-dimensional approximation space and solve the resulting (discrete) fixed point equation. Typically, piecewise (multi)linear approximation spaces are employed [1, 2]. Recently, however, an approximation space consisting of piecewise constant functions has been used for this purpose [8, 4, 3, 5, 6, 7]. The resulting discrete Bellman equation can be interpreted as a shortest path problem on a suitably defined graph and thus can readily be solved by fast shortest path algorithms.

The approach is particularly well suited for problems with highly irregular value function, complicated state constraints and naturally handles hybrid systems. Its extension to perturbed systems conceptually enables the treatment of discrete event systems.

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[1] The value function $V(x)$ of a state x of the system is the minimal cost accumulated along any controlled trajectory starting at x. The associated instantaneous cost may be given in the problem description or may have to be suitably defined.

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Robust Identification of Exogenous Disturbances

Hans-Wilhelm Knobloch

Given a system

$$\dot{x} = p(x) + G(x)w \tag{1}$$

with state x which is observed at sampling times $\nu\delta$, $\nu = 0, 1, ...$ Let $\bar{x}(t)$ be the measured state and assume

$$\|x(t) - \bar{x}(t)\| \leq \Delta, \tag{2}$$

$t \geq 0$, Δ known and fixed. $w(t)$ is an essentially unknown time signal, Lipschitz continuous, bounds for $\|w\|$ and the Lipschitz constant are known. Robust identification means: Construct an estimate $\bar{w}(t)$, $t \geq 0$, satisfying

$$\|w(t) - \bar{w}(t)\| = O(\Delta) + O(\delta) + \epsilon \text{ for } t \to \infty. \tag{3}$$

$\epsilon > 0$, arbitrary, O depending upon available information only. Special case: $w(t)$ is generated by a (deterministic) disturbance model, given by a differential equation $\dot{w} = s(w)$, "internal model principle", comprises the case of unknown parameters ($\dot{w} = 0$). Solution of the identification problem can be reduced to two standard problems of calculus

- Solve the linear PDE $V_x(p(x) + G(x)w) + V_w s(w) \leq 0$, $V > 0$.
- An equation of the form $V_i(x, w) = w_i$, $i = 1, \ldots, \dim(w)$, along $x = \bar{x}(t)$ should be solved.

Measures of Unobservability

Arthur J. Krener
(joint work with Kayo Ide)

An observed nonlinear dynamics is observable if the mapping from initial condition to output trajectory is one to one. The standard tool for checking observability is the observability rank condition but this only gives a yes or no answer. It does not measure how observable or unobservable the system is. Moreover it requires the ability to differentiate the dynamics and the observations. We introduce new tools, the local unobservability index and the local estimation condition number, to measure the degree of observability or unobservability of a system. To compute these one only needs the ability to simulate the system. We apply these tools to find the best location to put a sensor to observe the flow induced by two point vortices.

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The moment problem for rational measures: Convexity in the spirit of Krein

Anders Lindquist
(joint work with Christopher I. Byrnes)

The moment problem as formulated by Krein and Nudel'man [1] is a beautiful generalization of several important classical moment problems. Motivated by classical applications, in both finite and infinite dimensions, we recently formulated a new version of this problem that we call the moment problem for positive rational measures [2, 3]. The formulation reflects the importance of rational functions in signals, systems and control. While this version of the problem is decidedly nonlinear, the basic tools still rely on convexity. In particular, we present a solution to this problem in terms of a nonlinear convex optimization problem that generalizes the maximum entropy approach used in several classical special cases. See [3] for a full text.

Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a basis $(u_0, u_1, \cdots, u_n)$ of a (finite-dimensional) subspace $\mathfrak{P}$ of the Banach space $C[a, b]$ of complex-valued continuous functions defined on the real interval $[a, b]$, the *generalized moment problem* [1] is to find a positive measure $d\mu$ such that

$$\int_a^b u_k(t)d\mu(t) = c_k, \quad k = 0, 1, \cdots, n. \tag{1}$$

For $p \in \mathfrak{P}$, we set $P = \mathrm{Re}(p)$ and consider the closed convex cone $\mathfrak{P}_+ = \{p \in \mathfrak{P} \mid P \geq 0\}$. A standing hypothesis in [1], which we also assume here, is that there exists $p \in \mathfrak{P}_+$ such that $P > 0$ on $[a, b]$, so that the interior of $\mathfrak{P}_+$ satisfies $\mathring{\mathfrak{P}}_+ \neq \emptyset$. If $p \in \mathfrak{P}$, then we may write $p = \sum_{k=0}^{n} p_k u_k$. We define the dual cone $\mathfrak{C}_+$ as the set of sequences $c = (c_0, c_1, \cdots, c_n)$ such that

$$\langle c, p \rangle := \mathrm{Re}\left\{\sum_{k=0}^{n} p_k c_k\right\} = \int_a^b P \, d\mu \geq 0 \tag{2}$$

for all $p \in \mathfrak{P}_+$. We will call such a sequence *positive*.

In order to motivate our use of nonlinear methods for the constrained moment problems we need to solve for systems and signals, we will first briefly review the classical existence result proved by Krein et al [1] for this (generalized) moment problem, which makes crucial use of the convexity of the sets $\mathfrak{P}_+$, $\mathfrak{C}_+$ and the subset $\mathcal{M}_+ \subset C[a, b]^*$ of positive measures. Indeed, let

$$\mathfrak{M} : C[a, b]^* \to \mathbb{C}^{n+1} \tag{3}$$

be the continuous mapping defined via (5) for an arbitrary bounded measure $d\mu \in C[a, b]^*$. Then

$$\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+. \tag{4}$$

To prove the reverse containment, in [1] the convex conic hull $K(U)$ of the curve

$$U(t) = \begin{pmatrix} u_0(t) & u_1(t) & \dots & u_n(t) \end{pmatrix} \subset \mathbb{C}^{n+1}, \quad a \leq t \leq b,$$

is shown to satisfy $K(U) = \mathfrak{C}_+$. On the other hand, by choosing $d\mu = \delta_{t_0}$ for each $t_0 \in [a, b]$, it follows that $U \subset \mathfrak{M}(\mathcal{M}_+)$. In particular, to say that $\mathfrak{M}(\mathcal{M}_+)$ is closed is to say that $\mathfrak{C}_+ = K(U) \subset \mathfrak{M}(\mathcal{M}_+)$. Using the fact that $\mathring{\mathfrak{P}}_+ \neq \emptyset$, it is shown in [1] that $\mathfrak{M}(\mathcal{M}_+)$ is closed. Therefore

Theorem. [[1]] $\mathfrak{M}(\mathcal{M}_+) = \mathfrak{C}_+$, i.e. the (generalized) moment problem is solvable if and only if $c \in \mathfrak{C}_+$.

This formulation incorporates several important classical moment problems, including the power moment problem, the trigonometric moment problem and the moment problem arising in Nevanlinna-Pick interpolation. In both the power and the trigonometric moment problems one is led to consider the "polynomials," $P = \mathrm{Re}(p)$, for $p \in \mathfrak{P}$. For this reason, the functions $P := \mathrm{Re}(p)$, for $p \in \mathfrak{P}$ in an arbitrary generalized moment problem are referred to as "polynomials" for $\mathfrak{P}$.

Following this precedent, we refer to the ratio P/Q with $p, q \in \mathfrak{P}$ as "rational functions" for $\mathfrak{P}$, reflecting the importance of rational functions in describing models in systems, signals, estimation and control. Indeed, for the classical Nevanlinna-Pick interpolation problem, the elements of $\mathfrak{P}$ are rational functions σ/τ, where τ is fixed, and the "polynomials" are the real parts of elements in $\mathfrak{P}$. This of course implies that the "rational functions" are rational in the usual sense. A measure of the form $d\mu = \frac{P(t)}{Q(t)} dt$, where P and Q are positive polynomials for $\mathfrak{P}$, is a (generalized) *rational positive measure*.

Problem. Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a subspace $\mathfrak{P}$, the *(generalized) moment problem for rational measures* is to parameterize all positive rational measures $\frac{P(t)}{Q(t)}dt$ such that

$$\int_a^b u_k(t)\frac{P(t)}{Q(t)}dt = c_k, \quad k = 0, 1, \cdots, n. \tag{5}$$

Let $\mathcal{R}_+ \subset \mathcal{M}_+$ denote the subset of rational positive measures. For any $d\mu \in \mathcal{R}_+$, consider the sequence c defined by (5) and any $p = \sum_{k=0}^n p_k u_k \in \mathfrak{P}_+ \setminus \{0\}$. Then $\langle c, p\rangle > 0$ so that $c \in \mathring{\mathfrak{C}}_+$, the interior of $\mathfrak{C}_+$. Therefore, $\mathfrak{M}(\mathcal{R}_+) \subset \mathring{\mathfrak{C}}_+$, an enhancement of (4) for rational measures. In contrast, however, the subspace $\mathcal{R}_+ \subset \mathcal{M}_+$ is *not* convex, and our proof of the reverse inclusion follows along nonlinear lines, using instead the fact that $\mathring{\mathfrak{C}}_+$ is a connected manifold. Indeed, in [2] we show that $\mathfrak{M}|_{\mathcal{P}_+} \subset \mathring{\mathfrak{C}}_+$ is open by applying the Implicit Function Theorem. This argument also shows that, for any fixed $p \in \mathring{\mathfrak{P}}_+$, the restriction of the moment mapping $\mathfrak{M}$ to the submanifold

$$\mathcal{P}_+ = \{d\mu \in \mathcal{R}_+ : d\mu = \frac{P}{Q}dt, \;\; p \in \mathring{\mathfrak{P}}_+\} \tag{6}$$

has an everywhere invertible Jacobian matrix. Moreover, under the mild hypothesis that $\mathfrak{P}$ consists of Lipschitz continuous functions. we show in [2] that $\mathfrak{M}|_{\mathcal{P}_+}$ is a proper mapping and, as such, has a closed image. By connectivity, we see that $\mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+$. Furthermore, since $\mathring{\mathfrak{C}}_+$ is diffeomorphic to $\mathbb{R}^n$, by Hadamard's Global Inverse Function Theorem, we conclude [2] that $\mathfrak{M}|_{\mathcal{P}_+} : \mathcal{P}_+ \to \mathring{\mathfrak{C}}_+$ has a globally defined differentiable inverse. In particular, the moment problem for rational measures is well-posed in the sense of Hadamard.

This fundamental fact has several remarkable consequences. Defining the map $F^p : \mathring{\mathfrak{P}}_+ \to \mathring{\mathfrak{C}}_+$ such that $F^p(q) = \mathfrak{M}(\frac{P}{Q}dt)$, the generalized moment problem for rational measures takes the componentwise form $F_k^p(q) - c_k = 0, \quad k = 0, 1, \ldots, n.$ and we can fashion the 1-form

$$\omega_c = \mathrm{Re}\left\{\sum_{k=0}^n [c_k - F_k(q)]\, dq_k\right\} = \mathrm{Re}\sum_{k=0}^n c_k dq_k - \int_a^b \frac{P}{Q}dQdt$$

on $\mathring{\mathfrak{P}}_+$. Computing the exterior derivative we obtain $d\omega_c = 0$ so that by the Poincaré Lemma there exist a smooth function $\mathbb{J}_c$ obtained by integrating ω_c on any path between any two endpoints; i.e.,

$$\mathbb{J}_c(q) = \int \omega_c = \langle c, q\rangle - \int_a^b P \log Q \; dt. \tag{7}$$

establishing a "Dirichlet Principle" for the constrained moment problem, i.e. the moment equations are the critical point equations for a variational criterion $\mathbb{J}_c$ defined for $q = \sum_{k=0}^n q_k u_k$ in $\mathfrak{P}_+$. Furthermore, since the Jacobian, $\mathrm{Jac}(F^p)$, must satisfy $\mathrm{Jac}(F^p)^T = \mathrm{Jac}(F^p) = D^2(\mathbb{J}_c) > 0$, $\mathbb{J}_c$ is strictly convex. Moreover,

since $\mathbb{J}_c$ is a proper function, $\mathbb{J}_c$ always achieves its minimum at an interior point which at a solution of (5).

Theorem. $\mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+$, i.e. the moment problem for rational measures is solvable if and only if $c \in \mathring{\mathfrak{C}}_+$. Moreover, $F^p : \mathring{\mathfrak{P}}_+ \to \mathring{\mathfrak{C}}_+$ is a diffeomorphism. In particular, for $p \in \mathring{\mathfrak{P}}_+$ fixed and any $c \in \mathring{\mathfrak{C}}_+$ there is a unique solution which corresponds to a minimum of $\mathbb{J}_c$.

It can be shown that the problem to minimize (7) is the dual optimization problem, in the sense of mathematical programming, of the problem to maximize a certain relative entropy criterion subject to the moment conditions (1); see [4, 5] and references therein.

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Challenges in Model Order Reduction

Boris Lohmann

(joint work with Rudy Eid)

The precise and computer supported modeling of dynamical systems often leads to large sets of ordinary differential equations, complicating analysis, system design and control. *Model order reduction* aims at finding smaller sets of differential equations preserving or at least approximating the most important properties and the dynamic behavior of the original model. Subsequently, a short survey on the challenges in the development of new reduction methods is presented, together with some recent results from the author's research group.

1. Challenges in three categories

Although there are some families of methods available for the reduction of linear time-invariant models, like Balancing and Truncation [1], Modal Reduction [5], and Krylov Subspace Methods [1], as well as some methods for the reduction of nonlinear systems [4, 3, 6], a number of challenges is presently attracting attention. They can be grouped into three categories:

(1) **Problem classes:** Nonlinear systems and systems with various parameters to be preserved in the reduced model require dedicated new methods (section 3). Furthermore, the coupling between systems, possibly with high number of input and output variables requires specific methods, as well as the reduction of models involved in open-loop or closed-loop control systems.
(2) **Mathematical Aspects:** The quality of approximation is unclear in some important reduction methods; performance measures, error bounds, and guidelines on how to choose the parameters of the reduction procedures are to be found (section 2). Large-scale systems demand numerically robust algorithms that are able to handle models of order up to millions. In some applications, it is important to preserve structural properties, like stability, passivity, relative degree, or second order structure. Also, systems of differential algebraic equations need to be handled and reduced.
(3) **Applications:** Specific requirements on the reduction methods result from the different fields of technical application, like numerical and technical mechanics, systems theory and automatic control, micro and nano systems, and electromagnetic/electrodynamics.

2. A new model reduction approach in time-domain

R. Eid in his dissertation [2] develops a model reduction in time-domain which contributes to the important question of how to use the design parameters of a Krylov-based model reduction for a good approximation of the system's response to arbitrary input signals. The approach employs the scaled *Laguerre functions* with their free parameter α

$$\phi_i^\alpha(t) = \sqrt{2\alpha}\frac{e^{\alpha t}}{i!}\frac{d^i}{dt^i}(e^{-2\alpha t}t^i) \circ\!\!-\!\!\bullet\, \Phi_i^\alpha(s) = \mathcal{L}\phi_i^\alpha(t) = \frac{\sqrt{2\alpha}}{s+\alpha}\left(\frac{s-\alpha}{s+\alpha}\right)^i,$$

to express the impulse response of the original system as an infinite Laguerre series expansion of the form:

$$h(t) = \sum_{n=0}^{\infty} f_i \phi_i^\alpha(t).$$

It was first shown that if the projection matrices $\mathbf{V}$ and $\mathbf{W}$ are chosen as bases of the Krylov subspaces $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}, (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right)$, and $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{A}^T, (\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right)$ respectively, then $2q$ Laguerre coefficients of the impulse responses of the original and reduced order system match. Then, it was shown that reducing a state space model in *time-domain* by matching the Laguerre coefficients of the impulse response is equivalent to matching the moments of the transfer functions about $s_0 = \alpha$ in the *frequency-domain*. This result reformulates the open problem of how to choose the expansion point in the rational Krylov subspace reduction methods to the problem of choosing α in the Laguerre-based reduction method.

A solution is to minimize

$$\min_{\alpha} J(\alpha) = \sum_{i=0}^{\infty} i f_i^2,$$

leading to a basis where the first terms are almost enough to describe the impulse response. The optimal value of α can be analytically found to be $\alpha^* = \sqrt{\frac{M_2}{M_1}}$ where $M_1 = \frac{\int_0^\infty h^2(t)tdt}{\int_0^\infty h^2(t)dt}$ can be interpreted as the decay rate of the impulse response, and $M_2 = \frac{\int_0^\infty \dot{h}^2(t)tdt}{\int_0^\infty h^2(t)dt}$ describing its smoothness. When no impulse response data is available, M_1 and M_2 can be calculated by solving the following Lyapunov equations:

$$\mathbf{AX} + \mathbf{XA}^T + \mathbf{bb}^T = \mathbf{0}, \quad \mathbf{AY} + \mathbf{YA}^T + \mathbf{X} = \mathbf{0}.$$

In order to avoid solving them in the full order, an iterative *low order* algorithm is suggested: (i) Reduce the original system using an arbitrary α_0, (ii) Solve the corresponding Lyapunov equations for the reduced system to calculate $\mathbf{X}_r$ and $\mathbf{Y}_r$, (iii) calculate the new optimal parameter α_i, reduce the original system and go back to (ii). It was observed that the initial value α_0 does not affect the fast convergence (typically less than 4 iterations) of this algorithm. In addition, the calculated α^* results in an excellent approximation of the impulse response and of the peak region of the Bode diagram.

3. Nonlinear and parametric reduction with subspace separation

In the reduction of nonlinear and of parametric systems, a known technique is to first represent the model as a weighted sum of linear or non-parametric models, interpolating the original one at certain states $\mathbf{x}_i$ or parameter values p_i, and then apply a common order reducing projection $\mathbf{V}$ to these models. For instance, a linear *parametric* model $\dot{\mathbf{x}} = \mathbf{A}(p)\mathbf{x} + \mathbf{Bu},\ \mathbf{y} = \mathbf{Cx}$ can be approximated by

$$\dot{\mathbf{x}} = \sum_{i=1}^{s} \omega_i(p) \left[\mathbf{A}(p_i)\mathbf{x} + \mathbf{Bu}\right],\ \mathbf{y} = \mathbf{Cx},$$

where $\mathbf{A}(p)$ needs only to be known at s values p_i of p. The reduced model is

$$\dot{\mathbf{x}} = \sum_{i=1}^{s} \omega_i(p) \left[\mathbf{V}^T\mathbf{A}(p_i)\mathbf{V}\mathbf{x}_r + \mathbf{V}^T\mathbf{Bu}\right],\ \mathbf{y} = \mathbf{CV}\mathbf{x}_r.$$

A difficulty is the fact that the projector $\mathbf{V}$ must include the relevant subspace information from *all* the full order models simultaneously, increasing the order of the reduced model significantly. A *remedy* is to *separate the subspaces* to their corresponding local models, i.e. to apply to each local model only the (small) projector $\mathbf{V}_i$ designed for it. Thereby,

$$\dot{\mathbf{x}}_r^* = \sum_{i=1}^{s} \omega_i(p) \left[\mathbf{T}_i\mathbf{V}_i^T\mathbf{A}(p_i)\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^* + \mathbf{T}_i\mathbf{V}_i^T\mathbf{Bu}\right],\ \mathbf{y} = \sum_{i=1}^{s} \omega_i(p)\mathbf{CV}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*$$

The additional transformations $\mathbf{T}_i$ make sure that all local models generate derivatives that are allowed to be summed up, i.e. generating derivatives of the vector $\mathbf{x}_r^*$ with a common (physical) meaning. For this purpose, we define the q variables to be some linear combination of the original state variables, $\mathbf{x}^* = \mathbf{R}\mathbf{x} \approx \mathbf{R}\mathbf{V}_i\mathbf{x}_r$, where $\mathbf{R}$ is a (q,n)-matrix of full rank, resulting in the transformation matrices,

$$\mathbf{T}_i = \mathbf{R}\mathbf{V}_i.$$

The best conditioning of $\mathbf{T}_i$ occurs when $\mathbf{T}_i = \mathbf{R}\mathbf{V}_i \approx \mathbf{I}$. Hence, a possible choice is

$$[\mathbf{I} \ \cdots \ \mathbf{I}] \approx \mathbf{R}\,[\mathbf{V}_1 \ \cdots \ \mathbf{V}_s] \ \Rightarrow \ \mathbf{R}\,[\mathbf{I} \ \cdots \ \mathbf{I}]\,[\mathbf{V}_1 \ \cdots \ \mathbf{V}_s]^+ .$$

Similarly, *nonlinear* models $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$ can be reduced using locally linear models [4] together with the individual subspaces $\mathbf{V}_i$ introduced above (the weights ω_i are to be normalized to sum up to one),

$$\dot{\mathbf{x}}_r^* = \textstyle\sum_{i=1}^{s} \omega_i(\mathbf{V}_i\mathbf{T}_i^{-1}x_r^*)\mathbf{T}_i\mathbf{V}_i^T \left[\mathbf{f}(\mathbf{x}_i) - \mathbf{A}_i\mathbf{x}_i + \mathbf{A}_i\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^* + \mathbf{B}\mathbf{u}\right],$$
$$\mathbf{y} = \textstyle\sum_{i=1}^{s} \omega_i(\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*)\mathbf{C}\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*.$$

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Stabilisation of the periodic behaviour of discretely controlled continuous systems

Jan Lunze

(joint work with Axel Schild)

This paper presents a novel model-based design method for state-dependent switching policies of discretely controlled continuous systems. The design objective is to stabilise a prespecified periodic state trajectory of the continuous system by switching the operation mode of this system. As a key contribution, it is shown that the original design problem can be reformulated in terms of an equivalent periodic output-feedback problem. Hence, design methods for periodic systems can be applied to find the parameters of the switching surfaces in the state space, which are used to determine the switching times.

Discretely controlled continuous systems constitute an important subclass of hybrid (mixed discrete-continuous) systems [3], which consist of a continuous plant

in feedback connection with a discrete-event controller. Such systems have recently received much attention, because they are found throughout many application domains such as power electronics, manufacturing systems, process engineering, systems biology and robotics [1].

The control task of discretely controlled continuous systems is solved by switching the operation mode of the plant so as to meet specifications defined in terms of the continuous variables at stationary operation. As a central characteristic, the working principle of this system class requires to perpetually execute mode transition, which causes either periodic or chaotic behaviour.

This paper proposes a model-based design method for selecting the switching surfaces in the continuous state space. It is shown that it is sufficient to determine static event functions, the parameters of which are adapted to the deviation of the current continuous state at the switching times from the required limit cycle. The discretely controlled continuous systems is described by embedded maps [2], which make it possible to determine the continuous state and the switching time at the next switching instant in terms of the continuous state and switching time at the current switching instant.

Two methods for determining the parameters of the switching surfaces are proposed. The first method results in static switching surfaces which are proved to ensure the asymptotic local stability of the prespecified limit cycle [5]. The second method aims at adapting the switching surfaces to the effect of disturbances, which are measured as the distance of the current continuous state at the switching time from the prespecified limit cycle [6]. These methods have been successfully applied to design a generalised hysteresis controller for a DC-DC power converter [4].

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A Non-Stochastic Information Theory for Cooperative Networked Control

Girish N. Nair

In communications theory, unknown variables are almost always modelled as random variables (rv's), with concepts such as independence, entropy and information are defined in terms of the underlying probability distributions. This is in contrast with control theory, in which it is not uncommon to treat uncertainties and disturbances as bounded unknowns having no statistical structure. The emergence of networked control combines both fields and raises the question of whether it is possible to construct meaningful analogues of the important stochastic concepts of independence, Markovianness, Shannon entropy, Shannon information etc, *without assuming a probability space*. In this talk it is shown how this may be achieved, leading in particular to the definition of a novel, purely deterministic information function that satisfies many of the properties of Shannon's construction.

The key is to keep the probability theory convention of regarding an unknown variable X as an injective mapping X from some underlying, hidden space Ω to a set $\mathbf{X}$ of interest. Each $\omega \in \Omega$, representing e.g. a particular combination of initial plant state and noise trajectories, gives rise to a particular realisation $X(\omega)$, which is denoted by lower-case x. The variable X is called an *uncertain variable (uv)* and, as in probability theory, the dependence on ω is suppressed for conciseness. However, unlike probability theory, no measure is imposed on Ω but only on $\mathbf{X}$; the uncertainty structure exhibited by a collection of uv's will therefore be defined entirely in terms of uncertainty sets.

Given another uv Y in $\mathbf{Y}$, define with mild abuse of set notation

$$\begin{aligned} \{X\} &:= \{X(\omega) : \omega \in \Omega\} \subseteq \mathbf{X}, \\ \{X|y\} &:= \{X(\omega) : Y(\omega) = y, \omega \in \Omega\} \subseteq \{X\}, \\ \{(X,Y)\} &:= \{(X(\omega), Y(\omega)) : \omega \in \Omega\} \subseteq \{X\} \times \{Y\}. \end{aligned}$$

Call $\{X\}$ the *marginal uncertainty set* of X, $\{X|y\}$ its *uncertainty set conditional on* $Y = y$, and $\{(X,Y)\}$, the *joint uncertainty set* of (X,Y). We denote the family of all distinct, nonempty conditional uncertainty sets as

$$\{X|Y\} := \{\{X|y\} : y \in \mathbf{Y}\},$$

with empty sets omitted. Notice that $\cup_{\mathbf{A} \in \{X|Y\}} \mathbf{A} = \{X\}$, i.e. $\{X|Y\}$ is an $\{X\}$-*cover*. Further notice that

$$\{(X,Y)\} = \bigcup_{y \in \{Y\}} \{X|y\} \times \{y\},$$

i.e. the joint uncertainty set is fully determined by the conditional and marginal sets, similar to rv's.

In analogy with the notion of statistical independence, X, Z are said to be *unrelated* if

$$\{(X,Z)\} = \{X\} \times \{Z\},$$

which is equivalent to

$$\{X|z\} = \{X\}, \quad \forall z \in \{Z\}.$$

They are said to be *unrelated conditional on* Y if

$$\{(X,Z)|y\} = \{X|y\} \times \{Z|y\}, \quad \forall y \in \{Y\},$$

which is equivalent to

$$\{X|y,z\} = \{X|y\}, \quad \forall (y,z) \in \{(Y,Z)\}.$$

The uv's X, Y, Z are said to form a *Markov uncertainty chain* $X \leftrightarrow Y \leftrightarrow Z$ if X, Z are unrelated conditional on Y.

Quantify the *a priori* uncertainty associated with the $\mathbb{R}^n$-valued uv X by means of the so-called Renyi differential entropy of order 0,

$$h_0\{X\} = \ln \lambda_n\{X\} \in [-\infty, \infty],$$

assuming that $\{X\}$ is measurable w.r.t. Lebesgue measure λ_n. For conciseness, we call this *0-entropy*. Applying this notion to the family of uncertainty sets $\{X|Y\}$ in a worst-case way, the 0-entropy of X *conditional on* Y may be defined as

$$h_0\{X|Y\} := \sup_{y \in \{Y\}} h_0\{X|y\} = \ln \left(\sup_{y \in \{Y\}} \lambda_n\{X|y\} \right).$$

It would seem natural at this point to define the information gained about X from Y as the corresponding reduction in 0-entropy, $h_0\{X\} - h_0\{X|Y\}$; however such a construction can be shown to not satisfy a crucial sub-additivity property exhibited by Shannon information. This is overcome by defining *0-information* to be

$$I^*\{X|Y\} := \ln \left(\sup_{\mathbf{A} \subseteq \{X\}} \inf_{y \in \{Y\}} \frac{\lambda_n(\mathbf{A})}{\lambda_n(\mathbf{A} \cap \{X|y\})} \right)$$

where the supremum is over all λ_n-measurable subsets of $\{X\}$. That is, $I^*\{X|Y\}$ is a max-min log-ratio of X-uncertainty volumes before and after observing Y.

The novel conditional 0-entropy and 0-information just introduced above satisfy a number of inequalities that are crucial in deriving universal lower bounds on performance in distributed networked systems:

(1) 0-entropy bounds:

$$\begin{aligned} h_0\{X,Y\} &\leq h_0\{X\} + h_0\{Y\} \\ h_0\{X,Y|Z\} &\leq h_0\{X|Z\} + h_0\{Y|Z\} \\ h_0\{X|Y\} &\leq h_0\{X\} \end{aligned}$$

(2) $0 \leq I^*\{X|Y\} \leq \ln N^*\{X|Y\}$, where $N^*\{|Y\}$ is the minimal cardinality of all subfamilies of $\{X|Y\}$ that cover $\{X\}$ up to Lebesgue measure 0.

(3) More observations cannot hurt, i.e. $I^*\{X|Y_1, Y_2\} \geq I^*\{X|Y_1\}$.

(4) Data processing for Markov uncertainty chains - if $X \leftrightarrow Y \leftrightarrow Z$, then $I^*(X|Z) \leq I^*(X|Y)$.

(5) Superadditivity in unrelated unknowns - if $X_1 \in \mathbb{R}^m$, $X_2 \in \mathbb{R}^n$ are unrelated then $I^*\{X_1, X_2|Y\} \geq I^*\{X_1|Y\} + I^*\{X_2|Y\}$.

(6) Subadditivity in observations - if Z_1, Z_2 are unrelated and $Y_i = f_i(X, Z_i)$, $i \in \{1, 2\}$, then $I^*\{X|Y_1, Y_2\} \leq I^*\{X|Y_1\} + I^*\{X|Y_2\}$.

These inequalities lead in particular to a more elegant derivation of the results in [1], under the proviso that every link in the network lies on a so-called irreducible x_h-cycle. However, it is envisaged that the concepts introduced here may also prove useful in other areas, where notions of unrelatedness, Markovianness or information content are relevant but where a stochastic formulation is not desired.

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Structure and identification of stationary reciprocal processes

Giorgio Picci
(joint work with Francesca Carli)

In this talk we discuss stationary reciprocal processes defined on the finite interval $[1\,,\,N]$. These processes can be seen as a special class of Markov random fields restricted to one dimension. Non stationary reciprocal processes have been extensively studied in the past especially by Krener, Levy and co-workers see e.g. [4] and the references therein. However the specialization of the non-stationary results to the stationary case does not seem to have been pursued in sufficient depth in the litarature. Processes (and stochastic models) of this kind are especially useful for describing phenomena which naturally live in a finite region of the "time" (or space) line and estimation or identification of these models starting from observed data is a completely open problem which can in principle lead to many interesting applications in signal processing.

We show that a stationary m-dimensional reciprocal processes (of mean zero and finite variance) admits constant parameter descriptor-type representations of the following type

$$\sum_{k=-n}^{n} F_k\, \mathbf{y}(t+k) \;=\; \mathbf{d}(t)\,, \qquad t \in [1,\, N] \tag{1}$$

where the F_k's are $m \times m$ matrices with F_0 normalized to the identity and the model is associated to the cyclic boundary conditions:

$$\mathbf{y}(-k) \;=\; \mathbf{y}(N-k)\,; \qquad k = 0, 1, \ldots, n-1 \tag{2}$$

$$\mathbf{y}(N+k) \;=\; \mathbf{y}(k)\,; \qquad k = 1, 2, \ldots, n\,. \tag{3}$$

The process $\{\mathbf{d}(t)\}$ is the *conjugate process* [5] of $\mathbf{y}$, a stationary, finitely correlated of bandwidth n process with positive definite variance matrix $\mathbb{E}\,\mathbf{d}(t)\mathbf{d}(t)^\top := \Delta > 0$. The integer n is called the *index* of the process $\mathbf{y}$.

By definition of conjugate process, the following orthogonality condition holds

$$\mathbb{E}\,\mathbf{y}(t)\,\mathbf{d}(s)^\top = \Delta\delta(t-s)\,, \qquad t \neq s \in [1,\,N]\,, \tag{4}$$

where δ is the Kronecker symbol. The covariance matrix $\mathbf{R}_N$ of $\mathbf{y}$ defined by

$$\mathbf{R}_N = \left[\,\mathbb{E}\,\mathbf{y}(k)\,\mathbf{y}(j)^\top\,\right]_{k,j=1,\dots,N}$$

can be shown to be a block-circulant symmetric positive definite matrix. An easy consequence of property (4) is the following characterization.

Theorem. A full rank process is reciprocal of index n if and only if the inverse of its covariance matrix is a symmetric banded block-circulant matrix of bandwidth n.

Hence $\mathbf{M}_N := \mathbf{R}_N^{-1}$ has the following block-circulant **banded** structure

$$\mathbf{M}_N = \begin{bmatrix}
M_0 & M_1 & \dots & M_n & 0 & \dots & 0 & M_n^\top & \dots & M_1^\top \\
M_1^\top & M_0 & M_1 & \ddots & M_n & 0 & & 0 & \ddots & \vdots \\
\vdots & & \ddots & & & \ddots & & \vdots & & M_n^\top \\
M_n^\top & \dots & M_1^\top & M_0 & M_1 & \dots & M_n & \ddots & & 0 \\
0 & M_n^\top & & \dots & M_0 & \dots & & M_n & & \vdots \\
\vdots & & & \dots & & \ddots & & & & 0 \\
0 & & & \dots & & \dots & & & & M_n \\
M_n & & \ddots & & & & & & & \vdots \\
\vdots & \ddots & & \ddots & & \ddots & & & \ddots & M_1 \\
M_1 & \dots & M_n & 0 & \dots & 0 & M_n^\top & \dots & M_1^\top & M_0
\end{bmatrix}, \tag{5}$$

where $M_k = \Delta^{-1}F_k$, $k = -n, \dots, n$. It is seen that the matrix $\mathbf{M}_N$ defines the model (1) completely.It is called the *model matrix* of the process.

Note that this result makes the stochastic realization problem for reciprocal processes of index n completely trivial. In fact given a *bona-fide* covariance matrix $\mathbf{R}_N$ of the process, in other words, its *external description*, the model matrix $\mathbf{M}_N$ can be computed by simply inverting $\mathbf{R}_N$. This is the simplest answer one could hope for.This in turn leads to the following

Problem. Characterize the covariance matrix of a reciprocal process of index n. In other words, when does a (full rank) symmetric block-circulant covariance matrix have a symmetric banded block-circulant inverse of bandwidth n.

There is an easy indirect answer to this question in terms of spectral functions (Fourier transform) but a direct characterization in matrix terms is unknown.

Identification of reciprocal processes can be approached from several points of view. A simple approach is the classical method of moments well known in statistics. Assuming the process has index ν, one is led to solve a *bilateral Yule-Walker*

system of equations of the form

$$\sum_{j=-\nu}^{\nu} M_j\, R(k+j) \;=\; I\delta_0(k)\,, \qquad k = -N/2, \ldots, N/2-1 \tag{6}$$

which can be obtained directly from the model (1) using the orthogonality property (4).

Proposition. Assume we are given $2\nu+1$ covariance data $\{R_k\,;\, k = 0, 1, \ldots 2\nu\}$, then a unique reciprocal model of index (or bandwidth) ν can be computed by by solving the system of linear equations (6).

In practice a large number of covariance lags is undesirable since the $R(k)$'s will ultimately have to be estimated from the observation data and a large number of covariance estimates will introduce errors and unavoidably deteriorate the solution of the problem.

Under certain assumptions (in particular Gaussian distribution) it can be shown that the maximum likelihood estimation of reciprocal models requires exactly $\nu+1$ sample covariance estimates. The computation of the estimates of the matrix parameters M_k is a particular instance of a *Covariance selection problem* of the kind studied by the statistician M. Dempster [1] in the early seventies. In matrix terminology the covariance selection for stationary reciprocal models is equivalent to a special *matrix band extension problem* for block-circulant matrices. Matrix band extension problems for Toeplitz matrices have been studied by Dym, Gohberg and co-workers,[2], [3], but circulant matrices do not fit in the "banded algebra" framework used by these authors. This extension problem is still open and suprisingly tricky.

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Kazantzis-Kravaris-Luenberger observers

Laurent Praly

(joint work with V. Andrieu and L. Marconi)

We consider a dynamical system on $\mathbb{R}^n$

$$\dot{x} \;=\; f(x) \quad , \qquad y \;=\; h(x)$$

whose solution, issued from x, is denoted $X(x,t)$. To ease this presentation, we assume f and h are C^2 and there exists an invariant open subset $\mathcal{O}$ of $\mathbb{R}^n$ whose

closure $\mathtt{cl}(\mathcal{O})$ is compact. In this context, we want to design a triplet (m, ψ, ϕ) such that the solutions $(X(x,t), Z((x,z),t))$ in $\mathcal{O} \times \mathbb{R}^m$ of the extended system

$$\dot{x} \;=\; f(x) \quad , \qquad \dot{z} \;=\; \psi(z, h(x))$$

satisfy

$$\lim_{t \to +\infty} |\phi(Z((x,z),t)) - X(x,t)| \;=\; 0$$

For the linear case where $f(x) = Fx \ , \quad h(x) \;=\; Hx$, Luenberger wrote in IEEE MIL-8 (1964)

> *... Instead of requiring that the observer reconstruct the state vector itself, require that it reconstruct some ... transformation of the state vector. ... Assuming it were possible, ... it is clear that it would be possible to reconstruct the state vector itself, provided that the transformation were invertible.*
> *... It is first shown that it is relatively simple to build a system which will reconstruct some ... transformation of the state vector and then it is shown how to guarantee that the transformation obtained is invertible ...*

For the non linear case, in Systems & Control Letters 34 (1998), Kazantsis and Kravaris proposed to apply the same idea in the present context. This leads to a triplet $(m, \psi, \phi) = (n, (A, B), T^{-1})$ and the observer

$$\dot{z} \;=\; A\, z \;+\; B\, h(x) \quad , \qquad \hat{x} \;=\; T^{-1}(z)$$

with $m = n$ and T solution of

$$\frac{\partial T}{\partial x}(x)\, f(x) \;=\; A\, T(x) \;+\; B\, h(x) \ .$$

Unfortunately these authors and later on Krener and Xiao looked for a solution both too smooth and a diffeomorphism.

Without referring to (PDE) above, Kreisselmeier and Engel study, in IEEE Transactions on Automatic Control, 48-3, (2003) the function

$$T(x) \;=\; \int_{-\infty}^{0} \exp(-As) B(h(X(x,s)))ds \ .$$

and establish a relation between its injectivity with m large enough and observability but without giving any bound for m.

Continuing along these lines, we state a first result giving sufficient conditions for a triplet (m, ψ, ϕ) to be appropriate.

Theorem (Andrieu-Praly SICON 2006) : *Assume there exist an integer m, a Hurwitz matrix A in $\mathbb{C}^{m \times m}$, C^0 functions T and B and a class $\mathcal{K}_\infty$ function ρ such that T admits a Lie derivative $L_f T$ along f and we have*

$$\textit{(PDE)} \qquad \begin{array}{rcll} L_f T(x) & = & A\, T(x) \;+\; B(h(x))) & \forall x \in \mathcal{O} \ , \\ |x_1 - x_2| & \leq & \rho(|T(x_1) - T(x_2)|) & \forall x_1,\, x_2 \in \mathtt{cl}(\mathcal{O}) \ . \end{array}$$

Under this condition, there exists a C^0 function T^* such that, for each (x, z) in

$\mathcal{O} \times \mathbb{C}^{m\times p}$, the solution $(X(x,t), Z(x,z,t))$ of

$$\dot{x} \;=\; f(x) \quad , \qquad \dot{z} \;=\; A\,z \;+\; B(h(x)))$$

satisfies

$$\lim_{t\to+\infty} |T^*(Z(x,z,t)) - X(x,t)| \;=\; 0 \;.$$

This result leads to three questions :
1. When does there exist a solution to (PDE) ?
2. When is this solution uniformly injective?
3. Can we use an approximation ?

The answer to the existence question is *... is relatively simple ... (Luenberger).*

Theorem (Andrieu-Praly SICON 2006) : *For any Hurwitz complex matrix A in $\mathbb{C}^{m\times m}$ and any C^1 function $B : \mathbb{R}^p \to \mathbb{C}^{m\times p}$, the function $T : \mathtt{cl}(\mathcal{O}) \to \mathbb{C}^{m\times p}$, defined as*

$$T(x) \;=\; \int_{-\infty}^{0} \exp(-As)B(h(X(x,s)))ds \;.$$

is C^0 and admits a Lie derivative L_fT which satisfies (PDE). Moreover, there exists $\ell > 0$ such that T is C^1 if $\mathtt{real}(\mathtt{eigen}(A)) \;\leq\; -\ell$.

To answer the second question, we introduce the following definition.

Backward $\mathcal{O}$-distinguishability : *For each pair of distinct points x_1 and x_2 in $\mathcal{O}$, there exists a negative time t such that we have*

$$h(X(x_1,t)) \;\neq\; h(X(x_2,t)) \;.$$

Theorem (Andrieu-Praly SICON 2006) : *Assume the system is backward $\mathcal{O}$-distinguishable. Then, for any injective C^1 function $b : \mathbb{R}^p \to \mathbb{C}^p$, with picking $m = n + 1$, there exists $\ell > 0$ and a subset S of $\mathbb{C}^{n+1}$ with zero Lebesgue measure such that the function $T = (T_{\lambda_i}) : \mathtt{cl}(\mathcal{O}) \to \mathbb{C}^{(n+1)\times p}$ defined as*

$$T_{\lambda_i}(x) \;=\; \int_{-\infty}^{0} \exp(-\lambda_i s)\, b(h(X_m(x,s)))\, ds \;,$$

is uniformly (on the compact set $\mathtt{cl}(\mathcal{O})$) injective provided the $n+1$ complex numbers λ_i are (arbitrary) in $\mathbb{C}^{n+1} \setminus S$ and with real part smaller than $-\ell$.

Finally for the approximation, Andrieu and Praly proved that, if a function T_a is close enough to be a solution of (PDE), then the observer can be modified to make it exact while still solving our problem. They showed also under which conditions the expression below gives a good approximation

$$T_a(x) \;=\; -\sum_{i=1}^{p} \mathtt{diag}\left(\frac{1}{\lambda_1^i}, \ldots, \frac{1}{\lambda_m^i}\right) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} L_f^{i-1} b(h(x)) \;.$$

Actually, numerical approximations are easy to obtain. Let be given an appropriate pair (A, B) with A satisfying

$$|\exp(At)| \leq k \exp(-at) \;.$$

and leading to a function T with Lipschitz constant L_T on $\mathtt{cl}(\mathcal{O})$. Let ν_T be the desired tolerance on T. We choose :

- a finite grid of points $\{x_i\}_{i\in I}$ covering $\mathtt{cl}(\mathcal{O})$ with step size $\nu_x \leq \frac{\nu_T}{L_T}$,
- $t_* \geq \log\left(\frac{2k^2}{a\nu_T}\sup_{x\in\mathtt{cl}(\mathcal{O})}|B(h(x))|\right)$ $\quad\left(\approx \log\left(\frac{2k^2}{a\nu_T}\max_{i\in I}|B(h(x_i))|\right)\right)$

Then, for each i, we let $T_i = \mathcal{T}_a((x_i,0),-t^*))$ where $(X_a(x_i,t),\mathcal{T}_a((x_i,0),t))$ is an approximation via a numerical scheme of the solution $(X(x_i,t),\mathcal{T}((x_i,0),t))$ of

$$\dot{x} = f(x) \quad , \qquad \dot{\tau} \;=\; \exp(-At)\,B(h(x))$$

with a time step size small enough to satisfy

$$|\mathcal{T}_a((x_i,0),-t^*)) - \mathcal{T}((x_i,0),-t^*))| \;\leq\; \frac{\nu_T}{2}\ .$$

Proposition (Marconi-Praly IEEE TAC 2008) : *For all x in $\mathtt{cl}(\mathcal{O})$, there exists i in I so that we have $|T(x)-T_i|\leq\nu_T$.*

Since T is uniformly injective on $\mathtt{cl}(\mathcal{O})$ (compact), for each k in $\{1,\ldots,n\}$, there exists a concave function[1] ρ_k of class $\mathcal{K}_\infty$ satisfying

$$|x_{k1}-x_{k2}| \;\leq\; \rho_k(|T(x_1)-T(x_2)|) \qquad \forall x_1,x_2\in\mathtt{cl}(\mathcal{O})\ .$$

Proposition (Marconi-Praly IEEE TAC 2008) : *The approximation of the kth component T^*_k of T^* defined as*

$$T^*_{ak}(z) \;=\; \frac{1}{2}\left[\min_{i\in I}\{x_{ki} \;+\; \rho_k(|T_i-z|)\} \;+\; \max_{i\in I}\{x_{ki} \;-\; \rho_k(|T_i-z|)\}\right]$$

satisfies

$$|T^*_{ak}(z)-x_k| \;\leq\; 2\rho_k\left(2|T(x)-z|\right)+2[\nu_T+L_T\nu_x]) \;+\; \nu_x\ .$$

Fixed point analysis in biological networks of coupled feedback

Nicole Radde

I consider systems of ordinary differential equations with underlying graph structure and bounded trajectories, which are motivated by systems biology approaches for cellular processes. An algorithm for efficiently calculating fixed points based on feedback loop breaking is introduced and applied to two biological examples.

1. Introduction

Differential equation models based on chemical reaction kinetics are a standard model class in systems biology. In many real settings prior knowledge is available about potential interactions across cell components, while at the same time quantitative measurements of kinetic rates are missing. It is thus convenient to ask

[1] $\rho_k(s)\approx r_k s$ with $r_k=\max_{i\neq j}\frac{|x_{ki}-x_{kj}|}{|T_i-T_j|}$.

what can be learned from this qualitative information about the dynamics of the systems. Here I consider networks of n cell components described by

$$\dot{x}_i(t) = s_i(x(t)) - \gamma_i x_i(t) \quad i = 1, \ldots, n.$$

The state vector x contains concentrations of these components. The synthesis rate $s_i \geq 0$ is regulated by other network components, and degradation is described by a first order decay process. I assume s_i to be bounded, which implies that solutions are bounded as well and the system has a fixed point[1]. The system is supposed to have an underlying graph structure for which circuits, paths and semi-circuits are defined in the usual graph theoretical sense. I introduce an algorithm for determining all fixed points of the system by breaking circuits and calculating a one-dimensional characteristics, the *loop function*. This function can analytically be calculated for graphs which I call *leading node structures* (LNS), and it also provides information about stability. I discuss the loop function concept on two biological examples, the repressilator model and tryptophan regulation in *Escherichia coli*.

2. The loop breaking algorithm (LBA) for fixed point calculation

The idea of the LBA is that, given the model, the fixed point of a strongly connected component is completely characterized by the fixed point coordinate of one of the variables. This is used to construct a one-dimensional characteristic of the system, the loop function, whose zeros correspond to the fixed point coordinates. For a single loop of n components, the LBA works like follows: The loop is broken by setting the *leading node*, say x_n, to a fixed value κ. Then, the unique fixed point coordinates of this hierarchical open loop system are iteratively calculated for $x_1, x_2, \ldots, x_n$ as functions of κ. The loop is closed by releasing the leading node, which translates into calculating the zeros of the loop function,

$$\dot{x}_n = 0 = s_n(x_1^s(x_2^s(\ldots(x_{n-1}^s(\kappa))))) - \gamma_n \kappa.$$

The solutions $\hat{\kappa}$ of this equation are the fixed point coordinates of x_n, and coordinates for the other nodes are finally obtained again by inserting $\hat{\kappa}$ into the steady state equations for x_i, $i = 1, \ldots, n-1$. A similar concept was already introduced elsewhere [1], but is extended here for interrelated feedback loops.

2.1. The repressilator.

I consider a synthetic transcriptional network of three proteins and respective mRNAs, which form a negative feedback loop described by

$$\begin{aligned} \dot{m}_i &= -m_i + \frac{\alpha}{1 + p_j^n} + \alpha_0 =: -m_i + f(p_j) \\ \dot{p}_i &= -\beta(p_i - m_i), \end{aligned}$$

[1]The subsequent analysis also holds true for slightly relaxed conditions, existence of a bounded trapping region in the state space is the important issue here.

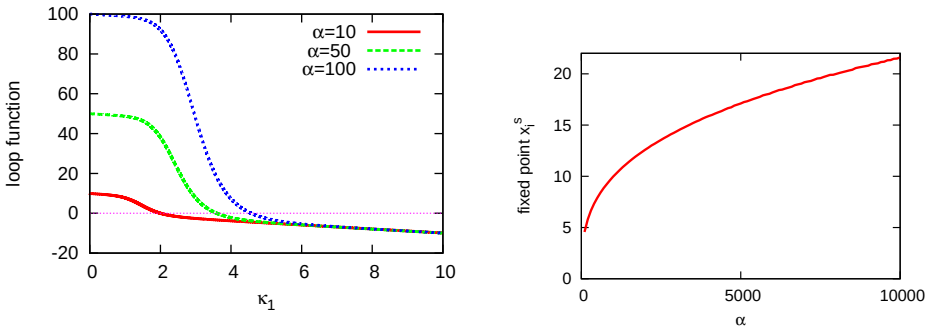

FIGURE 1. *Left:* Loop function of the repressilator model *Right:* Fixed point as a function of the parameter α.

with $i = lacI, tetR, cI$, $j = cI, lacI, tetR$, and m_i and p_i are mRNA and protein concentrations, respectively. This construct was used to create periodic expression in *Escherichia coli* [2]. The loop function is given by

$$lf(\kappa) = f(f(f(\kappa))) - \kappa,$$

which is shown in Figure 1 (*left*) for parameters $n = 2.1$, $\alpha_0 = 0$ and varying α, along with its zeros, which correspond to the fixed points of the system (*right*). In this example the loop function is strictly decreasing, since $f(\kappa)$ is strictly decreasing, which implies that the system cannot have more than one fixed point. This is conform with the well-known result that the vector field of systems with constant-sign Jacobian lacking positive circuits are injective. Moreover, the loop function is independent of the parameter β, which is shown in [2] to play a role for the stability of the fixed point. I conclude that, different from a one-dimensional vector field, a negative slope of the loop function at its zeros does not necessarily imply stability.

2.2. **Tryptophan regulation in** *Escherichia coli*. The LBA can be extended to systems which I denote *leading node structures* (LNS). These are characterized by the existence of a joint node in all circuits, such that circuits can be broken simultaneously by fixing this joint leading node. An example is the tryptophan regulation network described in [3]. It consists of four variables, free operator sites O_R, and mRNA, enzyme and tryptophan concentrations:

$$\begin{aligned}
\dot{O}_R &= k_1 O_t C_1(T) - (k_{d_1} O_R + \mu) O_R \\
\dot{\text{mRNA}} &= k_2 O_R C_2(T) - (k_{d_2} + \mu)\text{mRNA} \\
\dot{E} &= k_3 \text{mRNA} - \mu E \\
\dot{T} &= k_4 C_3(T) E - g \frac{T}{T + K_g} - \mu T
\end{aligned}$$

with sigmoidally decreasing functions $C_i(T)$. The corresponding graph is shown in Figure 2 (left). Tryptophan (T) is a leading node. The loop function (Figure 2 (right)) is again strictly decreasing, since all circuits in the graph are negative.

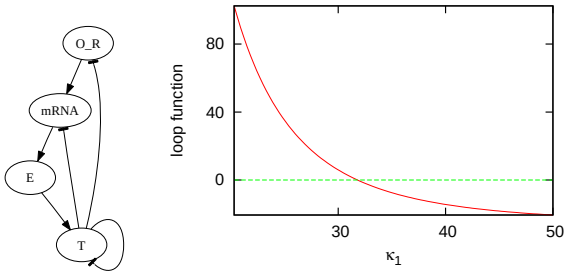

FIGURE 2. *Left:* The interaction graph of the tryptophan regulation network [3] is a leading node structure (LNS) and has an analytic loop function. *Right:* Loop function of the system

3. CONCLUSION

I am currently working on stability statements, and on extensions to arbitrary network structures, which is in the general case not analytically tractable any more. I am also grateful for interesting discussions during the workshop, which motivate me to have a closer look at I/O characteristics of controller systems. For instance, conditions for instability can directly be derived using criteria from I/O systems in the single loop case[2].

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Distributed Control using Decompositions and Games

ANDERS RANTZER

Many control applications have a decentralized structure, where each subunit has access to different information about the system state. Still, most control theory has been developed in a centralized setting, where all measurements are processed together to compute the control signals. This paradigm has conceptual advantages, but also inherent limitations in terms of complexity and integrity. The purpose of this lecture is to show how ideas from convex optimization and game theory may help to go beyond the traditional paradigm to support analysis and synthesis of distributed controllers.

[2]Personal communication with Prof. Eduardo Sontag, Rutgers University, Piscataway, NJ

In particular, we will reconsider well established methods for decomposition of large scale optimization problems by introduction of dual variables. These can be interpreted as prices in a market mechanism serving to achieve mutual agreement between different subproblems. The same idea can be used for decomposition of large scale control systems, with dynamics in both decision variables and prices. The dynamics bring interesting new phenomena. For example, expected future prices could be highly relevant for todays decisions.

Decomposition using price mechanisms is closely connected to game theory. A large-scale control system can be viewed as a game involving two types of players; subunits optimizing their local control action in response to prices, and intermediate players adjusting the prices to counteract disagreements between neighboring subunits. The game can be analysed under appropriate convexity assumptions and there is a close connection to the theory of potential games.

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When is a control system mechanical?

Witold Respondek
(joint work with Sandra Ricardo)

In this talk, we discuss the problem of when a control system of the form

$$\Sigma: \quad \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z),$$

where F and $G_1, \ldots, G_m$ are C^∞-vector fields on a $2n$-dimensional manifold M, is state equivalent to a mechanical control system.

To make that question precise, we will define the state equivalence and then we will explain what we mean by a mechanical control system. The systems Σ and

$$\tilde{\Sigma}: \quad \dot{\tilde{z}} = \tilde{F}(\tilde{z}) + \sum_{r=1}^{m} u_r \tilde{G}_r(\tilde{z}), \quad \tilde{z} \in \tilde{M}$$

are state equivalent, shortly *S-equivalent*, if there exists a diffeomorphism $\Phi: M \to \tilde{M}$ such that

$$\Phi_* F = \tilde{F} \qquad \text{and} \qquad \Phi_* G_r = \tilde{G}_r, \quad 1 \le r \le m.$$

We define a *mechanical control system* ($\mathcal{MS}$) as a 4-tuple $(Q, \nabla, \mathfrak{g}_0, d)$, in which

(i) Q is an n-dimensional *configuration manifold*;
(ii) ∇ is a symmetric affine connection on Q;
(iii) $\mathfrak{g}_0 = (g_0, g_1, \ldots, g_m)$ is an $(m+1)$-tuple of smooth vector fields on Q;
(iv) $d: TQ \to TQ$ is a fiber preserving map, linear on fibers.

A curve $\gamma : I \to Q,\ I \subset \mathbb{R}$, is a trajectory of $(\mathcal{MS})$ if it satisfies the equation

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = g_0(\gamma(t)) + d(\dot\gamma(t)) + \sum_{r=1}^{m} u_r g_r(\gamma(t))$$

or, equivalently, the second-order differential equation on the tangent bundle TQ

$$(\mathcal{MS}): \quad \begin{aligned} \dot{x}^i &= y^i \\ \dot{y}^i &= -\Gamma^i_{jk}(x)y^j y^k + d^i_j(x)y^j + g^i_0(x) + \sum_{r=1}^{m} u_r g^i_r(x), \end{aligned}$$

for $1 \le i \le n$, where $(x, y) = (x^1, \ldots, x^n,\ y^1, \ldots, y^n)$ are local coordinates on TQ (throughout, the summation convention is used). Above, $\Gamma^i_{jk}(x)$ are Christoffel symbols of the connection ∇, g_0 and $g_1, \ldots, g_m$ are positional forces, uncontrolled and controlled, respectively, and d correspond to dissipative- (or gyroscopic-) type forces.

We will be interested in the following problems:

(i) When is the control system Σ mechanical, that is, when it is state-equivalent to a mechanical system $(\mathcal{MS})$?
(ii) Is its mechanical structure (if exists) unique?
(iii) How to define invariants of a mechanical control structure?

Given a general control-affine system Σ, we consider a sequence of families of vector fields given as

$$\begin{aligned} \mathcal{H}_1 &= \{G_r \mid 1 \le r \le m\} \\ \mathcal{H}_2 &= \{[G_r, \mathrm{ad}_F G_s] \mid 1 \le r, s \le m\} \end{aligned}$$

and, inductively,

$$\mathcal{H}_i = \{[H_a, \mathrm{ad}_F H_b] \mid H_a \in \mathcal{H}_p,\ H_a \in \mathcal{H}_p,\ p + r = i\}.$$

Denote

$$\mathcal{H} := \bigcup_{i=1}^{\infty} \mathcal{H}_i.$$

Among all mechanical control systems a special class is distinguished: the class of *geodesically accessible mechanical control systems* $(\mathcal{GAMS})$, for which the smallest distribution containing the control vector fields $g_1, \ldots, g_m$ and closed with respect to the symmetric product is the whole TQ. Recall that the symmetric product of two vector fields g_i and g_j on Q, equipped with an affine connection ∇, is a new vector field on Q, defined as

$$\langle g_i : g_j \rangle = \nabla_{g_i} g_j + \nabla_{g_j} g_i.$$

If Σ admits a geodesically accessible mechanical structure $(\mathcal{GAMS})$, then there exists a one-to-one correspondence between the elements of $\mathcal{H}$ (defined on M) and input vector fields together with their iterative symmetric brackets (which span T_qQ for all $q \in Q$). This observation leads to to our first main result:

Theorem 1. *A system Σ, on a $2n$-dimensional manifold M, is locally, at $z_0 \in M$, S-equivalent to a $(\mathcal{GAMS})$ around a point of the form $(x_0, y_0) = (x_0, 0)$ if and only if*

(MS0) $F(z_0) \in \mathrm{Vect}_{\mathbb{R}}\mathcal{H}(z_0)$;
(MS1) $\dim \mathrm{Vect}_{\mathbb{R}}\mathcal{H}(z_0) = n$ *and* $\dim \mathrm{Vect}_{\mathbb{R}}\left(\mathcal{H} + [F, \mathcal{H}]\right)(z_0) = 2n$;
(MS2) $[\mathcal{H}, \mathcal{H}] = 0$.

If Σ is state equivalent to a mechanical system $(\mathcal{MS})$, then it always satisfies (MS2), independently of whether or not $(\mathcal{MS})$ is geodesically accessible (the last property being equivalent to (MS1)). The assumption of geodesic accessibility is a cornerstone for our approach. In fact, if a general control-affine system Σ admits a geodesically accessible mechanical structure then it is unique, up to an extended point transformation of the form $\Phi(x, y) = (\phi(x), \mathrm{D}\phi(x)y)$. The uniqueness fails if we consider mechanical structures that are not geodesically accessible. Indeed, we provide an example of a control systems that admits two nonequivalent mechanical structures (obviously, not geodesically accessible) thus proving that bi-mechanical control systems exist.

We show that conditions (MS1) and (MS2) of Theorem 1 encode enough information to get, in an invariant way (in terms of $\mathcal{H}$), all objects defining the canonical mechanical structure of the system Σ: the configuration manifold Q, the bundle structure of TQ on M, the connection ∇ on Q, the vector fields g_i and the linear map d. We describe the construction of that canonical mechanical structure on any control system Σ satisfying (MS0)-(MS2).

Then we deal with geodesically accessible affine connection systems, shortly $(\mathcal{GACS})$, that is, systems that are subject neither to dissipative-like (nor gyroscopic-like) forces nor uncontrolled forces. Consider Σ and $\tilde{\Sigma}$, control-affine systems which are locally state-equivalent to two $(\mathcal{GACS})$-systems. Given two frames $\mathcal{F}$ and $\tilde{\mathcal{F}}$ consisting of $2n = \dim M = \dim \tilde{M}$ independent vector fields, constructed for Σ with the help of $\mathcal{H}$ and for $\tilde{\Sigma}$ with the help of $\tilde{\mathcal{H}}$, we determine the structure functions $\alpha^s_{i_1 \ldots i_q}$, $\beta^s_{i_1 \ldots i_q}$ for system Σ and the structure functions $\tilde{\alpha}^s_{i_1 \ldots i_q}$, $\tilde{\beta}^s_{i_1 \ldots i_q}$ for system $\tilde{\Sigma}$.

Theorem 2. *Let Σ and $\tilde{\Sigma}$ be systems locally state equivalent, respectively at $z_0 \in M$ and $\tilde{z}_0 \in \tilde{M}$, to geodesically accessible $(\mathcal{GACS})$. Under a suitable constant rank condition satisfied by the structure functions, the following conditions are equivalent:*

(i) *Σ and $\tilde{\Sigma}$ are S-equivalent locally around z_0 and $\tilde{z}_0$;*
(ii) *Σ and $\tilde{\Sigma}$ are MS-equivalent locally around z_0 and $\tilde{z}_0$, that is, S-equivalent by an extended point transformation;*
(iii) *There exists a local diffeomorphism from a neighborhood of z_0 onto a neighborhood of $\tilde{z}_0$ such that*

$$\begin{aligned} (LAC) \quad \alpha^s_{i_1 \ldots i_q} &= \tilde{\alpha}^s_{i_1 \ldots i_q} \circ \varphi, \\ (SAC) \quad \beta^s_{i_1 \ldots i_q} &= \tilde{\beta}^s_{i_1 \ldots i_q} \circ \varphi. \end{aligned}$$

If Σ and $\tilde{\Sigma}$ are actually mechanical control systems $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$, respectively, then *(LAC)* and *(SAC)* of Theorem 2 take the following remarkable form.

The structure functions $\alpha^s_{i_1\dots i_q}$ and $\beta^s_{i_1\dots i_q}$ are equivalently defined by

$$\begin{aligned}\left[h_{i_q},\dots,[h_{i_3},[h_{i_2},h_{i_1}]],\dots,\right] &= (-1)^{q-1}\alpha^s_{i_1\dots i_q}\,h_s,\\ \langle h_{i_q}:\dots\langle h_{i_3}:\langle h_{i_2}:h_{i_1}\rangle\rangle\dots\rangle &= (-1)^{q}\beta^s_{i_1\dots i_q}\,h_s.\end{aligned}$$

where $h_1,\dots,h_n$ are independent elements of the family of symmetric vector fields. In this case, conditions (i), (ii) and (iii) of Theorem 2 are also equivalent to:

(iv) *There exists a local diffeomorphism ψ from a neighborhood of x_0 to a neighborhood of $\tilde{x}_0$ in Q and $\tilde{Q}$, respectively, such that*

$$\begin{aligned}\textit{(LAC)'}\quad \alpha^s_{i_1\dots i_q} &= \tilde{\alpha}^s_{i_1\dots i_q}\circ\psi,\\ \textit{(SAC)'}\quad \beta^s_{i_1\dots i_q} &= \tilde{\beta}^s_{i_1\dots i_q}\circ\psi.\end{aligned}$$

Now, *(LAC)'* and *(SAC)'* have a clear meaning that justifies their names: *(LAC)'* says that the Lie modules, generated by the symmetric vector fields of $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$, coincide (up to the conjugation by a diffeomorphism between the configuration manifolds Q and $\tilde{Q}$); *(SAC)'* says that the symmetric modules, generated by all symmetric vector fields of $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$, coincide (up to the conjugation by the same diffeomorphism).

Compositional Analysis and Control Techniques for Complex Dynamical Systems

Arjan van der Schaft
(joint work with Florian Kerber, Harsh Vinjamoor)

1. Introduction

Complexity and interconnection are keywords in many areas of science and engineering (micro- and nano-systems, embedded systems, systems biology, etc.). Furthermore, 'control' can be regarded as the addition of extra components and interaction couplings (feedback). In this talk we argue that mathematical systems theory should be the theory of general interconnected open dynamical systems. We draw inspiration from two conceptually related areas of expertise, namely *network modeling of physical systems*, and the theory of concurrent processes and formal verification from computer science. Major issues to be considered in a broader systems theory include the development of a notion of 'external equivalence' of open systems, the characterization of the achievable interconnected dynamics by adding control components, and the development of compositional analysis techniques.

2. Bisimulation and simulation

In [3] we have developed a notion of *bisimulation*, which extends the standard notion of bisimulation of labeled transition systems to the continuous domain. Restricting to linear continuous systems given in input-state-output form

$$\Sigma_i : \dot{x}_i = A_i x_i + B_i u_i + G_i d_i, \quad d_i \text{ auxiliary variables }, \quad y_i = C_i x_i,$$

we define a bisimulation relation between Σ_1 and Σ_2 to be a subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$, with the following property. Take any $(x_{10}, x_{20}) \in \mathcal{R}$ and any joint input $u_1(\cdot) = u_2(\cdot)$. Then for every $d_1(\cdot)$ there should exist a $d_2(\cdot)$ such that the resulting $x_1(\cdot)$, with $x_1(0) = x_{10}$, and $x_2(\cdot)$, with $x_2(0) = x_{20}$, satisfy

$$(x_1(t), x_2(t)) \in \mathcal{R}, \quad \text{for all } t \geq 0, \quad C_1 x_1(t) = C_2 x_2(t), \quad \text{for all } t \geq 0$$

and *conversely* (see also the work of Pappas [2]). Two systems Σ_1 and Σ_2 are called *bisimilar* ($\Sigma_1 \sim \Sigma_2$) if there exists a bisimulation relation $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the property $\pi_1(\mathcal{R}) = \mathcal{X}_1$, $\pi_2(\mathcal{R}) = \mathcal{X}_2$ with $\pi_i : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$, $i = 1, 2$, the canonical projections. By using tools from geometric control theory bisimulation relations can be easily characterized. The connection with classical equivalence theory of linear systems $\Sigma(A, B, C)$ is:

- There exists a bisimulation relation $\mathcal{R}$ between $\Sigma_1(A_1, B_1, C_1)$ and $\Sigma_2(A_2, B_2, C_2)$ if and only if their *transfer matrices* are the same.
- If Σ_1 and Σ_2 are *controllable* then they are bisimilar *if and only if* their transfer matrices are equal.
- The bisimulation relation is the graph of an *invertible mapping* if Σ_1 and Σ_2 are also *observable*.

The maximal bisimulation relation between two systems always exists, and can be computed using a variant of the *Maximal Controlled Invariant Subspace algorithm*. *Reduction* of a continuous system Σ is performed by *factoring out* the maximal bisimulation equivalence relation between Σ *and itself*. The one-sided version of bisimulation is called *simulation*, and there is also an algorithm for computing the maximal simulation relation.

Finally, by merging this theory of (bi-)simulation for continuous systems with the standard theory of (bi-)simulation of labeled transition systems we arrive at a structural notion of (bi-)simulation for *hybrid systems* [4].

3. Control by interconnection

Consider a system $\mathcal{P}$ with variables w and c which is interconnected to another system $\mathcal{C}$ though the $c-$variables. A natural question is: what are the 'achievable' interconnected systems, with respect to the w variables (up to bisimulation) ? Let the plant $\mathcal{P}$ be represented as

$$\begin{aligned} \dot{x}_P &= A_P x_p + B_P^u u_P + B_P^f f_P \\ \begin{bmatrix} z_P \\ y_P \end{bmatrix} &= \begin{bmatrix} C_P^z \\ C_P^y \end{bmatrix} x_P, \end{aligned}$$

while the desired system $\mathcal{S}$ is represented as

$$\dot{x}_S = A_S x_S + B_S f_S$$
$$z_S = C_S x_S$$

Consider now a controller system $\mathcal{C}$ given as

$$\dot{x}_C = A_C x_C + B_C u_C$$
$$y_C = C_C x_C$$

The problem statement is now: given $\mathcal{P}$ and $\mathcal{S}$, find necessary and sufficient conditions for the existence of a controller $\mathcal{C}$ such that $\mathcal{P} \parallel \mathcal{C}$ is bisimilar to $\mathcal{S}$, where $\mathcal{P} \parallel \mathcal{C}$ is given by an interconnection constraint

$$\begin{bmatrix} u_C \\ y_C \end{bmatrix} = \Pi \begin{bmatrix} u_P \\ y_P \end{bmatrix}$$

with Π a permutation matrix. (In a standard feedback interconnection the permulation matrix Π corresponds to $u_P = y_C, \quad y_P = u_C$.) The solution proceeds as follows. Based on $\mathcal{P}$ define the *zero-system* $\mathcal{N}$ as

$$\dot{x}_P = A_P x_P + B_P^f f_P$$
$$\begin{bmatrix} z_P \\ 0 \end{bmatrix} = \begin{bmatrix} C_P^z \\ C_P^y \end{bmatrix} x_p$$

Theorem 3.1: [5] $(\mathcal{N} \preccurlyeq \mathcal{S} \preccurlyeq \mathcal{P}) \quad \Leftrightarrow \quad (\exists \mathcal{C}$ such that $\mathcal{P} \parallel \mathcal{C} \sim \mathcal{S})$.

Note that the conditions $\mathcal{N} \preccurlyeq \mathcal{S}$ and $\mathcal{S} \preccurlyeq \mathcal{P}$ can be computationally checked using the algorithm for computing the maximal simulation relation.

For the *sufficiency part* of the proof we consider the *canonical controller* $\mathcal{C}_{can} := \mathcal{P} \parallel \mathcal{S}$ where the interconnection is with respect to the variables f and z. The result then follows by showing the simulations $\mathcal{S} \preccurlyeq \mathcal{C}_{can} \parallel \mathcal{P}$ and $\mathcal{C}_{can} \parallel \mathcal{P} \preccurlyeq \mathcal{S}$, based on the fact that for a linear continuous system $\Sigma_1 \preccurlyeq \Sigma_2, \Sigma_2 \preccurlyeq \Sigma_1 \Rightarrow \Sigma_1 \sim \Sigma_2$.

4. Compositional analysis

The aim of compositional analysis is to deduce properties of the interconnected system from properties of the component systems. Consider the *feedback interconnection* of linear input-output systems. $u_1 = -y_2 + e_1, u_2 = y_1 + e_2$, with e_1, e_2 external signals. It can be shown [1] that the problem of checking $\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$ is equivalent to checking

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \text{ and } \Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2}$$

For the case $e_1 = e_2 = 0$ this does not hold, but recourse can be taken to *Assume Guarantee Reasoning* rules. The *non-circular* Assume-Guarantee-Reasoning (AGR) rule

$$\begin{array}{lccc} (A) & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ (B) & \Sigma_{Q_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ \hline (C) & \Sigma_{P_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}$$

can be shown to be valid [1]. The *circular* AGR rule given as

$$\frac{\begin{array}{ll}(A) & \Sigma_{P_1}\|\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \\ (B) & \Sigma_{Q_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}\end{array}}{(C) \quad \Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}}$$

is valid for deterministic systems [1], and we conjecture it to be valid in general.

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Some Geometric Remarks on Flat Systems

Kurt Schlacher

(joint work with Markus Schöberl)

Since the introduction of flatness for lumped parameter systems into control about 15 years ago, see e.g. [3] and the citations therein, this approach has become quite popular in theory of lumped parameter systems. Also extensions to distributed parameter systems are known, particularly for trajectory planning. Apart from many progresses the "simple" problem, how to construct a flat output for general lumped parameter systems is still not totally solved, although necessary and sufficient conditions for their existence exists, see [6]. We introduce a new approach based on geometric properties, which are met by flat systems, such that a systematic construction of flat outputs is possible, see also [9].

We propose an elimination theory based approach to construct flat outputs for nonlinear lumped parameter systems with an arbitrary number of inputs. The main concept is the gradual reduction of the system to simpler ones such that one can decide for the final system, whether it admits a flat output. Since the simplest flat system is the empty system with m coordinates $(z^\alpha)_{\alpha=1}^m$ and an empty set of equations, this approach proves or disproves the equivalence of a system to the empty one. Here, we say two systems are equivalent, iff the solution of one system determines the solution of the other one unambiguously and vice versa. It is worth mentioning that this equivalence relation does involve neither the number of variables, equations, nor states or inputs.

Since elimination of variables leads to systems of implicit ordinary differential equations in general, this contribution deals with implicit dynamic systems, which define fibered submanifolds $\mathcal{S}$ of the first jet-manifold $J(\mathcal{Z})$ of the bundle $\mathcal{Z} \xrightarrow{\pi}$

$\mathcal{B}$ with base manifold $\mathcal{B}$, projection π and total manifold $\mathcal{Z}$, where (t) is the coordinate of $\mathcal{B}$ and $(t, z^\alpha)_{\alpha=1}^m$ are the coordinates of $\mathcal{Z}$, see e.g. [4]. We exclude also systems, which contain hidden constraints and confine ourselves to coordinate transformations, which leave (t) invariant. In this special case we have the relation $\mathcal{V}\left(J_0^1(\mathcal{E})\right) = J(\mathcal{E}) \times_{\mathcal{E}} \mathcal{V}(\mathcal{E})$, and therefore a natural map $\rho : \mathcal{V}\left(J_0^1(\mathcal{E})\right) \to J(\mathcal{E}) \times_{\mathcal{E}} \mathcal{V}(\mathcal{E})$. An important subclass of vector fields for the following are projectable vertical tangent vector fields on $\mathcal{S}$, they meet $\rho(v) \in \mathcal{V}(\mathcal{E})$, since they allow us to reduce the number of variables.

For the purpose of motivation, we apply the proposed algorithm to linear systems, and present afterwards an extension of this algorithm to nonlinear systems, see [9]. It turns out, that the existence of projectable vertical tangent vector fields is the key property for the system simplification. These fields exist in the linear case always, iff the system is reachable. This is true also for nonlinear systems, which are input to state linearizable by static feedback, see e.g. [5], [8] for the time invariant case. A test for the existence of these vector fields is presented, as well as a method for their construction. A geometric interpretation of this property can be given with the help of the manifold $\mathcal{S}_{(t,z)}$ over a point (t, z). If $\mathcal{S}_{(t,z)}$ contains a linear variety $\mathcal{V}_{(t,z)} \subset \mathcal{S}_{(t,z)}$, then one can eliminate variables and reduce the number of equations. In the case of flat linear or input to state linearizable systems $\mathcal{V}_{(t,z)}$ is always of maximal dimension and spanned by an involutive distribution. Here "always" means also after each elimination and reduction step. If the system is flat only, we have to distinguish mainly two cases. In the simple one $\mathcal{V}_{(t,z)}$ is still of maximal dimension, but involutivity may be absent. Still a system simplification step is possible. In the other one, one has to construct $\mathcal{V}_{(t,z)}$ explicitly by adding new relations, such that the extended set of equations describe $\mathcal{V}_{(t,z)}$ completely. In the language of flat systems this is called "dynamic extension". Again a system simplification step is possible. If $\mathcal{S}_{(t,z)}$ does not contain a linear $\mathcal{V}_{(t,z)}$, then the system is not flat. Summarizing, we can state, that the presented approach allows us to check equivalence of a given system to an empty one.

Finally, we apply the proposed algorithm to different examples like the one of [2], discuss the special case of systems with one input, see [1], or systems with m inputs and $m + 1$ states, see [7], where we reproduce the well known results not only, but achieve them as a straightforward specialization of the proposed general approach.

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The geometric mean of positive semidefinite matrices of fixed rank

Rodolphe Sepulchre
(joint work with Silvère Bonnabel)

This paper introduces a new distance and mean on the set of positive semi-definite matrices of fixed-rank. The proposed distance is derived from a well-chosen Riemannian quotient geometry that generalizes the reductive geometry of the positive cone and the associated natural metric. The resulting Riemannian space has strong geometrical properties: it is geodesically complete, and the metric is invariant with respect to all transformations that preserve angles (orthogonal transformations, scalings, and pseudo-inversion). The associated distance can be efficiently numerically computed via a simple algorithm based on SVD. The induced mean preserves the rank, possesses the most desirable characteristics of a geometric mean, and is easy to compute.

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Ergodicity of skew-product flows: an application of controllability theory on Lie groups

Héctor J. Sussmann
(joint work with Mahesh Nerurkar)

We study the ergodicity properties of skew-product flows obtained from a measure-preserving aperiodic base flow $\mathbf{T} = \{T_t\}_{t\in\mathbb{R}}$ on a compact metric space Ω endowed with a Borel probability measure m, by lifting the flow by means of a differential equation

$$g'(t) = A(T_t(\omega)) \cdot g(t)$$

on a compact connected Lie group G. Here A is a continuous map from Ω to the Lie algebra L of G. We want to consider maps A restricted to take values in some given closed convex subset S of L, and find conditions under which for a generic such A the resulting lifted flow is ergodic.

Precisely, we assume that

1. Ω is a compact metric space, endowed with a Borel probability measure m,
2. $\mathbf{T} = \{T_t\}_{t\in\mathbb{R}}$ is a jointly continuous, measure-preserving, aperiodic flow on Ω,
3. G is a compact connected Lie group,
4. L is the Lie algebra of G (realized as the tangent space to G at $e_{_G}$, the identity of G, and identified with the space of right-invariant vector fields on G),
5. S is a nonempty closed convex subset of L,
6. $\nu_{_G}$ is Haar measure on G (so $\nu_{_G}$ is bi-invariant), normalized so that $\nu_{_G}(G) = 1$.

We let $C^0(\Omega, S)$ be the space of all continuous maps $A : \Omega \mapsto S$. For each $A \in C^0(\Omega, S)$, the flow $\mathbf{T}$ gives rise to a *lifted flow* $\mathbf{T}^A = \{T_t^A\}_{t\in\mathbb{R}}$ on $G\times\Omega$, where the maps $T_t^A : G \times \Omega \mapsto G \times \Omega$ are defined as follows: for a given $(g, \omega) \in G \times \Omega$, we let $T_t^A(g, \omega) = (X^A(\omega, t)g, T_t(\omega))$, where $s \mapsto X^A(\omega, s)$ is the solution of the differential equation $\xi'(s) = A(T_s(\omega))(\xi(s))$ with initial condition $\xi(0) = e_{_G}$.

The product measure $\nu_{_G} \otimes m$ is a Borel probability measure on $G \times \Omega$, and each flow $\mathbf{T}^A$ is measure-preserving. We investigate under what conditions $\mathbf{T}^A$ is ergodic for generic $A \in C^0(\Omega, S)$. Precisely, let $C^0_{erg}(\Omega, S)$ be the set of all $A \in C^0(\Omega, S)$ such that the flow $\mathbf{T}^A$ is ergodic. We are interested in the properties

P1. $C^0_{erg}(\Omega, S)$ is nonempty,
P2. $C^0_{erg}(\Omega, S)$ is a residual subset of $C^0(\Omega, S)$ (that is, a subset that contains a countable intersection of open dense sets).

It turns out that P1 and P2 are equivalent. In order to state our necessary and sufficient condition for P1 and P2, we need to introduce a new flow, the "torus lift" of $\mathbf{T}$. First, we define Lie subalgebras $Lie(S, L)$, $Lie_0(S, L)$, of L by letting $Lie(S, L)$ be the smallest Lie subalgebra of L that contains S, and $Lie_0(S, L)$ be the smallest ideal of $Lie(S, L)$ that contains the differences $s_1 - s_2$, for all $s_1, s_2 \in S$. We observe that $Lie_0(S, L)$ is either equal to $Lie(S, L)$ or of codimension 1 in $Lie(S, L)$, so that, in particular, the quotient Lie algebra $Lie(S, L)/Lie_0(S, L)$ is Abelian. We then let $Gr(S, G)$, $Gr_0(S, G)$ be the connected Lie subgroups of G corresponding to $Lie(S, L)$ and $Lie_0(S, L)$, respectively. Then $Gr_0(S, G)$ is a normal subgroup of $Gr(S, G)$, and the quotient group $Gr(S, G)/Gr_0(S, G)$ is Abelian. It follows that the quotient $\tau^{S,G} = \overline{Gr(S, G)}/\overline{Gr_0(S, G)}$ of the closures of $Gr(S, G)$, $Gr_0(S, G)$ is also Abelian. Furthermore $\tau^{S,G}$ is compact and connected, so it is a torus.

On the torus $\tau^{S,G}$ there is a natural one-parameter subgroup $\{\gamma_t\}_{t\in\mathbb{R}}$ defined by letting $\gamma_t = [\exp(ts)]$, where "$[x]$" stands for "equivalence class of x modulo $\overline{Gr_0(S, G)}$," and s is any member of S. (It is easy to show that the class $[\exp(ts)]$ is the same for all $s \in S$.) The *torus lift* of $\mathbf{T}$ is then the flow $\mathbf{T}^{\tau^{S,G}} = \{T_t^{\tau^{S,G}}\}_{t\in\mathbb{R}}$

on $\tau^{S,G} \times \Omega$ defined by

$$T_t^{\tau^{S,G}}(u,\omega) = (\gamma_t u, T_t(\omega))\,.$$

We say that S has the *dense accessibility property* (DAP) if $\overline{Gr(S,G)} = G$, i.e., if the group $Gr(S,G)$ is dense in G. Since the closed set $\overline{Gr(S,G)} \times \Omega$ is $\mathbf{T}^A$-invariant for all $A \in C^0(\Omega, S)$, it is easy to see that it is necessary for P1 that S have the DAP.

Furthermore, if S has the DAP, and $\pi : G \times \Omega \mapsto \tau^{S,G} \times \Omega$ is the canonical projection map, defined by letting

$$\pi(g,\omega) = ([g],\omega)\,,$$

then it is necessary for $\mathbf{T}^A$ to be ergodic for some $A \in C^0(\Omega, S)$ that the image of the flow $\mathbf{T}^A$ under π be ergodic, that is, that the torus lift flow $\mathbf{T}^{\tau^{S,G}}$ be ergodic. In other words, a necessary condition for Property P1 is

P3. S has the dense accessibility property and the torus lift flow $\mathbf{T}^{\tau^{S,G}}$ is ergodic.

Our main theorem is then the following result, proved in [2]:

THEOREM NS. *Properties P1, P2 and P3 are equivalent.*

The proof of this result is based on the following result, proved in [2], which is a consequence of the main theorem of Lind [1].

THEOREM L. *Assume that Ω is a compact metric space, m is a Borel probability measure on Ω, and $T = \{T_t\}_{t\in\mathbb{R}}$ is a jointly continuous, m-preserving, aperiodic flow on Ω. Then for every positive $\Lambda \in \mathbb{R}$ and every $\varepsilon > 0$ there exists a compact subset $E \subseteq \Omega$ such that*

1. *The map $E \times [0,\Lambda] \ni (x,t) \mapsto H(x,t) = T_t x \in \Omega$ is a homeomorphism onto a compact subset $K(E,\Lambda)$ of Ω such that $m(K(E,\Lambda)) > 1-\varepsilon$.*
2. *There exists a unique Borel probability measure m^E on E having the property that the homeomorphism H is an isomorphism from the Borel probability space $(E \times [0,\Lambda], m^E \otimes bor_\Lambda)$ to the probability space obtained by endowing $K(E,\Lambda)$ with the probability measure arising by restricting m to $K(E,\Lambda)$ and normalizing it.*

(Here bor_Λ is Borel measure on the interval $[0,\Lambda]$, normalized in such a way that $bor_\Lambda([0,\Lambda]) = 1$.)

In the proof of Theorem NS, the key step is the following

LEMMA. *For any given $A \in C^0(\Omega, S)$, and any given $\delta > 0$, if f is a continuous real-valued function on $G \times \Omega$, $\varepsilon > 0$, and $L > 0$, then there exists an $L' > L$ and an $\tilde{A} \in C^0(\Omega, S)$ such that*

$$\|A - \tilde{A}\|_{sup} < \delta$$

(where the sup norm is computed using any norm on L), having the property that

$$\|TAv(f, T^{\tilde{A}}, L') - Av(f)\|_{L^2} < \varepsilon\,, \tag{1}$$

where $TAv(f, T^{\tilde{A}}, L')$ *is the time average of* f *along the flow* $T^{\tilde{A}}$ *over the interval* $[0, L']$, *defined by*

$$TAv(f, T^{\tilde{A}}, L')(g, \omega) = \frac{1}{L'} \int_0^{L'} f(T_t^{\tilde{A}}(g, \omega))dt\,,$$

and $Av(f)$ *is the space average of* f , *defined by*

$$Av(f) = \int_{G\times\Omega} f(g, \omega)\, d(\nu_{_G} \otimes m)(g, \omega)\,.$$

(If we let $S(f, L, \varepsilon)$ be the set of all $\tilde{A} \in C^0(\Omega, S)$ such that (1) holds for some $L' > L$, then it is easy to see that each $S(f, L, \varepsilon)$ is open in $C^0(\Omega, S)$, and $C^0_{erg}(\Omega, S)$ is the intersection of the sets $S(f, n, 2^{-n})$, taken over all positive integers n and all f in some countable dense subset of $C^0(G \times \Omega, \mathbb{R})$. The lemma implies that each $S(f, L, \varepsilon)$ is dense, so $C^0_{erg}(\Omega, S)$ is a residual subset of $C^0(\Omega, S)$.)

To prove the lemma, one uses Theorem L, with Λ chosen to be very large, in order to reduce the problem of finding a small perturbation $\tilde{A}$ with the desired properties to that of finding a small perturbation $\tilde{\eta}_\omega$ of each "open-loop control" η_ω, for $\omega \in E$, where $\eta_\omega(t) = A(T_t(\omega))$. (Naturally, these controls $\tilde{\eta}_\omega$ have to be "glued together" to obtain a continuous function $\tilde{A} : K(E, \Lambda) \mapsto S$, which can then be extended from $K(E, \Lambda)$ to all of Ω using the Tietze extension theorem). The perturbations $\tilde{\eta}_\omega$ must be such that the solution $[0, \Lambda] \ni t \mapsto \xi_\omega(t)$ of the Cauchy problem $\xi'(t) = \eta_\omega(t)(\xi(t))$, $\xi(0) = e_{_G}$, visits approximately every point of G a sufficient number of times. The existence of such a perturbation is guaranteed by results from controllability theory on Lie groups together with the ergodicity of the lifted flow.

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Uncertainty and robustness analysis of biochemical reaction networks

Steffen Waldherr

Biochemical reaction networks are commonly modelled by the differential equation

$$\dot{x} = Sv(x, p), \tag{1}$$

where $x \in \mathbb{R}^n$ is the concentration vector, $p \in \mathbb{R}^q$ the vector of reaction parameters, $v(x, p) \in \mathbb{R}^m$ the reaction rate vector, and $S \in \mathbb{Z}^{n\times m}$ the stoichiometric matrix. Such networks form the basis of most cellular processes like metabolism, signal transduction and gene expression. In this talk, we consider model uncertainties described by uncertain parameters p. Uncertainties in the models stem from insufficient biological knowledge about the considered processes, as well as

the experimental difficulties involved in measuring relevant variables *in vivo*. As a consequence, it is obligatory to study the effects of model uncertainty on the conclusions about the system that are drawn from the model.

We study three problems arising from parametric uncertainty in (1):

(1) The stationary uncertainty problem: Given a set $\mathcal{P} \subset \mathbb{R}^q$, compute the set of equilibrium points $\mathcal{X}^* = \{x \in \mathbb{R}^n \mid \exists p \in \mathcal{P} : Sv(x,p) = 0\}$.
(2) The stationary robustness problem: Given a set $\mathcal{X} \subset \mathbb{R}^n$, compute the set of allowable parameters $\mathcal{P}^* = \{p \in \mathbb{R}^q \mid Sv(x,p) = 0 \Rightarrow x \in \mathcal{X}\}$.
(3) The robust stability problem: Given nominal parameter values p_0, compute a robust stability radius ψ^* such that, for any $x \in \mathbb{R}^n$ and any $p \in \mathbb{R}^q$ satisfying $Sv(x,p) = 0$ and $\frac{p_{0,i}}{\psi^*} < p_i < p_{0,i}\psi^*$, $i = 1, \ldots, q$, the Jacobian $S\frac{\partial v}{\partial x}(x,p)$ is Hurwitz.

For typical biochemical networks, it is usually not possible to compute the sets $\mathcal{X}^*$ and $\mathcal{P}^*$ explicitly. We propose to compute bounding sets $\mathcal{X}_s$ and $\mathcal{P}_r$ such that $\mathcal{X}_s \supset \mathcal{X}^*$ and $\mathcal{P}_r \subset \mathcal{P}^*$, i.e. $\mathcal{X}_s$ contains all feasible equilibrium points and $\mathcal{P}_r$ contains only allowable parameter values.

Our solution to the stationary uncertainty and robustness analysis problems is based on an infeasibility certificate for the system of constraints

$$Sv(x,p) = 0, \quad x \in \hat{\mathcal{X}}, \quad p \in \hat{\mathcal{P}}, \tag{2}$$

where $\hat{\mathcal{X}} \subset \mathbb{R}^n$ and $\hat{\mathcal{P}} \subset \mathbb{R}^q$ are polytopic test regions that are chosen iteratively in a bisection algorithm to approximate $\mathcal{X}^*$ and $\mathcal{P}^*$, respectively. Infeasibility certificates for (2) may be obtained using a sum of squares representation of $v(x,p)$ and a convex relaxation to a semidefinite program [1, 2].

The algorithm for the robust stability problem presented in this talk uses a bisection on ψ to maximise a lower bound on the robustness radius ψ^*. For each test value $\hat{\psi}$, the stationary uncertainty analysis method is used to compute an outer bounding polytope for the uncertain Jacobian $S\frac{\partial v}{\partial x}(x,p)$ evaluated at an uncertain equilibrium point. Lyapunov techniques based on linear matrix inequalities such as the quadratic stability theorem [3] are then used to certify robust stability of all matrices within the polytope.

The proposed methods are applied to relevant models from biochemical signal transduction, e.g. the MAPK cascade and the NF-κB signalling pathway. We conclude that the methods are efficiently applicable to the analysis of such models and provide useful insights into the effects of large parametric uncertainties.

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Lyapunov functions for ISS networks

Fabian R. Wirth
(joint work with Sergey N. Dashkovskiy, Björn S. Rüffer)

1. Introduction

Small gain conditions are presented, which give sufficient conditions for the existence of an ISS Lyapunov function for a system obtained as the interconnection of many subsystems. These give Lyapunov versions of the results in [1]. The results are of interest in two ways. First, it is shown that a small gain condition is sufficient for input-to-state stability of the large-scale system in the Lyapunov formulation. Secondly, an explicit formula for an overall Lyapunov function is given. As the dimensions of the subsystems are essentially lower than the dimension of their interconnection, finding Lyapunov functions for them may be an easier task than for the whole system.

We consider a finite set of interconnected systems with state $x = \left(x_1^T, \dots, x_n^T\right)^T$, where $x_i \in \mathbb{R}^{N_i}$, $i = 1, \dots, n$ and $N := \sum N_i$. For $i = 1, \dots, n$ the dynamics of the i-th subsystem is given by

$$\Sigma_i : \ \dot{x}_i = f_i(x_1, \dots, x_n, u), \quad x \in \mathbb{R}^N, \ u \in \mathbb{R}^M, \ f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i} . \tag{1}$$

In order to state the stability definitions, we introduce three sets of comparison functions: $\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+, \gamma \text{ is continuous, strictly increasing, and } \gamma(0) = 0\}$ and $\mathcal{K}_\infty = \{\gamma \in \mathcal{K} : \gamma \text{ is unbounded}\}$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second argument.

A forward complete system $\dot{x} = f(x, u)$ with $x \in \mathbb{R}^N, u \in \mathbb{R}^M$ is called *input-to-state stable* [2] if there are $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that for all initial conditions $x_0 \in \mathbb{R}^N$ and all $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^M)$ we have

$$\|x(t; x_0, u(\cdot))\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty) . \tag{2}$$

It is known to be an equivalent requirement to ask for the existence of an ISS Lyapunov function, [3]. For our purposes, it will be convenient to require Lipschitz continuity away from the origin..

Definition: Consider a function $V : \mathbb{R}^N \to \mathbb{R}_+$, which is continuous, proper and positive definite and locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$, Then V is an ISS Lyapunov function for $\dot{x} = f(x, u)$, if there exist $\gamma \in \mathcal{K}$, and a positive definite function α such that in all points of differentiability of V we have

$$V(x) \geq \gamma(\|u\|) \quad \Longrightarrow \quad \nabla V(x) f(x, u) \leq -\alpha(\|x\|) . \tag{3}$$

2. Problem Statement

Consider the interconnected system (1) and assume that for each subsystem Σ_j there is a given function $V_j : \mathbb{R}^{N_j} \to \mathbb{R}_+$, which is continuous, proper and positive definite and locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$.

A continuous function $\mu : \mathbb{R}_+^n \to \mathbb{R}_+$ is called a monotone aggregation function (MAF) if (i) $\mu(s) \geq 0$ for all $s \in \mathbb{R}_+^n$ and $\mu(s) > 0$ if $s \geq 0$ and $s \neq 0$; (ii) if $x < y$, then $\mu(x) < \mu(y)$; (iii) if $\|x\| \to \infty$ then $\mu(x) \to \infty$.

For $i = 1, \ldots, n$ the function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is called an ISS Lyapunov function for Σ_i, if there exist $\mu_i \in \mathrm{M}AF_{n+1}, \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}, j \neq i, \gamma_{iu} \in \mathcal{K} \cup \{0\}$ and a positive definite function α_i such that

$$\begin{aligned} V_i(x_i) \geq \mu_i\left(\gamma_{i1}(V_1(x_1)), \ldots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(\|u\|)\right) \\ \implies \quad \nabla V_i(x_i) f_i(x,u) \leq -\alpha_i(\|x_i\|)\,. \end{aligned} \tag{4}$$

The functions γ_{ij} and γ_{iu} are called ISS Lyapunov gains.

In case f_i does not depend on x_j there is no influence of x_j on the state of Σ_i. In this case we define $\gamma_{ij} \equiv 0$. This allows us to collect the internal gains into a matrix

$$\Gamma := (\gamma_{ij})_{i,j=1,\ldots,n}\,. \tag{5}$$

If we add the external gains as the last column into this matrix then we denote it by $\overline{\Gamma}$. The function μ_i describes how the internal and external gains interactively enter in a common influence on x_i. The above definition motivates the introduction of the following nonlinear map

$$\overline{\Gamma}_\mu : \mathbb{R}_+^{n+1} \to \mathbb{R}_+^n, \quad \begin{bmatrix} s_1 \\ \vdots \\ s_n \\ r \end{bmatrix} \mapsto \begin{bmatrix} \mu_1(\gamma_{11}(s_1), \ldots, \gamma_{1n}(s_n), \gamma_{1u}(r)) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \ldots, \gamma_{nn}(s_n), \gamma_{nu}(r)) \end{bmatrix}. \tag{6}$$

Similarly we define $\Gamma_\mu(s) := \overline{\Gamma}_\mu(s, 0)$.

3. General Small Gain Results

A continuous path $\sigma \in \mathcal{K}_\infty^n$ will be called an Ω-*path with respect to* Γ_μ if

(1) for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;
(2) for every compact set $K \subset (0, \infty)$ there are constants $0 < c < C$ such that for all points of differentiability of σ_i^{-1} and $i = 1, \ldots, n$ we have

$$0 < c \leq (\sigma_i^{-1})'(r) \leq C\,, \quad \forall r \in K; \tag{7}$$

(3) $\Gamma_\mu(\sigma(r)) < \sigma(r)$, for all $r > 0$.

Now we can state the first main result, which regards the existence of Ω-paths.

Theorem Let $\Gamma \in \mathcal{K}_\infty^{n \times n}$ be a gain matrix and $\mu \in \mathrm{M}AF_n^n$. Assume that one of the following assumptions is satisfied

(1) Γ_μ is linear and the spectral radius of Γ_μ is less than one;
(2) Γ is irreducible and $\Gamma_\mu \not\geq \mathrm{i}d$;
(3) $\mu = \max$ and $\Gamma_\mu \not\geq \mathrm{i}d$;

Then there exists an Ω-path σ with respect to Γ_μ.

Theorem Consider an interconnected system Σ given by (1), with respective ISS Lyapunov function V_i and the corresponding gain operator $\overline{\Gamma}_\mu$. Assume there are an Ω-path σ with respect to Γ_μ and a function $\phi \in \mathcal{K}_\infty$ such that

$$\overline{\Gamma}_\mu(\sigma(r), \phi(r)) < \sigma(r)\,, \quad \forall\ r > 0 \tag{8}$$

is satisfied, then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i))\,. \tag{9}$$

In particular, for all points of differentiability of V we have the implication

$$V(x) \geq \max\{\phi^{-1}(\gamma_{iu}(\|u\|)) \mid i = 1, \dots n\} \implies \nabla V(x) f(x,u) \leq -\alpha(\|x\|)\,, \tag{10}$$

where α is a suitable positive definite function.

Corollary[Additive gain of external input **u**] Under the assumptions of the previous theorem assume additionally that the ISS-condition is additive in the gain of u, that is,

$$\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = \Gamma_\mu(V_1(x_1), \dots, V_n(x_n)) + \gamma_u(\|u\|)\,, \tag{11}$$

where $\gamma_u(\|u\|) = (\gamma_{1u}(\|u\|), \dots, \gamma_{nu}(\|u\|))^T$. If Γ_μ is irreducible and if there exists an $\alpha \in \mathcal{K}_\infty$ such that for $D = diag(\mathrm{id} + \alpha)$ the gain operator Γ_μ satisfies the strong small gain condition

$$D \circ \Gamma_\mu(s) \not\geq s$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (9), where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to $D \circ \Gamma_\mu$.

Corollary[Maximization w.r.t. external gain] Under the assumptions of the previous theorem assume additionally that u enters the ISS-condition via maximization, that is,

$$\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = \max\{\Gamma_\mu(V_1(x_1), \dots, V_n(x_n)), \gamma_u(\|u\|)\}\,, \tag{12}$$

where $\gamma_u(\|u\|) = (\gamma_{1u}(\|u\|), \dots, \gamma_{nu}(\|u\|))^T$. Then, if Γ_μ is irreducible and satisfies the small gain condition

$$\Gamma_\mu(s) \not\geq s$$

the interconnected system is ISS and an ISS Lyapunov function is given by (9), where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to Γ_μ.

The main difficulty in the proofs lies in the construction of the Ω paths, which relies on an application of the Kanster-Kuratowski-Mazurkiewicz (KKM) theorem.

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Convergency and Regulation

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(joint work with Nathan van de Wouw and Alexei Pavlov)

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1. Convergency

In many control problems such as the output regulation problem it is required that controllers be designed in such a way that all the solutions of the corresponding closed loop system 'forget' their initial conditions and converge to some steady state solution, which is only determined only by the input of the closed loop system. This input can be, for example, a command signal or a signal generated by a feed-forward part of the controller or as in the output regulation problem, it can be the signal generated by the exo-system. For asymptotically stable linear systems excited by inputs, this is a natural property. Indeed, due to the linearity of the system, every solution is globally asymptotically stable and therefore, all solutions of such a system 'forget' their initial conditions and converge to each other. After the transients have died out, the dynamics of the system are solely determined by the input. For nonlinear control systems in general global asymptotic stability of a system with zero input does not guarantee that all solutions of this system with a nonzero input 'forget' their initial conditions and converge to each other. This implies that for nonlinear systems the convergent dynamics property requires additional conditions. 'Forgetting' the initial conditions has been studied in a number of papers, most notably in the Russian literature V.A. Pliss, [1], and afterwards B.P. Demidovich, [2], have introduced the notion of convergent systems. In the context of a control system, the definition of a convergent system requires that there exists, for each (bounded) input function, a unique globally asymptotically stable solution that is bounded over the whole time axis. Other relevant contributions on similar notions of stability can be found in [3], [4], and in more recent times [5], [6], [7] and [8]. A concise review of the work of Demidovich, together with a sufficient condition for convergency is presented in [9].

2. Regulation

Regulation or rather output regulation serves as a central problem in control, and deals with the asymptotic tracking of prescribed reference signals and/or the asymptotic rejection of undesired disturbances in the output of a system. As such, many control problems can be cast in the frame of an output regulation problem. The linear output regulation problem was completely solved in the 1970s by B.A. Francis, W.M. Wonham, E.J. Davison and others, resulting in the famous 'internal model principle' in combination with the ubiquitous 'regulator equations',

see e.g. [10]. Several authors have addressed the nonlinear output regulation problem, with, amongst others, the seminal paper of C.I. Byrnes and A. Isidori, [11], describing the solvability in terms of the nonlinear regulator equations. Most of the work on the output regulator problem, and variations thereof, has concentrated on local or approximate versions of the problem, mostly due to the inherent difficulty to solve the regulator equations. Extensions of the problem to the global or semi-global case have only more recently been addressed, e.g. in [12]. By using the notion of convergency, a slightly different, but in some sense, more natural formulation of the output regulation problem is described. In particular, it is required that the solution of the regulation problem is such that the closed loop dynamics is convergent, so that a natural and unique steady state solution exists in all cases, and the initial conditions do not appear any longer in this solution. A detailed treatise on this approach to the output regulation problem is given in [13].

3. Frequency Response Functions

Convergent systems have the appealing property that periodic input functions give rise to periodic steady-state solutions, thus allowing for the particular discussion of what the steady state response is to harmonic input signals. In a linear context this information is usually captured in the frequency response plot or Bode plot describing both the amplitude and phase shift of the response as a function of the input amplitude and frequency. The (amplitude) frequency response function thus can be introduced for convergent control systems, and it forms a natural measure for which frequencies the system amplifies harmonic input signals. Although this is still far from a complete nonlinear frequency domain thinking, this approach serves as a first step towards a frequency domain based approach to controller design, in the sense that one may ask the closed loop dynamics to have specific amplitude shaping properties in certain frequency ranges. Obviously, this needs to be developed further since no analogy of 'phase' seems available in this manner. The subject of convergent dynamics based frequency response functions can be found in [13].

4. Conclusions

The notion of convergent dynamics provides a useful and valuable tool for studying nonlinear control systems. Control systems that belong to the class of convergent systems have a number of appealing properties, which resemble those of linear control systems. One could argue for this reason that convergent systems are a natural extension of linear systems.

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Open Problem: Almost input-to-state stability

David Angeli

The notion of Input-to-State Stability (see for instance [1]) is best formulated in terms of the so called comparison functions: a continuous function $\rho : [0, +\infty) \to [0, +\infty)$ is called positive definite if $\rho(0) = 0$ and $\rho(r) > 0$ for all $r > 0$. If in addition it is monotonically increasing, then it is a class $\mathcal{K}$ function. A class $\mathcal{KL}$ function $\beta : [0, +\infty)^2 \to [0, +\infty)$ is a continuous function for which $\beta(\cdot, r)$ is of class $\mathcal{K}$ for all fixed values of $r \geq 0$ and $\beta(s, \cdot)$ is decreasing to 0 for each fixed value of $s > 0$ as its argument tends to infinity. A nonlinear system

$$\dot{x} = f(x, d) \tag{1}$$

with x taking values in $\mathbb{R}^n$ is Input-to-State Stable (ISS) if, for all initial conditions $\xi \in \mathbb{R}^n$ and all input signals d, the corresponding solution $x(t, \xi, d)$ fulfills:

$$|x(t, \xi, d)| \leq \beta(|\xi|, t) + \gamma(\|d\|_\infty) \qquad \forall\, t \geq 0 \tag{2}$$

for some β of class $\mathbb{KL}$ and some γ of class $\mathcal{K}$. Notice that the definition implies $f(0, 0) = 0$, that is the origin is an equilibrium point for $d = 0$. In general $\|d\|_\infty$ denotes the (essential) infinity norm of the input signal d. A simple consequence of ISS is the so called *Asymptotic Gain* property, in particular:

$$\limsup_{t \to +\infty} |x(t, \xi, d)| \leq \gamma(\|d\|_\infty) \tag{3}$$

which holds for all $\xi \in \mathbb{R}^n$ and all Lebesgue measurable and essentially bounded inputs d. Such property is the most suitable candidate for a generalization to the study of systems on manifolds, [2]. Indeed, a system

$$\dot{x} = f(x, d)$$

with x taking value in some manifolds M and $f : M \times N \to T_xM$ is said to be almost Input-to-State Stable if

$$\begin{array}{rl} \exists\, \gamma \in \mathcal{K} : & \forall\, d(\cdot), \exists B_d \subset M : \mu(B_d) = 0 \\ : \forall\, \xi \notin B_d, & \limsup_{t\to+\infty} |x(t,\xi,d)| \leq \gamma(\|d\|_\infty). \end{array} \tag{4}$$

Notice that this is an almost global notion of asymptotic gain due to the zero-measure *bad set* B_d which is allowed, typically depending upon the selected input signal. Consider now the specific example:

$$\begin{array}{rcl} \dot{\theta} & = & \omega \\ \dot{\omega} & = & -\sin(\theta) - \omega + d \end{array} \tag{5}$$

representing a mathematical model of a pendulum with friction. In particular, the state vector $x = [\theta, \omega]'$ takes values on $M = \mathbb{S} \times \mathbb{R}$. We are now ready to state our open question.

Open problem: Is the system in (5) almost Input-to-State Stable?

Some additional remarks. It is well-known that for $d \equiv 0$, the equilibrium $[0, 0]$ is almost globally asymptotically stable, with a bad set B_d composed by a saddle-point $[\pi, 0]$ together with its 0-measure stable manifold. It is relatively easy to show that the following estimate holds globally:

$$\limsup_{t\to+\infty} |x(t,\xi,d)| \leq c + \gamma(\|d\|_\infty)$$

for some function $\gamma \in \mathcal{K}$ and some positive constant c. Hence, the true open issue only concerns disturbances of small amplitude. A similar system

$$\dot{\theta} = -\sin(\theta) + d$$

is known to possess the property; however, the tools available for proving almost ISS in this simpler case do not apply to the example in (5) due to the presence of a saddle point with negative divergence [3]. A more general question which probably needs to be answered along the solution of the above open problem could be whether or not hyperbolic saddle points have the potential for attracting sets of positive measure when forced by sufficiently small open loop disturbances.

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Open Problem: Existence of row-permutation matrices

Dietrich Flockerzi

Question 1:
Given a matrix $N \in \mathbb{R}^{(m+k)\times k}$ of full column rank k when does there exist a row-permutation matrix Π with

$$\Pi N = \begin{pmatrix} A \\ B \end{pmatrix}$$

having a regular $B \in \mathbb{R}^{k\times k}$ such that all row sums of the left-inverse $(I_{m\times m}, -AB^{-1})$ are positive? With the notation $e_n^{\mathrm{T}} = (1, ..., 1) \in \mathbb{R}^{1\times n}$:

$$e_m^{\mathrm{T}}(I_{m\times m}, -AB^{-1}) = (e_m^{\mathrm{T}}, \underbrace{-e_m^{\mathrm{T}}AB^{-1}}_{=:r^{\mathrm{T}}}) \quad \text{with} \quad r^{\mathrm{T}} > 0 \text{ ???} \tag{1}$$

Can one even choose $r^{\mathrm{T}} \geq e_k^{\mathrm{T}}$???

As a necessary condition for (1) one has for the positive r^{T} and $w^{\mathrm{T}} = (e_m^{\mathrm{T}}, r^{\mathrm{T}})$

$$w^{\mathrm{T}}\Pi N = e_m^{\mathrm{T}}A + r^{\mathrm{T}}B = e_m^{\mathrm{T}}[A - AB^{-1}B] = 0$$

so that $w^{\mathrm{T}}\Pi > 0$ (or $\geq e_n^{\mathrm{T}}$ resp.) is a left nullvector of N having at least m entries equal to 1.

Question 2:
Is the existence of a positive left nullvector q^{T} of N sufficient for a positive answer???

Answer: Yes.
For an algorithmic proof of the existence of such an $r^{\mathrm{T}} \geq e_k^{\mathrm{T}}$ see [1]. It is based on [2].

Question 3 – the open problem: Is there an 'old' reference for this result?

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Open Problem: Observability of quantized nonlinear systems

Lars Grüne

We consider a nonlinear discrete time dynamical system

$$x(t+1) = f(x(t))$$

with $x(t) \in \mathbb{R}^n$ and known map $f : \mathbb{R}^n \to \mathbb{R}^n$. Furthermore, we consider a finite quantization of a compact subset $X \subset \mathbb{R}^n$, i.e., a family of quantization regions $P_1, \ldots, P_N \subset \mathbb{R}^n$ satisfying

$$\bigcup_{i=0,\ldots,N} P_i = X \quad \text{and} \quad P_i \cap P_j = \emptyset \text{ for } i \neq j.$$

We assume that for an unknown trajectory $x(0), \ldots, x(T)$ contained in $X \subset \mathbb{R}^n$ we can measure the quantization regions $P(0), \ldots, P(T)$ satisfying $x(0) \in P(0), \ldots, x(T) \in P(T)$.

Problem: Under which conditions on f can we get nontrivial information about (at least some of) the $x(t)$?

Here "nontrivial" means more information than just $x(t) \in P(t)$.

An algorithmic way to obtain such information is the following:

(i) define $Q^+(0) := P(0)$, $Q^+(t) := f(Q^+(t-1)) \cap P(t)$, $t = 1, \ldots, T$
(ii) define $Q^-(T) := P(T)$, $Q^-(t-1) := f^{-1}(Q^-(t)) \cap P(t-1)$, $t = T, \ldots, 1$ (here $f^{-1}(Q) := \{x \in \mathbb{R}^n \mid f(x) \in Q\}$)
(iii) define $Q(t) := Q^+(t) \cap Q^-(t)$

It is easily seen that this construction implies $x(t) \in Q(t) \subseteq P(t)$. In fact, I conjecture that this is the "finest" information one can get from the $P(t)$. Using these sets $Q(t)$, the above problem can be reformulated as

Problem: Under which conditions on f are (at least some of) the $Q(t)$ small sets?

Clearly, some property of f is needed in order to retrieve information. For instance, if $f(x) = x$, one easily sees that $Q(t) = P(t)$ holds.

In order to build up some intuition, it is instructive to look at trajectories which lead to constant quantized measurements $P(t) \equiv P_i$, $t = 0, \ldots, T$, where P_i contains a fixed point of f as the unique invariant set. Then it is not too difficult to see that for T sufficiently large $Q(T)$ is a neighborhoodof the unstable manifold, $Q(0)$ is a neighborhood of the stable manifold and $Q([T/2])$ of the center manifold. In particular, if some of these manifolds are 0-dimensional, we can conclude that the respective sets form small neighborhoods of x^*, implying $x(T) \approx x^*$ if x^* is asymptotically stable and $x(0) \approx x^*$ if x^* is asymptotically stable in backward time.

This consideration suggests that some form of hyperbolicity of f might be a helpful property in order to give an answer to the problem.

Open Problem: Global Observability of Spaces of Negative Curvature

Arthur J. Krener

Consider an observed dynamical system.

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \\ x(0) &= x^0 \end{aligned} \tag{1}$$

where x are local coordinates on an n dimensional manifold and y is in $\mathbb{R}^p$. The system is observable over the interval $[0,T]$ if the mapping from initial state x^0 to output trajectory $y(0:T)$ is one to one. The notation $y(0:T)$ means the mapping $t \mapsto y(t)$ for $0 \le t < T$. It is locally observable over the interval $[0,T]$ if this mapping is locally one to one.

As a way of determining observability, consider the local linear approximating system around the the state trajectory $x^0(t)$ and output trajectory $y^0(t)$ starting at x^0,

$$\begin{aligned} \dot{\delta x} &= F(t)\delta x \\ \delta y &= H(t)\delta x \\ \delta x(0) &= \delta x^0 \end{aligned}$$

where

$$\begin{aligned} \delta x &\approx x - x^0(t) \\ \delta y &\approx y - y^0(t) \\ F(t) &= \frac{\partial f}{\partial x}(x^0(t)) \\ H(t) &= \frac{\partial h}{\partial x}(x^0(t)) \end{aligned}$$

The linear approximating system defines a linear mapping from small changes in the initial condition δx^0 to changes in the output $\delta y(0:T)$ that is tangent at x^0 to the corresponding nonlinear mapping from $x(0)$ to $y(0:T)$ defined by (1). Let $\Phi(t)$ be the fundamental matrix solution of the linear dynamics (2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\Phi(t) &= F(t)\Phi(t) \\ \Phi(0) &= I \end{aligned}$$

then the local observability gramian at x^0 is

$$P(x^0) = \int_0^T \Phi'(t)H'(t)H(t)\Phi(t)\mathrm{dt}$$

Suppose for some $T > 0$, the local observability gramian $P(x^0)$ is uniformly positive definite and uniformly bounded for all x^0. Then the system is locally observable over the interval $[0,T]$ and the local observability gramian defines a Riemannian metric on the state space. (Before computing the local observability gramian the output coordinates y should be properly scaled. If there is additive

noise present in the observation, the output coordinates y should be chosen so that the noise covariance is the identity.)

Then the obvious question is what does the curvature of the state space tell us about the observed system. We conjecture that if the curvature is uniformly negative and the manifold is geodesically complete then is it globally observable on $[0, T]$.

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An open problem about palindromic polynomials

Rodolphe Sepulchre
(joint work with Alexandre Mauroy)

The following is conjectured: Let $N \geq 2$ an even integer. The polynomial

$$S(z) = z^N + s_{N-1}z^{N-1} + \cdots + s_2 z^2 + s_1 z + 1$$

possesses N distinct roots on the unit circle if the coefficients $s_1 = s_{N-1}, s_2 = s_{N-2}, \ldots, s_{N/2}$ satisfy the ordering

$$1 > s_i > s_{i+1} > 0,\ i = 1, \ldots, \frac{N}{2}$$

A positive answer to the conjecture would provide a generalization of the stability result proven in [1]. The linear function $F(x) = S + \gamma x$ in that paper (Leaky integrate-and-fire model) could be replaced by the exponential $F(x) = e^{x^2}$, which is a good approximation of the quadratic integrate-and-fire model.

Note: At the time of publishing this report, it seems that the conjecture has just been solved by Julien Hendrickx and Alexander Megretski (MIT).

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