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Representations of Finite Groups

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Abstract. The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Markus Linckelmann (Aberdeen), Gunter Malle (Kaiserslautern) and Jeremy Rickard (Bristol). It covered a wide variety of aspects of the representation theory of finite groups and related objects like Hecke algebras. A particular focus was placed on the rapidly evolving area of fusion systems.

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Introduction by the Organisers

The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Markus Linckelmann (Aberdeen), Gunter Malle (Kaiserslautern) and Jeremy Rickard (Bristol). It was attended by 46 participants with broad geographic representation. It covered a wide variety of aspects of the representation theory of finite groups and related objects like Hecke algebras. A particular focus was placed on recent developments on fusion systems. This area, which sits somewhere between group theory, block theory and homotopy theory, has received a significant amount of attention since the breakthrough papers of Broto, Levi, Oliver in the last six or seven years.

In twelve longer lectures of 40 minutes each and seventeen shorter contributions of 30 minutes each, recent progress in representation theory was presented and interesting new research directions were proposed. Besides the lectures, there was plenty of time for informal discussion between the participants, either continuing ongoing research cooperation or starting new projects.

Several interesting new results were presented related to homological methods in representation theory.

Symonds sketched his recent proof, using equivariant cohomology, of a five-year old conjecture of Benson on the commutative algebra of the cohomology ring of a general finite group: that its Castelnuovo-Mumford regularity is always zero, a conjecture for which a lot of computational evidence had been accumulating recently.

Benson and Carlson both spoke on topics related to the Jordan type of modules for finite groups, a subject that originated with a finer study of the notion of rank varieties for modules, and which has recently found unexpected connections with other areas of mathematics. In particular, Benson gave a survey of conjectures and results on modules of constant Jordan type, including work of him and Pevtsova relating these to vector bundles on projective spaces.

Xu described his work on the cohomology of categories, including his surprisingly simple construction of a finite-dimensional algebra whose Hochschild cohomology is not finitely generated modulo nilpotent elements. This answers in the negative a question of Snashall, and has implications on extending the theory of cohomological varieties for representations of finite groups to representations of more general algebras.

Navarro and Tiep presented their proof of Brauer's longstanding height zero conjecture for 2-blocks of maximal defect. Eaton reported on the proof of the fact that every nilpotent block of a finite simple group must have abelian defect groups.

Geck described his construction of natural labels for modular principal series representations of finite groups of Lie-type which might point a way towards a proof of James' conjecture on decomposition numbers. Bonnafé showed that Lusztig's conjectures (P1)-(P15) on Hecke algebras with unequal parameters are compatible with the parametrizations of simple modules coming from Ariki's Theorem.

Amongst the talks on fusion systems, one of the highlights was the characterisation, by Ragnarsson and Stancu, of fusion systems in terms of a reciprocity property in double Burnside rings. This characterisation bypasses the ususal axiomatic description of fusion systems, and may well open new territory. One of the fundamental problems in block theory which can be formulated in terms of fusion systems without reference to blocks - the 2-cocycle gluing problem - was shown by Park to have more than one solution in certain cases (it remains an open question whether this problem has always at least one solution). The other fundamental open problem in this area is the question as to whether every fusion system has a centric linking system - and a conjecture of Oliver plays this back to finite p-groups. Mazza presented joint work with D. Green and L. Hethelyi proving special cases of Oliver's conjecture, implying in particular the existence of centric linking systems in those cases.

Bob Oliver reported on joint work with C. Broto and J. Møller, in which homotopy theoretic methods are used in order to obtain a sufficient criterion for two finite groups of Lie type to have equivalent fusion systems.

Workshop: Representations of Finite Groups

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Modules of constant Jordan type

David J. Benson

Let $E = \langle g_1, \dots, g_r\rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p-group and k be an algebraically closed field of characteristic p. Let $X_i = g_i - 1 \in kE$, so that $kE = k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p)$. If $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$, set

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r.$$

Following Carlson, Friedlander and Pevtsova, a finitely generated kE-module M is said to have *constant Jordan type* $[p]^{a_p} \dots [1]^{a_1}$ if this is the Jordan canonical form of X_α for all non-zero $\alpha \in \mathbb{A}^r(k)$. If we ignore the projective Jordan blocks of length p, we say that M has *stable constant Jordan type* $[p-1]^{a_{p-1}} \dots [1]^{a_1}$. One fundamental question is this: if $r \geq 2$, what stable constant Jordan types can occur? Last year in MSRI I showed that a single non-projective Jordan block can only occur if the length is 1 or $p-1$. Rickard conjectures that if $a_i = 0$ then $\sum_{j=i+1}^{p} a_j \equiv 0 \pmod p$, and Suslin conjectures that if $2 \leq i \leq p-1$ and $a_i \neq 0$ then either $a_{i+1} \neq 0$ or $a_{i-1} \neq 0$. Both of these conjectures seem to be out of reach at the moment. The simplest example which is not known is [3][1] in characteristic at least 5.

Given a module M of constant Jordan type, we produce algebraic vector bundles on projective space $\mathbb{P}^{r-1} = \operatorname{Proj} k[Y_1, \dots, Y_r]$ as follows. Let $\mathcal{O}$ be the structure sheaf on $\mathbb{P}^{r-1}$, and set $\tilde{M} = M \otimes_k \mathcal{O}$, a trivial bundle whose rank is equal to the dimension of M. We define $\theta\colon \tilde{M}(j) \to \tilde{M}(j+1)$ via $\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f$, and

$$\mathcal{F}_i(M) = \frac{\operatorname{Ker}\theta \cap \operatorname{Im}\theta^{i-1}}{\operatorname{Ker}\theta \cap \operatorname{Im}\theta^{i}}$$

as a subquotient of $\tilde{M}$. This is a vector bundle of rank a_i for $1 \leq i \leq p$ if and only if M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$. Intuitively, the vector bundle $\mathcal{F}_i(M)$ picks out the bottoms of the Jordan blocks of length i on M.

Theorem (Benson and Pevtsova, MSRI 2008). Given a rank s vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$, there exists a module M for $(\mathbb{Z}/p)^r$ of stable constant Jordan type $[1]^s$ such that:

(i) if $p = 2$ then $\mathcal{F}_1(M) \cong \mathcal{F}$

(ii) if p is odd then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$

where $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map.

There are obstructions to realising vector bundles in odd characteristic. For example, if M has stable constant Jordan type $[1]^s$ then

$$\text{for} \quad 1 \leq i \leq p-2 \quad \text{we have} \quad c_i(\mathcal{F}_1(M)) \equiv 0 \pmod p.$$

Blocks of characters at different primes

Christine Bessenrodt

Classically, blocks of a finite group are just sets of irreducible complex characters. In many decades of block theory, usually a prime p was fixed from the beginning and then the corresponding p-blocks were studied; also, viewing blocks as ideals in the block algebra in characteristic p does not invite a comparison at different primes. In the talk, some recent investigations looking at blocks as sets of characters and then comparing blocks at different primes were surveyed.

In 1997, G. Navarro and W. Willems took again the classical stand and conjectured in [9] that a p-block of irreducible characters of a group G can only be a q-block of irreducible characters of G when the blocks are of defect 0; they proved this for p-solvable groups. But it is not true in general as the example $G = 6 \cdot A_7$ at the primes $p = 5$, $q = 7$ shows; in joint work with G. Navarro, J. Olsson, P. H. Tiep [4] we showed, though, that the Navarro-Willems conjecture holds for principal blocks (this required the classification of finite simple groups).
Doing more general comparisons of blocks of characters at different primes just as sets has proved to be very fruitful. In a joint paper with G. Malle and J. Olsson [1], the idea of separability of characters by blocks at different primes was introduced; in [1], we have focussed on the investigation of quasi-simple groups and have shown that typically, the intersection of the principal blocks at various primes contains only the trivial character (and the cases where it does not hold are described quite explicitly). For the symmetric groups and their double covers, such separation problems and block comparisons are intricately connected with interesting combinatorial questions (see e.g. [2], [3], [10], [11]). Results on separation properties by Navarro, Turull and Wolf [8] and Turull and Wolf [12] show that also for solvable groups these notions are quite intricate.

In joint work with J. Zhang [5], we have investigated the separation of characters by principal blocks at different primes and block inclusions for general finite groups to deduce consequences for the structure of the corresponding groups. In particular, this has led to new criteria for the nilpotency and p-nilpotency of a group via separation properties for principal blocks.

In further recent work with J. Zhang [6], we have studied the covering of irreducible characters by principal blocks for finite groups. We have shown that the covering of all irreducible characters of a group by principal blocks is only possible when already one principal block suffices or the generalized Fitting subgroup has a very special structure (it is non-abelian simple or it is a special product of two non-abelian simple groups). The stronger restriction that any pair of irreducible characters should belong to some principal block even implies that the Fitting subgroup is on a finite list of non-abelian simple groups. These results involve a very detailed study of covering properties for simple groups and their products; for the simple groups of Lie type, it led to interesting questions on the intersection and the union of all principal blocks in the non-describing characteristics which could be answered by Hiss [7] and Tiep, respectively.

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Permutation resolutions for Specht modules

Robert Boltje

(joint work with Robert Hartmann)

We fix a natural number r and a commutative ring k. We denote by $\Lambda = \Lambda(r)$ the set of partitions of r and by $\Gamma = \Gamma(r)$ the set of compositions of r. We view Γ (and also the subset Λ) as a partially ordered set with respect to the dominance order $\trianglelefteq$.

For $\lambda \in \Gamma$ we denote the permutation kS_r-module M^λ as the free k-module over the set $\mathcal{T}(\lambda)$ of tableaux of shape λ whose entries along the rows are increasing. The group S_r acts on $\mathcal{T}(\lambda)$ from the left by applying a permutation to the entries of a tableau and then ordering the rows. The Specht module S^λ is defined as a submodule of the permutation module M^λ as usually: Let $\mathcal{T}^{\rm st}(\lambda) \subseteq \mathcal{T}(\lambda)$ denote the set of standard tableaux (with increasing rows and columns) and for $t \in \mathcal{T}^{\rm st}(\lambda)$ set $e_t := \sum_{\sigma\in C_t} \operatorname{sgn}(\sigma)\sigma t \in M^\lambda$, where C_t denotes the column stabilizer of λ. It is well-known that the elements e_t, $t \in \mathcal{T}^{\rm st}(\lambda)$, are k-linear independent and span over k a kS_r-submodule of M^λ, the Specht module S^λ. Note that $S^\lambda = 0$ if λ is not a partition.

It is also well-known that, for $\lambda, \mu \in \Gamma$, the k-module $\operatorname{Hom}_{kS_r}(M^\mu, M^\lambda)$ is free with basis θ_T, where T runs through the set $\mathcal{T}(\lambda,\mu)$ of generalized tableaux of shape λ and content μ which have inreasing row entries. We denote by $\mathcal{T}^\wedge(\lambda,\mu)$ the set of those $T \in \mathcal{T}(\lambda,\mu)$ whose entries in the i-th row are all greater or equal to i, for all $i \geq 1$. Next we define $\operatorname{Hom}^\wedge_{kS_r}(M^\mu, M^\lambda)$ as the k-span of the basis elements θ_T, $T \in \mathcal{T}^\wedge(\lambda, \mu)$. It is easy to verify that these sets of homomorphisms are stable under composition.

For $\lambda \in \Gamma$, we define a chain complex C_*^λ of kS_r-modules as follows. For $n \geq 0$ we define C_n^λ as the direct sum

$$\bigoplus_{\lambda=\lambda_0 \lhd \lambda_1 \lhd \cdots \lhd \lambda_n \in \Gamma} \mathrm{Hom}^\wedge_{kS_r}(M^{\lambda_0}, M^{\lambda_1}) \otimes \cdots \otimes \mathrm{Hom}^\wedge_{kS_r}(M^{\lambda_{n-1}}, M^{\lambda_n}) \otimes \mathrm{Hom}_k(M^{\lambda_n}, k).$$

Note that the above summand is isomorphic to a direct sum of copies of M^{λ_n}. For $i = 0, \ldots, n-1$, we define $d_{n,i} \colon C_n^\lambda \to C_{n-1}^\lambda$ on the above summand by composing the i-th and $i+1$-th homomorphism and viewing the result in the component with λ_{i+1} omitted from the chain. Next we set $d_n := \sum_{i=0}^n d_{n,i} \colon C_n^\lambda \to C_{n-1}^\lambda$ and obtain the chain complex C_*^λ. We extend C_*^λ by the map $\mathrm{Hom}_k(M^\lambda, k) \to \mathrm{Hom}_k(S^\lambda, k)$ which restricts a homomorphism from M^λ to the submodule S^λ, to obtain a chain complex $\widetilde{C}_*^\lambda$ and we have the following conjecture:

Conjecture The chain chain complex $\widetilde{C}_*^\lambda$ is exact for every partition λ of r.

We have the following partial results on $\widetilde{C}_*^\lambda$.

Theorem *If $\lambda \in \Lambda$ has at most two parts or is of the form $(\lambda_1, \lambda_2, 1)$ then $\widetilde{C}_*^\lambda$ is exact.*

Remark (a) Computations with GAP yielded that $\widetilde{C}_*^\lambda$ is exact for all partitions λ when $r \leq 5$. They also show that the Lefschetz character of $\widetilde{C}_*^\lambda$ vanishes for every $\lambda \in \Lambda$ when $r \leq 9$.

(b) The chain complex $\widetilde{C}_*^\lambda$ is not exact in general if λ is not a partition.

(c) One can show that $\widetilde{C}_*^\lambda$ is exact at $n = 0$ for all $\lambda \in \Gamma$.

(d) The maps in C_*^λ are *directed* in the sense that they are sums of maps $\mathrm{Hom}_k(M^\mu, k) \to \mathrm{Hom}_k(M^\lambda, k)$ with $\lambda \unlhd \mu$ in Γ.

(e) The construction of $\widetilde{C}_*^\lambda$ is independent of k in the sense that it is purely combinatorial. The version over the ring k results from the version over $\mathbb{Z}$ by extending scalars. Since all modules are free over the base ring, it suffices to show the exactness for $k = \mathbb{Z}$.

(f) In ongoing work on his Ph.D. thesis, Filix Maisch from Santa Cruz has extended many of the above results to the Iwahori Hecke algebra.

Kazhdan-Lusztig theory and Ariki's Theorem

Cédric Bonnafé
(joint work with Nicolas Jacon)

The modular representation theory of Hecke algebras of type B was first studied by Dipper-James-Murphy [5]: one of their essential tools was to construct a family of modules (called *Specht modules*) playing the same role as Specht modules in type A. Each of these new Specht modules have a canonical quotient which is zero or simple: one of the main problem raised by this construction is to determine which ones are non-zero. Later, Graham and Lehrer [8] developed the theory of *cellular algebras*, which contains, as a particular case, the construction of Dipper-James-Murphy. The problems of parametrizing the simple modules and computing the decomposition matrix of Specht modules were then solved by Ariki [1] using the canonical basis of Fock spaces of higher level. In fact, Ariki's Theorem provides different parametrizations of the simple modules of the Hecke algebra: only one of them (*asymptotic case*) has an interpretation in the framework of Dipper-James-Murphy and Graham-Lehrer. Recently, Geck showed that the Kazhdan-Lusztig theory with unequal parameters should provide a cell datum for each choice of a *weight function* on the Weyl group (if Lusztig's conjectures (P1)-(P15) hold [9, Conjecture 14.2]). Our main aim in this talk is to explain how Kazhdan-Lusztig theory *with unequal parameters* should lead to a unified approach for a better understanding of the representation theory of Hecke algebras: however, our result relies on Lusztig's conjectures. As a by-product, we should get an interpretation of *all* Ariki's parametrizations of simple modules and to an interpretation of the *v-decomposition numbers* using a classical idea (Jantzen's filtration).

More precisely, if Q and q are two indeterminates, if $\mathcal{H}_n$ denotes the Hecke A-algebra with parameters Q and q (here, $A = \mathbb{Z}[Q, Q^{-1}, q, q^{-1}]$) with standard basis $(T_w)_{w \in W_n}$, if ξ is a positive irrational number (!) and if r denotes the unique natural number such that $r \leq \xi < r+1$, then Kazhdan-Lusztig theory *should* provide a cell datum $\mathcal{C}^\xi = ((\mathrm{Bip}(n), \trianglelefteq_r), \mathcal{SBT}, C^\xi, *)$ where

- $\mathrm{Bip}(n)$ is the set of bipartitions of n and $\trianglelefteq_r$ is a partial order on $\mathrm{Bip}(n)$ depending on r;
- If $l \in \mathrm{Bip}(n)$, $\mathcal{SBT}(\lambda)$ denotes the set of standard bitableaux of (bi-)shape λ (filled with 1,..., n);
- If S, $T \in \mathcal{SBT}(\lambda)$ for some bipartition λ, $C^\xi_{S,T}$ is an element of $\mathcal{H}_n$ coming from a Kazhdan-Lusztig basis of $\mathcal{H}_n$ (it heavily depends on ξ);
- $* : \mathcal{H}_n \to \mathcal{H}_n$ is the A-linear anti-involution of $\mathcal{H}_n$ sending T_w to $T_{w^{-1}}$;

(see [3, Conjecture C]). If this conjecture holds then, by the general theory of cellular algebras, we can associate to each bipartition λ of n a *Specht module* S^ξ_λ endowed with a bilinear form ϕ^ξ_λ. If K is the field of fractions of A, then $KS^\xi_\lambda = K \otimes_A S^\xi_\lambda$ is the simple $K\mathcal{H}_n$-module associated to λ. Now, if Q_0 and q_0 are two elements of $\mathbb{C}^\times$ then, through the specialization $Q \mapsto Q_0$, $q \mapsto q_0$, we can construct the $\mathbb{C}\mathcal{H}_n$-module

$$D^\xi_\lambda = \mathbb{C}S^\xi_\lambda / \mathrm{Rad}(\mathbb{C}\phi^\xi_\lambda).$$

By the general theory of cellular algebras, it is known that the non-zero D^ξ_λ's give a set of representatives of simple $\mathbb{C}\mathcal{H}_n$-modules.

On the other hand, if we assume further that q_0^2 is a primitive e-th root of unity, if $Q_0^2 = -q_0^{2d}$ for some $d \in \mathbb{Z}$ (which is only well-defined modulo e), and if $s = (s_0, s_1) \in \mathbb{Z}^2$ is such that $s_0 - s_1 \equiv d \mod e$, then Ariki's Theorem provides a bijection between the set of Uglov's bipartitions $\mathrm{Bip}^s_e(n)$ and the set of simple $\mathbb{C}\mathcal{H}_n$-modules. Moreover, the decomposition matrix is given by

$\big(d^s_{\lambda\mu}(1)\big)_{\lambda\in\mathrm{Bip}(n),\mu\in\mathrm{Bip}^s_e(n)}$, where $\big(d^s_{\lambda\mu}(q)\big)_{\lambda,\mu\in\mathrm{Bip}(n)}$ is the transition matrix between the standard basis and Uglov-Kashiwara-Lusztig's canonical basis of the Fock space. Our main result is that Lusztig's conjectures (P1)-(P15) in [9, Conjecture 14.2] are "compatible" with Ariki's Theorem in the following sense:

Theorem. *Assume that* [9, Conjecture 14.2] *holds and assume that* $s_0 - s_1 \equiv d \mod e$ *and* $s_0 - s_1 \leq r < s_0 - s_1 + e$. *Then* $D^\xi_\lambda \neq 0$ *if and only* $\lambda \in \mathrm{Bip}^s_e(n)$ *and, if* $\lambda \in \mathrm{Bip}(n)$ *and* $\mu \in \mathrm{Bip}^s_e(n)$, *then* $[\mathbb{C}S^\xi_\lambda : D^\xi_\mu] = d^s_{\lambda\mu}(1)$.

In particular, if $[\mathbb{C}S^\xi_\lambda : D^\xi_\mu] \neq 0$, *then* $\lambda \trianglelefteq_r \mu$.

This theorem is stated in a slightly different form in [4, Introduction] but we have obtained some improvements (namely, that we only need Lusztig's conjectures) since its publication, thanks to [2] and the recent work of Pietraho [10].

Note that, in the *asymptotic case* (in other words, if $\xi > n-1$), then [9, Conjecture 14.2] holds (see [6]) and the cellular datum $\mathcal{C}^\xi$ is more or less equivalent to the one constructed by Dipper, James and Mathas (see the work of Geck, Iancu and Pallikaros [7]).

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Cohomology and complexity of cohomological Mackey functors

Serge Bouc

Let k be a field (of positive characteristic p). When G is a finite group, let $\mathsf{M}^c_k(G)$ denote the category of *cohomological Mackey functors* for G over k.

A finite group G is called a *poco* group over k if every finitely generated cohomological Mackey functor for G over k has a projective resolution with polynomial growth.

The main theorem of this talk is the following :

Theorem 1. *Let k be a field of characteristic $p > 0$, and G be a finite group. The following conditions are equivalent :*

(1) *The group G is a poco group over k.*
(2) *Let S be a Sylow p-subgroup of G. Then :*
  - *If $p > 2$, the group S is cyclic.*
  - *If $p = 2$, the group S has sectional rank at most 2, i.e. it is cyclic or metacyclic.*

The proof of this theorem uses the construction of functors between categories of cohomological Mackey functors for different groups, associated to finite bisets, to reduce the problem to the case where G is an elementary abelian p-group. In this case, the algebra of self extensions of the simple functor S_1 (defined by $S_1(1) = k$ and $S_1(H) = \{0\}$ for $1 < H \leq G$) can be described completely :

Theorem 2. *Let $G = (C_2)^m$, and k be a field of characteristic 2. The algebra $\mathcal{E} = Ext^*_{\mathsf{M}^c_k(G)}(S_1, S_1)$ has the following presentation :*

(1) *The generators γ_x are indexed by the elements of $G - \{0\}$. They have degree 2.*
(2) *The relations are as follows :*
  - *If $H < G$ with $|G : H| = 2$, then $\sum_{x \notin H} \gamma_x = 0$.*
  - *If x and y are distinct elements of $G - \{0\}$, then $[\gamma_x + \gamma_y, \gamma_{x+y}] = 0$.*

The Poincaré series $P(t) = \sum_{n \in \mathbb{N}} \dim_k Ext^n_{\mathsf{M}^c_k(G)}(S_1, S_1)\, t^n$ of $\mathcal{E}$ is equal to

$$P(t) = \frac{1}{(1-t^2)(1-3t^2)(1-7t^2)\cdots(1-(2^{m-1}-1)t^2)} \;.$$

When $p > 2$, this becomes

Theorem 3. *Let $G = (C_p)^m$, and k be a field of characteristic $p > 2$.*

- *The algebra $\mathcal{E} = Ext^*_{\mathsf{M}^c_k(G)}(S_1, S_1)$ is generated by elements γ_X of degree 2, for $X \leq G$ and $|X| = p$, and by elements τ_i of degree 1, for $1 \leq i \leq m$.*
- *The Poincaré series of $\mathcal{E}$ is equal to*

$$\frac{1}{(1-t)\big(1-t-(p-1)t^2\big)\big(1-t-(p^2-1)t^2\big)\cdots\big(1-t-(p^{m-1}-1)t^2\big)} \;.$$

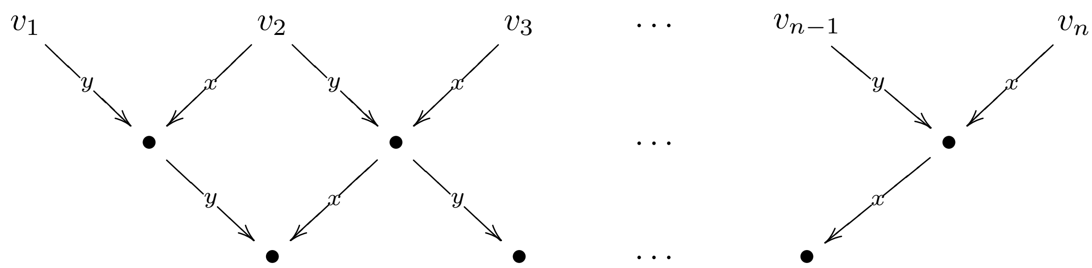
Theorem 3 was originally a conjecture, that I could only prove for $p = 3$. In a very recent joint work with Radu Stancu, we could prove this conjecture completely. We also obtained a presentation of $\mathcal{E}$ in terms of the generators γ_X and τ_i.

Generic kernels and other constructions

Jon F. Carlson
(joint work with Eric Friedlander, Julia Pevtsova and Andrei Suslin)

In this talk I will discuss the development of some very basic properties of modules and constructions of submodules. The aim is to give some structure to the category of modules over p-groups, even though many of the constructions can be applied to all finite groups. Throughout the lecture k is a field of characteristic $p > 0$ and G is a finite group. For the most part G is an elementary abelian p-group of rank r. In this case we note that the group algebra is a truncated polynomial ring and use the notation $kG = k[x_1, \ldots, x_r]/(x_1^p, \ldots, x_r^p)$. In the case that $r = 2$ or $r = 3$, we denote the group algebra generators x, y or x, y, z as appropriate.

Let's begin with an example. Let G be elementary abelian with $r = 2$ and assume that $p \geq 3$. We denote the following module $W_{n,3}$:



The diagram indicates that as a kG-module, $W_{n,3}$ is generated by $\{v_1, \ldots, v_n\}$ with relations generated by

$$xv_1 = 0 = yv_n = x^3 v_n : \quad x^3 v_i = 0 = yv_i - xv_{i+1}, \quad \text{for} \quad 1 \leq i < n-1.$$

This module has a very interesting property that illustrates why the category of kG-modules has wild representation type. Suppose that U and V are subspaces of the socle of $M = W_{n,3}$ (which is spanned by $x^2v_3, \ldots, x^2v_n$), then the kG-modules M/U and M/V are isomorphic if and only if U and V are exactly the same subspace. Hence there are a great many modules of this type.

The modules $W_{n,3}$, are a part of a class of modules that we call W modules. They satisify a condition which we call the equal images property [3] and they also have constant Jordan type [1]. The definitions of these term are as follows. Remember that the isomorphism type of a module over $K[t]/(t^p)$ is determined entirely by the Jordan type of the matrix of the element t on the module, that is, by the sizes of the Jordan blocks of the matrix of t on M.

Definition 1. *A kG-module M has the equal images property if for any extension K of k and any flat map $\alpha_K : K[t]/(t^p) \to KG$, we have that $\alpha_K(t)(K \otimes M) = \mathrm{Rad}(K \otimes M) = K \otimes \mathrm{Rad}(M)$.*

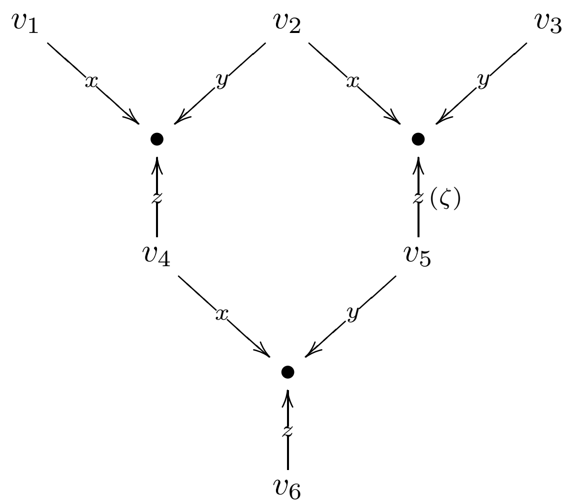
A kG-module M has constant Jordan type if for all extensions K of k and all flat maps $\alpha_K : K[t]/(t^p) \to KG$, we have that the Jordan type of the matrix of $\alpha_K(t)$ on $K \otimes M$ is constant.

The flat maps α_K are called π-points in [4]. There is an equivalence relation on the collection of π-points based on how they measure projectivity of kG-modules. The set of equivalence classes has the structure of a scheme and is homeomorphic to the spectrum of the cohomology ring $\mathrm{H}^*(G, k)$.

It is a fact that any module with the equal images property has constant Jordan type. Moreover, any quotient of a module having the equal images property has the equal images property. The above examples suggest that the category of modules with constant Jordan type might have wild representation type. This question is still open.

In the case that $r = 2$, the W modules play a particularly interesting role. First we observe that we can define other families of W modules with greater depth. That is, for any $2 \leq d \leq p$, define $W_{n,d}$ by exactly the same relations as above, except that we replace $x^3 v_i$ by $x^d v_i$ for all i. Then we can prove that every module having the equal images property is a homomorphic image of one of the modules $W_{n,d}$, for some n and some $d \leq p$.

In the case that $r > 2$, there seems to be no collection of modules playing this role. The point is that small changes in the modules can change the isomorphism type. This can be illustrated by the following family of modules. Let M_ζ be defined by the following diagram.

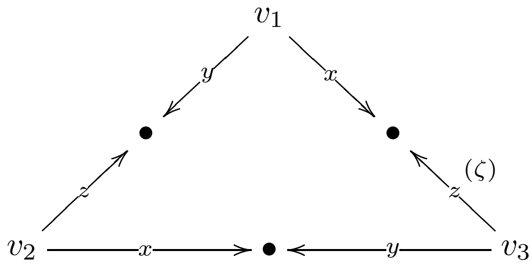


Here ζ is any element in k and the arrow marked with ζ is meant to indicate that the relation should be read as

$$xv_2 = \zeta z v_5$$

These modules all have the equal image property. If $\gamma \neq \zeta$ then the modules M_γ and M_ζ are not isomorphic. Moreover, every homomorphism of M_γ to M_ζ has its image in the radical of M_ζ.

We can introduce a variation on the equal images property with the following submodule of M_ζ. Let N_ζ be defined by the diagram:



Note that N_ζ has constant Jordan type only in the case that $\zeta = -1$. Otherwise, the nonmaximal support variety [5] of N_ζ is the union of the coordinate planes. These modules have the equal 2-images property, which we define as follows.

Definition 2. *A kG-module M has the equal 2-images property if for any extension K of k and any flat map $\alpha_K : K[t_1, t_2]/(t_1^p, t_2^p) \to KG$, we have that* $\text{Rad}(R_\alpha)(K \otimes M) = \text{Rad}(K \otimes M) = K \otimes \text{Rad}(M)$ *where R_α is the subalgebra which is the image of α_K.*

In a similar fashion we can define an equal s-images property for any $s < r$. Every module has submodules satisfying the equal s-images property. In the case that $s = 1$, and G is an elementary abelian group of rank $r = 2$, then a distinguished submodule with this property is the generic kernel which is defined as follows.

Definition 3. *Assume that the field k is infinite. Let M be a kG-module. For any $S \subset \mathbb{P}^1(k)$ let ${}_SM = \sum_{[a,b]\in S} \text{Ker}\{ax + by : M \to M\}$. The generic kernel of M is the intersection*

$$\mathcal{K}(M) = \bigcap_{S \subseteq \mathbb{P}^1(k),\ \textit{cofinite}} {}_SM.$$

When G is elementary abelian of rank 2, the generic kernel of a module M is the maximal submodule of M having the equal images property. Generic kernels can be defined for elementary abelian p-groups of rank greater than 2. The difference is that the intersection in the definition should be taken over open sets in the scheme of π-points. For general finite groups we can define a generic kernel for each component of the spectrum of the ring $\text{H}^*(G, k)$. We can also define a generic s-kernel for any $1 \leq s < r$ using multiple π-points as, for example, in Definition 2. For an elementary abelian p-group of rank r, the generic s-kernel of a module always has the equal $(r - s)$-images property, though it might not be the maximal submodule with the property.

There are also dual notions of the equal kernels property and the generic image of a module. The category of modules of constant Jordan type together with the collection of locally split sequences in the category form an exact category in the sense of Quillen [2]. Locally split means split on restriction along ever π-point.

We end with one example of the sort of theorem we get from the constructions. Assume that G is an elementary abelian group of rank 2.

Theorem 4. [3] *Let M be a module of constant rank and let $\mathcal{K}(M) \subset M$ be its generic kernel. Then we have an increasing filtration of M,*

$$x^{p-1}\mathcal{K}(M) \subseteq x^{p-2}\mathcal{K}(M) \subseteq \cdots \subseteq x\mathcal{K}(M) \subseteq \mathcal{K}(M) \subseteq x^{-1}\mathcal{K}(M) \\ \subseteq \cdots \subseteq x^{1-p}\mathcal{K}(M) \subseteq M$$

with the property that $x^i\mathcal{K}(M)$, for $i \geq 0$ has the equal images property and that $M/x^j\mathcal{K}(M)$ for $j \leq 1$ has the equal kernels property. Moreover, for any $\ell = ax+by$ with $[a,b] \in \mathbb{P}^1(k)$ and for all j, we have that $x^j\mathcal{K}(M) = \ell^j\mathcal{K}(M)$. Here, $x^j\mathcal{K}(M)$ denotes $(x^j)^{-1}\mathcal{K}(M)$ for $j < 0$.

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A survey on algebraic modules

David Craven

In this talk, I survey the theory of algebraic modules. Let M be a finite-dimensional KG-module, where K is a field of characteristic p and G is a finite group. The module M is called *algebraic* if it satisfies a polynomial $f(x)$ with integer coefficients. In 1979, Feit proved that all simple modules for all p-soluble groups are algebraic, and Alperin in the same year proved that the simple modules for $\mathrm{SL}_2(2^n)$ are algebraic; this was extended to $\mathrm{SL}_2(p^n)$ in characteristic p by Kovács a few years later.

More recently, I have proved a number of theorems in this area, some extending the results above, and some in a different direction. For example, I have extended Alperin's result, by proving that if G is a finite group with abelian Sylow 2-subgroups, then all simple modules are algebraic in characteristic 2.

A block-wise version of this – that all simple modules for 2-blocks with abelian defect group – is conjectural, but in joint work with Eaton, Kessar, and Linckelmann, this statement is shown to hold for the case where the block has Klein four defect groups.

Moving to indecomposable modules, we can isolate where algebraic modules are on the Auslander–Reiten quiver in some cases, and in particular prove that algebraic indecomposable modules of complexity at least 3 lie on the end of their component. In the case where the group is elementary abelian of order p^2, there is conjecturally a very strong relationship between being algebraic and the homological algebra: it seems that (for this group) an absolutely indecomposable module of dimension a multiple of p is algebraic if and only if it is periodic. This conjecture is interesting because it relates the tensor structure of the group algebra to the homological structure in a way that does not appear to have been considered before.

On vertices of simple modules for symmetric groups labelled by two part partitions

Susanne Danz
(joint work with Karin Erdmann and Burkhard Külshammer)

In 1958, J.A. Green introduced the notion of vertices and sources of indecomposable modules over group algebras. Given a finite group G and an algebraically closed field F of characteristic $p > 0$, a vertex of an indecomposble FG-module M is a group $P \leq G$ which is minimal with respect to the condition that $M \mid \mathrm{Ind}_P^G(\mathrm{Res}_P^G(M))$. Such a vertex is known to be a p-group, and it is unique up to G-conjugacy.

In this talk, I considered the simple modules for the symmetric group $\mathfrak{S}_n$ of degree $n \in \mathbb{N}$ over the field F. As is well-known, the isomorphism classes of simple $F\mathfrak{S}_n$-modules are parametrized by the p-regular partitions of n. The simple module corresponding to a partition λ of n is denoted by D^λ. However, there are only very few classes of simple $F\mathfrak{S}_n$-modules whose structure is generally well-understood. Amongst these is the class of simple $F\mathfrak{S}_n$-modules labelled by two part partitions. Therefore, in this talk, I focused on the following question:

Let $p = 2$. Given a (2-regular) partition $\lambda := (n-m, m)$ of n, what are the vertices of D^λ?

In [3], J. Müller and R. Zimmermann showed that, in the case where $m \leq 1$, the vertices of D^λ are always the defect groups of the block of $F\mathfrak{S}_n$ containing D^λ, unless $n = 4$ and $m = 1$. Namely, $D^{(3,1)}$ has the Sylow 2-subgroup of the alternating group $\mathfrak{A}_4$ as its vertex.

I presented two recent results on the vertices of simple $F\mathfrak{S}_n$-modules of the form $D^{(n-m,m)}$ which are stated below. If $m = \lfloor \frac{n-1}{2} \rfloor$ then we set $D(n) := D^{(n-m,m)}$ which is the basic spin $F\mathfrak{S}_n$-module.

Theorem 1. (S. Danz, B. Külshammer [2]) *Let $n \geq 2$, let $p = 2$, and let $n = \sum_{j=1}^{s} 2^{i_j}$ be the 2-adic expansion of n, for appropriate $s \geq 1$, $i_1 > \ldots > i_s \geq 0$. Let further P be a vertex of $D(n)$.*

(i) *If $n \equiv 2 \pmod 4$ then P is a Sylow 2-subgroup of $\mathfrak{S}_n$.*
(ii) *If $n \equiv 0 \pmod 4$ then P is a Sylow 2-subgroup of $\mathfrak{A}_n$.*
(iii) *If n is odd then P is conjugate to a Sylow 2-subgroup of $\mathfrak{A}_{2^{i_1}} \times \cdots \times \mathfrak{A}_{2^{i_s}}$.*

Theorem 2. (S. Danz, K. Erdmann [1]) *Let* $n \geq 6$*, and let* P *be a vertex of* $D := D^{(n-2,2)}$.

(i) *If* $n \equiv 3 \pmod 4$ *then* $P =_{\mathfrak{S}_n} P_{n-5} \times P_2 \times P_2$*, and* D *has trivial source.*

(ii) *Otherwise,* P *is a Sylow* 2*-subgroup of* $\mathfrak{S}_n$*. Moreover, if* n *is even then* $\mathrm{Res}_P^{\mathfrak{S}_n}(D)$ *is a source of* D*.*

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Nilpotent blocks of simple groups

Charles Eaton

(joint work with Jianbei An)

Generalising the idea of a p-nilpotent group, a p-block of a finite group is nilpotent if it gives rise to the smallest possible fusion system for its defect groups. The structure of such blocks is well-understood, but the frequency of their occurrence and behaviour with respect to normal subgroups are not. There are also open problems concerning their recognition, for example the conjecture of Puig concerning the number of simple modules for each B-subpair.

We study the nilpotent blocks of the finite simple groups. In a sense this extends the work of many authors determining the existence (or otherwise) of blocks of defect zero of simple groups.

We give characterisations of the nilpotent blocks of some classical groups, and use these to deduce that every nilpotent block of a finite simple group must have abelian defect groups.

The Lie module of the symmetric group

Karin Erdmann

(joint work with Kai Meng Tan)

Let $\mathfrak{S}_n$ be the symmetric group of degree n and let F be a field of characteristic p. The Lie module of $\mathfrak{S}_n$ can be defined as the right ideal $\mathrm{Lie}(n) := \omega_n F\mathfrak{S}_n$ generated by the Dynkin-Specht-Wever element,

$$\omega_n = (1 - c_2)(1 - c_3)\dots(1 - c_n)$$

where c_k is the k-cycle $(1\ 2\ \dots\ k)$ This is related to the free Lie algebra $L(V)$ over an F- vector space V of dimension $\geq n$; it is the image of the homogeneous part of degree n under the Schur functor.

The motivation of [3] comes from algebraic topology. In [4] certain coalgebra decompositions of the tensor algebra $T(V)$ are related to loop suspensions of p-torsion suspensions of some topological spaces. In this context one needs to understand a maximal projective submodule of $\mathrm{Lie}(n)$, called $\mathrm{Lie}^{\max}(n)$ in [4].

Write $\mathrm{Lie}(n) = \mathrm{Lie}(n)_{pf} \oplus \mathrm{Lie}(n)_{pr}$, where $\mathrm{Lie}(n)_{pr}$ is projective, and $\mathrm{Lie}(n)_{pf}$ does not have any non-zero projective summand. These are unique up to isomorphism, and then $\mathrm{Lie}^{\max}(n) \cong \mathrm{Lie}(n)_{pr}$. When p does not divide n, the module $\mathrm{Lie}(n)$ is projective. But otherwise, it has non-projective summands and projective summands, and in general not much is known. In [1], upper bounds for the dimension of $\mathrm{Lie}^{\max}(n)$ were found when $p = 2$ but it is not clear whether their methods generalize.

When $n = pk$ and p does not divide k, the summands of $\mathrm{Lie}(n)_{pf}$ have been parametrized in [2], via p-permutation modules. We use these results (in [3]) to find an upper bound for the dimension of $\mathrm{Lie}^{\max}(n)$. Namely we have

$$\dim(\mathrm{Lie}^{\max}(n)) \leq (n-1)! - \dim(\mathrm{Lie}(n)_{pf} \downarrow_P)$$

where P is a Sylow p-subgroup of $\mathfrak{S}_n$.

Based on [2], we obtain a recursive formula in terms of orbits of P on the coset space of a subgroup D isomorphic to $\mathfrak{S}_p \times \mathfrak{S}_k$ acting regularly. For example, D can be taken as $D_1 \times D_2$ where D_1 is the diagonal of the base group in $\mathfrak{S}_p \wr \mathfrak{S}_k$, and D_2 is the top group.

It was proved in [2] that there is a short exact sequence of $F\mathfrak{S}_n$-modules

$$0 \to \mathrm{Lie}(n) \to eF\mathfrak{S}_n \to S^p(\mathrm{Lie}(k)) \to 0$$

where e is an idempotent, and where $S^p(\mathrm{Lie}(k)) = \mathrm{Ind}_D^{\mathfrak{S}_n}(\Lambda)$ with $\Lambda := F \otimes \mathrm{Lie}(k)$, the external tensor product. We fix a Sylow p-subgroup P of $\mathfrak{S}_n$, and we show

Proposition *If $x \in \mathfrak{S}_n$ and $D_1^x \cap P \neq 1$ then $\mathrm{Ind}_{D^x \cap P}^{P}(\Lambda^x)$ has no projective summands; otherwise it is projective.*

Using the short exact sequence above, we get that the projective-free part $\mathrm{Lie}(n)_{pf}$ is isomorphic to $\mathrm{Ind}_D^{\mathfrak{S}_n}(\Omega(\Lambda))$ and $\Omega(\Lambda) \cong \Omega(F) \otimes \mathrm{Lie}(k)$. Furthermore, the $\mathfrak{S}_p$-module $\Omega(F)$ is the Specht module $S^{(p-1,1)}$ of dimension $p-1$, and one can deduce

$$\dim(\mathrm{Lie}(n)_P)_{pf} = (p-1)(k-1)! \sum_x |P : D^x \cap P|$$

where the sum is taken over double coset representatives x such that $D_1^x \cap P \neq 1$.

The problem is then to parametrize these double cosets, and detemine $|P : D^x \cap P|$, our result is a recursive formula.

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Modular Principal Series Representations

Meinolf Geck

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$ and $F\colon \mathbf{G} \to \mathbf{G}$ be the Frobenius map associated with an $\mathbb{F}_q$-rational structure on $\mathbf{G}$. Then $G := \mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ is a finite group of Lie type over $\mathbb{F}_q$. For example, the finite groups $G = \mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{GU}_n(\mathbb{F}_q)$ or $E_8(\mathbb{F}_q)$ arise in this way. Let k be an algebraically closed field of characteristic ℓ where either $\ell = 0$ or $\ell > 0$ is a prime not dividing q. Let $\mathrm{Irr}_k(G)$ be the set of irreducible representations of G over k (up to isomorphism). We wish to study $\mathrm{Irr}_k(G)$ for all G of a given "type", with q varifying. Here, the "type" of G is given by $(\mathbf{W}, \gamma)$ where $\mathbf{W}$ is the Weyl group of $\mathbf{G}$ and $\gamma\colon \mathbf{W} \to \mathbf{W}$ is the automorphism induced by F. Let $B \subseteq G$ be a Borel subgroup and set

$$\mathrm{Irr}_k(G \mid B) = \{\rho \in \mathrm{Irr}_k(G) \mid \rho \text{ admits non-zero vectors fixed by } B\}.$$

(This set is one piece in the partition of $\mathrm{Irr}_k(G)$ into Harish-Chandra series; see [4] where this partition and its connection with Hecke algebras is studied without restriction on ℓ.)

Characteristic 0. If $\ell = 0$, then, by classical results due to Bourbaki, Iwahori, Tits, ... ($\sim$ 1960's), there is a natural bijection $\mathrm{Irr}_k(G \mid B) \leftrightarrow \mathrm{Irr}_k(\mathbf{W}^\gamma)$, where $\mathbf{W}^\gamma$ is the group of fixed points of $\mathbf{W}$ under γ. (Note that $\mathbf{W}^\gamma$ is a finite Coxeter group.) In particular, $\mathrm{Irr}_k(G \mid B)$ is "independent of q". This is part of a more general picture where the "unipotent" representations of G are seen to be "independent of q" (Lusztig [9]).

The modular case. Now assume that $\ell > 0$. Then, following work of Fong–Srinivasan, Dipper–James, Broué–Malle–Michel, Hiss, ..., one expects that (at least for $\ell >> 0$) the "unipotent" ℓ-modular representations should only depend on e, the multiplicative order of q modulo ℓ. We have recently obtained some definite results in this direction concerning the set $\mathrm{Irr}_k(G \mid B)$.

Theorem. [2], *Assume that ℓ does not divide $|\mathbf{W}|$. Then there is a natural injection* $\mathrm{Irr}_k(G \mid B) \hookrightarrow \mathrm{Irr}_{\mathbb{C}}(\mathbf{W}^\gamma)$ *whose image only depends on e.*

This injection is defined using Dipper's Hom functors [1], the "cellularity" in the sense of Graham–Lehrer of the associated Hecke algebras (shown in [3]), and properties of the "unipotent support" of the irreducible representations of G in characteristic 0; see [10], [5]. The statement that this only depends on e under the given condition on ℓ essentially relies on a general result about the number of simple modules of Hecke algebras (see [7]).

General Version of James' Conjecture. ([8], [6]) *Assume that ℓ does not divide $|\mathbf{W}|$ and that k is the residue field of a discrete valuation ring $\mathcal{O}$ whose quotient* *field K is sufficiently large and of characteristic* 0*. Then the decomposition matrix (defined using ℓ-modular reduction via $\mathcal{O}$)*

$$D' = \big(d_{\chi\,\rho}\big)_{\chi\in\mathrm{Irr}_K(G|B),\,\rho\in\mathrm{Irr}_k(G|B)} \quad \textit{only depends on } e.$$

Note that, by the previous theorem, we know at least that the sets indexing the rows and columns of D' only depend on $\mathbf{W}$ and e. Using extensive computer calculations, the conjecture is now known to be true for G of exceptional type; see [6] and the references there.

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Zelevinsky involution in affine type A

Nicolas Jacon
(joint work with Cédric Lecouvey)

The Zelevinsky involution is a certain map which has its origins in the representation theory of p-adic groups. It can also be defined over the affine Hecke algebra $\widehat{H}$ of type A over $\mathbb{C}$ with parameter q. It has been studied by Moeglin-Waldspurger [5] and Leclerc-Thibon-Vasserot [4]. This map yields an involution τ on a certain set of simple $\widehat{H}$-modules parametrized by the "aperiodic multisegments". Hecke algebras of type A are quotients of affine Hecke algebras of type A and τ induces a "q-analogue" of the famous Mullineux involution on these finite type Hecke algebras. Hence, the Zelevinsky involution may be seen as a generalization of the Mullineux involution.

When q is specialized to an e-th root of 1, Leclerc, Thibon and Vasserot have shown that τ can be described using the crystal $B_e(\infty)$ of the negative part of the quantum algebra $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$.

In our recent work [3], we give an alternative and efficient procedure for computing this involution. The strategy is as follows. Let $\mathfrak{m}$ be an aperiodic multisegment labelling a simple module for the affine Hecke algebra $\widehat{H}$ of type A.

(1) We associate to $\mathfrak{m}$ a certain multipartition λ which labels a simple module for an Ariki-Koike algebra $\mathcal{H}$ (see [1]).
(2) We then apply an algorithm to compute the analogue of the Zelevinsky involution on Ariki-Koike algebras. This provides another multipartition μ labelling a simple $\mathcal{H}$-module (see [2]).
(3) We associate to μ an aperiodic multisegment $\mathfrak{m}'$ which labels a simple $\widehat{H}$-module.

The image of $\mathfrak{m}$ under the Zelevinsky involution is then $\mathfrak{m}'$.

All our computations can be made independently of the crystal structure on $B_e(\infty)$. Moreover, they do not require the determination of i -induction or i-restriction operations on simple modules. Finally, these results show a very simple relation between τ and the Kashiwara involution in affine type A.

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Springer basic sets

Daniel Juteau

This talk was about geometric methods to study modular representations of Weyl groups. I talked about a modular Springer correspondence that I defined in my thesis, and explained how this allows to define a basic set geometrically for Weyl groups. Springer basic sets should be helpful to determine explicitly the modular Springer correspondence in all types. The link between ordinary characters of Weyl groups and nilpotent orbits, established by Springer in 1976, led Lusztig to his theory of character sheaves, which allow to determine character values of finite reductive groups. Later, I hope to develop a theory of modular character sheaves.

1. Perverse sheaves

First I explained briefly and roughly what perverse sheaves look like. Let X be some complex algebraic variety endowed with a "good stratification" $X = \bigsqcup_{S\in\mathfrak{X}} S$. The strata are smooth connected, locally closed subvarieties, and the closure of a stratum is a union of strata. The strata are naturally ordered by $S_1 \geq S_2 \iff \overline{S}_1 \supset S_2$. Let us fix some prime ℓ and an ℓ-modular system $(\mathbb{K}, \mathbb{O}, \mathbb{F})$, and let $\mathbb{E}$ denote $\mathbb{K}$ or $\mathbb{F}$. A constructible complex $\mathcal{F}$ of $\mathbb{E}$-sheaves is a complex of $\mathbb{E}$-sheaves whose cohomology sheaves $\mathcal{H}^n\mathcal{F}$ are constructible. That is, we require that the restrictions $\mathcal{H}^n\mathcal{F}_{|S}$ be local systems. If we choose some base point $x_S \in S$, the functor "fiber at x_S" identifies local systems (with $\mathbb{E}$ coefficients) to finite dimensional representations of the fundamental group $\pi_1(S, x_S)$, over $\mathbb{E}$.

If G is a connected algebraic group, and X is a G-variety with finitely many orbits, we can use the orbits to stratify X, and we can consider G-equivariant objects: in that case, the representations of the fundamental groups have to factor through the finite quotients $\pi_1(S, x_S) \to A_G(x_S)$, where $A_G(x_S) = C_G(x_S)/C_G(x_S)^0$ is the finite group of components of the centralizer of x_S.

There is a Grothendieck-Verdier duality $\mathbb{D}$ on the derived category of constructible complexes. A constructible complex $\mathcal{F}$ is perverse if for all $S \in \mathfrak{X}$, we have $\mathcal{H}^n\mathcal{F}_{|S} = 0$ for all $n > -\dim S$, and a similar condition holds for $\mathbb{D}\mathcal{F}$. Perverse sheaves form an abelian category, which here ($\mathbb{E}$ is a field) is noetherian and artinian.

For $S \in \mathfrak{X}$, if $\mathcal{L}$ is an irreducible local system on S, then there is a unique perverse sheaf $\mathcal{F}$ supported by $\overline{S}$, whose restriction to S is $\mathcal{L}[\dim S]$, and such that for all $T < S$, we have $\mathcal{H}^n\mathcal{F}_{|T} = 0$ for all $n \geq -\dim T$, and similarly for $\mathbb{D}\mathcal{F}$. Then $\mathcal{F}$ is denoted $\mathcal{IC}(\overline{S}, \mathcal{L})$ and is called an intersection cohomology complex. The $\mathcal{IC}(\overline{S}, \mathcal{L})$ are the simple objects in the category of perverse sheaves.

All of the above makes sense for varieties over a base field of characteristic p different from ℓ, using the étale topology. There is also a version over $\mathbb{O}$, but there are some subtleties due to the torsion: one has to consider two perversities (hence two kinds of perverse sheaves), exchanged by the duality.

2. Springer correspondence

Let G be a connected reductive group over $k = \overline{\mathbb{F}}_p$, where p is sufficiently large and different from ℓ. Then G acts on its Lie algebra $\mathfrak{g}$ by the adjoint action, and there are only finitely many orbits in the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. We denote by $x_\mathcal{O}$ a representative of $\mathcal{O}$. We can actually assume that G is simple of adjoint type, in which case the finite groups $A_G(x_\mathcal{O})$ are either 2-elementary abelian, or symmetric groups S_3, S_4 or S_5. The set $\mathfrak{P}_\mathbb{E} = \{(x, \rho) \mid x \in \mathcal{N},\ \rho \in \operatorname{Irr}\mathbb{E}A_G(x)\}/{\sim_G}$ parametrizes the simple G-equivariant perverse sheaves with $\mathbb{E}$ coefficients on $\mathcal{N}$, which we denote by $\mathcal{IC}_\mathbb{E}(x, \rho)$, for $(x, \rho) \in \mathfrak{P}_\mathbb{E}$. Note that, if ℓ does not divide the orders of the finite groups $A_G(x)$, then we can identify $\mathfrak{P}_\mathbb{K}$ with $\mathfrak{P}_\mathbb{F}$.

Using a Fourier transform, one can define a Springer correspondence

$$\Psi_\mathbb{E} : \operatorname{Irr}\mathbb{E}W \hookrightarrow \mathfrak{P}_\mathbb{E}.$$

In characteristic zero, this was done by Springer in [Spr76] (without perverse sheaves), and then by Hotta-Kashiwara for D-modules in 1984, and by Brylinski with a Fourier-Deligne transform in 1986. In the modular case, this was done in [Jut07]. There are many other constructions in characteristic zero, which do not use a Fourier transform, and which give another parametrization, differing from this one by the sign character of W (Lusztig, Borho-McPerhson, Slodowy...).

In the case $G = GL_n$, the nilpotent orbits are parametrized by partitions (according to the sizes of the Jordan blocks), and we have $\mathcal{O}_\lambda \geq \mathcal{O}_\mu \iff \lambda \geq \mu$ for the usual dominance order of partitions. We denote by x_λ an element of $\mathcal{O}_\lambda$. Then all the $A_G(x_\lambda)$ are trivial, and we can identify both $\mathfrak{P}_\mathbb{K}$ and $\mathfrak{P}_\mathbb{F}$ with the set of partitions of n (the local systems have to be trivial). With this notation, we have $\Psi_\mathbb{K}(S^\lambda) = \lambda'$ (Hotta-Springer 1977), and if μ is ℓ-regular, we have $\Psi_\mathbb{F}(D^\mu) = \mu'$ too [Jut07], where the dash denotes conjugation of partitions.

3. Decomposition matrices

Recall how we define the decomposition matrix for a finite group (here a Weyl group) W: for $E \in \mathrm{Irr}\mathbb{E}W$ and $F \in \mathrm{Irr}\mathbb{F}W$, we define $d^W_{E,F}$ by $[\mathbb{F} \otimes_\mathbb{O} E_\mathbb{O} : F]$, where $E_\mathbb{O}$ is an integral form of E, and the decomposition matrix is $D^W = (d^W_{E,F})$.

We can do the same for G-equivariant perverse sheaves on the nilpotent cone: for $(x, \rho) \in \mathfrak{P}_\mathbb{K}$ and $(y, \sigma) \in \mathfrak{P}_\mathbb{F}$, we define $d^\mathcal{N}_{(x,\rho),(y,\sigma)}$ by $[\mathbb{F} \otimes^L_\mathbb{O} \mathcal{IC}_\mathbb{O}(x, \rho_\mathbb{O}) : \mathcal{IC}_\mathbb{F}(y, \sigma)]$, where $\rho_\mathbb{O}$ is some integral form of ρ, and $D^\mathcal{N} = (d^\mathcal{N}_{(x,\rho),(y,\sigma)})$ [Jut09].

Theorem 3.1. [Jut07] *We have*

$$d^W_{E,F} = d^\mathcal{N}_{\Psi_\mathbb{K}(E),\Psi_\mathbb{F}(F)}.$$

Hence D^W can be seen as a submatrix of $D^\mathcal{N}$. To compute the right-hand side, it would be enough to determine $\Psi_\mathbb{F}$ (this should be possible in general), and to determine the IC stalks with $\mathbb{F}_p$ coefficients for nilpotent orbit closures (this should be very difficult in general). The similar problems for characteristic zero coefficients have been solved a long time ago. I was able to compute, for all types, the stalk at a subregular element of $\mathcal{IC}(\mathcal{N}, \mathbb{F})$, and the stalk at 0 of $\mathcal{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$, where $\mathcal{O}_{\min}$ is the minimal non-trivial nilpotent orbit. In both cases, one can express the corresponding decomposition number in a uniform way, in terms of root systems [Jut08, Jut09].

For $G = GL_n$, we know that the ordinary and modular Springer correspondences are given by the conjugation of partitions, so that $[S^\lambda : D^\mu] = d^\mathcal{N}_{\lambda',\mu'}$. It follows that one can see James's row and column removal rule as a consequence of a smooth equivalence of nilpotent singularities obtained by Kraft and Procesi.

The matrix $D^\mathcal{N}$ is clearly unitriangular:

Proposition 3.2. *We have*

$$d^\mathcal{N}_{(x,\rho),(y,\sigma)} = \begin{cases} 0 & \textit{unless } \mathcal{O}_y \leq \mathcal{O}_x, \\ d^{A_G(x)}_{\rho,\sigma} & \textit{if } y = x. \end{cases}$$

(For small ℓ, we use the fact that the $D^{A_G(x)}$ are unitriangular themselves.) However, this does not imply directly that D^W is unitriangular (which was already known by other methods, for example by Geck-Rouquier). For this, one needs the additional observation (that I proved at MSRI in 2008; still unpublished):

Proposition 3.3. *If $(x,\rho) \notin \mathrm{Im}\Psi_{\mathbb{K}}$ and $d^{\mathcal{N}}_{(x,\rho),(y,\sigma)} \neq 0$, then $(y,\sigma) \notin \Psi_{\mathbb{F}}$.*

It follows that one can define a basic set using the Springer correspondence (for small ℓ, we also use the unitriangularity of the $D^{A_G(x)}$). Thus one can complete the following square (where the b_x define basic sets for the $A_G(x)$):

$$\begin{array}{ccc}
\mathrm{Irr}\mathbb{F}W & \xrightarrow{\Psi_{\mathbb{F}}} & \mathfrak{P}_{\mathbb{F}} = \bigsqcup_x \mathrm{Irr}\mathbb{F}A_G(x) \\
{\scriptstyle b}\,\vdots & & \big\downarrow {\scriptstyle \bigsqcup_x b_x} \\
\mathrm{Irr}\mathbb{K}W & \xrightarrow[\Psi_{\mathbb{K}}]{} & \mathfrak{P}_{\mathbb{K}} = \bigsqcup_x \mathrm{Irr}\mathbb{K}A_G(x)
\end{array}$$

and, using the order on $\mathrm{Irr}\mathbb{K}W$ induced by $\Psi_{\mathbb{K}}$ (and, for small ℓ, a natural order on the $\mathrm{Irr}\mathbb{K}A_G(x)$), we have $d^W_{E,F} = 0$ unless $E \geq b(F)$, and $d^W_{b(F),F} = 1$.

In a project with Karine Sorlin, we aim to use this fact to determine explicitly the modular Springer correspondence in all types.

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On stable equivalences and blocks with one simple module

Radha Kessar

(joint work with Markus Linckelmann)

Nilpotent blocks form an important example of blocks with one isomorphism class of simple modules-such blocks are Morita equivalent to the group algebra of a defect group of the block. However, there exist non-nilpotent blocks with one simple module, and the understanding of their structure is as yet incomplete.

We denote by $\mathcal{O}$ a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of prime characteristic p and quotient field K of characteristic zero. Given a finite group G, and a a block algebra B of $\mathcal{O}G$, we denote by $\ell(B)$ the number of isomorphism classes of simple $k \otimes_{\mathcal{O}} B$-modules. We prove the following result:

Let G be a finite group and B a block algebra of $\mathcal{O}G$ having a defect group of order at most p^2. Denote by C the Brauer correspondent of B and suppose that

K, k are splitting fields for B, C. If $\ell(C) = 1$ then $\ell(B) = 1$, the inertial quotient of B is abelian, the decomposition matrices of B and C are equal and there is a p-permutation equivalence between B and C inducing an isotypy between B and C all of whose signs are positive.

The main ingredient for proving the above is Rouquier's stable equivalence between B and its Brauer correspondent, obtained from "gluing" together various derived equivalences at local levels [4]. Since stable equivalences between block algebras preserve the character group $L^0(B)$ of generalised characters which vanish on p-regular elements, isometry arguments turn out to work particularly well for blocks with one simple module, because in that case the co-rank of $L^0(B)$ in the Grothendieck group over K of B is 1 and hence this subgroup contains enough information to reconstruct the number of irreducible characters of any block stably equivalent to B.

Broué's Abelian Defect Conjecture predicts more precisely that B and C are derived equivalent. If true, a result of Roggenkamp and Zimmermann would imply that B and C are actually Morita equivalent. This is known to hold if the defect groups of B are cyclic or Klein four because in that case the hypothesis of having a unique isomorphism class of simple modules implies that B and C are nilpotent, hence Morita equivalent to $\mathcal{O}P$. In order to prove the above we may therefore assume that p is odd, that a defect group P of B is elementary abelian of rank 2 and that the inertial quotient of B is non trivial. This forces the inertial quotient to be abelian, and hence the Brauer correspondent C is a quantum complete intersection [1], [2], [3]. In light of the above Theorem, one is led to ask the following:

Question: Let X be a quantum complete intersection over k and let Y be a finite dimensional symmetric k-algebra such that there is a stable equivalence of Morita type between X and Y. Suppose further that Y is local, $\dim_k(Y) = \dim_k(X)$ and $Z(X) \cong Z(Y)$ as k-algebras. Are X and Y isomorphic k-algebras?

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Blocks with metacyclic defect groups

Shigeo Koshitani
(joint work with Jürgen Müller, Miles Holloway, Naoko Kunugi)

Here we will present two results, namely, the first one is a joint work with **Jürgen Müller**, and the second one is that with **Miles Holloway** and **Naoko Kunugi**.

The results are on modular representation theory. We will use the following notation.

Let p be a prime and G a finite group. We denote by $(\mathcal{K}, \mathcal{O}, k)$ a splitting p-modular system for all subgroups of G. That is, $\mathcal{O}$ is a complete discrete valuation ring and $\mathcal{K}$ is the quotient field of $\mathcal{O}$, both of which have characteristic zero, and k is the residue field of $\mathcal{O}$, namely $k := \mathcal{O}/\mathrm{rad}(\mathcal{O})$ which has characteristic p. Moreover, the fields $\mathcal{K}$ and k are both splitting fields for all subgroups of G. We denote by C_n the cyclic group of order n for an integer $n \geq 1$.

Theorem 1 (J.Müller–S.Koshitani). *Assume that $p = 3$, $G := \mathsf{HN}$ is the Harada-Norton sporadic simple group, and A is a non-principal block algebra of the group algebra $\mathcal{O}G$ with defect group $P = C_3 \times C_3$ (actually, A is a unique such block algebra of $\mathcal{O}G$). Then, Broué's abelian defect group conjecture for A is true. Namely, the categories $\mathrm{D}^b(\text{mod-}A)$ and $\mathrm{D}^b(\text{mod-}B)$ are equivalent as triangulated categories, where B is the Brauer corresponding block algebra in $\mathcal{O}N_G(P)$, mod-A is the category of all finitely generated right $\mathcal{O}G$-modules which are free as $\mathcal{O}$-modules and which belong to A, and $\mathrm{D}^b(\text{mod-}A)$ is the bounded derived category of mod-A). In fact we get also that A and the principal 3-block algebra $B_0(\mathcal{O}[\mathsf{HS}])$ are Morita equivalent, where HS is the Higman-Sims sporadic simple group.*

Theorem 2 (M.Holloway–N.Kunugi–S.Koshitani). *Suppose that G has a Sylow p-subgroup P such that $P = M_{n+1}(p) = \langle x, y | x^p = y^{p^n} = 1, x^{-1}yx = y^{p^{n-1}+1} \rangle \cong C_{p^n}{:}C_p$, the semi-direct product of C_{p^n} by C_p for an integer $n \geq 2$, and that $A := B_0(\mathcal{O}G)$ is the principal block algebra of $\mathcal{O}G$. Then, it holds that*

$$\begin{aligned}
k_0(A) &= pe + p(p^{n-1} - 1)/e, \\
k_1(A) &= p^{n-2}(p-1)/e, \\
k(A) &= pe + (p^n + p^{n-1} - p^{n-2} - p)/e, \\
\ell(A) &= e,
\end{aligned}$$

where $e := |N_G(P)/P{\cdot}C_G(P)|$ (the inertial index for A), $k_i(A)$ is the number of all irreducible ordinary characters of G belonging to A which have height i, and $k(A)$ and $\ell(A)$ are the numbers of all irreducible ordinary and Brauer characters of G belonging to A, respectively. In addition, it turns out that a conjecture posed by S. Hendren [1, p.490] *holds for the principal blocks.*

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On Oliver's conjecture

Nadia Mazza
(joint work with D. Green and L. Héthelyi)

Bob Oliver's proof of the Martino-Priddy conjecture uses the classification of finite simple groups. For odd primes, this can be avoided, provided one shows a general vanishing result, which would imply the existence and uniqueness of linking systems associated to arbitrary saturated fusion systems.

This vanishing result is Oliver's conjecture and it turns out to be equivalent to showing that the containment

$$J(S) \leq \mathfrak{X}(S)$$

holds for any finite p-group S that may arise as a minimal strongly closed subgroup in a saturated fusion system. Here, p is an odd prime and $J(S)$ is the "elementary abelian version" of the Thompson subgroup of S; that is, the subgroup of S generated by all the elementary abelian subgroups of S of maximal order. The subgroup $\mathfrak{X}(S)$ is the Oliver subgroup of S, defined as the largest subgroup of S for which there is a series of normal subgroups Q_i of S such that $1 = Q_0 \leq Q_1 \leq \cdots \leq Q_n = \mathfrak{X}(S)$ and such that they satisfy the commutator relations

$$[\Omega_1(C_S(Q_{i-1}))\ ,\ Q_i\ ;\ p-1] = 1\ ,\ \forall\, 1 \leq i \leq n.$$

In his paper, Oliver reduces the question to finite simple groups and checks in each case, by hand, that the containment holds.

In a recent article, Green, Héthelyi and Lilienthal recast Oliver's conjecture in terms of $\mathbb{F}_pG$-modules and they show that $J(S) \leq \mathfrak{X}(S)$ if G has nilpotence class at most 2, where G is the factor group $S/\mathfrak{X}(S)$. This reformulation appeals to the study of the F-modules and quadratic offenders, arising in the problem of the failure of Thompson's factorisation. Hence, it turns out that Oliver's conjecture holds if and only if G has no F-module on which any central element of G of order p acts with minimal polynomial of degree p.

Since the result by Green, Héthelyi and Lilienthal, the first two authors and Mazza managed to prove that Oliver's conjecture holds in a few more cases, which are summarised as follows.

Theorem. Suppose that p is an odd prime and S is a p-group such that $G = S/\mathfrak{X}(S)$ satisfies any of the following conditions

(1) G has nilpotence class at most four;
(2) G is metabelian;
(3) G has p-rank at most p.

Then Oliver's conjecture $J(S) \leq \mathfrak{X}(S)$ holds for S.

As a consequence of this theorem and a compilation of older results from group theory, we obtain:

Corollary. Suppose that p is an odd prime and S is a p-group such that $G = S/\mathfrak{X}(S)$ satisfies any of the following conditions

(1) G has maximal nilpotence class;

(2) G is a regular 3-group.

Then Oliver's conjecture $J(S) \leq \mathfrak{X}(S)$ holds for S.

In this talk, we present a brief survey of Oliver's conjecture and its reformulation, and we discuss the approach and the results of the ongoing collaboration Green-Héthelyi-Mazza.

Dipper's hypothesis and self-injective endomorphism rings

Hyohe Miyachi

0.1. **Introduction and history.** R. Dipper [Dip90] made quite good assumptions on some endomorphism rings to classify simple modules over finite general linear groups in non-defining characteristic. Geck, Hiss and Malle [GHM96] followed his approach towards a classification of simple modules for the other types. But, later Geck and Hiss [GH97] noticed that the self-injectivity of Iwahori-Hecke algebras is useful for the classification using Sawada-Green approach [Gre7] instead of using Dipper's approach.

In my talk, first I removed a part of Dipper's assumptions in his paper "Quotient Hom functor II" [Dip98], namely, his one hypothesis implies the self-injectivity of an endomorphism ring in his set up.

Second, I talked about the use of self-injective endomorphism rings. The situation I have in my mind is different from Dipper and Geck-Hiss-Malle. The main examples are:

- (**Conjectural Correspondences**) Tackling De Visscher-Donkin conjecture [DVD05] on polynomial tilting injectives.
- (**Simple Specht Modules**) Obtaining simple Specht modules. (At March 2008 MSRI, I talked about this assuming something. This time, I removed this "something".)

The argument could be useful for any finite dimensional algebras such as group algebras, Hecke algebras of type A, Brauer algebras, its q-analogue Birman-Murakami-Wenzl algebras as long as behind the scene we have a "group" such as a Hopf algebra or more generally a Frobenius category in the sense of D. Happel [Hap88].

0.2. **Selfdual Dipper implies self-injectivity.** Let A be a finite dimensional algebra over a field $\Bbbk$. The following definition was the key for Dipper's work in [Dip90].

Definition 1 (Dipper)**.** *Let Y be an A-module, and let $\beta : P \to Y \to 0$ be the projective cover of Y. We say that the projective cover β satisfies Dipper's hypothesis if*

$$\mathrm{Hom}_A(P, Y) \cong \mathrm{Hom}_A(Y, Y).$$

We introduce a slightly new definition as follows:

Definition 2. *We say that* $P \to Y \to 0$ *satisfies selfdual Dipper's hypothesis if* $0 \to Y \to I$ *is an injective presentation of* Y*,* $I \cong P$ *and*

(1) $\mathrm{Hom}_A(P, Y) \cong \mathrm{Hom}_A(Y, Y)$ *and* $\mathrm{Hom}_A(Y, P) \cong \mathrm{Hom}_A(Y, Y)$

Remark 3. *Note that the first condition in (1) is equivalent to the original Dipper's hypothesis. And, the equation above is satisfied automatically in Dipper's main examples such as* $G = GL_n(\mathbb{F}_q)$ *in* [Dip98] *since* $\Bbbk[G]$ *is symmetric and all the unipotent cuspidals are selfdual.*

Definition 2 was motivated by the following lemma:

Lemma 4. *If* $\beta : P \to Y$ *satisfies selfdual Dipper's hypothesis, then* $\mathrm{End}_A(Y)$ *is self-injective.*

0.3. **A ring theoretical characterization of simple top Young modules.** Let $S(n, r)$ be the q-Schur algebra over $\Bbbk$ associated with the divided power quantum general linear group $U_{q^{1/2}}(\mathfrak{gl}_n)$ with unit parameter q and the r-fold tensor space $V^{\otimes r}$ of the natural representation V. We denote by $L_n(\lambda)$, $\nabla_n(\lambda)$, $I_n(\lambda)$ and $T_n(\lambda)$ the simple, the costandard (induced) module, the indecomposable injective polynomial module and the indecomposable tilting module over $S(n, r)$ corresponding to a highest weight λ. Write $S(r)$ for $S(r, r)$. If $r \leq n$, then we omit the subscript n of $L_n(\lambda), \nabla_n(\lambda)$ and $I_n(\lambda)$ since $S(r)$ and $S(n, r)$ are Morita equivalent and we may concentrate only on modules over $S(r)$ by excluding $S(n, r)$. There exists an idempotent φ in $S(r)$ such that $\mathcal{H}(r) := \varphi S(r) \varphi$ is the Iwahori-Hecke algebra of type A_{r-1}. We denote by F the Schur functor associated with ϕ from $S(r)$-modules to $\mathcal{H}(r)$-modules. $Y^\lambda := FI(\lambda)$ (resp. $S^\lambda := F\nabla(\lambda)$) is called the Young (resp. Specht) module corresponding to λ. For a partition λ write $l(\lambda)$ for the number of nonzero parts of λ.

Now, we state one of the main use of self-injective endomorphism rings in this report as follows:

Theorem 5. *For a partition* λ *of* r*, the following are equivalent:*

(1) $\mathrm{Top}(Y^\lambda)$ *is simple.*
(2) $\mathrm{End}_{\mathcal{H}(r)}(Y^\lambda)$ *is self-injective.*
(3) $\mathrm{End}_{S(n,r)}(I_n(\lambda))$ *is self-injective for some* $n \geq l(\lambda)$*.*
(4) *The projective cover* $\beta : P \to Y^\lambda \to 0$ *satisfies selfdual Dipper's hypothesis.*

Remark 6. *On* ((ii) $\Leftrightarrow$ (iii))*: Written in* [Gre80] *and* [Don98]*.* ((ii) $\Rightarrow$ (i)) *follows from* [Gre7] *(cf. Cabanes*[CE04]*). The others I couldn't find in the literatures.*

0.4. **De Visscher-Donkin.** The main reference of this section is [DVD05]. We omit the detail, see [DVD05] for the detail. De Visscher and Donkin asked the question "When do we get an indecomposable projective injective tilting module $T_n(\mu) \cong I_n(\lambda)$?". They defined two subsets Λ_n^i and Λ_n^t of polynomial dominant weights of $\mathfrak{gl}_n$ such that there exists a bijection $d : \Lambda_n^i \cong \Lambda_n^t$ and $I_n(\lambda) \cong T_n(d(\lambda))$. They conjectured that Λ_n^i is the largest subset with respect to the selfdual property.

So, the following conjecture seems to be reasonable.

Conjecture 7. $\mathrm{Top}(Y^\lambda)$ *is simple if and only if there exists* n *such that* $n \geq l(\lambda)$ *and* $I_n(\lambda)$ *is projective. More strongly,* $\mathrm{Top}(Y^\lambda)$ *is simple if and only if there exists* n *such that* $n \geq l(\lambda)$ *and* $\lambda \in \Lambda_n^i$.

Remark 8. *If* $I_n(\lambda)$ *is projective, then* $\mathrm{End}(I_n(\lambda)) \cong \mathrm{End}_{\mathcal{H}(r)}(Y^\lambda)$ *is self-injective, so by theorem 5,* $\mathrm{Top}(Y^\lambda)$ *is simple. The conjecture says that settling De Visscher Donkin conjecture is equivalent to the classification of simple top Young modules.*

0.5. **Simple Specht modules.** In this subsection, we shall introduce a new method to find "many" partitions λ such that S^λ is simple. As in the introduction, the main approach here is to import the self-injective endomorphism ring algebra structure from certain Frobenius categories side such as rational module categories to the other side such as Hecke algebra module categories. So, the main argument works for any $U_{q^{1/2}}(\mathfrak{g})$ of finite reductive type. If the reader is interested in the other types see [Jan03, p.463 E.9& p.528, H.15] for the tilting injective rational modules over $U_{q^{1/2}}(\mathfrak{g})$. (However, here in quantum groups unfortunately, we have to assume that the characteristic of $\Bbbk$ is zero since there is no projective object in rational module category in positive characteristic cases. So, to use De Visscher-Donkin has an advantage.)

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we denote by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ the conjugate partition of λ. I couldn't find the following theorem in the literatures.

Theorem 9 (Column Removal Epimorphisms)**.** *Assume that* λ *and* μ *are partitions of* r *and* $\sum_{i=1}^{s} \lambda'_i = \sum_{i=1}^{s} \mu'_i$ *for some* s. *For* $x \in \{\lambda, \mu\}$ *Let* $(x^L)' := (x'_1, x'_2, \ldots, x'_s)$ *and* $(x^R)' := (x'_{s+1}, x'_{s+2}, \ldots)$. *Further assume that for* $X \in \{L, R\}$ *there exists an epimorphism* $\nabla(\lambda^X) \to \nabla(\mu^X) \to 0$. *Then, there exists an epimorphism* $\nabla(\lambda) \to \nabla(\mu) \to 0$.

Remark 10. *The announcement of this theorem was given at a seminar, Leeds organised by R. Rouquier. Actually, similar to this theorem, one can make a stronger statement including an epimorphism of a direct sum of some costandard modules, but for the main use of this report the above theorem is good enough. The proof requires Ringel duality and Hecke algebras. H.H. Andersen suggested to show the vanishing of* $R^i ind(\lambda)$ *to obtain similar results for the other types and the full functorial construction. Interestingly the row removal analogue of this theorem is false (observed by Kai Meng Tan).*

Theorem 11. *Suppose for simplicity that the characteristic of* $\Bbbk$ *is zero. Let* λ *be a partition, at most* n *parts. Put* $\widetilde{\lambda} := \lambda + k(1^n)$ *for some* k. *Suppose that* $\nabla(\lambda)$ *is simple and* $\widetilde{\lambda} \in \Lambda_n^i$. *Then,* $S^{\widetilde{\lambda}}$ *is simple.*

Remark 12. *The condition that* $\nabla(\lambda)$ *is simple is very well known as Carter's criterion. For the proof we use Theorem 9 (which induces epimorphism* $Y^{\widetilde{\lambda}} \to S^{\widetilde{\lambda}} \to 0$*) , De Visscher-Donkin* [DVD05] *that* $I_n(\widetilde{\lambda})$ *is projective (which implies that* $\mathrm{End}(I_n(\lambda))$ *is self-injective via Nakayama functor) and Theorem 5.*

For the final form, we shall apply Brundan-Kleshchev branching theory and Kashiwara crystal base theory [Kas93, Lemma 5.1.1]. To describe this, we denote by E_i (resp. F_i) the i-restriction (resp. i-induction) functor. (See [Kle05],[Gro].)

Theorem 13 (An irreducible criterion). *Suppose that $\widetilde{\lambda}$ satisfies the condition in Theorem 11.*

(1) *Write $F_{i_1}^{(k_1)}F_{i_2}^{(k_2)}\cdots F_{i_m}^{(k_m)}S^{\tilde{\lambda}} \neq 0$ where $F_{i_s}^{(k_s+1)}F_{i_{s-1}}^{(k_s)}\cdots F_{i_m}^{(k_m)}S^{\tilde{\lambda}} = 0$ for $s = 1, \ldots, m$.*
Then, there exists a unique partition μ such that $F_{i_1}^{(k_1)}F_{i_2}^{(k_2)}\cdots F_{i_m}^{(k_m)}S^{\tilde{\lambda}} = S^{\mu}$ and S^{μ} is isomorphic to D^{μ^R}.

(2) *Write $E_{j_1}^{(l_1)}E_{j_2}^{(l_2)}\cdots E_{j_m}^{(l_m)}S^{\tilde{\lambda}} \neq 0$ where $E_{j_s}^{(l_s+1)}E_{j_{s-1}}^{(l_s)}\cdots E_{j_m}^{(l_m)}S^{\tilde{\lambda}} = 0$ for $s = 1, \ldots, m$. Then, there exists a unique partition ν such that $E_{j_1}^{(l_1)}E_{j_2}^{(l_2)}\cdots E_{j_m}^{(l_m)}S^{\tilde{\lambda}} = S^{\nu}$ and S^{ν} is isomorphic to D^{ν^R}.*

Here, α^R is James's e-regularization of α and e is the quantum characteristic associated with q. D^{α} is the simple top of S^{α} for e-regular α.

Remark 14. *For the complete classification of simple Specht modules, M. Fayers and S. Lyle* [FL09] *seemed to work in the complement of the condition Theorem 13. See* [FL09] *for the detail of the current improvement for the classification of simple Specht modules.*

We may import the result on simple Specht modules to the polynomial (rational) module categories as follows:

Lemma 15. *If S^{λ} is simple, then $L := \mathrm{Top}\nabla(\lambda)$ is simple and there is no e-regular composition factor in $\nabla(\lambda)$ except L where e is the quantum characteristic.*

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Involutions, Frobenius-Schur Indicators and Real Subpairs of 2-Blocks

John Murray

Let B be a real 2-block of a finite group G. Then there is a real element $g \in G$ such that g appears with nonzero multiplicity in the block idempotent of B and the central character of B does not vanish on the sum of the G-conjugates of g. The extended centralizer $C^*_G(g)$ of g is the stabilizer of the set $\{g, g^{-1}\}$ under G-conjugation. The conjugacy class of g is said to be a real defect class of B.

Let E be a Sylow 2-subgroup of $C^*_G(g)$ and let $D = E \cap C_G(g)$. Then the pair (D, E) is an extended defect couple of B. If B is principal then $g = 1_G$ and $D = E$ is a Sylow 2-subgroup of G. Otherwise $[E : D] = 2$. Gow (1988) showed that E is determined up to G-conjugacy.

We are interested in the influence of (D, E) on B. For example, we can show that (D, E) determines which B-subpairs are real: First, we can choose a Sylow B-subpair (D, b_D) so that $b_D^{EC(D)}$ is real with defect pair (D, E) (or equivalently $b_D = b_D^{eo}$, where e is such that $E = D\langle e\rangle$). Then a B-subpair (Q, b_Q) is real iff it is conjugate to (R, b_R), where $R \leq D$ and $E = DC_E(R)$. This in turn determines the number of real irreducible characters in B.

The involution module of G arises from the conjugation action of G on its involutions, over a field of characteristic 2. G. R. Robinson (1988) shows that a (2-modular) irreducible module occurs with composition multiplicity $\nu(\Phi)$ in this module, where Φ is the associated projective character. In particular $\nu(\Phi) \geq 0$, something that is not true in odd characteristic.

In (M. 2006) we show that the only projective components of the involution module are real and irreducible, and in particular belong to real 2-blocks of defect zero. Moreover, each irreducible in such a block occurs with multiplicity 1 in the involution module. We have a number of results linking (D, E) and the vertices of the components of the involution module that belong to B. For example, each vertex is contained in $\mathrm{C}_D(t)$, for some involution t such that $E = D\langle t\rangle$.

Now let x be a 2-element of G, let b be a 2-block of $C_G(x)$ such that $b^G = B$, and let θ be an irreducible Brauer character in b. We use the following 'global-local' result of Brauer:

$$\sum_{\chi\in\mathrm{Irr}(B)} \epsilon(\chi)d^{(x)}_{\chi,\theta} = \sum_{\psi\in\mathrm{Irr}(b)} \epsilon(\psi)d^{(x)}_{\psi,\theta}.$$

These methods allow us to enumerate the Frobenius-Schur indicators of the irreducible characters in B, when B has cyclic, Klein-four or dihedral defect group (M. 2008). The answer depends both on the Morita equivalence class of B and on (D, E). We also determine the vertices of the components of the involution module of B. In many cases we can even give composition multiplicities and Loewy series for these components. The other tame blocks could be dealt with similarly.

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Brauer Height Zero Conjecture for 2-blocks of Maximal Defect

Gabriel Navarro and Pham Huu Tiep

Suppose that G is a finite group, p is a prime and B is a p-block of G with defect group D. Richard Brauer's Height Zero Conjecture, one of the main conjectures in the Representation Theory of Finite Groups, states that all irreducible complex characters in B have height zero if and only if D is abelian. In 1984 Brauer's Height Zero Conjecture was proven for p-solvable groups by D. Gluck and T. Wolf [GW1], [GW2] with the "only if" part being extraordinary complicated. In 1988 the "if" implication was reduced to a question on quasisimple groups by T. Berger and R. Knörr [BK]. P. Fong and M. Harris proved the "if" direction of the conjecture for the principal 2-block in [FH]. (In fact, they proved the Broué Conjecture for those blocks.) Now, the recent advances on the McKay conjecture in [IMN], together with the recent and powerful results of M. Broué and J. Michel [BM] on unions of ℓ-blocks, of C. Bonnafé and R. Rouquier [BR] on Morita equivalences, and of course the Deligne-Lusztig theory [L], allow us to handle the full Brauer's Height Zero Conjecture for the 2-blocks of maximal defect.

Theorem A. *Let B be a 2-block of G with defect group $P \in \mathrm{Syl}_2(G)$. Then $\chi(1)$ is odd for all $\chi \in \mathrm{Irr}(B)$ if and only if P is abelian.*

Almost all of this work is devoted to prove the "only if" part of Theorem A, and we use the Classification of Finite Simple Groups. All of these ideas are relevant to the case of odd primes on which we are working now. We also stress that the methods and the ideas used here will definitely help to handle the general Height Zero Conjecture once the results in [IMN] are improved to general blocks to prove the Alperin-McKay conjecture, and the blocks of the quasisimple groups have been classified.

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Equivalences between fusion systems of finite groups of Lie type

Bob Oliver

(joint work with Carles Broto, Jesper M. Møller)

For any prime p and any finite group G, let $\mathcal{F}_p(G)$ denote the *p-fusion category* of G. This is the category whose objects are the p-subgroups of G, and where $\mathrm{Mor}_{\mathcal{F}_p(G)}(P,Q)$ is the group of all homomorphisms which are induced by conjugation in G.

When p is a prime and G_1 and G_2 are two finite groups, we say that $\mathcal{F}_p(G_1)$ and $\mathcal{F}_p(G_2)$ are *isotypically equivalent* (written $\mathcal{F}_p(G_1) \simeq \mathcal{F}_p(G_2)$) if there is an equivalence of categories $\mathcal{F}_p(G_1) \to \mathcal{F}_p(G_2)$ which commutes up to natural isomorphism with the forgetful functors from $\mathcal{F}_p(G_i)$ to the category of groups. Equivalently, $\mathcal{F}_p(G_1) \simeq \mathcal{F}_p(G_2)$ if and only if there are Sylow p-subgroups $S_i \in \mathrm{Syl}_p(G_i)$, and an isomorphism $\alpha\colon S_1 \xrightarrow{\cong} S_2$ which is *fusion preserving.* This last condition means that for all $P, Q \le S_1$ and all $\varphi \in \mathrm{Iso}(P,Q)$, φ is conjugation by an element of G_1 if and only if $\alpha\varphi\alpha^{-1} \in \mathrm{Iso}(\alpha(P), \alpha(Q))$ is conjugation by an element of G_2.

The following theorem is the main result in [1]:

Theorem A. *Fix a prime p, a connected reductive integral group scheme $\mathbb{G}$, and a pair of prime powers q and q' both prime to p. Then the following hold.*

(a) *If $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$.*

(b) *If $\mathbb{G}$ is of type A_n, D_n, or E_6, τ is a graph automorphism of $\mathbb{G}$, and $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $\mathcal{F}_p({}^\tau\mathbb{G}(q)) \simeq \mathcal{F}_p({}^\tau\mathbb{G}(q'))$.*

(c) *If $-\mathrm{Id}$ lies in the Weyl group of $\mathbb{G}$, and $\overline{\langle -1, q\rangle} = \overline{\langle -1, q'\rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $\mathcal{F}_p(\mathbb{G}(q)) \simeq \mathcal{F}_p(\mathbb{G}(q'))$ (and similarly for twisted groups).*

(d) *If $\mathbb{G}$ is of type A_n, D_{2m+1}, or E_6, and $\overline{\langle -q\rangle} = \overline{\langle q'\rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$, then $\mathcal{F}_p(\mathbb{G}^-(q)) \simeq \mathcal{F}_p(\mathbb{G}^+(q'))$*

The proof of Theorem A is based on homotopy theory. It seems likely that it can be shown using more algebraic methods, but as far as we can tell, no other proof of the result is currently known.

The starting point in the proof is a theorem by Martino and Priddy [3], that $\mathcal{F}_p(G_1) \simeq \mathcal{F}_p(G_2)$ if the p-completed classifying spaces $BG_1{}_p^\wedge$ and $BG_2{}_p^\wedge$ are homotopy equivalent. This last condition can also be formulated without defining p-completion: for any two spaces X and Y with finite fundamental group, $X_p^\wedge$ and $Y_p^\wedge$ are homotopy equivalent if and only if there is a third space Z and maps $X \longrightarrow Z \longleftarrow Y$ which induce isomorphisms in mod p homology. In fact, the converse to this result (conjectured by Martino and Priddy) is also true: if $\mathcal{F}_p(G_1) \simeq \mathcal{F}_p(G_2)$, then $BG_1{}_p^\wedge \simeq BG_2{}_p^\wedge$ ([4] and [5]). However, the proof of this last statement uses the classification of finite simple groups.

The proof of Theorem A thus depends on comparing the p-completed classifying spaces of different finite groups of the same Lie type. This uses a theorem of Eric Friedlander [2, Theorem 12.2], which describes $B\mathbb{G}(q)_p^\wedge$ as a "homotopy fixed set" of a certain self map of $B\mathbb{G}(\mathbb{C})$. The final piece of input is a theorem which says that under certain hypotheses on a space X, if two homotopy equivalences $\alpha, \beta\colon X \xrightarrow{\simeq} X$ generate the same closed subgroup of the group of all homotopy classes of self equivalences of X, then their homotopy fixed sets are homotopy equivalent [1, Theorem 2.4].

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Control of transfer and gluing problem for fusion systems

Sejong Park

We report two rather separate recent results on fusion systems, obtained during the author's stay in Oberwolfach as an Oberwolfach Leibniz Fellow in 2008–2009.

1. Control of transfer

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. We define two subgroups of P, the *$\mathcal{F}$-focal subgroup*

$$[P, \mathcal{F}] := \langle [Q, \mathrm{Aut}_{\mathcal{F}}(Q)] \mid Q \leq P \rangle$$

and the *$\mathcal{F}$-hyperfocal subgroup*

$$[P, O^p_*(\mathcal{F})] := \langle [Q, O^p(\mathrm{Aut}_{\mathcal{F}}(Q))] \mid Q \leq P \rangle.$$

A fusion subsystem $\mathcal{F}_0$ of $\mathcal{F}$ on $P_0 \leq P$ is said to be *of p-power index* in $\mathcal{F}$ if $P_0 \geq [P, O^p_*(\mathcal{F})]$ and $\mathrm{Aut}_{\mathcal{F}_0}(Q) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(Q))$ for all $Q \leq P_0$. It is a well-known fact [1, 4.3][9, 7.4] that for each $T \leq P$ with $T \geq [P, O^p_*(\mathcal{F})]$, there exists a unique saturated subsystem $\mathcal{F}_T$ of $\mathcal{F}$ on T which is of p-power index. It follows that $\mathcal{F}$ has a proper subsystem of p-power index if and only if $[P, \mathcal{F}] < P$. This generalizes the classical notions of focal and hyperfocal subgroups of finite groups.

The general question of control of transfer is as follows.

Control of Transfer. *For a saturated fusion system $\mathcal{F}$ on a finite p-group P, find $1 \neq W(P) \trianglelefteq P$ such that $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(W(P))]$.*

We present two recent results which generalize classical control of transfer theorems for finite groups due to Glauberman and Yoshida to saturated fusion systems. These are part of an ongoing joint work with A. Díaz, A. Glesser, N. Mazza, and R. Stancu.

Theorem 1.1 ([4, 1.2]; cf. [3, 12.4])**.** *Let $p \geq 5$ and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Then there is a group $K_\infty(P)$ such that $1 \neq K_\infty(P) \trianglelefteq P$ and $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(K_\infty(P))]$.*

Theorem 1.2 (cf. [11, 4.2][5, 10.1])**.** *Let P be a finite p-group with no homomorphic image isomorphic to $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z}$. For any saturated fusion system $\mathcal{F}$ on P, we have $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(P)]$.*

Theorem 1.2 was proved when the author invited Stancu and Díaz to Oberwolfach in 2008. We note that the proof of Theorem 1.2 uses the transfer map of the saturated fusion system $\mathcal{F}$ given by a P-P-biset Ω associated to $\mathcal{F}$.(cf. [2, 5.5]) Write

$$\Omega = \bigsqcup_{i \in I} P \times P / \Delta^{Q_i}_{\varphi_i}$$

where $Q_i \leq P$, $\varphi_i \in \mathrm{Hom}_{\mathcal{F}}(Q_i, P)$, and let A be an abelian group with trivial P-action. Then we define the transfer map

$$t_\Omega \colon H^*(P, A) \to H^*(P, A)$$

by

$$(1) \qquad t_\Omega = \sum_{i\in I} \mathrm{tr}_{Q_i}^P \mathrm{res}_{\varphi_i}$$

where tr and res denotes the usual transfer and restriction maps in group cohomology. The essential observation is that formula (1) can be viewed as a Mackey decomposition formula over the set of "P-P-double coset representatives" $\{\varphi_i \mid i \in I\}$ of the saturated fusion system $\mathcal{F}$.

2. Gluing problem for blocks of finite groups

We fix notation to be used throughout this section. Let G be a finite group, k an algebraically closed field of characteristic $p \mid |G|$, b a block of kG. Fix a maximal b-Brauer pair (P, e_P), and for each $Q \leq P$, let e_Q be the unique block of $kC_G(Q)$ such that $(Q, e_Q) \leq (P, e_P)$. Let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(G, b)$ be the saturated fusion system on P determined by the block b. Then $\mathcal{F}$ is an EI-category and $\mathcal{F}^c$ is a right ideal of $\mathcal{F}$, that is, a full subcategory of $\mathcal{F}$ such that whenever $Q, R \leq P$, $\mathrm{Hom}_{\mathcal{F}}(Q, R) \neq \emptyset$, $Q \in \mathcal{C}$, we have $R \in \mathcal{C}$.

Definition 2.1. Let $\mathcal{C}$ be an EI-category. Let $[\mathcal{C}]$ be the poset of isomorphism classes $[X]$ of objects X of $\mathcal{C}$. Let $S(\mathcal{C})$ be the *subdivision* of $\mathcal{C}$, that is, the category whose objects are the chains

$$\sigma = (X_0 \xrightarrow{\varphi_0} X_2 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} X_n)$$

of nonisomorphisms φ_i in $\mathcal{C}$, and for two chains $\sigma = (X_0 \to \cdots \to X_m)$ and $\tau = (Y_0 \to \cdots \to Y_n)$, the morphisms $\mu = (\mu_i)\colon \sigma \to \tau$ are collections of isomorphisms $\mu_i\colon X_i \to Y_{\alpha(i)}$ in $\mathcal{C}$ which make the obvious diagram commute. Finally, the poset $[S(\mathcal{C})]$ is called the *orbit space* of $\mathcal{C}$.

Linckelmann obtained the following five-term exact sequence using the contractibility of $[S(\mathcal{F}^c)]$.

Theorem 2.2 ([8, 1.1])**.** *There is an exact sequence of abelian groups*

$$0 \to H^1([S(\mathcal{F}^c)], \mathcal{A}^1) \to H^2(\mathcal{F}^c, k^\times) \xrightarrow{d} H^0([S(\mathcal{F}^c)], \mathcal{A}^2)$$
$$\to H^2([S(\mathcal{F}^c)], \mathcal{A}^1) \to H^3(\mathcal{F}^c, k^\times)$$

where $\mathcal{A}^i\colon [S(\mathcal{F}^c)] \to \mathrm{Mod}(\mathbb{Z})$ *is such that* $\mathcal{A}^i([\sigma]) = H^i(\mathrm{Aut}_{S(\mathcal{F}^c)}, k^\times)$*.*

By the work of Külshammer and Puig [6], each block b determines an element $\alpha^0 \in H^0([S(\mathcal{F}^c)], \mathcal{A}^2)$. Now we can state

Gluing Problem. *Find* $\alpha \in H^2(\mathcal{F}^c, k^\times)$ *such that* $d(\alpha) = \alpha^0$ *in the exact sequence of Theorem 2.2.*

If the Gluing Problem has a solution for every block, one can reformulate Alperin's weight conjecture as follows.

Theorem 2.3 ([7, 4.3])**.** *The following are equivalent.*

(1) *Alperin's weight conjecture holds for every block b.*

(2) *For every block b, we have*

$$\mathbf{k}(b) = \sum_{i \geq 0} (-1)^i \dim_k H^i([S(\mathcal{F}^c)], \mathcal{A})$$

where $\mathbf{k}(b)$ denotes the number of ordinary irreducible characters in the block b and $\mathcal{A}\colon [S(\mathcal{F}^c)] \to \mathrm{Mod}(k)$ is a covariant functor such that

$$\mathcal{A}([\sigma]) = \mathrm{Hom}_k(k_\alpha \mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma)/[k_\alpha \mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma), k_\alpha \mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma)], k).$$

Now we solve the Gluing Problem in some special cases.

Proposition 2.4. *Let $p = 2$, P be either a dihedral 2-group of order ≥ 4, a semidihedral 2-group of order ≥ 16, or a (generalized) quaternion 2-group of order ≥ 8. For any saturated fusion system $\mathcal{F}$ on P, we have*

$$H^2(\mathcal{F}^c, k^\times) = H^0([S(\mathcal{F}^c)], \mathcal{A}^2_{\mathcal{F}}) = 0.$$

Proposition 2.5. *Let p be odd, P an extraspecial p-group of order p^3 and exponent p, and $\mathcal{F}$ a saturated fusion system on P. Then we have $H^2(S(\mathcal{F}^c)], \mathcal{A}^1) = 0$. Furthermore, if $\mathcal{F} = \mathcal{F}_P(\mathrm{PSL}_3(\mathbb{F}_p))$, then*

$$H^1([S(\mathcal{F}^c)], \mathcal{A}^1) = \begin{cases} 0, & \text{if } p \not\equiv 1 \mod 3 \\ \mathbb{Z}/3, & \text{if } p \equiv 1 \mod 3. \end{cases}$$

In particular, the Gluing Problem for the principal p-block of $\mathrm{PSL}_3(\mathbb{F}_p)$ has a unique solution if $p \not\equiv 1 \mod 3$, and three solutions if $p \equiv 1 \mod 3$.

For Proposition 2.5, we use detailed fusion information which can be found in [10]. We hope to investigate the meaning of non-unique solutions of the Gluing Problem further in the future.

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On a Jackowski-McClure's theorem

Lluis Puig

Most of our research in the last six years is contained in the book *"Frobenius categories versus Brauer blocks"*, just appeared in the collection "Progress in Mathematics", published by Birkhäuser. Some of its main results already have been announced at Oberwolfach's meetings in 2003 and 2006.

On the one hand, this book introduces and develops the abstract setting of the *Frobenius categories* that we created fifteen years ago for a better understanding of what was loosely called the *local theory* of a finite group around a prime number p or around a *Brauer block*, and for the purpose of an eventual classification — a reasonable concept of *simple Frobenius category* arises.

On the other hand, the book develops in parallel this abstract setting with its application to the *Brauer blocks*, providing a framework for a deeper understanding of one of the central open problems in modular representation theory, namely Alperin's Weight Conjecture (AWC). Actually, this new framework suggests a more general form of AWC, and a significant result of the book is a *reduction theorem* of this form of AWC to quasi-simple groups.

The third part of the book deals with the so-called *localities* associated to a *Frobenius category*, giving some insight on the open question about the existence and the uniqueness of a *perfect locality* — also called *centric linking system* in the literature. A systematic appendix on the cohomology of categories states some useful and more or less known facts.

Carles Broto, Ran Levi and Bob Oliver consider the topological space $\mathfrak{B}(\mathcal{L})$ determined by a *perfect locality* $\mathcal{L}$ — associated with the full subcategory over the $\mathcal{L}$-selfcentralizing objects of a Frobenius P-category $\mathcal{F}$ — and prove that the cohomology group $\mathbb{H}^n(\mathfrak{B}(\mathcal{L}),k)$ of $\mathfrak{B}(\mathcal{L})$ — with coefficients in the prime field k of characteristic p — coincide with the *inverse limit* of the *contravariant* functor mapping any $\mathcal{F}$-selfcentralizing object Q on $\mathbb{H}^n(Q,k)$ and any $\mathcal{F}$-morphism on the corresponding restriction.

As a matter of fact, $\mathbb{H}^n(\mathfrak{B}(\mathcal{L}),k)$ *tautologically* coincides with $\mathbb{H}^n(\mathcal{L},k)$ — the cohomology group of the category $\mathcal{L}$ over the *trivial contravariant* functor — and then a question arises: could the above coincidence be proved *without* any topological consideration?

In our talk, we will show that the answer is in the affirmative *via* the general result below, which generalizes the meaningful part of a sixteenth years old Jackowski-McClure's statement. In our general situation, $\mathfrak{C}$ is a *small* category, $\tilde{\mathfrak{C}}$ is an *exterior quotient* of $\mathfrak{C}$ — the quotient of $\mathfrak{C}$ by a *coherent* family of *inner automorphisms* for the $\mathfrak{C}$-objects — and $\mathfrak{e} : \mathfrak{C} \to \tilde{\mathfrak{C}}$ is the canonical functor; actually, we consider the category $\tilde{\mathfrak{C}}$ augmented by a final object $+$ — denoted by $\tilde{\mathfrak{C}}^+$ — and the corresponding functor $\mathfrak{e}^+ : \mathfrak{C} \to \tilde{\mathfrak{C}}^+$. We set $\mathfrak{e}(\bullet) = \tilde{\bullet}$.

Then, we consider the $\mathfrak{e}^+$*-relative regular representation* of $\tilde{\mathfrak{C}}^+$, namely the functor from $\tilde{\mathfrak{C}}^+$ to the category of small categories $\mathfrak{CC}$

$$\mathfrak{e}^+\text{-}\mathfrak{rr} : \tilde{\mathfrak{C}}^+ \longrightarrow \mathfrak{CC}$$

sending $+$ to $\mathfrak{C}$ and any $\tilde{\mathfrak{C}}$-object $\tilde{A}$ to the small category $\mathfrak{C}_{\tilde{A}}$ where the objects are the pairs $(\tilde{h}, B)$ formed by a $\mathfrak{C}$-object B and a $\tilde{\mathfrak{C}}$-morphism $\tilde{h} : \tilde{B} \to \tilde{A}$, and where the morphisms from $(\tilde{h}, B)$ to a $\mathfrak{C}_{\tilde{A}}$-object $(\tilde{h}', B')$ are the $\mathfrak{C}$-morphisms $g : B \to B'$ fulfilling $\tilde{h}' \circ \tilde{g} = \tilde{h}$; of course, we consider in $\mathfrak{C}_{\tilde{A}}$ the composition induced by the composition in $\mathfrak{C}$; moreover, $\mathfrak{e}^+$-$\mathfrak{rr}$ maps the unique morphism $\tilde{A} \to +$ on the *forgetful* functor, and any $\tilde{\mathfrak{C}}$-morphism $\tilde{f} : \tilde{A} \to \tilde{A}'$ on the evident functor

$$\mathfrak{C}_{\tilde{f}} : \mathfrak{C}_{\tilde{A}} \longrightarrow \mathfrak{C}_{\tilde{A}'}$$

sending $(\tilde{h}, B)$ to $(\tilde{f} \circ \tilde{h}, B)$.

Similarly, we can define an $\mathfrak{e}^+$*-relative pull-back* in $\mathfrak{C}$ of a pair formed by a $\tilde{\mathfrak{C}}^+$-morphism $\tilde{f} : \tilde{B} \to \tilde{A}$ and by a $\mathfrak{C}_{\tilde{A}}$-object $(\tilde{f}', B')$ in the following way; we say that a pair formed by a $\mathfrak{C}_{\tilde{B}}$-object $(\tilde{g}, C)$ and by a $\mathfrak{C}$-morphism $g' : C \to B'$ is an $\mathfrak{e}^+$*-relative pull-back* of $\big(\tilde{f}, (\tilde{f}', B')\big)$ if it fulfills $\tilde{f} \circ \tilde{g} = \tilde{f}' \circ \tilde{g}'$ and, for any pair formed by a $\mathfrak{C}_{\tilde{B}}$-object $(\tilde{h}, D)$ and by a $\mathfrak{C}$-morphism $h' : D \to B'$ fulfilling $\tilde{f} \circ \tilde{h} = \tilde{f}' \circ \tilde{h}'$, there is a unique $\mathfrak{C}_{\tilde{B}}$-morphism $\ell : (\tilde{h}, D) \to (\tilde{g}, C)$ such that $g' \circ \ell = h'$ or, equivalently, there is a unique $\mathfrak{C}$-morphism $\ell : D \to C$ fulfilling the equalities

$$\tilde{g} \circ \tilde{\ell} = \tilde{h} \quad \text{and} \quad g' \circ \ell = h'.$$

The point is that, if any pair $\big(\tilde{f}, (\tilde{f}', B')\big)$ as above in $\mathfrak{C}$ admits an $\mathfrak{e}^+$*-relative pull-back*, then all the functors $\mathfrak{C}_{\tilde{f}} : \mathfrak{C}_{\tilde{A}} \to \mathfrak{C}_{\tilde{A}'}$ above and all the *forgetful* functors $\mathfrak{p}_{\tilde{A}} : \mathfrak{C}_{\tilde{A}} \to \mathfrak{C}$ have a *right adjoint* — noted $(\mathfrak{C}_{\tilde{f}})^{\mathfrak{a}}$ and $(\mathfrak{p}_{\tilde{A}})^{\mathfrak{a}}$ respectively — and we obtain the following result, where $\mathfrak{Ab}$ denotes the category of Abelian groups.

Proposition *With the notation and the hypothesis above, let* $\tilde{A}$ *be a* $\tilde{\mathfrak{C}}$*-object such that* $\tilde{\mathfrak{C}}(\tilde{A}, \tilde{B}) \neq \emptyset$ *for any* $\tilde{\mathfrak{C}}$*-object* $\tilde{B}$, $\mathfrak{m}_{\tilde{A}} : \mathfrak{C}_{\tilde{A}} \to \mathfrak{Ab}$ *a contravariant functor and* $\mathfrak{m} : \mathfrak{C} \to \mathfrak{Ab}$ *a direct summand of* $\mathfrak{m}_{\tilde{A}} \circ (\mathfrak{p}_{\tilde{A}})^{\mathfrak{a}}$. *Then,* $\mathbb{H}^n(\mathfrak{C}, \mathfrak{m})$ *is canonically isomorphic to the inverse limit of the contravariant functor* $\tilde{\mathfrak{C}} \to \mathfrak{Ab}$ *mapping any* $\tilde{\mathfrak{C}}$*-object* $\tilde{B}$ *on* $\mathbb{H}^n(\mathfrak{C}_{\tilde{B}}, \mathfrak{m} \circ \mathfrak{p}_{\tilde{B}})$.

Encoding fusion data in the double Burnside ring

Kári Ragnarsson
(joint work with Radu Stancu)

We discuss how fusion theory over a finite p-group can be encoded in its double Burnside ring via *characteristic elements.* A characteristic element for a fusion system $\mathcal{F}$ on a finite p-group S is an element in $A(S,S)$ or $A(S,S)_{(p)}$ with certain properties, formulated by Linckelmann–Webb, that mimic the properties of a finite group G with Sylow subgroup S when regarded as an (S,S)-biset. The existence of a characteristic element for a saturated fusion system was established by Broto–Levi–Oliver in [2].

Characteristic elements are by no means unique. Indeed, it is easy to show that a given fusion system has either zero or infinitely many characteristic elements. However, specializing to *characteristic idempotents*, that is, idempotents in $A(S,S)_{(p)}$ with the Linckelmann-Webb properties, I showed in [3] that every saturated fusion system $\mathcal{F}$ has a unique characteristic idempotent $\omega_{\mathcal{F}}$. Moreover, I showed that if Ω is a characteristic element or idempotent for $\mathcal{F}$, then $\mathcal{F}$ can be recovered as the *stabilizer fusion system*, defined as the largest fusion system on S that stabilizes Ω. Thus a fusion system is encoded in the double Burnside ring by its characteristic elements.

In recent work, joint with Radu Stancu, we have proved the converse of the aforementioned Broto–Levi–Oliver result, obtaining the following theorem.

Theorem. *If a fusion system has a characteristic idempotent, then it is saturated.*

This shows that the property of saturation can also be detected in the double Burnside ring. A similar result appears in unpublished work of Puig.

More importantly, while the Linckelmann–Webb properties of a characteristic element are defined in terms of the fusion system it characterizes, we give an intrinsic criterion for when an element in $A(S,S)_{(p)}$ is a characteristic idempotent for *some* fusion system. This criterion is the *Frobenius reciprocity relation*

$$(\Omega \times_{\Delta} \Omega) = (\Omega \times_{\Delta} 1) \circ \Omega,$$

where 1 is the unit in $A(S,S)_{(p)}$, $\circ$ is the standard composition of Burnside rings, and for (S,S)-bisets X and Y, $X \times_{\Delta} Y$ is the $(S, S \times S)$-biset obtained by letting S act on $X \times Y$ via the diagonal on one side.

Theorem. *An element in the (p-localized) double Burnside ring is characteristic for its stabilizer fusion system if and only if it satisfies Frobenius reciprocity.*

Combining this result with the work in [3], we obtain a striking result.

Theorem. *For a finite p-group S, saturated fusion systems over S are in bijective correspondence with nonzero idempotents in $A(S,S)_{(p)}$ that satisfy Frobenius reciprocity.*

The correspondence sends a saturated fusion system to its characteristic idempotent, and a Frobenius idempotent to its stabilizer fusion system.

This bijection gives us a completely new way to think of saturated fusion systems. Moreover it opens up new avenues of research, such as interpreting results on fusion systems in terms of characteristic idempotents, or classifying saturated fusion systems via characteristic idempotents. The result also has interesting consequences in algebraic topology, answering questions about stable splittings of classifying spaces, generalizing a variant of the Adams–Wilkerson theorem [1], and providing a tool to extend the results in [4] from elementary abelian p-groups to general p-groups.

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On the graded center of the stable category of a finite p-group

Radu Stancu
(joint work with Markus Linckelmann)

The graded center of a k-linear triangulated category $(\mathcal{C};\Sigma)$ over a commutative ring k is the graded k-module $Z^*(\mathcal{C}) = Z^*(\mathcal{C};\Sigma)$ which in degree $n \in \mathbb{Z}$ consists of all k-linear natural transformations $\varphi : \mathrm{Id}_{\mathcal{C}} \to \Sigma^n$ satisfying $\Sigma\varphi = (-1)^n\varphi\Sigma$; this becomes a graded commutative k-algebra with multiplication essentially induced by composition in $\mathcal{C}$. We refer to [6] for more details. By [2], the stable category $\underline{\mathrm{mod}}(A)$ of finitely generated modules over a finite-dimensional self-injective algebra A is a triangulated category with shift functor the inverse of the Heller operator. Its graded center has been calculated for Brauer tree algebras [4] and in particular also uniserial algebras [5]. These calculations suggest that Tate cohomology rings of blocks and the graded centers of their stable module categories are closely related. Since the Tate cohomology of a block is an invariant of the fusion system of the block, a good understanding of graded centers might shed some light on the question to what extent the fusion system of a block is determined by its stable module category. These calculations also suggest that in order to determine the graded center of the stable module category of a block one will need to do this first for a defect group algebra of the block, hence for finite p-group algebras. This is what motivates the present paper. The following result shows that $Z^0(\underline{\mathrm{mod}}(A))$ need not be finite-dimensional, answering a question raised in [6].

Theorem 1. *Let P be a finite 2-group of rank at least 2 and k an algebraically closed field of characteristic 2. Evaluation at the trivial kP-module induces a surjective homomorphism of graded k-algebras $Z^*(\underline{\mathrm{mod}}(kP)) \longrightarrow \hat{H}^*(P;k)$ whose kernel $\mathcal{I}$ is a nilpotent homogeneous ideal which is infinite-dimensional in each degree; in particular, $Z^0(\underline{\mathrm{mod}}(kP))$ has infinite dimension.*

For odd p we have a slightly weaker statement:

Theorem 2. *Let p be an odd prime, P a finite p-group of rank at least 2 and k an algebraically closed field of characteristic p. Evaluation at the trivial kP-module induces a surjective homomorphism of graded k-algebras $Z^*(\underline{\mathrm{mod}}(kP)) \longrightarrow \hat{H}^*(P;k)$ whose kernel $\mathcal{I}$ is a nilpotent homogeneous ideal which is infinite-dimensional in each odd degree.*

It is easy to see that the canonical map $Z^*(\underline{\mathrm{mod}}(kP)) \to \hat{H}^*(P;k)$ is surjective with nilpotent kernel $\mathcal{I}$. The point of the above theorem is that this kernel tends to have infinite dimension in each degree (if $p = 2$) and at least in each odd degree if $p > 2$. Note though that for p odd there is no known example with $Z^0(\underline{\mathrm{mod}}(kP))$ having infinite dimension. The proof shows more precisely that these dimensions have as lower bound the cardinality of the field k. Technically, the proofs of the above theorems are based on the fact that for A a symmetric algebra, an almost split sequence ending in an indecomposable non projective A-module U determines an *almost vanishing morphism* $\zeta_U : U \to \Omega(U)$ which in turn provides elements of degree -1 in the graded center of the stable category; see e.g. [6, Proposition 1.4]. Using modules with appropriate periods, these can then be "shifted" to all other degrees if the underlying characteristic is 2 and all other odd degrees if the characteristic is odd. The elements of $Z^*(\underline{\mathrm{mod}}(A))$ obtained in this way will be called *almost vanishing.* For Klein four groups, almost split sequences turn out to be the only way to obtain elements in the graded center of its stable module category beyond Tate cohomology. This can be seen using the classification of indecomposable modules over Klein four groups and leads to a slightly more precise statement.

Theorem 3. *Let P be a Klein four group and let k be an algebraically closed field of characteristic 2. Then the evaluation at the trivial kP-module induces a surjective homomorphism of graded k-algebras $Z^*(\underline{\mathrm{mod}}(kP)) \longrightarrow \hat{H}^*(P;k)$ whose kernel $\mathcal{I}$ is a homogeneous ideal which is infinite-dimensional in each degree. Moreover, we have $\mathcal{I}^2 = \{0\}$ and all elements in $\mathcal{I}$ are almost vanishing.*

This raises the question for which finite p-groups P is the graded center $Z^*(\underline{\mathrm{mod}}(kP))$ generated by Tate cohomology and almost vanishing elements. Another interesting question underlying some of the technical details below is the following. Given a periodic module U of period n of a symmetric algebra A over a field k, any isomorphism $\alpha : U \cong \Omega^n(U)$ induces an algebra automorphism of the stable endomorphism algebra $\underline{\mathrm{End}}_A(U)$ sending φ to $\alpha^{-1} \circ \Omega(\varphi) \circ \alpha$. When is this an inner automorphism? Equivalently, when can α be chosen in such a way that this automorphism is the identity? D. J. Benson observed that the answer is positive if α is induced by an element in Tate cohomology (or the Tate analogue of Hochschild cohomology) because then α is the evaluation at U of a natural transformation from the identity functor on $\underline{\mathrm{mod}}(A)$ to the functor Ω^n.

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A proof of Benson's regularity conjecture

Peter Symonds

We present a proof of Dave Benson's conjecture that the Castelnuovo-Mumford regularity of the mod-p cohomology of the cohomology of a profinite group is zero.

We start by discussing some consequences, such as a bound on the degrees of the generators and relations. Then we define regularity and present some of its basic properties. Finally we outline the proof, which uses equivariant cohomology of spaces following Quillen and a result of Duflot on p-toral actions.

Bisets stabilizing a simple module

Jacques Thévenaz
(joint work with Serge Bouc)

Let G be a finite group and k a field.

A kG-module L is said to be *stabilized* by a (G, G)-biset U if $kU \otimes_{kG} L \cong L$. If L is indecomposable, one can assume that U is transitive, hence of the form

$$U \cong \mathrm{Indinf}_{A/B}^{G} \mathrm{Iso}_{\phi} \mathrm{Defres}_{S/T}^{G} ,$$

where (A, B) and (S, T) are sections of G and $\phi : S/T \to A/B$ is an isomorphism. Such a biset is *minimal* if the size of the group A/B is minimal (among stabilizing bisets for L).

If G is a p-group and $k = \mathbb{Q}$, this phenomenon occurs in Bouc's theory of *genetic* subgroups associated to simple $\mathbb{Q}G$-modules (more precisely U has the special form $U = \mathrm{Indinf}_{S/T}^{G} \mathrm{Defres}_{S/T}^{G}$ with $S = N_G(T)$ and T is genetic). The present joint work is a first step towards the goal of generalizing this theory to arbitrary finite groups, arbitrary fields, and arbitrary indecomposable modules.

If U, as above, is a minimal biset stabilizing an indecomposable kG-module L, we prove the following :

1. The section (S, T) is *linked* to some conjugate $({}^{g}A, {}^{g}B)$ of (A, B), that is, we have isomorphisms induced by inclusions

$$({}^{g}A \cap S)/({}^{g}B \cap T) \cong {}^{g}A/{}^{g}B , \qquad ({}^{g}A \cap S)/({}^{g}B \cap T) \cong S/T .$$

2. With this property, the double coset SgA defined by g is unique.

3. If $U' = \mathrm{Indinf}_{A'/B'}^{G} \mathrm{Iso}_{\phi'} \mathrm{Defres}_{S'/T'}^{G}$ is another minimal biset stabilizing L, then the section (S', T') is linked to some conjugate $({}^{h}A, {}^{h}B)$ of (A, B). In particular the isomorphism type of S/T is unique. Moreover the isomorphism type of the corresponding $k[S/T]$-module $\mathrm{Defres}_{S/T}^{G}(L)$ is also unique.

4. If we assume further that L is a simple module, then the group S/T is a *Roquette group*, that is, all its normal abelian subgroups are cyclic.

Module correspondences in Clifford Theory

Alexandre Turull

Clifford Theory is at the heart of many important proofs in the representation theory of finite groups. Recently the author used it to prove a strengthened version of the McKay Conjecture for all p-solvable groups [3]. The strengthening involves, among other things, controlling certain rationality properties of the corresponding characters of p'-degree, involving fields of values and Schur indices. At the heart of the proof is the proof that certain Clifford Theories have to be isomorphic. The present research focuses on the properties of the module correspondences that are implied by this type of isomorphisms among Clifford Theories.

Let $\overline{G}$ be a finite group, and let $\pi_i : G_i \to \overline{G}$ be surjective homomorphisms with $\ker(\pi_i) = H_i$ for $i = 1, 2$. Let $\theta_i \in \mathrm{Irr}(H_i)$. It is well-known that, if θ_i is G_i-invariant, it determines a unique element $\alpha_i \in H^2(\overline{G}, \mathbf{C}^\times)$. If furthermore $\alpha_1 = \alpha_2$, then the Clifford theories above θ_1 and above θ_2 are *isomorphic*. In this case there is a bijection, with good properties, from the set of irreducible characters of G_1 above θ_1 to the set of irreducible characters of G_2 above θ_2. This bijection is not unique.

We are interested in module properties over fields in any characteristic which are not necessarily algebraically closed, and irreducible modules θ_i which are not necessarily G_i-invariant. For them, a good replacement for the group $H^2(\overline{G}, \mathbf{C}^\times)$ is the Brauer-Clifford group $\mathrm{BrClif}(\overline{G}, Z)$, which is isomorphic to $H^2(\overline{G}, \mathbf{C}^\times)$ in the classical case [1, 2]. If θ_i is an irreducible H_i-module over a field F, then it determines a commutative $\overline{G}$-algebra Z_i over F, and a unique element $\alpha_i \in \mathrm{BrClif}(\overline{G}, Z_i)$. The Clifford theories above θ_1 and above θ_2 are isomorphic if there exists a $\overline{G}$-algebra isomorphism $\beta : Z_1 \to Z_2$, which naturally induces the isomorphism

$$\widehat{\beta} : \mathrm{BrClif}(\overline{G}, Z_1) \to \mathrm{BrClif}(\overline{G}, Z_2),$$

and is such that $\widehat{\beta}(\alpha_1) = \alpha_2$. When this happens, there exists a bijection between the isomorphism classes of modules for subgroups of G_1 and those for subgroups of G_2 for modules over every extension field of the base field F. While this bijection has excellent compatibility properties with respect to subgroups and extensions of fields, it is also not unique.

While the non-uniqueness of these bijections is not a major problem when we want to study modules over fields all in the same characteristic, it is not desirable when we try to relate modules across different characteristics. A unique bijection can be obtained with just the choice of a particular $\overline{G}$-algebra isomorphism, and this unique bijection extends over all field extensions of F. We will discuss how this can be achieved.

Theorem 1. *With the notation and hypotheses as above, there exists a $\overline{G}$-algebra isomorphism $\beta : Z_1 \to Z_2$, which naturally induces the isomorphism*

$$\widehat{\beta} : \mathrm{BrClif}(\overline{G}, Z_1) \to \mathrm{BrClif}(\overline{G}, Z_2),$$

and is such that $\widehat{\beta}(\alpha_1) = \alpha_2$ if and only if there exist M_1 a θ_1-quasi-homogeneous FG_1-module and M_2 a θ_2-quasi-homogeneous FG_2-module, and an isomorphism of $\overline{G}$-algebras over F

$$\iota : \mathrm{End}_{FH_1}(M_1) \to \mathrm{End}_{FH_2}(M_2).$$

From each module M_i, we may construct a large category $\mathcal{C}(M_i)$ of modules for certain subgroups of G_i over field extensions of F. These categories are equivalent to the natural category associated to M_i by Clifford Theory.

Theorem 2. *Let M_i be a G_i-module over F such that $\mathrm{Res}^{G_i}_{H_i}(M_i)$ is completely reducible. Then the category $\mathcal{C}(M_i)$ is equivalent to the category of all modules N for subgroups of G_i that contain H_i over field extensions K of F such that $\mathrm{Res}_{H_i}(N)$ is completely reducible and each of the irreducible submodules of $\mathrm{Res}_{H_i}(N)$ is isomorphic to an irreducible submodule of $\mathrm{Res}^{G_i}_{H_i}(M_i \otimes_F K)$. (However, this equivalence is not unique).*

In view of the previous theorem, we may work, without loss, with the categories $\mathcal{C}(M_i)$. For them ι defines a unique *isomorphism of categories.*

Theorem 3. *For $i = 1, 2$, let M_i be a G_i-module such that $\mathrm{Res}^{G_i}_{H_i}(M_i)$ is completely reducible and let*

$$\iota : \mathrm{End}_{FH_1}(M_1) \to \mathrm{End}_{FH_2}(M_2)$$

be an isomorphism of $\overline{G}$-algebras over F. Then we have determined a unique isomorphism of categories *from $\mathcal{C}(M_1)$ to $\mathcal{C}(M_2)$.*

The above results are, of course, related to Morita equivalences in some cases. Suppose the field F has characteristic p, and the subgroups H_1 and H_2 are p'-groups. Fix K to be some field extension of F. Then the modules in $\mathcal{C}(M_i)$ which are KG_i-modules may be thought of as the modules for a certain algebra Λ_i. When looking at these modules up to isomorphism, the isomorphism of categories induces a Morita equivalence from the category of all modules for Λ_1 to the category of all modules for Λ_2. More precisely, in this situation, our result implies a family of Morita equivalences, one for each field extension of F and subgroup of $\overline{G}$.

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Cohomology rings of small categories and tensor identities

Fei Xu

Let k be a field and $\mathcal{C}$ a small category. In an earlier Oberwolfach report [4], we studied certain homological properties of the category algebra $k\mathbb{C}$ and established a surjective ring homomorphism from the Hochschild cohomology ring

$$\mathrm{HH}^*(k\mathcal{C}) := \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C})$$

to the ordinary cohomology ring

$$\mathrm{H}^*(\mathcal{C}; \underline{k}) := \mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k}) \cong \mathrm{H}^*(B\mathcal{C}, k),$$

where $B\mathcal{C}$ is the classifying space of $\mathcal{C}$ [3]. Here we give an alternative description of that map.

We begin with some general set-up and then specialize to functor categories. Let $(\mathfrak{T}, \otimes, \mathfrak{e}, a, l, r, T, \lambda, \rho)$ be a suspended monoidal category in the sense of Suarez-Alvarez [2], where $- \otimes - : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ is a bifunctor, $\mathfrak{e} \in \mathrm{Ob}\mathfrak{T}$ is the identity with respect to $- \otimes -$, $T : \mathfrak{T} \to \mathfrak{T}$ is an automorphism and the rest are various functors imposing all sorts of compatibilities of operations involving $\otimes, \mathfrak{e}, T$. For simplicity we will often abbreviate the notation to be $(\mathfrak{T}, \otimes, \mathfrak{e})$. In practice, the category $\mathfrak{T}$ will often be a triangulated category and T is the translation. We call $\mathfrak{e}$ the tensor identity of the suspended monoidal category.

Given a suspended monoidal category $(\mathfrak{T}, \otimes, \mathfrak{e})$, we examine the set of endomorphisms of $\mathfrak{e}$, $\mathrm{End}_{\mathfrak{T}}(\mathfrak{e}) := \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathfrak{T}}(\mathfrak{e}, T^p(\mathfrak{e}))$, which is defined and shown by Suarez-Alvarez to be a graded commutative ring. If $f : \mathfrak{e} \to T^p\mathfrak{e}$ and $g : \mathfrak{e} \to T^q\mathfrak{e}$, then $f \cdot g = T^q \circ g : \mathfrak{e} \to T^{p+q}\mathfrak{e}$. Suppose k is a field. When $\mathfrak{T}$ is a k-linear category, $\mathrm{End}_{\mathfrak{T}}(\mathfrak{e})$ is a graded commutative k-algebra.

We are mainly interested in the functor category $Vect_k^{\mathcal{E}}$ where $\mathcal{E}$ is a small category and $Vect_k$ is the category of k-vector spaces. Mitchell [1] showed there exists a fully faithful functor $Vect_k^{\mathcal{E}} \to k\mathcal{E}$-mod which means functors are modules. Especially when $\mathrm{Ob}\mathcal{E}$ is finite, the functor provides an equivalence of categories. It is often convenient to introduce a $k\mathcal{E}$-module as a functor. For instance, the trivial $k\mathcal{E}$-module $\underline{k}$ is defined as a constant functor which sends each object in $\mathcal{E}$ to the base field k and each morphism to the identity. Our functor category carries a natural tensor product among its objects, as described below, where we give two un-suspended categories equipped with tensor structure:

(1) $(Vect_k^{\mathcal{C}}, \boxtimes, \underline{k})$ where $\boxtimes$ is the point-wise tensor product over k of functors;
(2) $(A^e\text{-mod}, \otimes_A, A)$ where $\otimes_A$ is the usual tensor product of bimodules.

These categories lead to two suspended monoidal categories, in which the functor T we mentioned before is the suspension Σ:

(1) $(D^-(Vect_k^{\mathcal{C}}), \boxtimes, \underline{k})$ is a suspended monoidal category;
(2) $(D^-(A^e\text{-mod}), \otimes^{\mathbb{L}}_A, A)$ is a suspended monoidal category, where $\otimes^{\mathbb{L}}_A$ is the left derived functor of $\otimes_A$.

We note that when $A = k\mathcal{C}$ for a category $\mathcal{C}$, one may define two tensor structures on $D^-((k\mathcal{C})^e\text{-mod})$. The reason is that $(k\mathcal{C})^e \cong k\mathcal{C}^e$, where $\mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op}$.

The first tensor structure is given as in (1) with tensor product $\boxtimes$ and tensor identity $\underline{k} \in k\mathcal{C}^e$-mod, while the second one is given as in (2) with tensor product $\otimes^{\mathbb{L}}_{k\mathcal{C}}$ and tensor identity $k\mathcal{C}$.

The endomorphism ring of the tensor identity in each case is as follows.

(1) $\mathrm{End}_{D^-(k\mathcal{C}-mod)}(\underline{k}) \cong \mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k}) \cong \mathrm{H}^*(B\mathcal{C}, k)$, the ordinary cohomology ring of $\mathcal{C}$;

(2) We have two tensor structures on $D^-(k\mathcal{C}^e\text{-mod})$. With respect to $\boxtimes$ and $\underline{k}$ we have $\mathrm{End}^{\boxtimes}_{D^-(k\mathcal{C}^e-mod)}(\underline{k}) \cong \mathrm{Ext}^*_{k\mathcal{C}^e}(\underline{k}, \underline{k}) \cong \mathrm{H}^*(B\mathcal{C}^e, k)$. However with respect to the tensor product $\otimes^{\mathbb{L}}_{k\mathcal{C}}$ and tensor identity $k\mathcal{C}$ we just get the Hochschild cohomology ring $\mathrm{End}_{D^-(k\mathcal{C}^e-mod)}(k\mathcal{C}) \cong \mathrm{HH}^*(k\mathcal{C}) = \mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C})$. We will examine the second ring structure.

Let A be an algebra. Take $M \in A$-mod. It is well known that there is a ring homomorphism $\mathrm{Ext}^*_{A^e}(A, A) \to \mathrm{Ext}^*_A(M, M)$, induced by $- \otimes_A M$. If we fix $A = k\mathcal{C}$ and $M = \underline{k}$, the above map gives rise to a ring homomorphism from the Hochschild cohomology ring $\mathrm{HH}^*(k\mathcal{C})$ to the ordinary cohomology ring $\mathrm{H}^*(\mathcal{C}; \underline{k})$. We want to describe this map using the language of suspended monoidal categories (compare with [3, 4]).

Suppose $F(\mathcal{C})$ is the category of factorizations in $\mathcal{C}$ [3]. The objects in $F(\mathcal{C})$ are the morphisms in $\mathcal{C}$, and we have the following commutative diagram

$$\begin{array}{ccc} F(\mathcal{C}) & \xrightarrow{\ \tau\ } & \mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op} \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle pr} & \\ & \mathcal{C} & \end{array},$$

where pr is the projection onto the first component, t maps an object in $F(\mathcal{C})$ (that is, a morphism in $\mathcal{C}$) to its terminal, and τ takes an object in $F(\mathcal{C})$ to the pair of source and terminal of it as a morphism in $\mathcal{C}$. As we have seen in [3], the functors readily induce a commutative diagram of functor categories (for simplicity, we write them in the forms of module categories following Mitchell's observation)

$$\begin{array}{ccc} kF(\mathcal{C})\text{-mod} & \xrightarrow{\ LK_\tau\ } & k\mathcal{C}^e\text{-mod} \\ & {\scriptstyle LK_t}\searrow \quad \swarrow{\scriptstyle LK_{pr}} & \\ & k\mathcal{C}\text{-mod} & \end{array}$$

where LK_* are the corresponding left Kan extensions.

Since the left Kan extensions are right exact, we obtain a commutative diagram of derived categories

$$\begin{array}{ccc} D^-(kF(\mathcal{C})\text{-mod}) & \xrightarrow{\ \mathbb{LK}_\tau\ } & D^-(k\mathcal{C}^e\text{-mod}) \\ & {\scriptstyle \mathbb{LK}_t}\searrow \quad \swarrow{\scriptstyle \mathbb{LK}_{pr}} & \\ & D^-(k\mathcal{C}\text{-mod}) & \end{array},$$

in which the boldface $\mathbb{LK}_*$ are the derived functors of LK_*. This diagram makes sense because $LK_{pr} \circ LK_\tau \cong LK_t$ and each LK_*, as the left adjoint of an exact functor, sends every bounded above exact sequence of projectives to a bounded above exact sequence of projectives. We proved in [3] that LK_τ takes a projective resolution of $\underline{k} \in kF(\mathcal{C})$-mod to a projective resolution of $k\mathcal{C} \in k\mathcal{C}^e$-mod, that LK_t sends a projective resolution of $\underline{k} \in kF(\mathcal{C})$-mod to a projective resolution of $\underline{k} \in k\mathcal{C}$-mod, and that LK_{pr} maps any projective resolution of $k\mathcal{C} \in k\mathcal{C}^e$-mod to an exact sequence of $k\mathcal{C}$-modules whose rightmost non-zero term is $\underline{k}$. Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{D^-(kF(\mathcal{C})-mod)}(\underline{k}) & \xrightarrow{\mathbb{LK}_\tau} & \mathrm{End}_{D^-(k\mathcal{C}^e-mod)}(k\mathcal{C}) \\ & {\scriptstyle \mathbb{LK}_t}\searrow \quad \swarrow{\scriptstyle \mathbb{LK}_{pr}} & \\ & \mathrm{End}_{D^-(k\mathcal{C}-mod)}(\underline{k}) & \end{array} .$$

Since we also showed in [3] that $\mathbb{LK}_t$, induced by LK_t, is an algebra isomorphism, $\mathbb{LK}_{pr}$ must be a split surjection. In other words, the algebra homomorphism induced by LK_{pr} (or equivalently by $- \otimes_{k\mathcal{C}} \underline{k}$), $\mathrm{Ext}^*_{k\mathcal{C}^e}(k\mathcal{C}, k\mathcal{C}) \to \mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k})$ is split surjective.

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