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Computational Electromagnetism and Acoustics

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Abstract. The challenge inherent in the accurate and efficient numerical modeling of wave propagation phenomena is the common grand theme in both computational electromagnetics and acoustics. Many excellent contributions at this Oberwolfach workshop were devoted to this theme and a wide range of numerical techniques and algorithms were mustered to tackle this challenge.

Among these methods boundary integral equation methods received particular attention, both in frequency and time domain. Combined with phase modulation techniques they pave the way for the construction of frequency robust schemes for wave scattering. This is an exciting recent development, which is complemented by the invention of high-order spectral integral equation methods. In parallel, we also witnessed soaring interest in time-domain integral equation methods. The alternative volume based schemes for wave propagation problems can also boast innovations, most notably in the context of plane wave methods.

Though wave propagation is paramount, several other important aspects were addressed during the workshop ranging, among others, from the design of novel Galerkin schemes, asymptotic methods for geometrically singularly perturbed problems, analysis of spectral edge elements, to the design of preconditioners for quasi-static electromagnetics.

Mathematics Subject Classification (2000): 65Mxx, 65Nxx, 65Rxx, 78-04.

Introduction by the Organisers

This is the report about the third and last in a series of three Oberwolfach workshops on Computational Electromagnetics and Acoustics. The earlier events were held in 2004 and 2007. Again, the 2010 workshop attracted a distinguished

group of researchers in the field, the bulk of them with a background in numerical analysis, but also a considerable number of participants from engineering. A sizable number of PhD students and postdoctoral researchers attended the workshop and many of them were given the opportunity to present their work.

The workshop comprised a total of 28 presentations, six of which were meant to provide surveys about specific aspects and methods. The topics covered in the survey talks were

- *asymptotic analysis for wave propagation problems* by X. Claeys, page 438,
- *analysis of p-version edge element approximation of the Maxwell eigenvalue problem* by M. Costabel, page 443,
- *spectral-Galerkin surface integral methods for three Dimensional electromagnetic scattering* by M. Ganesh, page 449,
- *radiation boundary conditions for time-domain scattering problems* by T. Hagstrom, page 453,
- *plane wave discontinuous Galerkin methods* by I. Perugia, page 485, and
- *isogeometric analysis in electromagnetism* by R. Vazquez, page 506.

Looking back, the field of computational electromagnetics and acoustics has experienced rapid development in the past decade, has emerged as a core subject in numerical analysis, and, in the process, has matured as ever deeper and more comprehensive insights have been gained. With three workshops spanning six years we may try and trace some aspects of this evolution of the field.

An area that has reached a fairly mature state is the theory of "mimetic discretization" of the electromagnetic field equations based on co-chains and discrete differential forms. A final breakthrough was made by proving discrete compactness for spectral edge elements (see the extended abstract by M. Costabel on page 443), This closes a chapter in numerical analysis. The construction of fast solvers for low-frequency electromagnetic field problems has reached a similar maturity. With qualification, also the theoretical analysis of discontinuous Galerkin (DG) methods could successfully resolve most open problems in both time and frequency domain. Meanwhile, DG methods have become commonplace in computational engineering. The investigation of absorbing boundary conditions has also seen substantial progress (we refer to the abstract by T. Hagstrom on page 453) and many techniques have become standard tools. Of course, in all these areas, isolated open problems and specialized settings are still await further exploration.

The workshops saw attention shifting to the treatment of wave propagation problems at medium to high frequencies. Phase modulation techniques combined with boundary integral equation methods hold exciting promises (see the contribution by S. Langdon on page 472) as well as spectral methods (*cf.* the abstracts by M. Ganesh, page 449 and O. Bruno, page 434). Once obscure, time-domain integral equation methods have risen to prominence (see the articles by L. Banjai, page 421, J. Rodríguez, page 489, and F. Sayas, page 492) and we may see many more theoretical results and more algorithmic improvements in the near future. This list cannot be complete and the following abstracts will reveal many more promising directions for future research.

We also point out that the field of computational electromagnetics and acoustics is too broad to be covered by a single workshop. Important topics that were not represented by speakers include inverse scattering problems, multi-physics settings like plasmas, non-linear materials, and optimization. Thus computational electromagnetics and acoustics will continue to be an area of brisk and creative mathematical research, with an abundance of challenging questions and beautiful answers.

Workshop: Computational Electromagnetism and Acoustics

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Abstracts

Runge-Kutta convolution quadrature for acoustic scattering

Lehel Banjai
(joint work with Christian Lubich)

In this short report we discuss the efficient discretization of time domain boundary integral operators of acoustics and the solution of the arising discrete systems. The time domain boundary integral operators in question are all given in the form of a time convolution

$$K(\partial_t)g := \int_0^t k(t-\tau)g(\tau)d\tau, \tag{1}$$

The difficulty in computing such convolutions comes from the fact that the kernel $k(t)$ is always distributional and in many cases of practical interest, e.g., viscoelastodynamics and poroelastodynamics [10], even not known explicitly. However the Laplace transform of the kernel

$$K(s) := (\mathscr{L}k)(s) = \int_0^\infty k(t)e^{-st}dt \tag{2}$$

is always explicitly known and simpler. For this reason it is essential to be able to compute (1) by using only the Laplace transformed kernel $K(s)$. To make this dependence on the Laplace transformed kernel explicit, the operational calculus notation $K(\partial_t)g$ is used. The rationale behind this notation comes form the fact that for $K(s) = s$, $K(\partial_t)g = g'$, if $g(0) = 0$.

Convolution quadrature, introduced by Christian Lubich in 1988 for sectorial operators and extended to non-sectorial operators in 1994 [7], is a method to discretize time convolutions in time (!), but by using only the Laplace transformed kernel $K(s)$. The use of merely the operator $K(s)$ is the main reason for our interest in convolution quadratures. Other good methods exist for acoustics and electromagnetism that work purely in the time domain [1, 5, 9], these are however not as straightforward to extend to more complicated situtations such as dissipative media and above mentioned viscoelastodynamics and poroelastodynamics.

We introduce and give convergence results given for convolution quadrature by making the following assumption on the operator $K(s)$:

$$\begin{gathered} K(s) \text{ is analytic for } \operatorname{Re} s > 0 \text{ and bounded as} \\ |K(s)| \le C(\sigma_0)|s|^\mu, \text{ for } \operatorname{Re} s \ge \sigma_0 > 0. \end{gathered} \tag{3}$$

To make the connection to the time domain boundary integral operators let us note that single layer operator for the 3D wave equation with wave speed c is given by

$$\left(V(\partial_t)\varphi\right)(x,t) = \int_0^t \int_\Gamma \frac{\delta(t-\tau-|x-y|/c)}{4\pi|x-y|}\varphi(y,t)d\Gamma_y dt, \quad x \in \Gamma,$$

where V is the single layer operator in the Laplace domain:

$$(V(s)\phi)(x) := \int_\Gamma \frac{e^{-s\frac{|x-y|}{c}}}{4\pi|x-y|}\phi(y)d\Gamma_y. \tag{4}$$

Runge-Kutta based convolution quadrature. Let a Runge-Kutta method be given by its Butcher tableau $\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$ where $A \in \mathbb{R}^{m\times m}$, $b, c \in \mathbb{R}^m$. A Runge-Kutta method is said to be A-stable if the stability function

$$R(z) = 1 + zb^T(I - zA)^{-1}\mathbf{1} \tag{5}$$

is bounded as

$$|R(z)| \le 1, \quad \text{for } \mathrm{Re}\, z \le 0 \text{ and } I - zA \text{ is non-singular for all } \mathrm{Re}\, z \le 0. \tag{6}$$

To simplify expressions assume further that $b^TA^{-1} = (0, 0, \ldots, 1)$, i.e., that the method is stiffly accurate; this,in turn, implies that $c_m = 1$. For the convergence result we also need the technical assumption $|R(\mathrm{i}y)| < 1$ for $y \in \mathbb{R} \setminus \{0\}$. Radau IIA and Lobatto IIIC are examples of Runge-Kutta methods satisfying all of the above conditions.

In a Runga-Kutta method computations are done not only at the equally spaced points $t_j = j\Delta t$ but also at the stages $t_j + c_l\Delta t$, $l = 1, 2, \ldots, m$. Note that $c_m = 1$ implies $t_j + c_m\Delta t = t_{j+1}$. The Runge-Kutta based convolution quadrature approximation to $u(t_n + c_l\Delta t)$, $l = 1, \ldots, m$, is given by

$$\begin{pmatrix} u_{n1} \\ \vdots \\ u_{nm} \end{pmatrix} = K(\underline{\partial_t}^{\Delta t})g := \sum_{j=0}^{n} W_{n-j}^{\Delta t}(K) \begin{pmatrix} g(t_j + c_1\Delta t) \\ \vdots \\ g(t_j + c_m\Delta t) \end{pmatrix}. \tag{7}$$

Here the matrix convolution weights $W_j^{\Delta t}(K)$ are defined implicitly by

$$K\left(\frac{\Delta(\zeta)}{\Delta t}\right) = \sum_{j=0}^{\infty} W_j^{\Delta t}(K)\zeta^j, \tag{8}$$

with $\Delta(\zeta) = A^{-1} - \zeta A^{-1}\mathbf{1}b^TA^{-1}$, $\mathbf{1} := (1, 1, \ldots, 1)^T$.

If p is the order of the Runge-Kutta method and q is the stage order, then in [3] we could prove that the convergence order of the Runge-Kutta based convolution quadrature of (1) is given by $O(\Delta t^{q+1-\mu} + \Delta t^p)$, if the data g is sufficiently smooth and compatible. Numerical experiments for $K(s) = s^\mu/(1 - e^{-s})$ and the 3-stage Radau IIA based convolution quadrature confirm the convergence order predicted by our theory.

An open problem. In [1] it has been shown that

$$\|V^{-1}(s)\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} \le C(\sigma_0)\frac{|s|^2}{\mathrm{Re}\, s}, \quad \text{for } \mathrm{Re}\, s \ge \sigma_0 > 0. \tag{9}$$

Combining this estimate with our result would suggest a convergence of order $O(\Delta t^{q-1})$ for $V^{-1}(\underline{\partial_t}^{\Delta t})g$. Some preliminary numerical experiments, however, suggest a higher convergence order $O(\Delta t^q)$ both for the sphere $\Gamma = \mathbb{S}^2$ and a more

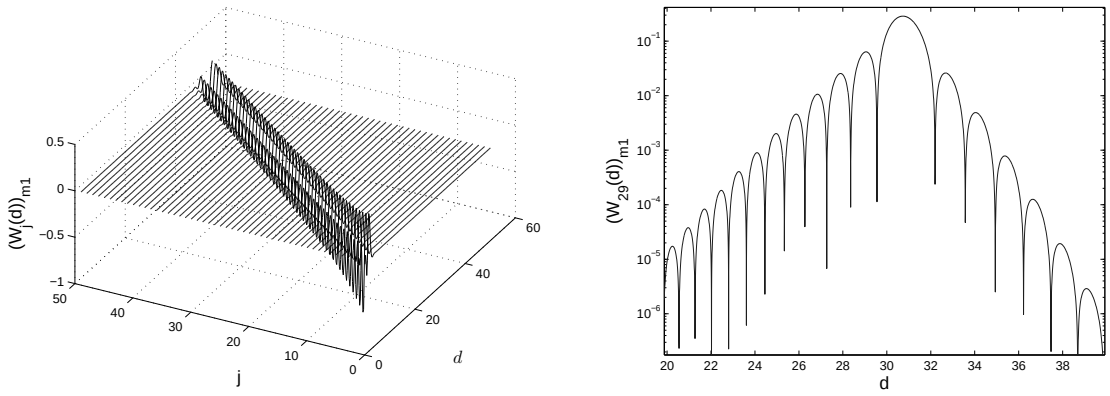

FIGURE 1. Kernel functions $(\widetilde{W_j^{\Delta t}}(d))_{m1}$ for the 3-stage Radau IIA (left) and the semilog plot of $(W_{29}(d))_{m1}$ (right).

complicated non-convex domain. The question then remains whether the estimate (9) is sub-optimal for some/all domains or if the theory should make use of the more complex bound (9) rather than the simpler (3) for which the analysis in [3] has been performed.

The convolution weights $W_j^{\Delta t}(V)$. If V is the single layer potential in the Laplace domain, see (4), then the convolution weights $W_j^{\Delta t}(V)$ have the form

$$(W_j^{\Delta t}(V)\varphi)(x) = \int_\Gamma \frac{\widetilde{W_j^{\Delta t}}\left(\frac{|x-y|}{c\Delta t}\right)}{4\pi|x-y|}\varphi(y)d\Gamma_y,$$

where $\widetilde{W_j^{\Delta t}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times m}$. In Figure 1 we show a plot of $(\widetilde{W_j^{\Delta t}}(d))_{m1}$ for $d \in [0.1, 50]$ and $j = 0, 1, \ldots, 50$. Also a plot of $(W_{29}(d))_{m1}$ is given that displays the exponential decay of this function away from the diagonal $d \approx 30$; this shows that up to exponentially small error the space discretization of the convolution quadrature weights have the familiar, though denser, sparsity pattern of, for example, space-time Galerkin discretizations of the time domain boundary integral operators.

Efficient implementation and numerical experiments. In applications we usually want to find $U_j = (u_{j1}, \cdots, u_{jm})^T$ such that $K(\underline{\partial_t}^{\Delta t})U = G_n$, with $G_n = (g(t_n + c_1\Delta), \cdots g(t_n + c_m\Delta))^T$. In [2] a modification of a recursive procedure of [6] is introduced which allows the solution of this discrete convolution equation without ever constructing the convolution weights. In this method the only operator that needs to be inverted is $W_0^{\Delta t}(V) = V\left(\frac{\Delta(0)}{\Delta t}\right)$. A number of non-trivial 3D numerical experiments have been presented in [2]. These show very good qualitative properties of the Runge-Kutta convolution quadrature in comparison to linear multistep based method.

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Hierarchical Matrix Preconditioners for the Curl-Curl Operator

Mario Bebendorf

(joint work with Jörg Ostrowski)

Solution of boundary value problems involving the curl-curl operator is paramount in computational electromagnetism. An example is the *ungauged vector potential based magnetostatic problem*

$$\mathbf{curl}\frac{1}{\mu}\mathbf{curl}\,\mathbf{u} = \mathbf{j}_0 \quad \text{in } \Omega, \tag{1a}$$

$$\mathbf{u}\times\mathbf{n} = 0 \quad \text{on } \partial\Omega\,, \tag{1b}$$

which we choose as our model problem with given source current $\mathbf{j}_0$. Here, $\mathbf{n}$ is the exterior normal at the boundary $\partial\Omega$ of the computational domain Ω, $\mu \in L^\infty(\Omega)$ with $\mu_0 \le \mu(x) \le \mu_1$ for some constants $\mu_0, \mu_1 \in \mathbb{R}$ is the *magnetic permeability*.

The curl-curl operator has a large kernel. An obvious idea to regularize the "magnetostatic operator" is to add a multiple of the identity. Hence, we consider the operator

$$\mathbf{L}_\alpha := \mathbf{curl}\frac{1}{\mu}\mathbf{curl} + \alpha\mathbf{I} \tag{2}$$

with constant $1/\mu_1 \le \alpha \in \mathbb{R}$ as a preconditioner for the magnetostatic operator $\mathbf{L}_0 = \mathbf{curl}\frac{1}{\mu}\mathbf{curl}$.

One of the most established methods for the iterative solution of electromagnetic problems are multigrid methods; see [6, 1]. Algebraic multigrid methods (AMG)

can be applied if no finite element grid hierarchy is available; see [8], and [3] for an improved version. However, they lack a comprehensive theoretical analysis. A major difference of the method in [8] and the method presented in this talk is that we do not regularize the problem itself. We rather use the regularized operator for generating preconditioners for the original problem (1), while in [8] an approximate solution which depends on the regularization parameter α is computed. See [7] for a preconditioning technique that relies on solvers for the discrete Poisson problem. In this talk we propose the use of hierarchical matrices ($\mathcal{H}$-matrices) [4, 5] due to their efficiency and robustness with respect to non-smooth coefficients in the differential operator.

1. Approximation by hierarchical matrices

Let $A_\alpha \in \mathbb{R}^{n\times n}$ arise from the $\mathbf{H}(\mathbf{curl})$-conforming edge element Galerkin discretization of $\mathbf{L}_\alpha$. The existence of $\mathcal{H}$-matrix approximations to inverse finite element stiffness matrices A_α^{-1} can be proved using the representation

$$(\mathbf{L}_\alpha^{-1}\mathbf{v})(x) = \int_\Omega \mathbf{G}(x,y)\mathbf{v}(y)\,\mathrm{d}y \quad \text{for all } \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega)$$

of the inverse containing the Green function $\mathbf{G}$ of $\mathbf{L}_\alpha$ and Ω. Degenerate approximations of $\mathbf{G}$ lead to low-rank matrices in A_α^{-1}. In [2] the following approximation result for the Green function $\mathbf{G}$ is proved:

Theorem 1. *Let $D_1 \subset \Omega$ and let $D_2 \subset \Omega$ be a convex polyhedron satisfying*

$$\eta \operatorname{dist}(D_1, D_2) \geq \operatorname{diam} D_2$$

for some $\eta > 0$. Then for any $\varepsilon > 0$ there is a separable approximation

$$\mathbf{G}_k(x,y) = \sum_{\ell=1}^{k} \mathbf{u}_\ell(x)\mathbf{v}_\ell(y)^T \quad \textit{with} \quad k \lesssim c_\eta^3 |\log\varepsilon|^4,$$

so that for all $x \in D_1$ we have $\|\mathbf{G}(x,\cdot) - \mathbf{G}_k(x,\cdot)\|_{\mathbf{L}^2(D_2)} \leq \varepsilon \|\mathbf{G}(x,\cdot)\|_{\mathbf{L}^2(\hat{D}_2)}$, where $\hat{D}_2 := \{y \in \Omega : 2\eta \operatorname{dist}(y, D_2) < \operatorname{diam} D_2\}$ and $c_\eta = 2c_A e(1+\eta)$.

2. Preconditioning the Curl-Curl operator

Let $0 = \lambda_1 = \cdots = \lambda_{m-1} < \lambda_m \leq \cdots \leq \lambda_n$ denote the eigenvalues of the symmetric positive semi-definite matrix A_0. Notice that the smallest non-vanishing eigenvalue λ_m is uniformly bounded away from zero. The regularization (2) leads to positive definite coefficient matrices $A_\alpha := A_0 + \alpha M$, where M denotes the mass matrix.

Let $C := L_\mathcal{H} L_\mathcal{H}^T$, $L_\mathcal{H} \in \mathcal{H}(P,k)$, be an approximate Cholesky decomposition of A_α satisfying

$$\|A_\alpha - C\|_2 \leq \varepsilon_\mathcal{H} \|A_\alpha\|_2.$$

Then an appropriate choice of the hierarchical matrix rounding precision $\varepsilon_\mathcal{H}$ guarantees that A_0 and C are spectrally equivalent on the orthogonal complement

$(\ker A_0)^\perp$ of the kernel of A_0. Let $\lambda'_1, \lambda'_n > 0$ denote the smallest and the largest eigenvalue of M. The choice

$$\varepsilon_{\mathcal{H}} := \frac{\alpha\lambda'_1}{2(\alpha\lambda'_n + \lambda_n)}$$

leads to spectral equivalence

$$\frac{1}{2}(1+\alpha\frac{\lambda'_1}{\lambda_n})x^T A_0 x \le x^T C x \le \frac{3}{2}(1+\alpha\frac{\lambda'_n}{\lambda_m})x^T A_0 x, \quad x \in (\ker A)^\perp,$$

of A_0 and C on $(\ker A_0)^\perp$. Note that the spectral equivalence on $(\ker A_0)^\perp$ is sufficient for a bounded number of iterations since the conjugate gradient method suppresses kernel components of the initial vector.

3. Numerical experiments

Numerical experiments were made on the test geometry of Fig. 1. The magnetic permeability jumps between a value of $1.3 \cdot 10^{-6}$ (Vs)/(Am) in the air and the coil to a value of $6.3 \cdot 10^{-4}$ (Vs)/(Am) in the core. The diameter of the coil is 2.45 cm.

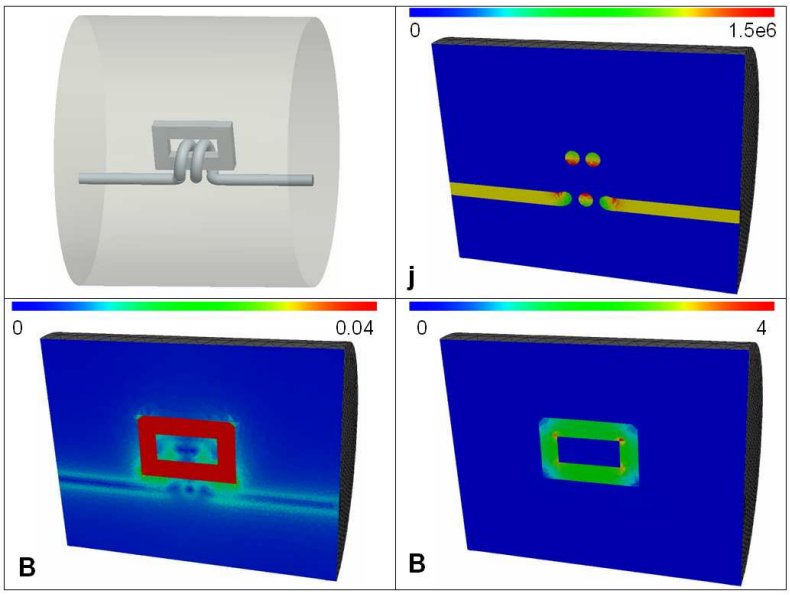

Figure 1. Results of the magnetostatic field computations.

Table 1 contains the time required to set up the hierarchical matrix preconditioner for various problem sizes n and different numbers of processors p, while Table 2 shows the time required for the iterative solution with residual error $\varepsilon_{\mathrm{CG}} = 10^{-4}$. For the parallelization of the methods see [2]. The columns labeled "E_p" contain the parallel efficiency. The results were obtained on a system consisting of two Intel Xeon 5160 processors (dual core, 3 GHz). In all tests we have used a multiple of the identity matrix for regularization, where the regularization parameter α was set to $2\pi/\mu_0$. Notice that the size of the matrix entries

in A is of the order $1/\mu_0$. The rounding accuracy $\varepsilon_{\mathcal{H}}$ of the hierarchical matrix Cholesky factorization was chosen 10^{-2}.

n	non-zeros	partition	$p=1$	$p=2$	E_2	$p=4$	E_4
163 693	2 679 725	3.4s	31.8s	16.1s	98.8%	9.7s	82.0%
297 302	4 884 262	7.2s	65.5s	33.5s	97.8%	17.8s	92.0%
420 881	6 909 745	11.3s	112.3s	59.3s	94.7%	33.9s	82.8%
523 989	8 626 747	14.2s	131.3s	66.4s	98.9%	40.5s	81.0%
664 539	10 921 019	20.0s	181.6s	91.1s	99.7%	50.0s	90.8%
742 470	12 192 476	22.3s	212.4s	115.6s	91.9%	60.3s	88.1%
810 412	13 284 530	25.2s	234.7s	131.2s	89.4%	70.6s	83.1%
955 968	15 715 398	29.5s	273.4s	159.6s	85.7%	82.7s	82.6%

TABLE 1. Cholesky factorization time on $p = 1, 2, 4$ processors.

n	#It	$p=1$	$p=2$	E_2	$p=4$	E_4
163 693	60	17.7s	9.3s	95%	6.2s	71%
297 302	75	42.0s	24.2s	87%	14.6s	72%
420 881	96	81.4s	43.5s	94%	28.8s	71%
523 989	92	93.9s	50.3s	93%	33.6s	70%
664 539	88	118.9s	63.6s	93%	42.2s	70%
742 470	77	117.9s	69.9s	84%	42.1s	70%
810 412	81	134.0s	74.0s	91%	47.3s	71%
955 968	85	163.3s	92.5s	88%	58.4s	70%

TABLE 2. Preconditioned CG solution on $p = 1, 2, 4$ processors.

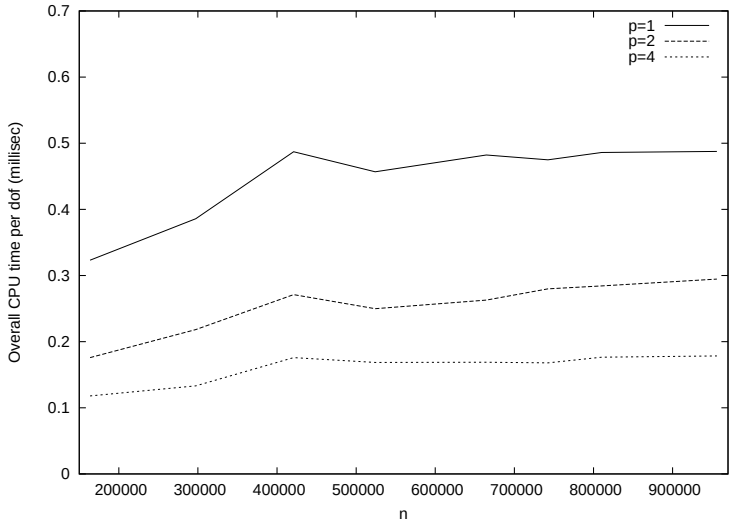

FIGURE 2. Total solution time per dof for $p = 1, 2, 4$ processors.

Apparently, the complexity scales almost linearly and the parallelization of the hierarchical LU factorization algorithm shows a competitive speedup. The number of iterations is bounded independently of n. The major part of the total solution time is used to construct the preconditioner. This ensures a quick computation of the magnetic field in case of varying exciting currents $\mathbf{j}_0$, i.e. for multiple right-hand sides.

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Some Preliminary Results on a Multi-Domain Boundary Element Method

Abderrahmane Bendali

(joint work with Xavier Claeys, Ralf Hiptmair, Carlos Jerez-Hanckes)

We consider the following scattering problem

$$
(1)\qquad \begin{cases} u \in H^1_{\mathrm{loc}}(\mathbb{R}^n) := \left\{ v \in \mathcal{D}'(\mathbb{R}^n);\ \varphi v \in H^1(\mathbb{R}^n),\ \forall \varphi \in \mathcal{D}(\mathbb{R}^n) \right\} \\ u_\ell = u|_{\Omega_\ell},\ \Delta u_\ell + \kappa_\ell^2 u_\ell = 0 \text{ in } \Omega_\ell\ (\ell = 0, \dots, m) \\ \lim_{r\to\infty} r^{(n-1)/2}(\partial_r(u_0 - u^{\mathrm{inc}}) - i\kappa_0(u_0 - u^{\mathrm{inc}})) = 0 \\ \displaystyle\sum_{\ell=0}^{m} \langle \partial_{\mathbf{n}_\ell} u_\ell, v \rangle_{\Gamma_\ell} = 0,\ \forall v \in H^1_{\mathrm{loc}}(\mathbb{R}^n) \end{cases}
$$

where

- $\{\Omega_\ell\}_{\ell=0}^{\ell=m}$ is a non overlapping decomposition of $\mathbb{R}^n$ such that Ω_ℓ $(\ell = 1, \dots, m)$ is a bounded domain,
- κ_ℓ complex constant such that $\mathrm{Re}\,\kappa_\ell > 0$, $\mathrm{Im}\,\kappa_\ell \geq 0$ $(\ell = 0, \dots, m)$,
- $u^{\mathrm{inc}}(x) = \exp(-i\kappa_0 x \cdot \Theta)$ with $\Theta \in \mathbb{R}^n$ such that $|\Theta| = 1$,
- $\Gamma_\ell = \partial\Omega_\ell$ and $\langle \cdot, \cdot \rangle_{\Gamma_\ell}$ is the duality bracket between $H^{-1/2}(\Gamma_\ell)$ and $H^{1/2}(\Gamma_\ell)$,

- $\mathbf{n}_\ell$ is the unit normal to Γ_ℓ (assumed to be at least Lipschitz) outwardly directed to Ω_ℓ.

The usual boundary integral formulations of problem (1) as those considered in [5] lead to a final linear system not well-adapted for an iterative solution. The aim of this study, for which some preliminary results are presented here, is to overcome this flaw.

Let us first present the boundary integral formulation. It has some similarities with the boundary integral domain decomposition introduced in [3] with the main difference that it involves no explicit determination for the Dirichlet-to-Neumann operator. Let us denoted by $\mathbb{H}^{\pm 1/2} := \prod_{\ell=0}^{n} H^{\pm 1/2}(\Gamma_\ell)$

$$X_D := \left\{ \mu \in \mathbb{H}^{1/2};\ \exists v \in H^1_{\mathrm{loc}}(\mathbb{R}^n),\ \mu_\ell = v|_{\Gamma_\ell}\ \ (\ell = 0, \ldots, m) \right\} \tag{2}$$

$$X_N := \left\{ p \in \mathbb{H}^{-1/2};\ \textstyle\sum_{\ell=0}^{m} \langle p_\ell, \mu_\ell \rangle_{\Gamma_\ell} = 0,\ \forall \mu \in X_D \right\}. \tag{3}$$

Clearly, the Cauchy data of problem (1)

$$\lambda = \{u_\ell|_{\Gamma_\ell}\}_{\ell=0}^{\ell=m}, \quad p = \{\partial_{\mathbf{n}_\ell} u_\ell|_{\Gamma_\ell}\}_{\ell=0}^{\ell=m}, \tag{4}$$

satisfy $\lambda \in X_D, \quad p \in X_N$. Using the usual integral representations of the solutions to the Helmholtz equation which either satisfy the radiation condition or are of finite energy for an absorbing ambient medium (cf., e.g., [4]), we can write

$$u_\ell(x) = u_\ell^{\mathrm{inc}}(x) + \Psi_{\mathrm{SL}}^{(\ell)} p_\ell(x) + \Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell(x),\ x \in \Omega_\ell\ (\ell = 0, \ldots, m) \tag{5}$$

where $\Psi_{\mathrm{SL}}^{(\ell)} p_\ell$ and $\Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell$ are the single- and the double-layer potentials created by the densities p_ℓ and λ_ℓ respectively

$$\left\{ \begin{array}{l} \Psi_{\mathrm{SL}}^{(\ell)} p_\ell(x) = \displaystyle\int_{\Gamma_\ell} G(\kappa_\ell, x, y) p_\ell(y) ds_y, \\ \Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell(x) = -\displaystyle\int_{\Gamma_\ell} \partial_{\mathbf{n}_\ell(y)} G(\kappa_\ell, x, y) \lambda_\ell(y) ds_y, \end{array} \right. \qquad (x \notin \Gamma_\ell) \tag{6}$$

The function G_ℓ is the Green kernel explicitly given by

$$G_\ell(\kappa_\ell, x, y) = \left\{ \begin{array}{ll} (i/4) H_0^{(1)}(\kappa_i\, |x - y|), & \text{for } n = 2, \\ \exp(i\kappa_\ell\, |x - y|)/4\pi\, |x - y|\,, & \text{for } n = 3. \end{array} \right. \tag{7}$$

For the convenience in the notation, we have set $u_\ell^{\mathrm{inc}}(x) = 0$ for $\ell \neq 0$ and $u_0^{\mathrm{inc}}(x) = u^{\mathrm{inc}}(x)$. The main feature, on which is based the class of boundary integral formulations presented here, is that, in (5), the densities p_ℓ and λ_ℓ are the Cauchy data of u_ℓ if and only if one succeeds to ensure that the right-hand side in this equation is zero in the complement $\Omega_\ell^c := \mathbb{R}^n \backslash \overline{\Omega_\ell}$ of Ω_ℓ. As a result, we can combine the approaches in [2] for removing spurious solutions due to internal resonances and in [1] for designing formulations which can be efficiently solved by means of an iterative method. Let M_ℓ be a smoothing positive operator compactly

acting from $H^{-1/2}(\Gamma_\ell)$ into $H^{1/2}(\Gamma_\ell)$ satisfying $\langle p_\ell, \overline{M_\ell p_\ell}\rangle_{\Gamma_\ell} > 0$ if $p_\ell \neq 0$. We hence get the class of formulations presented here

$$
(8) \quad \begin{cases} \text{find } (p,\lambda) \in \mathbb{H}^{-1/2} \times X_D, \\ \left\langle q_\ell, \beta \left(\Psi_{\mathrm{SL}}^{(\ell)} p_\ell + \Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell\right)^- + M_\ell \partial_{\mathbf{n}_\ell} \left(\Psi_{\mathrm{SL}}^{(\ell)} p_\ell + \Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell\right)^- \right\rangle_{\Gamma_\ell} \\ \qquad = -\left\langle q_\ell, \beta u_\ell^{\mathrm{inc}} + M_\ell \partial_{\mathbf{n}_\ell} u_\ell^{\mathrm{inc}} \right\rangle_{\Gamma_\ell}, \ \forall q_\ell \in H^{-1/2}(\Gamma_\ell) \ (\ell = 0, \ldots, m), \\ \sum_{\ell=0}^{\ell=m} \langle p_\ell, \mu_\ell \rangle_{\Gamma_\ell} = 0, \ \forall \mu \in X_D. \end{cases}
$$

with β a fixed parameter such that $\operatorname{Im} \beta > 0$.

The proof of the following theorem can be obtained in several ways.

Theorem *Let be given $t \in \mathbb{H}^{-1/2}$ and $\chi \in \mathbb{H}^{1/2}$. The problem*

$$
(9) \quad \begin{cases} \text{find } (p,\lambda) \in \mathbb{H}^{-1/2} \times X_D, \\ \left\langle q_\ell, \beta \left(\Psi_{\mathrm{SL}}^{(\ell)} p_\ell + \Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell\right)^- + M_\ell \partial_{\mathbf{n}_\ell} \left(\Psi_{\mathrm{SL}}^{(\ell)} p_\ell + \Psi_{\mathrm{DL}}^{(\ell)} \lambda_\ell\right)^- \right\rangle_{\Gamma_\ell} \\ \qquad = \langle q_\ell, \chi_\ell \rangle_{\Gamma_\ell}, \ \forall q_\ell \in H^{-1/2}(\Gamma_\ell) \ (\ell = 0, \ldots, m), \\ \sum_{\ell=0}^{\ell=m} \langle p_\ell, \mu_\ell \rangle_{\Gamma_\ell} = \langle t_\ell, \mu_\ell \rangle_{\Gamma_\ell}, \ \forall \mu \in X_D, \end{cases}
$$

admits one and only one solution satisfying the following estimate

$$
(10) \qquad \|(p,\lambda)\|_{\mathbb{H}^{-1/2} \times \mathbb{H}^{1/2}} \leq C \, \|(t,\chi)\|_{\mathbb{H}^{-1/2} \times \mathbb{H}^{1/2}} .
$$

The minus supscript indicates the tace on Γ_ℓ taken from the complement domain of Ω_ℓ.

As regards of the numerical approximation of problem (8), up to now we have partial results concerning only geometries not involving multiple points.

Theorem *If no boundary of the non-overlapping decomposition Γ_j intersects the interior of any domain Ω_ℓ, then, for a sufficiently small mesh size, the nodal continuous linear by element approximation of both the unknown and the testing functions yields a stable and convergent boundary element method for solving problem (8).*

In fact the main advantage of formulation (8) lies in the fact it can be solved by an iterative method, where each iteration is performed by solving a standard combined field integral equation in each domain Ω_ℓ. In a sense, the method boils down to a domain decomposition method. Plot 1 depicting the decrease of the GMRES-residual gives a clear indication on its efficiency.

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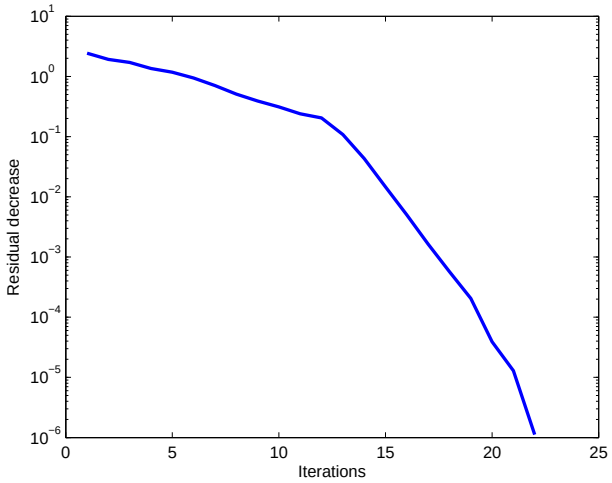

FIGURE 1. Decrease of the residual during GMRES iterations for a scattering problem involving three subdomains.

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Time-harmonic electromagnetism in presence of interfaces between classical materials and metamaterials

ANNE-SOPHIE BONNET-BEN DHIA

(joint work with Lucas Chesnel and Patrick Ciarlet)

In some adequate range of frequencies so-called metamerials display a behaviour which can be described by the usual Maxwell's equations with negative real values of ε and μ [1]. Such negative index materials have very attractive properties: for instance, a slice of metamaterial with $\varepsilon = -1$ and $\mu = -1$ could in principle help to build a perfect lens [2]. Our purpose is the study of mathematical and numerical properties of the time-harmonic Maxwell's equations in a bounded domain Ω which is the union of two subdomains Ω_1 and Ω_2, such that ε and μ take positive values in Ω_1 and negative values in Ω_2. Let us consider for instance a second order formulation for the electric field E:

$$(1) \qquad \begin{cases} \operatorname{curl}\left(\dfrac{1}{\mu}\operatorname{curl} E\right) - \omega^2 \varepsilon E = F & \text{in} \quad \Omega \\ E \times n = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where the source term $F \in L^2(\Omega)^3$ is such that $\operatorname{div} F = 0$. We have to face two difficulties: on one hand, as a consequence of the sign-switch of μ, the operator $\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\right)$ suffers from a lack of coerciveness; on the other hand, due to the

sign-switch of ε, the available results regarding the compactness of the embedding of the space of electric fields

$$V_\varepsilon = H_0(curl)(\Omega) \cap H(div, \varepsilon)(\Omega)$$

in $L^2(\Omega)^3$ do not apply. We discuss these two aspects and prove that well-posedness of (1) can be recovered under restrictive conditions on the contrasts in ε and μ between the two domains.

1. The coerciveness issue

Let us consider first the scalar problem

$$\left\{ \begin{array}{ll} \text{div } \left(\mu^{-1}\nabla u\right) + \omega^2 \varepsilon u = -f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \tag{2}$$

In the case where μ takes a constant value μ_i in each subdomain Ω_i and where the interface Σ between Ω_1 and Ω_2 is regular enough, one can prove [3] using an integral equation on Σ that problem (2) is Fredholm in $H_0^1(\Omega)$ if, and only if, $\kappa_\mu \neq -1$, where $\kappa_\mu = \frac{\mu_1}{\mu_2}$. On the other hand, the problem is ill-posed in $H_0^1(\Omega)$ if Σ has a right angle (see figure 1) and κ_μ lies in I_μ where the critical interval I_μ is defined by $I_\mu = [-3, -1/3]$ (see [4]).

In the general case of L^∞ coefficients (ε, μ and $1/\mu$) and Lipschitz interface Σ, we developed two approaches to prove that problem (2) is Fredholm in $H_0^1(\Omega)$ if one of the following two conditions is fulfilled:

(1) $\kappa_\mu^\uparrow > \kappa_\mu^{sup} \geq -1$ where $\kappa_\mu^\uparrow = \dfrac{\sup_{x\in\Omega_1} \mu_1}{\sup_{x\in\Omega_2} \mu_2}$

(2) $\kappa_\mu^\downarrow < \kappa_\mu^{inf} \leq -1$ where $\kappa_\mu^\downarrow = \dfrac{\inf_{x\in\Omega_1} \mu_1}{\inf_{x\in\Omega_2} \mu_2}$

where the constants κ_μ^{sup} and κ_μ^{inf} depend only on the geometry. The first approach consists in writing an augmented variational formulation of problem (2), whose unknowns are u and $\nabla u_{|\Omega_2}$, which is coercive+compact in $H_0^1(\Omega)$ [5]. The second approach consists in finding an isomorphism T of $H_0^1(\Omega)$ such that the bilinear form $a(u, v) = \int_\Omega \mu^{-1}\nabla u \cdot \nabla(Tv)$ is coercive on $H_0^1(\Omega)$. This allows to establish an inf-sup condition (and in a similar manner a discrete inf-sup condition) for the natural variational formulation of problem (2) (see [6]). The first approach has been extended to Maxwell equations in [7]. The extension of the second approach (T coerciveness) requires restrictive conditions on the regularity of μ and Σ. As a consequence, the convergence of usual finite element schemes for Maxwell equations in the general case of a Lipschitz interface is still an open question.

2. The compactness issue

As mentioned in the introduction, well-posedness of Maxwell's equations with sign-switching ε and μ requires in addition a compactness embedding result. Using the above results, we prove the

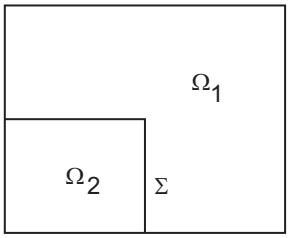

FIGURE 1. The corner case

Theorem 1. *The space* $V_\varepsilon = H_0(curl)(\Omega) \cap H(div, \varepsilon)(\Omega)$ *is compactly embedded in* $L^2(\Omega)^3$ *if*

$$\kappa_\varepsilon^{\uparrow} > \kappa_\varepsilon^{sup} \geq -1 \quad or \quad \kappa_\varepsilon^{\downarrow} < \kappa_\varepsilon^{inf} \leq -1$$

where the constants κ_ε^{sup} *and* κ_ε^{inf} *depend only on the geometry.*

Proof. Following Weber's proof, we consider a bounded sequence E_n of V_ε: classically, there exist bounded sequences $\varphi_n \in H_0^1(\Omega)$ and $\Psi_n \in H(curl)(\Omega) \cap H_0(div)(\Omega)$ such that $E_n = \nabla\varphi_n + \frac{1}{\varepsilon}\operatorname{curl}\Psi_n$. By hypothesis, the sequences $\operatorname{div}(\varepsilon\nabla\varphi_n) \stackrel{def}{=} f_n$ and $\operatorname{curl}\left(\frac{1}{\varepsilon}\operatorname{curl}\Psi_n\right) \stackrel{def}{=} F_n$ are bounded in L^2. The objective is then to find a subsequence such that $\nabla\varphi_n$ and $\operatorname{curl}\Psi_n$ converge in $L^2(\Omega)^3$. Consider for instance the φ_n. By Rellich theorem, there is a subsequence, still denoted by φ_n, which converges in $L^2(\Omega)$. Setting $\varphi_{nm} = \varphi_n - \varphi_m$, we have by linearity:

$$\int_\Omega \varepsilon\nabla\varphi_{nm} \cdot \nabla v = \int_\Omega f_{nm} v \quad \forall v \in H_0^1(\Omega)$$

We conclude by using the T coerciveness: taking $v = T\varphi_{nm}$, we get the Cauchy criterion for $\nabla\varphi_m$ in $L^2(\Omega)^3$. We proceed in the same way for the Ψ_n. □

Summing up, we prove that problem (1) is Fredholm if the following conditions hold:

$$\begin{array}{lll} 1) & \kappa_\mu^{\uparrow} > \kappa_\mu^{sup} \geq -1 \quad \text{or} & \kappa_\mu^{\downarrow} < \kappa_\mu^{inf} \leq -1 \\ 2) & \kappa_\varepsilon^{\uparrow} > \kappa_\varepsilon^{sup} \geq -1 \quad \text{or} & \kappa_\varepsilon^{\downarrow} < \kappa_\varepsilon^{inf} \leq -1 \end{array}$$

which read for constant ε_i and μ_i:

$$\kappa_\mu = \frac{\mu_1}{\mu_2} \notin I_\mu \quad \text{and} \quad \kappa_\varepsilon = \frac{\varepsilon_1}{\varepsilon_2} \notin I_\varepsilon$$

where I_μ and I_ε are critical intervals containing -1.

3. A way to overcome ill-posedness in the 2D corner case

We have investigated the problem of solving (2) when $\kappa_\mu \in I_\mu$, in the case of the geometry of Fig. 1.

For $\kappa_\mu \in \mathbb{C}\backslash I_\mu$ and $f \in L^2(\Omega)$, problem (2) is well-posed (except maybe for discrete real values of ω) and Mellin analysis [4] shows that the solution u is the sum of a regular part u_{reg} which is H^2 on each side of Σ, and a singular part proportional to the singular function S, which behaves near the singular point like

$$S(r, \theta) \sim r^\lambda \phi(\theta)$$

where the singular exponent λ has a strictly positive real part. When κ_μ tends to some limit value in I_μ, this real part tends to 0. The interval I_μ is exactly the interval of values of the contrasts κ_μ such that the singular function S has a pure imaginary singular exponent, so that $S \notin H^1(\Omega)$. More precisely, both the singular function S and the dual singular function S^* have a pure imaginary singular exponent:

$$S(r,\theta) = \overline{S}^*(r,\theta) \sim r^{i\gamma}\phi(\theta)$$

Using Mellin coordinates (t,θ) with $t = -\log r$ near the singular point, we can emphasize that there is a strong analogy between this problem and classical scattering problems in unbounded domains. Singular functions correspond to modes of the strip $\{(t,\theta); t \in \mathbb{R}; \theta \in [0,2\pi]\}$. For $\kappa_\mu \in \mathbb{C}\backslash I_\mu$, all modes are evanescent but for $\kappa_\mu \in I_\mu$, there exist two propagative modes, propagating in opposite directions t and $-t$. The strategy that we suggest is to establish a limiting absorption principle to select the "outgoing" mode, which corresponds to the correct singular behavior. This allows one to solve problem (2) in the space $H^1_0(\Omega) + \{S\}$. In addition classical numerical methods like PMLs could be used to solve (2) with finite elements.

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New methodologies for corner-edge treatment and efficient time evolution

Oscar P. Bruno

This note concerns recent progress in two important problems in numerical analysis and computational science, namely solution of Partial Differential Equations (PDE) with special attention to treatment of geometric singularities and accurate, fast solution of time-dependent problems.

Corner-edge problem. The problem of evaluating numerical solutions of Partial Differential Equations (PDE) under conditions that give rise to solution singularities (such as reduced differentiability and/or blow-up) is one of fundamental importance in science and engineering—yet, a wide variety of such problems have not been adequately addressed from a computational perspective. Here we consider a prototypical problem of this type, namely, solution of the Neumann problem for Laplace's equation in domains containing corners. For this problem, the associated integral-equation solutions, along with the physical fields, tend to infinity at the singular points. The high-order numerical methodology we propose [7], which is based on use of Nyström discretization of second-kind integral equations and exact cancellation of associated blow-up terms, applies to a variety of other problems for which integral formulations exist, including two- and three-dimensional problems concerning potential theory, scattering and diffraction in areas such as electromagnetics, acoustics and solid mechanics. In particular, our method exploits a new expression for the field near edges which makes it possible to avoid the two main difficulties arising in this context by existing Garlerkin approaches, namely, cancellation of infinities and expensive integrations.

A variety of high-order integral equation methods for two- and three-dimensional problems in domains with *smooth boundaries* have been available for some time [19, 3, 13, 23]. Both high- and low-order accurate integral equation methods for *non-smooth domains* have been put forward as well [21, 1, 11, 12, 17, 8, 9, 10, 18]; the references [21, 1] are representative of a significant portion of the (sizable) literature on *high order* integral equation approaches. The approach represented by the contribution [21] relies on use of first kind (singular or hypersingular) integral equations and high-order (Galerkin) boundary element methods; such approaches, which apply both to Dirichlet and Neumann problems and are theoretically sound, require costly evaluation of matrix elements and, for cases in which the integral equation solutions are unbounded, have, in practice, given rise to limited accuracies; see e.g. [16]. Focusing on previous Nyström methodologies, which are closer in spirit to the approach put forward in the present contribution, on the other hand, we mention [20, 2, 14]; the extensive literature in this area is discussed in [1, Chap. 8]. Each of these contributions employs special graded-mesh quadratures to achieve high-order accuracy in the solution of the *Dirichlet problem* in two-dimensional domains with corners by means of second-kind integral equations. A direct extension of this methodology to the Neumann case does not generally give rise to highly accurate solutions. A key theoretical and practical difference between the Dirichlet and Neumann problems in this regard, is that the solutions of the corresponding integral equations are bounded for the former (if the integral equations are selected appropriately), while they are generally unbounded (at corners) for the latter. The contribution [22], which, like [20, 2, 14], is based on use of changes-of-variables and graded meshes, considers solutions of both Dirichlet and Neumann problems and has produced results of significant accuracy. As demonstrated in [7], such an approach does not completely resolve the singular

corner behavior in the Neumann case, and thus it: 1) Cannot yield high accuracies for Neumann problems around corner points, unless expensive, highly-refined integration rules are used to evaluate integrals of the products of basis functions and the very highly peaked composition of the nearly non-integrable kernel and the graded-mesh change-of-variables; and 2) Owing to subtractive cancellations, it leads to diminishing accuracies as discretizations are refined beyond a certain level—yielding limited or no accuracy for problems which, like the sharp-angle problems mentioned below in this text, require fine sampling meshes.

Relying on analytical cancellation of singularities and special treatment of nearly non-integrable integrands, the approach [7] eliminates these difficulties and enables highly efficient high-order Nyström solution of the general *Neumann problems*. In this method, the leading singularity of the solution of the integral equation is treated separately, while the more regular remainder is handled using graded-mesh quadratures, so that cancellations errors are eliminated and high order accuracy is achieved without recourse to highly refined submeshes. Numerical results presented in [7] demonstrate the efficacy of this algorithm through applications to solution of Neumann problems for the Laplace operator over a variety of domains—including domains containing extremely sharp concave and convex corners, with angles as small as $\pi/100$ and as large as $199\pi/100$.

Efficient time-stepping: The FC family of evolution algorithms. We recently introduced a new methodology [4, 5] for the numerical solution of Partial Differential Equations (PDEs) in general spatial domains. This approach is based on use of the well-known Alternating Direction Implicit (ADI) methodology in conjunction with the "Fourier Continuation" method [6] for the resolution of the Gibbs phenomenon. Alternating direction algorithms can yield *unconditional stability* at approximately the same cost per time step as explicit (conditionally stable) finite difference formulations, and have thus been pursued aggressively over the last half century. The application of alternating direction methods has been hindered by a significant limitation however: previous alternating direction approaches could not be directly applied to PDEs on arbitrary (non-rectangular) domains without reducing the truncation error near the boundary to first order [4]. Our Fourier-Continuation Alternating-Direction (FC-AD) methodology [4, 5], in contrast, can produce high-order accuracies with unconditionally stable numerics for general geometries, for any spatial dimensionality, and in essentially linear time—of the order of a spatial Fast Fourier Transform per time-step. A variety of numerical examples presented in [4, 5] demonstrate the unconditional stability and high-order convergence of the proposed algorithm, as well the very significant improvements it can provide over corresponding accuracies and speeds resulting from other methods.

As noted in [4], over the more than fifty years since the introduction of the finite-difference-based ADI algorithm, many variants of this approach have been put forward, including methods for solution of a variety of linear and nonlinear PDEs as well as methods of high-order of spatial and temporal accuracy. As suggested above, previous unconditionally stable alternating-direction methods

can only achieve high-order accuracy in presence of a formulation of the given PDE on domains given by the union of a finite number of rectangular regions. The few unconditionally stable high-order ADI algorithms that have been applied to non-rectangular geometries rely upon domain mappings that translate the given problem into one posed on a rectangular geometry, to which a stable version of the algorithm is applicable. Unfortunately, however, the construction of such domain mappings is prohibitively laborious for most engineering and scientific applications. To the authors' knowledge, unconditionally stable high-order alternating direction algorithms for general domains without some form of domain mapping had not been produced prior to the contributions [4, 5].

Our use of the Fourier basis, which relies on Fourier approximation of non-periodic functions, requires resolution of a classical problem in numerical analysis: the Gibbs phenomenon. Briefly, our "FC(Gram) algorithm" [4, 5] for the resolution of the Gibbs phenomenon is an accelerated version of the "continuation method" [6] for accurate Fourier representation of nonperiodic functions. For a given d-dimensional (open) domain Ω, $d \geq 2$, our FC-AD algorithm uses a Cartesian grid given by the intersection of Ω with a Cartesian grid G in all space. The PDE is discretized in time and then split into sets of un-coupled spatial ODEs by means of an alternating direction technique. To complete the time-stepping algorithm, each one of the resulting spatial ODEs is then solved with high-order accuracy on the corresponding Cartesian line, call it L, by means of the FC(Gram) continuation method and one-dimensional grids in L. The one-dimensional grid in L equals the union of $L \cap G \cap \Omega$ and the set of boundary points $L \cap \partial\Omega$. Since the computational cost required for the FC(Gram) solution of each one of these spatial ODEs is proportional to that of a one dimensional FFT, the overall cost of a full FC-AD time-step is of the order of $\mathcal{O}(N \log(N))$ operations, where N denotes the size of the full d-dimensional spatial grid. As shown in [4, 5], these solvers can be thousands of times faster (and beyond, depending on the size of the problem) than other competing solvers; see e.g. Tables 1, 2 and 3 in the reference [4].

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Overview on a selection of recent works in asymptotic analysis for wave propagation problems

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In recent years, the study of wave propagation has induced a lot of research in relation with asymptotic analysis. While many types of problems may possibly be linked both to asymptotic analysis and scattering theory, the present overview focuses on recent advances concerned with wave propagation problems involving

a small perturbation of geometrical nature.

General issues Consider a well posed wave propagation problem (P_δ) with solution u_δ (called "exact") that depends on a parameter δ satisfying $k\delta << 1$ where k is the wave number. The problem (P_δ) is assumed to be a perturbed version of a limit problem (P_0) from which it differs only in a localised region that is as small as δ in one (or more) direction of space.

Our purpose is to design a numerical method for (P_δ) that would keep track of the influence of the perturbation at a reasonnable computational price. One idea is to find another well posed problem $(\widetilde{\mathrm{P}}_\delta)$ that satisfies two features:

- its dependency with respect to δ is easier to handle numerically,
- its unique solution $\widetilde{u}_\delta$ is a sufficiently accurate approximation of u_δ.

Deriving such an approximate model for (P_δ) is usually not a trivial issue. Suppose that we want to build $(\widetilde{\mathrm{P}}_\delta)$ such that $\|u_\delta - \widetilde{u}_\delta\| = O(\,\eta(\delta)\,)$ with $\eta(\delta) \to 0$. Consider the expansion

$$u_\delta(\mathbf{x}) = u_0(\mathbf{x}) + \lambda_1(\delta)u_1(\mathbf{x}) + \cdots + \lambda_N(\delta)u_N(\mathbf{x}) + O(\,\eta(\delta)\,)$$

One possible approach consists in identifying a (well posed) problem $(\widetilde{\mathrm{P}}_\delta)$ that would be satisfied by $\sum_{n=0}^{N} \lambda_n(\delta)u_n(\mathbf{x})$ up to a residual in $O(\eta(\delta))$, and proving that $\widetilde{u}_\delta$ satisfies a relevant error estimate noting that $\widetilde{u}_\delta = \sum_{n=0}^{N} \lambda_n(\delta)u_n + O(\,\eta(\delta)\,)$.

Although approximate models can be easier to handle, they may contain exotic features which would require specific numerical treatments. Once a numerical method has been proposed for $\widetilde{\mathrm{P}}_\delta$, a typical issue consists in proving that there exists $C > 0$ (independent of δ) and $\mu(h) \to 0$ such that $\|\widetilde{u}_\delta - \widetilde{u}_{\delta,h}\| \leq C\,\mu(h)$ for all $h, \delta \in (0,1)$ where $\widetilde{u}_{\delta,h}$ is the discrete solution.

Thin layer problems In the first category of problem that we examine, the perturbation has the structure of a thin layer of thickness δ located along smooth boundaries or smooth interfaces of the domain of propagation. In such a case the exact solution admits an expansion of the form: $u_\delta = u_0 + \delta u_1 + \delta^2 u_2 + \ldots$. A possible approximate model then consists in posing the wave equation in the limit geometry, as if there were no perturbation, and taking into account the thin layer by means of modified boundary conditions called Generalized Impedance Boundary Conditions (GIBC).

Scattering by objects with thin dielectric coating is a first exemple of such a case. On this subject, Engquist and Nédélec in [15] proposed a GIBC that was much more accurate than any of the modified impedance boundary conditions that had been previously introduced. Since then, a complete theoretical framework for thin coatings has been developped by Bendali, Lemrabet, Joly, Haddar and coworkers in [2, 3, 4, 14, 16], including derivation of full expansions and rigorous justifications of GIBCs of high order. Besides Poignard in [29] studied the case

a high contrasted thin coating, and Chun and Hestaven in [7] validated the good computational efficiency of GIBCs in a discontinuous Galerkin context.

Scattering by a highly absorbing obstacle is another case of thin layer problem. Assume that such an object admits an absorption coefficient $\sigma = k/\delta^2$ where $\delta \to 0$. A wave penetrating such an object keeps a significant amplitude only in a region of thickness δ concentrated along the exterior boundary of the obstacle: this is the skin effect. Whereas thin coating problems had been studied for long, there existed very few works on high absorption problems until recently, except [1, 32]. Haddar, Joly and Nguyen in [17, 18] proposed results comparable to what had been established in the case of thin coatings. Péron, Dauge and co-workers in [6, 27] also proposed a derivation of full expansions for obstacles with Lipschitz boundary (not just smooth) and studied precisely the influence of the geometry on the skin thickness. Finally Haddar and Lechleiter in [19] proposed a similar analysis in the context of scattering by an unbounded obstacle.

The case of a domain of propagation containing a dielectric layer of thickness δ is a third example that caught attention only recently. In [31] Schmidt and Tordeux proposed a full expansion and approximate trasmission conditions precise in $O(\delta^2)$ for a scalar problem. Poignard and Péron in [28, 30] derived an expansion in $O(\delta^3)$ of the exact solution to an electromagnetic scattering problem. Chun, Haddar and Hestaven in [8] formally derived an expansion for a time domain electromagnetic problem and studied computational efficiency of high order GIBCs in a discontinuous Galerkin framework.

Geometric singular perturbation problems In a second part of this presentation we focus on asymptotic problems for which the solutions admit an expansion with terms that may have a singular behaviour related to a singularity appearing in the geometry. In this situation, the exact solution may admit an expansion with more complex structure than for thin layer problems. On this type of asymptotic analysis for elliptic problems, the reference book is [26]. For such cases different asymptotic approaches are possible, we comment on two: multiscale expansion method and matched asymptotics. Clear presentation and comparison of those two approaches was the subject of [13]. Concerning matched asymptotics, a reference book is [20].

Multiscale expansion method and matched asymptotics share several common features. Both techniques involve far field functions expressed in standard variables, and near field functions that depend on scaled variables, and in both cases the expansion is obtained as an interpolation between the near and the far field terms. These methods differ on the cutt-off functions used to interpolate, and on the exact procedure used to construct the far field and near field terms. The multiscale expansion method provides sharper approximations but looks less intrinsic than matched asymptotics. The construction procedure associated to matched asymptotic requires to enforce an algebraic identity called "matching principle".

Several recent works dedicated to problems of this category deserve attention. Joly, Tordeux and co-workers in [12, 21, 22, 23, 24] applied matched asymptotics

and provided a full expansion and approximate models for the propagation of waves in thin slots. This work also brought deep insight on the matching principle for a whole class of problems. Dauge, Vial, Costabel and Caloz in [5, 33] studied a thin coating problem in the case of a domain whose boundary contains an angle. An important conclusion of this work is that the ansatz is directly related to the opening of the angle: this case is much more involved than the case of a smooth boundary. Besides in this situation, whether it is possible to derive high order GIBC remains an open question. The works of Tordeux and Vial have inspired further advances in many other situations including patch antennas, see [25], or diffraction by thin wires, see [9, 10, 11].

Open questions Concerning remaining open questions that we present, we would like to formulate two observations. First, asymptotic problems in a context of time domain scattering have received only few attention so far. Second, theoretical numerical analysis for approximate model in an asymptotic context still remains to be developed for most of the problems already studied from a purely analytical point of view.

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p version edge element approximation of the Maxwell eigenvalue problem

Martin Costabel

(joint work with Daniele Boffi, Monique Dauge, Leszek Demkowicz, Ralf Hiptmair, Alan McIntosh)

In the theory of finite element approximations of Maxwell's equations, it has been known for a long time that the spectrally correct approximation of the *eigenvalue problem* is harder to achieve than the approximation of the *source problem*. The finite element spaces have to satisfy additional requirements that are not always guaranteed. An example is the spectral ("p version") approximation on a single square element in $\mathbb{R}^2$ using $\mathbb{Q}^p$ vector fields, discretizing the variational formulation

$$\text{Find } \omega \neq 0 \text{ and } \boldsymbol{E} \in H_0(\mathbf{curl}, \Omega) \setminus \{0\} \text{ such that}$$
$$\forall \widetilde{\boldsymbol{E}} \in H_0(\mathbf{curl}, \Omega) \ : \quad \int_\Omega \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{E}} = \omega^2 \int_\Omega \boldsymbol{E} \cdot \widetilde{\boldsymbol{E}}$$

In this example, the Maxwell eigenvalues are well approximated, but with wrong multiplicities [8, Section 5.3]. As a consequence, the source problem with a frequency different from the eigenfrequencies is well approximated, but the approximation is not spectrally correct. The additional requirements (which are not satisfied in this example) can be cast in various different forms, a classical form being Kikuchi's *discrete compactness property* [16]. In the recently finished paper [4], we prove the discrete compactness property in a generalized setting of differential forms, which includes 2D and 3D Maxwell equations as special cases and covers several p version finite element approximations of the Maxwell eigenvalue problem where the question of spectrally correct approximation had been open, such as triangular meshes in 2D and tetrahedral and hexahedral meshes in 3D.

This proof uses two main ingredients that have become available only recently: On one hand, a class of projection operators onto the finite element spaces that enjoy certain p-uniform Sobolev estimates and satisfy the commuting diagram property ("cochain projections" for the De Rham complex, in the language of the survey article [2]). These are provided by the projection-based interpolants constructed and analyzed by Demkowicz et al [10, 11, 14]. On the other hand, a regularizing left inverse of the **curl** operator, given by the regularized Poincaré operator analyzed in [9]. This operator is now known to be a pseudodifferential operator of order -1 and has useful mapping properties between polynomial spaces. It belongs to a class of singular integral operators dual to a class generalizing the Bogovskiĭ operator [6, 12, 13, 17].

Embedding the **curl** operator into the De Rham complex and simultaneously embedding the finite element space used for discretizing the energy space $H_0(\mathbf{curl}, \Omega)$ into a discrete complex of finite element spaces has long been a successful recipe for constructing good approximations of Maxwell source and eigenvalue problems. This is an important theme in Hiptmair's survey article [15], and it has been elevated to the status of a dogma in the recent survey articles [1, 2] by Arnold, Falk

and Winther. In [2], a strong case is made for the claim that the right setting for finite element approximations of the Maxwell source problem are discrete subcomplexes of the L^2-based De Rham complex equipped with cochain projections that are uniformly bounded in the graph norm of the exterior derivative operator (for the approximation of $H_0(\mathbf{curl}, \Omega)$, this boundedness would correspond to a uniform bound in the $H(\mathbf{curl})$ norm.) For the eigenvalue problem, the stronger requirement of the availability of uniformly L^2-bounded cochain projections onto the discrete subcomplexes is introduced. In [1, 2], only the h version of the finite element method on simplicial meshes is considered, and for this situation, a constructive proof of the existence of h-uniformly L^2 bounded cochain projections is given.

Their construction is not applicable to the p version of the finite element method, and it is currently an ***open question*** whether p-uniformly L^2 bounded cochain projections on suitable discrete subcomplexes of the De Rham complex exist at all. My current conjecture is that they do *not* exist, in general. A motivation for this pessimistic view is based on the struggle in the papers [5, 3] to prove L^2-stability of a certain projection operator, to be used as a tool in the proof of the discrete compactness property. In [5], which considered the p and hp version on triangular meshes, it was suggested by numerical computations that such a stability would hold with a constant that grows like $\sqrt{p}$, and in [3] it was shown for rectangular meshes that this growth is indeed present. This not-quite-uniform estimate was sufficient to prove discrete compactness – and therefore spectral correctness of the approximation – in these cases, because the growth of the L^2 norm could be compensated by a sufficiently good approximation property in the same norm.

In the abstract framework of the paper [4], the **curl** operator is also embedded into the De Rham complex, and a commuting diagram plays an important role, as well as a piece of a subcomplex of the De Rham complex that serves as domain of definition of the required projection operators. But these projection operators are, as their concrete instances, the projection-based interpolation operators, only defined on some dense subspaces of the energy spaces, described by the graph norm in certain Sobolev spaces more regular than L^2, and they have p-uniformly bounded norms with respect to these smaller spaces.

The discrete compactness property appears at several places in the proof of the spectral correctness of the approximation of the eigenvalue problem. One place is its implication – together with a rather natural approximation property ("Completenes of the discrete kernel", see [7]) – of the discrete Friedrichs inequality (DFI). In its turn, (DFI) is easily seen to be equivalent to what is called the "spurious-free approximation" property in [7], that is, a finite gap between the eigenvalue zero and the rest of the discrete spectrum. In [2], it is shown that (DFI) (called "Poincaré inequality") is equivalent to the existence of uniformly graph-norm bounded cochain projections. The projections thus constructed are similar to those whose L^2-norm stability was studied in [5, 3].

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Convergent scattering algorithms

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Hybrid numerical methods based upon a combination of integral equations and asymptotic theories for the solution of high-frequency scattering problems have found an increased interest within the last two decades. Indeed, the methodologies

developed in this time span, that specifically concern scattering off a single two-dimensional smooth convex obstacle [2, 5, 6, 8], display the capability of predicting scattering returns within any prescribed accuracy utilizing a number of degrees of freedom independent of (or only mildly dependent on) the frequency.

This report concerns (i) the classification of Hörmander classes and asymptotic expansions of multiple scattering iterations for a collection of smooth convex obstacles that thereby allow for the extension of the single-scattering solvers in [5, 6, 8] to multiple-scattering configurations to accompany the algorithm in [3]; and (ii) the derivative estimates of multiple scattering iterations that are necessary for their rigorous numerical analysis and that facilitate the development of convergent scattering algorithms (for each fixed value of the wavenumber k) for the computation of each iterate (utilizing a number of degrees of freedom that depends only mildly on the frequency to attain a prescribed accuracy) based on the ideas in [5].

To present a summary of the relevant results we have recently developed in [1, 7], let us consider the problem of evaluating the scattering of an incident plane wave $u^{\rm inc}(x) = e^{ik\alpha\cdot x}$, $|\alpha| = 1$, from a compact impenetrable obstacle K with a smooth boundary ∂K. Throughout this note we concentrate on two-dimensional configurations wherein the relevant frequency-domain problem is modeled by the Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \backslash K,$$

where the scattered field u is required to satisfy the Sommerfeld radiation condition [4]; here, for definiteness, we shall assume Dirichlet boundary conditions on ∂K.

As is well known, this problem can be restated in the form of an integral equation in a variety of ways [4]; a convenient form for our purposes is that derived from the Green identities resulting in the equation

$$\eta(x) - \int_{\partial K} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta(y)\, ds(y) = 2\, \frac{\partial u^{\rm inc}(x)}{\partial \nu(x)}, \quad x \in \partial K \tag{1}$$

for the unknown density η (the normal derivative of the total field), where $\nu(y)$ denotes the vector normal to ∂K and exterior to K,

$$\Phi(x,y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|)$$

is the *outgoing* Green function, and $G = -2\Phi$. Since the solution of the integral equation (1) is not unique when the wavenumber k is an internal resonance, in practical implementations a "combined field" integral equation formulation must be used [4]. For the sake of simplicity, the derivations that follow, for the description of multiple scattering formulation of the scattering problem, are based upon the integral equation (1).

Let us now further suppose that the sound-soft obstacle K is decomposed into a finite collection of disjoint compact sub-scatterers $K = \bigcup_{\sigma\in\mathcal{I}} K_\sigma$. Then the integral equation (1) can be written as

$$(I - R)\,\eta = f \tag{2}$$

where $\eta(x) = (\eta_{\sigma_1}(x), \ldots, \eta_{\sigma_{|\mathcal{I}|}}(x))^t$ and $f(x) = (f_{\sigma_1}(x), \ldots, f_{\sigma_{|\mathcal{I}|}}(x))^t$ with η_σ and f_σ defined on ∂K_σ and

$$f_\sigma(x) = 2ik\, e^{ik\alpha\cdot x}\, \alpha \cdot \nu(x),$$

and the operator R is defined as

$$(R_{\sigma\tau}\eta_\tau)(x) = \int_{\partial K_\tau} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_\tau(y)\, ds(y), \qquad x \in \partial K_\sigma.$$

Inverting the diagonal part of (2) yields the equivalent relation

$$(I - T)\eta = g \tag{3}$$

with

$$g_\sigma = (I - R_{\sigma\sigma})^{-1} f_\sigma, \qquad \sigma \in \mathcal{I}$$

and

$$T_{\sigma\tau} = \begin{cases} (I - R_{\sigma\sigma})^{-1} R_{\sigma\tau} & \text{if } \sigma \neq \tau \\ 0 & \text{otherwise.} \end{cases}$$

The formulation (3) provides a convenient mechanism to account for multiple scattering since the m-th term in its Neumann series solution

$$\eta = \sum_{m=0}^{\infty} \eta^m = \sum_{m=0}^{\infty} T^m g \tag{4}$$

corresponds to contributions arising as a result of waves that have undergone m reflections. More precisely, we have

$$\eta^m\big|_{\partial K_\sigma} = \sum_{\substack{\tau_0, \cdots \tau_{m-1} \in \mathcal{I} \\ \sigma \neq \tau_{m-1}, \tau_j \neq \tau_{j-1}}} T_{\sigma\tau_{m-1}} T_{\tau_{m-1}\tau_{m-2}} \cdots T_{\tau_1\tau_0} g_{\tau_0}, \tag{5}$$

where each application of a $T_{\sigma\tau}$ entails an evaluation on ∂K_σ of a field generated by a current on ∂K_τ, and its use as an incidence for a subsequent solution of a single-scattering problem on ∂K_σ. Accordingly, equations (4) and (5) guarantee that η can be recovered as the superposition (over all infinite paths $\{K_m\}_{m\geq 0} \subset \{K_\sigma : \sigma \in \mathcal{I}\}$) of multiple scattering iterations η_m that recursively solve the integral equations

$$\eta_0(x) - \int_{\partial K_0} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_0(y)\, ds(y) = 2\, \frac{\partial u^{\mathrm{inc}}(x)}{\partial \nu(x)}\,, \quad x \in \partial K_0$$

and, for $m \geq 1$,

$$\eta_m(x) - \int_{\partial K_m} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_m(y)\, ds(y) = \int_{\partial K_{m-1}} \frac{\partial G(x,y)}{\partial \nu(x)}\, \eta_{m-1}(y)\, ds(y), \quad x \in \partial K_m$$

on the path $\{K_m\}_{m\geq 0}$.

Supposing now that the obstacles K_σ are strictly convex, under certain conditions, the multiple-scattering iterations η_m admit the factorizations

$$\eta_m(x) = e^{ik\varphi_m(x)}\, \eta_m^{\mathrm{slow}}(x), \quad x \in \partial K_m \tag{6}$$

wherein φ_m is the m-th geometrical optics phase, and where the asymptotic properties of the slow envelope η_m^{slow} are as follows (see [1, 7] for details):

Theorem 1 (Hörmander classes and asymptotic expansions of η_m^{slow}, [1, 7]) *The asymptotic characteristics of the slow densities η_m^{slow} specified by (6) are as follows:*
(i) *On the m-th illuminated region ∂K_m^{IL}, $\eta_m^{\text{slow}}(x) = \eta_m^{\text{slow}}(x,k)$ belongs to the Hörmander class $S^1_{1,0}(\partial K_m^{IL} \times (0,\infty))$ and admits the asymptotic expansion*

$$\eta_m^{\text{slow}}(x,k) \sim \sum_{j\geq 0} k^{1-j} a_{m,j}(x)$$

where $a_{m,j}(x)$ are complex-valued C^∞ functions. Accordingly, for any $N \in \mathbb{N}\cup\{0\}$, the difference

$$r_{m,N}(x,k) = \eta_m^{\text{slow}}(x,k) - \sum_{j=0}^{N} k^{1-j} a_{m,j}(x)$$

belongs to $S^{-N}_{1,0}(\partial K_m^{IL} \times (0,\infty))$ and thus satisfies the estimates

$$\left| D_x^\beta D_k^n r_{m,N}(x,k) \right| \leq C_{m,\beta,n,S}(1+k)^{-N-n}$$

on any compact subset S of ∂K_m^{IL} for any multi-index β and $n \in \mathbb{N}\cup\{0\}$.
(ii) *Over the entire boundary ∂K_m, $\eta^{\text{slow}}(x,k)$ belongs to $S^1_{2/3,1/3}(\partial K_m \times (0,\infty))$ and admits the asymptotic expansion*

$$\eta_m^{\text{slow}}(x,k) \sim \sum_{p,q\geq 0} k^{2/3-2p/3-q}\, b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

where $b_{m,p,q}(x)$ are complex-valued C^∞ functions, $Z_m(x)$ is a real-valued C^∞ function that is positive on the illuminated region ∂K_m^{IL}, negative on the shadow region ∂K_m^{SR}, and vanishes precisely to first order on the shadow boundary ∂K_m^{SB}, and the function Ψ is a certain contour integral of an Airy function (see [9]). Note specifically then, for any $P, Q \in \mathbb{N}\cup\{0\}$, the difference

$$R_{m,P,Q}(x,k) = \eta_m^{\text{slow}}(x,k) - \sum_{p,q=0}^{P,Q} k^{2/3-2p/3-q}\, b_{m,p,q}(x) \Psi^{(p)}(k^{1/3} Z_m(x))$$

belongs to $S^{-\mu}_{2/3,1/3}(\partial K_m \times (0,\infty))$, $\mu = \min\{2P/3, Q\}$, and thus satisfies the estimates

$$\left| D_x^\beta D_k^n R_{m,P,Q}(x,k) \right| \leq C_{m,\beta,n}(1+k)^{-\mu-2n/3+|\beta|/3}$$

for any multi-index β and $n \in \mathbb{N}\cup\{0\}$.

As we anticipated, the preceding theorem provides the necessary theoretical background for the extension of the single-scattering solvers [6, 6, 8] to multiple scattering configurations to accompany the algorithm in [3]. As a byproduct, we now present the derivative estimates of the slow envelopes η_m^{slow} that can be directly utilized for the numerical analysis of multiple scattering iterations η_m as is done in [5, 6] for a single convex obstacle.

Theorem 2 (Derivative estimates of η_m^{slow}, [7]) *Let $m \geq 0$, and denote by $y(s) = (y^1(s), y^2(s))$ the arc-length parametrization of ∂K_m. Then, for all $n \in \mathbb{N} \cup \{0\}$, there exist a constant $C_n > 0$ independent of k and s such that for all k sufficiently large,*

$$\left|D_s^n \eta_m^{\text{slow}}(y(s))\right| \leq k \begin{cases} C_n, & n = 0, 1, \\ C_n \left[1 + \sum_{j=2}^{n} k^{(j-1)/3}(1 + k^{1/3}|w(s)|)^{-(j+2)}\right], & n \geq 2, \end{cases}$$

where $w(s) = (s-a)(b-s)$ and $\partial K_m^{SB} = \{y(a), y(b)\}$ is the set of m-th shadow boundary points.

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Spectral-Galerkin Surface Integral Methods for Three Dimensional Electromagnetic Scattering

M. Ganesh

(joint work with S. C. Hawkins)

Many important physical processes involve scattering of electromagnetic waves by ensembles of deterministic and stochastic particles [5, 14, 15, 18, 20, 21, 22, 23, 24].

In particular, for applications such as light scattering in (i) the atmospheric sciences, with configurations consisting of computer models of atmospheric ice crystals and dust particles [12, 15, 18] with rough non-convex surfaces with unique stochastic description; and (ii) medical diagnostics [5, 14, 20, 22, 23, 24] involving, for example, several red blood cells; and in several other classes of wave propagation problems [21], it is efficient to develop algorithms that directly incorporate

local mapping properties of each obstacle in the configuration and use such mappings to reduce the computational complexity.

The spectral methods are one of three major classes of schemes in computational science. Three dimensional spectral-Galerkin algorithms critically depend on application-specific geometric properties for rapid high-order convergence (and hence reduced degrees of freedom) in computations compared to the other two major classes of grid based low-order finite-difference and boundary/finite-element methods.

In this short note, we discuss high-order spectral-Galerkin boundary integral algorithms with specific focus on simulating the scattering of electromagnetic waves by a collection of disjoint three-dimensional obstacles $D_1, D_2, \ldots, D_J$ situated in a homogeneous medium.

The general surface coordinates, in deterministic and stochastic computer modeling of various particles (described in the above applications and references) and in several other models, can be written locally in spherical coordinates as $(x, y, z) = (q_1^j(\theta, \phi), q_2^j(\theta, \phi), q_3^j(\theta, \phi))$, where (x, y, z) is a point on the surface of the j-th particle in the configuration, and (θ, ϕ) are the polar and azimuth angles of a point on the unit sphere S^2, leading to a diffeomorphic local mapping $\boldsymbol{q}_j : S^2 \to \partial D_j,\ j = 1, \ldots, J$, for the surface ∂D_j of the j-th obstacle D_j in the configuration. The local coordinate mappings q_1^j, q_2^j, q_3^j may be given by Fourier series, based on deterministic or stochastic descriptions of the particle surfaces, such as those described in [5, 14, 15, 18, 20, 21, 22, 23, 24].

Spectral boundary integral algorithms in three dimensions were investigated for potential and elasticity problems in [1, 2, 6, 7], for acoustic scattering in [4, 8, 13, 19] and for electromagnetic scattering in [9, 10, 11, 12, 16]. Naive generalization of spectrally accurate three dimensional acoustic scattering algorithms to electromagnetic scattering may destroy the advantages of the spectral algorithm. In particular, only quadratic convergence was obtained for electromagnetic scattering in [16], despite exploiting the mapping properties described above.

Spectrally accurate spectral-Galerkin surface integral algorithms for electromagnetic scattering for single and multiple particles were developed recently by the authors in [11, 12]. Due to the spectral accuracy of the algorithms, fewer unknowns are required to solve a given problem to a prescribed accuracy than low order boundary element methods. In particular, compared to industrial standard algorithms developed about a decade ago, our new algorithms require only 2 to 5% as many unknowns for particles that can be locally represented using the mapping properties described above. For example, the industrial standard Fast Illinois Solver Code [17] requires 2 408 448 unknowns to simulate scattering by a sphere of diameter 48 wavelengths. In contrast, our spectral algorithm requires only 48 670 unknowns to obtain similar accuracy [11]. Further increasing the number of unknowns by between one and three percent leads to very high-order accuracy in our spectral-Galerkin algorithms [11].

Several steps are crucial in the development of spectrally accurate multiple electromagnetic scattering algorithms, as described in [12].

The first step is the approximation of tangential surface currents on the surface of each scatterer in the ensemble, using non-polynomial basis functions. In this part of the algorithm, first the local spherical coordinate mappings are exploited, allowing use of a well-known high-order tangential basis on the fixed reference surface spanned by polynomials of degree, say not greater than n. Then, for the j-th connected surface, we develop an efficient transformation $\mathcal{F}_j$ with the property that $\mathcal{F}_j(\widehat{\boldsymbol{x}})\boldsymbol{Z}(\widehat{\boldsymbol{x}})$ is tangential at any point $\boldsymbol{q}_j(\widehat{\boldsymbol{x}})$ on ∂D_j whenever $\boldsymbol{Z}$ is a tangential function on the reference surface. In addition, we proved the $\mathcal{O}(n^{-r})$ best approximation property of the non-polynomial tangential basis, where r is the maximum smoothness of the particles in the configuration.

The second step, which is equally important as the first, is the spectrally accurate discretization of the several surface integral operators acting on the surface current in the multiple ensemble configuration, by analytically resolving singularities and approximating integrands in the J^2 surface integrals that describe interaction of the waves between any two particles in the configuration. Such approximations, as described in [12], also retain the spectral $\mathcal{O}(n^{-r})$ accuracy.

Combining these with several other technical details, due to the spectral accuracy, both direct and iterative surface decomposition approaches can be used for electromagnetic scattering simulations such as those given in the figure [12]. In addition to mathematically proving spectral accuracy of our algorithms, excellent comparison of simulation results with several experimentally measured radar cross section values (such as those in [3, 21]) have been established in [9, 10, 11, 12].

In the following table we demonstrate the high-order convergence of our algorithm by tabulating the accuracy of the backscattered RCS for a cluster of $J = 16$ stochastic ice crystal and dust atmospheric particles created by an incident wave with frequency ω. The relative errors reported below are obtained by matching digits of backscattered RCS values as the number of unknowns are increased, in comparison with the backscattered RCS value obtained using $n = 60$. The number of unknowns in our simulation is only NJ, where for $N = 7440$ for the case $n = 60$.

n	40	45	50	55
Relative error in backscattered RCS	3.429e-06	2.261e-07	1.726e-07	7.781e-08

In the figure below we visualize the shadow in the near field by plotting $|\mathcal{E}(\cdot, t)|$ in a vertical plane behind the 16 stochastic particles at time $t = \omega/4$.

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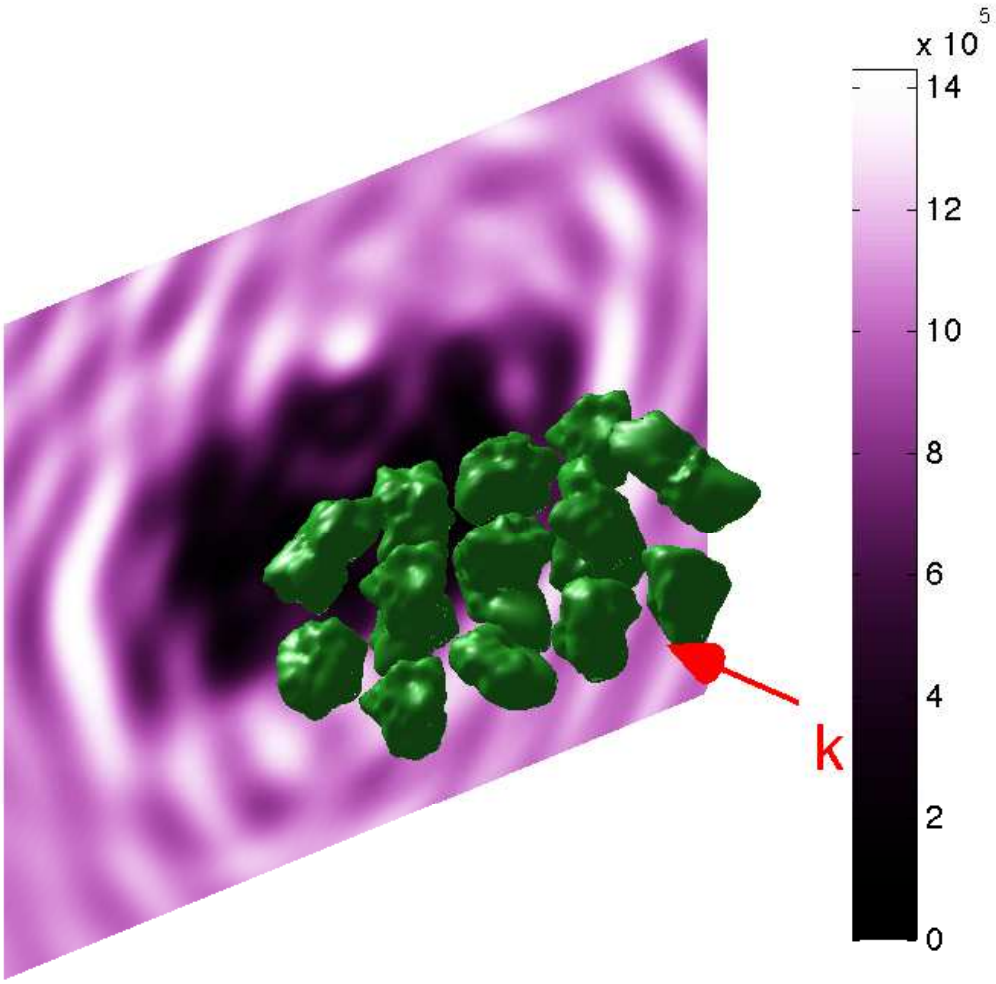

FIGURE 1. Total electric field $|\mathcal{E}(\cdot, t)|$ behind 16 stochastic ice crystal and dust particles at $t = \omega/4$, computed with $n = 60$.

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Radiation boundary conditions for time-domain scattering problems

Thomas Hagstrom

Although in the frequency domain the development of various fast solution algorithms leads, at least in my view, to the conclusion that integral equation formulations of scattering problems, when applicable, are more efficient than volume-based discretizations, the situation in the time domain is far less clear. In particular, volume-based approaches maintain the locality of the hyperbolic time-evolution operator and also allow the use of flexible and relatively well-understood approximation methods. In many cases, we contend that even the total memory requirements are decreased, as no extensive time history is required. However, for these potential advantages to be realized, rapidly convergent near-field radiation boundary conditions are required. The focus of this talk is the construction and analysis of optimal conditions for many of the standard models, including Maxwell's equations and the linearized Euler equations. We will also discuss some ideas for their extension to more complex cases.

We begin with the construction of boundary conditions in a half-space for hyperbolic systems. We consider

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + \sum_d B_d \frac{\partial u}{\partial y_d} = 0$$

where $B_d = B_d^T$ and, in preparation for deriving boundary conditions on the hyperplane $x = 0$, A has been block-diagonalized with positive-definite, negative-definite, and zero diagonal blocks, corresponding to a partition of the solution, $u = (u_-, u_+, u_0)^T$, in terms of incoming, outgoing, and tangential variables. For our error estimates we will assume that the scatterer is located a (small) distance δ to the left of the boundary. We formally solve the system by Fourier-Laplace (in $y - t$ - dual variables $k - s$) transformations. To properly label waves as causally incoming and outgoing, it is useful to take $\operatorname{Re} s > 0$. To develop approximations we will take

$$\operatorname{Re} s = \frac{1}{T}$$

where T is a time scale over which we want to guarantee accuracy - for example the simulation time. Note that we are explicitly introducing a time scale, which isn't directly involved in the operator we are approximating. In fact we will approximate a homogeneous symbol by an inhomogeneous one.

The solutions take the form

$$\hat{u} = e^{-\mu(s,k)x}\phi, \qquad \left(sI - \mu A + \sum ik_d B_d\right)\phi = 0$$

with waves labeled by the sign of $\operatorname{Re}\mu$. **Exact** radiation boundary condition at $x = 0$ simply encode the statement that there are no incoming waves. The idea behind our method is to interpolate the exact condition at some collection of eigenvalues. We attempt to develop uniform approximations along the entire inversion contour. To emphasize this fact we call the resulting boundary condition sequences **complete radiation boundary conditions**, or CRBCs for short. We choose the interpolants μ_j and $\bar{\mu}_j$ to be affine in s (and also possibly in k) corresponding respectively to eigenvalues with positive and negative real parts. We can realize these interpolants by evolving auxiliary variables along the boundary. The idea is to write down recursions which terminate for outgoing waves with $\mu = \mu_j$ or incoming waves with $\mu = \bar{\mu}_j$.

$$\begin{aligned}\frac{\partial \psi_j}{\partial t} - \left(c_j\frac{\partial}{\partial t} + \sigma_j\right)A\psi_j + \sum_d (B_d + \beta_{jd}A)\frac{\partial \psi_j}{\partial y_d} \quad &= \\ \frac{\partial \psi_{j+1}}{\partial t} + \left(\bar{c}_j\frac{\partial}{\partial t} + \bar{\sigma}_j\right)A\psi_j + \sum_d (B_d + \bar{\beta}_{jd}A)\frac{\partial \psi_j}{\partial y_d},\end{aligned}$$

for $j = 0, \ldots, q$ with $\psi_0 = u$. Truncate by:

$$\psi_{-,q+1} = 0,$$

with $\frac{\partial \psi_{+,0}}{\partial t}$ computed from the interior. Note that it is straightforward to reinterpret the recursions as a semidiscretized perfectly matched layer (PML) by reinterpreting the recursion indices as grid indices. See also the work of Guddati and coworkers [1, 2] for this connection.

What we've left out is the choice of nodes to guarantee rapid convergence. The fact is, we don't know how to find them in general. But we do know how to do it for isotropic systems, as well as a few (including the linearized Euler equations [3]) which are anisotropic. For the isotropic cases we have

$$\mu = \pm\gamma = c^{-1}\left(s^2 + c^2k^2\right)^{1/2}.$$

We first derive a convenient representation for γ and thus μ via some elementary computations.

$$\gamma = \cos\phi \cdot \frac{s}{c} + \frac{1}{cT} \cdot \frac{\sin^2\phi}{\cos\phi}$$

with $|\phi| < \frac{\pi}{2}$ everywhere on the inversion contour.

We then interpolate at an optimal set of nodes, ϕ_j. Precisely we prove in [4] that q interpolation nodes can be chosen which guarantee an error less than ϵ up to time T with

$$q \propto \ln\frac{1}{\epsilon} \cdot \ln\frac{cT}{\delta}.$$

The estimate is proven using estimates of the complex reflection coefficient adapting Newman's famous construction of exponentially convergent rational approximations to $|x|$ [5] combined with Parseval's relation. It is in using Parseval's relation for Laplace transforms that the restriction to $\operatorname{Re} s = O(T^{-1})$ arises.

Directly we show that the claimed estimate holds for approximations of the form:

$$\cos\phi_j = \left(2\frac{\delta}{cT\ln\frac{1}{\epsilon}}\right)^{j/q}, \quad j = 0, \ldots, q.$$

Computations of optimal nodes using the Remez algorithm show that the approximations can be made very efficient. For example, for $\frac{\delta}{cT} = 10^{-3}$, three digit accuracy can be guaranteed using 7 terms and six digits using 14.

To enable computations in exterior domains we construct corner closures by introducing multiply-indexed auxiliary variables at the corners which satisfy the recursions from each edge. We can combine the governing equations with the two sets of recursions to derive corner odes for the new variables, which in turn provide boundary conditions along the edges. (We can also view this construction as a standard corner layer using the PML analogy.) As yet we have no analysis of the corner closures but they work in practice; see numerical experiments in [4].

Although we believe the CRBCs provide an entirely satisfactory solution to the time-domain radiation boundary condition problem for the systems mentioned, a number of important mathematical and practical issues remain open.

Rigorous Error Analysis Including the Corner Conditions: One possibility is to write down an equivalent integral operator and try to adapt work on integral equations on nonsmooth domains. On the full plane we can write down the

equivalent integral operator using the Fourier-Laplace representation, but what is it on a polygon?

Simpler Corner Treatments: The implicit system on the corner can be large for 3d applications.A simpler corner treatment for the standard Padé approximants is implicit in recent work by Schmidt and coworkers (e.g. [6] on formulations of approximate boundary conditions using the pole condition. Can their method be adapted to the CRBCs? Can it be used to generalize conditions to curved-off corners (smooth boundaries)?

General hyperbolic systems: Do good interpolation nodes always exist, and if so how can we find them? It is straightforward to derive bounds on $\mu_\pm$ from the coefficient matrices, but are they too crude? Also, if the sign of the imaginary part of $\mu_\pm$ changes as the magnitute of $\operatorname{Im} s$ changes we are in the case of a phase velocity - group velocity mismatch. For dispersion relations following from quadratic equations we can always effectively transform to an isotropic problem. Then CRBCs work directly. See [7, 3]. What about more complicated cases, such as the problems in elasticity discussed by Bécache and coworkers in [8]?

In such cases we cannot choose complex μ_j, $\bar{\mu}_j$ so that each term in the product defining the reflection coefficient is less than 1. Purely real parameters, on the other hand, work, but may be very slowly convergent. It is interesting to note that the "damping layer" interpretation of the CRBCs in the case of purely real paramters is simply grid stretching. Grid stretching combined with damping is a popular technique in the engineering community. In [9] Appelö and Colonius show experimentally that this method can be quite accurate for elastic waves if high-order discretizations are used. Can a rigorous analysis be given?

Variable coefficients: Coefficients which are variable tangential to the boundary as in stratified media can be treated in the same way as constant coefficient problems. One only needs to estimate the location of the eigenvalues which can be done (perhaps crudely) using the coefficients. Variation in the normal direction is more difficult. That said, Ehrhardt and Zheng [10, 11] have recently characterized and approximated the exact conditions for periodic and decaying potentials. In the periodic case their formulation directly uses the function γ which we just approximated; thus the CRBCs are clearly relevant. More generally, the exact conditions formally follow from an operator Ricatti equation. It seems likely that this formulation could be combined with our interpolation scheme.

Alternatively, one could imagine combining the grid stretching method [9] mentioned earlier with some sort of goal-oriented grid adaption. We also mention a multiscale filtering approach for the Schrödinger equation with decaying potentials which has been proposed and analyzed by Sofer and Stucchio in [12].

Multiple scattering: Can we interface with a fast propagation algorithm or construct one out of the solution expression used to build the boundary conditions? The latter construction has recently been achieved by Sim and Grote [13], who use auxiliary functions from a high order boundary condition based on progressive wave expansions to directly to propagate the field.

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Discrete Lie-derivatives: Eulerian approach

Holger Heumann
(joint work with Ralf Hiptmair)

In this talk we present a novel approach to derive Eulerian discretizations of non-stationary generalized convection-diffusion equations. The calculus of differential forms permits us to express general non-stationary convective partial differential equation as

$$-\varepsilon(-1)^l d * d\omega(t) + *\partial_t\omega(t) + *L_{\boldsymbol{\beta}}\omega(t) = \varphi \quad \text{in } \Omega \subset \mathbb{R}^n \ . \tag{1}$$

This is an equation for an unknown l-form $\omega(t)$, $0 \leq l \leq n$, on the domain Ω. The symbol $*$ stands for the so-called Hodge operator mapping an l-form to an $(n-l)$-form, and d denotes the exterior derivative. Together they define the principal part

$d*d\omega$ of the differential operator. Lie-derivative $L_{\boldsymbol{\beta}}$, is a convection operator for a given velocity field $\boldsymbol{\beta}$. Differential forms can be modelled by means of functions and vector fields through so-called vector proxies [4, page 132]. For $n = 3$, in the case of $*$ induced by the Euclidean metric on $\mathbb{R}^3$, the operator $d*d\omega$ becomes $-\Delta$, $\nabla\times(\nabla\times\cdot)$, and $\nabla\cdot\nabla$ in vector proxy notation, for $l = 0, 1, 2$, respectively. The convection operators for vector proxies are $\boldsymbol{\beta}\cdot\nabla$, $\nabla(\boldsymbol{\beta}\cdot\cdot)-\boldsymbol{\beta}\times(\nabla\times\cdot)$ and $\boldsymbol{\beta}(\nabla\cdot)-\nabla\times(\boldsymbol{\beta}\times\cdot)$. We refer to [3] for more details and an introduction to the calculus of differential forms. For $0 < \varepsilon \ll 1$ we encounter in (1) a singularly perturbed boundary value problem. While a *stable* discretization for the case $l = 0$ has attracted immense attention in numerical analysis, see [5] and the many references cited therein, the cases $l = 2$ and $l = 3$ have been neglected. These cases are nevertheless relevant for numerical modeling, e.g. in magnetohydrodynamics. The magnetoquasistatic electrodynamic equations in moving media can be reformulated as a convection-diffusion problem for 1-forms in introducing the electromagnetic vector potential $\mathbf{A}$ [6, section 4]:

$$\partial_t\mathbf{A} + \varepsilon\nabla\times\nabla\times\mathbf{A} + (\nabla\times\mathbf{A})\times\boldsymbol{\beta} + \nabla(\mathbf{A}\cdot\boldsymbol{\beta}) = \mathbf{f}. \tag{2}$$

Here ε is the electrical resistivity σ^{-1}. In [6] we introduced Semi-Lagrangian methods for (1) based on the observation that:

$$\partial_t\omega(t) + L_{\boldsymbol{\beta}}\omega(t) = \lim_{\tau\to 0}\frac{\omega(t) - \boldsymbol{\Phi}^*_{-\tau}\omega(t-\tau)}{\tau}, \tag{3}$$

where $\boldsymbol{\Phi}^*_{-\tau}$ is the pullback induced by the flow $\boldsymbol{\Phi}_{-\tau}$ of the velocity $\boldsymbol{\beta}$ [2, p. 140]. In general, neither the conforming nor the non-conforming approximation spaces for discrete differential forms ω_h permit global continuity. Only the restriction of ω_h to elements T_i of the triangulation Ω_h is polynomial. Fully discrete Semi-Lagrangian schemes therefore require additional approximation steps. Either we use interpolation operators to map the pullback of discrete forms to the approximation space, or we decompose the domain into non-overlapping parts, where both the pullback of the discrete forms and the discrete forms are continuous. Both approaches can be used to derive Eulerian discretization. We will focus in the following on the second approach. Eulerian discretizations of (1) build on semi-discretization in space. While the spatial discretization of the diffusion operator is well known [3], we encountered difficulties for the convection operator. The Lie-derivative is not well defined for discontinuous discrete differential forms. But since our Semi-Lagrangian approach yields convergent schemes we postulated that for small perturbation parameter $\Delta t > 0$ the difference quotient

$$b_{\Delta t}(\omega_h, \eta_h) := \int_\Omega \frac{\omega_h - \boldsymbol{\Phi}^*_{-\Delta t}\omega_h}{\Delta t}\wedge *\eta_h, \tag{4}$$

should give a consistent discretization of the convection operator $L_{\boldsymbol{\beta}}$ and in particular the limit:

$$b_0(\omega_h, \eta_h) := \lim_{\Delta t}\int_\Omega \frac{\omega_h - \boldsymbol{\Phi}^*_{-\Delta t}\omega_h}{\Delta t}\wedge *\eta, \tag{5}$$

if it exists, will give a parameter-free consistent discretization. By means of the contraction/extrusion technique [7] we can show that this limit exists. Moreover $b_0(\omega_h, \eta_h)$ is a sum of volume integrals over elements T_i and face integrals over faces f_j. For vector proxies in $\mathbb{R}^3$ this is:

$$
\begin{aligned}
l = 0: \quad b_0(\omega_h, \eta_h) &\sim \sum_{T_i} \int_{T_i} \boldsymbol{\beta} \cdot \nabla u \, w \, d\mathbf{x} - \sum_{f_j} \int_{f_j} \boldsymbol{\beta} \cdot \mathbf{n}^+ (u^+ - u^-) \cdot w^+ d\mathbf{S} \\
l = 1: \quad b_0(\omega_h, \eta_h) &\sim \sum_{T_i} \int_{T_i} \nabla(\mathbf{u} \cdot \boldsymbol{\beta}) \cdot \mathbf{v} + (\nabla \times \mathbf{u} \times \boldsymbol{\beta}) \cdot \mathbf{v} \, d\mathbf{x} \\
&\quad - \sum_{f_j} \int_{f_j} \boldsymbol{\beta} \cdot \mathbf{n}^+ (\mathbf{u}^+ - \mathbf{u}^-) \cdot \mathbf{v}^+ d\mathbf{S} \\
l = 2: \quad b_0(\omega_h, \eta_h) &\sim \sum_{T_i} \int_{T_i} \nabla \times (\mathbf{u} \times \boldsymbol{\beta}) \cdot \mathbf{v} + (\nabla \cdot \mathbf{u} \, \boldsymbol{\beta}) \cdot \mathbf{v} \, d\mathbf{x} \\
&\quad - \sum_{f_j} \int_{f_j} \boldsymbol{\beta} \cdot \mathbf{n}^+ (\mathbf{u}^+ - \mathbf{u}^-) \cdot \mathbf{v}^+ d\mathbf{S} \\
\ell = 3: \quad b_0(\omega_h, \eta_h) &\sim \sum_{T_i} \int_{T_i} \nabla(u \cdot \boldsymbol{\beta}) \cdot \mathbf{v} \, d\mathbf{x} - \sum_{f_j} \int_{f_j} \boldsymbol{\beta} \cdot \mathbf{n}^+ (u^+ - u^-) \cdot w^+ d\mathbf{S}
\end{aligned}
$$

where the superscripts + and − indicate the trace from the upwind and downwind side. Note that the case $l = 3$ is the stabilized discontinuous Galerkin scheme [1]. It is possible to prove convergence for all these schemes, both for the conforming and the non-conforming approximations along the lines of the proof for $l = 3$ in [1]. Our numerical experiments confirm this.

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Multiple Traces Boundary Integral Formulation for Helmholtz Transmission Problems

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(joint work with Ralf Hiptmair)

We present a boundary formulation of the Helmholtz transmission problem over multiple penetrable subdomains that lends itself to operator preconditioning. Using interior Calderón projectors, the problem is cast in variational Galerkin form with a matrix operator whose diagonal is composed of block boundary integral operators. We show uniqueness of solutions, continuity and coercivity of the formulation in *ad hoc* functional spaces.

1. Transmission problem

1.1. **Geometry.** Let $\Omega_i \subset \mathbb{R}^d$ denote mutually disjoint, isotropic curvilinear Lipschitz polytopes possessing boundaries $\partial\Omega_i$ and individually homeomorphic to a sphere. These compose a scatterer Ω, i.e., $\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$, with boundary $\partial\Omega$. Denote the exterior isotropic unbounded domain by $\Omega_0 := \mathbb{R}^d \setminus \bar{\Omega}$. For each Ω_i, we write its complement as $\Omega_i^c := \mathbb{R}^d \setminus \bar{\Omega}_i$. Let $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, denote the common interfaces and eventually be equal to the empty set if the domains are not adjacent. Notice that $\Gamma_{ij} = \Gamma_{ji}$ and that each $\partial\Omega_i$ can be decomposed into its interfaces, $\partial\Omega_i = \bigcup_{j\in\Lambda_i} \bar{\Gamma}_{ij}$, where we have introduced the index set:

$$\Lambda_i := \{j = 0, \dots, N \,:\, j \neq i \quad \text{and} \quad \Gamma_{ij} \neq \emptyset\} \tag{1}$$

The union of the interfaces Γ_{ij} or *skeleton* is denoted by Σ and we set $\Sigma_0 := \Sigma \setminus \partial\Omega$, representing the union of only interior interfaces.

1.2. **Functional spaces.** Let Γ_{ij} have a boundary, then standard duality pairings are

$$\left(H^{1/2}(\Gamma_{ij})\right)' = \widetilde{H}^{-1/2}(\Gamma_{ij}) \quad \text{and} \quad \left(\widetilde{H}^{1/2}(\Gamma_{ij})\right)' = H^{-1/2}(\Gamma_{ij}),$$

where $\widetilde{H}^{1/2}(\Gamma_{ij})$ is the space of functions whose extension by zero over $\partial\Omega_i \setminus \bar{\Gamma}_{ij}$ lies in $H^{1/2}(\partial\Omega_i)$. Define the product trace space $\mathbf{V}_i := H^{1/2}(\partial\Omega_i) \times H^{-1/2}(\partial\Omega_i)$ equipped with the natural graph norm. Let $\mathbf{u}, \mathbf{v} \in \mathbf{V}_i$, identify $\mathbf{V}_i''$ with $\mathbf{V}_i$, and write their dual product as sum of component-wise dual products:

$$\langle \mathbf{u}\,,\, \mathbf{v} \rangle_{\times,i} := \langle \mathsf{Q}_i\, \mathbf{u}\,,\, \mathbf{v} \rangle_i = \langle \mathbf{u}\,,\, \mathsf{Q}_i\, \mathbf{v} \rangle_i\,, \quad \text{where } \mathsf{Q}_i := \begin{pmatrix} 0 & \mathsf{Id}_i \\ \mathsf{Id}_i & 0 \end{pmatrix}.$$

The associated sesquilinear form is $(\mathbf{u}\,,\, \mathbf{v})_{\times,i} := \langle \mathbf{u}\,,\, \overline{\mathbf{v}} \rangle_{\times,i}$ and we define the $\times$-*adjoint* of an operator $\mathsf{H}_i : \mathbf{V}_i \to \mathbf{V}_i$, through the relation

$$\left(\mathsf{H}_i^{\dagger}\, \mathbf{u}\,,\, \mathbf{v}\right)_{\times,i} := (\mathbf{u}\,,\, \mathsf{H}_i\, \mathbf{v})_{\times,i} \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}_i.$$

For multiple interfaces, we will require the tensor product spaces:

$$\widetilde{\mathcal{H}}^{-1/2}(\partial\Omega_i) := \bigotimes_{j\in\Lambda_i} \widetilde{H}^{-1/2}(\Gamma_{ij}) \quad \text{and} \quad \mathcal{H}^{1/2}(\partial\Omega_i) := \bigotimes_{j\in\Lambda_i} H^{1/2}(\Gamma_{ij}).$$

for which the chain of inclusions

$$H^{1/2}(\partial\Omega_i) \;\subset\; \mathcal{H}^{1/2}(\partial\Omega_i) \;\subset\; L^2(\partial\Omega_i) \;\subset\; \widetilde{\mathcal{H}}^{-1/2}(\partial\Omega_i) \;\subset\; H^{-1/2}(\partial\Omega_i)$$

holds. We also need the space $\widetilde{\mathbf{V}}_i := H^{1/2}(\partial\Omega_i) \times \widetilde{\mathcal{H}}^{-1/2}(\partial\Omega_i)$ and the subspaces defined by restriction over Γ_{ij}, $\mathbf{V}_{ij} := \mathbf{V}_i|_{\Gamma_{ij}}$ and $\widetilde{\mathbf{V}}_{ij} := \widetilde{\mathbf{V}}_i|_{\Gamma_{ij}}$ with associated dual product $\langle\cdot\,,\,\cdot\rangle_{\times,ij}$ and sesquilinear form $(\cdot\,,\,\cdot)_{\times,ij}$.

1.3. **Problem statement.** Let u in Ω_0 refer to the scattered wave while in Ω it represents the total wave. Define $\mathsf{P}_i := -(\Delta + \kappa_i^2)$ with $\kappa_i \in \mathbb{R}_+$ and introduce the trace operator $\boldsymbol{\gamma}^i u := (\gamma_{\mathrm{D}}^i u, \gamma_{\mathrm{N}}^i u) : H^1_{\mathrm{loc}}(\mathsf{P}_i, \Omega_i) \to \mathbf{V}_i$. Transmission conditions over each interface take the form (in distributional sense)

$$[\boldsymbol{\gamma} u]_{\Gamma_{ij}} = \big(\mathsf{X}_j\, \boldsymbol{\gamma}^j u \,-\, \boldsymbol{\gamma}^i u\big) \qquad \text{in } \Gamma_{ij}$$

where we introduce the orientation operator X_i for Neumann data. We seek $u \in H^1_{\mathrm{loc}}(\Omega \cup \Omega_0)$ such that

$$(2) \qquad \begin{cases} \mathsf{P}_i\, u = 0 & \text{in } \Omega_i \qquad i = 0, \ldots, N\,, \\ [\boldsymbol{\gamma} u] = \mathbf{g} & \text{on } \partial\Omega\,, \\ [\boldsymbol{\gamma} u] = \mathbf{0} & \text{on } \Sigma_0\,, \\ + \text{ radiation conditions} & \text{for } |\mathbf{x}| \longrightarrow \infty \end{cases}$$

where the boundary data $\mathbf{g} = (g_{\mathrm{D}}, g_{\mathrm{N}}) \in \mathbf{V}_i$ is given.

2. Multiple trace formulation

The idea relies on the weak enforcement of jump conditions across interfaces by doubling the number of trace unknowns in suitable functional spaces. These are Cartesian products of standard Dirichlet and special Neumann spaces, for which restriction and extension by zero operations are well defined.

2.1. **Restriction and extension operators.** Introduce the following operators acting on Dirichlet data:

$$\begin{aligned} \text{restriction:} \qquad & \mathsf{R}_{ij}^{\mathrm{D}} \;:\; \mathcal{H}^{1/2}(\partial\Omega_i) \longrightarrow H^{1/2}(\Gamma_{ij}), \\ \text{extension by zero:} \qquad & \mathsf{E}_{ij}^{\mathrm{D}} \;:\; H^{1/2}(\Gamma_{ij}) \longrightarrow \mathcal{H}^{1/2}(\partial\Omega_i)\,. \end{aligned}$$

Their dual adjoints are denoted

$$\begin{aligned} \mathsf{E}_{ij}^{\mathrm{N}} \;&:=\; \Big(\mathsf{R}_{ij}^{\mathrm{D}}\Big)' \;:\; \widetilde{H}^{-1/2}(\Gamma_{ij}) \longrightarrow \widetilde{\mathcal{H}}^{-1/2}(\partial\Omega_i), \\ \mathsf{R}_{ij}^{\mathrm{N}} \;&:=\; \Big(\mathsf{E}_{ij}^{\mathrm{D}}\Big)' \;:\; \widetilde{\mathcal{H}}^{-1/2}(\partial\Omega_i) \longrightarrow \widetilde{H}^{-1/2}(\Gamma_{ij}) \end{aligned}$$

where $\mathsf{E}_{ij}^{\mathrm{N}}$ is the "extension by zero" for Neumann data over $\partial\Omega_i$ while $\mathsf{R}_{ij}^{\mathrm{N}}$ is the restriction over Γ_{ij}. With these, one can define the operator over $\mathbf{V}_i$

$$\mathsf{R}_{ij}\,\boldsymbol{\varphi}^i \;:=\; \begin{cases} \begin{pmatrix} \mathsf{R}_{ij}^{\mathrm{D}} & 0 \\ 0 & \mathsf{R}_{ij}^{\mathrm{N}} \end{pmatrix} \boldsymbol{\varphi}^i & \text{if } j \in \Lambda_i, \\ \mathbf{0} & \text{any other case,} \end{cases} \qquad \forall\, j \in \{0, \ldots, N\}.$$

Its $\times$-adjoint, $\mathsf{R}_{ij}^{\dagger}$, is the formal extension by zero with diagonal terms $\mathsf{E}_{ij}^{\mathrm{D}}$ and $\mathsf{E}_{ij}^{\mathrm{N}}$, well defined for functions in $\widetilde{\mathbf{V}}_{ij}$. With it, we have the following result (in weak sense)

$$\sum_{j=0}^{N} \mathsf{R}_{ij}^{\dagger}\, \mathsf{R}_{ij} = \mathsf{Id}_i \;:\; \widetilde{\mathbf{V}}_i \longrightarrow \widetilde{\mathbf{V}}_i \;. \tag{3}$$

2.2. **Calderón projectors.** Recall the interior and exterior Calderón projectors:

$$\mathsf{C}_i = \frac{1}{2}\,\mathsf{Id} + \mathsf{A}_i \qquad \text{and} \qquad \mathsf{C}_i^c = \frac{1}{2}\,\mathsf{Id} - \mathsf{A}_i, \quad \text{where} \quad \mathsf{A}_i := \begin{pmatrix} -\,\mathsf{K}_i & \mathsf{V}_i \\ \mathsf{W}_i & \mathsf{K}_i' \end{pmatrix},$$

with V_i, K_i, K_i', W_i, denoting the standard single-layer, double-layer, adjoint double-layer, and hyper-singular boundary integral operators, respectively. If u is solution of the Helmholtz problem (2), it must hold

$$\boldsymbol{\gamma}^i u = \mathsf{C}_i\, \boldsymbol{\gamma}^i u = \left(\frac{1}{2}\,\mathsf{Id} + \mathsf{A}_i\right) \boldsymbol{\gamma}^i u \quad \Longrightarrow \quad \frac{1}{2}\boldsymbol{\gamma}^i u = \mathsf{A}_i\, \boldsymbol{\gamma}^i u \qquad \text{on } \partial\Omega_i. \tag{4}$$

2.3. **Multi-trace formulation.** Define $\mathbb{V}_N := \mathbf{V}_0 \times \cdots \times \mathbf{V}_N$ and equivalently for $\widetilde{\mathbb{V}}_N$. Expansion of the jump operator in (2) over each interface in Σ_0 in weak sense yields,

$$\left(\mathsf{R}_{0i}\,\boldsymbol{\gamma}^0 u \;-\; \mathsf{R}_{i0}\,\mathsf{X}_i\,\boldsymbol{\gamma}^i u\,,\, \mathsf{R}_{0i}\,\boldsymbol{\varphi}^0\right)_{\times,0i} \;=\; \left(\mathsf{R}_{0i}\,\mathsf{X}_0\,\mathbf{g}\,,\, \mathsf{R}_{0i}\,\boldsymbol{\varphi}^0\right)_{\times,0i}, \tag{5a}$$

$$\left(-\,\mathsf{R}_{0j}\,\mathsf{X}_0\,\boldsymbol{\gamma}^0 u + \mathsf{R}_{j0}\,\boldsymbol{\gamma}^j u\,,\, \mathsf{R}_{j0}\,\boldsymbol{\varphi}^j\right)_{\times,j0} \;=\; -\left(\mathsf{R}_{0j}\,\mathbf{g}\,,\, \mathsf{R}_{j0}\,\boldsymbol{\varphi}^j\right)_{\times,j0}, \tag{5b}$$

$$\left(\mathsf{R}_{ji}\,\boldsymbol{\gamma}^j u - \mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\gamma}^i u\,,\, \mathsf{R}_{ji}\,\boldsymbol{\varphi}^j\right)_{\times,ji} \;=\; 0\,, \tag{5c}$$

for $i \neq j \in \{1, \ldots, N\}$ and for all $\boldsymbol{\varphi} \in \widetilde{\mathbb{V}}_N$. We can extend conditions (5) in order to define transmission conditions over each $\partial\Omega_i$ as follows:

$$\left(\boldsymbol{\gamma}^0 u\,,\, \mathsf{R}_{0i}^{\dagger}\,\mathsf{R}_{0i}\,\boldsymbol{\varphi}^0\right)_{\times,0} - \left(\mathsf{R}_{0i}^{\dagger}\,\mathsf{R}_{i0}\,\mathsf{X}_i\,\boldsymbol{\gamma}^i u\,,\, \boldsymbol{\varphi}^0\right)_{\times,0} = \left(\mathsf{X}_0\,\mathbf{g}\,,\, \mathsf{R}_{0i}^{\dagger}\,\mathsf{R}_{0i}\,\boldsymbol{\varphi}^0\right)_{\times,0}, \tag{6a}$$

$$\left(\mathsf{R}_{j0}^{\dagger}\,\mathsf{R}_{0j}\,\mathsf{X}_0\,\boldsymbol{\gamma}^0 u\,,\, \boldsymbol{\varphi}^j\right)_{\times,j} - \left(\boldsymbol{\gamma}^j u\,,\, \mathsf{R}_{j0}^{\dagger}\,\mathsf{R}_{j0}\,\boldsymbol{\varphi}^j\right)_{\times,j} = \left(\mathsf{R}_{j0}^{\dagger}\,\mathsf{R}_{0j}\,\mathbf{g}\,,\, \boldsymbol{\varphi}^j\right)_{\times,j}, \tag{6b}$$

$$\left(\boldsymbol{\gamma}^j u\,,\, \mathsf{R}_{ji}^{\dagger}\,\mathsf{R}_{ji}\,\boldsymbol{\varphi}^j\right)_{\times,j} - \left(\mathsf{R}_{ji}^{\dagger}\,\mathsf{R}_{ij}\,\mathsf{X}_i\,\boldsymbol{\gamma}^i u\,,\, \boldsymbol{\varphi}^j\right)_{\times,j} \;=\; 0\,, \tag{6c}$$

for all $\boldsymbol{\varphi} \in \widetilde{\mathbb{V}}_N$. For simplicity, introduce the operator $\widetilde{\mathsf{X}}_{ji} := \mathsf{R}_{ji}^{\dagger}\,\mathsf{R}_{ij}\,\mathsf{X}_i$ mapping $\widetilde{\mathbf{V}}_i$ to $\widetilde{\mathbf{V}}_j$. Using (3) to simplify the sum of conditions (6a) with respect to i, one obtains:

$$\left(\boldsymbol{\gamma}^0 u\,,\, \boldsymbol{\varphi}^0\right)_{\times,0} - \sum_{i=1}^{N} \left(\widetilde{\mathsf{X}}_{0i}\boldsymbol{\gamma}^i u\,,\, \boldsymbol{\varphi}^0\right)_{\times,0} = \left(\mathsf{X}_0\,\mathbf{g}\,,\, \boldsymbol{\varphi}^0\right)_{\times,0}, \qquad \forall\, \boldsymbol{\varphi}^0 \in \widetilde{\mathbf{V}}_0, \tag{7}$$

while the sum (6c) + (6b) for i from one to N yields, for all $\boldsymbol{\varphi}^j \in \widetilde{\mathbf{V}}_j$,

$$\left(\boldsymbol{\gamma}^j u\,,\, \boldsymbol{\varphi}^j\right)_{\times,j} - \sum_{\substack{i=0 \\ i\neq j}}^{N} \left(\widetilde{\mathsf{X}}_{ji}\boldsymbol{\gamma}^i u\,,\, \boldsymbol{\varphi}^j\right)_{\times,j} = -\left(\mathsf{R}_{j0}^{\dagger}\,\mathsf{R}_{0j}\,\mathbf{g}\,,\, \boldsymbol{\varphi}^j\right)_{\times,j} \tag{8}$$

for fixed $j \in 1, \ldots, N$. Replacing (4) in (7) and (8), and identifying $\boldsymbol{\lambda}^i = \boldsymbol{\gamma}^i u$, gives an equivalent formulation to Problem (2), the *multi-trace formulation*: Seek $\boldsymbol{\lambda} \in \widetilde{\mathbb{V}}_N$ such that the variational form

$$(\mathbf{M}_N\,\boldsymbol{\lambda}\,,\,\boldsymbol{\varphi})_\times = \frac{1}{2}\left(\begin{pmatrix} \mathsf{X}_0\,\mathbf{g} \\ -\,\mathsf{R}_{10}^\dagger\,\mathsf{R}_{01}\,\mathbf{g} \\ \vdots \\ -\,\mathsf{R}_{N0}^\dagger\,\mathsf{R}_{0N}\,\mathbf{g} \end{pmatrix}\,,\,\boldsymbol{\varphi}\right)_\times \qquad \text{for all } \boldsymbol{\varphi} \in \widetilde{\mathbb{V}}_N \tag{9}$$

is satisfied for $\mathbf{g} \in \widetilde{\mathbf{V}}_0$ with

$$\mathbf{M}_N := \begin{pmatrix} \mathsf{A}_0 & -\frac{1}{2}\widetilde{\mathsf{X}}_{01} & \cdots & -\frac{1}{2}\widetilde{\mathsf{X}}_{0N} \\ -\frac{1}{2}\widetilde{\mathsf{X}}_{10} & \mathsf{A}_1 & \cdots & -\frac{1}{2}\widetilde{\mathsf{X}}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\widetilde{\mathsf{X}}_{N0} & -\frac{1}{2}\widetilde{\mathsf{X}}_{N1} & \cdots & \mathsf{A}_N \end{pmatrix} : \widetilde{\mathbb{V}}_N \longrightarrow \mathbb{V}_N\,. \tag{10}$$

Theorem 1. *The multi-trace formulation of the Helmholtz transmission problem is $\mathbb{V}_N$-coercive for $\boldsymbol{\lambda} \in \widetilde{\mathbb{V}}_N$ and continuous in $\widetilde{\mathbb{V}}_N$.*

The proof is based on the $\mathbf{V}_i$-coercivity of operators A_i and eliminating real parts of off-diagonal terms. This however constitutes a mismatch between continuity and coercivity preventing the use of standard results for stability.

Theorem 2. *The solution of the multi-trace formulation is unique.*

Future work includes numerical experiments and extension to Maxwell transmission problems.

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Electrodynamic Interface Conditions at a Moving Boundary

Stefan Kurz

For the mathematical modeling of electromagnetic boundary value and transmission problems, the interface conditions for electromagnetic fields are of utmost importance. What about moving and deforming bodies?

The standard textbook approach starts from a boundary at rest and employs Lorentz transformations ("frame hopping"), which is valid for uniform motion only [3, Ch. 5]. A more general analysis either considers Helmholtz' vector flux theorem (see [4] and the references cited therein) or starts from a model in four dimensions, and derives the (3+1)-dimensional interface conditions by a decomposition into "space" and time relative to an observer [5]. Still, the usual approach typically introduces coordinates, employs Ricci calculus, and utilizes metric heavily. On the other hand, the interface conditions are part of pre-metric electrodynamics, and should therefore be devoid of any metric [8, 9].

In this contribution, we derive the interface conditions in a pre-metric setting, based on minimal structures. Space-time is modeled as a bare manifold (no metric, no connection, no Lorentz transformation). Electromagnetic fields are modeled as cochains over the manifold (no coordinates, no localization).

Appropriate models for observers and interfaces in terms of foliations are introduced, and a relation between the observers' trace and projection operators is derived. The desired interface conditions are an immediate consequence.

Space-time. We model space-time as a four-dimensional differentiable manifold M, which shall be connected, orientable, and non-compact. We populate the manifold by introducing the space $\mathcal{C}_p(M)$ of chains. In geometric integration theory, this space is equipped with an appropriate norm. A Banach space of chains is obtained by a completion process. Then, integration operators, cochains, are defined as elements of the dual space $\mathcal{C}^p(M) = \mathcal{C}_p(M)^*$ [7, 14]. Since we model measuring probes by chains, we will model electromagnetic fields by cochains. The reading of the probe is then given by the dual pairing $\mathcal{C}^p(M) \times \mathcal{C}_p(M) \to \mathbb{R} : (\boldsymbol{\omega}, c) \mapsto \boldsymbol{\omega}\big|c$. For example, consider the electromagnetic field $F \in \mathcal{C}^2(M)$. For $c \in \mathcal{C}_2(M)$, the magnetic flux of F embraced by c is $F\big|c$.

Finally, we denote by $\chi(M)$ the space of smooth vector fields on M. In what follows, it is convenient but not required to talk about spacelike and timelike objects. To define such classes of objects, a causal structure is required on M [13].

Observer. We model "space" as a three-dimensional differentiable manifold E, which shall be connected, orientable and non-compact. The manifold E is embedded in M by a one-parameter family of embeddings $\Psi_t : E \to M$, $t \in \mathbb{R}$, smoothly depending on t, s.t. $\Psi_t(E)$ are spacelike hypersurfaces which fill space-time densely. We receive a foliation of space-time, also called slicing or hypersurface observer [2]. A point $Q \in E$ is mapped to a timelike smooth curve $\Psi_t(Q)$, which is called the world line of Q. This gives rise to a congruence of parameterized curves, densely filling space-time, which define a vector field $\mathbf{u} \in \chi(M)$. In a metric setting, this vector field could be normalized and would then be called four-velocity. Each point in space-time is uniquely contained in one hypersurface of the foliation (its instant

in time) and in one curve of the congruence (its point in "space"), respectively: The observer defines a diffeomorphism between the product manifold $E \times \mathbb{R}$ and space-time M.

We denote time-dependent cochains on E by $\mathcal{C}^p(E,\mathbb{R})$. The pullback map $\Psi_t^* : \mathcal{C}^p(M) \to \mathcal{C}^p(E,\mathbb{R})$ maps cochains from space-time into "space". The pullback of a cochain $\boldsymbol{\omega}$ and of its contraction $\mathbf{i}_\mathbf{u}\boldsymbol{\omega}$ with the vector field $\mathbf{u}$ [10], respectively, yields the horizontal and transversal pieces of observation, that are combined into the projection operator P [6, 12],

$$P : \mathcal{C}^p(M) \to \mathcal{C}^p(E,\mathbb{R}) \times \mathcal{C}^{p-1}(E,\mathbb{R}) : \boldsymbol{\omega} \mapsto \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \begin{pmatrix} \Psi_t^* \boldsymbol{\omega} \\ \Psi_t^* \mathbf{i}_\mathbf{u} \boldsymbol{\omega} \end{pmatrix}.$$

For example, the projection $PF = \big(B(t), -E(t)\big)^{\mathrm{T}}$ defines the magnetic flux density $B(t) \in \mathcal{C}^2(E,\mathbb{R})$ and the electric field $E(t) \in \mathcal{C}^1(E,\mathbb{R})$.

Interface in space-time. A point in "space" gives rise to a world line in space-time. Consequently, an interface in "space" gives rise to a (2+1)-dimensional entity in space-time (two spatial and one temporal dimensions). We let N be a (2+1)-dimensional connected and orientable manifold, dubbed "flatland" [1]. We parametrically embed "flatland" in space-time by a family of embeddings $\kappa_s : N \to M$, $s \in \mathbb{R}$, smoothly depending on s, where $s = 0$ coincides with the actual configuration. Embedded "flatland" might happen to be a hypersurface of discontinuity for the physical quantities. Therefore, we define single-sided trace operators by pullback, $\mathbf{t}^\pm : \mathcal{C}^p(M) \to \mathcal{C}^p(N) : \boldsymbol{\omega} \mapsto \mathbf{t}^\pm\boldsymbol{\omega} = \lim_{s\to\pm 0} \kappa_s^*\boldsymbol{\omega}$.

Maxwell-Faraday's law states flux conservation, i.e. $F\big|c = 0$ for all $c \in \mathcal{Z}_2(M)$, where $\mathcal{Z}_2(M) = \big\{c \in \mathcal{C}_2(M) : \partial c = 0\big\}$ is the space of 2-cycles. The continuity of the trace follows by standard methods, $(\mathbf{t}^+ - \mathbf{t}^-)F = 0$. In numerical methods, this type of essential interface condition is either included in the construction of the discrete spaces, or it is enforced by Lagrangian multipliers.

(3+1)-decomposition of interface conditions. To (3+1)-decompose such kind of interface conditions, the interplay of the observer with "flatland" has to be considered, see Fig. 1. It is assumed that for fixed t and s the intersection $\Psi_t(E) \cap \kappa_s(N)$ is a two-dimensional embedded submanifold. This situation is depicted in Fig. 1 for $t = t_1, t_2, t_3$ and $s = 0$. We choose a two-dimensional manifold S, connected and orientable, to model the surface. Moreover, we choose two families of embeddings $\lambda_{s,t} : S \to E$, $\Psi_{s,t}^N : S \to N$, smoothly depending on s, t, s.t. $\Psi_t \circ \lambda_{s,t} = \kappa_s \circ \Psi_{s,t}^N$ holds. For the pullback maps it follows that

$$\lambda_{s,t}^* \circ \Psi_t^* = \Psi_{s,t}^{N*} \circ \kappa_s^*. \tag{1}$$

Through each point in M there is a unique world line of a point Q fixed in "space" E and of a point R fixed in the surface S, both parameterized by t, hence vector fields $\mathbf{u}, \mathbf{u}' \in \chi(M)$. We let $\mathbf{v} = \mathbf{u}' - \mathbf{u}$ and note that $\mathbf{v}$ is everywhere tangential to the hypersurfaces $\Psi_t E$, same with $\mathbf{u}$ and $\kappa_s N$. By considering the pushforward of Ψ_t and κ_s, respectively, we write $\mathbf{v} = (\Psi_t)_* \hat{\mathbf{v}}(t)$ and $\mathbf{u} = (\kappa_s)_* \mathbf{u}_s^N$, and therefore

$$\mathbf{u} + (\Psi_t)_* \hat{\mathbf{v}}(t) = (\kappa_s)_* \mathbf{u}_s^N. \tag{2}$$

In a metric setting with four-velocity $\mathbf{u}$, $\hat{\mathbf{v}}(t) \in \chi(E,t)$ would be the ordinary velocity field of the moving interface with respect to the observer.

By using the rule $\Phi^* \circ \mathbf{i}_{\Phi_* \mathbf{w}} = \mathbf{i}_{\mathbf{w}} \circ \Phi^*$ we receive from (1) and (2) $\lambda^*_{s,t} \circ \big(\Psi^*_t \circ \mathbf{i}_{\mathbf{u}} + \mathbf{i}_{\hat{\mathbf{v}}(t)} \circ \Psi^*_t\big) = \Psi^{N*}_{s,t} \circ \mathbf{i}_{\mathbf{u}^N_s} \circ \kappa^*_s$. This can be combined with (1) into

$$\lambda^*_{s,t} \circ \begin{pmatrix} 1 & 0 \\ \mathbf{i}_{\hat{\mathbf{v}}(t)} & 1 \end{pmatrix} \circ \begin{pmatrix} \Psi^*_t \\ \Psi^*_t \circ \mathbf{i}_{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} \Psi^{N*}_{s,t} \\ \Psi^{N*}_{s,t} \circ \mathbf{i}_{\mathbf{u}^N_s} \end{pmatrix} \circ \kappa^*_s.$$

For $s \to \pm 0$ we obtain the main result

$$\boxed{\hat{\mathbf{t}}^{\pm}_t \circ \begin{pmatrix} 1 & 0 \\ \mathbf{i}_{\hat{\mathbf{v}}(t)} & 1 \end{pmatrix} \circ P = P^N \circ \mathbf{t}^{\pm},} \tag{3}$$

where $\hat{\mathbf{t}}^{\pm}_t$ denotes the time-dependent spatial trace operator, and P^N the "flatland" projection operator. Consider the trivial case $\hat{\mathbf{v}}(t) = 0$. Eq. (3) says that it does not matter if we first take the projection P from space-time M to "space" E and then the spatial trace $\hat{\mathbf{t}}_t$, or first the space-time trace $\mathbf{t}$ and then the projection P^N from "flatland" N to the surface S. In the general case, the contraction $\mathbf{i}_{\hat{\mathbf{v}}(t)}$ corrects for the effect of the motion.

Application to the electromagnetic field yields with $PF = \big(B(t), -E(t)\big)^{\mathrm{T}}$ and $(\mathbf{t}^+ - \mathbf{t}^-)F = 0$

$$(\hat{\mathbf{t}}^+_t - \hat{\mathbf{t}}^-_t)B(t) = 0,$$
$$(\hat{\mathbf{t}}^+_t - \hat{\mathbf{t}}^-_t)\big(E(t) - \mathbf{i}_{\hat{\mathbf{v}}(t)}B(t)\big) = 0.$$

If the cochains are localized as differential forms [14], and an Euclidean metric is chosen in E, then the differential forms can be represented by vector fields [11]. This yields the vectorial version

$$\mathbf{n} \cdot \big(\mathbf{B}^+(t) - \mathbf{B}^-(t)\big) = 0,$$
$$\mathbf{n} \times \Big(\mathbf{E}^+(t) - \mathbf{E}^-(t) + \hat{\mathbf{v}}(t) \times \big(\mathbf{B}^+(t) - \mathbf{B}^-(t)\big)\Big) = 0.$$

These interface conditions are valid for an arbitrarily moving and deforming interface, as long as it can be modeled as an embedded submanifold. The same derivation can be applied to the electromagnetic excitation $G \in \mathcal{C}^2(M)$, including the generalization to electromagnetic surface charge currents. A possible extension would consider discontinuous velocity fields, to allow for sliding contacts.

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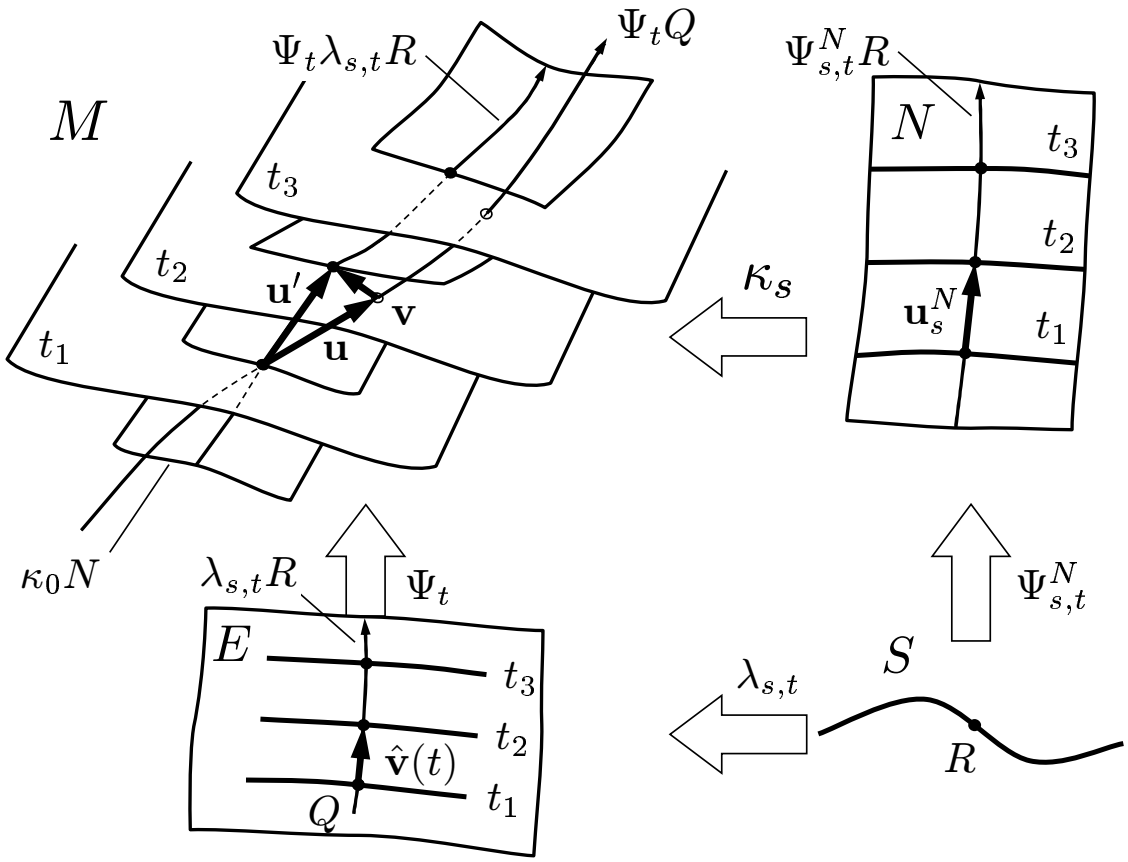

FIGURE 1. Space-time M, "space" E, "flatland" N and surface S are related by various embeddings.

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Advanced CEM Techniques for Solving Large Complex Electromagnetic Wave Problems

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(joint work with Zhen Peng, Xiaochuan Wang)

Finite element based non-overlapping domain decomposition (DD) methods have recently attracted considerable attention due to their ability to accurately and efficiently solve large and multi-scale electromagnetic radiation and scattering problems [1, 2, 3, 4, 5, 7, 8, 9, 10]. It is well known that the convergence of the DD algorithms depends strongly upon the nature of the transmission conditions (TC) that communicate information between adjacent sub-domains [5, 11, 12, 13]. Desprs showed [1] that the use of a complex Robin TC would lead to the iterative process converging quickly for propagating eigenmodes, though the evanescent modes are non-convergent. The optimized TCs of [2], by including a single second order transverse derivative, allow the convergence of both sets of modes. However, to obtain convergence of evanescent modes, the optimized TCs must trade in some performance of the propagating modes. Thus, the optimized TCs do not always provide better convergence against the conventional Robin TC.

In [5], a new type of SOTC, called SOTC-TE, which is shown to shift eigenvalues that correspond to transverse electric (TE) evanescent modes away from zero. While the DD method with SOTC-TE considerably improves the convergence w.r.t. the 1^{st} order Robin TC, it is only effective in preconditioning one set of problematic eigenvalues. The eigenmodes neglected by the SOTC-TE, namely the transverse magnetic (TM) evanescent modes, present the last impediment to solver convergence. We address these modes by introducing a full second order TC (SOTC-Full) that includes an additional term with a second order transverse derivative.

The SOTC-Full was first proposed and implemented in reference [4] for improving convergence in conformal DD methods. However, the incorporation of the SOTC-Full in the finite element implementation of conformal DD methods [4] results in a singular system matrix. This is due to the fact that the auxiliary variables, j, which represent the electric currents on the interfaces, were defined discontinuously over the interfaces. This gives rise to redundant basis functions on edges shared by two or more interfaces. Nonetheless, in the conformal DD methods, the excitation is kept in the range of the system matrix, and consequently the singular eigenvalues have little or no impact on the convergence. However, the scenarios are different once we extend the application of the SOTC-Full to the non-conformal DD methods. In the presence of non-conformal meshes, or non-matching triangulations, the original zero eigenvalues become small eigenvalues near zero. The occurrence of these small eigenvalues greatly affects negatively the convergence of the non-conformal DD methods. The detrimental effects of the non-conformal meshes on the zero eigenvalues are evidenced through many real-life examples, and were illustrated during the presentation. To mitigate the difficulty

encountered in the application of the SOTC-Full to non-conformal meshes, we examined the eigenvectors that correspond to zero eigenvalues in conformal DDM. What we found was that the zero eigenvectors do not satisfy the $\nabla \cdot \vec{B} = 0$ condition near corner edges. Consequently, a corner edge penalty term is introduced in the variational formulation. Numerical examples show clearly that the introduction of the corner edge penalty tern successfully circumvents the aforementioned difficulty with non-conformal DDM. In many large complex electromagnetic wave problems, the proposed non-conformal DDM with SOTC-Full injunction with corner edge penalty term result in an almost scaleable performance. The number of iterations is found to grow only logarithmically in the number of domains.

Moreover, we present our recent work on multi-solver domain decomposition methods for solving electromagnetic compatibility (EMC) effects of multiple antenna systems on large air platform. Although, theoretically, the scattering mechanism should be converging for scatterers that are not touching each other, in real computations the convergence is far from certain. Two scenarios are usually the cause for failure to converge in many practical numerical computations: scatterers that are very close to each other and the structures are near resonance. An effective way to combat the convergence issue is to adopt simply the Krylov solver. Finally, we proposed in the multi-solver DDM a special treatment for touching sub-domains, which avoids the troublesome self-integral terms.

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Universal Extension for Sobolev Spaces of Differential Forms $H^s(d, \Omega, \bigwedge^l(\mathbb{R}^d))$ and Applications

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(joint work with Ralf Hiptmair, Jun Zou)

The purpose of this report is to construct a family of universal extension operators for Sobolev spaces of differential forms $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$, for $d, l, k \in \mathbb{N}_0$, $d \geq 2$ and $0 \leq l \leq d$ in Lipschitz domains $\Omega \subset \mathbb{R}^d$ (the exact definition will be recalled later). In [5, Theorem 5, pp.181], E. M. Stein introduced the first universal extension operator for Lipschitz domains, which can be applied to function spaces $W^{m,p}(\Omega)$ for any $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. The Sobolev spaces of differential forms are fundamental for the theoretical analysis of, e.g., electromagnetic phenomena governed by Maxwell's equation [3, 4, 1], and fluid dynamics by Navier-Stokes equation [2]. The importance of universal extensions for Sobolev spaces of differential forms $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ lies in the fact that one can obtain a family of universal extension operators for all orders k of differentiability. The gist of the construction is the *commuting diagram property* between the pullback and the exterior derivative, and Stein's idea of *integral averaging* over the pullback of a *parametrized reflection mapping*. Here we will generalize Stein's extension for Lipschitz domains to Sobolev spaces of differential forms as follows:

Theorem 1. *Let Ω be a bounded Lipschitz domain or its complement, $k \in \mathbb{N}_0$ and $0 \leq l \leq d$. Then there exists a universal extension operator*

$$\mathscr{E}_l : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$$

satisfying

(1) $\mathscr{E}_l \boldsymbol{\omega} = \boldsymbol{\omega}$ *a.e. in* Ω, *and*
(2) *the extension operator is continuous*

$$\|\mathscr{E}_l \boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)} \leq C \|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)} \quad \forall\, \boldsymbol{\omega} \in \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l),$$

with the constant $C = C(\Omega, d, k, l)$ independent of the differential forms involved.

We point out in passing that the universal extension operators for $\boldsymbol{H}^k(\mathbf{curl};\Omega)$ and $\boldsymbol{H}^k(\mathrm{div};\Omega)$ in $\mathbb{R}^3$ are covered by our universal extension theorem as special cases for $l = 1, 2$, respectively.

Corollary 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $k \in \mathbb{N}_0$. Then there exist universal extension operators*

$$\begin{aligned}
&\mathscr{E}_0 : H^{k+1}(\Omega) \mapsto H^{k+1}(\mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_0 u = u, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_0 u\|_{H^k(\mathbb{R}^3)} \le C \|u\|_{H^k(\Omega)} ; \end{cases} \\
&\mathscr{E}_1 : \boldsymbol{H}^k(\mathbf{curl};\Omega) \mapsto \boldsymbol{H}^k(\mathbf{curl};\mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_1 \mathbf{u} = \mathbf{u}, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_1 \mathbf{u}\|_{\boldsymbol{H}^k(\mathbf{curl};\mathbb{R}^3)} \le C \|\mathbf{u}\|_{\boldsymbol{H}^k(\mathbf{curl};\Omega)} ; \end{cases} \\
&\mathscr{E}_2 : \boldsymbol{H}^k(\mathrm{div};\Omega) \mapsto \boldsymbol{H}^k(\mathrm{div};\mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_2 \mathbf{u} = \mathbf{u}, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_2 \mathbf{u}\|_{\boldsymbol{H}^k(\mathrm{div};\mathbb{R}^3)} \le C \|\mathbf{u}\|_{\boldsymbol{H}^k(\mathrm{div};\Omega)} ; \end{cases} \\
&\mathscr{E}_3 : H^k(\Omega) \mapsto H^k(\mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_3 u = u, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_3 u\|_{H^k(\mathbb{R}^3)} \le C \|u\|_{H^k(\Omega)} ; \end{cases}
\end{aligned}$$

with all the constants $C = C(k,\Omega)$ independent of the functions/fields involved.

Of considerable mathematical interest by itself, the newly established theoretical results have many potential important applications, for instance, its applications to the theory of interpolation and the proof of a generalized regular decomposition result reported here. In particular, the former application can further generalize the universal extension theorems from integer order $\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)$, $k \in \mathbb{N}_0$, to nonnegative real order $\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)$, $s \in \mathbb{R}_0^+$ by establishing the equivalence in

Lemma 3. *For $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, and $l \in \mathbb{N}_0$ with $0 \le l \le d$, it holds that*

$$\big[\, \boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l), \boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l) \,\big]_\theta = \boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l) \tag{1}$$

with equivalent norms, where $s = (1-\theta)s_0 + \theta s_1$ for $0 < \theta < 1$.

Moreover, the latter application provides a proof of the generalized regular decomposition lemma in

Theorem 4. *(Lifted regular decompositions) For every $k \in \mathbb{N}_0$, $1 \le l \le d$, there exist continuous maps $\mathsf{R} : \boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l) \mapsto \boldsymbol{H}^{k+1}(\Omega,\Lambda^l)$ and $\mathsf{N} : \boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l) \mapsto \boldsymbol{H}^{k+1}(\Omega,\Lambda^{l-1})$ such that*

$$\mathsf{R} + \boldsymbol{d} \circ \mathsf{N} = Id \qquad \textit{on} \quad \boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l) \,. \tag{2}$$

In addition, there are continuous maps $\mathsf{R}_0 : \boldsymbol{H}_0^k(\boldsymbol{d},\Omega,\Lambda^l) \mapsto \boldsymbol{H}_0^{k+1}(\Omega,\Lambda^l)$ and $\mathsf{N}_0 : \boldsymbol{H}_0^k(\boldsymbol{d},\Omega,\Lambda^l) \mapsto \boldsymbol{H}_0^{k+1}(\Omega,\Lambda^{l-1})$ such that

$$\mathsf{R}_0 + \boldsymbol{d} \circ \mathsf{N}_0 = Id \qquad \textit{on} \quad \boldsymbol{H}_0^k(\boldsymbol{d},\Omega,\Lambda^l) \,. \tag{3}$$

The generalized regular decomposition lemma can be used to present an alternative proof for the Nečas Lemma.

Corollary 5 (Generalized Nečas' Lemma)**.** *Let $s \in \mathbb{R}_0^+$ and $1 \le l \le d$. For a bounded Lipschitz domain $\Omega \in \mathbb{R}^d$ of full topological generality and all $\boldsymbol{\omega} \in$*

$\boldsymbol{d}\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^{l-1})$, then there is a $\boldsymbol{\eta}\in\boldsymbol{H}^{s+1}(\Omega,\Lambda^{l-1})$ and a positive constant C independent of $\boldsymbol{\eta}$ such that

$$\boldsymbol{d}\boldsymbol{\eta}=\boldsymbol{\omega}, \tag{4}$$

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^{l-1})}\le C\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^{s}(\Omega,\Lambda^{l})}\,. \tag{5}$$

Moreover, for all $\boldsymbol{\omega}\in\boldsymbol{d}\boldsymbol{H}^s_0(\boldsymbol{d},\Omega,\Lambda^{l-1})$ for $1\le l<d$, and $\int_\Omega\boldsymbol{\omega}=0$ if $l=d$, there is a $\boldsymbol{\eta}\in\boldsymbol{H}^{s+1}_0(\Omega,\Lambda^{l-1})$ and a positive constant C independent of $\boldsymbol{\eta}$ such that (4) *and* (5) *holds.*

It is natural to derive from Corollary 5 a similar Nečas-like result for the **curl** operator.

Corollary 6. *Assuming that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then*

(1) *there exists a positive constant C such that for all $\mathbf{v}\in\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl};\Omega)$, one can find $\mathbf{u}\in\boldsymbol{H}^1(\Omega)$ satisfying*

$$\mathbf{curl}\,\mathbf{u}=\mathbf{v}\quad\textit{and}\quad\|\mathbf{u}\|_{\boldsymbol{H}^1(\Omega)}\le C\,\|\mathbf{v}\|_{\boldsymbol{L}^2(\Omega)}\,; \tag{6}$$

(2) *if $\mathbf{v}\in\mathbf{curl}\,\boldsymbol{H}_0(\mathbf{curl};\Omega)$, we can find a $\mathbf{u}\in\boldsymbol{H}^1_0(\Omega)$ such that* (6) *holds.*

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Boundary element methods for high frequency scattering problems

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(joint work with Simon Chandler-Wilde)

We consider scattering of a time harmonic incident plane wave, $u^i(\mathbf{x})=\mathrm{e}^{\mathrm{i}k\mathbf{x}.\mathbf{d}}$, by a bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$, as modelled by the Helmholtz equation

$$\Delta u+k^2u=0,\quad\text{in }\mathbb{R}^2\backslash\Omega,$$

coupled with the Sommerfeld radiation condition and sound soft or impedance boundary conditions. Here $\mathbf{d}$ denotes the direction of the incident wave, and the wavenumber k is proportional to the frequency of the incident wave. Considering first the sound soft problem, with $u=0$ on the boundary Γ of Ω, Green's theorem allows us to reformulate the problem as a second kind boundary integral equation

$$\frac{1}{2}\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x})+\int_\Gamma\left(\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})}+\mathrm{i}\eta\Phi(\mathbf{x},\mathbf{y})\right)\frac{\partial u}{\partial\mathbf{n}}(\mathbf{y})\,\mathrm{d}s(\mathbf{y})=\frac{\partial u^i}{\partial\mathbf{n}}(\mathbf{x})+\mathrm{i}\eta u^i(\mathbf{x}),\quad\mathbf{x}\in\Gamma,$$

where $\mathbf{n}$ is the outward normal, $\Phi(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$ is the fundamental solution to the two-dimensional Helmholtz equation, and the coupling parameter $\eta \neq 0$ ensures that the integral equation has a unique solution for all k.

The conventional approach to solving this boundary integral equation, applying a Galerkin method in which $\partial u/\partial \mathbf{n}$ is approximated by piecewise polynomials, suffers from the restriction that the number of degrees of freedom required to achieve a prescribed level of accuracy grows at least linearly with respect to k. As an alternative, much recent work has focused on the development of hybrid schemes in which the unknown function ϕ (with $\phi = \partial u/\partial \mathbf{n}$, the complementary boundary data, in this case) is approximated by a combination of oscillatory and non-oscillatory functions, specifically:

$$\phi(\mathbf{x}) \approx \sum_{m=1}^{M} \exp(\mathrm{i}k\gamma_m(\mathbf{x}))V_m(\mathbf{x},k).$$

Here, the phase functions γ_m are chosen a-priori, in an attempt to represent the oscillatory nature of the solution explicitly, and the unknown functions V_m are then approximated by piecewise polynomials. The hope is that the functions V_m will be easier to approximate than ϕ.

Domain based methods, such as the Ultra Weak Variational Formulation or the Plane Wave Discontinuous Galerkin Method, use similar approximation spaces with large values of M, and are widely applicable to a range of scattering problems. However, in the absence of a clear motivation for the explicit choice of the phase functions γ_m, such schemes in general have a computational cost that scales at least linearly with k. Here, we focus in particular on simple problems for which one can deduce sufficient a-priori information regarding the oscillatory behaviour of ϕ such that one can choose γ_m in such a way that the functions V_m are non-oscillatory, even if M is small. For a range of two-dimensional convex scatterers, this enables the development of schemes with computational costs that are essentially independent of frequency, i.e. the computational cost required to achieve a prescribed level of accuracy does not grow as k increases.

For smooth convex obstacles, choosing $M = 1$ and $\gamma_1(\mathbf{x}) = \mathbf{x}.\mathbf{d}$ immediately reduces the computational cost from $O(k)$ to $O(k^{1/3})$, and a concentration of the degrees of freedom in the vicinity of the shadow boundary (where $\mathbf{n}.\mathbf{d} = 0$) can remove the k-dependence of the computational cost altogether [4]. For obstacles with corners, however, a slightly different approach is required, as the oscillations caused by diffraction from the corners are not well represented by the oscillations of the incident wave ($\mathrm{e}^{\mathrm{i}k\mathbf{x}.\mathbf{d}}$). Instead, for convex polygons, we write:

$$\phi(\mathbf{x}(s)) = \Psi(\mathbf{x}(s)) + v_+(\mathbf{x}(s))\mathrm{e}^{\mathrm{i}ks} + v_-(\mathbf{x}(s))\mathrm{e}^{-\mathrm{i}ks}, \quad \mathbf{x}(s) \in \Gamma, \tag{1}$$

where $\mathbf{x}(s)$ is an arclength parametrisation of the boundary, $\Psi(\mathbf{x}(s))$ represents the known physical optics solution (which behaves like $\mathrm{e}^{\mathrm{i}k\mathbf{x}(s).\mathbf{d}}$ on illuminated sides, as for smooth obstacles), and $v_\pm$ are smooth, non-oscillatory functions, that can be shown to be peaked at the corners of the polygon, and rapidly decaying away from corners. In particular, for the sound soft problem it is shown in [2] by

separation of variables near the corners that we have the behaviour

$$k^{-m}|{v_\pm}^{(m)}(s)| \leq Cm!\sqrt{m+1}(ks)^{-\alpha-m}, \quad \text{for } ks \leq 1,$$

where $\alpha = 1 - \pi/(\text{external angle at corner}) \in (0, 1/2)$, whilst away from corners we have the behaviour

$$k^{-m}|{v_\pm}^{(m)}(s)| \leq Cm!(ks)^{-1/2-m}, \quad \text{for } ks \geq 1.$$

This latter result is shown via consideration of a set of related half-plane problems, allowing us to identify the oscillatory nature of the unknown boundary data explicitly. For details we refer to [2]. These estimates can be used to design a hybrid approximation space consisting of the products of plane waves with piecewise polynomials supported on a graded mesh, with larger elements away from the corners of the polygon, leading to the result that, if V^+ is the best L_2 approximation from the approximation space,

$$k^{1/2}\|v_+ - V^+\|_2 \leq C_p \frac{n^{1/2}(1+\log(kL))^{1/2}}{N^{p+1}},$$

with a similar result holding for our approximation to v_-, where N is proportional to the total number of degrees of freedom, p is the degree of the polynomial approximation on each element, L is the maximum side length of the polygon, and n is the number of sides of the polygon. It is further demonstrated in [2] that the error in our Galerkin boundary element method solution satisfies a similar estimate, giving us a prescribed level of accuracy with a computational cost that grows only logarithmically with respect to k.

The extension of these ideas to convex curvilinear polygons is discussed in [5]. In this case the aim is to combine the ansatz for smooth convex obstacles with that for convex polygons (1), leading to the ansatz

$$\phi(\mathbf{x}(s)) = \mathrm{e}^{\mathrm{i}k\mathbf{x}(s)\cdot\mathbf{d}} w(\mathbf{x}(s)) + v_+(\mathbf{x}(s))\mathrm{e}^{\mathrm{i}ks} + v_-(\mathbf{x}(s))\mathrm{e}^{-\mathrm{i}ks}, \quad \mathbf{x}(s) \in \Gamma,$$

where w and $v_\pm$ are all to be found. Whereas for smooth obstacles it can be proven that w is slowly oscillating, and for convex polygons Ψ is known and $v_\pm$ can be proven to be slowly oscillating, as described above, for convex curvilinear obstacles such results have, to date, remained elusive. Numerical results in [5] seem to suggest however that the number of degrees of freedom required to approximate $\partial u/\partial \mathbf{n}$ to any given level of accuracy for convex curvilinear polygons grows only logarithmically as k increases.

The extension to convex polygons with impedance boundary conditions is described in [3]. In this case, we have $\phi = u$ as the unknown function to be approximated on Γ, rather than $\partial u/\partial \mathbf{n}$. Again we can express ϕ as a product of oscillatory and non-oscillatory functions on each side of the polygon, leading to the ansatz

$$\phi(\mathbf{x}(s)) = \text{physical optics} + v^+(\mathbf{x}(s))\mathrm{e}^{\mathrm{i}ks} + v^-(\mathbf{x}(s))\mathrm{e}^{-\mathrm{i}ks}, \quad x(s) \in \Gamma.$$

We prove in [3] that

$$k^{-m}|v^{\pm(m)}(s)| \leq \begin{cases} C_m \sup_{x\in\mathbb{R}^2\setminus\Omega} |u(x)|(ks)^{\hat{\alpha}-m}, & \text{for } ks \leq 1, \\ C_m \sup_{x\in\mathbb{R}^2\setminus\Omega} |u(x)|(ks)^{-1/2-m}, & \text{for } ks \geq 1, \end{cases}$$

where $\hat{\alpha} = \pi/(\text{external angle at corner}) \in (1/2, 1)$. We thus use a similar approximation space to that for sound soft convex polygons, albeit with a slightly different mesh grading, and this gives the result that if φ_N is the best L_2 approximation from the approximation space to $\varphi(s) := u(x(s))$, $x \in \Gamma$, it holds that

$$k^{1/2}\|\varphi_N - \varphi\|_2 \leq C_p \sup_{x\in\mathbb{R}^2\setminus\Omega} |u(x)| \frac{n^{1/2}(1+\log(kL))^{1/2}}{N^{p+1}},$$

demonstrating that we can again achieve any prescribed level of accuracy with a computational cost that depends only logarithmically on k.

For a much more detailed survey of the application of boundary element methods to the solution of high frequency scattering problems, we refer to [1].

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Calderón preconditioning of integral equations for perfectly electrically conductive and homogeneous penetrable objects

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(joint work with Francesco Andriulli, Hakan Bağcı, Kristof Cools, Felipe Valdés)

The use of Integral Equations (IE) for analyzing time-harmonic radiation and scattering from perfect electrically conducting (PEC) as well as penetrable objects has remained a popular choice among code developers and practitioners for almost four decades [1]. The popularity of IE-based methods stems from the fact that compared with their finite elements or finite differences counterparts, (i) they implicitly impose radiation conditions, and (ii) they do not require unknown fields to be discretized throughout homogeneous volumes. IEs are obtained by enforcing boundary conditions on the tangential component of the electric and/or magnetic field at every point on the surface of the scatterer. In contrast with the finite

elements and finite differences methods which lead to sparse systems of linear equations, IE-based methods give rise to full matrices. Thus, when dealing with increasingly large problems the use of direct solvers is impractical and iterative solvers are called for. When solving IEs with iterative solvers, the computational cost scales multiplicatively with the cost of multiplying the system matrix with a trial solution vector, and the number of iterations required to reach a specified residual error. Nowadays, the use of accelerators such as adaptive integral or multilevel fast multipole method are widely used to reduce the cost of a matrix-vector multiplication. The required number of iterations typically is proportional to the system matrix's condition number, viz. the ratio of largest and smallest singular values. For an ill-conditioned system, iterative methods require so many iterations that the numerical solution becomes prohibitively expensive.

The literature abounds with IEs for analyzing scattering from homogeneous penetrable objects. Dual source techniques, which are by far the most popular, solve a coupled pair of electric, magnetic, or mixed/combined field integral equations for electric and magnetic surface currents. Single source techniques on the other hand, solve one electric, magnetic, or combined field integral equation (EFIE, MFIE, and CFIE) for an electric or magnetic surface current density. Equations of the first kind (first presented in [2] and [3]) involve hypersingular operators which lead to ill-conditioned matrices when discretized, therefore are susceptible to dense mesh and low frequency breakdown. Moreover, they exhibit resonances; that is, their solution is not unique at a set of discrete frequencies that grows increasingly dense as the electrical size of the scatterer increases. Second kind equations [4, 5] on the other hand, do not suffer from dense mesh nor low frequency breakdown, but they are still susceptible to resonances and hence problematic when applied to the analysis of electrically large scatterers. A linear combination of first and second kind single source equations has been proposed in [4], yielding a resonant free combined field IE. Unfortunately, this equation still contains a hypersingular electric field integral operator rendering the entire equation hypersingular and susceptible to dense mesh breakdown. We have presented a Calderón-preconditioned CFIE (CP-CFIE) which is obtained as the linear combination of a second kind single source MFIE and a Calderón-preconditioned first kind single source EFIE [6]. As such, the equation is free from spurious resonances and it is not susceptible to dense mesh breakdown. The proposed single source equation contains double and triple operator products, the discretization of which is achieved by multiplying system matrices arising from the discretization of the various (standalone) operators involved using carefully chosen basis and testing functions. Specifically, Rao-Wilton-Glisson (RWG) functions are used alongside Buffa-Christiansen (BC) functions to stably discretize operator products as they give rise to a well-conditioned Gram matrix and guarantee the numerical annihilation of products of discretized hypersingular operator components [7]; similar techniques are used to discretize triple operator products. A preliminary implementation of this approach has been tested on a sphere of radius 1m. Figure 1(a) shows the relative residual error versus iteration count achieved during the iterative solution of the matrix

systems obtained by discretizing CFIE and CP-CFIE. The simulation is repeated for five discretizations with minimum edge size δ ranging from 1/400 to 1/2000 of a wavelength. For the CFIE, the convergence rate deteriorates as $\delta \to 0$, and the number of iterations needed to reach a relative residual error of 10^{-5} increases. The CP-CFIE exhibits a convergence rate that is independent of the discretization and a low number of iterations. The condition number of the impedance matrices obtained discretizing four single source equations (EFIE, MFIE, CFIE, and CP-CFIE) is plotted versus frequency in Figure 1(b). As expected, the matrices obtained by discretizing the EFIE and MFIE become ill-conditioned in the vicinity of the resonant frequencies of a spherical PEC cavity of radius 1 m. On the other hand, those obtained by discretizing the CFIE and CP-CFIE are free from these resonances.

Efficient and stable discretization of operator products like the ones present in Calderón-preconditioned EFIEs (CP-EFIE) for PEC, as well as in CP-CFIE for penetrable objects, relies on the properties of RWG and BC basis functions. In the last decade, high-order IE solvers have increasingly gained attention in the computational electromagnetics community. These solvers exploit the use of either high-order representation of the geometry, high-order representation of the unknown current densities (high-order basis functions), or both. Among high-order basis functions, a common choice is the set of Graglia-Wilton-Peterson (GWP) basis functions [8], which are constructed as the product of a set of scalar polynomials (complete up to a given order p) and the RWG basis functions. For a desired accuracy on the solutions, solvers using high-order basis functions have shown to be more accurate as well as CPU and memory efficient than their zeroth-order predecessors leveraging RWG functions. Unfortunately, the BC basis functions are zeroth-order in nature, and only can be used in conjunction with RWG basis functions. This limitation imposes a severe constraint on the accuracy and efficiency of present Calderón preconditioned solvers, either for PEC or for penetrable objects. We have developed a set of high-order div- and quasi curl-conforming basis functions which extend the properties of the BCs [9]. These functions are high-order in nature and are designed to be used alongside GWP functions in the discretization of operator products like the ones present in CP-EFIE for PEC, as well as in the CP-CFIE for penetrable objects. The proposed basis functions are constructed as the orthogonal projection of the range of the EFIE operator (pertinent to PEC objects) onto div-conforming GWPs defined on a barycentrically refined mesh. We have tested the set of basis functions on a PEC sphere of radius 1 m. The relative residual error versus iteration count achieved during the iterative solution of the matrix systems obtained by discretizing a diagonal-preconditioned EFIE and CP-EFIE is shown in Figure 2(a). The simulation, performed with basis functions of order 5, is repeated for three discretizations with decreasing minimum edge size δ. For the diagonally-preconditioned EFIE, the convergence rate deteriorates as $\delta \to 0$, and the number of iterations needed to reach a relative residual error of 10^{-5} increases. The CP-EFIE exhibits a convergence rate that is independent of the discretization and a low number of iterations. Figure 2(b) puts in evidence the

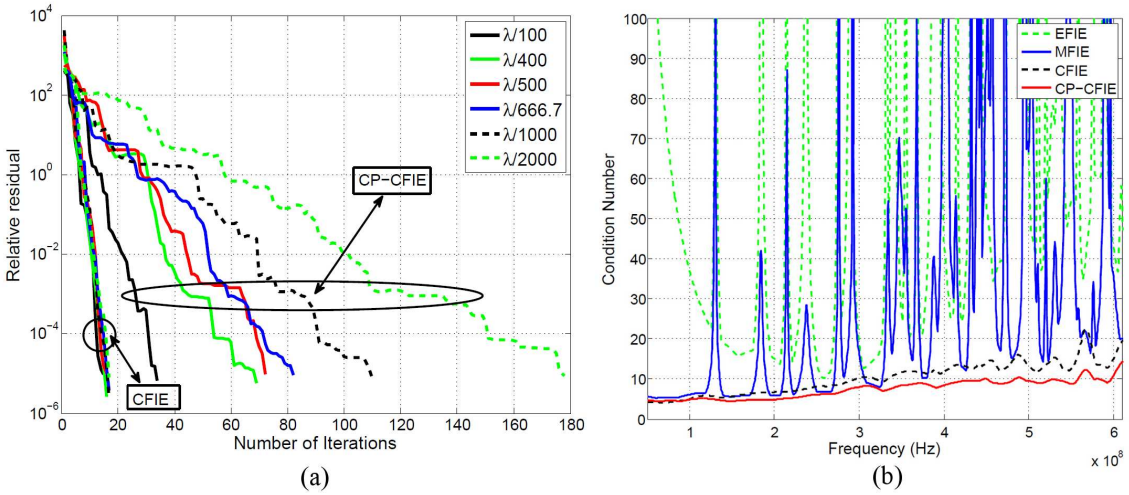

FIGURE 1. (a) Relative residual error versus iteration count achieved during the iterative solution of CFIE and CP-CFIE. (b) Condition number versus frequency of matrix obtained discretizing four different single source equations.

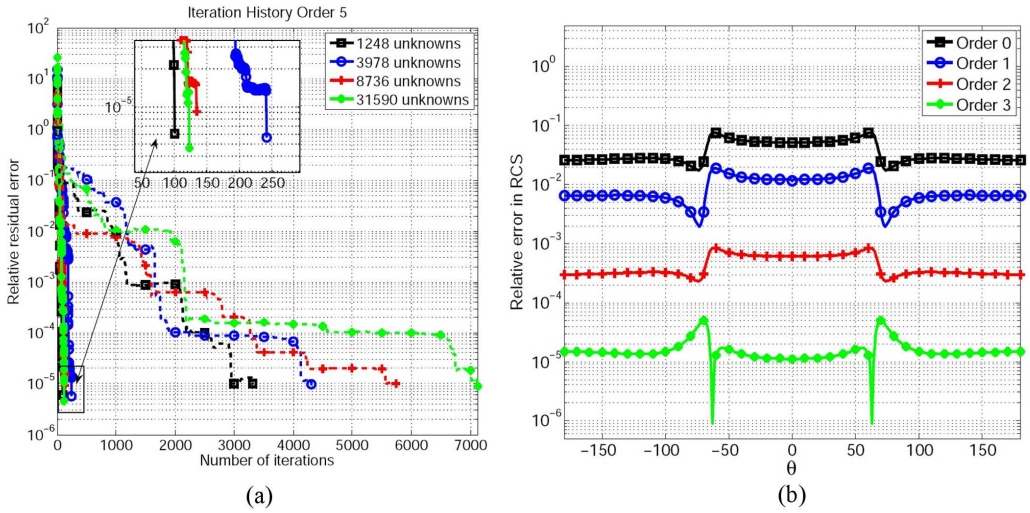

FIGURE 2. (a) Relative residual error versus iteration count achieved during the iterative solution of diagonally-preconditioned and CP-EFIE for a PEC sphere. (b) Relative error in the bistatic RCS of a PEC sphere. The RCS solutions are computed with basis functions of order $p = 0, 1, 2, 3$, and compared to Mie series solution.

high-order nature of the proposed basis functions. For a fixed discretization of the PEC sphere, increasing of the order in the basis functions increases the accuracy in the solution.

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Approximation by plane waves

Andrea Moiola

(joint work with Ralf Hiptmair, Ilaria Perugia)

The Trefftz methods are special finite element methods where the trial and test functions are solutions of the underlying PDE in the interior of each element. In the case of the homogeneous Helmholtz equation

$$-\Delta u - \omega^2 u = 0,$$

plane waves ($x \mapsto e^{i\omega x\cdot d}$) or circular/spherical waves are usually used as basis functions. These methods have been developed in order to cope with the problems that arise in the numerical simulation of the propagation and interaction of acoustic and electromagnetic waves (e.g., numerical dispersion, pollution effect); see for instance [1], [2] and [3].

One of the main step in the convergence analysis of any plane wave Trefftz method is the proof of a *best approximation estimate* for the trial space: given a solution $u \in H^{K+1}(D)$ of the homogeneous Helmholtz equation, it is possible to approximate u in Sobolev norms with a linear combination of p plane waves of

given directions $\{d_k\}_{k=1,\dots,p}$

$$\inf_{\alpha\in\mathbb{C}^p} \left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega d_k\cdot x} \right\|_{H^j(D)} \leq C\ \epsilon(h,p)\ \|u\|_{H^{K+1}(D)}\,,$$

where $\epsilon(h,p)$ goes to zero for decreasing h (the diameter of the domain D) or increasing p, with a certain order of convergence.

The only best approximation estimates for plane waves space available in literature (see [5] and [1]) are limited to two dimensional domains and are not completely satisfactory. We can prove best approximation estimates with algebraic order of convergence in h and p (exponential in p if the solution can be extended outside the domain), both in two and three dimensional domains. The dependence of every bounding constant on the wavenumber is always made explicit. The order of convergence in h is sharp, as verified by numerical experiments, while the one in p is sharp only in the two dimensional case.

The proof is split in two parts. In the first part we show how u can be approximated by the so-called *generalized harmonic polynomials*, i.e. the circular waves

$$x = re^{i\psi} \mapsto e^{il\psi}\ J_l(\omega r), \qquad\qquad l\in\mathbb{Z},$$

in two dimensions and the spherical waves

$$x \mapsto Y_{l,m}\big(\tfrac{x}{|x|}\big)\ j_l(\omega|x|), \qquad\qquad 0\leq|m|\leq l\in\mathbb{N},$$

in three dimensions. Here J_l, j_l and $Y_{l,m}$ are the Bessel functions, the spherical Bessel functions and the spherical harmonics, respectively.

The main tools used are the Vekua transforms for the N–dimensional Helmholtz equation:

$$V_1, V_2 : L^\infty(D) \to L^\infty(D),$$
$$V_j[\phi](x) := \phi(x) + \int_0^1 M_j(x,t)\phi(tx)\,\mathrm{d}t\ , \quad \text{a.e. } x\in D,\ j=1,2;$$

where the functions $M_1, M_2 : D\times[0,1]\to\mathbb{R}$ are defined as

$$M_1(x,t) = -\frac{\omega|x|}{2}\ \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}}\ J_1(\omega|x|\sqrt{1-t}),$$
$$M_2(x,t) = -\frac{i\omega|x|}{2}\ \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}}\ J_1(i\omega|x|\sqrt{t(1-t)}).$$

These integral operators map Helmholtz solutions into harmonic functions on the same domain and viceversa. This allows to reduce the problem to the simpler case of the approximation of harmonic functions by harmonic polynomials. The orders of convergence in h are proved using a suitable version of the Bramble–Hilbert theorem, the ones in p in two dimensions using the result provided by [6] while the ones in p in two dimension are new. In this case however the speed of convergence is not explicit but depends on the shape of the domain in an unknown way. This result provides also a best approximation estimate for all the Trefftz methods based on circular and spherical waves (Fourier-Bessel functions).

In the second part we approximate these functions with plane waves. The link between plane waves and the circular/spherical waves is provided by the Jacobi–Anger expansion. We truncate the expansion and give a bound on the solution of the linear system obtained; in three dimensions this requires a careful choice of the propagating directions of the plane waves.

The final result states that for any solution $u \in H^{K+1}(D)$, $D \subset \mathbb{R}^N$ starshaped with diameter h, $N = 2, 3$,

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x \cdot d_k} \right\|_{j,\omega,D} \leq C\, h^{K+1-j} q^{-\lambda(K+1-j)}\, \|u\|_{K+1,\omega,D}$$

where the Sobolev norms are weighted with ω, the constant C depend on ωh (explicitly), j, k, $\{d_k\}$ and the shape of D. The number p of plane waves is linked to the parameter q as $p = 2q + 1$ in two dimensions and as $p = (q + 1)^2$ in three dimensions. The positive parameter λ depends only on the shape of D: in two dimensions it can be chosen arbitrarily close to 1 while in three dimensions we are not able to provide an appropriate lower bound. At the present time this is the main gap in the theory: it prevents from proving a sharp order of convergence in p in three dimensions.

Most of these results are described in details and proved in the report [4].

The proof of analogous bounds for Maxwell equations

$$\operatorname{curl} \operatorname{curl} u - \omega^2 u = 0$$

is more difficult because the Vekua operators are not appropriate for these equations. We use Herglotz functions and vector spherical harmonics to separate the vectorial generalized harmonic polynomials that are solution of Maxwell equations from the ones that are not divergence free. With these tools we can prove a h-estimate for spherical waves with the same order of convergence proved for the Helmholtz case but using a few more basis functions.

In order to estimate the approximation by plane waves we have proved a special Jacobi-Anger formula for vector spherical harmonics that is well-adapted for Maxwell equations.

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Weighted Poincaré inequalities for high contrast coefficients

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(joint work with Robert Scheichl)

In this short note we consider iterative solvers for second-order elliptic problems with highly varying coefficients and the robustness analysis of these solvers.

Our main motivation is the following nonlinear magnetostatic problem in two dimensions. Assuming isotropic material, neglecting hysteresis, and considering the transverse magnetic mode, we obtain

$$-\mathrm{div}\big[\nu(|\nabla u|)\nabla u\big] \;=\; J_3\,.$$

Here, u is the third component of the vector potential $\mathbf{A}$, J_3 is the third component of the current density, and the magnetic reluctivity ν is given by $\nu(s) = g^{-1}(s)/s$, where g is the B-H-curve such that $|\mathbf{B}| = g(|\mathbf{H}|)$. Note that $\mathbf{B} = (\partial_2 u, -\partial_1 u, 0)$.

This nonlinear PDE is discretized using continuous piecewise linear finite elements. We solve the discrete nonlinear system by a Newton or fixed point method. In either case, the linear system to be solved in the k-th step is of variational type, and its coefficient is either $\nu(|\nabla u^{(k)}|)$ or a closely related matrix-valued coefficient whose local anisotropy is negligibly small in many applications. However, the coefficient can exhibit large jumps, e. g. of order 10^3 between iron and air. Additionally, the solution u will have singularities at (re-entrant) material corners. Therefore, the coefficient $\nu(|\nabla u^{(k)}|)$, can vary strongly within ferromagnetic materials, also within a magnitude of 10^3, see e.g. [9].

When the number of unknowns becomes large, the use of direct solvers is not feasible. Standard analyses of iterative methods, however, depend on the global variation of the coefficient. In order to carry out a more careful analysis, weighted Poincaré type inequalities turn out to play a central role, especially in the analysis of domain decomposition and multigrid methods. In the following, we consider the simplified model problem $-\mathrm{div}\big[\alpha\,\nabla u\big] = f$ with the scalar coefficient α. First, we would like to give a brief overview on existing work, but we restrict ourselves to two-level overlapping Schwarz and FETI type domain decomposition solvers.

Two-level overlapping Schwarz (coefficient α resolved by the coarse mesh). The known results here are based on the weighted L^2-projections, see [1]. The so-called *quasi-monotone* case is well-treated in [2], and the bounds are independent of the values of α. If α is not quasi-monotone, it is necessary to resort to other ("exotic") coarse spaces, see [2, 15]. Related results, where effective condition numbers are bounded, can be found in [5, 19].

FETI type methods (coefficient α resolved by subdomain partitioning). In [7] it is shown that the condition number grows as $C\,(1+\log(H/h))^2$, where H is the

subdomain diameter, h the mesh size, and the constant C is independent of H, h, and α, see also [8] for FETI-DP methods.

Two-level overlapping Schwarz (non-resolved case). Here, to the best of our knowledge, the only paper on the standard piecewise linear coarse space is [3], where Galvis and Efendiev prove weighted Poincaré inequalities for the case of a finite number of low-valued inclusions in a high-valued background medium. The Poincaré constant there is independent of the values of α, but depends on the number of inclusions. New results that apply to more general situations can be found in the upcoming paper [17]. The rest of the literature (see e.g., [4, 6, 11, 16] and the references therein), resorts to problem-adapted coarse spaces based on energy minimization or solution of local eigenproblems.

FETI type methods (non-resolved case). In [10], we showed that if the coefficient is constant in the *boundary layer* of width η of each subdomain, the condition number of FETI is bounded by $C\left(\frac{H}{\eta}\right)^2(1+\log(H/h))^2$, independently of α. If the values of α in the subdomain interior is larger or equal to the value in the boundary layer, the quadratic dependency even reduces to a linear one. This result can be generalized to coefficients that vary mildly and smoothly in the boundary layer [10, 12]. In order to treat cases where the coefficient jumps *along* the interface, we made use of special Poincaré type inequalities, see [12].

The key tool in the analyses of the non-resolved case above are weighted Poincaré type inequalities of the form

$$\inf_{c\in\mathbb{R}}\int_\Omega \alpha(x)\,|u(x)-c|^2\,dx \;\le\; C\int_\Omega \alpha(x)\,|\nabla u(x)|^2\,dx \qquad \forall u\in H^1(\Omega).$$

By a variational principle, one easily shows that the infimum above is attained at

$$c \;=\; \frac{\int_\Omega \alpha(x)\,u(x)\,dx}{\int_\Omega \alpha(x)\,dx}.$$

Clearly, C will in general depend on α. However, there are a lot of situations, where C is at least independent of high contrast in α and depends just on the geometry of the isolines in the coefficient. In order to classify such situations, we have generalized in [13] the notion of quasi-monotonicity that has been introduced by Sarkis et al., cf. [2, 14].

Definition. Let the coefficient $\alpha:\Omega\to\mathbb{R}^+$ be piecewise constant with respect to a partition $\{\Omega_k\}$ of Ω and let $X^*\subset\overline{\Omega}$ be an m-dimensional manifold. Then we call α *type-m X^*-quasi-monotone* if for every Ω_ℓ there is an index k with $X^*\subset\overline{\Omega}_k$ and a path through the subregions of the partition joining Ω_ℓ and Ω_k such that (i) the interface between two subsequent subregions is an m-dimensional manifold, and (ii) the coefficient is non-decreasing along the path (starting from Ω_ℓ).

Lemma. Let α be type-$(d-1)$ X^*-quasi-monotone on a bounded domain $\Omega\subset\mathbb{R}^d$ and set $\overline{u}^{X^*} := \mathrm{meas}(X^*)^{-1}\int_{X^*} u\,dX^*$. Then,

$$\int_\Omega \alpha\,|u-\overline{u}^{X^*}|^2\,dx \;\le\; C\,\mathrm{diam}(\Omega)^2\int_\Omega \alpha\,|\nabla u|^2\,dx \qquad \forall u\in H^1(\Omega),$$

where the constant C is independent of the values of α.

If α is type-m with $m < d-1$ and the domain Ω exhibits some regularity, then we can show similar discrete inequalities for finite element functions, albeit with an additional factor of $(1+\log(\eta/h))$ if $m = d-2$ and of η/h if $m = d-3$, where η is the maximum diameter of any of the subregions Ω_k, cf. [13]. Standard inequalities with such additional factors are well summarized in [18, Sect. 4.6].

The dependence of the constant C on the geometry of α can be made explicit in some sense. In [13], we provide examples in two dimensions including domains with long thin subregions, inclusions, checkerboard distributions, etc.

Using the above lemma, robustness statements can be derived for both overlapping Schwarz and FETI methods. For two-level overlapping Schwarz, the coefficient needs to be type-m quasi-monotone in each of the patches associated with a coarse element, cf. [3, 17]. This can theoretically be achieved by adapting the coarse mesh to the coefficient. However, ideas to *automate* this by e. g. using element agglomeration are still under development [17].

For FETI, the coefficient needs to be type-m quasi-monotone in each subdomain, or, more generally, in the boundary layer of each subdomain. For the magnetostatic problem above, it can be shown (and also seen in numerical experiments) that it pays off to choose the subdomain partition such that it *does not resolve material interfaces*, but *isolates material corners* in the interior of subdomains, see [11, 12, 13].

Comparable robustness results for problems in $H(\mathrm{curl})$ are to the best of our knowledge not yet available in the unresolved case. In $H(\mathrm{curl})$, the analogue of Poincaré type inequalities are (stable) Helmholtz decompositions. A very promising result is the one by Hu and Zou [20], where a discrete Helmholtz decomposition is constructed that is orthogonal in a weighted L^2-inner product. However, the authors do not consider the case of non-resolved variation of the coefficient in the principal curl-curl term, but only non-resolved variation in the zero-order term.

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Plane Wave Discontinuous Galerkin Methods

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(joint work with Ralf Hiptmair, Andrea Moiola)

Plane wave discontinuous Galerkin methods (PWDG) are a class of Trefftz-type methods for the spatial discretization of wave problems in frequency domain.

As a model problem, consider the Helmholtz equation with impedance boundary conditions: given a bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a fixed wave number $\omega > 0$, and $g \in L^2(\partial\Omega)$, find u such that

$$\begin{aligned} -\Delta u - \omega^2 u &= 0 && \text{in } \Omega\,, \\ \nabla u \cdot \boldsymbol{n} + i\omega\, u &= g && \text{on } \partial\Omega\,, \end{aligned} \tag{1}$$

where $\boldsymbol{n}$ is the outer normal unit vector to $\partial\Omega$, and i is the imaginary unit.

The oscillatory behavior of solutions to (1) renders low order finite element methods inefficient already in medium-frequency regimes because of the *pollution effect* (see [3]): though for sufficiently small ωh, h being the mesh size, an accurate approximation of u is possible, the Galerkin procedure fails to provide it.

Attempts to remedy this have focused on incorporating extra information in the form of plane wave functions $\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d}\cdot\boldsymbol{x})$, $|\boldsymbol{d}| = 1$, into the discretization spaces. We denote by *plane wave methods* the numerical schemes arising from this approach, among which we recall: *i)* the *Plane Wave Partition of Unity Methods* (see [2]) which are conforming methods with approximating spaces constituted by modulated plane waves; *ii)* the *Ultra Weak Variational Formulation* (see [9, 5, 6]) derived from a variational formulation whose unknowns are impedance traces along interelement boundaries, and using approximating spaces made of plane waves only; *iii)* the *Discontinuous Enrichment Method* (see [10, 1, 25, 11]) with plane wave basis functions and interelement continuities enforced by means of Lagrange multipliers; *iv)* the *Wave Based Prediction Technique* (see [8]) with global wave approximation functions and boundary conditions imposed in an integral way; *v)* the *Variational Theory of Complex Rays* (see [24]) with approximating spaces made by plane waves with amplitude depending on the wave vector. We focus on the *Ultra Weak Variational Formulation* (UWVF) and variants of it.

The UWVF has recently received new interest: in a series of papers by Monk, Collino, Huttunen, Kaipio and Malinen [20, 19, 18, 16, 17, 7], algorithmic aspects, as well as new applications, have been investigated and a commercial simulation software based on it has been developed (see http://www.waveller.com/). This has motivated investigations into its theoretical foundations.

The UWVF can be recast as a *discontinuous Galerkin* (DG) method with *plane wave basis functions*. This was done in [18] and in [12] in the hyperbolic context, and in [4] and in [14] in the elliptic context. We denote by *plane wave discontinuous Galerkin methods* (PWDG) the class of general DG methods with plane wave basis functions; the methods in the PWDG class differ from each other for the choice of the numerical fluxes in their definition.

We outline the derivation of PWDG methods for (1), referring to [14, 15] for details. Introduce an auxiliary variable $\boldsymbol{\sigma}$ and rewrite problem (1) as

$$
\begin{aligned}
& i\omega\,\boldsymbol{\sigma} = \nabla u && \text{in } \Omega\,, \\
& i\omega\, u - \nabla\cdot\boldsymbol{\sigma} = 0 && \text{in } \Omega\,, \\
& i\omega\boldsymbol{\sigma}\cdot\boldsymbol{n} + i\omega\, u = g && \text{on } \partial\Omega\,.
\end{aligned}
\tag{2}
$$

Consider a partition $\mathcal{T}_h = \{K\}$ of Ω with granularity h. Write an element-by-element variational formulation of (2) in the (discontinuous) discrete spaces $V_p(\mathcal{T}_h)^d$ and $V_p(\mathcal{T}_h)$, with p denoting the elemental space dimension, and replace boundary traces by *numerical fluxes*. The auxiliary variable $\boldsymbol{\sigma}_{h,p}$ can be eliminated and a further integration by parts gives

$$
\int_K (\overline{-\Delta v_{hp} - \omega^2 v_{hp}})\, u_{hp}\,\mathrm{d}V + \int_{\partial K} \widehat{u}_{hp}\,\overline{\nabla v_{hp}\cdot\boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega\widehat{\boldsymbol{\sigma}}_{hp}\cdot\boldsymbol{n}\,\overline{v}_{hp}\,\mathrm{d}S = 0\,. \tag{3}
$$

By choosing $V_p(\mathcal{T}_h)$ locally made by linear combination of p plane waves of frequency ω in different directions, the volume term in (3) vanishes. For the numerical fluxes, denoting by $\{\!\{\cdot\}\!\}$ and $[\![\cdot]\!]_N$, with standard DG notation, the averages and jump operators, respectively, we define $\widehat{\boldsymbol{\sigma}}_{h,p} = \frac{1}{i\omega}\{\!\{\nabla_h u_{h,p}\}\!\} - \alpha\,[\![u_{h,p}]\!]_N$ and

$\widehat{u}_{h,p} = \{\!\{u_{h,p}\}\!\} - \frac{\beta}{i\omega} [\![\nabla_h u_{h,p}]\!]_N$ on interior faces (we refer to [14, 15] for the definition on boundary faces). The classical UWVF is recovered by setting $\alpha = \beta = 1/2$, while one can enhance stability by choosing hp–dependent α and β (see [14, 15]).

Assume Ω convex, $\mathcal{T}_h$ shape-regular and quasi-uniform, $p = 2m + 1$ in 2D or $p = (m + 1)^2$ in 3D, with integer $m \geq 1$ (i.e., p is the dimension of the harmonic polynomial space of degree m in 2 or 3 variables). We can summarize the theoretical results available on PWDG methods in the following table:

	$2D$	$3D$	pollution effect	energy-error q-opt. estim.	L^2–error q-opt. estim.
h–version*	✓	✓	yes	✓	✓
p–version	✓	missing**	no	✓	open pbl.

* for hp–dependent flux parameters;
** only best approximation estimates are missing.

For the h–version error analysis, slightly suboptimal error estimates in the L^2–norm have been proved in [4] for the UWVF, while in [14] low order convergence (1 in energy-norm and 2 in L^2–norm) has been proved for the PWDG method with hp–flux parameters applied to the inhomogeneous Helmholtz problem (i.e., right-hand side $f \in L^2(\Omega)$ in the first equation of (1)), by using a duality argument and under a threshold condition of the type "$\omega^2 h$ small enough". These estimates are sharp, as demonstrated in [13]. The same theory has been applied in [21] to get quasi-optimal h–convergence in both 2D and 3D for the homogeneous Helmholtz problem (order m in energy-norm and $m + 1$ in L^2–norm), exploiting best approximation properties for homogeneous Helmholtz solutions by means of plane wave spaces proved in [22] (for generic functions, plane wave spaces only have low order approximation capability, independently of p).

The error analysis for the p–version of the PWDG method has been developed in [15]. Even if the theory covers both the two- and three-dimensional cases, the final estimates have been obtained for the 2D case only, since p–version best approximation estimates (see [22]) are not available, at the moment, in 3D. Both the cases of constant and special hp–flux parameters are considered, thus including the case of the standard UWVF. A standard duality argument is no longer applicable since plane wave do not have high order approximation properties for general functions, thus the same approach as the one in [4], based on a result contained in [23], has been adopted. Quasi-optimal error estimates in energy-norm only have been obtained (and exponential convergence for smooth analytical solutions).

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Discontinuous Galerkin methods and retarded potentials for time dependent wave propagation problems on unbounded domains

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(joint work with Toufic Abboud, Patrick Joly and Isabelle Terrasse)

This work has been motivated by computational acoustics. More precisely, in aeroacoustics, it is of interest to compute acoustic wave propagation in the presence of a uniform flow that is *locally* perturbed by a scatterer. Assuming that this flow is steady and given, one could use the linearized Euler equations (LEEs) to model the problem. From the numerical point of view, this gives rise to two major difficulties: i) due to the presence of convective terms in the equations, there is no natural variational formulation of the problem; thus the finite element method can not be directly applied, ii) one needs to account the unbounded nature of the computational domain.

Discontinuous Galerkin methods (DGM) have recently gained attention for the resolution of time dependent wave propagation problems [10]. These methods enjoy of a great flexibility in terms of $h - p$ adaptivity and can easily handle heterogeneities. Moreover, they are well adapted to solve equations containing convective terms such as aeroacoustics.

When the problem is posed on an unbounded domain, one needs to artificially bound the computational domain. Many approaches to tackle this problem have been proposed in the literature. Under some assumptions on the external boundary one can use local/explicit methods such as absorbing boundary conditions [7] or perfectly matched layers. However, both techniques are not exact and can meet stability issues when applied to the advective wave equation (see however [4, 12] for some works on the stabilization of the second technique). The increasing computational power together with the progress of rapid algorithms [5] like the fast multipole method make possible, at least when the exterior domain is homogeneous, the use of exact or transparent boundary conditions. Such conditions rely on the explicit representation of the solution on the exterior domain Ω_e in terms of its traces on the boundary $\Gamma = \overline{\Omega_i} \cap \overline{\Omega_e}$ using the so-called retarded potential representations (RP). Among these techniques we can distinguish two types: i) those based on collocation techniques [5], easier to implement but might meet some stability issues [3]; ii) those based on a Galerkin approach [9], leading to more robust methods.

For these reasons we have investigated the question of coupling a discontinuous Galerkin method in the interior domain with a Galerkin retarded potential method on the boundary. To start with, we have considered the simplified model of the scalar wave equation. Even if there are many contributions to the coupling of the finite element method (FEM) and the boundary element method (BEM) for elliptic and harmonic problems [11], there are much less works for time dependent problems and they mainly concern collocation RP methods [2, 13]. We should mention some works concerning the coupling of FEM and Galerkin RP in different frameworks [8, 1]. During the talk, we have introduced an algorithm allowing to

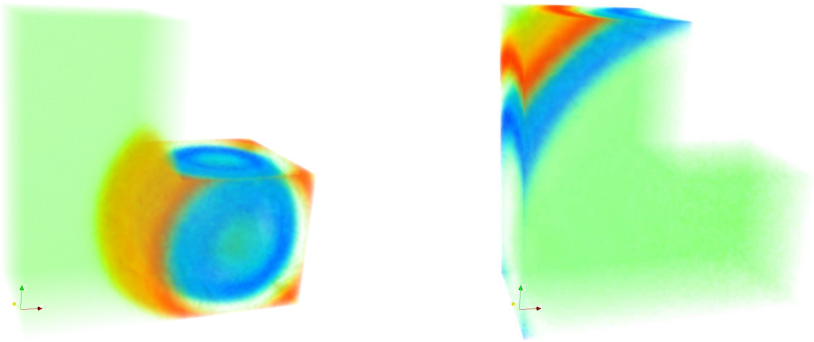
couple a discontinuous Galerkin method combined with explicit finite differences in time in the interior domain with a Galerkin retarded potential approximation on the external boundary. The coupling, based on the ideas introduced in [8, 6], permits to use both methods with *quasi*-optimal discretization parameters in the sense that will be explained later. The stability of the global discretization is obtained when the usual CFL condition in the interior domain is satisfied by means of a discrete energy identity enforced by the coupling formula. We have shown some numerical experiments on academic problems that show the feasibility of the whole discretization procedure. The algorithm can be generalized to other symmetric hyperbolic systems such as LEEs.

Let us provide more details on the construction of the coupling algorithm. We start by rewriting the problem as a transmission problem between the interior domain Ω_i and the exterior domain Ω_e. The equations on the exterior are replaced by two integral equations on the boundary Γ involving the four standard integral operators for the transient wave equation [9]. Both systems are coupled by the usual transmission conditions on the same boundary. Appropriate manipulations allow us to derive a variational formulation of the coupled problem with the following properties (this is the first key point of the method): i) it is well suited for the discretization with DG methods in space with central fluxes and explicit finite differences in time in Ω_i, ii) it is also well adapted to a Galerkin approximation using retarded potentials on the boundary Γ, iii) the transmission conditions are directly included on the formulation. As a consequence, using the solutions on the exterior and in the interior as the test functions, one obtains an equivalent of the usual energy identity for the wave equation providing a priori stability estimates.

The second step on the construction of the numerical method is the discretization of the terms in the global variational formulation that are not involved on the coupling. We use a Galerkin space-time approximation for the integral equations with a typical time step given by Δt, with $c\Delta t/h$ of the order of 1 to use the RP method with quasi-optimal settings (this choice is a good trade-off between the numerical cost and the accuracy). In the interior we discretize first in space employing a DG method by using typically a tetrahedric mesh of the domain and piecewise discontinuous P_k elements. The resulting differential system is discretized using a second order finite difference scheme (leap frog scheme). The time step needs to be rather small in the interior domain in order to satisfy the CFL condition introduced by the explicit scheme. That is why we use a discretization parameter given by $\Delta t/p$ where p is a given positive integer. This allows to perform the computations in the interior and on the boundary with quasi-optimal discretization parameters. The key point at this stage is that the coupling terms have not been discretized yet.

The third step consists on establishing a discrete energy identity (similar to the one obtained with the continuous variational formulation) which will drive the construction of the scheme for the coupling terms. This identity involves some terms associated to the coupling (not yet discretized). From this identity we propose a coupling formula that does not introduce any energy dissipation or amplification.

In this way, the global discretization procedure is well-posed and stable under the usual CFL condition in the interior domain. Moreover there is almost no additional cost due to the coupling. The coupling scheme, being stable by construction (under the usual CFL condition in Ω_i), is very robust. However, most of the coupling terms propose by this method are only first order consistent, which generate high frequency spurious oscillations in the interior domain (aliasing phenomena). Following the ideas on [14], we can post-process the interior solution by time averaging, which restores the second order accuracy of the interior scheme. Moreover, one can build a new coupling algorithm allowing to compute the post-processed unknowns directly. We have tested this coupling algorithm in some academical configurations obtaining promising results. We can see in the figure two snapshots of a spherical wave propagating on the free space. We have considered a (non-convex) L-shaped domain as the interior domain. No visible spurious reflexions on the artificial boundary can be seen.



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Energy estimates for space discretizations of Time–Domain Integral Equations

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In the mathematical literature, time–domain boundary integral equations methods can be traced to the seminal work of Alain Bamberger and Tuong Ha Duong [2], [3], although the key to them is the classical Kirchhoff formula for wave propagation around obstacles. Bamberger and Ha Duong's proof of ellipticity of the single and double layer boundary integral operators for the acoustic wave equation sparked extensive work that dealt with exterior and transmission problems for acoustic, elastic and electromagnetic waves (see [1], [6], [12], to name just a few). This wide and deep literature in the French school of numerical analysis, as well as many other recent approaches, is summarized in [11]. Note that, unfortunately, many of the results remained in doctoral dissertations and have not been widely available, which has limited recognition of this first wave of results. The methods that were explored in this first group of publications are space and time Galerkin methods. To the best of the author's knowledge, in all of them the finite dimensional spaces are tensor products of time and space discretizations. Some time later, based on his convolution quadrature method (first introduced and analyzed in [18] and [19]), Christian Lubich proposed a new way of approaching the discretization of the single layer acoustic retarded integral equation [20]. Space discretization is still of Galerkin type, but the time variable is dealt with by using the Laplace transform. The collocation method has also been explored in [9]. All these approaches have been the object of systematic and serious practical testing in the engineering literature: see [10] and [24], as well as their many sequels. Recently, the mathematical interest in this family of methods, that never abandoned the electrical and mechanical engineering community, has been stirred again and several research groups have started proposing new solutions to unsolved problems both in the area of time–Galerkin discretization and with Convolution Quadrature methods: [4, 5, 7, 13, 16, 17].

The object of this note is the presentation of some new results concerning the behavior of the energy as a function of time when some of the time domain boundary integral equations (or coupled systems of these with variational formulations of wave propagation problems in interior heterogeneous obstacles) are discretized

in space using Galerkin methods. The results are then pertinent to both time–Galerkin and convolution quadrature discretizations. No restriction is imposed on the choice of finite element spaces for the space variables discretizations.

The results use elementary arguments. On the one hand, we rely on C_0-groups of isometries and their relation with second order initial value problems in Hilbert spaces, that is, on the adaptation of Lumer–Phillips theorem [21] to wave equations. On the other hand, we develop the idea of recasting discretized integral equations and boundary–field problems (BEM and BEM–FEM problems) as exotic transmission problems in the entire space. This technique was devised by the author of this talk in collaboration with Antonio Laliena in [17]. The trick has been later employed in [23] and [22] to solve some open questions on coupling BEM and FEM with only one integral equation on polyhedral interfaces.

With these tools, we can prove some interesting preliminary results. For instance, the space discretization of the indirect time domain integral equation associated to the scattering of acoustic waves by a sound soft obstacle preserves the total energy (kinetic plus potential) as a function of time. The total energy has to be computed in the exterior as well as in the interior of the obstacle, which shows that space discretization produces leakage of energy to the interior of the obstacle. The same fact holds for the indirect approach for the sound hard case.

With direct methods (discretizations of the Kirchhoff formula) this is not the case any more. In addition to the interior leakage of energy, there is an oscillation term that changes energy as a function of time and that decreases as space discretization becomes richer. The understanding of this term is not full yet. It can be vaguely described as a function that scatters (both inside and outside the obstacle) the part of the normal derivative of the incident wave that cannot be approximated by the space used for BEM discretization of the normal derivative of the scattered wave. The fact that we are scattering a function that does not appear explicitly in the numerical method just adds some mystery to the problem.

In the case of non–homogeneous obstacles, the BEM–FEM formulation *à la Costabel–Han* ([8], [14]) has a principal component that preserves energy as well as an additional oscillation term. An interesting fact is that the Johnson–Nédélec coupling (with only one integral equation [15]) is associated to an evolution equation that is not energy conservative. This fact might be a hint of the practical impossibility of using this non–symmetric coupling method for time dependent problems and on the need of using symmetric formulations.

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A Posteriori Error Analysis for Hybridized Interior Penalty Discontinuous Galerkin Method for H(curl)-Elliptic Problems

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(joint work with R. H. W. Hoppe and T. Warburton)

We develop and analyze an adaptive hybridized interior penalty discontinuous Galerkin (IPDG-H) method for H(curl)-elliptic boundary value problems in 3D arising from a semi-discretization of the eddy current equations in time. The IPDG-H method is derived from a mixed formulation. We study a residual-type error estimator and establish its reliability within the unified framework developed in [3]. The performance of the method is illustrated by a numerical example.
Given a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset$, we consider the H(curl)-elliptic problem

$$\begin{aligned} \mathbf{curl}\ \mu^{-1}\ \mathbf{curl}\ \mathbf{u} + \sigma \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \boldsymbol{\gamma}_t(\mathbf{u}) := \mathbf{u} \wedge \mathbf{n} &= \mathbf{g}_1 \quad \text{on } \Gamma_D, \\ \boldsymbol{\pi}_t(\mu^{-1}\mathbf{curl}\ \mathbf{u}) := \mathbf{n} \wedge (\mu^{-1}\mathbf{curl}\ \ \mathbf{u} \wedge \mathbf{n}) &= \mathbf{g}_2 \quad \text{on } \Gamma_N, \end{aligned}$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{g}_1 \in \mathbf{L}^2(\Gamma_D)$ and $\mathbf{g}_2 \in \mathbf{H}(\mathbf{curl}^0_{\Gamma_N}; \Gamma_N)$. We further assume that μ is symmetric positive definite and $\sigma \geq 0$, both being elementwise constant with respect to a given coarse simplicial triangulation $\mathcal{T}_H(\Omega)$. We set $\mathbf{V} := \{\mathbf{v} \in \mathbf{H}(\mathrm{curl}; \Omega) \mid \boldsymbol{\gamma}_t(\mathbf{u}) = \mathbf{g}_1\}$, $\mathbf{Q} := \mathbf{L}^2(\Omega)$, and denote by $\mathbf{V}_0$ the subspace of $\mathbf{V}$ with zero tangential trace on Γ_D. Introducing $\mathbf{p} := \mu^{-1}\mathbf{curl}\ \mathbf{u}$ as an additional variable, the mixed formulation amounts to the computation of $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}$ such that

$$\mathcal{A}(\mathbf{u}, \mathbf{p}) = \ell^{(1)} + \ell^{(2)}. \tag{1}$$

Here, $\mathcal{A} : \mathbf{V} \times \mathbf{Q} \to \mathbf{Q}^* \times \mathbf{V}_0^*$ is the operator $(\mathcal{A}(\mathbf{u}, \mathbf{p}))(\mathbf{v}, \mathbf{q}) := a(\mathbf{p}, \mathbf{q}) - b(\mathbf{u}, \mathbf{q}) + b(\mathbf{v}, \mathbf{p}) + c(\mathbf{u}, \mathbf{v})$, where $a(\mathbf{p}, \mathbf{q}) := \int_\Omega \mu\ \mathbf{p} \cdot \mathbf{q}\ dx$, $b(\mathbf{u}, \mathbf{q}) := \int_\Omega \mathbf{curl}\ \mathbf{u} \cdot \mathbf{q}\ dx$, and $c(\mathbf{u}, \mathbf{v}) := \int_\Omega \sigma\ \mathbf{u} \cdot \mathbf{v}\ dx$. Moreover, the functionals $\ell^{(i)}, 1 \leq i \leq 2$, are given by $\ell^{(1)}(\mathbf{q}) := 0$ and $\ell^{(2)}(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dx + \langle \mathbf{g}_2, \boldsymbol{\gamma}_\mathbf{t}(\mathbf{v}) \rangle$.

Theorem 0.1. *The operator $\mathcal{A}$ is a continuous, bijective linear operator. Hence, for any $\ell^{(1)}, \ell^{(2)} \in \mathbf{Q}^* \times \mathbf{V}_0^*$ equation (1) admits a unique solution $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}$ such that*

$$\|(\mathbf{u}, \mathbf{p})\|_{\mathbf{V} \times \mathbf{Q}} \lesssim \|\ell^{(1)}\|_{\mathbf{Q}^*} + \|\ell^{(2)}\|_{\mathbf{V}_0^*}. \tag{2}$$

IPDG Method: DG methods are based on the approximation of $(\mathbf{u}, \mathbf{p})$ by elementwise polynomials $(\mathbf{u}_H, \mathbf{v}_H) \in \mathbf{V}_H \times \mathbf{Q}_H$, where $\mathbf{V}_H = \mathbf{Q}_H := \{\mathbf{v}_H \in \mathbf{L}^2(\Omega) \mid \mathbf{v}_H|_T \in \mathbf{\Pi}_k(T)), T \in \mathcal{T}_H(\Omega)\}$ and $\mathbf{\Pi}_k(T), k \in \mathbb{N}$, stands for the linear space of vector fields whose components are polynomials of degree k. We denote by a_H, b_H and c_H the mesh-dependent bilinear forms as given by $a_H(\mathbf{p}_H, \mathbf{q}_H) := \sum_{T \in \mathcal{T}_H(\Omega)} \int_T \mu \mathbf{p}_H \cdot \mathbf{q}_H\ dx$ etc. and introduce $d_H : \mathbf{V}_H \times \mathbf{Q}_H \to \mathbb{R}$ according to $d_H(\mathbf{u}_H, \mathbf{q}_H) := \sum_{F \in \mathcal{F}_H(\bar{\Omega})} \int_F \boldsymbol{\gamma}_\mathbf{t}(\mathbf{u}_H) \cdot \boldsymbol{\pi}_\mathbf{t}(\mathbf{q}_H)\ d\tau$, where $\mathcal{F}_H(\bar{\Omega})$ refers to the set

of faces of the triangulation. We further define $\ell_H^{(1)} : \mathbf{Q}_H \to \mathbb{R}$ and $\ell_H^{(2)} : \mathbf{V}_H \to \mathbb{R}$ by $\ell_H^{(1)}(\mathbf{q}_H) = 0$ and $\ell_H^{(2)}(\mathbf{v}_H) := \sum_{T\in\mathcal{T}_H(\Omega)} \int_T \mathbf{f}\cdot\mathbf{v}_H \; dx + \sum_{F\in\mathcal{F}_H(\Gamma_N)} \int_F \mathbf{g}_2 \cdot \boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{v}_H)\; d\tau$. DG methods amount to the computation of $(\mathbf{u}_H, \mathbf{p}_H) \in \mathbf{V}_H \times \mathbf{Q}_H$ such that

$$a_H(\mathbf{p}_H, \mathbf{q}_H) + b_H(\mathbf{q}_H, \mathbf{u}_H) + d_H(\widehat{\mathbf{u}}_H, \mathbf{q}_H) = \ell_H^{(1)}(\mathbf{q}_H) \quad , \quad \mathbf{q}_H \in \mathbf{Q}_H, \tag{3a}$$

$$b_H(\mathbf{p}_H, \mathbf{v}_H) - d_H(\mathbf{v}_H, \widehat{\mathbf{p}}_H) + c_H(\mathbf{u}_H, \mathbf{v}_H) = \ell_H^{(2)}(\mathbf{v}_H) \quad , \quad \mathbf{v}_H \in \mathbf{V}_H, \tag{3b}$$

where the non-conformity of the approach is taken care by the proper specification of the numerical fluxes $\widehat{\mathbf{u}}_H$ and $\widehat{\mathbf{p}}_H$. In particular, if the numerical fluxes are chosen by means of

$$\boldsymbol{\gamma}_{\mathbf{t}}(\widehat{\mathbf{u}}_{\mathbf{H}}) := \begin{cases} \{\boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{u}_H)\} \;, & F \in \mathcal{F}_H(\Omega), \\ 0 \qquad\quad , & F \in \mathcal{F}_H(\Gamma) \end{cases} , \tag{4a}$$

$$\boldsymbol{\pi}_{\mathbf{t}}(\widehat{\mathbf{p}}_{\mathbf{H}}) := \begin{cases} \{\boldsymbol{\pi}_{\mathbf{t}}(\mu^{-1}\mathbf{curl}\ \mathbf{u}_H)\} - \alpha\ h_F^{-1}\ [\boldsymbol{\gamma}_{\mathbf{t}}(\mathbf{u}_H)] \;, & F \in \mathcal{F}_H(\Omega) \\ 0 \qquad\qquad\qquad\qquad\qquad\quad , & F \in \mathcal{F}_H(\Gamma) \end{cases} , \tag{4b}$$

with $\alpha > 0$ denoting a suitably chosen penalty parameter, we obtain the symmetric IPDG method for H(curl)-elliptic problems as studied in [1].

Hybridization of the IPDG Method: The idea of hybridization is to enforce the continuity of the tangential traces of $\mathbf{u}_H$ across the interior edges of the triangulation by a Lagrange multiplier in the space $\mathbf{M}_H := \{\boldsymbol{\mu}_H \in L^2(\mathcal{F}_H) \;\big|\; \boldsymbol{\mu}_H|_F \in \boldsymbol{\Pi}_k(F),\ F \in \mathcal{F}_H(\bar{\Omega})\}$. Choosing a numerical flux function $\widehat{\mathbf{p}}_H$, not necessarily the same as above, the IPDG-H method is to find $(\mathbf{u}_H, \mathbf{p}_H, \boldsymbol{\lambda}_H) \in \mathbf{V}_H \times \mathbf{Q}_H \times \mathbf{M}_H$ such that

$$a_H(\mathbf{p}_H, \mathbf{q}_H) - b_H(\mathbf{q}_H, \mathbf{u}_H) + d_H(\boldsymbol{\lambda}_H, \mathbf{q}_H) = \ell_H^{(1)}(\mathbf{q}_H) \quad , \quad \mathbf{q}_H \in \mathbf{Q}_H, \tag{5a}$$

$$b_H(\mathbf{p}_H, \mathbf{v}_H) - d_H(\mathbf{v}_H, \widehat{\mathbf{p}}_H) + c_H(\mathbf{u}_H, \mathbf{v}_H) = \ell_H^{(2)}(\mathbf{v}_H) \quad , \quad \mathbf{v}_H \in \mathbf{V}_H, \tag{5b}$$

$$d_H(\boldsymbol{\mu}_H, \widehat{\mathbf{p}}_{\mathbf{H}}) = 0 \quad , \quad \boldsymbol{\mu}_H \in \mathbf{M}_H. \tag{5c}$$

We eliminate the vector fields $\mathbf{u}_H$ and $\mathbf{p}_H$ by static condensation and this results in a global variational system only in the Lagrange multiplier $\boldsymbol{\lambda}_H$. Once we have computed $\boldsymbol{\lambda}_H$, we obtain $\mathbf{u}_H$ and $\mathbf{p}_H$ by the solution of low-dimensional, local subproblems. To this end, we follow the unified framework from [2].

A Posteriori Error Analysis: It is an immediate consequence of (2) that if $(\tilde{\mathbf{u}}_H, \tilde{\mathbf{p}}_H) \in \mathbf{V} \times \mathbf{Q}$ is an approximation of the solution $(\mathbf{u}, \mathbf{p})$ of (1), then

$$\|(\mathbf{u} - \tilde{\mathbf{u}}_H, \mathbf{p} - \tilde{\mathbf{p}}_H)\|_{\mathbf{V}\times\mathbf{Q}} \;\lesssim\; \|\mathrm{Res}_1\|_{\mathbf{Q}^*} + \|\mathrm{Res}_2\|_{\mathbf{V}_0^*},$$

where the residuals $\mathrm{Res}_i, 1 \le i \le 2$, are given by $\mathrm{Res}_1(\mathbf{q}) := \ell^{(1)}(\mathbf{q}) - a(\tilde{\mathbf{p}}_H, \mathbf{q}) + b(\tilde{\mathbf{u}}_H, \mathbf{q})$ and $\mathrm{Res}_2(\mathbf{v}) := \ell^{(2)}(\mathbf{v}) - b(\mathbf{v}, \tilde{\mathbf{p}}_H) - c(\tilde{\mathbf{u}}_H, \mathbf{v})$. Now, if $(\mathbf{u}_H, \mathbf{p}_H, \boldsymbol{\lambda}_H)$ is the solution of (5a)-(5b), we choose $\tilde{\mathbf{p}}_H = \mathbf{p}_H$ and $\tilde{\mathbf{u}}_H$ as the unique minimizer of the consistency error

$$\xi := \min_{\tilde{\mathbf{v}}_H \in \mathbf{V}} \Big(\sum_{T\in\mathcal{T}_H(\Omega)} \int_T (|\mathbf{u}_H - \tilde{\mathbf{v}}_H|^2 + |\mathbf{curl}(\mathbf{u}_H - \tilde{\mathbf{v}}_H)|^2)\; dx \Big)^{1/2}.$$

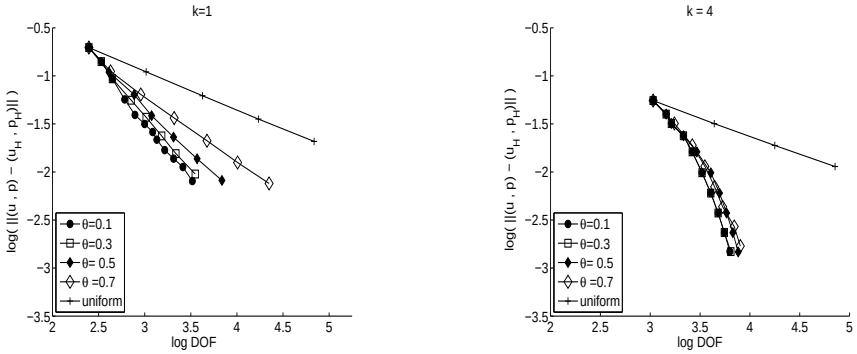
The norms of the residuals $\mathrm{Res}_i, 1 \le i \le 2$, and the consistency error ξ can be estimated by a residual-type a posteriori error estimator η consisting of the element residuals $\eta_{T,1} := \|\mu\, \mathbf{p_H} - \mathbf{curl}\ \mathbf{u_H}\|_{0,T}, \eta_{T,2} := h_T\|\mathbf{f} - \mathbf{curl}\ \mathbf{p}_H - \sigma\mathbf{u}_H\|_{0,T}, \eta_{T,3} := h_T\|\nabla\cdot(\mathbf{f}-\sigma\mathbf{u}_H)\|_{0,T}, T \in \mathcal{T}_H(\Omega)$, and the face residuals $\eta_{F,1} := h_F^{1/2}\|[\boldsymbol{\pi}_t(\mathbf{p}_H)]\|_{0,F}$, $\eta_{F,2} := h_F^{1/2}\|\mathbf{n}_F \cdot [\mathbf{f} - \sigma\mathbf{u}_H)]\|_{0,F}$, $\eta_{F,3} := h_F^{-1/2}\|[\gamma_t(\mathbf{u}_H)]\|_{0,F}, F \in \mathcal{F}_H(\Omega)$, and $\eta_{F,4} := h_F^{-1/2}\|\mathbf{g}_1 - \gamma_t(\mathbf{u}_H)\|_{0,F}, F \in \mathcal{F}_H(\Gamma_D), \eta_{F,5} := h_F^{1/2}\|\mathbf{g}_2 - \boldsymbol{\pi}_t(\mathbf{p}_H)\|_{0,F}, \eta_{F,6} := h_F^{1/2}\|\mathbf{n}_F \cdot (\mathbf{f} - \sigma\mathbf{u}_H))\|_{0,F}, F \in \mathcal{F}_H(\Gamma_N)$.

Theorem 0.2. *Let* $(\mathbf{u}, \mathbf{p}) \in \mathbf{V} \times \mathbf{Q}$ *and* $(\mathbf{u_H}, \mathbf{p_H}, \boldsymbol{\lambda_H}) \in \mathbf{V_H} \times \mathbf{Q_H} \times \mathbf{M_H}$ *be the solutions of the mixed formulation* (1) *and the IPDG-H method* (5a)-(5c)*, respectively. Then, there holds*

$$\|(\mathbf{u}, \mathbf{p}) - (\mathbf{u}_H, \mathbf{p}_H)\| \ \lesssim\ \eta,$$

where $\|(\mathbf{u}, \mathbf{p}) - (\mathbf{u}_H, \mathbf{p}_H)\| := (\|\mathbf{p} - \mathbf{p}_H\|_{0,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_H\|_{curl,H,\Omega}^2)^{1/2}$ *and* $\|\cdot\|_{curl,H,\Omega}$ *refers to the broken* $H(curl, \Omega)$*-norm.*

Numerical Results: For a 2D H(curl)-elliptic boundary value problem on the L-shaped domain $\Omega = (0,1)^2 \setminus ((0,1) \times 0) \cup (0 \times (0,1))$ with Dirichlet data on $\Gamma_D := (0 \times (0,1) \cup (0,1) \times 0)$ and Neumann data on $\Gamma_N := \Gamma \setminus \Gamma_D$ such that $\mathbf{u} = \mathbf{grad}(r^{\frac{2}{3}} \sin(\frac{2}{3}\phi))$ is the exact solution, the figure below shows the decrease of the error as a function of the degrees of freedom (DOF) on a logarithmic scale in case $k = 1$ (left) and $k = 4$ (right) for both uniform refinement and adaptive refinement using various values of the universal constant $0 < \theta < 1$ in the standard bulk criterion that has been implemented for marking elements and edges for refinement.



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Propagation of acoustic waves in fractal networks

Adrien Semin
(joint work with Patrick Joly)

1. Introduction

We are interested in solving the wave equation in fractal domains such as human lungs, that can be modeled modulo some approximation as infinite dyadic trees (as in [4, 6]). As it is not possible to do numerical computations on the whole geometry, the idea is to truncate resolution of wave generation to a finite number of generations, and to replace remaining generations by adapted DtN operators, which is possible if one assumes that the cut subtrees are self-similar.

2. Theoretical aspects

For this part, we consider that we work in $\mathbb{R}^d$ (with $d = 2$ or $d = 3$).

2.1. Self-similar p-adic tree.

We can define a self-similar p-adyc tree by

- a finite closed segment Σ given by $\Sigma = \{(t, 0), 0 \leq t \leq 1\}$ (in $\mathbb{R}^2$) or $\Sigma = \{(t, 0, 0), 0 \leq t \leq 1\}$ (in $\mathbb{R}^3$),
- p strictly contracting direct similitudes $(s_i)_{0\leq i<p}$ of ratio $\alpha_i < 1$ such that $s_i(\mathbf{0}) = (\mathbf{1})$ for any i.

where $\mathbf{0}$ is the origin and $\mathbf{1} = (1, 0)$ (in $\mathbb{R}^2$) or $\mathbf{1} = (1, 0, 0)$ (in $\mathbb{R}^3$).

With these datas, we build the tree $\mathcal{T}$ by induction: we define $\mathcal{T}^0 = \Sigma$; given $n \in \mathbb{N}$, we define $\mathcal{T}^{n+1} = \mathcal{T}^n \cup s_0(\mathcal{T}^n) \cup \cdots \cup s_{p-1}(\mathcal{T}^n)$; finally, we define $\mathcal{T} = \bigcup \mathcal{T}^n$.

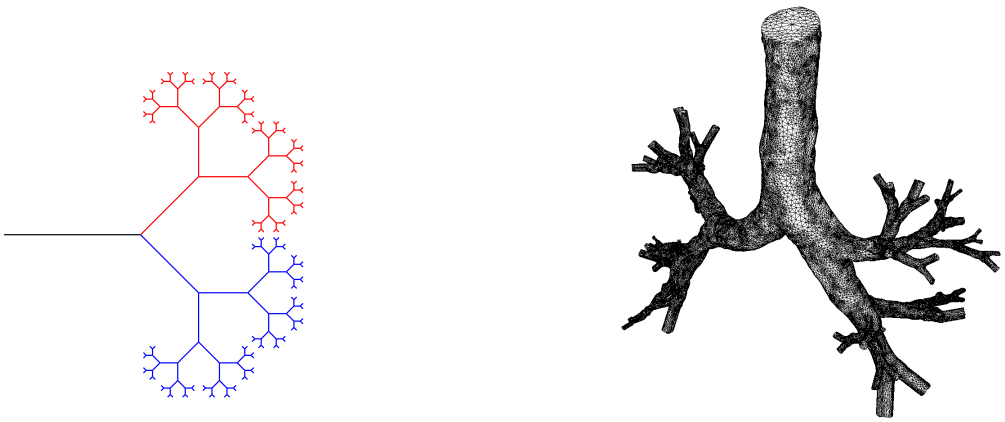

Figure 1. On the left: the tree $\mathcal{T}$ (whole figure), and subtrees $\mathcal{T}_0$ and $\mathcal{T}_1$. On the right: mesh associated to the partial 3D lung.

We shall denote by $\mathbb{E}(\mathcal{T})$ the set of edges of $\mathcal{T}$ and by $\mathbb{V}(\mathcal{T})$ the set of interior vertices of $\mathcal{T}$. We also define subtrees $\mathcal{T}_i$ of $\mathcal{T}$ as $\mathcal{T}_i = s_i(\mathcal{T})$ (see figure 1 for an example of configuration with $p = 2$).

2.2. **Variational spaces and Helmholtz equations.** On $\mathcal{T}$, given $\{\mu_i > 0,\ 0 \leq i < p\}$, we define the unique piecewise constant weight function $\widetilde{\mu} : \mathcal{T} \to \mathbb{R}_+^*$ such that:

$$\begin{cases} \widetilde{\mu} = 1 \quad \text{on } \Sigma, \\ \widetilde{\mu} \circ s_i = \mu_i\, \widetilde{\mu} \end{cases}$$

and we denote by $\widetilde{\mu}_e$ the value of $\widetilde{\mu}$ on the edge $e \in \mathbb{E}(\mathcal{T})$.

We define then the weighted "broken" norms depending of μ as

$$\|u\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} = \sum_{e\in\mathbb{E}(\mathcal{T})} \widetilde{\mu}_e \, \|u\|^2_{\mathrm{L}^2(e)} \qquad |u|^2_{\mathrm{H}^1_\mu(\mathcal{T})} = \sum_{e\in\mathbb{E}(\mathcal{T})} \widetilde{\mu}_e \, \|u'\|^2_{\mathrm{L}^2(e)} \ ,$$

where u' on $e \in \mathbb{E}(\mathcal{T})$ is the derivative of u with respect to the curvilinear abscissa along e, and the associated Sobolev spaces

$$\begin{aligned} \mathrm{H}^1_\mu(\mathcal{T}) &= \Big\{ v \text{ continuous such that } \|u\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} + |u|^2_{\mathrm{H}^1_\mu(\mathcal{T})} < \infty \Big\} \\ \mathrm{H}^1_{\mu,0}(\mathcal{T}) &= \text{closure of } \left\{ v \in \mathrm{H}^1_\mu(\mathcal{T}) \text{ such that } \exists\, n \in \mathbb{N},\ v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}_n \right\} . \end{aligned}$$

Moreover we define the following Besov spaces

$$\begin{aligned} \mathcal{H}^1_\mu(\mathcal{T}) &= \Big\{ v \text{ continuous such that } |v(\mathbf{0})|^2 + |u|^2_{\mathrm{H}^1_\mu(\mathcal{T})} < \infty \Big\} \\ \mathcal{H}^1_{\mu,0}(\mathcal{T}) &= \text{closure of } \left\{ v \in \mathcal{H}^1_\mu(\mathcal{T}) \text{ such that } \exists\, n \in \mathbb{N},\ v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}_n \right\} . \end{aligned}$$

We also define the Helmholtz problem with "Neumann" or "Dirichlet" condition at infinity: find $u \in \mathrm{H}^1_\mu(\mathcal{T})$ (resp. $u \in \mathrm{H}^1_{\mu,0}(\mathcal{T})$) such that $u(\mathbf{0}) = 1$ and, for any test function $v \in \mathrm{H}^1_\mu(\mathcal{T})$ (resp. $v \in \mathrm{H}^1_{\mu,0}(\mathcal{T})$):

$$\int_{\mathcal{T}} \widetilde{\mu} u' v' - \omega^2 \int_{\mathcal{T}} \widetilde{\mu} u v = 0, \qquad \text{where } \omega \in \mathbb{C} \text{ is the wave pulsation.} \tag{1}$$

This formulation automatically implies a homogeneous wave equation on each edge $e \in \mathbb{E}(\mathcal{T})$, and standard Kirchhoff conditions on each interior vertex $v \in \mathbb{V}(\mathcal{T})$. Standard Kirchhoff conditions are detailed in [5, 3].

Remark. *The particular choice $\mu_i = \alpha_i^{d-1}$ (the associated tree is called a d-geometric tree) is obtained by considering $\mathcal{T}$ as the limit of $\mathcal{T}^\varepsilon$ when ε tends to 0, where $\mathcal{T}^\varepsilon$ is built as $\mathcal{T}$ - the only difference is that Σ^ε is a d-dimensional domain which tends to Σ when ε tends to 0. Then (1) appears as the limit model for the solution of the d-dimensional homogeneous Helmholtz equation on $\mathcal{T}^\epsilon$.*

2.3. **Results.** When $\mathrm{Im}(\omega) \neq 0$, problem (1) admits a unique solution $u_{\mathfrak{n}} \in \mathrm{H}^1_\mu(\mathcal{T})$ (resp. $u_{\mathfrak{d}} \in \mathrm{H}^1_{\mu,0}(\mathcal{T})$). Moreover, one has

$$u_{\mathfrak{n}} \neq u_{\mathfrak{d}} \iff \mathrm{H}^1_\mu(\mathcal{T}) \neq \mathrm{H}^1_{\mu,0}(\mathcal{T}) \iff \sum \frac{\mu_i}{\alpha_i} > 1 \,. \tag{2}$$

In the following, we assume that (2) is satisfied (this is the interesting case). By denoting $\Lambda_{\mathfrak{n}}(\omega)$ the value of $u'_{\mathfrak{n}}(\mathbf{0})$ (resp. $\Lambda_{\mathfrak{d}}(\omega)$ the value of $u'_{\mathfrak{d}}(\mathbf{0})$), we can replace the Helmoltz equation on $\mathcal{T}$ by a transparent DtN condition

$$u'(\mathbf{0}) = \Lambda_{\mathfrak{n}}(\omega) u(\mathbf{0}) \qquad (\text{resp. } u'(\mathbf{0}) = \Lambda_{\mathfrak{d}}(\omega) u(\mathbf{0})) \,. \tag{3}$$

Proposition 2.1. *$\Lambda_{\mathfrak{n}}$ and $\Lambda_{\mathfrak{d}}$, as functions of ω, satisfy the following quadratic relation (obtained by looking at the problem satisfied on each subtree $\mathcal{T}_i$, this approach is similar to the approach done in* [1]*)*

$$\Lambda(\omega)\cos(\omega) - \omega\sin(\omega) = \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \left(\cos(\omega) + \frac{\Lambda(\omega)}{\omega} \sin(\omega) \right) \Lambda(\alpha_i \omega) \,. \tag{4}$$

For $\omega = 0$, (4) becomes

$$\Lambda(0) = \Lambda(0)\Big(1 + \Lambda(0)\Big) \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \,, \tag{5}$$

whose solutions are $\Lambda_{\mathfrak{n}}(0) = 0$ and $\Lambda_{\mathfrak{d}}(0) = (1 - \sum \mu_i/\alpha_i)/(\sum \mu_i/\alpha_i)$.

Conjecture 2.1. *There exists at most two homeomorphic functions Λ satisfying (4), and the unicity is given knowning $\Lambda(0)$.*

If the length of Σ is ℓ instead of 1, one has the following scaling formulas for the traces $\Lambda_{\mathfrak{n}}(\ell, \omega)$ and $\Lambda_{\mathfrak{d}}(\ell, \omega)$ of "Neumann" and "Dirichlet" solutions of (1):

Proposition 2.2. *One has*

$$\Lambda_{\mathfrak{n}}(\ell, \omega) = \frac{1}{\ell} \Lambda_{\mathfrak{n}}(\ell\, \omega) \quad \textit{and} \quad \Lambda_{\mathfrak{d}}(\ell, \omega) = \frac{1}{\ell} \Lambda_{\mathfrak{d}}(\ell\, \omega) \,.$$

So if we want to solve problem (1) on $\mathcal{T}^n$ instead on $\mathcal{T}$, one has to replace Helmholtz equation on each subtree by the DtN condition (3), where we set length of Σ equal to ℓ_n. Since $\ell_n \sim \alpha^n$, with $\alpha = \max(\alpha_i)$, for large n it is sufficient to get a good approximation of $\Lambda(\omega)$ for small ω which can be done with Taylor expansions.

Proposition 2.3. *For ω small, one has*

$$\begin{aligned}
\Lambda_{\mathfrak{n}}(\omega) &= \frac{1}{1 - \sum \mu_i \alpha_i} \omega^2 + O(\omega^4) \\
\Lambda_{\mathfrak{d}}(\omega) &= \frac{1 - \sum \mu_i/\alpha_i}{\sum \mu_i/\alpha_i} + \frac{1 + \sum \mu_i/\alpha_i + \left(\sum \mu_i/\alpha_i\right)^2}{3\left(\left(\sum \mu_i/\alpha_i\right)^2 - \sum \mu_i \alpha_i\right)} \omega^2 + O(\omega^4) \,.
\end{aligned}$$

2.4. **Back to the time-domain wave equation.** Neglecting the $O(\omega^4)$ term in formulas of proposition 2.3 allows us to write $\Lambda_{\mathfrak{n}}(\omega)$ (resp. $\Lambda_{\mathfrak{d}}(\omega)$) under the form

$$\Lambda_{\mathfrak{n}}(\omega) = \lambda^0_{\mathfrak{n}} + \lambda^2_{\mathfrak{n}}\,\omega^2 \qquad \big(\text{resp. } \Lambda_{\mathfrak{d}}(\omega) = \lambda^0_{\mathfrak{d}} + \lambda^2_{\mathfrak{d}}\,\omega^2\big)\ . \tag{6}$$

Injecting (6) in (3) and going back to time-domain leads to the following DtN operator

$$u'(t,\mathbf{0}) = \lambda^0\,u(t,\mathbf{0}) - \lambda^2\,\frac{\partial^2 u}{\partial t^2}(t,\mathbf{0})\ . \tag{7}$$

If we want to ensure stability for the time-domain wave equation with this condition, one has to check $\lambda^0 \leq 0$ and $\lambda^2 \geq 0$:

- for $\Lambda_{\mathfrak{d}}$, under hypothesis (2), one always has $\lambda^0_{\mathfrak{d}} \leq 0$ and $\lambda^2_{\mathfrak{d}} \geq 0$,
- for $\Lambda_{\mathfrak{n}}$, one has $\lambda^0_{\mathfrak{n}} = 0$, and one has $\lambda^0_{\mathfrak{n}} \geq 0$ if and only if one has $\sum \mu_i\alpha_i < 1$, i.e. if and only if the constant function $\mathbb{1}$ belongs to $\mathrm{L}^2_\mu(\mathcal{T})$.

3. Numerical results

To validate results of previous section, we solve time-domain wave equation on $\mathcal{T}_n$ for various values of n with outgoing condition at $\mathbf{0}$ and different conditions at outer boundary of n (with coefficients computed thanks to proposition 2.2):

- Dirichlet condition,
- First order impedance condition $u'(t,\cdot) = \lambda^0 u(t,\cdot)$,
- Second order impedance condition given by (7).

Numerical tests validate the condition (7) and show accuracy of this condition with respect to number of generations we consider. These results are work in progress for general case.

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Coercivity of boundary integral equations in high frequency scattering

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(joint work with Timo Betcke, Simon N. Chandler-Wilde, Ivan G. Graham, Valery P. Smyshlyaev)

Whether continuity and coercivity hold for variational formulations of several classical PDEs is well known. However, determining the conditions under which coercivity holds for boundary integral equation operators in acoustic scattering is still an open problem. Consider the problem of time-harmonic acoustic scattering from a sound-soft bounded obstacle $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$ with Lipschitz boundary $\Gamma := \partial\Omega$. That is, we are looking for the solution u of the problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^d \backslash \overline{\Omega}\,, \quad u = 0 \quad \text{on } \partial\Omega\,, \tag{1}$$

$$\frac{\partial u_s}{\partial r} - iku_s = o(r^{-(d-1)/2}), \tag{2}$$

where $u = u_{inc} + u_s$ is the total field, u_{inc} is a solution of (1) in a neighborhood of Ω, such as an incident plane wave, u_s is the scattered field, and r is the radial coordinate.the solution u is given by

$$u(x) = u_{inc}(x) - \int_\Gamma \Phi(x, y) u_n(y) ds(y), \quad x \in \mathbb{R}^d \backslash \overline{\Omega},$$

where u_n is the outward pointing normal derivative of u and $\Phi(x, y)$ stands for the standard free-space Green's function. To compute u_n one can solve the boundary integral equation

$$A_{k,\eta} u_n = 2\frac{\partial u_{inc}}{\partial n} - 2i\eta u_{inc}\,, \quad A_{k,\eta} := I + K' - i\eta S, \tag{3}$$

where $\eta \in \mathbb{R}\backslash\{0\}$, I is the identity, and K' and S are defined by

$$K'u(x) := 2\int_\Gamma \frac{\partial \Phi(x, y)}{\partial n(x)} u(y) ds(y), \quad Su(x) := 2\int_\Gamma \Phi(x, y) u(y) ds(y), \quad x \in \Gamma\,.$$

The corresponding sesquilinear form is defined as $a_{k,\eta}(u, v) := \langle A_{k,\eta} u, v\rangle$, with $\langle u, v\rangle := \int_\Gamma u(y)\overline{v(y)} ds(y)$ being the standard L^2-inner product. It was recently shown by Chandler-Wilde and Langdon in [10] that the operator $A_{k,\eta}$ is bijective with bounded inverse in the Sobolev spaces $H^{s-1/2}(\Gamma)$ for $|s| \le \frac{1}{2}$ and $\eta \in \mathbb{R}\backslash\{0\}$ (see also the book by Colton and Kress [12] for unique solvability of (3) in $C(\Gamma)$ with C^2 boundary).

The common choice for the coupling parameter η is to take η proportional to k for k large, and η constant for k small. This has been based on theoretical studies for the case of Γ a circle or sphere [17, 16, 1, 2], and also on computational experience [7]. Recently this choice has been backed up as near optimal for conditioning for more general domains by the analysis of [9].

An integral operator closely related to $A_{k,\eta}$ is

$$A'_{k,\eta} := I + K - i\eta S \tag{4}$$

where K is the double layer integral operator. This operator appears in the classic indirect boundary integral formulation due to Brakhage and Werner [5], Leis [19] and Panič [20]. ("Indirect" refers to the fact that this integral operator does not arise from Green's integral representation, whereas the so-called "direct" integral operator from (3) does.) The operator $A'_{k,\eta}$ is the adjoint of $A_{k,\eta}$ with respect to the real inner product $\langle u, v\rangle_{\mathbb{R}} := \int_\Gamma u(y)v(y)ds(y)$, thus, $\|A_{k,\eta}\| = \|A'_{k,\eta}\|$, where the norm is that induced by the standard L^2-inner product. If continuity and coecrivity hold for $A_{k,\eta}$ then they also hold for $A'_{k,\eta}$ with the same constants.

In acoustic scattering the key question is thus not only whether $a(\cdot,\cdot)$ is continuous and coercive, but also how the continuity constant $C > 0$ and coercivity contant $\gamma > 0$ depend on the wavenumber k. Indeed, this is the main motivation for studying the variational form of (3). The classical theory of second kind integral equations such as (3), which is based on the fact that for sufficiently smooth domains the integral operator from (3) is a compact perturbation of the identity, gives asymptotically quasi-optimal error estimates. However, these error estimates have the following two disadvantages: The first is that they are not explicit in the wavenumber k; i.e. they do not say how either the constant in the estimates, or the dimension of the approximation space N, depend on k [3].

The second is that much research effort has been focused recently on determining novel approximation spaces which take into account the high oscillation of the solution as k increases [8], and it does not appear that the classical theory can be used to prove error estimates for numerical methods using these subspaces. On the other hand, if continuity and coercivity of $a(\cdot,\cdot)$ can been established with constants explicit in k, then quasi-optimality estimates are valid for $\mathcal{V}^{(h)}$ *any* finite dimension subspace of $L^2(\Gamma)$.

Continuity of $a_{k,\eta}(\cdot,\cdot)$ is much more easy to establish than coercivity: by Cauchy-Schwartz, $|t(u,v)| \leq C\|u\|\|v\|$ holds for the bilinear form involving $A_{k,\eta}$ with $C = \|A_{k,\eta}\|$, and this is seen to be sharp by letting $v = A_{k,\eta}u$. The question of bounding $\|A_{k,\eta}\|$ with bounds explicit in k and η was investigated in detail in [9]. A first result on the coercivity of $a_{k,\eta}(\cdot,\cdot)$ was given in [13], where it was shown that with Γ the unit circle (in 2-d) and the unit sphere (in 3-d), with $\eta = k$, $a_{k,k}(\cdot,\cdot)$ is coercive for sufficiently large k with $\gamma \geq 1$. However, the question of coercivity and of $k-$dependence of the coercivity constant γ is still unanswered for more complicated domains. Although nothing is known directly about the coercivity constant γ for domains other than the circle/sphere, results on the norm of the inverse of $A_{k,\eta}$ can be used to deduce information about γ using the fact that if $A_{k,\eta}$ is coercive then $\gamma \leq \frac{1}{\|A^{-1}_{k,\eta}\|}$. Chandler-Wilde, Graham, Langdon and Linder [9] proved that if a part of Γ is C^1 then $\|A^{-1}_{k,\eta}\| \geq 1$ and, hence, $\gamma \leq 1$. Thus the bound obtained for γ for the circle in [13] is sharp. This follows from the fact that S and K are smoothing operators on smooth parts of Γ. In the same paper the authors constructed an example of a non-convex, non-starlike "trapping" domain in 2-d for which there exists an increasing sequence k_n where $\|A^{-1}_{k,\eta}\|$ grows as k_n increases. Indeed, for this domain, when $\eta = k$, $\|A^{-1}_{k_n,k_n}\| \gtrsim k_n^{9/10}$, where B is

independent of k. It is not known whether $A_{k,k}$ is coercive for this domain or not, but this example shows that if it is coercive, it cannot be uniformly coercive in k since $\gamma \lesssim k_n^{-9/10}$, which tends to zero as $k_n \to \infty$.

The final result on $\|A_{k,\eta}^{-1}\|$ which is relevant for coercivity was obtained by Chandler-Wilde and Monk in [11]. Their result implies that if Γ is Lipschitz , C^2 in a neighborhood of almost every $x \in \Gamma$, and starlike with respect to the origin, that is

$$\operatorname*{ess\,inf}_{x\in\Gamma} \; x \cdot n(x) > 0,$$

then for $\eta \gtrsim k$

$$\|A_{k,\eta}^{-1}\| \lesssim 1.$$

Thus, the "blow-up" of $\|A_{k,\eta}^{-1}\|$ for the "trapping" domain in [9] cannot occur when Ω is starlike.

Recall the definition of "trapping" and "non-trapping" from the epilogue of [18]: consider all the rays starting in the exterior of Ω inside some large ball of finite radius. Continue all the rays according to the law of reflection (angle of incidence equals angle of reflection) whenever they hit $\partial\Omega$, until they finally leave the large ball. We call Ω trapping if there are arbitrary long paths or closed paths of this kind; otherwise Ω is non-trapping. (Note that there are subtleties associated with rays hitting the boundary at a tangent, and also for domains with non-smooth boundaries.) In scattering theory, both for the Helmholtz equation (1) and for the time dependent wave equation, the geometry of the domain, and in particular whether it is trapping or not, plays a key role [18]. The connection between trapping and coercivity is discussed more in [4].

From the definition of the numerical range (field of values, [14]) we have the following equivalent characterisation of coercivity: The sesquilinear form $t(u, v) := \langle Tu, v\rangle$ associated with a linear operator T on a Hilbert space is coercive if and only if $0 \notin \overline{W(T)}$. Furthermore, if $t(\cdot, \cdot)$ is coercive then the coercivity constant γ is given by $\gamma = d(0, W(T))$, where d is the usual set distance. Thus, we can determine coercivity by computing the numerical range of the operator $A_{k,\eta}$, which is a well studied problem in the numerical linear algebra literature for matrices acting on $\mathbb{C}^n$. Coercivity for several interesting polygonal and smooth domains in two dimensions is investigated in this way in [4] for the most common choice of coupling constant $\eta = k$. Since in practice one works with Galerkin discretizations of $a_{k,\eta}(\cdot, \cdot)$, [4] gives convergence estimates of the numerical range based on Galerkin discretizations with standard piecewise constant boundary element discretizations.

The numerical results in [4] demonstrate that coercivity of the direct combined boundary integral operator $A_{k,k}$ seems to hold uniformly on a wide range of domains. This is surprising since for standard domain based variational formulations of the underlying Helmholtz equation only a weaker Gårding inequality, with k dependent perturbation term, holds [15]. Some rigorous results on proving coercivity of $A_{k,\eta}$ in certain situations, as well as proving coercivity of other related integral operators, have recently been obtained.

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Isogeometric analysis in electromagnetism: analysis and testing

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(joint work with A. Buffa, G. Sangalli)

1. Introduction

Isogeometric Analysis (IGA) is a method for the discretization of partial differential equations, that was recently introduced by T.J.R. Hughes et al. in [7]. The concept of IGA was born with the aim of making easier the interaction between the Computer Aided Design (CAD) software and the numerical solver. In order to describe the geometry, CAD tools make use of different kind of functions, the most popular of them being non-uniform rational B-splines (NURBS) (see [10]). The main idea of IGA is to use NURBS basis functions as the trial and test functions in the discrete variational formulation. The main features of IGA compared to finite elements are the following:

- The geometry is defined exactly, in the sense that the NURBS description given by CAD is preserved.
- The computed solution is smoother, due to the higher continuity of NURBS basis functions.

Thanks to this second property, in IGA we can perform the so-called k-refinement, which consists on increasing the degree and the continuity of the basis functions (see [7]). This refinement provides better convergence in terms of degrees of freedom than p-refinement, and it has been found of particular interest in the approximation of the entire spectrum of harmonic and biharmonic problems [8].

In this work we present the application of IGA to electromagnetic problems, and in particular to the Maxwell eigenproblem: *Find* $\omega \in \mathbb{R}, \omega \neq 0$ *and* $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$, $\mathbf{u} \neq \mathbf{0}$ *such that*

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} = \omega^2 \int_\Omega \mathbf{u} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega). \tag{1}$$

The method we describe was introduced in [5], and analyzed in [4]. Our method is based on B-splines, but the domain Ω may be defined with NURBS, thus the CAD representation of the geometry is preserved. In the next section we give a brief overview about B-splines, using the same notation of [5], and refer the reader to [3] and [11] for extended works on the subject.

2. B-splines and NURBS

Given two positive integers p and n, we define the open knot vector $\Xi := \{0 = \xi_1, \xi_2, \ldots, \xi_{n+p+1} = 1\}$, where the term *open* means that the first and last knots are repeated $p+1$ times. Univariate B-splines basis functions are defined from the knot vector following the algorithm described in [3]. To characterize the space they span, we define the vector $\{\zeta_1, \ldots, \zeta_m\}$ of knots without repetitions. The space spanned by these basis functions is formed by piecewise polynomials of degree p having α_i continuous derivatives at the points ζ_i, with $\alpha_i = p - r_i$, and r_i the

multiplicity of the knot ζ_i. We will denote this space of B-splines as $S^p_{\boldsymbol{\alpha}}(\Xi) \equiv S^p_{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} := \{\alpha_1, \dots \alpha_m\}$ is the vector of continuities. It is worth to note that the derivatives of B-splines are also B-splines, and in fact

$$\left\{ \frac{d}{dx} v : v \in S^p_{\boldsymbol{\alpha}} \right\} \equiv S^{p-1}_{\boldsymbol{\alpha}-1}. \tag{2}$$

The definition is generalized to the trivariate case (also bivariate) by tensor products. Given the integers p_d and n_d, for $d = 1, 2, 3$, we introduce the knot vectors Ξ_d as before, and also the corresponding vectors of knots without repetitions and continuities. Notice that the vectors Ξ_d define a Cartesian partition of the unit cube $\widehat{\Omega} = (0, 1)^3$, that is, a mesh. The space of B-splines associated to these knots is denoted by $S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3}$, and is formed by piecewise polynomials with interelement regularity given by the vectors $\boldsymbol{\alpha}_d$ (see [5] and [4] for details).

NURBS basis functions and geometrical entities are then defined from these B-splines. Roughly speaking, to each B-spline basis function $B_{\mathbf{i}}$, where $\mathbf{i}$ is a three-dimensional multi-index (or two-dimensional for bivariate splines), we associate a positive weight $w_{\mathbf{i}}$, and define the corresponding NURBS basis function as

$$N_{\mathbf{i}}(\mathbf{x}) = \frac{w_{\mathbf{i}} B_{\mathbf{i}}(\mathbf{x})}{\sum_{\mathbf{j}} w_{\mathbf{j}} B_{\mathbf{j}}(\mathbf{x})}.$$

In order to describe the geometry, we associate to each NURBS basis function $N_{\mathbf{i}}$ a control point $\mathbf{C}_{\mathbf{i}} \in \mathbb{R}^3$. The domain is then defined by the parametrization

$$\begin{array}{rcl} \mathbf{F} : \widehat{\Omega} & \longrightarrow & \Omega \\ \mathbf{x} & \longmapsto & \mathbf{F}(\mathbf{x}) := \sum_{\mathbf{j}} N_{\mathbf{j}}(\mathbf{x}) \mathbf{C}_{\mathbf{j}}. \end{array} \tag{3}$$

For further details and examples we defer the reader to [10] and [7]. In the following we will assume that our physical domain Ω is open, Lipschitz, simply connected and with connected boundary, and that it can be described through a parametrization as (3).

3. Commuting De-Rham diagram

It is well known that, in order to discretize (1), it is necessary to find a set of discrete spaces that mimic the following exact sequence [1, 9]:

$$H^1_0(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}_0(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}_0(\mathrm{div}; \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)/\mathbb{R}. \tag{4}$$

In finite elements the discrete sequence is formed by nodal, edge and face finite elements. A sequence of B-spline spaces can be also constructed thanks to property (2). Let us define the spaces in the parametric domain $\widehat{\Omega}$

$$\hat{X}^0 := S^{p_1,p_2,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3}, \quad \hat{X}^1 := S^{p_1-1,p_2,p_3}_{\boldsymbol{\alpha}_1-1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3} \times S^{p_1,p_2-1,p_3}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2-1,\boldsymbol{\alpha}_3} \times S^{p_1,p_2,p_3-1}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3-1},$$
$$\hat{X}^2 := S^{p_1,p_2-1,p_3-1}_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2-1,\boldsymbol{\alpha}_3-1} \times S^{p_1-1,p_2,p_3-1}_{\boldsymbol{\alpha}_1-1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3-1} \times S^{p_1-1,p_2-1,p_3}_{\boldsymbol{\alpha}_1-1,\boldsymbol{\alpha}_2-1,\boldsymbol{\alpha}_3},$$
$$\hat{X}^3 := S^{p_1-1,p_2-1,p_3-1}_{\boldsymbol{\alpha}_1-1,\boldsymbol{\alpha}_2-1,\boldsymbol{\alpha}_3-1},$$

and let us denote with the subindex 0 the spaces with homogeneous boundary conditions (tangential for $\hat{X}_0^1$, normal for $\hat{X}_0^2$, null average value for $\hat{X}_0^3$). It can be proved that the following sequence in $\widehat{\Omega}$ is exact:

$$\hat{X}_0^0 \xrightarrow{\mathbf{grad}} \hat{X}_0^1 \xrightarrow{\mathbf{curl}} \hat{X}_0^2 \xrightarrow{\text{div}} \hat{X}_0^3. \tag{5}$$

The exact sequence in the physical domain Ω is then constructed by applying suitable push-forwards (see, e.g., [1, 4]). For instance, the discrete space for approximating $\mathbf{H}_0(\mathbf{curl};\Omega)$ is defined as

$$X_0^1 := \{\mathbf{u}_h \in \mathbf{H}_0(\mathbf{curl};\Omega) : \iota^1(\mathbf{u}_h) \in \hat{X}_0^1\},$$

where ι^1 is the curl-conserving pull-back, defined as

$$\iota^1(\mathbf{u}) := (D\mathbf{F})^T(\mathbf{u}\circ\mathbf{F}),$$

and $\mathbf{F}$ is the same parametrization given by (3).

In order to develop the numerical analysis of our discrete spaces, it is necessary to construct a commuting diagram between the continuous and the discrete spaces. This is done by defining suitable projectors onto our discrete spaces. To do it briefly, the projectors in the reference domain $\widehat{\Omega}$ are formed by tensor products of the univariate projectors $\widehat{\Pi}_{0,S}^p$, analyzed in [11, 2], and $\widehat{\Pi}_{0,A}^{p-1}$, defined in [4]. Both projectors are L^2-stable and local, and they satisfy

$$\widehat{\Pi}_{0,A}^{p-1}\frac{d}{dx}u = \frac{d}{dx}\widehat{\Pi}_{0,S}^p u \quad \forall u \in H_0^1(0,1).$$

For instance, we define the projector $\widehat{\Pi}_0^1 : \mathbf{H}_0(\mathbf{curl};\widehat{\Omega}) \to \widehat{X}_0^1$ as

$$\widehat{\Pi}_0^1 := (\widehat{\Pi}_{0,A}^{p_1-1}\otimes\widehat{\Pi}_{0,S}^{p_2}\otimes\widehat{\Pi}_{0,S}^{p_3}) \times (\widehat{\Pi}_{0,S}^{p_1}\otimes\widehat{\Pi}_{0,A}^{p_2-1}\otimes\widehat{\Pi}_{0,S}^{p_3}) \times (\widehat{\Pi}_{0,S}^{p_1}\otimes\widehat{\Pi}_{0,S}^{p_2}\otimes\widehat{\Pi}_{0,A}^{p_3-1}).$$

Then, the projector in the physical domain is constructed by push-forward:

$$\iota^1(\Pi_0^1\mathbf{u}) = \widehat{\Pi}_0^1(\iota^1(\mathbf{u})), \quad \forall \mathbf{u} \in \mathbf{H}_0(\mathbf{curl};\Omega).$$

With similar choices for the three other spaces, the diagram can be proved to be commutative. Moreover, as the projectors are defined from $\widehat{\Pi}_{0,S}^p$ and $\widehat{\Pi}_{0,A}^{p-1}$ they are also L^2-stable and local. Once again, we address the reader to [4] for the proofs.

4. Application to Maxwell eigenvalue problem

The discrete spaces presented above have been applied to the approximation of problem (1). The discrete version of the problem reads: *Find* $\omega \in \mathbb{R}$, $\omega \neq 0$, *and* $\mathbf{u}_h \in X_0^1$, $\mathbf{u}_h \neq \mathbf{0}$ *such that*

$$\int_\Omega \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h = \omega^2 \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h, \quad \forall \mathbf{v}_h \in X_0^1.$$

Since the continuous and discrete spaces satisfy a commuting diagram, with L^2-stable and local projectors, we know that our discretization is spurious-free, and that the discrete solutions converge to the continuous one when refining in terms of the mesh size h (see [1]).

In order to test the behavior of the method, it has been applied to several benchmark problems. The numerical results can be found in [4, 5, 6]. The method has proved to be spurious-free, and capable to approximate singular functions in non-convex domains. The convergence rate in terms of the mesh size h always agrees with the theoretical value. Moreover, the convergence in terms of the degrees of freedom is better than with standard finite elements.

Finally, it is worth to note that the method provides the two main properties of IGA: the geometry is exactly described using very few elements, and the computed solution is smoother than the one given by finite elements. In particular, if the geometry is smooth enough, the computed solution may be continuous, and its divergence can be computed. We refer the reader again to [4] for the study of the divergence and some numerical results.

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BETI-methods for acoustic scattering problems

Markus Windisch

(joint work with Olaf Steinbach)

We consider the Helmholtz problem with Neumann boundary conditions

$$\begin{aligned} \Delta u(x) + \kappa^2 u(x) &= 0 \qquad \text{for } x \in \Omega \subset \mathbb{R}^3 \\ \frac{\partial}{\partial n} u(x) &= g(x) \quad \text{for } x \in \Gamma = \partial\Omega \end{aligned} \tag{1}$$

which we solve by using a domain decomposition approach, more precisely a tearing and interconnecting method (see [3]), and by using boundary element discretizations for the local subproblems. This has several advantages:

- κ can be chosen piecewise constant (constant on each subdomain).
- The BEM can be easily exchanged in specific domains by other methods like finite element methods.
- The domain decomposition approach offers a natural parallelization for the implementation.

The boundary value problem (1) is equivalent to the local boundary value problems

$$\Delta u_i(x) + \kappa_i^2 u_i(x) = 0 \qquad \text{for } x \in \Omega_i, \tag{2}$$

$$\frac{\partial}{\partial n_i} u_i(x) = g(x) \quad \text{for } x \in \Gamma_i = \partial\Omega_i \cap \Gamma, \tag{3}$$

together with the transmission or interface boundary conditions

$$u_i(x) = u_j(x) \quad \text{for } x \in \Gamma_{ij}, \tag{4}$$

$$\frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{5}$$

To avoid non–uniqueness in the solution of either local Dirichlet or Neumann boundary value problems, instead of the Neumann transmission boundary condition in (4) we consider a Robin type interface condition given as

$$\frac{\partial}{\partial n_i} u_i(x) + \frac{\partial}{\partial n_j} u_j(x) + i\eta_{ij} R_{ij}[u_i(x) - u_j(x)] \;=\; 0 \quad \text{for } x \in \Gamma_{ij}, i < j, \tag{6}$$

together with the Dirichlet transmission condition

$$u_i(x) = u_j(x) \quad \text{for } x \in \Gamma_{ij}. \tag{7}$$

Note that $R_{ij} : H^{1/2}(\Gamma_{ij}) \to \widetilde{H}^{-1/2}(\Gamma_{ij})$ is assumed to be self–adjoint and $H^{1/2}(\Gamma_{ij})$–elliptic, and $\eta_{ij} \in \mathbb{R}\backslash\{0\}$. In this case, the equivalence of the interface transmission conditions (6) and (7) with (4) and (5) follows immediately.

The local subdomain boundary $\Gamma_i = \partial\Omega_i$ of a subdomain Ω_i is considered as the union

$$\Gamma_i = (\Gamma_i \cap \Gamma) \cup \bigcup_{\Gamma_{ij}} \Gamma_{ij} \ ,$$

where $\Gamma_i \cap \Gamma$ corresponds to the original boundary where Neumann boundary conditions are given, while Γ_{ij} denotes the coupling boundary with an adjacent subdomain. We define

$$(R_i u_{|\Gamma_i})(x) \;:=\; (R_{ij} u_{|\Gamma_{ij}})(x) \quad \text{for } x \in \Gamma_{ij} \tag{8}$$

and

$$\eta_i(x) \;:=\; \begin{cases} \eta_{ij} & \text{for } x \in \Gamma_{ij},\ i < j, \\ -\eta_{ij} & \text{for } x \in \Gamma_{ij},\ i > j, \\ 0 & \text{for } x \in \Gamma_i \cap \Gamma\,. \end{cases} \tag{9}$$

We assume, that $\eta_i(x)$ for $x \in \Gamma_i$ does not change its sign. This can be guaranteed either when considering a checker board domain decomposition [1], or when enforcing Robin type boundary conditions only on a part of the local boundary Γ_i, i.e. setting $\eta_{ij} = 0$ on some coupling boundaries Γ_{ij}.

Since we want to use a boundary integral approach for the local problems, we have to introduce first the local representation formulae, see e.g. [4]

$$u_i(x) = \int_{\Gamma_i} U^*_{\kappa_i}(x,y) t_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_i(y) ds_y \quad \text{for } x \in \Omega_i, \tag{10}$$

where

$$U^*_{\kappa_i}(x,y) = \frac{1}{4\pi} \frac{e^{i\kappa_i |x-y|}}{|x-y|}, \quad t_i(y) := \frac{\partial}{\partial n_y} u_i(y), \; y \in \Gamma$$

are the fundamental solution of the Helmholtz equation and the associated normal derivative of the solution u_i, respectively. By taking the Dirichlet and Neumann traces of the representation formulae (10) we obtain a system of local boundary integral equations

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I - K_{\kappa_i} & V_{\kappa_i} \\ D_{\kappa_i} & \frac{1}{2} I + K'_{\kappa_i} \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix}, \tag{11}$$

where we have used the standard notations for boundary integral operators. The mapping properties of all boundary integral operators as introduced above are well known, see, e.g. [2]

If we reformulate now the problem (2),(3),(6),(7) with the help of boundary integral equations, discretize the resulting system and apply a tearing and interconnecting approach, then we end up with the discrete formulation (see [5])

$$\begin{pmatrix} V_{\kappa_1,h} & -\widetilde{K}_{\kappa_1,h} & & & & \\ \widetilde{K}'_{\kappa_1,h} & D_{\kappa_1,h} + i\eta_1 R_{1,h} & & & & -B_1^\top \\ & \ddots & & & & \vdots \\ & & V_{\kappa_p,h} & -\widetilde{K}_{\kappa_p,h} & & \\ & & \widetilde{K}'_{\kappa_p,h} & D_{\kappa_p,h} + i\eta_p R_{p,h} & & -B_p^\top \\ & B_1 & \dots & B_p & & \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{u}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{g}_1 \\ \vdots \\ \underline{0} \\ \underline{g}_p \\ \underline{0} \end{pmatrix},$$

where $\tilde{K}$ is defined by $\tilde{K} := (\frac{1}{2} I + K)$, and B_i are boolean matrices which ensure the continuity of the Dirichlet data. It can be proven that this discrete formulation omits a unique solution, if (1) has a unique solution and if the discretization is fine enough.

After eliminating the primal degrees of freedom we end up with the Schur complement system

$$F\underline{\lambda} = \sum_{i=1}^{p} \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h} + i\eta_i R_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{0} \\ B_i^\top \underline{\lambda} \end{pmatrix}$$
$$= -\sum_{i=1}^{p} \begin{pmatrix} 0 & B_i \end{pmatrix} \begin{pmatrix} V_{\kappa_i,h} & -\widetilde{K}_{\kappa_i,h} \\ \widetilde{K}'_{\kappa_i,h} & D_{\kappa_i,h} + i\eta_i R_{i,h} \end{pmatrix}^{-1} \begin{pmatrix} \underline{0} \\ \underline{g}_i \end{pmatrix}.$$

Numerical examples and possible preconditioning strategies can be found in [6].

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Efficient high-order Maxwell solvers via discrete space splittings

Sabine Zaglmayr

1. Problem formulation

In order to guarantee the unique solvability for vector potential formulations of magnetostatic problems, additional constraints, so called gauging conditions, have to be imposed. In particular, we consider the Coulomb gauge, which enforces orthogonality of the magnetic vector potential to gradient fields. In fact, we consider the problem of finding a vector field $A : \Omega \to \mathbb{R}^3$ such that

$$\text{curl } \nu \text{curl } A = j \quad \text{in } \Omega \tag{1a}$$

$$\text{div } A = 0 \quad \text{in } \Omega \tag{1b}$$

$$A \times n = 0 \quad \text{on } \partial\Omega. \tag{1c}$$

We assume $\nu \in L_\infty(\Omega)$ with $0 < \nu_0 \leq \nu$ and compatible right hand side $\text{div}\, j = 0$. For simplicity of presentation only, we assume Dirichlet boundary conditions (1c) and the domain Ω to be contractible. Problem (1) arises also in iterative solvers of the Maxwell Eigenvalue problem as well as in alternative formulations of the Stokes problem.

In this presentation, we investigate one possibility to realize the divergence-free constraint, and we propose a two-step strategy for an efficient realization by *hp*-finite element methods based on a careful construction of the involved high order Nédélec finite element spaces.

2. hp-discretization

A crucial ingredient in the construction of conforming hp-finite elements is the following discrete exactness property

$$\begin{array}{ccccccc} H_0^1(\Omega) & \xrightarrow{\nabla} & H_0(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & H_0(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L_{2,0}(\Omega) \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ \mathbb{W}_{hp} & \xrightarrow{\nabla} & \mathbb{V}_{hp} & \xrightarrow{\mathrm{curl}} & \mathbb{Q}_{hp} & \xrightarrow{\mathrm{div}} & \mathbb{S}_{hp}, \end{array}$$

see e.g. [3, 2]. Exactness of the de Rham sequence means that the gradient fields span the kernel of the curl operator and the curl fields span the kernel of the div operator. The second line indicates that the same property should be satisfied also on the discrete level. Discrete exactness ($\ker(\mathrm{curl},\mathbb{V}_{hp}) = \nabla\mathbb{W}_{hp+1}$) and a discrete Friedrichs' inequality [5] for $H_0(\mathrm{curl})$ already imply the stability of the discrete variational problem corresponding to (1): Find $A_{hp} \in \mathbb{V}_{hp}$ and $\Phi_{hp} \in \mathbb{W}_{hp}$ such that

$$(\nu\mathrm{curl}A_{hp}, \mathrm{curl}v_{hp}) + (\nabla\Phi_{hp}, v_{hp}) = (j, v_{hp}) \qquad \forall v_{hp} \in \mathbb{V}_{hp} \tag{2a}$$

$$(A_{hp}, \nabla\psi_{hp}) = 0 \qquad \forall \psi_{hp} \in \mathbb{W}_{hp}. \tag{2b}$$

In the following we show how to solve this mixed problem efficiently by utilizing special constructions of the $H(\mathrm{curl})$-conforming high-order finite element space.

3. Discrete space splitting

We start from a hierarchic representation

$$\mathbb{W}_{hp} := \mathbb{W}_h \oplus \mathbb{W}_p \quad \subset H_0^1(\Omega)$$

of the H^1-conforming finite element space as direct sum of a low-order space $\mathbb{W}_h$ and a completing space $\mathbb{W}_p$ of higher order polynomials. According to the discrete de Rham sequence the gradients of functions in $\mathbb{W}_{hp}$ shall be included in the $H(\mathrm{curl})$-conforming finite element space $\mathbb{V}_{hp}$. This can be accomplished by the following construction [10, 11]

$$\mathbb{V}_{hp} := \mathbb{V}_h \oplus \nabla\mathbb{W}_p \oplus \widetilde{\mathbb{V}}_p \quad \subset H_0(\mathrm{curl},\Omega),$$

where $\mathbb{V}_h$ denotes the classical low-order Nédélec space [8] and $\nabla\mathbb{W}_p$ are the higher order gradient fields. The completion space $\widetilde{\mathbb{V}}_p$ is linearly independent to *all* gradient fields $\nabla H^1(\Omega)$ and provides the approximation properties of $\mathbb{V}_{hp}$. The required discrete exactness property is thus fulfilled by construction. Moreover, the largest part of the kernel space, i.e. the higher order gradient fields can be represented explicitly. This property will be utilized in the following.

4. A two-step gauging strategy

For solving (2) we proceed as follows: First, we solve a mixed problem on a reduced space $(\mathbb{V}_h \oplus \widetilde{\mathbb{V}}_p) \times \mathbb{W}_h$ dropping higher order gradient and scalar fields. In a second step, we establish the full solution of problem (2) by post-processing. Both subproblems are much smaller and better conditioned than the original mixed problem. In detail, the algorithm looks as follows:
(A) Find $A_h \in \mathbb{V}_h$, $\widetilde{A}_p \in \widetilde{\mathbb{V}}_p$, and $\Phi_h \in \mathbb{W}_h$ such that

$$\begin{aligned}\big(\nu \mathrm{curl}\,(A_h + \widetilde{A}_p), \mathrm{curl}\,(v_h + \widetilde{v}_p)\big) + (\nabla\Phi_h, v_h) &= (j, v_h + \widetilde{v}_p)\\ (A_h, \nabla\psi_h) &= 0\end{aligned}$$

for all $v_h \in \mathbb{V}_h$, $\widetilde{v}_p \in \widetilde{\mathbb{V}}_p$, and $\psi_h \in \mathbb{W}_h$. Note that the second equation is needed only to ensure uniqueness of A_h. Since $\widetilde{\mathbb{V}}_p$ is linearly independent to gradient fields, the curl-curl operator is positive definite on $\widetilde{\mathbb{V}}_p$.
(B) Find $w_{hp} \in \mathbb{W}_{hp}$ such that

$$(\nabla w_{hp}, \nabla\psi_{hp}) = (A_h + \widetilde{A}_p, \nabla\psi_{hp}) \qquad \forall \psi_{hp} \in \mathbb{W}_{hp}.$$

In this second step, we compute the projection of $A_h + \widetilde{A}_p$ onto the gradient space $\nabla\mathbb{W}_{hp}$. Note that $\widetilde{\mathbb{V}}_p$ is linearly independent but not necessarily orthogonal to gradient fields.

Theorem 1. *Let* $(A_h, \widetilde{A}_p)$ *be the solution of* (A) *and* w_{hp} *be the solution of* (B), *then* $A_{hp} := A_h + \widetilde{A}_p - \nabla w_{hp}$ *and* $\Phi_{hp} := 0$ *solve* (2).

If one is interested only in *curl* A_{hp}, then the post-processing step (B) can be skipped.

5. Discussion

The proposed two-step strategy requires the solution of linear systems which are much smaller and better conditioned than that of the original mixed problem. For three dimensional computations, the dimension of the reduced space $(\mathbb{V}_h + \widetilde{\mathbb{V}}_p) \times \mathbb{W}_h$ used in subproblem (A) is approximately 50 % smaller than that of the full space $\mathbb{V}_{hp} \times \mathbb{W}_{hp}$. Problem (A) consists of two coupled subproblems: an elliptic high-order part and a low-order mixed problem. Efficient solvers for the low-order problem are readily available, e.g. via tree/cotree gauging [2] or penalization [9], see also [6, 1]. The high-order part can be addressed by local block smoothers [11]. Problem (B) is a scalar elliptic problem for which efficient solvers are well-known. The assembly of $A_{hp} := \widetilde{A}_{hp} - \nabla w_{hp}$ does not require any further computation, if the gradients of the basis functions of $\mathbb{W}_p$ are used directly as basis functions for $\nabla\mathbb{W}_p$ in the $H(\mathrm{curl})$ finite element space.

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