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Algebraic Groups

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Abstract. The workshop dealt with a broad range of topics from the structure theory and the representation theory of algebraic groups (in the widest sense). There was emphasis on the following areas:

- classical and quantum cohomology of homogeneous varieties,
- representation theory and its connections to orbits and flag varieties.

Mathematics Subject Classification (2000): 14Lxx, 14Mxx, 14Nxx, 17Bxx, 20Gxx.

Introduction by the Organisers

The workshop continued a series of Oberwolfach meetings on algebraic groups, started in 1971 by Tonny Springer and Jacques Tits. This time, the organizers were Michel Brion and Jens Carsten Jantzen.

During the last years, the subject of algebraic groups (in a broad sense) has seen important developments in several directions, also related to representation theory and algebraic geometry. The workshop aimed at presenting some of these developments in order to make them accessible to a "general audience" of algebraic group-theoretists, and to stimulate contacts between participants.

Several series of talks were dedicated to areas of research that have recently seen decisive progress :

- classical and quantum cohomology of homogeneous varieties (Chaput, Perrin, Tamvakis)
- representation theory and its connections to orbits and flag varieties (Goodwin, Riche, Rumynin, Vasserot)
- intersection cohomology in positive characteristics (Fiebig, Juteau)
- geometry and classification of spherical varieties (Avdeev, Gandini).

Other talks introduced to several recent advances in different areas: classical questions on the subgroup structure and the representations of reductive groups (Hille, Littelmann, Ressayre, Roehrle), Schubert and Deligne-Lusztig varieties (Goertz, Kuttler), generalizations of Newton polytopes (Kiritchenko), versal actions of algebraic groups (Reichstein), geometry of symmetric Lie algebras (Bulois), quantum homogeneous spaces (Lehrer).

In order to leave enough time for fruitful discussions, the number of talks (generally of one hour) was limited to five per day, and to 21 altogether.

Besides the scientific program, the participants enjoyed a piano recital on Thursday evening, by Pierre-Emmanuel Chaput, Peter Fiebig and Harry Tamvakis.

The workshop was held under special circumstances: due to lasting disruptions of the airplane traffic, 15 registered participants could not make the trip to MFO, and had to cancel their participation. This includes several mathematicians who very likely would have given a talk. As partial replacements, 6 participants (from universities in Germany or France) could join the workshop at the last minute.

There were 40 participants, coming mainly from Europe and North America. This includes 6 young researchers who participated as Oberwolfach Leibniz Graduate Students. The organizers are grateful to the Leibniz-Gemeinschaft for this support, and to the MFO for providing excellent working conditions.

Workshop: Algebraic Groups

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Abstracts

On solvable spherical subgroups of semisimple algebraic groups

Roman Avdeev

This report contains a new approach to classification of solvable spherical subgroups of semisimple algebraic groups. This approach is completely different from the approach by D. Luna [1] and provides an explicit classification.

Let G be a connected semisimple complex algebraic group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. We denote by U the maximal unipotent subgroup of G contained in B. The Lie algebras of G, B, U, ... are denoted by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{u}$, ..., respectively. Let $\Delta \subset \mathfrak{X}(T)$ be the root system of G with respect to T (where $\mathfrak{X}(T)$ denotes the character lattice of T). The subsets of positive roots and simple roots with respect to B are denoted by Δ_+ and Π, respectively. For any root $\alpha \in \Delta_+$ consider its expression of the form $\alpha = \sum\limits_{\gamma \in \Pi} k_\gamma \gamma$. We put $\operatorname{Supp} \alpha = \{\gamma \in \Pi \mid k_\gamma > 0\}$. The root subspace of the Lie algebra $\mathfrak{g}$ corresponding to a root α is denoted by $\mathfrak{g}_\alpha$. The symbol $\langle A \rangle$ will denote the linear span of a subset $A \subset \mathfrak{X}(T)$ in $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $H \subset B$ be a connected solvable algebraic subgroup of G and $N \subset U$ its unipotent radical. We say that the group H is *standardly embedded* into B (with respect to T) if the subgroup $S = H \cap T$ is a maximal torus of H. Obviously, in this case we have $H = S \leftthreetimes N$.

Suppose that $H \subset G$ is a connected solvable subgroup standardly embedded into B. Then we can consider the natural restriction map $\tau : \mathfrak{X}(T) \to \mathfrak{X}(S)$. Put $\Phi = \tau(\Delta_+) \subset \mathfrak{X}(S)$. Then we have $\mathfrak{u} = \bigoplus\limits_{\lambda \in \Phi} \mathfrak{u}_\lambda$, where $\mathfrak{u}_\lambda \subset \mathfrak{u}$ is the weight subspace of weight λ with respect to S. Similarly, we have $\mathfrak{n} = \bigoplus\limits_{\lambda \in \Phi} \mathfrak{n}_\lambda$, where $\mathfrak{n}_\lambda = \mathfrak{u}_\lambda \cap \mathfrak{n} \subset \mathfrak{u}_\lambda$. Denote by c_λ the codimension of $\mathfrak{n}_\lambda$ in $\mathfrak{u}_\lambda$.

Recall that a subgroup $H \subset G$ is called *spherical* if the group B has an open orbit in G/H. The following theorem provides a convenient criterion of sphericity for connected solvable subgroups of G.

Theorem 1. *Suppose $H \subset G$ is a connected solvable subgroup standardly embedded into B. Then the following conditions are equivalent:*

(1) *H is spherical in G;*

(2) *$c_\lambda \leqslant 1$ for every $\lambda \in \Phi$, and the weights λ with $c_\lambda = 1$ are linearly independent in $\mathfrak{X}(S)$.*

Now we suppose that H is spherical.

Definition. A root $\alpha \in \Delta_+$ is called *marked* if $\mathfrak{g}_\alpha \not\subset \mathfrak{n}$.

An important role of marked roots in studying solvable spherical subgroups is clear from the theorem below.

Theorem 2. *Up to conjugation by elements of T, the subgroup H is uniquely determined by its maximal torus $S \subset T$ and the set $\Psi \subset \Delta_+$ of marked roots.*

Remark 1. The subgroup H is explicitly recovered from S and Ψ.

Marked roots have the following property: if $\alpha \in \Psi$ and $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta_+$, then exactly one of two roots β, γ is marked. Taking this property into account, we say that a marked root β is *subordinate* to a marked root α, if $\alpha = \beta + \gamma$ for some root $\gamma \in \Delta_+$. Given a marked root α, we denote by $C(\alpha)$ the set consisting of α and all marked roots subordinate to α. Further, we say that a marked root α is *maximal* if it is not subordinate to any other marked root. Let M denote the set of all maximal marked roots.

Proposition 1. *For any marked root α there exists a unique simple root $\pi(\alpha) \in \operatorname{Supp}\alpha$ with the following property: if $\alpha = \beta+\gamma$ for some roots $\beta, \gamma \in \Delta_+$, then the root β is marked iff $\pi(\alpha) \notin \operatorname{Supp}\beta$ (and so the root γ is marked iff $\pi(\alpha) \notin \operatorname{Supp}\gamma$).*

Definition. If α is a marked root, then the simple root $\pi(\alpha)$ appearing in Proposition 1 is called the *simple root associated with the marked root α.*

From Proposition 1 we see that for any marked root α the set $C(\alpha)$ is uniquely determined by the simple root $\pi(\alpha)$. Therefore, the whole set Ψ is uniquely determined by the set M and the map $\pi\colon \mathrm{M} \to \Pi$.

Theorem 3 (Classification of marked roots)**.** *All possibilities for a marked root α and the simple root $\pi(\alpha)$ are presented in Table 1.*

Notations used in Table 1. The symbol $\Delta(\alpha)$ denotes the root system generated by Supp α, i. e. $\Delta(\alpha) = \langle \operatorname{Supp} \alpha \rangle \cap \Delta$. We suppose that $\operatorname{Supp} \alpha = \{\alpha_1, \ldots, \alpha_n\}$. The numeration of simple roots of simple Lie algebras is the same as in [2].

TABLE 1. Marked roots

	type of $\Delta(\alpha)$	α	$\pi(\alpha)$
1	any of rank n	$\alpha_1 + \alpha_2 + \ldots + \alpha_n$	$\alpha_1, \alpha_2, \ldots, \alpha_n$
2	B_n	$\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} + 2\alpha_n$	$\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$
3	C_n	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$	α_n
4	F_4	$2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	α_3, α_4
5	G_2	$2\alpha_1 + \alpha_2$	α_2
6	G_2	$3\alpha_1 + \alpha_2$	α_2

For any root $\alpha \in \Delta_+$ consider the (connected) Dynkin diagram $D(\alpha)$ of the set Supp α. We say that a root $\delta \in \operatorname{Supp} \alpha$ is *terminal with respect to* Supp α if the node of $D(\alpha)$ corresponding to δ is connected by an edge (possibly, multiple) with exactly one other node of $D(\alpha)$.

Now we introduce some conditions on a pair of marked roots (α, β). These conditions will be used later.

(D0) $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta = \varnothing$

(D1) $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta = \{\delta\}$, δ is terminal with respect to both $\operatorname{Supp}\alpha$ and $\operatorname{Supp}\beta$, $\pi(\alpha) \neq \delta \neq \pi(\beta)$

(D2) the Dynkin diagram of the set $\operatorname{Supp}\alpha \cup \operatorname{Supp}\beta$ has the form shown on Figure 1 (for some $p, q, r \geqslant 1$), $\alpha = \alpha_1 + \ldots + \alpha_p + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\beta = \beta_1 + \ldots + \beta_q + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\pi(\alpha) \notin \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$, $\pi(\beta) \notin \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$

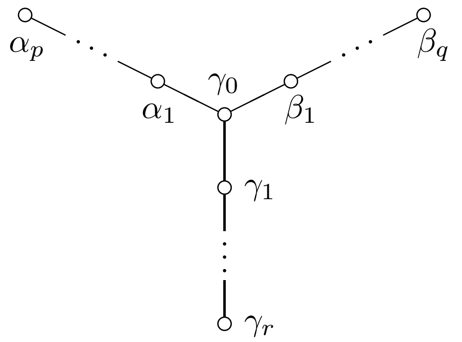

Figure 1.

(E1) $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta = \{\delta\}$, δ is terminal with respect to both $\operatorname{Supp}\alpha$ and $\operatorname{Supp}\beta$, $\delta = \pi(\alpha) = \pi(\beta)$, $\alpha - \delta \in \Delta_+$, $\beta - \delta \in \Delta_+$

(E2) the Dynkin diagram of the set $\operatorname{Supp}\alpha \cup \operatorname{Supp}\beta$ has the form shown on Figure 1 (for some $p, q, r \geqslant 1$), $\alpha = \alpha_1 + \ldots + \alpha_p + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\beta = \beta_1 + \ldots + \beta_q + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\pi(\alpha) = \pi(\beta) \in \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$

Next, we need to introduce an equivalence relation on M. For any two roots $\alpha, \beta \in \mathrm{M}$ we write $\alpha \sim \beta$ iff $\tau(\alpha) = \tau(\beta)$. Having introduced this equivalence relation, to each connected solvable spherical subgroup H standardly embedded into B we assign the set of combinatorial data $(S, \mathrm{M}, \pi, \sim)$.

Theorem 4. *The above assignment is a one-to-one correspondence between the following two sets:*

(1) *the set of all connected solvable spherical subgroups standardly embedded into B, up to conjugation by elements of T;*

(2) *the set of all sets $(S, \mathrm{M}, \pi, \sim)$, where $S \subset T$ is a torus, M is a subset of Δ_+, $\pi\colon \mathrm{M} \to \Pi$ is a map, $\sim$ is an equivalence relation on M, and the set $(S, \mathrm{M}, \pi, \sim)$ satisfies the following conditions:*

(A) $\pi(\alpha) \in \operatorname{Supp}\alpha$ *for any $\alpha \in \mathrm{M}$, and the pair $(\alpha, \pi(\alpha))$ is contained in Table 1;*

(D) *if $\alpha, \beta \in \mathrm{M}$ and $\alpha \not\sim \beta$, then one of conditions* (D0), (D1), (D2) *holds;*

(E) *if $\alpha, \beta \in \mathrm{M}$ and $\alpha \sim \beta$, then one of conditions* (D0), (D1), (E1), (D2), (E2) *holds;*

(C) *for any $\alpha \in \mathrm{M}$ the condition* $\operatorname{Supp}\alpha \not\subset \bigcup\limits_{\beta \in \mathrm{M}\setminus\{\alpha\}} \operatorname{Supp}\beta$ *holds;*

(T) $\operatorname{Ker}\tau|_R = \langle \alpha - \beta \mid \alpha, \beta \in \mathrm{M}, \alpha \sim \beta \rangle$, *where* $R = \bigl\langle \bigcup\limits_{\gamma \in \mathrm{M}} \operatorname{Supp}\gamma \bigr\rangle$.

Remark 2. The unipotent radical $N \subset U$ of a connected solvable spherical subgroup H standardly embedded into B is uniquely (up to conjugation by elements of T) determined by the set $(\mathrm{M}, \pi, \sim)$ satisfying conditions (A), (D), (E), (C).

To complete the classification of connected solvable spherical subgroups of G up to conjugation, it remains to determine all sets of combinatorial data that correspond to one conjugacy class of such subgroups. Consider again a connected solvable subgroup $H \subset G$ standardly embedded into B. We say that a marked root α is *regular* if the projection of $\mathfrak{n}$ to the root space $\mathfrak{g}_\alpha$ is zero. Choose any regular marked simple root $\alpha \in \Delta_+$ and fix an element $n_\alpha \in N_G(T)$ such that its image in the Weyl group W is the simple reflection r_α corresponding to α (here $N_G(T)$ is the normalizer of T in G). Obviously, the group $n_\alpha H n_\alpha^{-1}$ is also standardly embedded into B. Its set of combinatorial data is $(n_\alpha S n_\alpha^{-1}, \mathrm{M}', \pi', \sim')$ for some M', π', and $\sim'$. In order to determine M', π', and $\sim'$, we consider two cases:

(1) if $\alpha \in \operatorname{Supp}\delta$ for some $\delta \in r_\alpha(\mathrm{M}\backslash\{\alpha\})$, then $\mathrm{M}' = r_\alpha(\mathrm{M}\backslash\{\alpha\})$, $\pi'(\beta) = \pi(r_\alpha(\beta))$ for any $\beta \in \mathrm{M}'$, $\beta \sim' \gamma$ iff $r_\alpha(\beta) \sim r_\alpha(\gamma)$ for any $\beta, \gamma \in \mathrm{M}'$;

(2) if $\alpha \notin \operatorname{Supp}\delta$ for any $\delta \in r_\alpha(\mathrm{M}\backslash\{\alpha\})$, then $\mathrm{M}' = r_\alpha(\mathrm{M}\backslash\{\alpha\}) \cup \{\alpha\}$, $\pi'(r_\alpha(\beta)) = \pi(\beta)$ for any $\beta \in \mathrm{M}\backslash\{\alpha\}$, $\pi'(\alpha) = \alpha$, $r_\alpha(\beta) \sim' r_\alpha(\gamma)$ iff $\beta \sim \gamma$ for any $\beta, \gamma \in \mathrm{M}\backslash\{\alpha\}$, $\alpha \not\sim' \beta$ for any $\beta \in \mathrm{M}'\backslash\{\alpha\}$.

Transformations of the form $H \mapsto n_\alpha H n_\alpha^{-1}$ described above are called *elementary transformations.*

Theorem 5. *Suppose $H_1, H_2 \subset G$ are two connected solvable spherical subgroups standardly embedded into B, and $H_2 = gH_1g^{-1}$ for some $g \in G$. Then:*

(1) $H_2 = \sigma H_1 \sigma^{-1}$ for some $\sigma \in N_G(T)$;

(2) H_2 can be obtained from H_1 by applying a finite sequence of elementary tramsformations.

Thus, Theorems 4 and 5 give a complete classification of connected solvable spherical subgroups of G.

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Sheets of semisimple symmetric Lie algebras

Michaël Bulois

When H is an algebraic group acting on a variety X, the H-sheets of X are the irreducible components of the sets of the form

$$X^{(m)} := \{x \in X \mid \dim H.x = m\} \text{ for some } m \in \mathbb{N}.$$

The Lie algebra case. In [1], Borho and Kraft studied H-sheets when X is a vector space and H acts linearly on it. They also gave a parameterization when $X = \mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra and $H = G$ is its adjoint group [1, 2]. This parameterization mainly relies on the knowledge of Jordan classes. The Jordan class of an element $x \in \mathfrak{g}$ can be defined by

$$J_G(x) := \{y \in g \mid \exists g \in G,\, g.\mathfrak{g}^x = \mathfrak{g}^y\}.$$

Jordan classes are equivalence classes and one can show that $\mathfrak{g}$ is a finite disjoint union of these classes. Then, it is easily seen that a G-sheet is the union of Jordan classes. So the goal is to explain which Jordan classes are embeded in the closure of another Jordan class. This is what Borho and Kraft did and their method is heavily based on the induction of nilpotent orbits.

Katsylo proved in [5] the existence of a geometric quotient S/G for any G-sheet S of $\mathfrak{g}$. More recently, Im Hof showed that the G-sheets are smooth when $\mathfrak{g}$ is of classical type [4]. The parametrization of G-sheets used in [5, 4] differs from the one given in [1, 2] by the use of "Slodowy slices". More precisely, let S be a G-sheet containing a nilpotent element e and embed e into an $\mathfrak{sl}_2$-triple (e,h,f). Following the work of Slodowy [6, §7.4], the associated Slodowy slice $e+X$ of S is defined by

$$e + X := (e + \mathfrak{g}^f) \cap S.$$

Then, one has $S = G.(e+X)$ and S/G is isomorphic to a the quotient of $e+X$ by a finite group [5]. Furthermore, since $G \times (e+X) \to S$ is smooth [4], the geometry of S is closely related to that of $e+X$.

The symmetric Lie algebra case. A symmetric Lie algebra is a couple $(\mathfrak{g},\theta)$ where $\mathfrak{g}$ is a (finite dimensional complex semisimple) Lie algebra and θ is a Lie algebra involution of $\mathfrak{g}$. It yields an eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated to respective eigenvalues $+1$ and -1. The connected subgroup $K \subset G$ satisfying $\mathfrak{Lie}(K) = \mathfrak{k}$ acts linearly on the vector space $\mathfrak{p}$. Moreover the G-action on $\mathfrak{g}$ can be seen as a particular case of this construction.

The talk aimed to present recent results obtained on K-sheets of symmetric Lie algebras in [3].

First, the notion of Jordan class have a symmetric analogue. The Jordan K-class of $x \in \mathfrak{p}$ is defined in [7, §39] by

$$J_K(x) := \{y \in \mathfrak{p} \,|\, \exists k \in K,\ k.\mathfrak{p}^x = \mathfrak{p}^y\}.$$

Again, it is possible to show that a K-sheet is a finite union of Jordan classes. However, the parameterization of G-sheets of [1] does not seem generalizable to K-sheets of symmetric Lie algebras.

In order to get some parameterization results, the use of Slodowy slices seems more appropriate. If $e \in \mathfrak{p}$ is a nilpotent element of a K-sheet S_K, we embed e in a normal $\mathfrak{sl}_2$-triple (e,h,f) (i.e. $e, f \in \mathfrak{p}$, $h \in \mathfrak{k}$) and we define a $\mathfrak{p}$-Slodowy slice of S_K by

$$e + X_{\mathfrak{p}} := (e + \mathfrak{p}^f) \cap S_K.$$

In type A, the result is the following.

Theorem 1. *If* $\mathfrak{g} \cong \mathfrak{gl}_n$ *then*

$$S_K = \overline{K.(e + X_{\mathfrak{p}})}^{reg}.$$

More generally, it is possible to provide some conditions on S_K that are sufficient to get this result in the general case. It seems manageable to say whether these conditions are satisfied or not when $\mathfrak{g}$ is classical.

The main difference with the Lie algebra case is the necessity of considering the closure of $K.(e + X_{\mathfrak{p}})$. Therefore a result similar to Katsylo's geometric quotient seems hard to reach with the $\mathfrak{p}$-Slodowy slice. However, we can hope gluing together several $K.(e + X_{\mathfrak{p}})$ for different nilpotent elements $e \in S_K$ in order to get the whole variety S_K. Whence the following question (at least in type A).

Question 1. *Do K-sheets have a geometric quotient? If yes, can we describe this quotient by means of the $\mathfrak{p}$-Slodowy slice?*

As a by-product of the proof of Theorem 1, we get several informations on Jordan K-classes. For example

Theorem 2. *If $x \in \mathfrak{p}$ then $J_G(x) \cap \mathfrak{p}$ is equidimensional and its irreducible component are some Jordan K-classes $J_K(x_i)$. Moreover we can assume that the elements $x_i \in \mathfrak{p}$ have the same semisimple part than x.*

Finally, a fixed point theorem yield the following

Theorem 3. *K-sheets of classical symmetric Lie algebras are smooth.*

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Littlewood-Richardson rule for minuscule Schubert cells

Pierre-Emmanuel Chaput

In general, I am interested in "all that can be said" about rational homogeneous spaces, and particularly exceptional homogeneous spaces. Recently, I have studied, with Nicolas Perrin and Laurent Manivel, the classical and quantum cohomology of some spaces which seem more tractable, that we call (co)minuscule spaces and (co)adjoint spaces. Computing the quantum cohomology was a good motivation to understand the geometry of the rational curves in those homogeneous spaces.

More recently, I started a project with Laurent Evain on the equivariant cohomology of the Hilbert schemes of $\mathbb{A}^2$.

I am also interested in the geometry of subvarieties of homogeneous spaces. The idea is to replace the study of embeddings $X \subset \mathbb{P}^n$ by the embeddings $X \subset G/P$. Questions such as the existence of a connexity theorem in this context are attractive to me.

1. Quantum cohomology : algebraic aspects

Laurent Manivel, Nicolas Perrin and I studied the quantum cohomology of so-called (co)minuscule varieties, namely, the Grassmannians (homogeneous under SL_n), the spinor varieties ($Spin_{2n}$), the Lagrangian Grassmannians (Sp_{2n}), the quadrics (SO_n), and two exceptional examples homogeneous under groups of type E_6 or E_7.

Here we gave insight into the quantum cohomology and the ordinary cohomology of other homogeneous spaces under semi-simple groups, and even under Kac-Moody groups. First of all, we gave a combinatorial formula allowing one to compute some intersection numbers in such general spaces. The most striking example of such a formula is the celebrated Littlewood-Richardson rule computing these coefficients for Grassmannians using jeu de taquin. This rule was conjectured by D.E. Littlewood and A.R. Richardson in [9] and proved by M.P. Schützenberger in [12]. Generalisation to minuscule and cominuscule homogeneous spaces of classical types were proved by D. Worley [14] and P. Pragacz [11]. Recently, this rule has been extended to exceptional minuscule homogeneous spaces by H. Thomas and A. Yong [13].

In [5], we largly extended their rule to any homogeneous space X under a Kac-Moody group but only for certain cohomology classes called Λ-minuscule. For X minuscule, any cohomology class is Λ-minuscule.

Let us once again confess that this rule, very efficient to compute some particular intersection numbers, does not compute all of them. However, in [6], we considered the case when X is the closed G-orbit in the projectivisation of the Lie algebra $\mathfrak{g}$ of G (which we call adjoint varieties), and we showed that our rule yields all the intersection coefficients of X up to half the dimension of X plus one, and that this is enough to get a presentation of $H^*(X)$.

While performing the two works [5, 6], we developped a software, written in Java, which allows one to make the combinatorial computations, as well as the algebraic computations, involved to get a presentation of the cohomology. We hope that this software could be usefull to the mathematical community. It is available at the webpage `www.math.sciences.univ-nantes.fr/~chaput/quiver-demo.html`.

2. Quantum cohomology : geometric aspects

If quantum cohomology attracts much of the recent mathematical research for itself, for us it is also creating some challenging interesting questions about the geometry of rational curves. In our study of quantum cohomology of homogeneous spaces, the above results are the consequence of some geometric constructions we were able to perform.

For example the so-called recursion formula is one of our main ingredients in order to prove the Littlewood-Richardson rule. It relies on geometric properties of the Bott-Samelson resolution of Schubert varieties and intersections therein, see [5, Lemma 2.24].

In the adjoint case, this follows from an explicit decomposition of $\mathfrak{g}$ as an L-module, where L is the subgroup of G stabilizing two generic points in X. In fact, using this decomposition we are able to produce explicitly infinitly many curves through 3 fixed points.

In the coadjoint case, X is a hyperplane section of another homogeneous space that we call a Scorza variety. We show using the geometry of this Scorza variety that the locus covered by all rational curves of degree 3 passing through 2 fixed points is in all these cases a particular horospherical variety, such that through a third point pass infinitly many curves of degree 3.

We also show that Gromov-Witten invariants of degree 1 can be computed as classical invariants in the variety of lines in X, which is itself a homogeneous space under G. This is a comparison result in the spirit of [3].

Another intriguing question was raised by Buch and Mihalcea in their study of the K-theoretical Gromov-Witten invariants. Let $d_{\max}$ denote the least integer d such that through any two points in X there passes a curve of degree d. They made the following

Conjecture 1. *(***Buch-Mihalcea***) Let X be a cominuscule variety and let $d > d_{\max}$. If x_1, x_2, x_3 are general points in X, then the Gromov-Witten variety $GW_d(x_1, x_2, x_3)$ is rational.*

This conjecture is true in type A by [1, Corollary 2.2], and we show that it is true for any cominuscule space except exactly in the case where $X = E_6/P_1$ and $d = 3$. In fact, in this case the Gromov-Witten variety is not rational but empty.

3. Dual varieties of subvarieties of homogeneous spaces

When $X \subset \mathbb{P}V$ is a subvariety of a projective space, then the variety of tangent hyperplanes to X is a subvariety of the dual projective space denoted $X^* \subset \mathbb{P}V^\vee$. The biduality theorem $(X^*)^* = X$ holds, and as a consequence tangency loci are linear. In [2], I define an analogue for subvarieties of some homogeneous spaces. For example, if X is a subvariety of a Grassmannian then there is a natural notion of a dual variety X^* in the dual Grassmannian, with the property $(X^*)^* = X$. This also holds for subvarieties of odd spinor varieties and subvarieties of E_6/P_1 and E_6/P_3. In [2], the dual varities of all Schubert varieties are computed.

4. Equivariant cohomology of Hilbert schemes

I have interest in questions such as: understand better the connection between $H^*_T(Hilb_n(\mathbb{A}^2))$ and symmetric polynomials, compute the equivariant product of two elements in the Bialinicki-Birula base of $H^*((Hilb_n(\mathbb{A}^2))$, find a model for $H^*(Hilb_n(\mathbb{P}^2))$, understand the quantum cohomology of $Hilb_n(\mathbb{P}^2)$.

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The p-smooth locus of Schubert varieties

Peter Fiebig
(joint work with Geordie Williamson)

Let X be an irreducible complex variety and let k be a field.

Definition. (1) X is *k-smooth* if the local cohomology of X with coefficients in k behaves at any point as if it was a smooth point of X, i.e. if for any $y \in X$ we have

$$H^d(X, X \setminus \{y\}, k) \cong \begin{cases} k, & \text{if } d = 2\dim_{\mathbb{C}} X, \\ 0, & \text{if } d \neq 2\dim_{\mathbb{C}} X. \end{cases}$$

(2) The *k-smooth locus* of X is the largest open k-smooth subvariety of X.

We have various results on rational smooth (i.e., the $\mathbb{Q}$-smooth) loci (see, for example,[1, 4]). In our paper [6] we determine the p-smooth (i.e., the $\mathbb{F}_p$-smooth) locus of certain varieties X that carry additional structure: we assume that there is a torus $T \cong (\mathbb{C}^\times)^r$ acting on X and that we are given a (finite) stratification $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$. We assume that these data satisfy the following hypotheses:

- There is only a finite number of T-fixed points in X and each fixed point is attractive.
- There is only a finite number of one-dimensional T-orbits and the closure of each is smooth.

- X admits a covering of open affine T-stable subvarieties, each of which contains an attractive fixed point.
- For each $\lambda \in \Lambda$ there is a T-equivariant isomorphism $X_\lambda \cong \mathbb{C}^{n_\lambda}$, where $\mathbb{C}^{n_\lambda}$ carries a linear T-action.
- The T-equivariant derived category $D_{T,\Lambda}(X,k)$ of Λ-constructible sheaves of k-vector spaces is preserved by Verdier duality.

Note that our assumptions imply that each X_λ contains a unique T-fixed point. We denote this point by λ, which gives us an identification of the set of fixed points with Λ. The closure relation on strata induces a partial order on Λ: we set $\lambda \leq \mu$ if X_λ is contained in the closure of X_μ.

To such data we assign a *moment graph* $\mathcal{G}$ as follows. Its set of vertices is the set of T-fixed points in X. We connect the fixed points λ and μ by an edge if $\lambda \neq \mu$ and if there is a one-dimensional T-orbit E in X that contains λ and μ in its closure. Then we choose a T-equivariant isomorphism $E \cong \mathbb{C}^\times_{\alpha_E}$, where on the right T acts via the character $\alpha_E \in X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times)$. We finally label the edge corresponding to E by the character α_E (note that α_E is well-defined up to a sign).

In order to formulate our main result, we have to put some restrictions on the field k of coefficients.

Definition. We say that $(\mathcal{G}, k)$ is a GKM-pair if for any vertex λ the labels of any two distinct edges adjacent to λ are k-linearly independent.

More precisely, the GKM-condition means that for any two distinct edges $E\colon \lambda \overset{\alpha}{\text{——}} \mu$, $E'\colon \lambda \overset{\alpha'}{\text{——}} \mu'$ we have that $\alpha \otimes 1 \notin k(\alpha' \otimes 1)$ as elements in the vector space $X^*(T) \otimes_{\mathbb{Z}} k$. Clearly, this is a condition on the characteristic of k. The main result in [6] is:

Theorem 1. *Suppose that X is a projective variety that satisfies the assumptions above, and that k is a field such that $(\mathcal{G}, k)$ is a GKM-pair. Then the k-smooth locus of X is $\bigsqcup_{\lambda \in \Omega} X_\lambda$, where*

$$\Omega = \left\{ \lambda \in \Lambda \;\middle|\; \begin{array}{c} \text{for any } \mu \geq \lambda \text{ there are exactly} \\ \dim_{\mathbb{C}} X \text{ edges adjacent to } \mu \end{array} \right\}.$$

In the following we sketch a proof of the above result. We first need a version of the *localization theorem* of Goresky, Kottwitz and MacPherson (cf. [7]) for the case that the field of coefficients $\mathbb{C}$ is replaced by k. For this, we follows the proof of the localization theorem in [3] closely.

For $\mathcal{F} \in D_T(X,k)$ let us consider the localization map $\mathrm{H}_T^\bullet(\mathcal{F}) \to \mathrm{H}_T^\bullet(\mathcal{F}_{X^T}) = \bigoplus_{\lambda \in \Lambda} \mathrm{H}_T^\bullet(\mathcal{F}_\lambda)$ (here, and in the following, we write $\mathcal{F}_Y$ for the restriction of $\mathcal{F}$ to a subvariety $Y \subset X$). Then we get

Theorem 2. *Suppose that $\mathrm{H}_T^\bullet(\mathcal{F})$ is free over $S_k = S(X^*(T) \otimes_{\mathbb{Z}} k)$. Then the localization map $\mathrm{H}_T^\bullet(\mathcal{F}) \to \bigoplus_{\lambda \in \Lambda} \mathrm{H}_T^\bullet(\mathcal{F}_\lambda)$ is injective and becomes an isomorphism after inverting all the labels α_E of $\mathcal{G}$.*

In the second step, we determine the image of the localization. For this it is convenient to use the language of *sheaves on moment graphs*.

Definition. A *k-sheaf* $\mathscr{M} = (\{\mathscr{M}^\lambda\}, \{\mathscr{M}^E\}, \{\rho_{\lambda,E}\})$ on $\mathcal{G}$ is given by

- a graded S_k-module $\mathscr{M}^\lambda$ for any vertex λ,
- a graded S_k-module $\mathscr{M}^E$ for any edge E such that $\alpha_E \mathscr{M}^E = 0$,
- a graded S_k-module homomorphism $\rho_{\lambda,E} \colon \mathscr{M}^\lambda \to \mathscr{M}^E$ whenever the vertex λ is adjacent to the edge E.

To any $\mathcal{F} \in D_T(X,k)$ we can assign a k-sheaf $\mathbb{W}(\mathcal{F})$ on $\mathcal{G}$ as follows: we set $\mathbb{W}(\mathcal{F})^\lambda := \mathrm{H}^\bullet_T(\mathcal{F}_\lambda)$, $\mathbb{W}(\mathcal{F})^E := \mathrm{H}^\bullet_T(\mathcal{F}_E)$ and we define $\rho_{\lambda,E}$ as the map

$$\mathrm{H}^\bullet_T(\mathcal{F}_\lambda) \xleftarrow{\sim} \mathrm{H}^\bullet_T(\mathcal{F}_{E\cup\lambda}) \longrightarrow \mathrm{H}^\bullet_T(\mathcal{F}_E).$$

For any k-sheaf $\mathscr{M}$ on $\mathcal{G}$ and any subset I of Λ we define the space of sections of $\mathscr{M}$ over I as

$$\Gamma(I,\mathscr{M}) := \left\{ (m_\lambda) \in \bigoplus_{\lambda\in I} \;\middle|\; \begin{array}{c} \rho_{\lambda,E}(m_\lambda) = \rho_{\mu,E}(m_\mu) \\ \text{for all } \lambda,\mu \in I \\ \text{that are connected by } E \end{array} \right\}.$$

The GKM-condition is crucial for the following result:

Theorem 3. *Suppose that* $\mathcal{F} \in D_T(X,k)$ *is such that* $\mathrm{H}^\bullet_T(\mathcal{F})$ *and* $\mathrm{H}^\bullet_T(\mathcal{F}_\lambda)$ *for any* λ *are free* S_k*-modules. Then* $\mathrm{H}^\bullet_T(\mathcal{F}) = \Gamma(\Lambda, \mathbb{W}(\mathcal{F}))$ *as subspaces in* $\bigoplus_{\lambda\in\Lambda} \mathrm{H}^\bullet_T(\mathcal{F}_\lambda) = \bigoplus_{\lambda\in\Lambda} \mathbb{W}(\mathcal{F})^\lambda$.

Now let $\mathcal{P} \in D_T(X,k)$ be the T-equivariant *parity sheaf* on X that restricts to the constant sheaf on the open stratum (note that the definition of a T-equivariant parity sheaf is analogous to the definition of parity sheaves [8], cf. Daniel Juteau's talk at this conference). Let $\mathscr{B}$ be the Braden–MacPherson k-sheaf on $\mathcal{G}$ (cf. [2]). We show:

Theorem 4. *Suppose* $\mathcal{P}$ *exists. Then* $\mathbb{W}(\mathcal{P}) \cong \mathscr{B}$.

We have the following *multiplicity one* theorem for Braden–MacPherson sheaves:

Theorem 5 ([5]). *We have* $\mathscr{B}^\lambda \cong S_k$ *if and only if* $\lambda \in \Omega$.

Here, again, the GKM-assumption is crucial. We deduce from the above two theorems that $\mathcal{P}_\lambda \cong k_\lambda$ if and only if $\lambda \in \Omega$. Using the Verdier self-duality of $\mathcal{P}$ we then prove that the stalk at λ of the IC-complex on X (with coefficients in k) is of dimension 1 if and only if $\lambda \in \Omega$. Finally, we prove that this is the case if any only if X_λ is contained in the k-smooth locus of X. This finishes the proof of our main theorem.

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Spherical orbit closures in simple projective spaces

Jacopo Gandini

Let G be a semisimple simply connected algebraic group over $\mathbf{C}$, fix a maximal torus T and a Borel subgroup $B \supset T$. Denote R the root system of G associated to T and $S \subset R$ the basis associated to B. If $G_i \subset G$ is a simple factor, denote $S_i \subset S$ the corresponding subset of simple roots. If λ is a dominant weight, denote V_λ the associated simple module and define its *support* as follows

$$\mathrm{Supp}(\lambda) = \{\alpha \in S : \langle \alpha^\vee, \lambda \rangle \neq 0\}.$$

Suppose that $Gx_0 \subset \mathbf{P}(V_\lambda)$ is a *spherical orbit*: this means that B has an open orbit in Gx_0. Then we are interested in its closure $X = \overline{Gx_0}$, and in particular in the normality of X.

Particular cases are that of the adjoint group $G_{\mathrm{ad}} \simeq (G \times G)/N_G(\mathrm{diag}(G))$, regarded as a $(G \times G)$-space, which is spherical because of the Bruhat decomposition, and more generally that of a symmetric space, i. e. of the shape $G/N_G(G^\sigma)$, where $\sigma : G \to G$ is an algebraic involution, which is spherical because of the Iwasawa decomposition.

1. **The case of the adjoint group.** If $\mathrm{Supp}(\lambda) \cap S_i \neq \varnothing$ for every i, then G_{ad} is identified with the orbit of the identity line in $\mathbf{P}(\mathrm{End}(V_\lambda))$; since $\mathrm{End}(V_\lambda)$ is a simple $(G \times G)$-module, the situation is the one considered above. In joint work with P. Bravi, A. Maffei and A. Ruzzi, we gave a complete classification of the normality of the of the associated compactification $X_\lambda = \overline{(G \times G)[\mathrm{Id}]}$, proving the following theorem:

Theorem 1 (see [1]). *The variety X_λ is normal if and only if λ satisfies the following condition, for every connected component $S_i \subset S$:*

(N) *If $\mathrm{Supp}(\lambda) \cap S_i$ contains a long root, then it contains also the short simple root that is adjacent to a long simple root.*

A main tool in the proof of Theorem 1 is the multiplication map between sections of globally generated line bundles on the wonderful completion of G_{ad}: such completion coincides with the variety associated as above to any regular dominant weight and it was studied by C. De Concini and C. Procesi in [5] in the more general setting of a symmetric space. Unlike the general case of a wonderful variety, in the case of the group such map is explicitly described; moreover it was proved to be surjective by S. Kannan in [7] and more generally by R. Chirivì and A. Maffei in [4] in the case of a wonderful symmetric variety. These facts allow to describe a set of generators of the projective coordinate ring of the normalization of X_λ and they allow to give a criterion of normality which turns out to be equivalent to condition (N).

Moreover we gave an explicit characterization of the smoothness of X_λ, proving the following theorem:

Theorem 2 (see [1]). *The variety X_λ is smooth if and only if, for every connected component $S_i \subset S$, λ satisfies condition* (N) *of Theorem 1 together with the following conditions:*

(QF1) $\mathrm{Supp}(\lambda) \cap S_i$ *is connected and, in case it contains a unique element, then this element is an extreme of S_i;*

(QF2) $\mathrm{Supp}(\lambda) \cap S_i$ *contains every simple root which is adjacent to three other simple roots and at least two of the latter ones.*

(S) $S \smallsetminus \mathrm{Supp}(\lambda)$ *is of type* A.

While conditions (QF1) and (QF2) characterize $\mathbf{Q}$-factoriality following a theorem given by M. Brion in [2] which holds for a general spherical variety, condition (S) follows by a theorem given by D. Timashev in [10] which holds for a projective group embedding. However Theorem 2 holds in a similar way for any simple normal completion of a symmetric space (see [1]).

Even if X_λ is non-normal, actually it is homeomorphic to its normalization. This follows considering the more general case of a symmetric orbit, which was considered by A. Maffei in [9], where it is proved that the corresponding orbit closure X is always homeomorphic to its normalization.

2. **The model case.** A very different behaviour, somehow opposite to the one which occurs in the symmetric case, occurs in the model case, i. e. if the considered orbit is of the shape $G/N_G(H)$, where G/H is a model space: a model space for G is an homogeneous space G/H such that every simple G-module occurs with multiplicity one in $\mathbf{C}[G/H]$. Model spaces were classified by D. Luna in [8], where it is defined a wonderful variety M_G^{mod} (called the *wonderful model variety* of G) whose orbits naturally parametrize up to isomorphism the model spaces for G: more precisely any orbit in M_G^{mod} is of the shape $G/N_G(H)$ where G/H is a model space, and this correspondence gives a bijection up to isomorphism. This construction highlights a *principal model space*, namely the model space which dominates the open orbit in M_G^{mod}.

If $G_i \subset G$ is a simple factor of type B or C, number the simple roots in $S_i = \{\alpha_1^i, \ldots \alpha_{r(i)}^i\}$ starting from the extreme of the Dynkin diagram of G_i where the double link is; define moreover $S_i^{\text{even}}, S_i^{\text{odd}} \subset S_i$ as the subsets whose element index is respectively even and odd; set

$$N_i^{\text{even}}(\lambda) = \min\{k \leqslant r(i) \,:\, \alpha_k^i \in \operatorname{Supp}(\lambda) \cap S_i^{\text{even}}\},$$

$$N_i^{\text{odd}}(\lambda) = \min\{k \leqslant r(i) \,:\, \alpha_k^i \in \operatorname{Supp}(\lambda) \cap S_i^{\text{odd}}\}.$$

Finally, if G_i is of type F_4, number the simple roots in $S_i = \{\alpha_1^i, \alpha_2^i, \alpha_3^i, \alpha_4^i\}$ starting from the extreme of the Dynkin diagram which contains a long root. Then we proved the following theorem:

Theorem 3 (see [6])**.** *Let $x_0 \in \mathbf{P}(V_\lambda)$ be such that $\operatorname{Stab}(x_0) = N_G(H)$, where G/H is the principal model space of G. Then X is homeomorphic to its normalization if and only if following conditions are fulfilled, for every connected component $S_i \subset S$:*

(i) *If S_i is of type B, then either $\alpha_1^i \in \operatorname{Supp}(\lambda)$ or $\operatorname{Supp}(\lambda) \cap S_i^{\text{even}} = \varnothing$;*
(ii) *If S_i is of type C, then $N_i^{\text{odd}}(\lambda) \geqslant N_i^{\text{even}}(\lambda) - 1$;*
(iii) *If S_i is of type F_4 and $\alpha_2^i \in \operatorname{Supp}(\lambda)$, then $\alpha_3^i \in \operatorname{Supp}(\lambda)$ as well.*

3. **The strict case.** Let's go back to a generic spherical orbit $Gx_0 \subset \mathbf{P}(V_\lambda)$ and set $H = \operatorname{Stab}(x_0)$. It has been shown by P. Bravi and D. Luna in [3] that such an orbit admits a wonderful completion M; this allows us to describe the orbits of X and those of its normalization from a combinatorial point of view in terms of their *spherical system*, which is a triple of combinatorial invariants that D. Luna attached to a spherical homogeneous space which admits a wonderful completion and which uniquely determines it.

Suppose moreover that M is *strict*, i. e. that the stabilizer of any point $x \in M$ is self-normalizing: this includes the symmetric case as well as the model case. Then, following the description of the orbits of X and of those of its normalization, we get a complete classification of the simple modules V_λ endowed with an embedding $G/H \hookrightarrow \mathbf{P}(V_\lambda)$ (which, if it exists, it is unique) which gives rise to an orbit closure homeomorphic to its normalization (Theorem 6.9 in [6]). The classification is based on a combinatorial condition on $\operatorname{Supp}(\lambda)$ which is easily read off by the *spherical diagram* of G/H, which is a very useful tool to represent its spherical system starting by the Dynkin diagram of G. Such condition of bijectivity is substantially deduced from the model case, where the classification is expressed by Theorem 3, whereas it is always fulfilled if H is a symmetric subgroup or if G is simply laced.

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Representations of finite W-algebras

Simon Goodwin

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and let $e \in \mathfrak{g}$ be nilpotent. The finite W-algebra $U(\mathfrak{g}, e)$ associated to the pair $(\mathfrak{g}, e)$ is a finitely generated algebra obtained from $U(\mathfrak{g})$ by a certain quantum Hamiltonian reduction. Finite W-algebras were introduced to the mathematical literature by Premet in [15], where he showed that $U(\mathfrak{g}, e)$ can be viewed as a quantization of the Slodowy slice through the nilpotent orbit of e; see also [8].

There is a close connection between finite dimensional irreducible representations of $U(\mathfrak{g}, e)$ and primitive ideals of $U(\mathfrak{g})$ stemming from Skryabin's equivalence, [19]. This link was investigated further by Premet Losev and Ginzburg in [16, 17, 11, 12, 9] culminating in [12, Thm. 1.2.2], which says that there is a bijection between: the primitive ideals of $U(\mathfrak{g})$ whose associated variety is the closure of the adjoint orbit of e; and the orbits on the component group of the centralizer of e on the isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules. This also gives rise to a relationship between 1-dimensional $U(\mathfrak{g}, e)$-modules and completely prime ideals of $U(\mathfrak{g})$. The former have been shown to exist for $\mathfrak{g}$ of classical type in [11] and, using a reduction to the case of rigid nilpotents from [18], $\mathfrak{g}$ exceptional not of type E_8 in [10].

Further motivation for the study of finite W-algebras comes from: noncommutative deformations of singularities, see [15]; representation theory of modular reductive Lie algebras, see [15, 17, 18]; and representation theory of degenerate cyclotomic Hecke algebras, see [7].

In [4], Brundan, Kleshchev and the author developed a highest weight theory for finite W-algebras including a definition of Verma modules for $U(\mathfrak{g}, e)$. Each Verma module has an irreducible head and any finite dimensional irreducible module for $U(\mathfrak{g}, e)$ is isomorphic to the head of a Verma module. This gives rise to a strategy for classifying finite dimensional irreducible $U(\mathfrak{g}, e)$-modules in the usual way. An analogue $\mathcal{O}(e)$ of the BGG category $\mathcal{O}$ for finite W-algebras is also defined. It was conjectured in [4], and subsequently proved by Losev in [13], that $\mathcal{O}(e)$ is equivalent to a certain category of *generalized Whittaker modules*; this category has been studied in the literature, most recently by Milicic and Soergel, [14].

For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, the representation theory of $U(\mathfrak{g}, e)$ was studied by Brundan and Kleshchev in [5, 6]. They obtained a classification of finite dimensional irreducible modules along with numerous other results. Their classification is stated nicely in terms of the *pyramid* associated to e. Recent work of Brown gives a classification of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules when $\mathfrak{g}$ is of classical type and e is *rectangular*, see [1, 2].

In recent joint work with Brown, we have been working on the representation theory of $U(\mathfrak{g}, e)$ when $\mathfrak{g}$ is of classical type. We have obtained a classification finite dimensional irreducible $U(\mathfrak{g}, e)$-modules with integral central character when e is *even multiplicity*, [3]. In analogy to the type A situation this classification is nicely stated in terms of the *pyramid* associated to e. The proof depends on the aforementioned relationship of $\mathcal{O}(e)$ with a category of generalized Whittaker modules. This is a step towards the major open problem of understanding the structure and representation theory of finite W-algebras associated to classical Lie algebras.

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Reduction of Shimura varieties and Deligne-Lusztig varieties

Ulrich Görtz

We discuss some analogies, and direct relations, between Deligne-Lusztig varieties and Kottwitz-Rapoport strata in the reduction of Siegel modular varieties with Iwahori level structure.

1. Deligne-Lusztig varieties

Let G_0 be a connected reductive group over $\mathbb{F}_q$, the finite field with q elements. Fix a maximal torus and a Borel subgroup $T_0 \subset B_0$ in G_0, both defined over $\mathbb{F}_q$. Denote by k an algebraic closure of $\mathbb{F}_q$, and by G, B, T the base change to k. Let W be the absolute Weyl group, and σ the Frobenius automorphism. In [2], Deligne and Lusztig defined the following locally closed subvarieties of G/B, which nowadays are called Deligne-Lusztig varieties:

$$X(w) = \{gB \in G/B;\ g^{-1}\sigma(g) \in BwB\}.$$

We have the following "local model diagram":

$$\begin{array}{ccccccc}
G/B & \xleftarrow{\text{proj.}} & G & \xrightarrow{L} & G & \xrightarrow{\text{proj.}} & G/B \\
\uparrow & & \uparrow & & & & \uparrow \\
X(w) & \longleftarrow & \widetilde{X(w)} & & \longrightarrow & & C(w)
\end{array}$$

where L is the Lang map $g \mapsto g^{-1}\sigma(g)$, $C(w) = BwB/B$ denotes the Schubert cell attached to w, and $\widetilde{X(w)}$ is equal to the inverse image of $X(w)$ under the projection, and at the same time to the inverse image of $C(w)$ under the composition proj. $\circ\, L$. All horizontal arrows in this diagram are smooth and surjective, and we obtain a similar diagram by replacing $X(w)$ and $C(w)$ by their closures in G/B (and replacing $\widetilde{X(w)}$ accordingly). Therefore we obtain as a direct corollary:

Proposition 1. *The Deligne-Lusztig variety $X(w)$ is smooth. Its dimension is $\dim X(w) = \dim C(w) = \ell(w)$, the length of w. Its closure $\overline{X(w)}$ is smoothly equivalent to the Schubert variety $\overline{C(w)}$, and in particular is normal and Cohen-Macaulay. We have*

$$\overline{X(w)} = \bigsqcup_{v \leq w} X(v) \qquad \textit{(as sets)},$$

where $\leq$ denotes the Bruhat order on W.

Haastert showed that all Deligne-Lusztig varieties are quasi-affine. Furthermore, we note the following result (see [4]):

Proposition 2. *Let S' be a subset of the set S of simple reflections such that S' meets every σ-orbit. Then*

$$\bigcup_{s \in S'} X(s) \cup X(1)$$

is connected.

The proposition is a refinement of Lusztig's criterion for the connectedness criterion which says that $X(w)$ is connected if and only if $S' := \{s \in S;\ s \leq w\}$ satisfies the condition of the proposition.

2. Reduction of Shimura varieties

Fix an integer $g \leq 1$ and a prime number p. We consider the moduli space $\mathcal{A}$ of g-dimensional principally polarized abelian varieties over k, an algebraic closure of $\mathbb{F}_p$. It is a smooth k-variety, and we use a level structure away from p such that it is connected. Furthermore, we consider the moduli space $\mathcal{A}_I$ of abelian varieties with Iwahori level structure at p, which parametrizes chains

$$A_0 \to A_1 \to \cdots \to A_g$$

of isogenies of order p of g-dimensional abelian varieties with principal polarizations λ_0, λ_g on A_0, A_g, such that the pull-back of λ_g to A_0 is $p\lambda_0$.

Denote by Λ_i the following chain of k-vector spaces: $\Lambda_i = k^{2g}$, $i = 0, \dots, g$, and we fix maps $\alpha_i := \operatorname{diag}(0, \dots, 0, 1, 0, \dots, 0)\colon \Lambda_i \to \Lambda_{i+1}$, where the 1 is in the $(i+1)$-th position, $i = 0, \dots, g-1$. We equip Λ_0 and Λ_g with the standard symplectic pairing.

We recall the definition of the "local model" à la de Jong [1], Deligne/Pappas and Rapoport/Zink:

$$M^{\mathrm{loc}}(S) = \{(\mathcal{F}_i)_i \in \prod_{i=0}^{g} \operatorname{Grass}_g(\Lambda_i)(S);\ \alpha_i(\mathcal{F}_i) \subseteq \mathcal{F}_{i+1},$$
$$\mathcal{F}_0, \mathcal{F}_g \text{ totally isotropic}\}.$$

The local model can be identified with a union of Schubert varieties in the affine flag variety for the group GSp_{2g}. and the corresponding local model diagram:

$$\begin{array}{ccccc} \mathcal{A}_I & \longleftarrow & \widetilde{\mathcal{A}}_I & \longrightarrow & M^{\mathrm{loc}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{A}_w & \longleftarrow & \widetilde{\mathcal{A}}_w & \longrightarrow & C(w) \end{array}$$

Here $\widetilde{\mathcal{A}}_I$ is the space of pairs $((A_\bullet)_\bullet, \Psi)$, where $(A_\bullet)_\bullet \in \mathcal{A}_I$ and Ψ is an isomorphism of chains $H^1_{DR}(A_\bullet/S) \xrightarrow{\sim} \Lambda_\bullet \otimes \mathcal{O}_S$. The morphism $\widetilde{\mathcal{A}}_I \to \mathcal{A}_I$ is a torsor for the automorphism group scheme of the chain $\Lambda_\bullet$ which is smooth, and using the theory of Grothendieck and Messing, one can show that the morphism $\widetilde{\mathcal{A}}_I \to M^{\mathrm{loc}}$ is smooth, too. In the lower row of the diagram, w is an element of the extended affine Weyl group such that the corresponding Schubert cell $C(w)$ lies in M^{loc}, and $\widetilde{\mathcal{A}}_w$ is the inverse image of $C(w)$ in $\widetilde{\mathcal{A}}_I$.

The locally closed subvarieties $\mathcal{A}_w \subset \mathcal{A}_I$ are defined by the above diagram; they are called Kottwitz-Rapoport (KR) strata, and by the definition the local structure of these strata and their closures is the same as the structure of the corresponding Schubert cells and Schubert varieties in the affine flag variety.

Using that the Hodge bundle on $\mathcal{A}_g$ is ample, one can show that all KR strata are quasi-affine, [7] Theorem 5.4. It turns out that KR strata are usually connected:

Theorem 1 ([7] Theorem 7.4, Corollary 7.5)**.** *Every KR-stratum which is not contained in the supersingular locus of $\mathcal{A}_I$ is connected.*

One of the ingredients of the proof is the above mentioned result on Deligne-Lusztig varieties (in the special case of certain unitary groups, where it was first proved by Ekedahl and van der Geer). The structure of those KR strata that are contained in the supersingular locus can be made very explicit. We have

Theorem 2 ([6] §6, [7] Corollary 7.5)**.** *Let $\mathcal{A}_w$ be a KR stratum which is contained in the supersingular locus. Then there exists $0 \leq i \leq \frac{g}{2}$ such that for every point $(A_\bullet)_\bullet \in k$, the abelian varieties A_i and A_{g-i} are superspecial. Furthermore $\mathcal{A}_w$ is the union of copies of Deligne-Lusztig varieties for a (non-split) group whose Dynkin diagram is obtained from the extended Dynkin diagram of type $\tilde{C}_2$ by omitting the vertices i and $g-i$.*

As a corollary of these results and a computation of the dimension of the p-rank 0 locus in $\mathcal{A}_I$, one obtains

Corollary 1 ([7] Theorem 1.1)**.** *If g is even, then the supersingular locus in $\mathcal{A}_I$ has dimension $g^2/2$. If g is odd, then its dimension lies between $g(g-1)/2$ and $(g+1)(g-1)/2$.*

This should be compared with the result of Li and Oort [8] that the dimension of the supersingular locus in $\mathcal{A}_g$ is $[g^2/4]$. Note also that th supersingular locus in $\mathcal{A}_I$ is not equidimensional as soon as $g \geq 2$. In joint work with Maarten Hoeve we investigated the relationship between KR strata (also in the general parahoric case) to the Ekedahl-Oort stratification, see [5]. Compare also the work of Ekedahl and van der Geer [3].

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Prehomogeneous spaces, the volume of a tilting module, and parabolic group actions

Lutz Hille

1. The Action of Gl_N on a product of flag varieties $\mathrm{Gl}_N/P_1 \times \mathrm{Gl}_N/P_1 \times \ldots \times \mathrm{Gl}_N/P_s$

We consider the General Linear Group Gl_N and parabolic subgroups P_i for $i = 1, \ldots, s$. Then the group Gl_N acts on the product of quotients $\mathrm{Gl}_N/P_1 \times \mathrm{Gl}_N/P_1 \times \ldots \times \mathrm{Gl}_N/P_s$ via left multiplication (see also [5]). Any parabolic subgroup P_i is the stabilizer of a flag $V_1^i \subset V_2^i \subset \ldots \subset V_{r(i)}^i$, with $\dim V_j^i = a_j^i$. We denote the vector $(N, a_1^1, \ldots, a_{r(1)}, a_1^2, \ldots, a_{r(s)}^s)$ just by $\underline{a}$. Moreover, we denote the product of the flag varieties by $X(a)$ and consider it as a Gl_N–variety.

For such an action we are interested in the following main questions:
(M1) Does Gl_N act with a finite number of orbits on $X(a)$?
(M2) Does Gl_N act with a dense orbit on $X(a)$?
(M3) If Gl_N acts with a dense orbit on $X(a)$ find a representative of the dense orbit.

In fact one should study the set of all vectors $\underline{a}$ so that the answer to (M1) repectively (M2) is Yes. It turns out that there is an equivalent formulation in terms of quivers.

Consider the quiver Q with vertices $q_{i,j}$ with $i = 1, \ldots, s$ and $j = 1, \ldots, r(i) - 1$ and one central vertex 0 identified with $q_{i,r(i)}$. The quiver Q has arrows $(i,j) \longrightarrow (i, j+1)$ for $j = 1, \ldots, r(i)$. Thus Q becomes a star like quiver with s arms of length $r(1), r(2), \ldots, r(s)$, respectively.

For the quiver Q and a dimension vector $\underline{a} = (N, a_1^1, \ldots, a_{r(s)}^s)$ we define the representation space

$$\mathcal{R}(Q, \underline{a}) := \bigoplus_{\alpha:(i,j)\longrightarrow(i,j+1)\in Q_1} \mathrm{Hom}(k^{a_i^j}, k^{a_i^{j+1}}).$$

On this affine space we consider the action of the group $\mathrm{Gl}(\underline{a}) := \prod \mathrm{Gl}(a_i^j) \times \mathrm{Gl}_N$ via base change. We say $\underline{a}$ is injective, if $a_j^i \leq a_{j+1}^i$ for all possible pairs (i,j) (with our convention above we have in particular $a_j^i \leq N$). Note that $\underline{a}$ is injective for the action of Gl_N on $X(a)$ by definition. It turns out that all questions above for an injective dimension vector have the same answer (however we do not claim that there exists a bijection between the orbits, this does only hold if we restrict the action to the subspace in $\mathcal{R}(Q, \underline{a})$ consisting of injective maps).

Lemma 1. *Let $\underline{a}$ be an injective dimension vector.*
(1) The group Gl_N acts with a finite number of orbits on $X(a)$ precisely when $\mathrm{Gl}(\underline{a})$ acts with a finite number of orbits on $\mathcal{R}(Q, \underline{a})$.
(2) The group Gl_N acts with a dense orbit on $X(a)$ precisely when $\mathrm{Gl}(\underline{a})$ acts with a dense orbit on $\mathcal{R}(Q, \underline{a})$.

The proof just uses the fact that for maps that are not injective we can write them as a direct sum of some injective maps with some non-injective maps and the non-injective maps only consist of finitely many isomorphims classes.

This equivalence motivates to consider the three main questions for any quiver Q in the next section.

2. The Action Associated to a Quiver

In this section we consider the three main questions for quiver representations and the associated action of a product of General Linear Groups on the representation space. For, let Q be a quiver (with t vertices), with set of vertices Q_0 and set of arrows Q_1 and let $d = (d_1, \ldots, d_t)$ be a dimension vector. Then the group $G(d) := \prod_{q \in Q_0} \mathrm{Gl}(d_q)$ acts via base change on the representation space $\mathcal{R}(Q, d) := \oplus_{\alpha \in Q_1} \mathrm{Hom}(k^{d(s(\alpha))}, k^{d(t(\alpha))})$. It turns out that an element x in a dense orbit corresponds to a module x over the path algebra without selfextensions. Such a module can be completed to a tilting module T (it has no selfextensions and t pairwise non-isomorphic direct summands). Thus $T = \oplus_{i=1}^t T_i^{a_i}$ with some multiplicities $a_i > 0$. Then we define a cone and the volume

$$\sigma_T := \text{ convex hull of } \{\underline{\dim} T_i\}_{i=1}^t, \qquad \mathrm{vol}(T) = \prod_{i=1}^t 1/\dim T_i.$$

The definition of the cone makes sense for all representations of Q, however we use it only for representations x with $\mathrm{Ext}^1(x, x) = 0$. Then we define a set of cones associated to Q

$$\Sigma_Q := \{\sigma_x \mid \mathrm{Ext}^1(x, x) = 0\}.$$

We formulate the main theorem and explain the terminology afterwards. With $\mathcal{T}$ we denote the set of isomorphism classes of tilting modules, that contain each indecomposable direct summand only with multiplicity one.

Theorem 1. *a) The set of cones Σ_Q is a smooth, quasi-complete (in $\mathbb{R}^t_{\geq 0}$), pureley t–dimensional fan in the positive quadrant $\mathbb{R}^t_{\geq 0}$.*
b)The group $G(d)$ acts with a dense orbit on $\mathcal{R}(Q, d)$ precisely when d is a lattice point in the support of Σ_Q, that is d is a dimension vector of a rigid representation of Q.
c) The fan is finite precisely when Q is a Dynkin quiver. In this case $\sum_{T \in \mathcal{T}} \mathrm{vol}(T) = 1$ is a finite sum.
d) The fan is infinite with $\sum_{T \in \mathcal{T}} \mathrm{vol}(T) = 1$ precisely when Q is tame, but not Dynkin.
e) $\sum_{T \in \mathcal{T}} \mathrm{vol}(T) < 1$ precisely when Q is wild. In this case the fan is also infinite.

f) Assume we have a subset $\mathcal{S}$ of $\mathcal{T}$ with $\sum_{T\in\mathcal{S}} \mathrm{vol}(T) = 1$ then $\mathcal{T} = \mathcal{S}$ and Q is tame.

Note that we allow infinite fans. Except for this generalization, our notion of a fan coincides with the notion in toric geometry (see e.g. [2]). A fan is smooth if each cone is generated by a part of a $\mathbb{Z}$–basis. A fan is purely t–dimensional, if each cone is contained in a t–dimensional cone. For such a fan smooth just means that each t–dimensional cone is generated by a $\mathbb{Z}$–basis. The fan is quasi-complete if for each t–dimensional cone σ in the fan and each facet μ of σ either there exists precisely one cone τ with $\tau \cap \sigma = \mu$ or μ is already contained in the hyperplane defined by $d_i = 0$ (it is in the boundary of the positive quadrant $\mathbb{R}^t_{\geq 0}$).

The theorem allows to answer question (M2) and (M3) in a recursive way. One starts with some tilting modules and, using that Σ is quasi-complete, one can recursively construct all other tilting modules. Using induction on the boundary (that corresponds to the tilting modules over a quiver with $t-1$ vertices) one can in fact construct all d so that $G(d)$ acts with a dense orbit on the representation space. Any such tilting module, considered as an element in the representation space, is a representative of the dense orbit. Finally, also the answer to question (M1) is known for Dynkin and tame quivers. Only for wild quivers there are some dimension vectors were the answer is not known.

3. Parabolic Group Actions

The main motivation for the techniques developed in the previous section is an open question for parabolic group actions. The methods therin cannot be used, here we need to consider quivers with relations. However, the results for parabolic group actions are quite similar. In this section we fix a natural number $t > 1$. For any flag

$$V_0 = \{0\} \subseteq V_1 \subseteq \ldots V_{t-1} \subseteq V_t$$

of length t and dimension vector $d := (\dim V_1, \dim V_2 - \dim V_1, \ldots, \dim V_t - \dim V_{t-1})$ we can define several groups and Lie algebras as follows. The parabolic group $P(d)$ is the stabilizer of the flag. It acts on the Lie algebra of its unipotent radical

$$\mathfrak{p}_u(d) := \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{i-1}\}$$

via conjugation. Moreover, we have the $P(d)$–invariant Lie subalgebras

$$\mathfrak{n}_h(d) := \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{h(i)}\},$$

where h is any function $h : \{1, 2, \ldots, t\} \longrightarrow \{0, 1, \ldots, t-1\}$ satisfying $h(i) < i$ for all $i = 1, 2, \ldots, t$. The instances where $P(d)$ acts with a finite number of orbits on $\mathfrak{p}_u(d)$, respectively on its derived Lie algebras $\mathfrak{p}_u(d)^{(l)}$ for all d were classified in [4] and [1] (in fact even for the other classical groups as well). By Richardson's result $P(d)$ acts always with a dense orbit on $\mathfrak{p}_u(d)$. In contrast, the set

$$D_h := \{d \in \mathbb{Z}^t_{\geq 0} \mid P(d) \text{ acts with a dense orbit on } \mathfrak{n}_h(d)\}$$

is only known for some functions of h. It is, for example, not known for $h(i) = i-2$ and $t > 9$.

In our main result that follows from [3] and the previous work on quasi-hereditary algebras associated to bimodule problems, we claim that D_h consists of lattice points in a smooth fan. The cones in this fan are defined for some modules without selfextensions over a certain quasi-hereditary algebra associated to h. Moreover, the normal form of a representative of the dense orbit does only depend on the cone σ containing d.

Open problem: The main open problem is whether the fan defining the set D_h is connected via facets to the boundary. To be precise, does there exist for each cone σ a sequence of cones σ_i so that $\sigma_i \cap \sigma_{i+1}$ is a facet of both, $\sigma_1 = \sigma$ and one facet of σ_r is defined by $d_i = 0$ for some $i = 1, \dots, t$.

A positive answer to this problem for parabolic group actions, like it is already known for quivers, solved the (algorithmic) construction of all cones covering D_h. Thus we could describe all actions with a dense orbit at least recursively.

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Parity sheaves

Daniel Juteau

(joint work with Carl Mautner, Geordie Williamson)

1. Motivation

One can often translate a problem in representation theory into a problem about perverse sheaves on a related algebraic variety equipped with a stratification $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$. For example, the stalks of the simple perverse sheaves (which are also called intersection complexes) can encode the answer of the problem. This approach has been very successful in characteristic zero, that is if we consider ordinary representation theory and sheaves with coefficients in characteristic zero. The first example is the proof (due to Beilinson-Bernstein and Brylinski-Kashiwara) of the Kazhdan-Lusztig conjectures about representations of semi-simple Lie algebras.

The main reason why this is useful is because intersection complexes are computable, mostly due to the decomposition theorem. For example, one can often compute their stalks inductively because a given intersection complex will appear as a direct summand in a direct image whose stalks are computable (these stalks are just the cohomology groups of the fibers), and the other summands will be intersection complexes on lower strata whose stalks are known by induction, appearing with a computable multiplicity (for example, if the map is semi-small, a condition which ensures that the direct image is a perverse sheaf, then one just has to study the top cohomology of the fibers).

Unfortunately, the decomposition theorem does not hold anymore if we consider sheaves with coefficients in a field k of characteristic $p > 0$. There are still relations between modular representation theory and perverse sheaves, like the version of the geometric Satake correspondence due to Mirkovic and Vilonen (see the last part). So it would be very desirable to compute the stalks of intersection complexes with k coefficients, but it becomes very quickly extremely difficult.

Taking advantage of the fact that the varieties appearing in representation are not random but usually have very particular properties, we propose in [1] to consider a new class of "sheaves" (we actually mean complexes in the derived constructible category). Namely, the fibers of the resolutions that we consider usually have odd-vanishing cohomology over $\mathbb{Q}$, and hence also over $\mathbb{F}_p$ for p large enough, or better, very often the cohomology is free and concentrated in even degrees over $\mathbb{Z}$ (like Bott-Samelson resolutions of Schubert varieties, or the Springer resolution of the nilpotent cone). A resolution whose fibers have even cohomology over k will be called k-even (or just even if the cohomology is even and free over $\mathbb{Z}$). It is interesting to note that by Kaledin's results, a symplectic resolution is semi-small and $\mathbb{Q}$-even, and actually $\mathbb{F}_p$-even for p greater than twice the dimension of the variety, because there is a resolution of the diagonal in K-theory, and one can use the Chern character to go to cohomology.

Since a direct image under a resolution is no longer a semi-simple complex, one can instead study its indecomposable summands. However, whereas simple objects are parametrized by pairs consisting of a stratum and an irreducible local system, it is not clear how to classify all the indecomposable summands appearing in direct images of resolutions. In [2], Soergel undertook this program for Bott-Samelson resolutions in finite flag varieties (stratified by Bruhat cells), and found that the indecomposable summands that appear, which he called special sheaves, are parametrized by the Weyl group, just as the simple perverse sheaves (Bruhat cells are simply connected so only trivial local systems can appear). His proof goes through representation theory. Moreover, he shows that specific instances of Lusztig's conjectures about modular representations of reductive groups (for weights "around the Steinberg weight") is equivalent to the decomposition theorem to hold for certain morphisms between generalized flag varieties. [1]

[1]Fiebig later pursued this idea to give a new proof of Lusztig's conjectures for big primes, using the affine flag variety and sheaves on moment graphs. He actually found the first general explicit bound, though it is huge compared to the Coxeter number. He also got the multiplicity one case for all p greater than the Coxeter number.

We take another point of view. We remark that direct summands of the direct image of the constant sheaf under an even resolution are parity complexes, by which we mean the following: their stalks and costalks are concentrated in degrees of a fixed parity. It turns out that this is enough to characterize them, in the situations we typically consider.

2. Definition and first properties

Assumption. From now on, we assume that the cohomology of the strata X_λ with coefficients in any local system $\mathcal{L}$ is concentrated in even degrees.

Let us point out that sometimes we actually need to consider the equivariant setting (and thus use equivariant cohomology, and the equivariant derived category). For example, the nilpotent cone of $\mathfrak{gl}_n$ stratified by the GL_n-orbits satisfies the assumption in the GL_n-equivariant setting, but not in the classical setting.

Theorem 1. *Suppose the assumption above is satisfied.*

(1) *Let X_λ be a stratum and let $\mathcal{L}$ be an irreducible local system on it. Then there is at most one indecomposable parity complex $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ supported on $\overline{X}_\lambda$ and whose restriction to X_λ is $\mathcal{L}[\dim X_\lambda]$.*

(2) *Moreover, any indecomposable parity complex is isomorphic to $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})[n]$ for some $\lambda \in \Lambda$, some irreducible local system $\mathcal{L}$ and some integer n.*

We call the $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ parity sheaves. To prove the existence of a parity sheaf for a given pair, the only method we know is to use proper even pushforwards. We can prove existence and uniqueness in the following situations: (Kac-Moody) Schubert varieties (and notably the affine Grassmannian), toric varieties (with torus action), and the nilpotent cone of $\mathfrak{gl}_n$ (with GL_n action).

While intersection complexes quickly become horribly difficult to compute in characteristic p, one can in principle compute stalks of parity sheaves inductively using direct images of resolutions, but this involves computing the ranks of certain intersection forms, which is not easy to do in practice. However, in some situations including the case of Schubert varieties, Fiebig and Williamson proved that the moment graph algorithm of Braden and MacPherson, applied with characteristic p coefficients, gives the stalks of parity sheaves.

3. Parity sheaves and tilting modules

Let G be a simple, simply connected reductive group over k, and let $G^\vee$ be its dual over $\mathbb{C}$. We fix a maximal torus $T \subset G$ and a Borel subgroup containing T. We denote by $\Lambda = X(T)^+$ the dominant characters associated to this choice. Let $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$. We consider the affine Grassmannian $G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$, stratified by the $G^\vee(\mathcal{O})$-orbits which are parametrized by Λ. It is an ind-variety. The strata are simply-connected, so that only trivial local systems can occur. We have the existence and uniqueness of a parity sheaf $\mathcal{E}(\lambda)$ for each $\lambda \in \Lambda$. Recall that the geometric Satake correspondence is an equivalence of tensor categories between the representation category of a reductive algebraic group scheme equipped with the usual tensor product and a category of perverse sheaves on the affine Grassmannian of the Langlands dual group, equipped with a convolution product.

Theorem 2. *Assume that $p > 2$ in types B_n or D_n, $p > n$ in type C_n, $p > 3$ in types G_2, F_4 or E_6, $p > 19$ in type E_7, and $p > 31$ in type E_8. Then, for $\lambda \in \Lambda$, the parity sheaf $\mathcal{E}(\lambda)$ is perverse, and corresponds to the tilting module $T(\lambda)$ under the geometric Satake correspondence.*

The proof of this theorem goes through representation theory. We have an understanding of the geometric situation in the case of a minuscule weight and in the highest short root case (for which we have a minimal nilpotent singularity). Then we use the fact that convolutions of parity complexes are again parity complexes, and that tensor products of tilting modules are again tilting modules, and the game is to generate all fundamental tilting modules using the ones mentioned above. For bad primes, we know that parity sheaves may fail to be perverse. This is related to the torsion in the stalks of standard sheaves. It is well possible that a better bound is possible in types E_7 and E_8. However, in type C_n, for p less than n, Donkin pointed out that one cannot always generate all fundamental tilting modules in this way, so either the result fails for those primes, or one would have to find a new idea to prove the result in those cases.

In the cases we know where $\mathcal{E}(\lambda)$ is not perverse, it turns out that the 0-th perverse cohomology sheaf is the one corresponding to the tilting module. This leads to conjecture that for *any* prime, and any $\lambda \in \Lambda$, the tilting module with highest weight λ corresponds to ${}^p\mathcal{H}^0\mathcal{E}(\lambda)$ under the geometric Satake equivalence.

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From moment polytopes to string bodies

Valentina Kiritchenko

In toric geometry, a central role is played by moment (or Newton) polytopes of projective toric varieties. In the past decades, various analogs of Newton polytopes for other reductive group actions were constructed culminating in a recent construction of *string bodies* (special Newton-Okounkov convex bodies). My talk was mostly devoted to this construction [3]. A future objective is to use string bodies to study geometry of varieties with a reductive group action (as in the toric case). Below we discuss such an application in a non-toric example.

String bodies for the varieties of complete flags are just string polytopes (e.g. Gelfand–Zetlin polytopes in the case of $GL_n(\mathbb{C})$). Together with Evgeny Smirnov and Vladlen Timorin we develop a new approach to the Schubert calculus on the variety of complete flags in $\mathbb{C}^n$ using the volume polynomial on Gelfand–Zetlin polytopes. This approach allows us to compute the intersection product of Schubert cycles on the flag variety by intersecting faces of the Gelfand–Zetlin polytope. The Gelfand–Zetlin polytope thus gives a combinatorial model for the intersection theory on the flag variety.

First recall the definition of the volume polynomial. Let P be a convex polytope in $\mathbb{R}^n$. Two convex polytopes are called *analogous* if they have the same normal fan. The polytopes analogous to P form a semigroup with respect to *Minkowski sum*. We can embed this semigroup into its Grothendieck group V_P, which is a real vector space (its elements are called *virtual polytopes*). The *volume polynomial* vol is a homogeneous polynomial of degree n on the vector space V_P such that its value $vol(Q)$ on any convex polytope $Q \in V_P$ is equal to the volume of Q.

The volume polynomial on the space V_P was used by Pukhlikov and Khovanskii to describe the cohomology rings of smooth toric varieties. Recall that each integrally simple lattice polytope P (that is, only n edges meet at every vertex, and the primitive lattice vectors on these edges form a basis in $\mathbb{Z}^n \subset \mathbb{R}^n$) defines a smooth polarized toric variety X_P. The Chow ring of X_P (or equivalently, the cohomology ring $H^*(X_P, \mathbb{Z})$, which lives only in even degrees) is isomorphic to the quotient R_P of the ring of differential operators on V_P with constant integer coefficients. To get R_P we quotient by the operators that annihilate the volume polynomial. This description is functorial. It is clear that the ring R_P lives only in degrees up to n (since the volume polynomial has degree n) and that R_P has a non-degenerate pairing (Poincaré duality) defined by $(D_1, D_2) := D_1 D_2(vol) \in \mathbb{Z}$ for any two homogeneous operators D_1 and D_2 of complementary degrees. In fact, the Poincaré duality on the ring R_P is the key ingredient in the proof of the isomorphism between R_P and $H^{2*}(X_P, \mathbb{Z})$ (see [2] for more details).

Note that if P is not simple, we can still define the ring R_P, which will still live in degrees up to n and satisfy Poincaré duality. However, its relation to the Chow ring of (now singular) toric variety X_P is unclear. On the other hand, the ring R_P for non-simple polytopes is sometimes related to the Chow rings of smooth non-toric varieties.

We now consider the ring R_P for the *Gelfand–Zetlin polytope* $P = P_\lambda$ (which is not simple) associated with a strictly dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of the group $GL_n(\mathbb{C})$. Recall that the Gelfand-Zetlin polytope P_λ is a convex lattice polytope in $\mathbb{R}^d$, where $d = n(n-1)/2$ (see e.g. [4] for more details). Note that Gelfand–Zetlin polytopes P_λ and P_μ are analogous for any two strictly dominant weights λ and μ, and hence define the same space V_P and the same ring R_P. The ring R_P is isomorphic to the Chow ring (or to the cohomology ring) of the complete flag variety X for $GL_n(\mathbb{C})$ (note that $\dim\ X = d$) so that the differential operators $\frac{\partial}{\partial\lambda_1}, \ldots, \frac{\partial}{\partial\lambda_n}$ get mapped to the first Chern classes of the tautological line bundles on X. This follows immediately from the results of Kaveh [2] and can also be deduced directly from the Borel presentation for the cohomology ring $H^*(X, \mathbb{Z})$ using that the volume of P_λ (regarded as a function of λ) is equal to $\prod_{i<j}(\lambda_i - \lambda_j)$ times a constant.

We now discuss an important feature of the isomorphism $R_P \simeq CH^*(X)$: the isomorphism allows us to identify the algebraic cycles on X with the linear combinations of the faces of P. We first recall the easier case of simple polytopes [6, §2]. If P is simple then the dimension of the space V_P is equal to the number l of facets of P (since we can move independently by parallel transport each of the hyperplanes containing the facets of P). Note that for non-simple P the dimension of V_P is strictly less than l (e.g. if P is an octahedron, then V_P is just one-dimensional). For simple P, the space V_P has natural coordinates $(H_1, \ldots, H_l)$ called the *support numbers*. They are defined by fixing l covectors $h_1, \ldots, h_l$ on $\mathbb{R}^n$ such that the facet Γ_i of P (for each $i = 1, \ldots, l$) is contained in the hyperplane $h_i(x) = H_i(P)$ for some constant $H_i(P)$ and the polytope P satisfies the inequalities $h_i(x) \le H_i(P)$. Then any collection of real numbers $(H_1, \ldots, H_l)$ uniquely defines a (possibly virtual) polytope in V_P by the inequalities $h_i(x) \le H_i$. The ring R_P then has multiplicative generators $\partial_1 := \frac{\partial}{\partial H_1}, \ldots, \partial_l := \frac{\partial}{\partial H_l}$. We now assign to each product $\partial_{i_1} \ldots \partial_{i_k}$ (for distinct $i_1, \ldots, i_k$) the face $\Gamma_{i_1} \cap \ldots \cap \Gamma_{i_k}$ of P (if we identify R_P with the Chow ring of the smooth toric variety X_P then this becomes the well-known correspondence between the (cycles of) torus orbits in X_P and the faces of P).

It easy to check that all linear relations between $\partial_1, \ldots, \partial_l$ have form $h_1(v)\partial_1 + \ldots + h_l(v)\partial_l = 0$, where $v \in \mathbb{Z}^n \subset \mathbb{R}^n$ (because the volume of a polytope does not change if the polytope is parallely transported by the vector v). Using these linear relations we can always reduce any monomial in $\partial_1, \ldots, \partial_l$ to the linear combination of monomials containing only pairwise distinct ∂_i. Geometrically, this corresponds to computing the intersection product of the closures of torus orbits by using linear equivalence relation on the closures of codimension one orbits. Polytope P and ring R_P allows one to make these computations more explicit by using geometric invariants of P (such as volume of P, integer distances to the facets etc.).

If P is not simple, then things become more complicated. I now state our results in this case. It is still possible (though less straightforward) to identify each element of R_P with a linear combinations of faces of P, but not every face of P would correspond to an element of R_P. Namely, we embed the ring R_P into a certain R_P-module M_P whose elements can be regarded as linear combinations of arbitrary faces of P modulo some relations. The module M_P depends on the choice of a simple resolution $\widetilde{P}$ of P (that is, $\widetilde{P}$ is obtained from P by generic parallel transports of the hyperplanes containing the facets of P), and is also defined using the volume polynomial. The product of an element in M_P by an element of R_P can again be computed by intersecting faces (and applying linear relations if necessary to make the faces transverse). While all of these applies to any convex polytope P it is especially interesting to study the case where $P = P_\lambda$ is a Gelfand–Zetlin polytope due to the isomorphism $R_P \simeq CH^*(X)$ for the flag variety X. Recall that $CH^*(X)$ (as a group) is a free abelian group with the basis of Schubert cycles. We now give the answer to the following natural question: how to express Schubert cycles as linear combinations of faces of the Gelfand–Zetlin polytope?

The relation between Schubert cycles and faces of the Gelfand–Zetlin polytope was first investigated in [5], and then by different methods also in [6] and [4]. We noticed that the ring R_P and its realization by faces via the module M_P provide the uniform setting for all previously known results on the cycle-face correspondence as well as for some new results. In particular, we proved the following formula, which is formally similar to the Fomin–Kirillov theorem on Schubert polynomials and uses the correspondence between *rc-graphs* (or *reduced pipe-dreams*) and certain faces of the Gelfand–Zetlin polytope described in [5]. Denote by X_w the Schubert cycle corresponding to the permutation w as in [6, §4]. Then the following identity holds in M_P:

$$X_w = \sum_{w(\Gamma)=w} \Gamma, \tag{1}$$

where the sum is taken over all *rc-faces* (see [6, §4] for the definition) of P_λ with permutation w. Note that (1) can not be deduced from the Fomin–Kirillov theorem because the faces Γ will not usually belong to R_P (only to M_P) and hence can not be identified with the monomials in the corresponding Schubert polynomial. Our proof of (1) uses simple convex geometry arguments.

Once we have identity (1) it is easy to get many other presentations of Schubert cycles via faces by applying to (1) the relations in R_P. We have described all linear relations between facets, which turned out to be quite simple and used them to represent each Schubert cycle as a sum of faces that are transverse to all rc-faces. Hence, the intersection of any two Schubert cycles can also be written as the sum of faces (that is, with nonnegative coefficients). We hope that further investigation will lead to a transparent Littlewood–Richardson rule (different from the one in [1]) for the varieties of complete flags. A simple example (for $n = 3$) illustrating our approach to Schubert calculus via Gelfand–Zetlin polytopes can be found in [4, §4].

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Singularities of affine Schubert varieties

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(joint work with V. Lakshmibai)

Let $A = \mathbb{C}[[t]]$ be the ring of formal power series and $F = \mathbb{C}((t))$ be its field of fractions (the field of formal Laurent series). The *affine Grassmannian* is defined as $X = \mathcal{G}/\mathcal{P}$ where $\mathcal{G} = G(F)$ and $\mathcal{P} = G(A)$ and $G = \mathrm{SL}_n$ (of course this definition makes sense for other algebraic groups). To be more precise, X is defined as a specific projective ind variety whose $\mathbb{C}$-valued points are given by $G(F)/G(A)$; as such it is a direct limit (directed union) of finite dimensional irreducible projective varieties. $G(F)$ can be thought of as an algebraic analogue of the loop group LG of holomorphic maps from $\mathbb{C}^*$ to $G(\mathbb{C})$. X on the other hand is also a homogeneous space for the Kac-Moody group associated to the affine root system of type $\hat{A}_{n-1}$, and it is the "right" object to study the analogues of Schubert varieties for this group.

Let $B \subseteq G$ be the group of lower triangular matrices and let $\mathcal{B} \subseteq \mathcal{P}$ be the pre-image of B under the evaluation map $e\colon G(A) \to G = G(\mathbb{C})$ that sends the matrix (g_{ij}) to $(g_{ij}(0))$. An *affine Schubert variety* is a $\mathcal{B}$-orbit closure in X, they are parameterized by certain elements of the affine Weyl group; we write $X(w)$ for the Schubert variety $\overline{\mathcal{B} \cdot w\mathcal{P}} \subset X$. These varieties have been studied by many authors from various viewpoints. We focus on the following question:

> Determine the singular locus of $X(w)$ (which itself is a union of Schubert varieties).

Let $\mathcal{T}$ be the torus of rank n obtained as $T \times S$ where T is the diagonal torus in SL_n and $S = \mathbb{C}^*$ acts by "rotating the loops." Every Schubert variety is $\mathcal{T}$-stable. In the following we will use the concrete ind-variety structure as outlined in Kumar [4] and Magyar [7]: the points of X may be identified with certain A-lattices in F^n which in turn admit an identification with a subset of the infinite Grassmannian $\mathrm{Gr}(\infty)$. The upshot is that X is the direct limit of varieties of the form $G(d,N)^u$ where u is a certain unipotent element in $\mathrm{GL}_N(\mathbb{C})$ and $G(d,N)$ is the Grassmannian of d-planes in N-space. $\mathcal{B}$ acts on $G(d,N)$ linearly and commutes with u, and every Schubert variety in X has the form $X(w) = Y^u$ where Y is a Schubert variety in $G(d,N)$ for the lower triangular Borel subgroup of GL_N (here d, N, and Y of course depend on w). The $\mathcal{B}$-orbits in $G(d,N)^u$ then are parameterized by sequences $1 \leq i_1 < i_2 < \cdots < i_d \leq N$ such that the following holds: if an integer i appears in such a sequence then either $i+n$ also appears, or $i+n > N$. (The sequence $x = (x_1, x_2, \ldots, x_d)$ corresponds to the $\mathcal{T}$-stable subspace $e_x := \langle e_{i_1}, e_{i_2}, \ldots, e_{i_d} \rangle$ spanned by those coordinate lines appearing in the sequence. The associated $\mathcal{B}$-orbit is then $\mathcal{B}e_x$.) Let I^u be the set of all sequences with this property; by abuse of notation we denote by $X(w) = \overline{\mathcal{B}e_w}$ the Schubert variety associated to $w \in I^u$ (no confusion should arise with the earlier convention). For $w \in I^u$, let $L(w) = (l_1, l_2, \ldots, l_{2n})$ where $l_i = |\{x_j \mid x_j \equiv i \mod n\}|$ if $i \leq n$, and $l_i = l_{i-n} + 1$ if $i < n$. w is uniquely determined by $L(w)$.

We construct two types of "patterns" in $L(w)$ that give rise to singularities similar to the classical setting of the full flag variety in type A (cf. [6]). As an example, one of these patterns is defined as follows: there exist $1 \leq i < g < j < k \leq 2n$ such that $l_i \geq l_j > l_g \geq l_k$ (moreover $i \leq n$, $j < i + n$ and $k < g + n$). In addition there are two degenerate forms of these patterns, and finally what we call an "imaginary" pattern. This last pattern gives rise to singularities as it implies the existence of tangent vectors of purely imaginary $\mathcal{T}$-weight (ie. a weight that is zero on T), that cannot be tangent to any $\mathcal{T}$-stable curve. The other patterns yield singularities by forcing a specific point in $X(w)$ (depending on the pattern) to have too many $\mathcal{T}$-stable curves through it (see [5] for details).

Recently, Billey and Mitchell completely classified all the (globally) smooth and rationally smooth Schubert varieties in the affine Grassmannians of all types (see [1]). They showed that for each given type, there are only finitely many smooth Schubert varieties. Even in type A however, the list of rationally smooth Schubert varieties is strictly bigger than the list of smooth ones (it is in fact infinite).

Recall that in the classical setting, whenever a Schubert variety in the simply laced types ADE is given, a point is smooth if and only if it is rationally smooth (this is Peterson's ADE-Theorem, cf. [3]). Moreover, by results of Carrell-Peterson, a point x in $X(w)$ is rationally smooth, if and only if the number of curves stable under the maximal torus through the given point x, equals the dimension and the same holds at all points which ly above in the Bruhat-Chevalley order ([2]). They also show that these curves correspond bijectively to the collection of all reflections $s_{\hat{\alpha}}$ in the affine Weyl group for which $x \neq s_{\hat{\alpha}} x \leq w$ (here we write $\hat{\alpha} = \alpha + h\delta$ for a root (of the affine system) with α an ordinary root and δ the generator of the character group $X(S)$).

Together with a student of mine (Valerie Cheng) we seem to be able to show that $x \leq w$ is a smooth point, if and only if x is rationally smooth, and if for all points y such that $x \leq y \leq w$, and all $s_{\hat{\alpha}}$ for which $y < s_{\hat{\alpha}} y \leq w$ we have $h = 0$, or $h = 1$ and $\alpha > 0$. The main point here is that this is a completely combinatorial criterion involving only the (decorated) Bruhat graph.

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Equivariant K-theory of non-commutative quantum spaces

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Summary. This is a report of joint work with R. Zhang. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathrm{U}_q(\mathfrak{g})$ be the corresponding quantised enveloping algebra (quantum group), as defined by Drinfeld. Regarding a (noncommutative) space with $\mathrm{U}_q(\mathfrak{g})$-symmetry as a $\mathrm{U}_q(\mathfrak{g})$-module algebra A, we may think of equivariant vector bundles on A as projective A-modules with compatible $\mathrm{U}_q(\mathfrak{g})$-action. We construct an equivariant K-theory of such quantum vector bundles using Quillen's exact categories, and provide means for its computation. The equivariant K-groups of quantum homogeneous spaces and quantum symmetric algebras of classical type are computed. This work was motivated in part by the 1987 work of Bass and Haboush [1] on the G-equivariant K-theory of algebraic G-varieties, where G is a reductive linear algebraic group, of which our work is a quantum generalisation.

Notation. We regard $\mathrm{U}_q(\mathfrak{g})$ as a $\mathbf{k}$-Hopf algebra, where $\mathbf{k}$ is the function field $\mathbb{C}(q^{\frac{1}{2}})$, and write the comultiplication as $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. The permutation map P takes $a \otimes b \in A \otimes B$ to $b \otimes a$, so that $\Delta' := P\Delta$ is the opposite comultiplication. The Drinfeld R-matrix [4] will be regarded as an element R in $\mathrm{U}_q\widetilde{(\mathfrak{g})\otimes}\mathrm{U}_q(\mathfrak{g})$, a completion of $\mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})$. We shall use U and $\mathrm{U}_q(\mathfrak{g})$ interchangably.

The category $\mathrm{U}_q(\mathfrak{g})$-mod of $\mathrm{U}_q(\mathfrak{g})$ modules which we shall consider is the category of locally finite $\mathrm{U}_q(\mathfrak{g})$-modules of type $(1, 1, \ldots, 1)$. Note that if M, N are $\mathrm{U}_q(\mathfrak{g})$-modules, then $\check{R} := P \circ R$ is a $\mathrm{U}_q(\mathfrak{g})$-homorphism : $M \otimes N \to N \otimes M$. A $\mathrm{U}_q(\mathfrak{g})$ *module algebra* (cf. [3, 9]) is a $\mathbf{k}$-algebra, which is also a $\mathrm{U}_q(\mathfrak{g})$-module, and satisfies $x(ab) = \sum x_{(1)} a x_{(2)} b$ (for $a, b \in A$ and $x \in \mathrm{U}_q(\mathfrak{g})$). It is known that if A, B are $\mathrm{U}_q(\mathfrak{g})$ module algebras, then so is $A \otimes B$, provided multiplication is defined by $a \otimes b \cdot a' \otimes b' = \sum a a'_{(1)} \otimes b_{(2)} b'$, where $\check{R}(b \otimes a') = \sum a'_{(1)} \otimes b_{(2)}$. Examples of module algebras are $T(V) = \oplus_{r \geq 0} V^{\otimes r}$, where V is a $\mathrm{U}_q(\mathfrak{g})$-module. See [2, 5] for other examples.

In our situation, the notion of '$\mathrm{U}_q(\mathfrak{g})$-equivariant vector bundle' will be interpreted as a $\mathrm{U}_q(\mathfrak{g})$-equivariant A-module M, for appropriate A modules. Such an M, which we refer to as an $A - \mathrm{U}$-module, is a $\mathbf{k}$-module with A-action α and $\mathrm{U}_q(\mathfrak{g})$-action μ such that the two obvious maps $\mathrm{U}_q(\mathfrak{g}) \otimes A \otimes M \to M$ coincide.

Some module categories. Denote the category of $A-\mathrm{U}$-modules by $A-\mathrm{U}$-mod, its full subcategory of those objects which are finitely generated as A-modules by $\mathcal{M}(A, \mathrm{U})$, and by $\mathcal{P}(A, \mathrm{U})$ the full subcategory of projectives in $\mathcal{M}(A, \mathrm{U})$. Recall that a left Noetherian algebra A is (left) regular if every finitely generated left A-module has a finite resolution by finitely generated projective A-modules.

Theorem 1. *If A is a left regular algebra which has the structure of a $\mathrm{U}_q(\mathfrak{g})$-module algebra, then every object in $\mathcal{M}(A, \mathrm{U})$ admits a finite $\mathcal{P}(A, \mathrm{U})$-resolution.*

Quillen's theory and $\mathrm{U}_q(\mathfrak{g})$-equivariant vector bundles. In [7], Quillen explains how to associate with certain categories, a type of homotopy theory, called higher algebraic K-theory. An *exact category* is an additive category $\mathcal{M}$ with a class E of short exact sequences which satisfy certain axioms. Any abelian category (with E taken to be all exact sequences) is exact, as are many subcategories of abelian categories. An important example for us is the full subcategory of finitely generated projective left R-modules of the category of R-modules, where R is any ring.

To any exact category $\mathcal{M}$, we associate its Quillen category $Q\mathcal{M}$, whose objects are those of $\mathcal{M}$ but whose morphisms $(M \to M')$ are diagrams of the form

$$M' \overset{j}{\leftarrow} N \overset{i}{\rightarrow} M,$$

where i, j are respectively injective and surjective maps in E. One then forms the classifying space $\mathcal{B}Q\mathcal{M}$ (assuming $\mathcal{M}$ is small), which is a CW-complex, and then the K-groups $K_i(\mathcal{M})$ are defined by $K_i(\mathcal{M}) = \pi_{i+1}(\mathcal{B}Q\mathcal{M})$.

Accordingly, if A is a $\mathrm{U}_q(\mathfrak{g})$-module algebra, thought of as the ring of functions on a non-commutative space, define its $\mathrm{U}_q(\mathfrak{g})$-equivariant K-groups as $K_i^{\mathrm{U}}(A) := K_i(\mathcal{P}(A, \mathrm{U}))$. It is known that $K_0^{\mathrm{U}}(A)$ is the Grothendieck group of $\mathcal{P}(A, \mathrm{U})$.

Theorem 2. *Let A be a left regular algebra with the structure of a $\mathrm{U}_q(\mathfrak{g})$-module algebra. Then there are isomorphisms $K_i^{\mathrm{U}}(A) \overset{\sim}{\rightarrow} K_i(\mathcal{M}(A, \mathrm{U}))$ for $i = 0, 1, 2, \ldots$*

The proof of the above theorem uses Quillen's resolution theorem for the embedding $\mathcal{P}(A, \mathrm{U}) \to \mathcal{M}(A, \mathrm{U})$.

This theorem may be effectively applied to filtered module algebras by dévissage. Suppose S is a $\mathrm{U}_q(\mathfrak{g})$-module algebra with a filtration $0 = F_{-1}S \subset F_0S \subset \ldots$ such that F_iS is $\mathrm{U}_q(\mathfrak{g})$-stable, $1 \in F_0S$ and $F_iSF_jS \subseteq F_{i+j}S$. Then $gr_iS := F_iS/F_{i+1}S$ and $grS := \oplus_{i \geq 0} gr_iS$ are $\mathrm{U}_q(\mathfrak{g})$-modules, and $A := F_0S$ and grS are $\mathrm{U}_q(\mathfrak{g})$-module algebras, the latter being graded.

Theorem 3. *Assume that grS is left noetherian and A-flat (meaning that $S \otimes_A --$ is exact). If $A(= F_0S)$ has a finite projective grS-resolution, then there exist isomorphisms : $K_i(\mathcal{M}(A, \mathrm{U})) \overset{\sim}{\rightarrow} K_i(\mathcal{M}(grS, \mathrm{U}))$. Further if A is regular, then S is regular, and we have isomorphisms $K_i^{\mathrm{U}}(A) \overset{\sim}{\rightarrow} K_i^{\mathrm{U}}(S)$.*

Application to quantum symmetric algebras. In practice many $\mathrm{U}_q(\mathfrak{g})$-module algebras arise as quadratic algebras, and therefore are amenable to Koszul resolution methods. Let V be a $\mathrm{U}_q(\mathfrak{g})$-module, and let I be a subspace of $V \otimes V$. Define $\mathbf{k}(V, I) := T(V)/(I)$, where (I) is the ideal of $T(V)$ generated by I. We say $\mathbf{k}(V, I)$ is a quantum symmetric algebra if it has a PBW basis, or equivalently if it has Poincaré series $(1 - t)^{-\dim V}$. For examples of such algebras see [5]. These may arise by considering the action of the R-matrix endomorphism $\check{R}$ on $V \otimes V$. The quantum analogues of the symmetric spaces of the natural representations of the quantum groups of classical type are incuded in this class of examples. We then have the following analogue of the theorem of Bass-Haboush.

Theorem 4. *Let $\mathbf{k}(V, I)$ be a quantum symmetric algebra, and assume that A is left Noetherian. Then*

(i) A is regular.

(ii) $K_i^{\mathrm{U}}(A) :\cong K_i(\mathcal{P}(A, \mathrm{U})) \xrightarrow{\sim} K_i(\mathrm{U}_q(\mathfrak{g}) - mod)$

The proof involves proving that $\mathbf{k}$ (the trivial $\mathrm{U}_q(\mathfrak{g})$-module) has a finite projective resolution by A-modules. Such a resolution is constructed from the Koszul complex. Our argument proves the following statement.

Theorem 5. *Let $A = \mathbf{k}(V, I)$ be a quantum symmetric algebra, with $A^!$ its Koszul dual. Then*

(i) A is Koszul, i.e. $\mathrm{Ext}^{\cdot}(\mathbf{k}, \mathbf{k}) \cong A^!$ *as graded algebra.*

(ii) The Koszul complex provides a projective A-resolution of $\mathbf{k}$ *of length* $\dim V$.

An immediate consequence is

Theorem 6. *Let A be a quantum symmetric algebra. Then $K_i^{\mathrm{U}}(A) \xrightarrow{\sim} K_i^{\mathrm{U}}(\mathbf{k}) \xrightarrow{\sim} K_i(\mathrm{U}_q(\mathfrak{g}) - mod)$.*

Quantum homogeneous spaces. We now apply the above results to quantum analogues of the homogeneous spaces G/K. Recall that the finite dual $\mathrm{U}_q(\mathfrak{g})^*$ of $\mathrm{U}_q(\mathfrak{g})$ is a Hopf algebra with multiplication defined by $fg(x) = \sum f(x_{(1)})g(x_{(2)})$ and comultiplication defined by $\Delta(f)(x_1 \otimes x_2) = f(x_1 x_2)$. Let $A_{\mathfrak{g}}$ be the subalgebra of $\mathrm{U}_q(\mathfrak{g})^*$ generated by all coefficient functions of all $\mathrm{U}_q(\mathfrak{g})$-modules. There are standard ways of defining left and right actions of $\mathrm{U}_q(\mathfrak{g})$ on $A_{\mathfrak{g}}$: $L_x f = \sum \langle f_{(1)}, S(x)\rangle f_{(2)}$, while $R_x f = \sum \langle f_{(2)}, x\rangle f_{(1)}$.

Let $\mathrm{U}_q(\mathfrak{l})$ be a 'Levi subalgebra' of $\mathrm{U}_q(\mathfrak{g})$, generated by a subset of the e_i, f_i, and all of the $k_i^{\pm 1}$. We then define $A := A_{\mathfrak{g}}^{L(\mathrm{U}_q(\mathfrak{l}))}$. This has a right $\mathrm{U}_q(\mathfrak{g})$ action, and is a $\mathrm{U}_q(\mathfrak{g})$ module algebra analogous to $\mathbb{C}[G/K]$ in the classical case. It is what we refer to as a quantum homgeneous space. The next result is the quantum analogue of a well known result in classical equivariant K-theory.

Theorem 7. *We have isomorphisms $K_i^{\mathrm{U}}(A) \xrightarrow{\sim} K_i(\mathrm{U}_q(\mathfrak{l}) - mod)$ for all i.*

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Bott-Samelson varieties, one-skeleton galleries and the path model

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(joint work with Stéphane Gaussent)

The aim of our work is twofold: we want to give a direct *geometric interpretation* of the path model for representations and the associated Weyl group combinatorics [8], and we want to get a *geometric compression* for Schwer's formula for Hall-Littlewood polynomials [10].

Concerning the connection with the path model, a first step in this direction was done in [1]. The advantage of the present approach (in comparison with [1]) is that galleries in the one-skeleton of the apartment can directly be identified with piecewise linear paths running along the one-skeleton in the standard apartment, and they can be concatenated. The goal now is to show that the original approach by Lakshmibai, Musili and Seshadri [4, 5] towards what later became the path model has an intrinsic geometric interpretation in the geometry of the affine Grassmannian, respectively in the geometry of the associated affine building.

To give a more precise description of both aims and of the results, let G be a semisimple algebraic group defined over $\mathbb{C}$, fix a Borel subgroup B and a maximal torus T. Let U^- be the unipotent radical of the opposite Borel subgroup. Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of complex formal power series and let $\mathcal{K} = \mathbb{C}((t))$ be the quotient field. For a dominant coweight λ and an arbitrary coweight μ consider the following intersection in the affine Grassmannian $G(\mathcal{K})/G(\mathcal{O})$:

$$Z_{\lambda,\mu} = G(\mathcal{O}).\lambda \cap U^-(\mathcal{K}).\mu.$$

Let $\mathbb{F}_q$ be the finite field with q elements and replace the field of complex numbers by the algebraic closure K of $\mathbb{F}_q$. Assume that all groups are defined and split over $\mathbb{F}_q$. Replace $\mathcal{K}$ by $\mathcal{K}_q = \mathbb{F}_q((t))$ and $\mathcal{O}$ by $\mathcal{O}_q = \mathbb{F}_q[[t]]$; the Laurent polynomials $L_{\lambda,\mu}$ defined by $L_{\lambda,\mu}(q) = |Z^q_{\lambda,\mu}|$ show up as coefficients in the Hall-Littlewood polynomial: $P_\lambda = \sum_{\mu\in X^\vee_+} q^{-\langle\rho,\lambda+\mu\rangle} L_{\lambda,\mu} m_\mu$ (see [3, 10]).

Based on the description of $Z_{\lambda,\mu}$ in [1], Schwer gives a decomposition $Z^q_{\lambda,\mu} = \bigcup S_\delta$, where the δ are certain galleries of alcoves in the standard apartment of the associated affine building. The structure of the S_δ is quite simple and hence $|S_\delta|$ is easy to compute, but the decomposition has the disadvantage that the sum $|Z^q_{\lambda,\mu}| = \sum |S_\delta|$ has many terms.

For G of type $\mathtt{A}_n$, there are other formulas for the coefficients of the Hall-Littlewood polynomials. For example, there is a combinatorial formula using semistandard Young tableaux in Macdonald's book [9], or one can specialize the Haglund-Haiman-Loehr formula for Macdonald polynomials. By analyzing the combinatorics involved in the Haglund-Haiman-Loehr formula, Lenart [7] has shown that in type $\mathtt{A}_n$ certain terms in Schwer's formula can be naturally grouped together such that the resulting formula coincides with the specialization of the Haglung-Haiman-Loehr formula, he calls this the compression phenomenon.

Our approach to "*compression*" is geometric and independent of the type of the group, but, of course, the aim still is to recover with these geometric methods in the type $\mathtt{A}_n$-case for example the formula in Macdonalds book.

We replace the desingularization of the Schubert variety X_λ in [1] by a Bott-Samelson type variety Σ which is a fibred space having as factors varieties of the form H/Q, where H is a semisimple algebraic group and Q is a maximal parabolic subgroup. In terms of the affine building, a point in this variety is a sequence of parahoric subgroups of $G(\mathcal{K})$ reciprocative contained in each other.

More precisely, in terms of the faces of the building, a point in Σ is a sequence of closed one-dimensional faces, where successive faces have (at least) a common zero-dimensional face (i.e. a vertex). So if the sequence is contained in an apartment, then the point in Σ corresponds to a piecewise linear path in the apartment joining the origin with a special vertex.

We introduce the notion of a minimal one-skeleton gallery (which always lies in some apartment) and of a positively folded combinatorial gallery in the one-skeleton. The points in Σ corresponding to the points in the open orbit $G(\mathcal{O}).\lambda \subset X_\lambda$ are exactly the minimal galleries. Since Σ is smooth, by choosing a generic one parameter subgroup of T in the anti-dominant Weyl chamber, we get a Białynicki-Birula decomposition, the centers δ of the cells C_δ correspond to combinatorial one-skeleton galleries δ (i.e. the galleries lying in the standard apartment). We show that $C_\delta \cap G(\mathcal{O}).\lambda \neq \emptyset$ if and only if δ is positively folded.

The Białynicki-Birula decomposition of Σ can be used to define a decomposition $Z_{\lambda,\mu} = \bigcup_\delta (Z_{\lambda,\mu} \cap C_\delta)$, the indexing set of the strata are positively folded one-skeleton galleries. To see the *geometric compression* compared to the decomposition in [1], consider the case for G of type $\mathtt{A}_n$. It is known that $Z_{\lambda,\mu}$ has at least $\dim V(\lambda)_\mu$ irreducible components. Now in the $\mathtt{A}_n$-case the galleries can be translated into the language of Young tableaux, and the positively folded galleries ending in μ correspond exactly to the semi-standard Young tableaux of shape λ and weight μ. In this sense the new decomposition can be viewed as the optimal geometric decomposition for type $\mathtt{A}_n$. The general feature of the new approach is that there are much less non-LS-galleries (see below) than in the old approach. For example in the case of type $\mathtt{A}_n$, all positively folded galleries are LS-galleries.

To investigate the intersection $Z_{\lambda,\mu} \cap C_\delta$ we need to *unfold* the (possibly) folded gallery δ. As a consequence of the unfolding procedure we present the formula for the coefficients of the Hall-Littlewood polynomials, the summands below counting the number of points in the intersection of $Z^q_{\lambda,\mu} \cap C_\delta$ for δ being positively folded:

Theorem 1. [2]

$$L_{\lambda,\mu}(q) = \sum_{\delta\in\Gamma^+(\gamma_\lambda,\mu)} q^{\ell(w_{D_0})} \left(\prod_{j=1}^{r} \sum_{\mathbf{c}\in\Gamma^+_{\mathfrak{s}^j_{V_j}}(\mathbf{i}_j,op)} q^{t(\mathbf{c})}(q-1)^{r(\mathbf{c})} \right).$$

To explain what this formula means let γ_λ denote the starting gallery joining the origin with the dominant weight λ and let $\Gamma^+(\gamma_\lambda,\mu)$ denote the set of all one-skeleton galleries of the same type that are positively folded. Each such gallery is a sequence of pairs of edges having a common vertex, say

$$(V_0 = 0 \subset E_0 \supset V_1 \subset E_1 \supset \ldots \subset E_r \supset V_{r+1}).$$

Each triple $(E_{i-1} \supset V_i \subset E_i)$ corresponds to one of the factors in the product in the formula. Presently we have a second sum within the product, we hope that in the near future we will be able to ameliorate this part. In fact, in the $\mathtt{A}_n$ case, this can be avoided and one gets the formula in Macdonalds book [9].

To explain now the second sum, for each triple $(E \supset V \subset F)$ of edges E, F having a common vertex V let $\mathfrak{s}$ be a sector containing E and let $w_{\mathfrak{s}} = w(C_V^-, \mathfrak{s}_V)$ be the element in the local Weyl group W_V^v at the vertex V that sends the residue class C_V^- of the anti-dominant Weyl chamber to $\mathfrak{s}_V$. Let D be the closest chamber to C_V^- containing F_V. Since $(E \supset V \subset F)$ is positively folded, $w_{-\mathfrak{s}_V} = w(C_V^-, -\mathfrak{s}_V) \leq w_D = w(C_V^-, D)$. Fix a reduced decomposition of $w_D = s_{i_1} \cdots s_{i_r}$ in W_V^v and denote its type by $\mathbf{i} = (i_1, ..., i_r)$. The set $\Gamma^+_{\mathfrak{s}_V}(\mathbf{i}, op)$ denotes the set of all galleries $\mathbf{c} = (C_V^-, C_1, ..., C_r)$ of residue chambers of type $\mathbf{i}$ which are positively folded with respect to $\mathfrak{s}_V$ and have the property that the face $F'_V \subset C_r$ of the same type as F_V forms a "minimal pair" (i.e. are contained in opposite sectors) with E_V in the local apartment $\mathbb{A}_V$.

The positively folded one-skeleton galleries having $q^{\langle \lambda+\mu, \rho \rangle}$ as a leading term in the counting formula for $|Z^q_{\lambda,\mu} \cap C_\delta|$, are called *LS-galleries*; this is an abbreviation for Lakshmibai-Seshadri galleries. These galleries play a special role and are connected with the indexing system by generalized Young tableaux introduced by Lakshmibai, Musili and Seshadri in a series of papers, see for example [4, 5, 6]. Recall that these papers were the background for the path model theory started in [8]. An important notion introduced in the theory of standard monomials is the defining chain [4, 5], which was a breakthrough on the way for the definition of standard monomials and generalized Young tableaux. In the context of the crystal structure of the path theory this notion again turned up to be an important combinatorial tool to check whether a concatenation of paths is in the Cartan component or not. Still, the definition had the air of an ad hoc combinatorial tool. But in the context of Białynicki-Birula cells, the folding of a minimal gallery by the action of the torus occurs naturally: during the limit process (going to the center of the cell) the direction (= the sector in the language of buildings) attached to a minimal gallery is transformed into the weakly decreasing sequence of Weyl group elements, the defining chain for the positively folded one-skeleton gallery in the center of the cell.

The connection between the path model theory and the one-skeleton galleries is summarized in the following corollary. For a fundamental coweight ω let π_{ω_i} : $[0,1] \to X^\vee_{\mathbb{R}}$, $t \mapsto t\omega$ be the path which is just the straight line joining $\mathfrak{o}$ with ω and let γ_ω be the one-skeleton gallery obtained as the sequence of edges and vertices lying on the path.

Corollary 2. *Write a dominant coweight $\lambda = \omega_{i_1} + \ldots + \omega_{i_r}$ as a sum of fundamental coweights, write $\underline{\lambda}$ for this ordered decomposition. Let $\mathcal{P}_{\underline{\lambda}}$ be the associated* *path model of LS-paths of shape* $\underline{\lambda}$ *defined in* [8] *having as starting path the concatenation* $\pi_{\omega_{i_1}} * \ldots * \pi_{\omega_{i_r}}$. *For a path* π *in the path model denote by* γ_π *the associated gallery in the one-skeleton of* $\mathbb{A}$ *obtained as the sequence of edges and vertices lying on the path. The one-skeleton galleries* γ_π *obtained in this way are precisely the LS-galleries of the same type as* $\gamma_{\omega_{i_1}} * \cdots * \gamma_{\omega_{i_r}}$.

In fact, the notion of a *defining chain for LS-paths* coincides in this case with the notion of a defining chain for the associated gallery.

Since the number of the LS-galleries is the coefficient of the leading term of $L_{\lambda,\mu}$, and since $P_\lambda \to s_\lambda$ for $q \to \infty$, we get as an immediate consequence of Theorem 1 the following character formula. This provides a geometric proof of the path character formula, first conjectured by Lakshmibai (see for example [6]) and proved in [8]:

Corollary 3. $\mathrm{Char}\, V(\lambda) = \sum_\delta e^{target(\delta)}$, *where the sum runs over all LS-galleries of the same type as* γ_λ.

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On the quantum *K*-theoretic product for some homogeneous spaces

Nicolas Perrin

(joint work with A. Buch, P.-E. Chaput and L. Mihalcea)

The quantum K-theory is a generalisation of both K-theory and quantum cohomology. It has been introduced by A. Givental in [4] for rational homogeneous spaces G/P where G is a semisimple algebraic group over $\mathbb{C}$ and P is a parabolic subgroup. In a common work with A. Buch, P.-E. Chaput and L. Mihalcea, we concentrate on the case where P is a maximal parabolic subgroup or equivalently on the case where the Picard group $\mathrm{Pic}(G/P)$ is $\mathbb{Z}$.

The group structure $QK(G/P)$ is simply the tensor product of the K-theoretic group $K(G/P)$ of G/P by a power series ring in a quantum parameter q. However, one of the main differences with the quantum cohomology is that the K-theoretic quantum product is defined by an infinite sum. Indeed, there is a natural basis of the K-theory for G/P given by $([\mathcal{O}_w])_{w\in W^P}$ where $[\mathcal{O}_w]$ is the class of the structure sheaf of the Schubert variety $X(w)$ in G/P and W^P is an index set for Schubert varieties. The product is given by

$$[\mathcal{O}_u] * [\mathcal{O}_v] = \sum_{d\geq 0} \sum_{w\in W^P} N_{u,v}^w(d) q^d [\mathcal{O}_w]$$

where the $N_{u,v}^w(d)$ are defined using the K-theoretic ring of the moduli space of stable maps. Contrarily to the quantum cohomology case, the structure constants $N_{u,v}^w(d)$ do not, *a priori*, vanish for large degrees d. Therefore A. Givental had to define the product in the completed group $QK(G/P) = K(G/P) \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]$ rather than in the polynomial one $K(G/P) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The main question we adress here is the following:

Question 1. *Is the quantum K-theoretic product defined over the tensor product $K(G/P) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$? In other words, is the quantum K-theoretic product polynomial?*

In the paper [1], A. Buch and L. Mihalcea computed Pieri and Giambelli formulas for the quantum K-theoretic ring of Grassmannian varieties. In particular they proved that in that case the quantum K-theoretic product is polynomial. We generalise this last result.

Let us denote by $M_{d,3}(x,y,z)$ the locus, in the moduli space of stable maps of degree d with 3 marked points, of the maps sending the three marked points respectively on x, y and z. Our main result in the

Theorem 1. *Assume that for d large enough, and for (x,y,z) general in $(G/P)^3$, the variety $M_{d,3}(x,y,z)$ is rationaly connected, then the quantum K-theoretic product for G/P is polynomial.*

Using the results of [3], we obtain unconditional results for some homogeneous spaces.

Corollary 1. *Assume that G/P is homogeneous under a classical group, cominuscule or adjoint of type different from A or G_2, then the quantum K-theoretic product for G/P is polynomial.*

We call adjoint variety the projectivisation of the closed orbit of a semisimple group G in its adjoint representation, these varieties are also called minimal nilpotent orbits. Cominuscule homogeneous spaces are natural generalisations of

Grassmannian varieties. Here is a list of them.

Type	*Variety*	*Dimension*	*Index*
A_{n-1}	$\mathbb{G}(k,n)$	$k(n-k)$	n
B_n	$\mathbb{Q}^{2n-1}$	$2n-1$	$2n-1$
C_n	$\mathbb{G}_\omega(n,2n)$	$\frac{n(n+1)}{2}$	$n+1$
D_n	$\mathbb{Q}^{2n-2}$	$2n-2$	$2n-2$
D_n	$\mathbb{G}_Q(n,2n)$	$\frac{n(n-1)}{2}$	$2n-2$
E_6	$\mathbb{OP}^2$	16	12
E_7	E_7/P_7	27	18

Cominuscule varieties.

In the above list, we denoted by $\mathbb{G}(k,n)$ (resp. $G_Q(n,2n)$, resp. $\mathbb{G}_\omega(n,2n)$) the Grassmann variety of k-dimensional subspaces in a n-dimensional vector space (resp. a connected component of the Grassmann variety of isotropic n-dimensional subspaces in a $2n$-dimensional vector space with a non degenerate symmetric form, resp. the Grassmann variety of isotropic n-dimensional subspaces in a $2n$-dimensional vector space with a non degenerate symplectic form). We denoted by $\mathbb{Q}^m$ the m-dimensional smooth quadric by $\mathbb{OP}^2 = E_6/P_1$ the Cayley plane and by E_7/P_7 the Freudenthal variety.

We also have explicit lower bounds on d for the vanishing of $N^w_{u,v}(d)$. Let $d_{\rm rc}$ be an integer such that for all $d \geq d_{\rm rc}$, $M_{d,3}(x,y,z)$ is rationally connected, and let $d_{\rm cl}$ be the minimal degree of a chain of lines connecting any two points in G/P. We prove

Theorem 2. *For $d \geq d_{\rm rc} + d_{\rm cl}$, we have $N^w_{u,v}(d) = 0$.*

For cominuscule homogeneous varieties we are even able to prove a better (and even sharp) bound for the vanishing of $N^w_{u,v}(d)$. It is the same bound as in the quantum cohomology. Denote by $d_{\max}$ the smallest interger d such that any two points of G/P are connected by a degree d rational curve

Theorem 3. *For $d > d_{\max}$, we have $N^w_{u,v}(d) = 0$.*

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Versal actions with a twist

Zinovy Reichstein

We will work over a base field k of characteristic 0.

Recall that an action of a linear algebraic group G on an algebraic k-variety X is called *generically free* if X has a dense G-invariant open subset U such that the stabilizer $\mathrm{Stab}_G(x) = \{1\}$ for every $x \in U(\overline{k})$. Here $\overline{k}$ denotes an algebraic closure of k. Equivalently, X has a G-invariant dense open subset X_0 (possibly smaller than U), which is the total space of a G-torsor

$$\pi \colon X_0 \to Y \, .$$

Suppose K/k is a finitely generated field extension. Then elements of $H^1(K, G)$ can be interpreted as birational isomorphism classes of generically free G-varieties (i.e., k-varieties with G-action) equipped with a k-isomorphism of fields $k(X)^G \simeq K$. An element of $H^1(K, G)$ corresponding to X is represented by the G-torsor π, as above, restricted to the generic point $\mathrm{Spec}(K) \to Y$ of Y.

Elements of $H^1(K, G)$ often have an alternative interpretation as algebraic objects of a certain type, defined over K. The type depends on G. For example, if $G = O_n$, these objects are non-degenerate n-dimensional quadratic forms and if $G = \mathbf{PGL}_n$ then these objects are central simple algebras. This makes it possible, at least in principle, to study such objects by geometric means, i.e., by studying actions of G on algebraic varieties, up to birational isomorphism.

An important notion in this context is that of a versal action. We say that a generically free G-variety X is *versal* if for every generically free G-variety Z and every dense open G-invariant subset $U \subset X$, there exists an equivariant map $Z \to U$. Informally speaking, this means that every G-torsor over a field extension of k can be obtained from $\pi \colon X_0 \to Y$ (as above) by pull-back; cf. [4, Section 5].

Versal G-varieties X carry a great deal of information about the algebraic group G. One can often prove assertions of the form "all generically free G-actions have property $\mathcal{P}$" by checking that a single versal G-action has property $\mathcal{P}$. For this reason special cases and variants of this notion have played an important role in Galois theory under the name of "generic field extensions" (here G is a finite group), in the theory of central simple algebras, under the name of "universal division algebras" (here $G = \mathrm{PGL}_n$), and in the theory of quadratic forms (here $G = O_n$).

Versal G-varieties are not unique, and versal varieties with special properties are often of particular interest. For example, the existence of a G-variety X such that the field $k(X)^G$ of G-invariant rational functions on X is related to the Noether problem for the group G, and the existence of low-dimensional versal varieties is related to the problem of computing the *essential dimension* of G; see [5].

We are interested in constructing (or more precisely, recognizing) versal actions. Specifically, given a generically free G-variety, how do we tell whether or not it is versal?

I will now state some partial answers to this question obtained jointly with Alex Duncan. The starting point is the following simple lemma. Part (a) was suggested by the anonymous referee of [1].

Lemma 1. *(a) (cf.* [1, Lemma 3.4]*) Consider a generically free action of an algebraic group G on an al algebraic variety X defined over k. If L-points are dense in ${}^TX$ for every field extension L/k and every G-torsor $T \to \mathrm{Spec}(L)$ then the G-action on X is versal.*

(b) The converse is true if X is a (pseudo)-homogeneous space for some linear algebraic group Γ containing G.

Here ${}^TX$ denotes "twist of X by T", which is defined as the geometric quotient of $X \times T$ by the natural (diagonal) action of G. For details of this construction, see, e.g., [3, Section 2] or [1, Section 2]. Note that ${}^TX$ is an algebraic variety defined over the field L. It is, in fact, an L-form of X, i.e., it is isomorphic to X over the algebraic closure of L. We also remark that there is no natural G-action on X; we lose the G-action in the process of constructing ${}^TX$. However, ${}^TX$ carries a natural action of the twisted group ${}^TG$.

Example 1. *(*[1, Proposition 3.3]*) Suppose G acts on a linear algebraic group Γ by group automorphisms. If this action is generically free then it is versal.*

Indeed, as we mentioned above, in this case the twisted variety ${}^TG$ is a linear algebraic group over L. Hence, by a theorem of Chevalley, L-points are dense in ${}^TG$. □

Theorem 1. *Let Γ be a linear algebraic group, G, H be closed subgroups, and $X = \Gamma/H$. Suppose the natural G-action on X is generically free. Then this action is versal if and only if the image of the natural map $H^1(L, G) \to H^1(L, \Gamma)$ lies in the image of the map $H^1(L, H) \to H^1(L, \Gamma)$ for every field extension L/k.*

Example 2. *(*[5, Proposition 7.1], [4, Ex. I.5.4]*) Every generically free linear action of G on a vector space V is versal. Here we think of V as a homogeneous space for the group $\Gamma = \mathbf{GL}(V)$, and $H^1(L, \Gamma)$ is trivial for every L by Hilbert's Theorem 90.*

Example 3. *$H = \{1\}$. The translation action of a subgroup G on Γ is versal if and only if the map $H^1(L, G) \to H^1(L, \Gamma)$ is trivial. The same is true for the translation G-action on Γ/H if H is a special group. (This means that $H^1(L, H)$ is trivial for every L/k.)*

Example 4. *(*[2, Corollary 3.3]*) Let G be a finite subgroup of $\mathbf{PGL}_n$. Then the natural action of G on $\mathbb{P}^{n-1}$ is versal if and only if G lifts to $\mathbf{GL}_n$.*

Example 5. *$H = G$. If the translation action of G on Γ/G is generically free then it is versal.*

Example 6. *$H = N$ = normalizer of a maximal torus in Γ. If the translation action of G on Γ/N is generically free then it is versal.*

This follows from a well-known fact (originally due to Springer) that the map $H^1(L, N) \to H^1(L, G)$ is surjective for every L.

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The Horn conjecture and related questions

Nicolas Ressayre

The basic question we consider is:

What can be said about the eigenvalues of a sum of two Hermitian matrices, in terms of the eigenvalues of the summands?

If A is a Hermitian n by n matrix, we will denote by $\lambda(A) = (\lambda_1 \geq \cdots \geq \lambda_n)$ its spectrum. Consider the following set:

$$\mathrm{Horn}(n) = \{(\lambda(A), \lambda(B), \lambda(C)) \ : \ \begin{array}{l} A, B, C \text{ are 3 Hermitian matrices} \\ \text{s.t. } A + B + C = 0\}. \end{array}$$

Let $\mathcal{P}(r, n)$ denote the set of parts of $\{1, \cdots, n\}$ with r elements. Let $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r, n)$. We set: $\lambda_I = (i_r - r, i_{r-1} - (r-1), \cdots, i_2 - 2, i_1 - 1)$. We will denote by 1^r the vector $(1, \cdots, 1)$ in $\mathbb{R}^r$. In 1962, Horn conjectured the following inductive description of $\mathrm{Horn}(n)$:

Conjecture 1. *Let (λ, μ, ν) be a triple of nonincreasing sequences of n real numbers. Then, $(\lambda, \mu, \nu) \in \mathrm{Horn}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \tag{1}$$

and for any $r = 1, \cdots, n-1$, for any $(I, J, K) \in \mathcal{P}(r, n)^3$ such that

$$(\lambda_I, \lambda_J, \lambda_K - (n-r)1^r) \in \mathrm{Horn}(r), \tag{2}$$

we have:

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0. \tag{3}$$

A consequence of the Horn conjecture is that $\mathrm{Horn}(n)$ is **convex**. This fact and another Horn's result (namely the Schur-Horn Theorem) motivated several important works on convexity in Hamiltonian geometry. B. Kostant in 1970 interprets the Schur-Horn Theorem as a special case of a more general theorem for compact Lie groups. In 1982, M. Atiyah, and independently V. Guillemin and S. Sternberg proved a wider generalization to Hamiltonian action of compact tori.

Finally, F. Kirwan in 1984 obtained a generalization to an Hamiltonian action of any compact Lie group. Kirwan's theorem allows to prove that Horn(n) is convex.

For I in $\mathcal{P}(r,n)$, we denote by σ_I the corresponding Schubert class in the cohomology of the Grassmanian of r-dimensional subspaces of $\mathbb{C}^n$. We have explained how to use the Rayleigh to prove that if $\sigma_I.\sigma_J.\sigma_K \in \mathbb{N}^*[pt]$ then inequality (3) holds. Then, we state the much harder

Theorem 1 (Klyachko, 1998)**.** *Conjecture 1 holds after replacing condition* (2) *by* $\sigma_I.\sigma_J.\sigma_K \in \mathbb{N}^*[\mathrm{pt}]$.

We are going to explain one of the ingredients used by Klyachko. The irreducible representations V_λ of GL_n correspond bijectively with the set Λ_n^+ the nonincreasing sequences λ of n integers (using the notion of dominant weight). In this context, the basic question is

Given two irreducible representations λ and μ of GL_n, what are the irreducible subrepresentations of $V_\lambda \otimes V_\mu$?

We define the Littlwood-Richardson coefficients by the following

$$V_\lambda \otimes V_\mu = \sum_\nu c_{\lambda\mu}^\nu V_\nu. \tag{4}$$

We set

$$\mathrm{LR}(\mathrm{GL}_n) = \{(\lambda,\mu,\nu) \in (\Lambda_n^+)^3 \ : \ (V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n} \neq \{0\}\}.$$

The relation with the Horn problem is

Theorem 2. *Let* (λ,μ,ν) *be a triple of nonincreasing sequences of* n *rational numbers. Then,* $(\lambda,\mu,\nu) \in \mathrm{Horn}(n)$ *if and only if* $(k\lambda, k\mu, k\nu) \in \mathrm{LR}(n)$ *for some positive integer* k.

The counterpart of the convexity of Horn(n) is the fact that $\mathrm{LR}(\mathrm{GL}_n)$ is stable by addition. Now, the existence of an invariant in $V_{k\lambda} \otimes V_{k\mu} \otimes V_{k\nu}$ for some positive integer k can be interpreted in terms of semistability in GIT. Then, inequalities (3) are interpreted as semistability conditions.

The inductive nature of Horn(n) is now explained by a classical Lesieur Theorem which asserts that if $\sigma_I.\sigma_J.\sigma_K \in c_{IJK}[\mathrm{pt}]$, for some integer c_{IJK} then

$$c_{IJK} = \dim\left((V_{\lambda_I} \otimes V_{\lambda_J} \otimes V_{\lambda_K - (n-r)1^r})^{\mathrm{GL}_{n-r}}\right). \tag{5}$$

By equation (5), Theorems 1 and 2, the Horn conjecture is a consequence of the following saturation conjecture

$$c_{k\lambda\ k\mu}^{k\nu} \neq 0 \Rightarrow c_{\lambda\ \mu}^{\nu} \neq 0. \tag{6}$$

In 1999, Knutson-Tao proved this conjecture ending the proof of the Horn conjecture.

It turns out that the Horn conjecture gives redundant inequalities. Indeed, Belkale proved that a list of inequalities extracted to Horn's one (or Theorem 1) is sufficient to characterize Horn(n).

Theorem 3 (Belkale, 1999)**.** *Inequalities* (3) *coming from I, J and K such that* $\sigma_I.\sigma_J.\sigma_K = [\mathrm{pt}]$ *give a complete list of inequalities.*

Moreover, we cannot improve the Belkale Theorem:

Theorem 4 (Knutson-Tao-Woodward, 2004)**.** *The list of inequalities determined by Belkale to be sufficient is in fact minimal.*

Let $G \subset \hat{G}$ be two connected reductive groups. Generalizing the above basic question about the tensor product decomposition of GL_n, we now consider the following one:

What irreducible representations of G appear in a given irreducible representation of $\hat{G}$?

The story of this question also start with a finitely generated semigroup namely $\mathrm{LR}(G, \hat{G})$ and the generated cone $\sqcup LR$. The first complete list of inequalities was determined by Berenstein-Sjamaar. For $\mathrm{LR}(G, G^s)$, Belkale-Kumar obtained in 2006 a smaller list and still sufficient of inequalities. In 2010, R. obtained the minimal list of inequalities for $\sqcup LR(G, \hat{G})$, proving in particular that the Belkale-Kumar's list is minimal.

Then, we have discussed some recent results of Kapovich-Millson, Belkale-Kumar about the saturation problem for $\mathrm{LR}(G, G^2)$.

Finally, we have discussed the condition " $\sigma_{w_1} \odot_0 \sigma_{w_2} \odot_0 \sigma_{w_3} = [\mathrm{pt}]$" which parametrizes the inequalities for $\sqcup LR(G, G^2)$.

Geometry of the Steinberg variety, affine Hecke algebras and modular representations of semisimple Lie algebras

Simon Riche
(joint work with Roman Bezrukavnikov)

Let G be a connected, simply-connected, semisimple algebraic group over an algebraically closed field $\Bbbk$ of positive characteristic. Let $B \subset G$ be a Borel subgroup of G, and let $\mathfrak{b} \subset \mathfrak{g}$ be their respective Lie algebras. Consider the flag variety $\mathcal{B} = G/B$, its cotangent bundle

$$\widetilde{\mathcal{N}} \ := \ T^*\mathcal{B} \ \cong \ \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X_{|g \cdot \mathfrak{b}} = 0\}$$

(called the Springer resolution) and the Grothendieck resolution

$$\widetilde{\mathfrak{g}} \ := \ \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X_{|g \cdot [\mathfrak{b}, \mathfrak{b}]} = 0\}.$$

Let $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ be the open subset of regular semisimple elements, and let $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ be its inverse image in $\widetilde{\mathfrak{g}}$. It is well-known that there exists an action of the Weyl group W of G on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$. For $w \in W$, we denote by Z_w the closure (in $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$) of the graph of the action of w. We denote by Z'_w the scheme-theoretic intersection $Z_w \cap (\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}})$.

Let also B_{aff} be the extended affine Weyl group attached to G. It has a natural set of generators $\{T_w,\, w \in W\} \cup \{\theta_x,\, x \in X^*(T)\}$ (see e.g. [7, Appendix]). Finally, let $S \subset W$ be the set of simple reflections. Our main result is the following.

Theorem 1. (i) *Assume $p \neq 2$ if G has a component of type $\mathbf{F}_4$, and $p \neq 3$ if G has a component of type $\mathbf{G}_2$. Then there exists a unique action of the group B_{aff} on the category $D^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, resp. $D^b\mathrm{Coh}(\widetilde{\mathcal{N}})$, such that*

- *T_s acts by convolution with $\mathcal{O}_{Z_s}$, resp. $\mathcal{O}_{Z'_s}$, for any $s \in S$;*
- *θ_x acts by tensor product with the line bundle $\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)$, resp. $\mathcal{O}_{\widetilde{\mathcal{N}}}(x)$, naturally attached to x.*

(ii) *Assume $p = 0$, or p is bigger than the Coxeter number of G. Then the action of T_w is given by the convolution with $\mathcal{O}_{Z_w}$, resp. $\mathcal{O}_{Z'_w}$.*

A proof of (i), under stronger assumptions on p, has appeared in [7]. A simpler proof will appear in [4]. This action can be considered as an extension, to the whole of $\widetilde{\mathfrak{g}}$, of the W-action on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$.

A proof of (ii), based on Representation Theory, will also appear in [4]. It would be desirable to find a geometric proof of this result. To find such a proof, it would certainly be necessary to understand the geometric properties of the varieties Z_w better. For example, it follows from our proof that, if $p = 0$ or if p is bigger than the Coxeter number of G, then Z_w is Cohen-Macaulay. In many examples, Z_w is also normal, but we could not find a general proof of this property.

Let us explain the representation-theoretic meaning of this result. First, assume for simplicity that $p = 0$. It is well-known (see [5, 6]) that the extended affine Hecke algebra $\mathcal{H}_{\mathrm{aff}}$ (i.e. the quotient of the group algebra of B_{aff} over $\mathbb{Z}[v, v^{-1}]$ by the quadratic relations) is isomorphic, as an algebra, to $K^{G \times \Bbbk^\times}(\widetilde{\mathcal{N}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}})$, the $G \times \Bbbk^\times$-equivariant K-theory of the Steinberg variety $\widetilde{\mathcal{N}} \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}$, endowed with the convolution product. The same holds for the variety $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$. Our result provides, for any element $b \in B_{\mathrm{aff}}$, an object $\mathcal{K}_b$ in $D^b\mathrm{Coh}^{G\times\Bbbk^\times}_{\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}}(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}})$, defined up to isomorphism. Here $\mathcal{K}_b$ is the kernel by which b acts. Passing to the Grothendieck group, one can check (see [7, §6]) that the assignment $b \mapsto [\mathcal{K}_b]$ induces the isomorphism constructed in [6]. In other words, statement (i) of our result "lifts" the geometric construction of the affine Hecke algebra to the level of (derived) categories. In this context, statement (ii) gives a geometric description, in terms of coherent sheaves, of the standard basis in $\mathcal{H}_{\mathrm{aff}}$ given by the products $T_w\theta_x$ ($w \in W$, $x \in X^*(T)$).

Now if p is bigger than the Coxeter number of G, our action is the geometric counterpart of the action contructed in [3] on some derived categories of representations of the Lie algebra $\mathfrak{g}$ over our field $\Bbbk$ of positive characteristic, under the equivalences of [2] (see [7, §5]). This interpretation is crucial for our proof of statement (ii). The geometric description of this action is used in [1] to characterize the t-structure on $D^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ arising from the equivalences of derived categories proved in [2], and to prove Lusztig's conjectures from [6].

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Serre's notion of G-complete reducibility: recent results

Gerhard Röhrle
(joint work with Michael Bate, Benjamin Martin and Rudolf Tange)

I gave a brief overview of some of the main results from [2], [4], and [5].

1. Serre's notion of complete reducibility

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$. Following Serre [11], we say that a (closed) subgroup H of G is *G-completely reducible* (G-cr) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup of P; for an overview of this concept see for instance [10] and [11]. In the case $G = \mathrm{GL}(V)$ (V a finite-dimensional k-vector space) a subgroup H is G-cr exactly when V is a semisimple H-module, and likeweise for other classical groups for $p \neq 2$, [11, Ex. 3.2.2(b)]. So this faithfully generalizes the notion of complete reducibility from representation theory. If H is a G-cr subgroup of G, then H^0 is reductive, [11, Prop. 4.1]. If $p = 0$, also the converse holds, [11, Prop. 4.2].

2. Richardson's Philosophy and relative G-cr subgroups

Let G act diagonally on G^n by simultaneous conjugation:

$$g \cdot (x_1, \ldots, x_n) = (gx_1g^{-1}, \ldots, gx_ng^{-1}).$$

For $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$ let $H = \overline{\langle x_1, \ldots, x_n \rangle}$, be the algebraic subgroup of G generated by (the terms of) $\mathbf{x}$. In [8], Richardson characterized the closed G-orbits in G^n by means of his notion of *strong reductivity.* In [1, Thm. 3.1] we showed that this is equivalent to Serre's concept of G-complete reducibility. As a consequence, we get the following geometric interpretation of the latter, [1, Cor. 3.7]:

Theorem 1. *Let H be a subgroup of G. Let $\mathbf{h} \in G^n$ be a generating tuple of H. Then H is G-cr if and only if the G-orbit $G \cdot \mathbf{h}$ of $\mathbf{h}$ in G^n is closed.*

In [5], we study the closed H-orbits in G^n for an arbitrary reductive subgroup H of G, generalizing Richardson's work [8]. This in turn leads to a generalization of Serre's concept of G-complete reducibility, as follows. Let $Y(G)$ denote the set of cocharacters of G, i.e., the set of homomorphisms $\lambda : k^* \to G$. For a subgroup H of G let $Y(H)$ denote the set of cocharacters of H. Clearly, we have $Y(H) \subseteq Y(G)$. Recall that for any pair (P, L) of a parabolic subgroup P of G and a Levi subgroup L of P there is a $\lambda \in Y(G)$ such that $P = P_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)\cdot g \text{ exists}\}$ and $L = L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)\cdot g = g\}$. Let $c_\lambda : P_\lambda \to L_\lambda$, by $g \mapsto \lim_{a\to 0} \lambda(a)\cdot g$, cf. [8, 2.3]. Let $K, H \leq G$ with H reductive. We say that K *is relatively G-completely reducible with respect to H* provided whenever $K \leq P_\lambda$ for $\lambda \in Y(H)$, there exists a $\mu \in Y(H)$ so that $P_\mu = P_\lambda$ and $K \leq L_\mu$. For $H = G$ we recover Serre's concept of G-complete reducibility; also, if K is a subgroup of H, then K is relatively G-cr with respect to H if and only if K is H-cr.

In [5, Thm. 1.1] we obtain the following generalization of Theorem 1:

Theorem 2. *Let H be a reductive subgroup of G. Let $\mathbf{k} \in G^n$ and let K be the algebraic subgroup of G generated by $\mathbf{k}$. Then $H \cdot \mathbf{k}$ is closed in G^n if and only if K is relatively G-cr with respect to H.*

Theorem 2 characterizes the closed H-orbits in G^n algebraically and for $H = G$ it recovers Theorem 1. Unlike in the case of G-cr subgroups, there is no known simple characterization of relative G-cr subgroups when $p = 0$, not even in the case $G = \mathrm{GL}(V)$. This notion is more sublte that it appears at first glance. In general, G-cr-ness does not imply, nor is it implied by relative G-cr-ness, [5, Rem. 3.2]. For a further discussion, see [5].

3. Uniform S-instability and complete reducibility

Theorem 1 allows to employ methods from GIT in the study of G-cr subgroups of G. However, there are several drawbacks in this construction. For instance, the associated destabilizing parabolic subgroup of a non-G-cr subgroup H of G depends on the choice of a generating tuple and not on H itself.

In [4], we strengthen the optimality results of Kempf–Rousseau [7, 9] by combining them with ideas of Hesselink [6]: Let V be an affine G-variety, S a non-empty closed G-invariant subset and X a non-empty subset of V. Extending [6], we say that X *is uniformly S-unstable* provided there exists $\lambda \in Y(G)$ such that $\lim_{a\to 0} \lambda(a) \cdot x$ exists and belongs to S for every $x \in X$. In [4, Thm. 4.2], we prove the analogue of Kempf's key instability theorem [7, Thm. 4.2] in this setting. In particular, there always exists an *optimal class* $\Omega(X, S)$ of cocharacters of G which uniformly destabilze X into S and this class gives rise to a unique *optimal destabilizing parabolic subgroup* $P(X, S)$ of G.

Let $H \leq G$ and let $\lambda \in Y(G)$ with $H \leq P_\lambda$. Let $M = c_\lambda(H)$. Suppose that H is not G-cr. Then H and M are not G-conjugate [4, Thm. 5.8]. Suppose that H^n admits a generating tuple of H. Setting $S = \overline{G \cdot M^n}$, we see that H is uniformly S-unstable in the sense above. Our next theorem ([4, Thm. 5.16]) is an application of our strengthening of Kempf's result ([4, Thm. 4.2]) to G-cr subgroups of G.

Theorem 3. *Let $H \leq G$ and $n \in \mathbb{N}$ such that H^n contains a generating tuple of H. Let $\lambda \in Y(G)$ with $H \subseteq P_\lambda$ and set $M = c_\lambda(H)$. Set $S = \overline{G \cdot M^n}$ and put $\Omega(H, M) := \Omega(H^n, S)$. Then the following hold:*

(i) *$P_\mu = P_\nu$ for all $\mu, \nu \in \Omega(H, M)$. Let $P(H, M)$ denote the unique parabolic subgroup of G so defined. Then $H \subseteq P(H, M)$ and $R_u(P(H, M))$ acts simply transitively on $\Omega(H, M)$.*

(ii) *We have $\Omega(gHg^{-1}, gMg^{-1}) = g \cdot \Omega(H, M)$ and $P(gHg^{-1}, gMg^{-1}) = gP(H, M)g^{-1}$ for any $g \in G$. In particular, $N_G(H) \leq P(H, M, k)$.*

(iii) *If $\mu \in \Omega(H, M)$, then $\dim C_G(c_\mu(H)) \geq \dim C_G(M)$. If M is G-conjugate to H, then $\Omega(H, M) = \{0\}$ and $P(H, M) = G$. If M is not G-conjugate to H, then H is not contained in any Levi subgroup of $P(H, M)$.*

Note, Theorem 3(ii) shows that $N_G(H) \leq P(H, M)$. Moreover, $P(H, M)$ only depends on H and not on the choice of a generating tuple for H. For further consequences and results concering this notion of uniform S-instability, see [4, §4].

4. Rationality Questions

In this section let k be any field, let $\overline{k}$ be its algebraic closure. Assume that G is defined over k. Following Serre again [11] we say that a subgroup H of G is *G-completely reducible over k* if whenever H is contained in a k-defined parabolic subgroup P of G, there exists a k-defined Levi subgroup of P containing H.

Our first aim here is to give a "geometric" characterization of this notion analogous to Theorem 1. For that we require the following definition, [4, Def. 3.8]. Let V be a k-defined affine G-variety. Let $v \in V$. We say that the $G(k)$-orbit $G(k) \cdot v$ is *cocharacter-closed over k* if for any k-defined cocharacter $\lambda \in Y(G)$ such that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists, v' is $G(k)$-conjugate to v. Note that we do not require v to be a k-point of V.

It follows from the Hilbert-Mumford Theorem that $G \cdot v$ is closed if and only if $G(\overline{k}) \cdot v$ is cocharacter-closed over $\overline{k}$. We thus consider the $G(k)$-orbits that are cocharacter-closed over k as a generalization to non-algebraically closed fields of the closed G-orbits in V. In [4, Thm. 5.9] we obtain the desired analogue to Theorem 1 in this setting:

Theorem 4. *Let H be a subgroup of G and let $\mathbf{h} \in H^n$ be a generating tuple of H. Then H is G-cr over k if and only if $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k.*

For further results on this notion of cocharacter-closed $G(k)$-orbits, see [4, §3]. In particular, note the subtlety that the cocharacter-closed $G(k)$-orbits in $V(k)$ need not coincide with the Zariski closed $G(k)$-orbits in $V(k)$, [4, Rem. 3.9]. Our next rationality result, conjectured by Serre, is proved in [2, Thm. 1.1].

Theorem 5. *Let k_1/k be a separable field extension, let G be defined over k, and $H \leq G$ be k-defined. Then H is G-cr over k if and only if H is G-cr over k_1.*

Remarks. (i). For $\mathrm{GL}(V)$, Theorem 5 is readily seen to hold, [4, Ex. 5.12].

(ii). Theorem 5 was proved in [1, Thm. 5.8] for k perfect, by passing back and forth between k and its algebraic closure $\overline{k}$ and between k_1 and $\overline{k}$. In general this approach fails, because the extension $\overline{k}/k$ need not be separable.

(iii). There are examples showing that each implication in Theorem 5 fails without the separability assumption on k_1/k; see [1, Ex. 5.11] and [3, Ex. 7.22].

(iv). The reverse implication in Theorem 5 is proved in [4, Thm. 5.11]. The proof of [4, Thm. 5.11] rests on a general rationality result, [4, Thm. 3.1], concerning G-orbits in an affine variety. The proof of the forward direction in [2] is based on the recently established Tits Centre Conjecture, see [2] for a statement and the relevant references of various parts of the proof.

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New age subregular representations of F_4

Dmitriy Rumynin

Let $\mathfrak{g}$ be a simple Lie algebra of type F_4 over an algebraically closed field of characteristic $p \geqslant 5$ (or even 12). Let $e \in \mathfrak{g}$ be a nilpotent element of type $\mathrm{F}_4(a_3)$ and $S_4 = C(e)/C_0(e)$. The reduced enveloping algebra $\mathcal{U}_e = \mathcal{U}_e(\mathfrak{g})$ splits into blocks

$$\mathcal{U}_e = \bigoplus_{[\lambda]\in(\Lambda/p\Lambda)/W} A_\lambda .$$

We discuss the problem of computing $\mathrm{Irr} A_\lambda$ as an S_4-set of centrally extended points.

Giambelli formulas for classical G/P spaces

Harry Tamvakis

The classical Giambelli formula writes any Schubert class in the cohomology ring of the Grassmannian X as a polynomial in the special Schubert classes, which are the Chern classes of the universal quotient bundle over X. We will describe an analogue of this result which holds in the cohomology ring of any classical G/P space. The answer involves some new algebraic expressions using raising operators which interpolate between Schur determinants and Pfaffians. Much of this work is in collaboration with Anders Buch and Andrew Kresch.

1. The type A Grassmannian

We begin with the usual Grassmannian $\mathrm{G}(m, N)$ which parametrizes linear subspaces Σ of $\mathbb{C}^N$ with $\dim(\Sigma) = m$. Let $n = N - m$. The cohomology ring $\mathrm{H}^*(\mathrm{G}(m, N), \mathbb{Z})$ is a free abelian group with basis given by the Schubert classes σ_λ, one for each partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ whose Young diagram fits inside an $m \times n$ rectangle. Now σ_λ is the class of a Schubert variety X_λ of complex codimension $|\lambda| = \sum_i \lambda_i$. When $\lambda = p$ is a positive integer, we get the special Schubert varieties $X_p = \{\Sigma \mid \Sigma \cap \mathbb{C}^{n+1-p} \neq 0\}$, where $\mathbb{C}^k = \mathbb{C}^k \times 0 \subset \mathbb{C}^N$. The special Schubert classes $\sigma_p = [X_p]$ for $1 \leq p \leq n$ generate the ring $\mathrm{H}^*(\mathrm{G}(m, N), \mathbb{Z})$; moreover, σ_p is equal to the Chern class $c_p(Q)$ of the universal rank n quotient vector bundle $Q \to \mathrm{G}(m, N)$. The classical Giambelli formula is the statement that

$$\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \leq i, j \leq \ell} \tag{1}$$

in $\mathrm{H}^*(\mathrm{G}(m, N), \mathbb{Z})$. Here and in the rest of this note we understand that $\sigma_0 = 1$ and $\sigma_p = 0$ if $p < 0$.

2. The Giambelli problem for G/P

The Schubert calculus on $\mathrm{G}(m, N)$ has an analogue on any homogeneous space G/P, where G is a complex reductive Lie group and P a parabolic subgroup of G. We wish to generalize (1) to a corresponding formula which is true on any G/P space. The Bruhat decomposition of the Lie group G induces a natural decomposition of G/P into Schubert cells, which in turn gives rise to the $\mathbb{Z}$-basis of Schubert classes for the cohomology of G/P. However there appears to be no uniform way to define the notion of a *special* Schubert class in this generality.

When G is a *classical* Lie group, one can define special Schubert class generators for the cohomology ring $\mathrm{H}^*(G/P, \mathbb{Z})$ uniformly, as follows. In this situation, the variety G/P parametrizes partial flags of subspaces of a vector space, which in types B, C, and D are required to be isotropic with respect to an orthogonal or symplectic form. First, the special Schubert varieties on any Grassmannian are defined as the locus of (isotropic) linear subspaces which meet a given (isotropic or coisotropic) linear subspace nontrivially, following [2]. The special Schubert classes are the cohomology classes determined by these Schubert varieties. Finally, the special Schubert classes on a partial flag variety G/P are the pullbacks of special

Schubert classes on Grassmannians. This agrees with the conventions used in Lie type A. In most cases, these special classes are equal to the Chern classes of the universal quotient vector bundles over G/P, up to a factor of two.

3. Symplectic Grassmannians

We describe here our answer to the Giambelli problem in the case of the Grassmannians which are quotients of the symplectic group Sp_{2n}. Equip $\mathbb{C}^{2n}$ with a symplectic form and fix an integer k with $0 \leq k \leq n$. Let $\mathrm{IG}(n-k, 2n)$ denote the symplectic Grassmannian which parametrizes isotropic subspaces Σ of $\mathbb{C}^{2n}$ with $\dim(\Sigma) = n-k$. Say that a partition λ is k-strict if it has no repeated parts larger than k. The Schubert varieties X_λ in IG are indexed by the k-strict partitions λ whose diagrams fit inside an $(n-k) \times (n+k)$ rectangle. More precisely, we have

$$X_\lambda = \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap \mathbb{C}^{p_j(\lambda)}) \geq j,\ 1 \leq j \leq \ell(\lambda)\},$$

where

$$p_j(\lambda) = n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}.$$

The cohomology $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$ is a free abelian group on the basis of Schubert classes $\sigma_\lambda = [X_\lambda]$. The special Schubert classes satisfy $\sigma_p = [X_p] = c_p(Q)$, as in type A.

For any $i < j$ and integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite support, we define $R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots)$. A raising operator R is any finite monomial in these R_{ij}'s. We set $m_\alpha = \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots$ and $R\,m_\alpha = m_{R\alpha}$ for any raising operator R. For any k-strict partition λ, consider the formal expression

$$R^\lambda = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j > 2k + j - i} (1 + R_{ij})^{-1}$$

where the first product is over all pairs $i < j$ and second product is over pairs $i < j$ such that $\lambda_i + \lambda_j > 2k + j - i$.

Theorem 1 ([3]). *For any k-strict partition λ, we have $\sigma_\lambda = R^\lambda m_\lambda$ in the cohomology ring $\mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$.*

The Giambelli formula of Theorem 1 agrees with (1) when $\lambda_i \leq k$ for all i. Moreover, the Giambelli problem for the Lagrangian Grassmannian (case $k = 0$) was first solved by Pragacz [5], who expressed the answer using a Schur Pfaffian.

4. Type A partial flag varieties

Suppose now that $X = \mathrm{GL}_N/P$ parametrizes partial flags of subspaces

$$0 \subset E_1 \subset \cdots \subset E_r \subset \mathbb{C}^N$$

with $\dim(E_j) = d_j$ for $1 \leq j \leq r$. Let S^P denote the set of permutations $w \in S_N$ such that $w(i) < w(i+1)$ for each i not in $\{d_1, \ldots, d_r\}$. For every $w \in S^P$, we have a Schubert class $\sigma_w \in \mathrm{H}^{2\ell(w)}(X, \mathbb{Z})$, where $\ell(w)$ denotes the length of w. The classes σ_w for $w \in S^P$ form a $\mathbb{Z}$-basis of $\mathrm{H}^*(X, \mathbb{Z})$. If $Q_j = \mathbb{C}^N/E_{r+1-j}$, then the Chern classes $c_p(Q_j)$ are the special Schubert classes on X.

Given two sequences $c = \{c_i\}_{i\geq 0}$ and $d = \{d_i\}_{i\geq 0}$ with $c_0 = d_0 = 1$, let $\{h_i\}_{i\geq 0}$ be the sequence defined by the equation of formal power series

$$\sum_{r=0}^{\infty} h_r t^r = (1 - c_1 t + c_2 t^2 - \cdots)^{-1}(1 - d_1 t + d_2 t^2 - \cdots)$$

and set $s_\lambda(c-d) = \det(h_{\lambda_i+j-i})_{i,j}$. Given any two vector bundles E and F over the same base X, we let $s_\lambda(E-F) = s_\lambda(c(E) - c(F))$.

Theorem 2 ([4]). *For any permutation $w \in S^P$, we have*

$$\sigma_w = \sum_{\underline{\lambda}} c_{\underline{\lambda}}^w \, s_{\lambda^1}(Q_1) s_{\lambda^2}(Q_2 - Q_1) \cdots s_{\lambda^r}(Q_r - Q_{r-1}) \tag{2}$$

in $\mathrm{H}^*(X, \mathbb{Z})$, *where the sum is over all r-tuples $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of partitions, and $c_{\underline{\lambda}}^w$ is a nonnegative integer.*

We note that the coefficient $c_{\underline{\lambda}}^w$ in (2) is a 'quiver coefficient', and that there exist several combinatorial interpretations for these integers. Formula (2) was inspired by the work of Buch and Fulton [1] on degeneracy loci of type A quivers.

5. Type C partial flag varieties

In this section we let $X = \mathrm{Sp}_{2n}/P$ parametrize partial flags of subspaces

$$E_\bullet \; : \; 0 \subset E_1 \subset \cdots \subset E_r \subset \mathbb{C}^{2n}$$

with $\dim(E_j) = d_j$ for $1 \leq j \leq r$ and E_r isotropic with respect to the symplectic form on $\mathbb{C}^{2n}$. Let W^P denote the set of signed permutations w in the Weyl group W of type C_n whose descent positions are not included among the d_j. We have a Schubert class $\sigma_w \in \mathrm{H}^{2\ell(w)}(X, \mathbb{Z})$ for any $w \in W^P$, and the quotient bundles Q_j and special classes $c_p(Q_j)$ are defined as in §4. Finally, let $k = n - d_r$.

Theorem 3 ([6]). *For every element $w \in W^P$, we have*

$$\sigma_w = \sum_{\underline{\lambda}} e_{\underline{\lambda}}^w \, \Theta_{\lambda^1}(Q_1) s_{\lambda^2}(Q_2 - Q_1) \cdots s_{\lambda^r}(Q_r - Q_{r-1}) \tag{3}$$

in $\mathrm{H}^*(X, \mathbb{Z})$, *where the sum is over all r-tuples $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of partitions with λ^1 k-strict, $e_{\underline{\lambda}}^w$ is a nonnegative integer, and $\Theta_{\lambda^1}(Q_1) = R^{\lambda^1} c_{\lambda^1}(Q_1)$.*

Remarks. 1) The *theta polynomial* Θ_λ in (3) is the type C analogue of a Schur polynomial in type A. The coefficients $e_{\underline{\lambda}}^w$ have a combinatorial interpretation. Moreover, the mixed nature of the ingredients in (3) is compatible with the geometry. Indeed, the morphism which sends $E_\bullet$ to E_r realizes X as a fiber bundle over $\mathrm{IG}(n-k, 2n)$ with fiber equal to a type A partial flag variety.

2) There are more general versions of Theorems 2 and 3 which hold for the torus-equivariant cohomology of any classical G/P space; this includes the case when G is an orthogonal group. Furthermore, our Giambelli formulas for Grassmannians have extensions which are valid in the small quantum cohomology ring.

3) It would be interesting to have answers to the following natural questions: (i) Can the exceptional groups be included in this story? (ii) Does the Giambelli formula of Theorem 1 appear in the theory of group representations?

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Canonical bases and Affine Hecke algebras of classical types

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(joint work with Michela Varagnolo)

A new family of graded algebras, called KLR algebras, has been recently introduced in [9], [11]. These algebras yield a categorification of $\mathbf{f}$, the negative part of the quantized enveloping algebra of any type. In particular, one can obtain a new interpretation of the canonical bases, see [12]. In type A or $A^{(1)}$ the KLR algebras are Morita equivalent to the affine Hecke algebras and their cyclotomic quotients. Hence they give a new way to understand the categorification of the simple highest weight modules and the categorification of $\mathbf{f}$ via some Hecke algebras of type A or $A^{(1)}$. One of the advantages of KLR algebras is that they are graded, while the affine Hecke algebras are not. This explain why KLR algebras are better adapted than affine Hecke algebras to describe canonical bases. Indeed one could view KLR algebras as an intermediate object between the representation theory of affine Hecke algebras and its Kazhdan-Lusztig geometric counterpart in term of perverse sheaves. This is central in [12], where KLR algebras are proved to be isomorphic to the Ext-algebras of some complex of constructible sheaves.

In the other hand, the (branching rules for) affine Hecke algebras of type B have been investigated quite recently, see [4], [5], [6], [7], [8]. Lusztig's description of the canonical basis of $\mathbf{f}$ in type $A^{(1)}$ in [10] implies that this basis can be naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type A. This identification was used in [3]. More precisely, there is a linear isomorphism between $\mathbf{f}$ and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [3] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on $\mathbf{f}$. In [4], [5], [6], [7] a similar behavior is conjectured and studied for affine Hecke algebras of type B. Here $\mathbf{f}$ is replaced by an explicit module $V(\lambda)$ over an explicit algebra $\mathbf{B}$. First, it is conjectured that $V(\lambda)$ admits a canonical basis. Next, it is conjectured that this basis is naturally identified with the set of isomorphism classes of simple objects of a category of modules of the affine Hecke algebras of type B. Further, in this identification the branching rules of the affine Hecke algebras of type B are given by the $\mathbf{B}$-action on $V(\lambda)$. The first conjecture has been proved in [4] under the restrictive assumption that $\lambda = 0$. Here we prove the whole set of conjectures. Indeed, our construction is slightly more general, see the appendix.

Roughtly speaking our argument is as follows. In [4] a geometric description of the canonical basis of $V(0)$ was given. This description is similar to Lusztig's description of the canonical basis of $\mathbf{f}$ via perverse sheaves on the moduli stack of representations of some quiver. It is given in terms of perverse sheaves on the moduli stack of representations of a quiver with involution. First we give a analogue of this for $V(\lambda)$ for any λ. This yields the existence of a canonical basis $B(\lambda)$ for $V(\lambda)$ for arbitrary λ. Then we compute explicitely the Ext-algebras between complexes of constructible sheaves naturally attached to quivers with involutions. These complexes enter in a natural way in the definition of $B(\lambda)$. This computation yields a new family of graded algebras $\mathbf{R}_m$ where m is a nonnegative integer. We prove that the algebras $\mathbf{R}_m$ are Morita equivalent to the affine Hecke algebras of type B. Finally we describe $V(\lambda)$ and the basis $B(\lambda)$ in terms of the Grothendieck group of $\mathbf{R}_m$.

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