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Deformation Methods in Mathematics and Physics

Organised by
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Abstract. Deformations of mathematical structures play an important role in most parts of mathematics but also in theoretical physics. In this interdisciplinary workshop, different aspects of deformations and their applications were discussed. The workshop was attended by experts in the fields, but also by quite a number of young post-docs and PhD students. One of the goals was to foster interactions between different communities.

Mathematics Subject Classification (2000): 16xx, 17xx, 53xx, 70xx, 81xx, 83xx.

Introduction by the Organisers

The workshop *Deformation Methods in Mathematics and Physics*, was organised by Alice Fialowski(Budapest), Jürg Fröhlich(Zürich), and Martin Schlichenmaier (Luxembourg) and took place from September 25 to October 1st, 2010 at the Mathematisches Forschungsinstitut Oberwolfach (MFO).

Deformation theory plays an important role in many branches of mathematics and physics. In mathematics, deformation theoretical methods are crucial for constructing and for studying classifying spaces (moduli spaces). Furthermore, by deformation one obtains new interesting mathematical objects from known ones.

In physics, the mathematical theory of deformations is a powerful tool to construct new theories of physical reality from known ones. The concepts of *symmetry* and *deformations* are considered to be two fundamental guiding principle for further developing physical theory.

In 2006, there was a precursor workshop in Oberwolfach (2006/3) with the title "Deformations and Contractions in Mathematics and Physics", where both mathematicians and physicists participated. The Workshop was an enlightening experience for the participants and turned out to be very successful. Both groups - mathematicians and physicists - benefited from the week. For a more detailed description of the talks presented, see Oberwolfach Reports 2006/3.

Based on the success of the workshop, the organizers were invited by the Editor of the International Journal of Theoretical Physics to prepare a special volume (Vol. 46, No. 11, 2007) dedicated to the topics presented at the workshop .

To a certain extent the actual workshop took up the challenges and open problems of the 2006 workshop. But, equally important, it evolved into new directions. The infinite-dimensional case was more in the center of interest and deformations of higher order algebraic structures played a prominent role.

The following is a (non-exhaustive) list of topics discussed at the workshop.

(1) Formulations of formal deformations in the context of differential graded Lie algebras, Maurer Cartan elements, higher structures, (curved) A_∞ algebras, operads, graph complexes, in particular also the deformation of diagrams.
(2) Constructions of moduli spaces, versal families for a given deformation problem, in particular also the discussion of global versus formal deformations and the question of rigidity. There exist (infinite-dimensional) algebras which are formally rigid but admit nontrivial (non-formal) deformations (sometimes called parameters).
(3) The deformation quantization of symplectic and Poisson manifolds, in particular also the question to find subalgebras for which the deformation quantization converges, furthermore the behaviour of deformation quantizations under reduction by a group action, Drinfeld associators.
(4) Deformed Geometry and Gravity, with the help of fuzzy space geometries, large N limits of Yang-Mills matrix models, Anti-de-Sitter space time.
(5) Quantum Field Theory, in particular the deformation of the local observable algebra, renormalisation and regularisation of QFT, family of Dirac operators.

The talks were supplemented by two talks of overview character on deformation quantization and on the deformation philosophy in physics.

The workshop was attended by 49 participants from all over the world. The official program consisted of 21 lectures.

On Thursday night a Young Researchers Session took place. Five advanced PhD students and post-docs gave short presentations on results obtained during their PhD research. This activity was well received by the speakers and by the audience.

Beside the official program, there was ample time for further activities of the participants, such as self-organised sessions and discussion groups.

Workshop: Deformation Methods in Mathematics and Physics

Table of Contents

Abstracts

The deformation philosophy, historical background, mathematical frameworks and physical applications

Daniel Sternheimer

1. Presentation

This opening talk[1] at the Workshop *Deformations in Mathematics and Physics* aims at giving the "flavor" of deformation quantization and of some aspects of noncommutative geometry, their background and perspectives. It also exemplifies that it is advisable to have an idea of *why* one develops some theories. Indeed an often ignored issue in the interaction between mathematics and physics is that, when they at all care about mathematical issues, physicists usually explain to mathematicians *what* they are doing, sometimes also *how* (at their level of rigor), but not *why* they are dealing with such questions in such a manner – maybe because they do not ask themselves the question, following what is done in the community, and/or because they are convinced that mathematicians should only be asked to provide a "toolbox". Mathematical physicists try to speak the mathematical language of physics with both accents and grammar.

The presentation summarizes an approach developed over many joint works (some, in progress) that would not have been possible without the deep insight on the role of deformations in physics of Moshe Flato, my friend and coworker for 35 years, a true mathematical physicist and physical mathematician.

Deformations in physics and mathematics are part of a deformation philosophy, promoted in mathematical physics in joint work with Moshe Flato since the 70's. The main conceptual advances in 20^{th} century physics, relativity and quantization, manifest it. In deformation quantization (including its realization on manifolds), quantization is understood as deformations of commutative algebra structures into non commutative algebra structures (which includes quantum groups). One may also think of objects dual to noncommutative algebras, the so-called quantum spaces, as deformations of classical spaces, the objects dual to commutative algebras (that is the essence of noncommutative geometry). Deforming Minkowski space-time leads to a fruitful object which together with its group of symmetries is referred as AdS or "anti de Sitter space". The study of AdS has significant physical consequences (e.g. composite massless particles, AdS/CFT correspondence). Combining all this leads to an ongoing program in which AdS would be quantized in some regions related to black holes, with possible implications in particle physics and cosmology. In particular we speculate that this could explain a universe in accelerated expansion and maybe baryogenesis. Hopefully this broad picture will inspire some junior attendants and readers of this Report.

2. Deformations: the physical philosophy, their ancient origins, modern and some possible future developments

Physical theories have domains of applicability defined by the relevant distances, velocities, energies, etc. involved. The passage from one domain (of distances, etc.) to another doesn't happen in an uncontrolled way: experimental phenomena appear that contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. A relatively easy to read exposition of that philosophy in the present context can be found in the recent [5].

In physics the first example is the discovery that the Earth is not flat. Much later Einstein's theory of relativity can be viewed as a deformation of Newtonian mechanics and its Galilean symmetry. In mathematics Riemann surface theory is probably the first example, followed in the late 50's by the Kodaira–Spencer theory of deformations of complex structures and in the early 60's by the Gerstenhaber theory of deformations of algebras, at the time when Flato came to Paris (from the Racah school in Jerusalem): we immediately realized that the above "physical" deformations are naturally cast in that formulation. From there we felt that the intuitive idea of quantization should also fall in that framework. But it took another decade before we could realize the mathematical developments showing that quantization *is* a deformation, and develop what is now called deformation quantization. We shall not enter into any of its many details nor go beyond what was said in the previous section about its manifold avatars, referring to the founding papers [1] and e.g. the comprehensive survey of the "state of the art" about 10 years ago in [2], recent reviews such as [5], and references quoted therein.

In the past decade there has been an extremely wide array of developments and applications of deformation quantization, both in mathematics and in physics (including algebraic geometry and string theory). A particularly active area of research deals with "singular spaces", while until the 90's mostly manifolds (real or complex) were considered: new phenomena appear, and new tools are needed.

The deformation of (1+3 dimensional) Minkowski space-time to AdS has, among its physical consequences, the fact that massless particles can be treated, in a way compatible with quantum electrodynamics for the photon, as composite of more elementary objects, the singletons (discovered by Dirac in 1963), massless particles in (1+2) dimensions (a manifestation of AdS/CFT). That was extended by Frønsdal in 2000 to treat leptons as composite of singletons, massified by interaction with some Higgs. See e.g. a review in [3]. Deforming further the AdS symmetry group to the quantum group $\mathrm{SO}_q(3,2)$ is then natural and (e.g. for q root of unity) brings challenging new phenomena such as finite-dimensional unitary irreducible representations. Combining all this brought us [4] to realize quantized anti de Sitter space (qAdS) black holes, building a Lorentzian version of Connes' spectral triples based on universal deformation quantization formulae obtained from an oscillatory integral kernel on an appropriate symplectic symmetric space.

Based on that study a cosmological Ansatz would then be that space-time is, in some small regions at the edge of our Universe, not only deformed (to AdS with tiny negative curvature ρ, which does not exclude at cosmological distances to have a positive curvature or cosmological constant, e.g. due to matter) but also "quantized" to some qAdS. These regions could be considered, in a sense to make more precise (e.g. with some measure or trace) as having "finite" (possibly "small") volume (for q even root of unity) and behave like black holes. At the "border" of these one would have, for most practical purposes at "our" scale, the Minkowski space-time, obtained by $q\rho \to 0$. From these, "q-singletons" could emerge, create massless particles that would be massified by interaction with dark matter or dark energy. That could (and should, otherwise there would be manifestations closer to us, that were not observed) occur mostly at or near the "edge" of our universe in accelerated expansion. These "qAdS black holes" ("inside" which one might find compactified extra dimensions) could be a kind of "shrapnel" resulting from the Big Bang (in addition to background radiation) and maybe provide a clue to baryogenesis. At this stage these are mere speculations, but the many mathematical and physical problems suggested by that application of our deformation philosophy are challenging, worth studying independently, and could lead to yet unexpected important developments.

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Time and dynamics in physics. The quest for well-defined mathematical models

Olav Arnfinn Laudal

If we want to study a natural phenomenon, called **P**, we would, in the present scientific situation, want to describe **P** in some mathematical terms, say as a mathematical *object*, X, depending upon some parameters, in such a way that the changing aspects of **P** would correspond to altered parameter-values for X.

X would be a *model for* **P** if, moreover, X with any choice of parameter-values, would correspond to some, possibly occurring, aspect of **P**.

Two mathematical objects X(1), and X(2), corresponding to the same aspect of **P**, would be called equivalent, and the set, **M**, of equivalence classes of these objects should be called the *moduli space* of the models, X. The study of the natural phenomenon **P**, would then be equivalent to the study of the *structure* of **M**. In particular, the notion of *time* would, in agreement with Aristotle and St. Augustin, see [4], be a *metric* on this space.

With this philosophy in mind I have embarked on the study of the *dynamics* of moduli spaces of representations of associative algebras, see [3]. For any associative k-algebra A, k a field, we have, in [4], and [5], defined a *phase space* $Ph(A)$, i.e. a universal pair of a morphism $\iota : A \to Ph(A)$, and an ι- derivation, $d : A \to Ph(A)$, such that for any morphism of algebras, $A \to R$, any derivation of A into R decomposes into d followed by an A- homomorphism $Ph(A) \to R$, see [4] and [5]. Iterating this construction we obtain a limit morphism $\iota^n : Ph^n(A) \to Ph^\infty(A)$ with image $Ph^{(n)}(A)$, and a universal derivation $\delta \in Der_k(Ph^\infty(A), Ph^\infty(A))$, the *Dirac*-derivation.

This Dirac derivation will, as we shall see, create the dynamics in our different geometries, on which we shall build our theory. A *dynamical structure*, defined for a *space*, or for any associative k-algebra A, is now an ideal $(\sigma) \subset Ph^\infty(A)$, stable under the Dirac derivation. The quotient algebra $\mathbf{A}(\sigma) := Ph^\infty(A)/(\sigma)$, together with the induced Dirac derivation, will be called a dynamical system.

Recall now that for any k-algebra A, and right A-modules V, W, there is an exact sequence, $Hom_k(V,W) \to Der_k(A, Hom_k(V,W) \to Ext^1_A(V,W) \to 0$, where the image of, $Hom_k(V,W) \to Der_k(A, Hom_k(V,W))$ is the sub-vectorspace of trivial (or inner) derivations. Recall also that $Ext^1_A(V,V)$ is the tangent space of the deformation functor of the A-module V.

The basic notions of non-commutative deformations of families of modules, and the resulting affine non-commutative algebraic geometry, have been treated in several texts, see in particular [1] and [2]. Given a finitely generated k-algebra A, k a field, there is a commutative algebra $C(n)$, and an open subvariety $U(n) \subseteq Spec(C(n))$ forming an étale covering of the set of isomorphism classes, $Simp_n(A)$, of simple n-dimensional representations. Moreover there exists a versal family, $\tilde{\rho} : A \longrightarrow End_{C(n)}(\tilde{V})$, inducing all isoclasses of simple n-dimensional A-modules.

Suppose, in line with our philosophy, that we have uncovered the moduli space of the mathematical models of our subject, and that A is the *affine k-algebra* of this space, assumed to contain all the parameters of our interest. Assume moreover that we have guessed a dynamical system $\mathbf{A}(\sigma)$, with Dirac derivation δ, and a metric defining of time. Notice first that any right $\mathbf{A}(\sigma)$-module V is also a $Ph^\infty(A)$-module, and therefore corresponds to a family of $Ph^n(A)$-module-structures on V, for $n \geq 1$, i.e. to V, considered as an A-module, together with a sequence $\{\xi_n\}$, of a tangent, or a *momentum*, ξ_0, an acceleration vector, ξ_1, and any number of higher order *momenta* ξ_n. Thus, specifying a point $v \in Simp_n(A(\sigma))$ implies specifying a formal curve through v_0, the base-point, of the miniversal deformation space of the A-module V.

Knowing the dynamical structure, (σ), and the state of our *object* V at a *time* τ_0, i.e. knowing the structure of our *representation* V of the algebra $\mathbf{A}(\sigma)$, at that time (which is a problem that we should return to), this makes it reasonable to believe that we, from this, may deduce the state of V at any *later* time τ_1. This assumption, on which all of science is based, is taken for granted in most textbooks in modern physics. The mystery is, of course, why Nature seems to be parsimonious, in the sense of Fermat and Maupertuis, giving us a chance of guessing dynamical structures.

Any family of components of $Simp(\mathbf{A}(\sigma))$, with its versal family $\tilde{V}$, will, in the sequel, be called a *family of particles*. A section ϕ of the bundle $\tilde{V}$, is now a function on the moduli space $Simp(A)$, not just a function on the *configuration space*, $Simp_1(A)$, nor on $Simp_1(\mathbf{A}(\sigma))$. The value $\phi(v) \in \tilde{V}(v)$ of ϕ, at some point $v \in Simp_n(A)$, will be called a *state* of the particle, at the *event* v.

$End_{C(n)}(\tilde{V})$ induces also a bundle, of *operators*, on the étale covering $U(n)$ of $Simp_n(\mathbf{A}(\sigma))$. A section, ψ of this bundle should be called a *quantum field*. In particular, any element $a \in \mathbf{A}(\sigma)$ will, via the versal family map, $\tilde{\rho}$, define a quantum field, and the set of quantum fields form a k-algebra.

Physicists will tend to be uncomfortable with this use of their language. A classical quantum field for any traditional physicist is, usually, a *function* ψ, defined on some *configuration space*, (which is not our $Simp_n(\mathbf{A}(\sigma))$, with values in the polynomial algebra generated by certain *creation* and *annihilation*-operators in a *Fock-space*. This interpretation may, however, be viewed as a special case of our general set-up.

Let $v \in Simp_n(\mathbf{A}(\sigma))$ correspond to the right $\mathbf{A}(\sigma)$-module V, with structure homomorphism $\rho_v : \mathbf{A}(\sigma) \to End_k(V)$, then the Dirac derivation δ composed with ρ_v, gives us an element, $\delta_v \in Der_k(\mathbf{A}(\sigma), End_k(V))$. Modulo the trivial (inner) derivations, δ_v defines a class, $\xi(v) \in Ext^1_{\mathbf{A}(\sigma)}(V,V)$, i.e. a tangent vector to $Simp_n(\mathbf{A}(\sigma))$ at v. The Dirac derivation δ therefore defines a unique one-dimensional distribution in $\Theta_{Simp_n(\mathbf{A}(\sigma))}$, which, once we have fixed a versal family, defines a vector field, $\xi \in \Theta_{Simp_n(\mathbf{A}(\sigma))}$, and in good cases, a (rational) derivation, $\xi \in Der_k(C(n))$, inducing a derivation, $[\delta] \in Der_k(\mathbf{A}(\sigma), End_{C(n)}(\tilde{V}))$, lifting ξ, and, in the sequel, identified with ξ. By definition of $[\delta]$, there is now a *Hamiltonian* operator $Q \in M_n(C(n))$, satisfying the fundamental equation $\delta = [\delta] + [Q, \tilde{\rho}(-)]$. This equation means that for an element (an observable) $a \in \mathbf{A}(\sigma)$ the element $\delta(a)$ acts on $\tilde{V} \simeq C(n)^n$ as $[\delta](a) = \xi(\tilde{\rho}_V(a))$ plus the Lie-bracket $[Q, \tilde{\rho}_V(a)]$.

The *dynamics of the system* is now given in terms of the Dirac vector-field $[\delta]$, generating the vector field ξ on $Simp_n(\mathbf{A}(\sigma))$. An integral curve γ of ξ is a *solution* of *the equations of motion*. Let γ start at $v_0 \in Simp_n(\mathbf{A}(\sigma))$ and end at $v_1 \in Simp_n(\mathbf{A}(\sigma))$, with length $\tau_1 - \tau_0$. This is only meaningful for ordered fields k, and when we have given a metric (time) on the moduli space $Simp_n(\mathbf{A}(\sigma))$. Assume this is the situation. Then, given a *state*, $\phi(v_0) \in \tilde{V}(v_0) \simeq V_0$, of a *particle*, there is a canonical evolution map, $U(\tau_0, \tau_1)$ transporting $\phi(v_0)$ from time τ_0, i.e. from the point representing V_0, to time τ_1, i.e. corresponding to some point representing V_1, along γ. It is given as, $U(\tau_0, \tau_1)(\phi(v_0)) = exp(\int_\gamma Q d\tau)(\phi(v_0))$, where $exp(\int_\gamma)$ is the non-commutative version of the classical action integral. There are analogies of the S-matrix, of perturbation theory, and so also of Feynman-integrals and diagrams. In particular, Planck's Constants and Fock space pop up in a natural way.

In [4], I sketched a physical *toy model*, of the physical systems composed of an observer and an observed, both sitting in Euclidean 3-space. The corresponding moduli space, $\mathbf{Hilb}^{(2)}(\mathbf{E}^3)$, is easily computed. Provided with a natural metric, i.e. with time, it was called the *time-space* of the model. A relative velocity in $\mathbf{E}^3$ is now seen to be an oriented line in the tangent space of a point of $\underline{\tilde{H}}$. Thus the space of velocities is compact. This lead to a *physics* where there are no infinite velocities, and where the principle of relativity comes for free. Moreover, he operators C, P, T of classical physics, and the three fundamental *gauge groups* of the standard model, $U(1)$, $SU(2)$ and $SU(3)$ are part of the structure of $\mathbf{Hilb}^{(2)}(\mathbf{E}^3) = \underline{\tilde{H}}/Z_2$, where $\underline{\tilde{H}}$ is the space of ordered pairs of points of $\mathbf{E}^3$, blown up along the diagonal $\underline{\Delta}$. Experimenting with natural metrics defined in $\underline{\tilde{H}}$, we see a promising possibility of defining notions like mass and charge, of different colors, related to this structure. A catchy way of expressing this would be: Every point in our *real world*, $\mathbf{E}^3 \simeq \underline{\Delta}$, corresponds to a "black hole", the exceptional fiber of the blow up, outfitted with mass and charge.

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Quantum deformation theory

Alexander A. Voronov

Quantum deformation theory is based on the *Quantum Master Equation* (*QME*), also known as the *Batalin-Vilkovisky* (*BV*) *Master Equation*:

$$dS + \hbar \Delta S + \frac{1}{2}\{S, S\} = 0,$$

inasmuch as classical deformation theory is based on the *Classical Master Equation* (*CME*), a.k.a. the *Maurer-Cartan Equation*:

$$dS + \frac{1}{2}[S, S] = 0.$$

The QME is defined in a space $V[[\hbar]]$ of formal power series with values in a (differential graded) dg BV algebra V, whereas the CME is defined in a dg Lie algebra $\mathfrak{g}$.

In classical deformation theory, there are two sides of the story: abstract deformation theory, coming from the works of Deligne, Schlessinger, Stasheff, Goldman, Millson, Kontsevich, and Soibelman, and concrete deformation theories, such as deformations of complex structures (Kodaira-Spencer), associative algebras (Gerstenhaber), and many others. Abstract deformation theory takes the dg Lie algebra $\mathfrak{g}$ as a primary object and studies the CME, the associated deformation functor, and its moduli space. Concrete deformation theory presents a dg Lie algebra governing the deformation problem and uses the specifics of the concrete situation to understand the local structure of the moduli space, such as smoothness, formality, obstructions, virtual dimension, etc.

In quantum deformation theory, just a tip of the iceberg is beginning to appear. There are a few papers, [1, 2, 3], which may be viewed as making first steps in abstract quantum deformation theory. In the paper [4], Terilla puts forward a program of quantizing deformation theory.

There is no general theory of quantum deformations yet, and it is not understood what quantum deformations are in concrete examples. Further steps in quantum deformation theory have been discussed in the talk.

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Construction of Moduli Spaces of Lie, Associative and Infinity Algebras through Extensions and Deformation Theory

Michael Penkava

(joint work with Alice Fialowski, Joshua Frinak, Austen Ott)

An A_∞-algebra is an odd codifferential on the tensor coalgebra of the parity reversion of a $\mathbb{Z}_2$-graded vector space. In other words, it is an odd coderivation of this coalgebra whose square is zero. Similarly an L_∞-algebra is an odd codifferential on the symmetric coalgebra of the parity reversion of a $\mathbb{Z}_2$-graded space. The parity reversion $W = \Pi V$ is the same same underlying space, but with the parity of homogeneous elements reversed. Associative algebras are examples of A_∞-algebras and (graded) Lie algebras are examples of L_∞- algebras.

Let $A(W)$ denote either the tensor or symmetric coalgebra of W, and d be a codifferential on it. If d' is a codifferential on $A(W')$, then a coalgebra morphism $g : A(W) \to A(W')$ is said to be a morphism of infinity algebras if $g \circ d = d' \circ g$. However, this definition has a problem, because the image of an infinity algebra under a morphism will not be in general an infinity algebra. The problem is that this definition really is the definition of a morphism of coalgebras equipped with a codifferential, and infinity algebras are more specialized coalgebras. To modify the definition, it is simply necessary to require that the kernel of the morphism be a "standard coideal", and the image be an infinity algebra. Conditions for this are easy to state, and all examples that arise in the literature, including minimal models, satisfy this more restricted definition.

There are two features that allow the theory of extensions of an associative or Lie algebra by another such algebra to be expressed in a nice form, which make it possible to construct moduli spaces of higher dimensional algebras from lower dimensional ones using extensions. The first feature is a classification theorem of algebras in terms of simple algebras. In the Lie case, an algebra is either semisimple, solvable, or has a unique maximal solvable ideal, and the quotient of the algebra by this solvable ideal is semisimple. This means that any finite dimensional Lie algebra is either solvable, or an extension of a semisimple algebra by a solvable algebra. For associative algebras, there is a unique maximal nilpotent ideal, and if the algebra is not itself nilpotent, the quotient of the algebra by this maximal nilpotent is semisimple. For both Lie and associative algebras, the complex simple algebras are completely classified.

The second feature is that there is a complete theory of extensions of Lie and associative algebras. Given an algebra structure δ on W, and an algebra structure μ on M, an algebra structure d on $V = M \oplus W$, which extends the structures δ and μ is given by a "module" structure λ, which can be thought of as a map $\lambda : W \to \hom(M, M)$ and a "cocycle" ψ, which is a map $\psi : W \otimes W \to M$, which satisfy certain conditions, namely

$$\begin{aligned}
&[\mu, \lambda] = 0\\
&[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0\\
&[\delta + \lambda, \psi] = 0
\end{aligned}$$

These conditions can be unraveled to say that the extensions are classified by a certain cohomology H_μ on the space of coderivations of the coalgebra on V. This cohomology is also a graded Lie algebra, and there is a cohomology operator on H_μ giving a space $H_{\mu,\delta+\lambda}$. The conditions for (δ, ψ) to be determine an extension, and the classification of such extensions, can be given in terms of these two cohomology spaces.

An infinity algebra is given by a codifferential $d = d_1 + d_2 + \cdots$, where $d_k : W^k \to W$, and W^k is either $T^k(W)$ or $S^k(W)$, depending on whether we are considering an A_∞ or L_∞ algebra (or others). If n is the least integer such that

$d_n \neq 0$, then d_n is itself a codifferential, and the structure of d can be studied as a deformation problem for the codifferential d_n. Thus the first step in classifying codifferentials on W is to study the moduli space of degree n codifferentials.

To classify them, we consider the space $C^{k,l}$ of linear maps $V^{k+l} \to M$ which vanish unless there are k elements from M and l elements from W. If $\lambda^{k,l} \in C^{k,l}$, then to classify extensions of a degree n codifferential δ on W by a degree n codifferential μ on M, we consider $\lambda = \lambda^{n-1,1} + \cdots + \lambda^{1,n-1}$, and $\psi \in C^{0,n}$. If we set $\delta_0 = \mu$, $\delta_k = \lambda^{n-k,k}$ for $1 \leq k < n-1$, $\delta_{n-1} = \delta + \lambda^{1,n-1}$ and $\delta_n = \psi$ then the condition for $d = \delta + \mu + \lambda + \psi$ to be a codifferential is $\sum_{i+j=k}[\delta_i, \delta_j] = 0$, for $k = 0, \ldots 2n$. This means that we obtain $2n+1$ equations which need to be satisfied to obtain an extension.

There is a sequence of coboundary operators $D_0, , \cdots , D_n$, where D_0 is defined on the whole space $C^{\bullet,\bullet}$ of coderivations, giving a cohomology H_0, and D_{k+1} defined on H_k, giving a descending sequence of cohomology spaces. These spaces are involved in the classification of extensions. In fact, the first three of these spaces arise in the classical problem of extensions of associative or Lie algebras.

With Alice Fialowski and some undergraduate researchers, we have been studying moduli spaces of infinity algebras of fixed degree n for some low dimensional spaces V and low degrees n. We have determined a complete classification of these moduli spaces, and have studied which algebras arise as extensions. Unlike the Lie or associative algebra case, most algebras are not extensions, and the ones which fail to be extensions do not have some of the properties one might hope for if they were to play the role of simple algebras. For example, they may deform into other algebras. Thus the classification of infinity algebras is not as easy as the associative or Lie case, because we cannot build higher dimensional spaces by the theory of extensions.

Contractions, Relaxations and Deformations of the hyperbolic plane

Pierre Bieliavsky

(joint work with S. Detournay and Ph. Spindel)

Contractions As it is well known, the group $SL_2(\mathbb{R})$ "contracts" onto the (1+1) Poincaré group $\mathbb{P} := SO(1,1) \ltimes \mathbb{R}^2$ inducing a "curvature contraction" of the $SL_2(\mathbb{R})$-co-adjoint orbits. Focusing first on the Hermitean co-adjoint orbit $\mathbb{D} := SL_2(\mathbb{R})/SO(2)$, we realize the contraction just by observing that the Iwasawa factor $\mathbb{S} := NA \simeq ax+b$ of $G := SL_2(\mathbb{R})$ simply transitively acts on both $\mathbb{D}$ and $\mathbb{M} := SO(1,1) \ltimes \mathbb{R}^2/\mathbb{R}$, the generic co-ajoint orbit of the (1+1) Poincaré group $\mathbb{P}$. As a consequence of this observation one gets two $\mathbb{S}$-equivariant symplectic identifications: $\mathbb{D} = \mathbb{S} = \mathbb{M}$.

The symplectic homogeneous space $\mathbb{M}$ admits a unique structure of symplectic symmetric space i.e. a unique $\mathbb{P}$-invariant symplectic connection $\nabla^{\mathbb{M}}$ whose associated geodesic symmetries are global affine transformations. Realizing both symmetric space geometries, the hyperbolic planar one on $\mathbb{D}$ and the one on $\mathbb{M}$

induced by $\nabla^{\mathbb{M}}$, at the level of the *same* space $\mathbb{S}$, one realizes $\nabla^{\mathbb{M}}$ as a $\mathbb{S}$-equivariant contraction of $\nabla^{\mathbb{D}}$.
Solvable symplectic symmetric spaces such as $\mathbb{M}$ admit invariant strict deformation quantizations defined by oscillatory three-point kernels [Bi02]. To explain this, we use Severa's canonical map on a symplectic symmetric space. Assume $M = G/K$ is a symplectic symmetric space for which the action of G is strongly Hamiltonian. As a G-symplectic space, M is thus locally isomorphic to a co-adjoint orbit in the dual $\mathfrak{g}^\star$ of the Lie algebra $\mathfrak{g}$ of G. At every point x of $\mathbb{M}$, the symmetric space structure induces a canonical linear injection $\iota_x : T_x(M) \to \mathfrak{g}$ that dualizes as a linear projection $\iota_x^\star : \mathfrak{g}^\star \to T_x^\star(M)$. The Severa map around x is then simply defined as the local subimmersion:$\pi_x := \iota_x^\star \circ J : M \to T_x^\star(M)$, where J denotes the moment map [Se06]. When Severa's map happens to be a global diffeomorphism (e.g. for $\mathbb{M}$ and $\mathbb{D}$), we define the following canonical three-point function [BDS09] that we call *Severa's area:*$S^M : M \times M \times M \to \mathbb{R} : (x, y, z) \mapsto \omega_x(\pi_x y, \pi_x z)$ where ω_x denotes the symplectic structure on $T_x^\star(M)$.
Severa's area enjoys the remarkable property of being totally skewsymmetric in the three points x, y and z [Bi02]. In the case of the solvable space $M = \mathbb{M}$ (and many others that are solvable), Severa's area coincides with the symplectic area of the oriented geodesic triangle that admits x, y and z as midpoints of its geodesic edges [Bi02]. Denoting by $m(x, y)$ the midpoint on the geodesic arc between x and y, the map: $(x, y, z) \mapsto (m(x, y), m(y, z), m(z, x))$ actually defines a global diffeomorphism of $\mathbb{M} \times \mathbb{M} \times \mathbb{M}$ onto itself (this does not hold for the hyperbolic plane $\mathbb{D}$). We denote by $\Phi : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \to \mathbb{M} \times \mathbb{M} \times \mathbb{M}$ the *inverse* of the above described diffeomorphism.

The full "space of quantizations" is then described as follows. We denote by Θ the set of smooth functions valued in the Schwartz operator multipliers $\mathcal{P} : \mathbb{R} \to \mathcal{O}_M(A)$ $(A \simeq \mathbb{R})$ such that $\lim_{\theta \to 0} \mathcal{P}_\theta \equiv 1$.
Theorem 1: [BBM03, Bi07] Let u and v be compactly supported continuous functions on $\mathbb{M}$.

(i) Let $\mathcal{P} \in \Theta$. There exists a pre-Hilbert space structure on $\mathcal{D}(\mathbb{M}) := C_c^\infty(\mathbb{M})$ such that the formula[1]

$$u \star_\theta^{\mathcal{P}} v(x) = \frac{1}{\theta^2} \int_{\mathbb{M}\times\mathbb{M}} \sqrt{\mathrm{Jac}_\Phi(x, y, z)}\, e^{\frac{i}{\theta} S^{\mathbb{M}}(x,y,z)} \, \frac{\mathcal{P}_\theta(a_x - a_y)\mathcal{P}_\theta(a_y - a_z)}{\mathcal{P}_\theta(a_x - a_z)} \, u(y)\, v(z)\, \mathrm{d}y\, \mathrm{d}z$$

extends to the Hilbert completion $\mathcal{H}_{\theta,\mathcal{P}} := \overline{\mathcal{D}(\mathbb{M})}$ as an associative Hilbert algebra structure on which the automorphism group $\mathrm{Aut}(\mathbb{M})$ acts by unitary automorphisms.

(ii) For every element $\mathcal{P} \in \Theta$, the asymptotic expansion: $u \star_\theta^{\mathcal{P}} v(x) \sim \sum_k \theta^k C_k^{\mathcal{P}}(u, v)(x) =: u \tilde{\star}_\theta^{\mathcal{P}} v(x)$ defines a $\mathrm{Aut}(\mathbb{M})$-invariant formal $\star$-product on $C^\infty(\mathbb{M})$ $[[\theta]]$.

(iii) Every $\mathrm{Aut}(\mathbb{M})$-invariant formal $\star$-product on $C^\infty(\mathbb{M})$ $[[\theta]]$ is of the form $\tilde{\star}_\theta^{\mathcal{P}}$ for some element $\mathcal{P}$ of Θ.

[1] For every x in $\mathbb{M}$, we set $x = n_x a_x$ according to the decomposition $\mathbb{M} = \mathbb{S} = NA$.

In other words, the space of quantizations is realized by the space Θ. All this extends to the higher dimensional situation [BM01, Bi07].

Relaxations The quantizations described in the preceding subsection are Poincaré invariant but never $SL_2(\mathbb{R})$-invariant. We now explain a process that yields every $SL_2(\mathbb{R})$-invariant quantization on $\mathbb{S}$ (hence on $\mathbb{D}$) from a given one at the contracted level (i.e. on $\mathbb{M}$).

Let us start by considering the Lie algebra $\mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n}$ of NA and view it as a subalgebra of both $\mathbb{P}$ and G. Let $\star$ be *any* $\mathrm{Aut}(\mathbb{M})$-invariant formal $\star$-product on $C^\infty(\mathbb{M})[[\theta]]$. Consider the natural linear injection $\mathfrak{s} \to \mathfrak{Der}(\star)$ into the derivation algebra $\mathfrak{Der}(\star)$ of $\star$.

Proposition 1: [BDS09] Let $D \in \mathfrak{Der}(\star)$ be any derivation that generates together with $\mathfrak{s}$ an $\mathfrak{sl}_2$-algebra of derivations of $\star$: $\mathbb{R}D \oplus \mathfrak{s} \simeq \mathfrak{sl}_2(\mathbb{R})$. Then:

(i) Denoting by $\mathcal{F}_N$ the partial Fourier transform on $\mathbb{S}$ w.r.t the N-variable, the operator $\mathcal{F}_N \circ D \circ \mathcal{F}_N^{-1} =: \Box_D$ is a second-order differential operator.

(ii) Up the a real multiple, the principal symbol $\Box_D \mapsto \sigma(\Box_D) =: \sigma(\Box)$ is independent of the choice of D and $\star$.

(iii) The principal symbol $\sigma(\Box)$ is the one of the Laplace operator associated to an anti- de Sitter metric.

Let us now consider a generator $F \in \mathfrak{g} := \mathfrak{sl}_2(\mathbb{R})$ of the line $\tau\mathfrak{n}$ where τ is a Cartan involution that anti-fixes $\mathfrak{a}$. Note that the associated fundamental vector field $F^\star$ does not act as a derivation of $\star$, however, one has:

Proposition 2: [BDS09] Consider the natural Hopf algebra $(C^\infty(\mathbb{S}), ., \Delta, \epsilon)$ induced by the Lie group structure on $\mathbb{S}$. Then, every formal weak solution $u \in \mathcal{D}'(\mathbb{S})[[\theta]]$ of $-(D \otimes I)(\epsilon \otimes I)\Delta(u) = (I \otimes F^\star)(\epsilon \otimes I)\Delta(u)$ whose associated convolution operator $\ell_u : \varphi \mapsto u \times \varphi$ on $\mathcal{D}(\mathbb{S})[[\theta]]$ is invertible determines an $SL_2(\mathbb{R})$-invariant associative product $\sharp_u$ by transporting $\star$ under ℓ_u. Moreover, every $SL_2(\mathbb{R})$-invariant $\star$-product on $\mathbb{D} = \mathbb{S}$ is of the form $\sharp_u$ for some solution u.

We call *relaxation* such a convolution operator ℓ_u. Note that, since the operator $D \otimes I$ commutes with the vector field $I \otimes F^\star$, determining the evolution essentially amounts to solving an *evolution equation* for D or equivalently (by intertwining under the partial Fourier transform $\mathcal{F}_N$) to determining the evolutions of the second order operator $\Box_D$.

Proposition 3: [BDS09] The evolution equation can be explicitly solved by separation of variables (SOV).

Having the explicit solution of the evolution of D at our disposal, we are now able to explicitly determine the quantizations. We focused on two examples. The first one concerns the hyperbolic plane. We describe it below. The second one deals with Zagier's construction of Rankin-Cohen deformations of the modular algebra [Za94]. The geometry involved in the latter is flat [BTY07].

Unterbeger type solutions In [UU88], A. and J. Unterberger defined what they called Bessel's symbolic calculus on the hyperbolic plane as a curved analogue of Weyl's calculus. The composition formula of symbols they end with does not, as they observe, yield a $\star$-product in the sense that the semiclassical limit is not defined. Disposing of the full space of such products, we produced Unterberger

type quantizations that possess the right semiclassical limit. The resulting integral kernel involves, again, Severa's area $S^{\mathbb{D}}$ on the hyperbolic plane.
Theorem 2: Consider the following special function: $K_\theta(\varpi) := \frac{1}{16\pi^3\theta^4}\int_0^\infty t^2 J_{\frac{1}{\theta}}\left(\frac{t}{\theta}\right) e^{\frac{i}{\theta}t\varpi}\,\mathrm{d}t$. Then, there exists a pre-Hilbert space of functions $\mathcal{H}^{\mathbb{D}}_\theta \subset C^\infty(\mathbb{D})$ that closes as an associative Hilbert algebra under the integral product formula:

$$u\,\sharp_\theta\, v\,(x) := \int_{\mathbb{D}\times\mathbb{D}} K_\theta\big(S^{\mathbb{D}}(x,y,z)\big)\; u(y)\,v(z)\,\mathrm{d}y\,\mathrm{d}z\ .$$

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Involutions and Representations for Reduced Quantum Algebras

Simone Gutt and Stefan Waldmann

The results that we presented in our two talks can be found with details and references in [10].

Some mathematical formulations of quantizations are based on the algebra of observables and consist in replacing the classical algebra of observables $\mathcal{A}$ (typically complex-valued smooth functions on a Poisson manifold M) by a non commutative one $\boldsymbol{\mathcal{A}}$. Formal deformation quantization was introduced in [1]; it constructs the quantum observable algebra by means of a formal deformation (in the sense of Gerstenhaber) of the classical algebra. Given a Poisson manifold M and the classical algebra $\mathcal{A} = C^\infty(M)$ of complex-valued smooth functions, a star product on M is a $\mathbb{C}[[\lambda]]$-bilinear associative multiplication on $\mathcal{C}^\infty(M)[[\lambda]]$ with

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g), \tag{1}$$

where $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$, where the C_r are bidifferential operators so that $1 \star f = f = f \star 1$ for all $f \in \mathcal{C}^\infty(M)[[\lambda]]$. The algebra of quantum observables is $\boldsymbol{\mathcal{A}} = (\mathcal{C}^\infty(M)[[\lambda]], \star)$.

An important classical tool to "reduce the number of variables", i.e. to start from a "big" Poisson manifold M and construct a smaller one M_{red}, is given by reduction: one considers an embedded coisotropic submanifold in the Poisson manifold, $\iota : C \hookrightarrow M$ and the canonical foliation of C which we assume to have a nice leaf space M_{red}. In this case one knows that M_{red} is a Poisson manifold in a canonical way.

We consider here the particular case of the Marsden-Weinstein reduction: let $\mathsf{L} : G \times M \longrightarrow M$ be a smooth left action of a connected Lie group G on M by Poisson diffeomorphisms and assume we have an ad^*-equivariant momentum map $J : M \longrightarrow \mathfrak{g}^*$. The constraint manifold C is chosen to be the level surface of J for momentum $0 \in \mathfrak{g}^*$ (thus we assume, for simplicity, that 0 is a regular value). Then $C = J^{-1}(\{0\})$ is an embedded submanifold which is coisotropic. The group G acts on C and the reduced space is the orbit space of this group action of G on C (in order to guarantee a good quotient we assume that G acts freely and properly).

Given a mathematical formulation of quantization, one studies then a quantized version of reduction and how "quantization commutes with reduction". This has been done in the framework of deformation quantization by various authors [3, 8, 7]. We shall use here the approach proposed by Bordemann [2] . Since the emphasis is put in our quantization scheme on the observable algebra, recall that at the classical level if $\iota : C \hookrightarrow M$ is an embedded coisotropic submanifold, one considers $\mathcal{J}_C = \{f \in \mathcal{C}^\infty(M) \mid \iota^* f = 0\} = \ker \iota^*$ the vanishing ideal of C. It is an ideal in the associative algebra $\mathcal{C}^\infty(M)$ and a Poisson subalgebra of $\mathcal{C}^\infty(M)$. One defines $\mathcal{B}_C = \{f \in \mathcal{C}^\infty(M) \mid \{f, \mathcal{J}_C\} \subseteq \mathcal{J}_C\}$, and assuming that the canonical foliation of C has a nice leaf space M_{red} (i.e. a structure of a smooth manifold such that the canonical projection $\pi : C \longrightarrow M_{\mathrm{red}}$ is a submersion); then

$$\mathcal{B}_C / \mathcal{J}_C \ni [f] \ \mapsto \ \iota^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}}) = \mathcal{A}_{\mathrm{red}} \tag{2}$$

induces an isomorphism of Poisson algebras.

The fact that this is an isomorphism in our setting of Marsden Weinstein reduction can be seen using the Koszul resolution as follows. The Koszul complex is $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}) = \mathcal{C}^\infty(M) \otimes \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}$ with Koszul differential defined by

$$\partial x = \mathrm{i}(J)x. \tag{3}$$

Since the group G acts properly on M one can find (see for instance [3]) a nice G-invariant tubular neighborhood M_{nice} of C and a G-equivariantly diffeomorphism

$$\Phi : M_{\mathrm{nice}} \longrightarrow U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^* \tag{4}$$

so that $U_{\mathrm{nice}} \cap (\{p\} \times \mathfrak{g}^*)$ is star-shaped $\forall p \in C$ and $J_{|M_{\mathrm{nice}}} = \mathrm{pr}_2 \circ \Phi$. This allows to define a G-equivariant prolongation map from functions on C to functions on M

$$\mathcal{C}^\infty(C) \ni \phi \mapsto \mathrm{prol}(\phi) = (\mathrm{pr}_1 \circ \Phi)^* \phi \in \mathcal{C}^\infty(M_{\mathrm{nice}}) \tag{5}$$

and homotopies h_i for the Koszul complex, explicitly given on the nice neighbourhood by

$$(h_k x)(p) = e_a \wedge \int_0^1 t^k \frac{\partial(x \circ \Phi^{-1})}{\partial \mu_a}(c, t\mu)\, \mathrm{d}\, t, \tag{6}$$

for $x \in \mathcal{C}^\infty(M_{\mathrm{nice}}, \Lambda^k_{\mathbb{C}}\mathfrak{g})$.

In particular $\ker i^* = \operatorname{im} \partial$ and the quotient space $\mathcal{B}_C \big/ \mathcal{J}_C$ is isomorphic to $\mathcal{C}^\infty(M_{\mathrm{red}})$ via the mutually inverse maps $\mathcal{B}_C \big/ \mathcal{J}_C \ni [f] \mapsto \iota^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}})$ and $\mathcal{C}^\infty(M_{\mathrm{red}}) \ni u \mapsto [\mathrm{prol}(\pi^* u)] \in \mathcal{B}_C \big/ \mathcal{J}_C$. The Poisson bracket on M_{red} is defined through this bijection

$$\pi^*\{u, v\}_{\mathrm{red}} = \iota^*\{\mathrm{prol}(\pi^* u), \mathrm{prol}(\pi^* v)\} \qquad u, v \in \mathcal{C}^\infty(M_{\mathrm{red}}). \tag{7}$$

Passing to a deformation quantized version of phase space reduction, one starts with a formal star product $\star$ on M. The associative algebra $\boldsymbol{\mathcal{A}} = (\mathcal{C}^\infty(M)[[\lambda]], \star)$ is playing the role of the quantized observables of the big system. A good analog of the vanishing ideal $\mathcal{J}_C$ will be a left ideal $\boldsymbol{\mathcal{J}}_C \subseteq \mathcal{C}^\infty(M)[[\lambda]]$ such that the quotient $\mathcal{C}^\infty(M)[[\lambda]] \big/ \boldsymbol{\mathcal{J}}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to the functions $\mathcal{C}^\infty(C)[[\lambda]]$ on C. Then we define $\boldsymbol{\mathcal{B}}_C = \{a \in \boldsymbol{\mathcal{A}} \mid [a, \boldsymbol{\mathcal{J}}_C] \subseteq \boldsymbol{\mathcal{J}}_C\}$, i.e. the normalizer of $\boldsymbol{\mathcal{J}}_C$ with respect to the commutator Lie bracket of $\boldsymbol{\mathcal{A}}$, and consider the associative algebra $\boldsymbol{\mathcal{B}}_C \big/ \boldsymbol{\mathcal{J}}_C$ as the reduced algebra $\boldsymbol{\mathcal{A}}_{\mathrm{red}}$. Of course, this is only meaningful if one can show that $\boldsymbol{\mathcal{B}}_C \big/ \boldsymbol{\mathcal{J}}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ in such a way, that the isomorphism induces a star product $\star_{\mathrm{red}}$ on M_{red}.

Starting from a strongly invariant star product on M, there is a method to construct a good left ideal inspired by the BRST approach in [3], simpler as we only need to deform the Koszul part of the BRST complex. The quantized Koszul operator $\boldsymbol{\partial} : \mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})[[\lambda]] \longrightarrow \mathcal{C}^\infty(M, \Lambda^{\bullet-1}_{\mathbb{C}}\mathfrak{g})[[\lambda]]$ is defined by

$$\boldsymbol{\partial} x = \mathrm{i}(e^a) x \star J_a + \frac{\mathrm{i}\lambda}{2} C^c_{ab} e_c \wedge \mathrm{i}(e^a)\,\mathrm{i}(e^b) x + \frac{\mathrm{i}\lambda}{2}\,\mathrm{i}(\Delta) x, \tag{8}$$

where $\{e_a\}$ is a basis of $\mathfrak{g}$, $C^c_{ab} = e^c([e_a, e_b])$ are the structure constants of $\mathfrak{g}$ and $\Delta(\xi) = \mathsf{tr}\,\mathrm{ad}(\xi)$ is the modular one-form $\Delta \in \mathfrak{g}^*$ of $\mathfrak{g}$.

The good left ideal is the image of the Koszul differential $\boldsymbol{\mathcal{J}}_C = \operatorname{im} \boldsymbol{\partial}_1$. Then $\boldsymbol{\mathcal{J}}_C = \ker \boldsymbol{\iota}^*$ where $\boldsymbol{\iota}^* = \iota^* \left(\mathsf{id} + (\boldsymbol{\partial}_1 - \partial_1)\, h_0\right)^{-1} : \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow \mathcal{C}^\infty(C)[[\lambda]]$. The quotient algebra $\boldsymbol{\mathcal{B}}_C \big/ \boldsymbol{\mathcal{J}}_C$ is isomorphic to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ via the mutually inverse maps $\boldsymbol{\mathcal{B}}_C \big/ \boldsymbol{\mathcal{J}}_C \ni [f] \mapsto \boldsymbol{\iota}^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ and $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]] \ni u \mapsto [\mathrm{prol}(\pi^* u)] \in \boldsymbol{\mathcal{B}}_C \big/ \boldsymbol{\mathcal{J}}_C$. The induced star product $\star^{(\kappa)}_{\mathrm{red}}$ on $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ is

$$\pi^*(u \star^{(\kappa)}_{\mathrm{red}} v) = \boldsymbol{\iota}^* \left(\mathrm{prol}(\pi^* u) \star \mathrm{prol}(\pi^* v)\right). \tag{9}$$

1. Involutions for the reduced quantum algebra

The algebra of quantum observables is not only an associative algebra but it has a *-involution; in the usual picture, where observables are represented by operators, this *-involution corresponds to the passage to the adjoint operator. In the framework of deformation quantization, a way to have a *-involution on $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ is to ask the star product to be Hermitian, i.e. such that

$\overline{f \star g} = \overline{g} \star \overline{f}$ and the *-involution is then just given by complex conjugation. The problem we presented in the first talk is how to get in a natural way a *-involution for the reduced algebra, assuming that $\star$ is a Hermitian star product on M. We want a construction coming from the reduction process itself; we start with a left ideal $\boldsymbol{\mathcal{J}} \subseteq \boldsymbol{\mathcal{A}}$ in some algebra and take $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$ as the reduced algebra, where $\boldsymbol{\mathcal{B}}$ is the normaliser of $\boldsymbol{\mathcal{J}}$ in $\boldsymbol{\mathcal{A}}$. If now $\boldsymbol{\mathcal{A}}$ is in addition a *-algebra we have to construct a *-involution for $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$. From all relevant examples in deformation quantization one knows that $\boldsymbol{\mathcal{J}}$ is only a left ideal, hence can not be a *-ideal and thus $\boldsymbol{\mathcal{B}}$ can not be a *-subalgebra. Consequently, there is no obvious way to define a *-involution on the quotient.

The main idea here is to use a representation of the reduced quantum algebra and to translate the notion of the adjoint. Observe that $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$ can be identified (with the opposite algebra structure) to the algebra of $\boldsymbol{\mathcal{A}}$-linear endomorphisms of $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{J}}$. We shall use an additional positive linear functional i.e. a $\mathbb{C}[[\lambda]]$-linear functional $\omega : \boldsymbol{\mathcal{A}} \longrightarrow \mathbb{C}[[\lambda]]$ such that $\omega(a^*a) \geq 0$ for all $a \in \boldsymbol{\mathcal{A}}$, where positivity in $\mathbb{C}[[\lambda]]$ is defined using the canonical ring ordering of $\mathbb{R}[[\lambda]]$. Defining the Gel'fand ideal of ω by $\boldsymbol{\mathcal{J}}_\omega = \{a \in \boldsymbol{\mathcal{A}} \mid \omega(a^*a) = 0\}$, one can construct a *-representation (the GNS representation), of $\boldsymbol{\mathcal{A}}$ on $\mathcal{H}_\omega = \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{J}}_\omega$ with the pre Hilbert space structure defined via $\langle \psi_a, \psi_b \rangle = \omega(a^*b)$ where ψ_a denotes the equivalence class of $a \in \boldsymbol{\mathcal{A}}$. Then the algebra of $\boldsymbol{\mathcal{A}}$-linear endomorphisms of $\mathcal{H}_\omega$ (with the opposite structure) is equal to $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}_\omega$. Hence, to define a *-involution on our reduced quantum algebra, the main idea is now to look for a positive linear functional ω such that the left ideal $\boldsymbol{\mathcal{J}}$ we use for reduction coincides with the Gel'fand ideal $\boldsymbol{\mathcal{J}}_\omega$ and such that all left $\boldsymbol{\mathcal{A}}$-linear endomorphisms of $\mathcal{H}_\omega$ are adjointable. In this case $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$ becomes in a natural way a *-subalgebra of the set $\mathfrak{B}(\mathcal{H}_\omega)$ of adjointable maps. Up to here, the construction is entirely algebraic and works for *-algebras over rings of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathrm{i}^2 = -1$ and an ordered ring R, instead of $\mathbb{C}[[\lambda]]$ and $\mathbb{R}[[\lambda]]$.

A formal series of smooth densities $\sum_{r=0}^{\infty} \lambda^r \mu_r \in \Gamma^\infty(|\Lambda^{\mathrm{top}}| T^*C)[[\lambda]]$ on the coisotropic submanifold C such that $\overline{\mu} = \mu$ is real, $\mu_0 > 0$ and so that μ transforms under the G-action as $\mathsf{L}^*_{g^{-1}}\mu = \frac{1}{\Delta(g)}\mu$, where Δ is the modular function, yields a positive linear functional

$$\omega_\mu(f) = \int_C \boldsymbol{\iota}^*(f)\mu \qquad \text{for } f \in \mathcal{C}^\infty_0(M)[[\lambda]]. \tag{10}$$

The corresponding GNS representation of $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ is defined on the pre Hilbert space $\mathcal{C}^\infty_0(C)[[\lambda]]$ with scalar product $\langle \phi, \psi \rangle_\mu = \int_C \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \mu$ as the left module structure $\bullet$

$$f \bullet_\kappa \phi = \boldsymbol{\iota}^*(f \star \mathrm{prol}(\phi)) \quad \phi \in \mathcal{C}^\infty(C)[[\lambda]], f \in \mathcal{C}^\infty(M)[[\lambda]. \tag{11}$$

For any $u \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ there exists a unique $u^* \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ such that $\langle \phi, \psi \bullet_{\mathrm{red}} u \rangle_\mu = \langle \phi \bullet_{\mathrm{red}} u^*, \psi \rangle_\mu$ for all $\phi, \psi \in \mathcal{C}^\infty_0(C)[[\lambda]]$, and the map $u \mapsto u^*$ is a

*-involution for $\star_{\rm red}$ of the form

$$u^* = \overline{u} + \sum_{r=1}^{\infty} \overline{I_r(u)} \tag{12}$$

with differential operators I_r on $M_{\rm red}$.

In our Marsden-Weinstein reduction context, complex conjugation is also a *-involution for $\star_{\rm red}$. Studying whether the *-involution corresponding to a series of densities μ is the complex conjugation yields a new notion of quantized modular class in the following way. Formal series of densities μ correspond to formal series of densities Ω on $M_{\rm red}$ (locally $\Phi^* \left(\mu_{|_{\pi^{-1}U}}\right) = \Omega_{|_U} \boxtimes \mathbb{D}^{\rm left} g$) and one has

$$\int_{M_{\rm red}} v \star_{\rm red} u \,\Omega = \int_{M_{\rm red}} \overline{u^*} \star_{\rm red} v \,\Omega \qquad u, v \in \mathcal{C}_0^\infty(M_{\rm red})[[\lambda]]. \tag{13}$$

The *-involution corresponding to a formal series μ, hence to a formal series Ω, is the complex conjugation iff the automorphism of $\star_{\rm red}$ defined by

$$I_\Omega : \mathcal{C}_0^\infty(M_{\rm red})[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]] : u \mapsto \overline{u^*} \tag{14}$$

is equal to the identity, and this is true iff the map τ_Ω defined by

$$\tau_\Omega : \mathcal{C}_0^\infty(M_{\rm red})[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]] : u \mapsto \tau_\Omega(u) := \int_{M_{\rm red}} u\,\Omega \tag{15}$$

is a trace functional. The existence of a trace density for $\star_{\rm red}$ is non-trivial: it implies that Ω_0 is a Poisson trace (i.e. τ_{Ω_0} vanishes on Poisson brackets) and this implies that the Poisson structure of $M_{\rm red}$ is *unimodular* [12]. Given another series of densities $\Omega' = \varrho\Omega$ on $M_{\rm red}$ there is a unique $\boldsymbol{\varrho} \in \mathcal{C}^\infty(M_{\rm red})[[\lambda]]$ with $\boldsymbol{\varrho}_0 = \varrho_0 > 0$ such that

$$\tau_{\Omega'}(u) = \tau_\Omega(\boldsymbol{\varrho} \star_{\rm red} u) \qquad \forall u \in \mathcal{C}_0^\infty(M_{\rm red})[[\lambda]] \tag{16}$$

and the *-involutions * and $^{*'}$ are related by the inner automorphism

$$u^{*'} = \overline{\boldsymbol{\varrho}} \star_{\rm red} u^* \star_{\rm red} \overline{\boldsymbol{\varrho}}^{-1}. \tag{17}$$

Observe that the automorphism I_Ω coincides with the identity at order 0 in λ so that one can write

$$I_\Omega = \exp(D_\Omega) \tag{18}$$

for a derivation $D_\Omega = \sum_{r=1}^\infty \lambda^r D_\Omega^{(r)}$ of $\star_{\rm red}$. Given two series of densities on $M_{\rm red}$, the difference $D_\Omega - D_{\Omega'}$ is an inner derivation of $\star_{\rm red}$ so that the Hochschild cohomology class of D_Ω is independent of Ω; we call it the *quantum modular class*. One sees that $D_\Omega^{(1)} = \mathrm{i}\Delta_{\Omega_0}$, where Δ_{Ω_0} is the modular vector field of $M_{\rm red}$ with respect to Ω_0.

2. Representations of the reduced quantum algebra

The question we addressed in the second talk is the study of the representations of the reduced algebra with the *-involution given by complex conjugation. We want to relate the categories of modules of the big algebra and the reduced algebra. The usual idea is to use a bimodule and the tensor product to pass from modules of one algebra to modules of the other. In the context of quantization and reduction this point of view has been pushed forward by Landsman [11], mainly in the context of geometric quantization. Contrary to his approach, we have, by construction of the reduced star product, a bimodule structure on $\mathcal{C}^\infty(C)[[\lambda]]$. We want more properties to have a relation between the *-representations of our algebras on inner product modules. The notions are transferred, following [6, 4], from the theory of Hilbert modules over C^*-algebras to our more algebraic framework.

In order to build such a bimodule we look at

(19)
$$\mathcal{C}^\infty_{\mathrm{cf}}(C) = \left\{\phi \in \mathcal{C}^\infty(C) \;\middle|\; \mathrm{supp}(\phi)\cap\pi^{-1}(K) \text{ is compact for all compact} K \subseteq M_{\mathrm{red}}\right\}.$$

The space $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ is then a sub-bimodule for the left $(\mathcal{C}^\infty(M)[[\lambda]],\star)$- and right $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]],\star_{\mathrm{red}})$-module structure on $\mathcal{C}^\infty(C)[[\lambda]]$. We define on it a $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product by the explicit formula

$$\pi^* \langle \phi, \psi\rangle_{\mathrm{red}} = \int_G \boldsymbol{\iota}^*(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi))\mathbb{D}^{\mathrm{left}}g, \tag{20}$$

where we use the left invariant Haar measure on G. The bimodule structure and inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ gives a strong Morita equivalence bimodule between $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ and the finite rank operators on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. The crucial point is to show the complete positivity of the inner product. In some sense, the resulting equivalence bimodule can be viewed as a deformation of the corresponding classical limit which is studied independently in the context of the strong Morita equivalence of the crossed product algebra with the reduced algebra. If G is not finite, the finite rank operators do not have a unit, thus we have a first non-trivial example of a strong equivalence bimodule for star product algebras going beyond the unital case studied in [5].

The *-algebra $(\mathcal{C}^\infty(M)[[\lambda]],\star)$ acts on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ in an adjointable way with respect to the $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product and we obtain a Rieffel induction functor from the strongly non-degenerate *-representations of $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]],\star_{\mathrm{red}})$ on pre-Hilbert right $\mathcal{D}$-modules to those of $(\mathcal{C}^\infty(M)[[\lambda]],\star)$, for any auxiliary coefficient *-algebra $\mathcal{D}$ over $\mathbb{C}[[\lambda]]$. Thus we obtain a functorial induction only for the direction “bottom-up”. It does not seem to be possible to get a functorial induction of the other direction.

As example, we study the geometrically trivial situation $M = M_{\mathrm{red}} \times T^*G$ where on M_{red} a Poisson bracket and a corresponding star product $\star_{\mathrm{red}}$ is given while on T^*G we use the canonical symplectic Poisson structure and the canonical star product $\star_{\mathrm{G}}$ from [9]. Up to the completion issues, the Rieffel induction with $\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)[[\lambda]]$ simply consists in tensoring the given *-representation of $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with the Schrödinger representation on $\mathcal{C}^\infty_0(G)[[\lambda]]$.

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Deformation quantization and flat connections

Anton Alekseev

The KZ connection comes from the WZW model in 2D CFT,

$$A^n_{KZ} = \sum_{i<j} t_{ij}\ \ln(z_i - z_j) \in \Omega^1(\mathbb{C}^n \setminus \mathrm{diag}, t_n)$$

where $t_n = \mathrm{freeLie}(t_{ij},\ i,j = 1,\dots,n)/\mathrm{relations}$, the relations being

- $t_{i,i} = 0$
- $t_{i,j} = t_{j,i}$
- $[t_{i,j}, t_{k,l}] = 0$
- $[t_{i,j} + t_{i,k}, t_{j,k}] = 0$

t_n is the Lie algebra of pure braids.

The holonomy

$$\Phi_{KZ} = \mathrm{Hol}(A^3_{KZ}, \cdot_{z_1} \xrightarrow{z_2} \cdot_{z_3})$$

is a Drinfeld associator. That is,

(1) $\Phi = 1 + \frac{1}{24}[t_{1,2}, t_{2,3}] + \cdots$
(2) Φ satisfies pentagon equation.

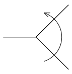
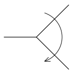
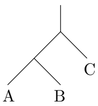
It turns out that the Torossian connection in 2D BF theory gives rise to another solution of associator conditions. This connection has the form

$$A_T^n = \sum_{\Gamma \in \text{trees}} W_\Gamma \cdot \langle \Gamma \rangle$$

where W_Γ is the Feynman weight of a tree and $\langle\Gamma\rangle$ is the image of a tree in the Lie algebra

$$\text{tree}_n = \left\langle \begin{array}{c} \text{trivalent planar oriented trees} \\ \text{with leaves colored in elements} \\ \text{of } \{1, 2, \ldots, n\} \end{array} \right\rangle / \text{relations}$$

the relations being

-  $= -$ 
- 

$+\, cycl = 0$

One can prove

$$\Phi_T = \text{Hol}(A_T^3, \cdot \to \cdot) \neq \Phi_{KZ}$$

giving a new solution of associator axioms.

Curved infinity-algebras and their characteristic classes

Andrey Lazarev
(joint work with Travis Schedler)

Kontsevich has associated certain characteristic classes to finite-dimensional L_∞- or A_∞-algebras equipped with an invariant inner product, [Kon93, Kon94]. These are expressed in terms of the homology of certain complexes spanned by graphs with some additional structures. This construction is by now well-understood both from the point of view of Lie algebra homology and topological conformal field theory; see, for example, [HL08].

In this note, we explore a natural generalization of this construction to the case of *curved* algebras, introduced by Positselski in [Pos93]. It turns out that a complete description of these classes, and of the homology of the associated graph complexes, is possible. We show that these are all obtainable from one-dimensional algebras, and that these classes are zero for algebras with zero curvature. This contrasts with the corresponding problem for conventional graph complexes, which is still widely open.

As explained by Kontsevich, these graph complexes can be viewed as computing the stable homology of Lie algebras of symplectic vector fields on a vector space W (in the A_∞ case, one should take *noncommutative* symplectic vector fields). This motivates us to consider the stability maps. In this direction, we prove that the map from the Lie algebra of symplectic vector fields on W vanishing at the origin to the homology of the Lie algebra of *all* vector fields on $W \oplus \mathbb{C} \cdot w$, where

w is an odd vector, is zero. Similarly, we show the same for the Lie algebra of noncommutative symplectic vector fields.

The precise relation to the previous result is as follows. Any cyclic L_∞-algebra structure on V defines an unstable characteristic class in the homology of the Lie algebra of symplectic vector fields on the shifted vector space $W = \Pi V$. As $\dim V \to \infty$, the homology of this Lie algebra converges to the graph homology, and the image of the unstable characteristic class under the stability maps gives, in the limit, the aforementioned (stable) characteristic class. Hence, our result above says that the unstable curved characteristic class of an algebra with zero curvature already maps to zero under the first stability map $W := \Pi V \hookrightarrow W \oplus \mathbb{C} \cdot w$.

A related observation is the following: if A is a curved (associative or Lie, or more generally A_∞- or L_∞-) algebra with nonzero curvature, then A is gauge equivalent (i.e., homotopy isomorphic) to the algebra with the same curvature and zero multiplication, in a sense we will recall below. In the case of *cyclic* curved algebras, we also compute the gauge equivalence classes, which are less trivial: nontrivially curved algebras are gauge equivalent to the direct sum of a curved algebra of dimension at most two of a certain form (but having nontrivial multiplications in general), with a zero algebra.

This observation hints at a triviality of curved infinity-algebras from a homological point of view, at least when the cyclic structure is not considered. A similar result on the triviality of the corresponding derived categories was obtained recently in [KLN10].

Finally, we generalize these results to the operadic setting, i.e., to types of algebras other than associative and Lie algebras. In particular, we can apply it to Poisson, Gerstenhaber, BV, permutation, and pre-Lie algebras. For the most part, the generalization is straightforward, and we restrict ourselves with giving only an outline of arguments in this section. There is, however, one important aspect which is less visible in the special cases of commutative and ribbon graphs: a curved graph complex associated with a cyclic (or even modular) operad $\mathcal{O}$ is quasi-isomorphic to a variant of the *deformation complex of a curved $\mathcal{O}$-algebra on a one-dimensional space*. Therefore, this graph complex supports the structure of a differential graded Lie algebra. This differential Lie algebra, and its Chevalley-Eilenberg complex, appeared in various guises in the works of Zwiebach-Sen, Costello and Harrelson-Voronov-Zuniga on quantum master equation, [SZ96, Cos09, HVZ07].

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The four-punctured sphere and the Weyl algebra: two exercises in algebraic deformation theory

Murray Gerstenhaber

Classically, the deformation theory of algebras identifies the space of infinitesimal deformations of an algebra A over a commutative unital ring k with the cohomology group $H^2(A, A)$. When this vanishes A is called *absolutely rigid* and no non-trivial deformation is possible. Nevertheless, A may depend in an essential way on one or more parameters. (The same problem arises for non-compact complex analytic manifolds.) This paper examines (i) the algeba of the sphere with four marked points ("punctures") $k[1/(x-a_1), 1/(x-a_2), 1/(x-a_3), 1/(x-a_4)]$ and (ii) the Weyl algebra $k[x, y]/(xy - qyx - 1)$, both of which are absolutely rigid but where the more general cohomology theory of presheaves of algebras over a small category detects the existence of non-trivial infinitesimals. In both one considers instead of A alone, how it is "put together" from the inclusion of smaller subalgebras whose images generate A and which "move" relative to each other. (The theory is patterned after that for complex manifolds where if a covering is given then its sets are the objects of the category and the inclusions its morphisms; to each set of the covering one assigns its ring of holomorphic functions.) From every presheaf of associative algebras over a small category one can build a single algebra whose cohomology and deformation theories are identical with those of the entire presheaf, but the examples here differ. In the first one must use only cocycles which vanish at the marked points to insure that they remain deleted; these form the subcomplex whose cohomology detects the infinitesimals; in the second there is no restriction.

Geometric and algebraic structures for general crossed modules

Christoph Schweigert
(joint work with Jennifer Maier, Thomas Nikolaus)

A finite crossed module consists of two finite groups H and G, an action $\mu : G \to \mathrm{Aut}(H)$ by group automorphisms and a boundary map $\partial : H \to G$ compatible with the adjoint action:

$$\partial(g.h) = \mathrm{Ad}_g(\partial h) \qquad (\partial h).h' = \mathrm{Ad}_h(h')$$

for all $g \in G$ and $h, h' \in H$. To a finite crossed module, one can associate a finite-dimensional complex braided Hopf-Algebra $\mathcal{D}(H \stackrel{\partial}{\to} G)$ whose representation category is a premodular category [1].

This category is a modular tensor category, if and only if the boundary map ∂ is an isomorphism of groups. Its failure to be modular is encoded by a Tannakian subcategory for the finite group $J := \ (\ker\partial)^* \rtimes_{\hat{\mu}} \mathrm{coker}\partial$. The regular representation of J is a commutative symmetric Frobenius algebra in the Tannakian subcategory. Induction with respect to this algebra gives a modularization in the sense of Bruguières [2]. One can show that this modularization is [6] equivalent, as a braided tensor category, to the representation category of the Drinfeld double $\mathcal{D}(X)$ of the finite group

$$X := \ \mathrm{im}\partial \cong H/\ker\partial \ .$$

Hence finite crossed modules do not lead to new modular tensor categories. Still, as we have shown in this talk, they lead to interesting algebraic structure: We show that a finite crossed module allows to construct a J-equivariant modular tensor category [4, 7] whose neutral component is the modular tensor category $\mathcal{D}(X) - mod$. A J-equivariant modular tensor category $\mathcal{C}$ allows to construct a J-equivariant three-dimensional TFT, i.e. a tensor functor from a cobordism category with J-covers to the symmetric monoidal category of finite-dimensional vector spaces. Moreover, by the so-called orbifold construction, one can associate to the J-equivariant modular tensor category $\mathcal{C}$ a modular tensor category $\mathcal{C}/J$. In our case, the orbifold category of the untwisted sector of the J-equivariant category is just Bantay's premodular tensor category.

Our constructions rely on a geometric realization of the representation category of the Drinfeld double. As a target space, we take the groupoid BX with one object that is associated to the finite group X. Loop space is then modelled by the inertia groupoid, i.e. the functor category

$$\Lambda BX = [B\mathbb{Z}, BX] = X//X$$

which turns out to be equivalent to the action groupoid for the adjoint action of X on itself. The representation category is then the category $[X//X, \mathrm{vect}] \cong \mathcal{D}(X) - mod$ of X-equivariant vector bundles on X. Fusion and braiding are then obtained using the techniques of [5, 8] via pull-push constructions.

For injective boundary map ∂, the construction of the J-equivariant category is based on Schreier theory. It asserts that for finite groups X, J, exact sequences of groups

$$0 \to X \to G \to J \to 0$$

with a fixed set-theoretic section $s : J \to G$ are in bijection with *weak* two-functors from BJ to the automorphism 2-category $\mathrm{AUT}(BX)$. In this way, we get (categorified) J-actions on the target space BX, the configuration space $X//X$ and finally on the category $[X//X, \mathrm{vect}] \cong \mathcal{D}(X) - mod$ of vector bundles. Orbifoldizing by this action gives Bantay's category in the case of injective boundary map. Twisted sectors are obtained by constructing vector bundles on twisted loop spaces.

They turn out to be vector bundles over twisted action groupoids $X//^jX$, where the adjoint action is twisted by the automorphism $j \in \mathrm{AUT}(X)$ to $\mathrm{Ad}_x^j(y) := xyj(x)^{-1}$.

Our results fit into the general picture developped in [3]: J-equivariant categories with neutral component $X//X - mod$ are in bijection to morphisms to the Brauer-Picard 2-group $BrPic(X//X - mod)$. The latter has as objects invertible module categories over the modular tensor category $[X//X, \mathrm{vect}]$, as 1-morphisms invertible functors of module categories and as 2-morphisms invertible natural transformations of such functors. Our construction amounts to a factorization

$$\begin{array}{ccc} J & \longrightarrow & BrPic(X//X - mod) \\ & \searrow \quad \nearrow & \\ & \mathrm{AUT}(BX) & \end{array}$$

where the left arrow is the categorified action and the right arrow maps $j \in \mathrm{AUT}(X)$ to the module category $X//^jX - mod$ over the tensor category $X//X - mod$.

The general case uses 2-gerbes on the groupoid BX and implements an action of J by 2-gerbe endomorphisms. In the case of surjective boundary map, the equivariance group acts by tensoring with equivariant line bundles over X.

Our results illustrate in particular the usefulness of higher categorical notions and techniques (as developped e.g. in [5, 8]) for algebraic constructions.

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Deformations of algebras and their diagrams

Martin Markl

We will work over a fixed characteristic zero field $\mathbf{k}$. Everyone knows that deformations of an associative algebra (A, μ) are *controlled* by the Hochschild cohomology. By 'controlled by' we usually mean that

– $H^1(A, A)$ classifies infinitesimal deformations and

– $H^2(A, A)$ contains obstructions for their extensions.

But more is true: the Hochschild cochain complex $C^*(A, A)$ carries the *Gerstenhaber bracket* $[-, -]$ which turns it into a dg-Lie algebra

$$\mathfrak{g} := (C^*(A, A), [-, -], \delta).$$

Let $L := \mathfrak{g} \otimes (t) \subset \mathfrak{g} \otimes \mathbf{k}[[t]]$. If one considers the solutions of the *Maurer-Cartan equation* with the associated Lie group

$$\mathrm{MC}(\mathfrak{g}) := \{s \in L^1;\ \delta s + \frac{1}{2}[s, s] = 0\},\quad \mathrm{G}(\mathfrak{g}) := \exp(L^0),$$

then the moduli space of formal deformations of μ equals the quotient $\mathfrak{Def}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g})/\mathrm{G}(\mathfrak{g})$.

Our aim is to show that the same scheme holds for a wide class of algebras and their diagrams, though instead of dg-Lie one sometimes needs an L_∞*-algebra*. We will show how to construct, for a (diagram of) algebra(s) A belonging to a specified class of structures, an L_∞-algebra $\mathfrak{g} = (C^*(A, A), \delta = l_1, l_2, \ldots)$ governing its deformations.

We will focus on explicit calculations and examples. We, in particular, show that deformations of morphisms are controlled by a fully-fledged L_∞-structure. We give an example where a 'curved' (= with l_0-term) L_∞-algebra occurs. We also demonstrate that L_∞-deformation algebras are crucial for deformations of exotic structures.

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Let us explain some points mentioned above. By a 'class of structures' we mean algebras over a (colored, in the case of diagrams) $\mathbf{k}$-vector space operad $\mathcal{P}$. The construction of the L_∞-deformation complex

$$\mathfrak{g} = (C^*(A, A), \delta = l_1, l_2, \ldots)$$

goes in two steps:

Step 1: Finding the *minimal*, or at least *cofibrant*, model $\alpha : (\mathbb{F}(E), \partial) \to (\mathcal{P}, 0)$ of the operad $\mathcal{P}$. By definition, α is a homology isomorphism, $\mathbb{F}(E)$ the free operad on a collection E, and the minimality means that $\partial(E)$ consists of decomposable elements of $\mathbb{F}(E)$. This step is nontrivial. A rich theory of minimal models exists, but it will not be discussed here.

Step 2: The minimal model determines $\mathfrak{g}$ via a straightforward procedure which will be illustrated on several examples.

Conclusion. By analyzing examples and observing some general rules, we conclude that:

– Deformations of a single reasonable algebra are always governed by a dg-Lie algebra. Here 'reasonable' means an algebra over a Koszul quadratic operad, while the anthropic principle says that all algebras one might find in Nature are of this type. For algebras over a non-Koszul operad the deformation complex, however, carries nontrivial higher l_n's.

– The situation dramatically changes when one considers deformations of *diagrams* of algebras. The corresponding deformation complex carries nontrivial higher operations even for the diagram consisting of a single morphism between algebras.

– The situation develops even further when one considers diagrams with loops. The L_∞-deformation complex then contains a non-trivial *curvature* term l_0.

History and references. The classical deformation theory for associative algebras was worked out in a series of seminal papers [2, 3, 4]. The standard reference for operads is [10], their minimal models were introduced in [7]. The standard reference for L_∞-algebras is [5].

The deformation cohomology based on a resolution of the corresponding operad was first considered in the proceedings [6] of the Winter School 'Geometry and Physics,' Zdíkov, Bohemia, January 1993. The L_∞-deformation complex was constructed by van der Laan in [11]. The explicit description used in the talk was obtained in [8], its colored version in [1]. An example of the L_∞-deformation complex of a non-Koszul algebra was given in [9].

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The Deformation Philosophy - An Example: Atomism as Quantization

Jürg Fröhlich

20^{th} century physics has been characterized by the confirmation of a paradigm and three revolutions. The paradigm is the atomistic structure of matter, the revolutions are quantum theory and the relativity theories. It is argued that all the new theories can be viewed as arising from precursor theories by deformations (of kinematical algebras and/or symmetries). This point of view is illustrated by showing (on examples) that certain atomistic theories of matter arise by quantization of continuum theories of matter. As an example, it is shown that the Newtonian mechanics of identical point particles interacting through 2-body potentials can be viewed as the quantization of Vlasov theory.

Emergent (noncommutative) geometry and gravity from Yang-Mills Matrix Models

Harold Steinacker

A introductory review to emergent noncommutative gravity within Yang-Mills Matrix models is presented. Space-time is described as a noncommutative brane solution of the matrix model, i.e. as submanifold of R^D. Fields and matter on the brane arise as fluctuations of the bosonic resp. fermionic matrices around such a background, and couple to an effective metric interpreted in terms of gravity.

Our starting point is the identification of a gravity sector within noncommutative gauge theory. NC gauge theory has been considered previously as a deformation of Yang-Mills gauge theory, living on NC space. From that point of view, it is well-known that the $U(1)$ sector of $U(n)$ gauge theory on the Moyal-Weyl quantum plane R^n_θ (which is the simplest example of a NC space) plays a special role: it does not decouple from the remaining $SU(n)$ degrees of freedom, and its quantum effective action is drastically different from its commutative counterpart due to UV/IR mixing. These and other "strange" features have been viewed as obstacles for the physical application of NC gauge theory, and a relation to gravity has been widely conjectured.

In order to have a well-defined framework, we will focus on matrix models of Yang-Mills type. These models have non-commutative spaces or space-time as solutions, i.e. quantized Poisson manifold. Thus space-time and geometry are dynamical rather than put in by hand, and the models should be considered as background independent. $U(1)$ fluctuations of the matrices around NC space-time describe geometrical deformations such as gravitons, while $SU(n)$ fluctuations describe nonabelian gauge fields. The kinetic terms of these fields arises from the commutators in the matrix model, and encodes an effective metric which is essentially universal for all fields and matter. Since this metric is dynamical, it must be interpreted in terms of gravity. This leads to an intrinsically non-commutative mechanism for gravity, combining the metric and the Poisson structure in a specific way. It provides a natural role for non-commutative or quantized space-time in physics.

Space-time is described as a 3+1-dimensional NC brane $M_\theta \subset R^D$ (possibly with compactified extra dimensions), which carries a Poisson tensor $\theta^{\mu\nu}(x)$. All matter and gauge fields live on this space-time brane, and there are no physical fields propagating in the ambient D-dimensional space unlike in other braneworld scenarios. An effective metric $G^{\mu\nu} \sim \theta^{\mu\mu'}\theta^{\nu\nu'}g_{\mu'\nu'}$ arises on this space-time brane, which governs the kinetic term of all fields more-or-less as in general relativity (GR). This metric is dynamical, however it is not a fundamental degree of freedom: it is determined by the embedding $M_\theta \subset R^D$, and the Poisson tensor $\theta^{\mu\nu}$ describing noncommutativity. Hence the fundamental degrees of freedom are different from GR, and can be interpreted alternatively in terms of NC gauge theory. This makes the dynamics of emergent NC gravity somewhat difficult to disentangle, and the effective metric is not governed in general by the Einstein equations.

Furthermore, we will identify higher-order terms in the matrix model which incorporate the intrinsic curvature of the NC manifold, similar to the Einstein-Hilbert action.

We will identify 2 classes of solutions: in the "Einstein branch", solutions of the Einstein equations can be realized as embedded submanifolds for $D \geq 10$. Since the Einstein-Hilbert action arises upon quantization and is not part of the bare matrix model, the model must be free of UV/IR mixing above a scale Λ identified as Planck scale. This singles out the IKKT model [4] or close relatives, with $D = 10$ and maximal supersymmetry above Λ. In contrast, the solutions in the "harmonic branch" are governed by the brane tension rather than the induced Einstein-Hilbert term.

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Progress in Solving a 4 dimensional NCQFT

Harald Grosse
(joint work with Raimar Wulkenhaar)

This report is based on [1].

In previous work [3] we have proven that the following action functional for a ϕ^4-model on four-dimensional Moyal space gives rise to a renormalisable quantum field theory:

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi \Big)(x) \ . \tag{1}$$

Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation and becomes self-dual at $\Omega = 1$. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [6, 7]. The vanishing of the β-function at $\Omega = 1$ was next proven in [8] at three-loop order and finally by Disertori, Gurau, Magnen and Rivasseau [9] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible. The Landau ghost problem is solved.

The action functional (1) is most conveniently expressed in the matrix base of the Moyal algebra [3]. For $\Omega = 1$ it simplifies to

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi) \ , \tag{2}$$

$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big) \ , \qquad V(\phi) = \frac{Z^2\lambda}{4} \sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} \ , \tag{3}$$

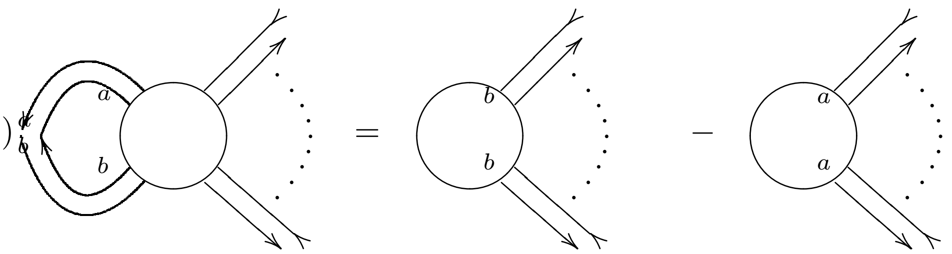
The model only needs wavefunction renormalisation $\phi \mapsto \sqrt{Z}\phi$ and mass renormalisation $\mu_{bare} \to \mu$, but no renormalisation of the coupling constant [9] or of $\Omega = 1$. All summation indices $m, n, \ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$, and $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size.

The key step in the proof [9] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs one special insertion vertex

(4) $$V_{ab}^{ins} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$$

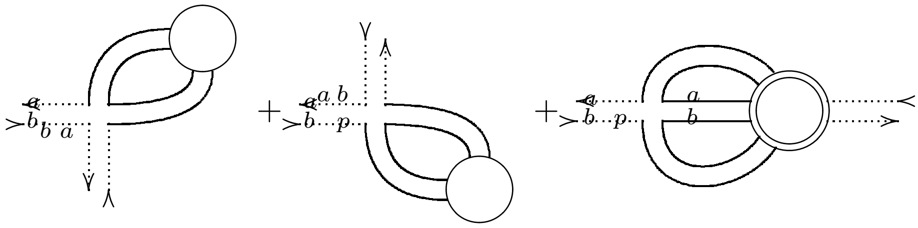
is the same as the difference of graphs with external indices b and a, respectively, $Z(|a|-|b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$:

(5) $Z(|a|-|b|)$



The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

(6) $\Gamma_{ab} =$



The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

(7) $$\Gamma_{ab} = Z^2\lambda \sum_p \Big(G_{ap} + G^{-1}_{ab} G^{ins}_{[ap]b}\Big) = Z^2\lambda \sum_p \Big(G_{ap} - G^{-1}_{ab} \frac{G_{bp} - G_{ba}}{Z(|p|-|a|)}\Big)$$
$$= Z^2\lambda \sum_p \Big(\frac{1}{H_{ap} - \Gamma_{ap}} + \frac{1}{H_{bp} - \Gamma_{bp}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p|-|a|)}\Big)\,.$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare} in H (3). Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (7) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2 \frac{\alpha}{1-\alpha}$, $|p| =: \mu^2 \frac{\rho}{1-\rho}$ and

(8) $$\Gamma^{ren}_{ab} =: \mu^2 \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \Big(1 - \frac{1}{G_{\alpha\beta}}\Big)\,,$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [1]:

Theorem 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$
\begin{aligned}
G_{\alpha\beta} = 1 + \lambda\Big(& \frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \\
& + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\
& - \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Big) ,
\end{aligned}
\tag{9}
$$

where $\alpha, \beta \in [0,1)$,

$$
\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \quad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho} , \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} ,
$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function [1].

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. We obtain

$$
\begin{aligned}
G_{\alpha\beta} = 1 &+ \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\
&+ \lambda^2\Big\{AB\big((I_{\substack{\alpha\\ \bullet}} - \alpha) + (I_{\substack{\beta\\ \bullet}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)\big) \\
&\quad + A\big(\beta I_{\substack{\beta\\ \bullet}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
&\quad + B\big(\alpha I_{\substack{\alpha\\ \bullet}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big)\Big\} + \mathcal{O}(\lambda^3) ,
\end{aligned}
\tag{10}
$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

$$
\begin{aligned}
I_\alpha &:= \int_0^1 dx\, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha) , \\
I_{\substack{\alpha\\ \bullet}} &:= \int_0^1 dx\, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2 .
\end{aligned}
\tag{11}
$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labelled by rooted trees.

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Courant bracket contractions and generalized geometries

Janusz Grabowski

Dirac structures on manifolds provide a geometric setting for Dirac's theory of constrained mechanical systems. To formulate the integrability condition defining Dirac structure, Courant [3] introduced a natural skew-symmetric bracket operation on sections of $\mathcal{T}M = TM \oplus T^*M$. The Courant bracket does not satisfy the Leibniz rule with respect to multiplication by functions nor the Jacobi identity. These defects disappear upon restriction to a Dirac subbundle because of the isotropy condition. Particular cases of Dirac subbundles are graphs of closed 2-forms (presymplectic forms) and Poisson bivector fields on M.

The nature of the Courant bracket itself remained unclear until several years later when it was observed by Liu, Weinstein and Xu [10] that $\mathcal{T}M$ endowed with the Courant bracket plays the role of a 'double' object, in the sense of Drinfeld [5], for a pair of Lie algebroids over M. Thus, in complete analogy with Drinfeld's Lie bialgebras, in the category of Lie algebroids there also exist 'bi-objects', Lie bialgebroids, introduced by Mackenzie and Xu [9] as linearizations of Poisson groupoids. On the other hand, every Lie bialgebra has a double which is a Lie algebra. This is not so for general Lie bialgebroids. Instead, Liu, Weinstein and

Xu [10] show that the double of a Lie bialgebroid is a more complicated structure they call a *Courant algebroid*, $\mathcal{T}M$ with the Courant bracket being a special case.

There is also another way of viewing Courant algebroid as a generalization of Lie algebroid. This requires a change in the definition of the Courant bracket to produce its non-skewsymmetric version, the *Courant-Dorfman bracket*, so that the traditional Courant bracket becomes skew-symmetrization of the new one [4, 12]. This change replaces one of the defects with another one: a version of the Jacobi identity is satisfied, while the bracket is no-longer skew-symmetric. Such algebraic structures have been introduced by Loday [11] under the name *Leibniz algebras*, but they are often called now *Loday algebras*. This approach allows us to describe the corresponding Courant-Dorfman bracket as a derived bracket of a symplectic Poisson bracket in a supergeometric setting [12].

On any Courant algebroid E one can consider analogs of Nijenhuis tensors, as the latter concept can be applied to a large class of algebraic structures on vector bundles [1, 2]. Nijenhuis tensors N which are compatible with the symmetric form on E being a part of the Courant algebroid structure are called *Corant-Nijenhuis tensors* and they lead naturally to contractions of the Courant-Dorfman bracket. Being a Courant-Nijenhuis tensor is a strong condition which, in the case of the classical Courant algebroid $\mathcal{T}M$, forces that any orthogonal Courant-Nijenhuis tensor has its square proportional to the identity. We therefore distinguish three main cases: $N^2 = -I$, $N^2 = I$, and $N^2 = 0$. They are Courant analogs of complex, product, and tangent structures on the manifold M. The case $N^2 = -I$ has been introduced by Hitchin and studied by his student Gualtieri [7, 8] under the name *generalized complex structure*. This generalized complex geometry unifies complex geometry with symplectic geometry, like the concept of a Dirac structure unifies Poisson geometry with the presymplectic one.

All this can be presented in a supergeometric setting in which a Courant algebroid is a graded supermanifold equipped with a homological vector field [12]. Generalized geometries have also a nice supergeometric description [6].

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Twist deformation quantization, noncommutative gravity and Einstein spaces

Paolo Aschieri

The study of the structure of spacetime at Planck scale, where quantum gravity effects are non-negligible, is one of the main open challenges in fundamental physics. Since the dynamical variable in Einstein general relativity is spacetime itself (with its metric structure), and since in quantum mechanics and in quantum field theory the classical dynamical variables become noncommutative, one is lead to conclude that noncommutative spacetime is a feature of Planck scale physics. A first question to be asked in this context is whether one can consistently deform Riemannian geometry into a noncommutative Riemannian geometry. In [1] we address this question by considering deformations of the algebra of functions on a manifold obtained via a quite wide class of $\star$-products. In this framework we successfully construct a noncommutative version of differential and of Riemannian geometry, and we obtain the noncommutative version of Einstein equations. In [2] we show existence and uniqueness of the Levi-Civita connection on a noncommutative Riemannian manifold, and study a class of noncommutative Einstein manifolds.

The $\star$-products we consider are associated with a deformation by a twist $\mathcal{F}$ [3] of the Lie algebra of infinitesimal diffeomorphisms on a smooth manifold M . Since $\mathcal{F}$ is an arbitrary twist, we can consider it as the dynamical variable that determines the possible noncommutative structures of spacetime. Examples of the noncommutative spacetime structures we obtain include the Moyal-Weil (or θ-constant) noncommutative space and the quantum (hyper)plane $xy = qyx$. The twists $\mathcal{F}$ is an element $\mathcal{F} \in U\Xi \otimes U\Xi$, where $U\Xi$ is the universal enveloping algebra of the Lie algebra of vector fields (infinitesimal local diffeomorphisms). Since vector fields act on functions, forms and tensor fields, using the twist $\mathcal{F}$ we canonically deform these spaces into the $\star$-noncommutative spaces of functions, forms and tensor fields. The Lie algebra of vector fields is similarly deformed to a $\star$-Lie algebra: a quantum Lie algebra in the spirit of [4]. Furthermore we show that this deformed Lie algebra has a deformed action on the noncommutative spaces of functions, forms and tensor fields. We have thus constructed a tensor calculus that is covariant under infinitesimal noncommutative diffeomorphisms. In the special case of θ-constant noncommutativity, if we choose the preferred coordinate system $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ we recover the results of [5]. Next the $\star$-covariant derivative is then defined in a global coordinate independent way. Locally the $\star$-covariant derivative is completely determined by its coefficients $\Gamma^{\rho}_{\mu\nu}$. Using the deformed Leibniz rule for vector fields we extend the $\star$-covariant derivative to all type of tensor fields. Having the covariant derivative it is easy to guess the expression for the noncommutative curvature and torsion. Then one has to show that these operators are well defined noncommutative tensors. This is done by showing that they are (left) $A_\star$ -linear maps on vector fields, where $A_\star$ is the space of noncommutative functions. Also the noncommutative Ricci tensor is singled out by requiring it to be a (left) $A_\star$ -linear map. Finally the metric is an arbitrary $\star$-symmetric element in the $\star$-tensor product of 1-forms $\Omega \otimes_\star \Omega_\star$. The scalar curvature can then be defined and Einstein equations on $\star$-noncommutative space are obtained. The requirement of $A_\star$-linearity fixes the possible ambiguities arising in the noncommutative formulation of Einstein gravity theory.

This noncommutative gravity is a minimal deformation of usual gravity, based on considering a noncommutative spacetime structure and implementing the equivalence principle via noncommutative general covariance.

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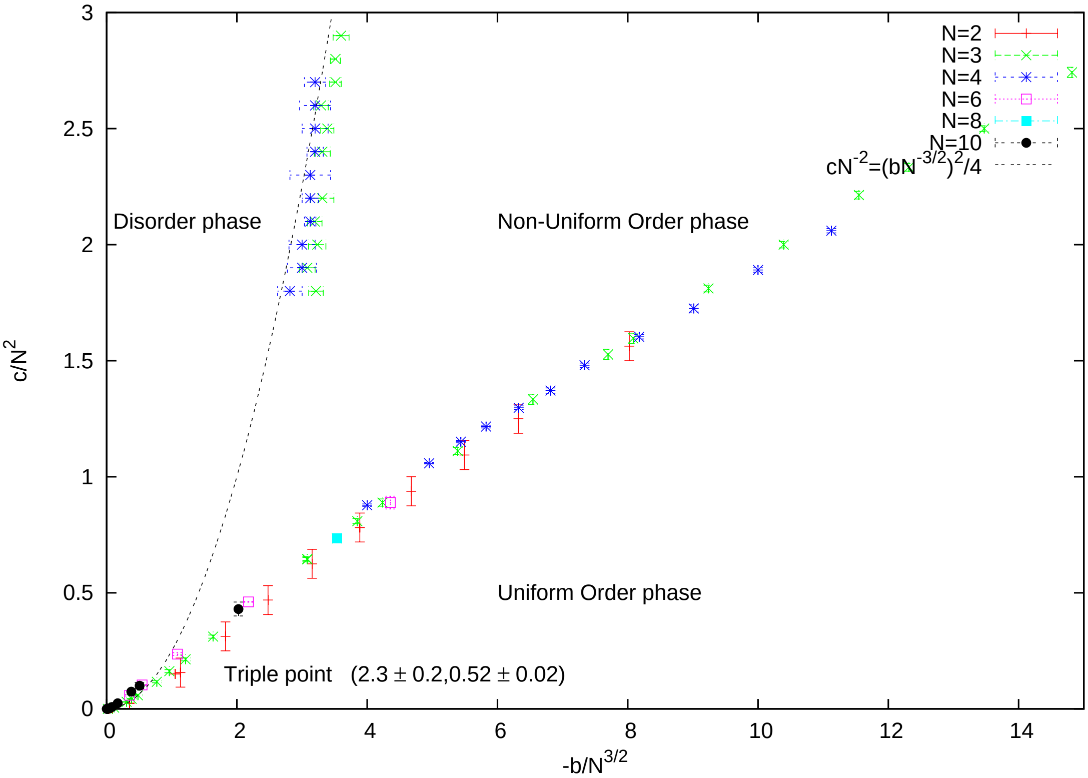
Equivariant Vector Bundles on Fuzzy Spaces

Denjoe O'Connor

Fuzzy spaces are usually taken to be sequences of finite dimensional matrix approximations to the algebra of functions of a commutative space. The geometry is imposed on the matrix algebra by prescribing either a Dirac or Laplace-Beltrami operator. There is now a large variety of such spaces [1, 2], mostly based on adjoint orbits of classical groups, but many other examples have been constructed [3]. However, the ability to retain rotational and higher symmetries is of significant advantage in applications especially in the non-perturbative study of field theories.

One application of such spaces is as an alternative regularisation of quantum field theory suitable for non-perturbative studies [1]. The standard non-perturbative regularisation is a lattice approximation to the theory. This of necessity breaks spacetime symmetries. The advantage of adjoint orbit fuzzy spaces is that they typically preserve the spacetime symmetries. It is also hoped that such models can guide us towards new models for the microstructure of spacetime.



The simplest field theory is that of a real scalar field. When regulated on a fuzzy space it becomes a Hermitian matrix model in the presence of external fixed matrices. For the fuzzy sphere the Euclidean action is given by $S_{\rm N}(\Phi; a, b, c) = Tr(-a[L_a, \Phi]^2 + b\Phi^2 + c\Phi^4)$ where L_a are generators in the N dimensional irreducible representation of $su(2)$ and Φ an N $\times$ N matrix. The Euclidean Field theory for this model is given by the probability measure

$$\mu(\Phi) = \frac{\mathrm{e}^{-S_{\rm N}(\Phi)}}{Z} d[\Phi] \qquad \text{where} \qquad Z = \int_{\mathrm{Mat_N}} [d\Phi]\mathrm{e}^{-S_{\rm N}(\Phi)} \ .$$

The action $S_N(\Phi; a, b, c)$ converges for $N \to \infty$ to the action of a scalar field ϕ on the round commutative sphere $S(\phi, r, \lambda) = \int_{S^2} d^2x\sqrt{g}(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{r}{2}\phi^2 + \frac{\lambda}{4!}\phi^4)$ provided $\Phi = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} c_{lm}\hat{Y}_{lm}$ where $\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_{lm}$ prescribes the commutative field to be approximated, with $\hat{Y}_{lm}$ polarization tensors (eigenvectors of the fuzzy Laplacian) and Y_{lm} are spherical harmonics. More precisely one has:

$$\lim_{\mathrm{N}\to\infty} \left| S(\phi, r, \lambda) - S_{\rm N}(\Phi, \frac{1}{2\mathrm{N}}, \frac{r}{2\mathrm{N}}, \frac{\lambda}{4!\mathrm{N}}) \right| \to 0 \ .$$

In the fluctuating theory with the measure $\mu(\Phi)$ the physics is richer than in the commutative case. The phase diagram of the model (found by Monte Carlo simulation [4]) is shown in the figure. A full theoretical determination of the phase lines is still elusive, though, there has been progress in recent perturbative efforts to calculate the transition lines [5]. The phase diagram is controlled by the triple point (a Lifshitz point). To recover the commutative theory we must add a term to modify the action which moves the triple point off to infinity in both variables.

To describe fuzzy spinor and gauge field theories it is useful to have a construction of fuzzy vector bundles that can serve as a guide to the construction of suitable action functionals. This can be achieved by following a Fock space construction along the lines of lines of [6] where equivariant line bundles for S^2_F were constructed. The construction [7] uses the Fock space generated by the $N(N+1)$ oscillators ${a^\dagger}_\imath^\alpha = (a^\imath_\alpha)^\dagger$, denoted $\mathcal{F}^{Total}$. These oscillators carry the anti-fundamental representation of $u(N+1)$ and the fundamental representation of $u(N)$. The $u(N+1)$ generators $\hat{J}^\alpha_\beta = {a^\dagger}_\imath^\alpha a^\imath_\beta$ and $u(N)$ generators $\hat{J}_\imath{}^\jmath = {a^\dagger}_\imath^\alpha a^\jmath_\alpha$ mutually commute and have the common $u(1)$ generator $\hat{N} = {a^\dagger}_\imath^\alpha a^\imath_\alpha = N\hat{\mathcal{N}}$. The Fock space $\mathcal{F}^{Total}$ carries a representation of $su(N+1)\times su(N)\times u(1)$ and $\mathcal{F}^{Total} = \oplus_{\mathcal{R}}\mathcal{F}_{\mathcal{R}}$, where the sum is over all irreducible representations $\mathcal{R}$ of $u(N)$. Identical oscillators and Schur-Weyl duality guarantee that each representation occurs precisely once in the decomposition.

The $su(N)$ singlet representations are generated by the pseudo-oscillators

$$A^\dagger_\alpha := \frac{1}{\sqrt{c_N(\hat{\mathcal{N}})}}\frac{1}{N!}\epsilon_{\overline{\alpha}\overline{\theta}_1\cdots\overline{\theta}_N}\epsilon_{\imath_1\cdots\imath_N}a^{\imath_1}_{\theta_1}\cdots a^{\imath_N}_{\theta_N} \qquad \text{with} \qquad c_N(L) = \frac{(N+L-1)!}{L!}\ .$$

$A^\dagger_\alpha$ and A^α *obey the Heisenberg commutation relations* on reduced Fock space (the subspace of singlet representations) $\mathcal{F}$, which is orthogonal to the remainder so that we have $\mathcal{F}^{Total} = \mathcal{F}\oplus\mathcal{F}^\perp$. and the space $\mathcal{F}^\perp$ can further be decomposed under $su(N)$ with the leading representation the fundamental of $su(N)$, carried by the index $\imath$ on a single oscillator ${a^\dagger}_\imath^\alpha$.

The natural generalization of operators introduced in [6] is then given by

$$\begin{array}{rclcrcl}
\hat{K}_\imath := & (A^\dagger_\alpha)^{\mathrm{L}}((a^\dagger)_\imath^\alpha)^{\mathrm{R}} & : & \mathcal{F}_L\otimes\mathcal{F}^*_L & \longmapsto & \mathcal{F}_{L+1}\otimes\mathcal{F}^*_{L,\imath} \\
\hat{K}_{\bar\imath} := & (A^\alpha)^{\mathrm{L}}(a^\imath_\alpha)^{\mathrm{R}} & : & \mathcal{F}_L\otimes\mathcal{F}^*_L & \longmapsto & \mathcal{F}_{L-1}\otimes\mathcal{F}^*_{L,\bar\imath} \\
\hat{K}_0 := & \frac{1}{2}(\hat{\mathcal{N}}^{\mathrm{L}}_A - \hat{\mathcal{N}}^{\mathrm{R}}) & : & \mathcal{F}_L\otimes\mathcal{F}^*_L & \longmapsto & 0\ ,
\end{array}$$

where $\hat{\mathcal{N}}_A = A^\dagger_\alpha A^\alpha$ and we have denoted the subspace of Fock space spanned by vectors of the form $(a^\dagger)_\imath^\alpha A^\dagger_{\alpha_1}\dots A^\dagger_{\alpha_L}|0\rangle$ by $\mathcal{F}_{L,\imath} = \mathcal{F}^{\bar\imath}_L$.

The algebra $\mathcal{F}_L\otimes\mathcal{F}^*_L$ is annihilated by $[\hat{K}_\imath,\hat{K}_{\bar\jmath}]$ and non-square matrices $\mathbf{M}_q \in \mathcal{F}_L\otimes\mathcal{F}^*_{L-q}$ represent equivariant line bundles, when the geometry is specified by the Laplacian $\Delta_K = \frac{1}{2}\left(\hat{K}_\imath\hat{K}_{\bar\imath} + \hat{K}_{\bar\imath}\hat{K}_\imath\right) + \frac{2N}{N+1}\hat{K}^2_0$. We can re-express Δ_K in the form $\Delta_K = (\hat{L}^{\mathrm{L}}_a - \hat{J}^{\mathrm{R}}_a)^2$, where $\hat{L}_a = A^\dagger\frac{\lambda_a}{2}A$ and $\hat{J}_a = (a^\dagger)_\imath\frac{\overline{\lambda_a}}{2}a^\imath$. With the Laplacian $\Delta_K\mathbf{M}_{\mathcal{R}} = (\hat{L}^{\mathrm{L}}_a - \hat{J}^{\mathrm{R}}_a)^2\mathbf{M}_{\mathcal{R}} = (\hat{J}^{\mathrm{L}}_a - \hat{J}^{\mathrm{R}}_a)^2\mathbf{M}_{\mathcal{R}}$, where $(\hat{J}^{\mathrm{L}}_a - \hat{J}^{\mathrm{R}}_a)^2$ is the $su(N+1)$ quadratic Casmir. Equivariant vector bundles over $\mathbb{CP}^N_F$ are represented by $\mathbf{M}_{\mathcal{R}}\in\mathcal{F}_L\otimes\mathcal{F}^*_{\mathcal{R}}$ where $\mathcal{R}$ is any irreducible representation of $u(N)$ The eigenspaces of Δ_K are the irreducible representations in the decomposition of $\mathcal{F}_L\otimes\mathcal{F}^*_{\mathcal{R}}$. The $su(N+1)$ quadratic Casimir in this representation give the eigenvalues.

The Fermionic oscillators $\gamma^{\imath}$ and $\gamma^{\bar\imath} = (\gamma^{\imath})^\dagger$ have vacuum $|\,\Omega\,\rangle$ and generate the Clifford algebra $\{\gamma^{\imath},\, \gamma^{\bar\jmath}\} = \delta^{\imath\bar\jmath}$. A universal massless Dirac operator [8] for $\mathbb{CP}^N_F$ is

$$\not{D} := \gamma^{\bar\imath}\hat K_{\bar\imath} + \gamma^{\imath}\hat K_{\imath} = (A^\alpha)^{\mathrm{L}}(a^{\imath}_{\alpha})^{\mathrm{R}}\gamma^{\bar\imath} + (A^\dagger_\alpha)^{\mathrm{L}}(a^{\dagger}{}^{\alpha}_{\imath})^{\mathrm{R}}\gamma^{\imath}\ ,$$

with a noncommutative spinor on $\mathbb{CP}^N_F$ given by

$$\Psi = \sum_{k=0}^{n}\frac{1}{k!}\boldsymbol{\psi}_{\bar\imath_1\cdots\bar\imath_k}\gamma^{\bar\imath_1}\cdots\gamma^{\bar\imath_k}\,|\,\Omega\,\rangle = \sum_{k=0}^{n}\frac{1}{k!}\mathcal{F}_{L-k}\otimes\mathcal{F}^*_{R,\bar\imath_1\cdots\bar\imath_k}\gamma^{\bar\imath_1}\cdots\gamma^{\bar\imath_k}\,|\,\Omega\,\rangle$$

where $\boldsymbol{\psi}_{\bar\imath_1\cdots\bar\imath_k}$ are equivariant fuzzy vector bundles as described above. In field theory these will have coefficients which are Grassmann variables. The construction gives spinors for odd N and *spin*^{c} more generally. The spectrum of $\not{D}$ coincides precisely with the commutative one except that the higher modes are cutoff.

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Formality and free algebras

Olivier Elchinger

(joint work with Martin Bordemann, Abdenacer Makhlouf, Simone Gutt)

An associative algebra is called formal if its Hochschild complex equipped with the Gerstenhaber graded Lie structure is quasi-isomorphic in the L_∞ sense to its Hochschild cohomology.

In 1997, in his paper [9] on deformation quantization on a Poisson manifold, Kontsevich showed that the symmetric algebra of a vector space is formal.

We consider here the case of free algebras. We will show that except in dimension 0 and 1, free algebras are not formal.

1. Definitions

We consider an associative algebra $(\mathcal{A},\mu_0)$ over $\mathbb{K}$ with $\mathrm{car}(\mathbb{K})=0$.
We consider the Hochschild cochains of degree k :

$$\mathfrak{A}^0:=\mathcal{A},\quad \mathfrak{A}^k:=\mathbf{C}^k_H(\mathcal{A},\mathcal{A})=\mathrm{Hom}(\mathcal{A}^{\otimes k},\mathcal{A}),\quad \mathfrak{A}:=\bigoplus_{k=0}^{\infty}\mathfrak{A}^k$$

Equipped with the Gerstenhaber bracket, the shifted space $(\mathfrak{A}[1],[\ ,\ ]_G)$ is a graded Lie algebra.

We consider the differential $b=[\mu,\ ]_G$, the Hochschild cohomology groups : $\mathfrak{a}^k:=\mathbf{H}^k_H(\mathcal{A},\mathcal{A})$ and $\mathfrak{a}:=\bigoplus_{k\in\mathbb{Z}}\mathfrak{a}^k$. We call ϕ an HKR map if :

$$\mathfrak{A}\supset Z\mathfrak{A}\underset{\phi}{\overset{p}{\rightleftarrows}}\mathfrak{a}\quad\text{with}\quad b\phi=0\quad\text{and}\quad p\circ\phi=id_{\mathfrak{a}} \tag{1}$$

For $\xi,\eta\in\mathfrak{a}$, we have

$$\phi([\xi,\eta]_s)=[\phi(\xi),\phi(\eta)]_G+b\big(\phi_2(\xi,\eta)\big) \tag{2}$$

so that ϕ is not a morphism of graded Lie algebras $(\mathfrak{a}[1],[\ ,\ ]_s)\to(\mathfrak{A}[1],[\ ,\ ]_G)$ in general.

Definition 1.1 ([9])**.** $(\mathcal{A},\mu_0)$ is called *formal* if there is a morphism of differential graded coalgebras (of degree 0) $\Phi:\mathcal{S}(\mathfrak{a}[2])\to\mathcal{S}(\mathfrak{A}[2])$ such that the restriction Φ_1 of Φ to $\mathfrak{a}[2]$ is an HKR map. Φ is called a formality map or an L_∞-morphism.

Proposition 1.2 ([2])**.** *We consider a sequence of linear maps $\{\phi_k\}_{k\in\mathbb{N}^\star}$, with $\phi_k:\mathfrak{a}[1]^{\otimes k}\to\mathfrak{A}[1]$, satisfying*

(i) ϕ_1 is an HKR map
(ii) ϕ_k is of degree $1-k$
(iii) ϕ_k is graded antisymmetric

and such that for each $k\in\mathbb{N}$, $\forall\ x_1,\dots,x_{k+1}\in\mathfrak{a}$

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant k+1}&\epsilon_{ij}(x_1,\dots,x_{k+1})\phi_k([x_i,x_j]_s,x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+1})\\ &=(-1)^{\frac{k(k+1)}{2}}b\phi_{k+1}(x_1,\dots,x_{k+1})\\ +\frac{1}{2}\sum_{a=1}^{k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant k+1}&\omega_a(x_1,\dots,x_{k+1})\cdot\\ &[\phi_a(x_{i_1},\dots,x_{i_a}),\phi_{k-a+1}(x_1,\dots,\hat{x_{i_1}},\dots,\hat{x_{i_a}},\dots,x_{k+1})]_G,\end{aligned} \tag{3}$$

where ε_{ij} and ω_a are signs dependant on the x_i.
Then $\Phi=\sum_{k\geqslant 1}\phi_k$ is a formality map.

2. Settings for free algebras

Let V be a $\mathbb{K}$-vector space, $\mathcal{A} := TV = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ the free algebra on V.

For $\dim V = 0$, $TV = \mathbb{K}$ is formal, and for $\dim V = 1$, $TV \cong \mathbb{K}[x]$ is formal too.

Theorem 2.1. *For* $\dim V \geqslant 2$, *the Hochschild cohomology of* TV *is*

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}^0 \oplus \mathfrak{a}^1 \\ &= TV^{TV} \oplus \mathrm{Der}(TV, TV)/\,\mathrm{Inder}(TV, TV) \\ &= \mathbb{K}1 \oplus \mathrm{Hom}_{\mathbb{K}}(V, TV)/TV^+. \end{aligned} \tag{4}$$

Lemma 2.2. *If* TV *is formal, then* ϕ_k *have to vanish for* $k \geqslant 3$.

We have $TV = TV^+ \oplus \varepsilon(TV)$ with ε the counit. $(\mathrm{Hom}(V, TV), [\ ,\ ]_D)$ is a Lie algebra with $[\psi, \xi]_D = \bar{\psi}\xi - \bar{\xi}\psi$ where $\bar{\psi}$ is the derivation associated to ψ. We decompose $\mathrm{Hom}(V, TV) = \mathcal{H} \oplus TV^+$ by choosing a graded complement $\mathcal{H}^k$ in each degree; we note $P_k : \mathrm{Hom}(V, V^{\otimes k+1}) \to \mathcal{H}^k$ the canonical projection and $Q_k : \mathrm{Hom}(V, V^{\otimes k+1}) \to V^{\otimes k}$ such that $1 - P_k = bQ_k$.

3. Non formality

Let $\phi_1 : \mathfrak{a} \to \mathfrak{A}$ be the HKR map according to this decomposition. To see if TV is formal, we have to verify the equation (3) for $0 \leqslant k \leqslant 2$. The level $k = 0$ is the condition for ϕ_1 to be an HKR map. At the level $k = 1$, working in $\mathcal{H}$, we obtain a condition on ϕ_2 and an arbitrary map $q : \mathcal{H} \wedge \mathcal{H} \to \mathbb{K}$. With this condition, the equation at the level $k = 2$ rewrites

$$\begin{aligned} &\circlearrowleft_{x_1,x_2,x_3} \ \phi_2(P[x_1, x_2]_D, x_3) - [x_1, \phi_2(x_2, x_3)]_D \\ =&\circlearrowleft_{x_1,x_2,x_3} \ q([x_1, x_2]_s, x_3) + \varepsilon(x_1(Q[x_2, x_3]_D)). \end{aligned} \tag{5}$$

For TV to be formal, the right-hand side should vanish.

$$\begin{aligned} \text{Define} \quad \sigma \ &: \wedge^3\mathfrak{a} \to \mathbb{K} \\ (x_1, x_2, x_3) &\mapsto \circlearrowleft_{x_1,x_2,x_3} \varepsilon\big(\mathsf{pr}_{-1}(x_1)(Q[x_2, x_3]_D)\big) \end{aligned}$$

Proposition 3.1. $\circlearrowleft_{x_1,x_2,x_3} \ q([x_1, x_2]_s, x_3) = -\delta_{CE} q$ *is a scalar* 3*-coboundary of the Chevalley-Eilenberg cohomology of* $\mathfrak{a}$.
σ *is a scalar* 3*-cocycle of the Chevalley-Eilenberg cohomology of* $\mathfrak{a}$.

σ is a 3-cocycle but not a 3-coboundary, so there is no map q such that the right-hand side of (5) vanishes. The algebra TV is not formal for $\dim V \geqslant 2$.

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Nichols algebras over nilpotent groups

Simon Lentner

Hopf algebras present a common generalization of groups and Lie algebras, such that representations can still be **tensored**, and appear as natural **deformations** thereof, like in noncommutative spacetime. The most prominent example is $U_q(sl_2)$ and it's truncations acting on the quantum plane, followed by other $U_q(\ell)$.

I am concerned with classifying **pointed** Hopf algebras over $\mathbb{C}$, meaning their (co)semisimple part (coradical) is just a groupring $k[G]$. Like the above, they further contain Lie algebra elements V (primitives) and their enveloping **Nichols algebra** $B(V)$ is deformed by G and thus sometimes happens to break off at finite dimension. Over abelian groups, Scheider and Andruskiewitsch classified in [AS10] most cases, but over the nonabelian groups this seem a rather rare phenomenon.

While there's some progress especially for simple groups (e.g. [MS00], [AFGV09], [AFGV10]), I construct new examples for centrally extended groups $G \leftarrow E \leftarrow H$ (or rule out such) by H-**orbifoldizing** a known Nichols algebra $B(V)$ over G. This is a technique I adopted from Quantum Field Theory, namely the algebra

$$H \to H^2(G, \mathbb{C}^*) \quad \Rightarrow \quad B(\bar{V}) := \left(\bigoplus_{\sigma \in Im\ H} B(V)_\sigma \right)^H$$

is again a Nichols algebra, this time over E! Here we map H to a 2-cocycle subgroup corresponding to the extension, sum over all twists (Bigalois objects, see [S03]) and take the H-stabilizer to eliminate the new coradical $k[Im\ H]$.

I first $\mathbb{Z}_2$-orbifoldized successfully **new examples** like $\mathbb{Z}_2^2 \leftarrow Q_8$ (also the known D_4 in [MS00]) and $S_4 \leftarrow GL_2(\mathbb{Z}_3)$. Another example might be obtained by $\mathbb{Z}_4$-orbifoldizing the only open case $[(123)(45)] \subset S_5$ in [AFGV09] to $GL_2(\mathbb{Z}_5)$.

Currently I aim the **classification of nilpotent** G as my dissertation by inductively (de-)orbifoldizing. This works now for odd orders, where only abelian groups can appear, and also produces new examples over the 2-extraspecial groups. Still, some irregular cases like the quasidihedrals $\tilde{D}_8$ are to be treated. At the conference, we noticed a striking similarity to the concept presented by Prof. Schweigert.

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Quantum Field Theory on Noncommutative Curved Spacetimes

Alexander Schenkel

(joint work with Thorsten Ohl)

Motivations for a NC spacetime geometry are provided by string theory, quantum gravity and Gedanken experiments combining quantum mechanics with general relativity. Recently, there has been a lot of progress in QFT on NC flat space, but only little has been done on QFT on NC curved spacetimes, which however is essential for cosmology and black hole physics. To fill this gap we propose an approach to QFT on NC curved spacetimes [1, 2] based on formal deformation quantization using Drinfel'd twists. Let $(\mathcal{M}, g)$ be a time-oriented, connected and globally hyperbolic Lorentzian manifold and let $\mathcal{F} \in U\Xi[[\lambda]] \otimes U\Xi[[\lambda]]$ be an abelian Drinfel'd twist, where Ξ is the Lie algebra of vector fields on $\mathcal{M}$ and λ is the deformation parameter. We define a deformed action functional for a real scalar field using the formalism of [3]

$$S_\star = -\frac{1}{2}\int_{\mathcal{M}} \big(\langle d\Phi, \langle g^{-1_\star}, d\Phi\rangle_\star\rangle_\star + \Phi\star\Phi\big)\star \mathrm{vol}_\star \ . \tag{1}$$

In [1] we have shown that the deformed wave operator $P_\star$ obtained by varying the action (1) has a unique retarded and advanced Green's operator $\Delta_{\pm\star}$, provided we assume a support condition on $P_\star$. Moreover, we have shown that the space of real solutions of the wave equation $P_\star\Phi = 0$ is isomorphic to the factor space $V_\star := H/P_\star[C_0^\infty(\mathcal{M}, \mathbb{R})[[\lambda]]]$, where

$$H := \big\{\varphi \in C_0^\infty(\mathcal{M})[[\lambda]] : (\Delta_{\pm\star}[\varphi])^* = \Delta_{\pm\star}[\varphi]\big\} \ . \tag{2}$$

The space $V_\star$ can be equipped with a symplectic structure

$$(3)\qquad \omega_\star : V_\star \otimes_{\mathbb{R}} V_\star \to \mathbb{R}[[\lambda]],\ ([\varphi],[\psi]) \mapsto \omega_\star([\varphi],[\psi]) = \int_{\mathcal{M}} \varphi^* \star \Delta_\star[\psi] \star \mathrm{vol}_\star\ ,$$

where $\Delta_\star := \Delta_{+\star} - \Delta_{-\star}$, and can be quantized canonically in terms of the $*$-algebra (over $\mathbb{C}[[\lambda]]$) of field polynomials $\mathcal{A}_{(V_\star,\omega_\star)}$. We have established a $*$-algebra isomorphism $\mathfrak{S} : \mathcal{A}_{(V_\star,\omega_\star)} \to \mathcal{A}_{(V[[\lambda]],\omega)}$ between the NC QFT $\mathcal{A}_{(V_\star,\omega_\star)}$ and the formal power series extension of the commutative QFT $\mathcal{A}_{(V[[\lambda]],\omega)}$. In [2] we have investigated examples of convergent deformations of QFTs and have studied the similarities and differences to formal deformation quantization. Besides the expected features of convergent deformations, e.g. nonlocality, we have found that the relation between the deformed and the undeformed QFT changes. More precisely, we have obtained an injective, but not surjective, $*$-homomorphism $\mathfrak{S} : \mathcal{A}_{(V_\star,\omega_\star)} \to \mathcal{A}_{(V,\omega)}$.

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Gerstenhaber-Schack diagram cohomology from operadic point of view

Martin Doubek

Developped in a series of papers by M. Markl (e.g. [3]), there is a construction of a cohomology theory for any type of algebras described by an operad $\mathcal{A}$. To rougly describe this construction, let A be an algebra over the operad $\mathcal{A}$. This is equivalent to give a morphism $\mathcal{A} \to \mathcal{E}nd_A$, where $\mathcal{E}nd_A$ is the endomorphism operad of the underlying vector space of A. Given a free (or cofibrant) resolution $\mathcal{R} \xrightarrow{\sim} \mathcal{A}$, the composition $\mathcal{R} \to \mathcal{A} \to \mathcal{E}nd_A$ makes $\mathcal{E}nd_A$ an operadic $\mathcal{R}$-module and we can consider the space of operadic derivations $\mathrm{Der}(\mathcal{R}, \mathcal{E}nd_A)$ with a natural differential. Then the operadic cohomology is

$$H^*(A, A) = H^*(\mathrm{Der}(\mathcal{R}, \mathcal{E}nd_A))$$

In fact this is André-Quillen cohomology, but performed on the level of operads and with coefficients in $\mathcal{E}nd_A$. In standard case, one recovers Hochschild cohomology ($\mathcal{A} = \mathcal{A}ss$), Chevalley-Eilenberg cohomology ($\mathcal{A} = \mathcal{L}ie$) and many others.

Of course, the problem is to construct $\mathcal{R}$ explicitly and as small as possible. In many cases of interest, this has been solved in a satisfying way by Koszul duality theory (e.g. [4]).

We can consider $\mathcal{A}$ to be a coloured operad and this allows us to write down the cohomology for diagrams of morphisms between algebras. Then however we are no longer in the Koszul case and convenient resolutions are hard to come by. A single morphism between algebras over Koszul operad is probably the only well understood case. What we are interested in is a completely general diagram D of algebras over a fixed operad. Cohomology for this diagram is already known, it was invented by Gerstenhaber and Schack [1]. Understanding this from the operadic point of view would allow us to equip the corresponding complex with L_∞ structure governing the deformations of the diagram [2].

Although this is not yet achieved, we have shown that Gerstenhaber and Schack cohomology is isomorphic to the operadic cohomology. To do so, we have followed ideas of M. Markl in [2] to show that the operadic cohomology can be in general computed as an Ext functor in the category of operadic $\mathcal{A}$-modules,

$$H^*(D, D) = \mathrm{Ext}^*(\mathcal{MDA}, \mathcal{E}nd_D)$$

for certain module $\mathcal{MDA}$. This reduces the problem of constructing cohomology for $\mathcal{A}$-algebras to making explicit resolutions of $\mathcal{MDA}$ in the abelian category of $\mathcal{A}$-modules. So far resolutions of $\mathcal{A}$-modules are very little explored.

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Infinitesimal deformations of unital algebraic structures

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(joint work with Joan Millès)

Given a unital associative algebra A, one may study its deformations *as a unital associative* algebra, or forget the fact that it has a unit and study its deformations as an associative algebra [1]. It is known, however, that deformations of A as a unital associative algebra, and as an associative algebra, are identical, because every associative deformation of a unital associative algebra is itself unital.

Many familiar algebraic structures have reasonable notions of "unit." For example, one can define unital commutative associative algebras, unital Gerstenhaber algebras, or unital BV algebras. In each case, one can study the deformations of a unital algebra A within the unital category, or forget that it has a unit and study its deformations among all algebras of its type.

In this talk, we propose a method for studying this phenomenon for general "unital algebraic structures," as defined in [2]. Given $\mathcal{P}$ an operad, and $u\mathcal{P}$ an operad encoding a unital version of P, we combine the explicit resolution of $u\mathcal{P}$ given in [2] and the study of operadic cohomology theories in [3] to analyze the relationship between infinitesimal unital-$\mathcal{P}$ deformations and infinitesimal $\mathcal{P}$ deformations.

In the case when the infinitesimal deformations are isomorphic, the speaker wonders whether one can extend this to an isomorphism of general deformations by means of the associated $\mathcal{L}ie_\infty$-algebras.

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Derived Brackets Approach to Deformations of Morphisms

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(joint work with Marco Zambon)

In [1], a L_∞ algebra governing simultaneous deformations of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and a morphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ between them has been constructed.

On the one hand, Lie algebra morphisms can be understood as a linear version of coisotropic submanifolds since the graph of ϕ^* (the linear dual of ϕ) is a coisotropic submanifold of the Poisson manifold $\mathfrak{g}^* \times \mathfrak{h}^*$ equipped with the Poisson bivector coming from the Lie algebra brackets.

On the other hand, in [2] has been constructed a L_∞ algebra governing deformations of coisotropic submanifolds of a Poisson manifold. But simultaneous deformations of the coisotropic submanifolds and the background Poisson structure were not considered. The original motivation of our work was to consider simultaneous deformations in such a geometrical context and exhibit a L_∞ algebra governing them.

The operadic methods used in [1] do not apply to the geometric case, this is why our strategy was to reconsider the linear problem of [1] with the tools used in the geometrical setting [2], namely the derived bracket construction of T. Voronov [3]. The hope was that we could understand how to deal with simultaneity within the derived bracket approach, and then later tackle the geometrical problem thanks to this knowledge.

This strategy worked well, since it turned out that it suffices to replace the tools of [3] by their extensions given in [4]. We have formulated a convenient theoretical setting enabling other applications in the theory of Courant algebroids and Dirac structures, and also in generalized complex geometry. But in the following we will limit ourselves to recalling Voronov's results and show how they can be applied in the linear case. For the sake of simplicity of exposition, we consider associative algebras and their morphisms rather than their Lie analogues. But one should have in mind that all that is done here also works for algebras over an arbitrary quadratic Koszul operad.

Definition 0.1. *A V-data consists in a a quadruple $(L, \mathfrak{a}, P, \Delta)$ where $(L, [\cdot,\cdot])$ is a graded Lie algebra, $\mathfrak{a}$ an abelian sub-Lie algebra, $P\colon L \to \mathfrak{a}$ a projection whose* *kernel is a Lie subalgebra of L and Δ an element of degree one in $Ker(P)$ such that $[\Delta, \Delta] = 0$.*

Theorem 0.1. [3, Thm. 1] & [4, Thm. 2] *Let $(L, \mathfrak{a}, P, \Delta)$ be a V-data, then*
1) $\mathfrak{a}$ is a $L_\infty[1]$ algebra for the multibrackets ($n \geq 1$)

$$\{a_1, \ldots, a_n\} = P[\ldots[[\Delta, a_1], a_2], \ldots, a_n]. \tag{1}$$

2) the space $L[1] \oplus \mathfrak{a}$ is a $L_\infty[1]$-algebra for the differential

$$d(x[1], a) := (-(Dx)[1], P(x + Da)),$$

the binary bracket

$$\{x[1], y[1]\} = [x, y][1](-1)^{|x|},$$

and for $n \geq 1$:

$$\{x[1], a_1, \ldots, a_n\} = P[\ldots[x, a_1], \ldots, a_n],$$
$$\{a_1, \ldots, a_n\} = P[\ldots[Da_1, a_2], \ldots, a_n].$$

Here $a_1, \ldots, a_n \in \mathfrak{a}$. All the remaining multibrackets vanish.

Notation 0.1. *We will denote by $\mathfrak{a}^P_\Delta$ and by $(L[1] \oplus \mathfrak{a})^P_\Delta$ the $L_\infty[1]$-algebras produced by the previous theorem.*

Let us now consider a morphism $\Phi : U \to V$ between two associative algebras (U, μ) and (V, ν). It is well known that the space $L := \oplus L^i$ with

$$L^i := T^{i+1}(U \oplus V)^\star \otimes (U \oplus V)$$

is a graded Lie algebra for the Gerstenhaber bracket. Let us define $\mathfrak{a} = \oplus \mathfrak{a}^i$ with

$$\mathfrak{a}^i := T^{i+1}U^\star \otimes V.$$

It is an abelian subalgebra.

In order to define a projection P onto the abelian subalgebra $\mathfrak{a}$, let us introduce the notation $T^{I,J}(U, V)$, where $I \amalg J = \{1, \ldots n\}$, by saying that an element $x_1 \otimes \cdots \otimes x_n$ of $T^n(U \oplus V)$ belongs to $T^{I,J}(U, V)$ if x_i belongs to U when $i \in I$, x_i belongs to V otherwise. One has the natural decomposition

$$L^i = \bigoplus_{I \amalg J = \{1,\ldots,i+1\}} T^{I,J}(U^\star, V^\star) \otimes U \quad \bigoplus_{I \amalg J = \{1,\ldots,i+1\}} T^{I,J}(U^\star, V^\star) \otimes V$$

and one recognizes $\mathfrak{a}^i$ as the term $T^{\{1,\ldots,i+1\},\emptyset}(U^\star, V^\star) \otimes V$. Denoting by P the projection onto $\mathfrak{a}$ given by this decomposition, one defines the projection P_Φ onto $\mathfrak{a}$ by

$$P_\Phi := P \circ e^{[\cdot,\Phi]}.$$

The main result of this note is

Proposition 0.1. *The set $(L, \mathfrak{a}, P_\Phi, \Delta)$ forms a V-data, and in particular theorem 0.1 applies, leading to the $L_\infty[1]$ algebra $(L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta$ on $L[1] \oplus \mathfrak{a}$*

To make the connection with the problem of simultaneous deformations of morphisms and associative algebras, one considers the graded subspace L' of L defined by $L'^n = T^{n+1}U^\star \otimes U \oplus T^{n+1}V^\star \otimes V$:

Lemma 0.1. *$L'[1] \oplus \mathfrak{a}$ is a sub $L_\infty[1]$ algebra of $L[1] \oplus \mathfrak{a}$. In other words, the brackets of $(L[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$, when restricted to elements of $L'[1] \oplus \mathfrak{a}$, take values in $L'[1] \oplus \mathfrak{a}$. One denotes by $(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$ this sub-$L_\infty[1]$ algebra.*

One can compare $(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$ to the $L_\infty[1]$ algebra constructed in [1], whose Maurer-Cartan elements are simultaneous deformations of the associative algebras μ and ν and of the morphism Φ. Since these two $L_\infty[1]$ algebras are isomorphic, one obtains

Corollary 0.1. *$(L'[1] \oplus \mathfrak{a})_\Delta^{P_\Phi}$ governs the simultaneous deformations of the associative algebras μ and ν and of the morphism Φ.*

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Families of Dirac operators and quantum affine groups

Jouko Mickelsson

The motivation for this work comes from the following problem: Twisted equivariant K-theory on compact Lie groups can be constructed in Fredholm operator realization using the representation theory of loop groups. The construction can be done using a quantum field theory model in 1+1 space-time dimensions, namely the supersymmetric Wess-Zumino-Witten model. In a moral sense the Fredholm family of operators can be thought of as a family of Dirac operators on a loop group. Although this idea cannot (yet) be made precise analytically, it makes sense algebraically through representation theory of loop groups. The problem is now whether it is possible to deform the family of Fredholm operators, transforming covariantly under the loop group, to a family of operators transforming covariantly under an affine quantum group. It is clear that the construction must be done completely algebraically, already because of the fact that a compact quantum group is not a manifold.

In the undeformed case, for a compact Lie group G, twisted K-theory is defined by an element of $H^3(G, \mathbb{Z})$, the Dixmier-Douady class of a gerbe. It turns out that in the Fredholm operator realization this class corresponds precisely to *level* of an irreducible highest weight representation of the central extension of the loop group. In particular, when G is compact and simple Lie group we have $H^3(G, \mathbb{Z}) = \mathbb{Z}$ and the relation to the level is given by $k + \kappa =$ the Dixmier-Douady class. Here κ is the dual Coxeter number of G.

The Dirac operator Q is acting in $H_f \otimes H_b$ where H_f is the q-fermionic Fock space and H_b carries another highest weight representation of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$, where $\hat{\mathfrak{g}}$ is the affine algebra defined by a simple Lie algebra $\mathfrak{g}$. The action of the nontrivial central extension is seen in the action of the element $K_0K_1 \dots K_\ell$, which is no more equal to the unit operator but a power of q where the exponent depends on the level of the representation. The K_i's are the Cartan elements (which are group like) of $U_q(\hat{\mathfrak{g}})$.

$$Q = i \sum \psi_a^n \otimes T_a^{-n} + i\frac{1}{3} \sum \psi_a^n K_a^{-n} \otimes 1$$

where T_a^n are basis vectors of the adjoint module, acting as linear operators in the space H_b. The labels are: $n \in \mathbb{Z}$ corresponds to the Fourier index in a loop algebra and $1 \leq a \leq N$ labels the basis in an adjoint module of $U_q(\mathfrak{g})$. We need also another copy of the adjoint module, acting in the space H_f. The components are denoted by K_a^n. The vectors ψ_a^n are elements in a quantum Clifford algebra acting as operators in a q-Fock space H_f.

The adjoint action of an element $a \in U_q(\hat{\mathfrak{g}})$ is defined by $x \mapsto \sum_{(a)} a'xS(a'')$, in Sweedler's notation for the coproduct $\Delta(a) = \sum_{(a)} a' \otimes a''$. Here S is the antipode.

In contrast to the undeformed case, the operators K_a^n, T_a^n do not satisfy the defining relations of the algebra $U_q(\hat{\mathfrak{g}})$.

The construction of the defining relations of the quantum Clifford algebra generated by the ψ_a^n's involves the R-matrix in the adjoint representation of $U_q(\mathfrak{g})$ and a generalized Hecke algebra. The reason for the need of the generalization of the Hecke algebra is due to the fact that in the adjoint representation the twisted R-matrix $\check{\mathrm{R}} = \sigma R$ has more than two eigenvalues; σ is the transposition of components in a tensor product. In the defining represention the eigenvalues are $-q, q^{-1}$, corresponding to generalized antisymmetric and symmetric tensors in the braiding relations and for this reason the minimal polynomial for R is quadratic. But already in the case of $U_q(\widehat{\mathfrak{sl}}(2))$ there are three different eigenvalues $-q^{-2}, q^2, q^{-4}$ and therefore the quadratic relation $(\check{\mathrm{R}} + q)(\check{\mathrm{R}} - q^{-1}) = 0$ has to be replaced by $(\check{\mathrm{R}} + q^{-2})(\check{\mathrm{R}} - q^2)(\check{\mathrm{R}} - q^{-4}) = 0$.

Finally, we can define a family of operators $Q_A = Q + A$ transforming covariantly under the quantum adjoint action. Here A is linear in the Clifford algebra generators ψ_a^n and is an element of the adjoint module for $U_q(\hat{\mathfrak{g}})$. It plays the role of a vector potential on a unit circle in the undeformed case ($q = 1$).

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Local and covariant deformations of observable algebras in QFT

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(joint work with D. Buchholz, S. J. Summers)

Quantum field theory is a rich playground for deformation theory: For studying quantization of classical field theories, the connection between non-relativistic and relativistic systems, or the relation between interaction-free and interacting quantum field theories, one can study deformations taking the quantum of action, the inverse speed of light, or the coupling constant as deformation parameter. Formal deformations in coupling constants (perturbation theory) still provide the main predictions of high energy physics. Motivated by the long-standing problem of constructing interacting quantum field theories in four space-time dimensions beyond the perturbative level, this talk reviews a new approach to convergent deformations of quantum field theories, emphasizing the operator-algebraic aspects of the subject.

Starting with the simple example of a chiral quantum field theory on a lightray, it is explained how locality and covariance lead to the involved structure of the algebra of observables in quantum field theory. In this setting, one considers a strongly continuous unitary representation U of the translation group $(\mathbb{R}, +)$ on a separable Hilbert space $\mathcal{H}$, a U-invariant vector $\Omega \in \mathcal{H}$, representing the vacuum, and a von Neumann algebra $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$, representing the observables localized in the right half line $\mathbb{R}_+$. These data are required to satisfy the compatability conditions

- Ω is cyclic and separating for $\mathcal{R}$,
- $U(x)\mathcal{R}U(x)^{-1} \subset \mathcal{R}$ for $x \geq 0$,

expressing covariance and the Reeh-Schlieder property of the vacuum.

If the above conditions are satisfied, a full quantum field theory, *i.e.*, a local, covariant net of von Neumann algebras $I \mapsto \mathcal{A}(I)$ associated with intervals $I \subset \mathbb{R}$, can be reconstructed from the data $(\mathcal{R}, U, \mathcal{H})$, Hence deformations of quantum field theories can be obtained from deformations of such triples $(\mathcal{R}, U, \mathcal{H})$. In this context, it is interesting to note that under very general assumptions [1, 2], the structure of the algebra is severely restricted (type III_1 factor), and in most situations even uniquely determined (hyperfinite type III_1 factor). So $\mathcal{R}$ is rigid in the sense of deformation theory, and one has to consider deformations of the inclusions $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ instead of deformations of $\mathcal{R}$ alone.

In this very general setting, deformations can be obtained from the action of a symmetry group. Whereas the case of chiral quantum field theories, where the non-abelian translation-dilation group acts, is currently under investigation [3], higher-dimensional theories deformed by an action of $\mathbb{R}^d$, $d \geq 2$, are better understood [4], and will be reviewed here.

The examples of deformations of quantum field theories to be discussed here rely on the technique of *warped convolution* [5] and Rieffel's closely related deformations of C^*-algebras with $\mathbb{R}^d$-action [6]. For the formulation of warped convolutions, we consider a Hilbert space $\mathcal{H}$ with a unitary, strongly continuous representation U

of $\mathbb{R}^d$, and a $(d \times d)$-matrix Q which is antisymmetric w.r.t. a chosen bilinear form (p, x) on $\mathbb{R}^d$, as deformation parameter. A smooth operator A on $\mathcal{H}$ is then deformed according to

$$A_Q := (2\pi)^{-d} \int dp \int dx \, e^{i(p,x)} U(Qp) A U(x - Qp) \,, \tag{1}$$

where the integral is defined in an oscillatory sense. Algebras of warped operators form representations of Rieffel-deformed C^*-algebras, and the deformation map $A \mapsto A_Q$ is linear, compatible with taking adjoints, preserves the unit, and commutes with the adjoint action of U.

Further properties of warped convolutions are explained by considering products of deformed operators $A_Q B_Q$, the interplay with an invariant vector, and connections to Tomita-Takesaki modular theory. Moreover, the covariance of $A \mapsto A_Q$ w.r.t. representations extending U to the semidirect product of $\mathrm{GL}(d)$ and $\mathbb{R}^d$ is described, and necessary conditions for commutators of the form $[A_Q, B_{-Q}]$ to vanish, are given.

With this tools at hand, an example of a local, covariant deformation of quantum field theories on four-dimensional Minkowski space is discussed. Adopting an appropriately generalized, causal notion of “left” and “right” in terms of wedges rather than in terms of half lines, a description of quantum field theories in terms of triples $(\mathcal{R}, U, \mathcal{H})$ similar to the one in the introductory example is given. It is then explained how warped convolution can be applied to deform such triples, making use of the spectrum condition (positivity of the energy), and adjusting the deformation parameter to the geometry of Minkowski space.

This discussion shows that to deform a quantum field theory in a local, covariant manner, one has to deal not with a single deformation of one global algebra, but rather has to take into account a whole family of deformations, corresponding to subalgebras localized in different regions in spacetime.

Finally, some properties of the emerging deformed quantum field theories and the relation of the described procedure to theories on non-commutative Minkowski space [7, 8] is explained. Further applications of this method to field theories on curved spacetimes and conformal field theories have been studied in [9] and [10].

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