Mathematisches Forschungsinstitut Oberwolfach

Report No. 56/2010

DOI: 10.4171/OWR/2010/56

Mini-Workshop: Wellposedness and Controllability of Evolution Equations

Organised by
Birgit Jacob, Wuppertal
Jonathan Partington, Leeds
Sandra Pott, Lund
Hans Zwart, Enschede

December 12th – December 18th, 2010

Abstract. This mini-workshop brought together mathematicians engaged in partial differential equations, operator theory, functional analysis and harmonic analysis in order to address a number of current problems in the wellposedness and controllability of infinite-dimensional systems.

Mathematics Subject Classification (2000): Primary: 47D06, 93B05, 93C20; Secondary: 93B28, 47A57, 47B35, 42B37.

Introduction by the Organisers

The mini-workshop *Wellposedness and controllability of Evolution Equations*, organised by Birgit Jacob (Wuppertal), Jonathan Partington (Leeds), Sandra Pott (Lund), and Hans Zwart (Twente) was held December 12th–18th, 2010. This meeting was well attended with 16 participants with broad geographic representation.

Systems modelled by linear ordinary differential equations have long been studied and there exists a wide body of theory and design algorithms dealing with control of these systems. The state describing such a systems lies in a finite-dimensional vector space. This setting has its limitations, as many systems of interest do not fall into this class. A more interesting generalization is that to systems with an infinite-dimensional state-space. This class includes delay systems, and systems modelled by functional differential equations and partial differential equations, generally called evolution equations. Motivated by applications in such diverse fields as aeronautics, electrical engineering and biology, evolution equations with boundary control and boundary observation are of particular interest. The basic object under study is therefore a linear semigroup system specifed by the equations

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t), \qquad (t \geq 0), \\ y(t) &= Cx(t), \qquad \text{with} \quad x(0) = x_0, \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of operators on a Hilbert space H with domain $D(A)$, and $B : U \to H_{-1}$ and $C : D(A) \to Y$ are respectively the control and observation operators of the system, which are in general unbounded with respect to H (hence H_{-1} is in general an extrapolation Hilbert space containing H). This very powerful formulation enables one to study delay systems and systems specified by partial differential equations in the same framework.

The talks can be grouped into three main themes:

- Harmonic analysis and operator theory
- Well-posedness of evolution equations
- Observability of evolution equations.

In the first theme the following participants gave talks: Tuomas Hytönen, Rainer Nagel, Jonathan Partington, Lutz Weis, Brett Wick, Christian Wyss. Furthermore, Birgit Jacob, Bernd Klöss, Mark Opmeer, Olof Staffans, and George Weiss were the speakers for the second theme. The last theme was covered by Bernhard Haak, Luc Miller, Marius Tucsnak, and Hans Zwart. Although we have grouped them according to our themes, there was significant overlap between the approaches which stimulated many productive discussions.

The organizers and participants thank the *Mathematisches Forschungsinstitut Oberwolfach* for providing an inspiring setting for this mini-workshop, which allowed us to concentrate on the mathematics. In the following we include the abstracts in alphabetical order.

Mini-Workshop: Wellposedness and Controllability of Evolution Equations

Table of Contents

Abstracts

Exact observability, square function estimates and spectral theory

Bernhard H. Haak
(joint work with El Maati Ouhabaz)

The aim of this talk is twofold. First we give some necessary spectral conditions on the semigroup of a linear control system to have the backward-forward conditioning property. This property extends the notion of zero-class admissibility discussed in the literature. We obtain stronger results under weaker assumptions.

In the second part we prove a new sufficient condition for exact observability in Hilbert spaces for generators of contractions semigroups.

1. Introduction

In the article [1] we study exact observability of linear systems (A, C) on Banach spaces of the form

$$\begin{cases} x'(t) + Ax(t) &= 0 \\ x(0) &= x_0 \\ y(0 &= Cx(t) \end{cases}$$

We suppose throughout this article that $-A$ is the generator of a strongly continuous semigroup $T(t)_{t\geq 0}$ on a Banach space X. Let Y be another Banach space and suppose that the observation operator $C : X \to Y$ is linear and closed on X but bounded only on $\mathcal{D}(A)$ when endowed with the graph norm.

Definition 1. We say that C is L^2-admissible in time $\tau > 0$ (for A or for $T(t)_{t\geq 0}$) if there exists a constant $M(\tau) > 0$ such that

$$\sup_{x\in\mathcal{D}(A),\|x\|=1} \int_0^\tau \|CT(t)x\|_Y^2\, dt =: M(\tau)^2 < \infty.$$

We say that C is exactly L^2-observable for A (or for $T(t)$) in time $\eta > 0$ if there exists a constant $m(\eta) > 0$ such that

$$\inf_{x\in\mathcal{D}(A),\|x\|=1} \int_0^\eta \|CT(t)x\|_Y^2\, dt =: m(\eta)^2 > 0.$$

In a first part we will tackle the question under which conditions the semigroup $T(t)_{t\geq 0}$ that admits an admissible and exactly observable observation necessarily extends – or dilates – to a strongly continuous group. In this direction we extend and complete former results of [4].

A second part is devoted to a sufficient condition for exact observability. The criterion is clearly true for bounded analytic contraction semigroups on Hilbert spaces, but the first part reveals this to be impossible unless A is bounded. The charm of the criterion and its proof is therefore how not to make use of analyticity of the semigroup.

2. The backward-forward conditioning (BFC) property

Definition 2. An admissible observation operator C fro A is called *zero-class admissible*, if $\lim_{\tau\to 0+} M(\tau) = 0$.

Consider the linear operator $\widetilde{\Psi}_\tau : X \to L^2(0,\tau;Y)$ defined by $\widetilde{\Psi}_\tau x = CT(\cdot)x$. Then admissibility (i.e., $M(\tau) < \infty$) means that $\widetilde{\Psi}_\tau$ is a bounded operator. If in addition $m(\tau) > 0$, then $\widetilde{\Psi}_\tau$ is injective and has closed range. Therefore, we may consider the operator $\Psi_\tau : X \to \mathcal{R}(\widetilde{\Psi}_\tau)$, $\Psi_\tau = \widetilde{\Psi}_\tau$. We have $\|\Psi_\tau^{-1}\| = \frac{1}{m(\tau)}$.

Definition 3. We say that the system (A, C) has the backward-forward conditioning property or shortly that (A, C) is a BFC-system if there exists some $0 < \eta < \tau$ such that C is admissible and exactly observable in time τ and if

$$\text{(BFC)} \qquad \|\Psi_\tau^{-1}\|\,\|\Psi_\eta\| < 1.$$

If C is exactly observable in some time τ and of zero-class the system is BFC. Zero-class admissibility is introduced and studied in [4]. In [2] a concrete example is given that turns out to be BFC-system without being zero-class. Our next aim is to study spectral properties of BFC-systems. We will extend some results which have been proved in [4] in the context of zero-class operators. We note also that related ideas and results were obtained previously by Nikolski [3] in the particular case of bounded observation operators C on X. Let us introduce the classical function $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\epsilon(t) := \inf_{\|x\|=1} \|T(t)x\|$. It is clear that $\epsilon(t)$ is strictly positive for all $t > 0$ if this holds for a single $t_0 > 0$.

Lemma 4. *If (A, C) is a BFC-system, then $\epsilon(t) > 0$.*

Lemma 5. *Suppose that (A, C) is a BFC-system. Then $T(t)^*$ is injective for one (and thus all) $t > 0$ if and only if $T(t)$ extends to a group on X.*

For a closed operator S on X denote its point spectrum by $\sigma_P(S)$, the approximate point spectrum by $\sigma_A(S)$ and the residual spectrum by $\sigma_R(S)$. It is easy to see that $\sigma_R(S) = \sigma(S)\backslash\sigma_A(S)$. Of course, $\sigma_P(S) \subseteq \sigma_A(S)$.

Proposition 6. *Let (A, C) be an admissible BFC-system. Then there exist no approximate point spectrum of A with arbitrary large real parts.*

Corollary 7. *Let (A, C) be an admissible BFC-system. Then*

$$Re(\partial\sigma_A(A)) := \{Re(\lambda), \lambda \in \sigma_A(A)\}$$

is bounded.

Proposition 8. *Let (A, C) be an admissible BFC-system. If $\sigma_R(A) \cap [0, +\infty)$ is bounded, then $(T(t))_{t\geq 0}$ extends to a group on X.*

Corollary 9. *Assume that (A, C) is an admissible BFC-system. If $T(t)$ is compact for some $t > 0$, then X has finite dimension.*

3. Sufficient conditions for exact observability

Proposition 10. *Let X and Y be Hilbert spaces. Then, if (A,C) is exactly observable and admissible in infinite time, $T(t)_{t\geq 0}$ is similar to a contraction semigroup.*

Proof. Denote by $\langle x,y\rangle_Y$ the scalar product of Y and define for $x,y\in\mathcal{D}(A)$

$$\langle x,y\rangle_X^{\sim} := \int_0^\infty \langle CT(t)x, CT(t)y\rangle_Y\, dt.$$

It is easy to see that this is an equivalent scalar product. With respect to the new norm, $T(t)_{t\geq 0}$ is a contraction semigroup. □

Theorem 11. *Let $-A$ be the generator of a semigroup of contractions $(T(t))_{t\geq 0}$ on a Hilbert space H and assume that A has dense range. Assume (A,C) is infinite-time admissible.*

Then, if $\|CA^{-\frac{1}{2}}x\| \geq \delta\|x\|$ for all $x\in\mathcal{D}(A)\cap\mathcal{R}(A)$ and some $\delta>0$, C is infinite-time exactly observable for A.

Proof. (sketch). By von Neumann's inequality A has a bounded H^∞ functional calculus on the closed right half-plane $\overline{S_{\frac{\pi}{2}}}$. This yields square-function estimates of the type $C_\phi\|x\|^2 \leq \|\phi(tA)x\|^2_{L^2(\mathbb{R}_+,\frac{dt}{t})}$ for a certain class of bounded holomorphic functions. We shall use this for the special choice $\psi(z)=z^{-\frac{1}{2}}(e^{-2z}-e^{-z})$. Notice $\psi(tA)=(tA)^{-\frac{1}{2}}(T(2t)-T(t))$. For $x\in\mathcal{D}(A)\cap\mathcal{R}(A)$,

$$\begin{aligned}
\|x\|^2 &\leq K^2\int_0^\infty \|(tA)^{-\beta}(T(2t)-T(t))x\|^2\,\tfrac{dt}{t}\\
&\leq \tfrac{K^2}{\delta^2}\int_0^\infty \|CA^{-\frac{1}{2}}(tA)^{-\frac{1}{2}}(T(2t)-T(t))x\|_Y^2\,\tfrac{dt}{t}\\
&= \tfrac{K^2}{\delta^2}\int_0^\infty \|CA^{-1}(T(2t)-T(t))x\|_Y^2\,\tfrac{dt}{t^2}\\
&\leq \tfrac{K^2}{\delta^2}\int_0^\infty \|\int_t^{2t} CT(s)x\,ds\|_Y^2\,\tfrac{dt}{t^2}\\
\text{(Cauchy Schwarz)} &\leq \tfrac{K^2}{\delta^2}\int_0^\infty\int_t^{2t} \|CT(s)x\|_Y^2\,ds\,\tfrac{dt}{t}\\
&= \tfrac{K^2}{\delta^2}\int_1^2\int_0^\infty \|CT(tu)x\|_Y^2\,dt\,du\\
&= \log(2)\tfrac{K^2}{\delta^2}\int_0^\infty \|CT(r)x\|_Y^2\,dr.
\end{aligned}$$

We have proved the desired inequlity for $x\in\mathcal{D}(A)\cap\mathcal{R}(A)$. An approximation argument using admissibility extend it for all $x\in\mathcal{D}(A)$. □

References

[1] Bernhard H. Haak and El Maati Ouhabaz *Exact observability, square function estimates and spectral theory*, preprint.

[2] Birgit Jacob, Jonathan R. Partington, and Sandra Pott, *Zero-class admissibility of observation operators*, Systems Control Lett. **58** (2009), no. 6, 406–412.
[3] Nikolai Nikolski, *Rudiments de la thorie du contrle : vision operationnelle*, Publication de l'Ecole Doctorale de mathmatiqes de Bordeaux, 1996, Lecture notes of a course given 1994-1995 in Bordeaux.
[4] Gen Qi Xu, Chao Liu, and Siu Pang Yung, *Necessary conditions for the exact observability of systems on Hilbert spaces*, Systems Control Lett. **57** (2008), no. 3, 222–227.

L^p-variants of Carleson's embedding arising from Kato's square root problem

Tuomas Hytönen

My talk dealt with some inequalities of Harmonic Analysis with potential interest as *tools* in the area of the workshop. Except for this opening paragraph, the word *wellposedness* will be mentioned only once, and *controllability*—not at all.

The classical embedding. Let $\mathcal{D} := \{2^{-k}([0,1)^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}$ be the collection of *dyadic cubes* in $\mathbb{R}^n$, and denote the related averaging operators by

$$\mathbb{E}_Q f := 1_Q \langle f \rangle_Q := 1_Q \frac{1}{|Q|} \int_Q f dx.$$

The dyadic version of Carleson's classical embedding theorem states that, for a given sequence of numbers $(\lambda_Q)_{Q\in\mathcal{D}}$, we have

$$\Big(\sum_{Q\in\mathcal{D}} |\lambda_Q \langle f \rangle_Q|^2 \Big)^{1/2} \leq A \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^n) \tag{1}$$

if and only if this estimate holds for all $f = 1_R$, $R \in \mathcal{D}$, if and only if

$$\Big(\sum_{Q\subseteq R} |\lambda_Q|^2 \Big)^{1/2} \leq a|R|^{1/2} \quad \forall R \in \mathcal{D},$$

with best constants $A \simeq a$.

From L^2 to L^p. A simple L^p-variant of this result is obtained by replacing all 2's by p's, with $p \in (1,\infty)$. This variant is correct, and can be proven by an immediate modification of the L^2 result; however, from the point of view of certain applications, it is not the "right" extension. Rather, there arises the need for a Carleson embedding of the type

$$\|Bf\|_{L^p} := \Big\| \Big(\sum_{Q\in\mathcal{D}} |b_Q \mathbb{E}_Q f|^2 \Big)^{1/2} \Big\|_{L^p} \leq A \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^n) \tag{2}$$

under reasonable conditions on the multiplying *functions* (not constants!) b_Q. For $p = 2$, this reduces to the classical embedding (1) with $\lambda_Q = \|b_Q\|_{L^2}$, but such a simplification is unavailable in general, since the norms of ℓ^2 and L^p cannot be commuted for $p \neq 2$.

The form of the left side of (2) can be understood by comparison to *Littlewood–Paley theory*, which gives rise to quadratic expressions of the similar type with an ℓ^2 norm inside the L^p norm.

Background for the L^p-embedding. The need for the L^p embedding (2) arose from the author's attempt, with McIntosh and Portal [4], to generalize parts of the solution of the *Kato square root problem* from L^2 to L^p. This problem concerns a second-order, divergence-form operator $L = -\operatorname{div} A\nabla$, where $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ satisfies the ellipticity condition $\operatorname{Re}(\xi, A(x)\xi) \geq \delta|\xi|^2$ for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{C}^n$. We think of L as an unbounded operator in $L^2(\mathbb{R}^n)$; its square root $\sqrt{L}$ can be defined as another unbounded operator in the same space. Kato's conjecture claimed that the domain of definition of the square-root satisfies

$$\mathsf{D}(\sqrt{L}) = \mathsf{D}(\nabla) = W^{1,2}(\mathbb{R}^n), \quad \|\sqrt{L}u\|_{L^2} \simeq \|\nabla u\|_{L^2} \quad \forall u \in \mathsf{D}(\sqrt{L}).$$

This was confirmed by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [2] in what might be considered a culmination of McIntosh's theory of *functional calculus.* The proof consisted of a combination of Operator Theory and Harmonic Analysis, and invoked the classical Carleson embedding (1) in the estimation of the *principal part* of the operator. And in trying to adapt this proof to L^p, the variant formulated in (2) was needed.

Since the resolution of Kato's conjecture, the underlying methods have been further developed—by Auscher, Axelsson and McIntosh, see e.g. [1], among others—to a new approach to *wellposedness* of boundary value problems with L^2 data. Our L^p techniques give some prospects for the extension of these results to the L^p setting.

Positive results on the L^p-embedding. As in the classical L^2 case, we may test the estimate (2) with $f = 1_R$ to see that the boundedness of the following quantity is *necessary* for (2):

$$\|\{b_Q\}_{Q\in\mathcal{D}}\|_{\mathrm{Car}^p} := \sup_{R\in\mathcal{D}} \Big(\frac{1}{|R|}\int_R \Big[\sum_{Q\subseteq R} |b_Q|^2\Big]^{p/2} dx\Big)^{1/p}.$$

The classical theorem confirms that it is also sufficient for $p = 2$. The key elements of the proof are the *level sets* $\mathcal{R}_k := \{Q \in \mathcal{D} : |\langle f\rangle_Q| > 2^k\}$, the application of the Carleson testing condition to the *maximal cubes* $R \in \mathcal{R}_k$, and the observation that $\sum |R| \leq |\{Mf > 2^k\}|$, where the sum is over these maximal R, and M is the maximal operator.

In the L^2 case, the required rearrangements of the summations and integration for this argument can be easily performed. This is not the case for $p \neq 2$, and some identities have to be replaced by inequalities. Here, a dichotomy between $p \in (1, 2]$ and $p \in (2, \infty)$ arises. In the former case, we (i.e., [4]) make use of the bounded embedding $\ell^{p/2} \hookrightarrow \ell^1$ and in the latter, the triangle inequality in $L^{p/2}$. With appropriate modifications of the classical proof, we then deduce that

$$\|Bf\|_{L^p} \leq C_p \|\{b_Q\}_{Q\in\mathcal{D}}\|_{\mathrm{Car}^p} \times \begin{cases} \|f\|_{L^p}, & p \in (1, 2], \\ \|f\|_{L^{p,2}}, & p \in (2, \infty). \end{cases}$$

where $L^{p,2}$ is the *Lorentz space*. Observing the bounded embedding $\mathrm{Car}^{p+\epsilon} \hookrightarrow \mathrm{Car}^{p-\epsilon}$, the latter estimate may be interpolated by the real method between $p \pm \epsilon$ to give

$$\|Bf\|_{L^p} \leq C_{p,\epsilon}\|\{b_Q\}_{Q\in\mathcal{D}}\|_{\mathrm{Car}^{p+\epsilon}}\|f\|_{L^p}, \quad p \in (2,\infty), \quad \epsilon > 0.$$

Only for $p \in (1,2]$ do our necessary and sufficient conditions meet. So is the $\epsilon > 0$ really needed for $p > 2$?

Negative result on the L^p-embedding. The above results are optimal in the following sense: the embedding

$$\|Bf\|_{L^p} \leq C_p\|\{b_Q\}_{Q\in\mathcal{D}}\|_{\mathrm{Car}^p}\|f\|_{L^p} \tag{3}$$

is false for any $p \in (2,\infty)$. A counterexample showing this was first sketched by M. Lacey (personal communication, Edinburgh, September 2009) for $p = 4$. The details for all $p \in (2,\infty)$ were worked out by myself and M. Kemppainen [3].

The counterexample is surprisingly simple. It suffices to consider dimension $n = 1$ and $b_{[0,2^{-j})} := 2^{(N-j)/p}1_{[0,2^{-N})}$ for all $j = 0,\ldots,N$, and $b_Q := 0$ for all other dyadic intervals Q. Here N is a large auxiliary number. One easily checks that $\|\{b_Q\}_{Q\in\mathcal{D}}\|_{\mathrm{Car}^p} \leq C$, independent of N.

For any given sequence of numbers $x = (x_j)_{j=0}^N$, we can then construct a function f with $\|f\|_{L^p} \simeq \|x\|_{\ell^p}$ while $\|Bf\|_{L^p} \simeq \|x\|_{\ell^2}$, so the Carleson embedding (3) would imply the bounded embedding $\ell^p_N \hookrightarrow \ell^2_N$, uniformly in N—an impossibility for $p > 2$. It is a direct computation to check that the following choice works:

$$f := y_N 1_{[0,2^{-N})} + \sum_{j=0}^{N-1}(2y_j - y_{j+1})1_{[2^{-j-1},2^{-j})}, \quad y_j := 2^{j/p}x_j.$$

The point is to impose the desired averages $\langle f\rangle_{[0,2^{-j})} = y_j$ for $j = 0,1,\ldots,N$.

Vector-valued issues. All the results of the two papers [3, 4] discussed here are actually formulated more generally for *vector-valued functions* $f : \mathbb{R}^n \to X$, where X is an infinite-dimensional Banach space. A Carleson embedding theorem relevant in this context involves a *randomized* formulation of the square function, and the relevant maximal function needed in the estimate is a new *Rademacher maximal function* introduced in [4]. It might or might not be bounded from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n)$; if it is bounded, the space X is said to have the *RMF property.* It turns out (see [3, 4]) that the vector-valued embedding $\|Bf\|_{L^p_X} \leq C^X_{p,\epsilon}\|\{b_Q\}_{Q\in\mathcal{D}}\|_{\mathrm{Car}^{p+\epsilon}}\|f\|_{L^p_X}$ holds if and only if X has RMF; with $\epsilon = 0$, the characterizing condition is RMF and *type p*.

References

[1] P. Auscher, A. Axelsson, A. McIntosh, *Solvability of elliptic systems with square integrable boundary data.* Ark. Mat. **48** (2010), no. 2, 253–287.

[2] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, P. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on* $\mathbb{R}^n$. Ann. of Math. (2) **156** (2002), no. 2, 633–654.

[3] T. Hytönen, M. Kemppainen, *On the relation of Carleson's embedding and the maximal theorem in the context of Banach space geometry.* Preprint, arXiv:1002.2876 (2010).
[4] T. Hytönen, A. McIntosh, P. Portal, *Kato's square root problem in Banach spaces.* J. Funct. Anal. **254** (2008), no. 3, 675–726.

Infinite-dimensional port-Hamiltonian systems

Birgit Jacob
(joint work with Hans Zwart)

Modeling of dynamical systems with a spatial component leads to lumped parameter systems, when the spatial component may be denied, and to distributed parameter systems otherwise. The mathematical model of distributed parameter systems will be a partial differential equation. Examples of dynamical sytems with a spatial component are, among others, temperature distribution of metal slabs or plates, and the vibration of aircraft wings.

We will study distributed parameter port-Hamiltonian systems. Let $P_1 \in \mathbb{K}^{n\times n}$ be invertible and self-adjoint, let $P_0 \in \mathbb{K}^{n\times n}$ be skew-adjoint, i.e., $P_0^* = -P_0$, and let $\mathcal{H} \in L^\infty([a,b];\mathbb{K}^{n\times n})$ such that $mI \le \mathcal{H}(\zeta) \le MI$ for a.e. $\zeta \in [a,b]$ and constants $m, M > 0$ independent of ζ. We equip the Hilbert space $X := L^2([a,b];\mathbb{K}^n)$ with the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^*\mathcal{H}(\zeta)f(\zeta)\,d\zeta. \tag{1}$$

Then the differential equation

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)x(\zeta,t)\right) + P_0\left(\mathcal{H}(\zeta)x(\zeta,t)\right). \tag{2}$$

is called a *linear, first order port-Hamiltonian system.* The associated *Hamiltonian* $E : [a,b] \to \mathbb{K}$ is given by

$$E(t) = \frac{1}{2}\int_a^b x(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{3}$$

This class contains the above mentioned examples. We equip a port-Hamiltonian system with boundary conditions of the form

$$\tilde{W}_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} = 0, \qquad t \ge 0, \tag{4}$$

where $\tilde{W}_B$ is an $n \times 2n$ matrix of rank n. In [1] it is shown the operator

$$Ax := P_1\frac{d}{d\zeta}\left(\mathcal{H}x\right) + P_0\left(\mathcal{H}x\right) \tag{5}$$

with domain

$$D(A) = \{x \in L^2([a,b];\mathbb{K}^n) \mid \mathcal{H}x \in H^1([a,b];\mathbb{K}^n),\ \tilde{W}_B \left[\begin{smallmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{smallmatrix}\right] = 0\}. \tag{6}$$

is the infinitesimal generator of a contraction semigroup on X if and only if $W_B P_0^{-1} \Sigma P_0^{-*} W_B^* \geq 0$. Here

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}. \tag{7}$$

Further, it is possible to determine which boundary variables are suitable as inputs and outputs [2], and how the system can be stabilized via the boundary [3].

References

[1] Y. Le Gorrec, H. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. Control Optim.*, 44 (2005) 1864-1892.

[2] J.A. Villegas, H. Zwart, Y. Le Gorrec, and B. Maschke. Exponential stability of a class of boundary control systems. *IEEE Trans. Automat. Control* 54 (2009) 142-147.

[3] H. Zwart, Y. Le Gorrec, B. Maschke, and J.A. Villegas. Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain. ESAIM: Control, Optimisation and Calculus of Variations 16(4) (2010) 1077-1093.

Control & Stability of Difference Semigroups

Bernd Klöss

(joint work with K.-J. Engel, M. Kramar-Fijavvz, R. Nagel and E. Sikolya)

1. Wellposedness of Difference Semigroups

We consider the operator

$$A := \operatorname{diag}\left(c_j \frac{d}{dx}\right)_{j=1}^{m}, D(A) := \left\{g \in \mathrm{W}^{1,p}([0,1], \mathbb{C}^{\mathrm{m}}) | \;\; Kg(1) = Lg(0)\right\},$$

where $K,\ L \in M_{m\times m}(\mathbb{C})$ such that $\operatorname{rank}(K|L) = m$. This operator describes a transport process on a graph consisting of m edges where the structure and dynamical behavior is coupled via the boundary conditions appearing in $D(A)$. The following theorem holds.

Theorem. *Assume that* $\operatorname{rank}(K|L) = m$. *Then the operator A generates a C0-semigroup if and only if K is invertible. In that case* $f(1) = \underbrace{K^{-1}L}_{=:\mathbb{B}} f(0)\ \forall f \in$

$D(A)$.

In this case we refer to A as a *difference operator.* If $c_j = 1$, then the following representation formulae is valid

$$(T(t)f)(s) = \mathbb{B}^n f(s+t-n) \text{ for } n \leq s+t < n+1, \;\; n \in \mathbb{N}_0,$$

$s \in [0,1]$, $t \geq 0$ and $f \in \mathrm{L}^p([0,1], \mathbb{C}^m)$. This formulae basically tells us that the process is coded in the matrix $\mathbb{B}$. Using standard semigroup theory one can prove the following for the case of general speeds.

Theorem. *For a difference operator A on* $\mathrm{L}^2([0,1], \mathbb{C}^m)$ *the following holds.*

(1) *A generates a contraction semigroup if and only if $\mathbb{B}$ is contractive.*

(2) *A generates a group if and only if $\mathbb{B}$ is invertible.*
(3) *A generates a unitary group if and only if $\mathbb{B}$ is unitary.*

2. Control of Difference Semigroups

We want to determine the reachable states of a difference process when a feedback is applied in the vertices of the graph. For this purpose we use the following theory of abstract boundary control systems following [2].

Let X, ∂X and U be Banach spaces referred to as the **state**, **boundary** and **control space**, respectively. On these spaces consider

(1) a linear closed and densely defined **system operator** $A_m : D(A_m) \subset X \to X$,
(2) a **boundary operator** $Q \in \mathcal{L}([D(A_m)], \partial X)$ and
(3) a **control operator** $B \in \mathcal{L}(U, \partial X)$.

Then we consider the **abstract Cauchy problem with boundary control**

$$\begin{cases} \dot{x}(t) &= A_m x(t), \quad t \geq 0, \\ Qx(t) &= Bu(t), \quad t \geq 0, \\ x(0) &= 0, \end{cases} \tag{1}$$

where $x : \mathbb{R}_+ \to X$ is the state trajectory and $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+, U)$ is a control function. We are interested in classical solutions for this system and make the following assumptions.

Assumptions. *1. Q is surjective.*
2. $A := (A_m)_{|\ker Q}$ generates a C0-semigroup $(T(t))_{t \geq 0}$.

Under these assumptions, G. Greiner showed that for every $\lambda \in \rho(A)$ the operator $Q_{|\ker(\lambda - A_m)}$ has a bounded inverse $Q_\lambda : \partial X \to X$, called the **Dirichlet operator** ([3]). Using this operator and the extrapolated operator A_{-1} one can characterize classical solutions of (1) via Balakrishnan's variation of parameters formula.

Theorem. *Let $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}_+, U)$ and $\lambda \in \rho(A)$. If $x(\cdot)$ is a classical solution of system (1), then*

$$x(t) = (\lambda - A_{-1}) \int_0^t T(t-s) Q_\lambda B u(s) ds, \quad t \geq 0. \tag{2}$$

Based on (2) we define controllability.

Definition. *The **controllability map** $\mathcal{B}_t \in \mathcal{L}(\mathrm{L}^p([0,t], U), X_{-1})$ is*

$$\mathcal{B}^{BC}_t u(\cdot) := (\lambda - A_{-1}) \int_0^t T(t-s) Q_\lambda B u(s) ds, \quad t \geq 0.$$

*Moreover, the **reachability space in time** $t > 0$ is* $\mathcal{R}^{BC}_t := \mathrm{Ran}\left(\mathcal{B}^{BC}_t\right)$.

We can now examine the abstract Cauchy problem associated to a difference operator with speeds $c_j = 1$ for all $j = 1, \dots, m$ on boundary controllability. More precisely, we consider the transport system

$$\begin{cases} \dot{z}_j(t,s) &= z_j'(t,s), & s \in [0,1], \quad t \geq 0, \quad j = 1, \dots, m, \\ z(t,1) &= \mathbb{B} z(t,0) + u(t) \cdot u_0, & t \geq 0, \\ z(0,s) &= 0, & s \in [0,1]. \end{cases} \tag{3}$$

Then one can use the above theory to prove the following result.
Theorem. *For $t \geq m$ the reachability space of system (3) is*

$$\mathcal{R}_t^{BC} = \mathrm{L}^p([0,1],\mathbb{C}) \otimes \operatorname{span}\left\{u_0, \mathbb{B}u_0, \ldots, \mathbb{B}^{m-1}u_0\right\}$$

3. Stability for Difference Semigroups

In the last part of the talk we examine difference semigroups on stability. A fundamental result is the principle of linear stability.
Theorem. *Let $(A, D(A))$ be a difference operator on $\mathrm{L}^2([0,1],\mathbb{C}^m)$. Then A satisfies*

$$s(A) = w_0(A).$$

Now, using the determinant function

$$\chi : \mathbb{C} \to \mathbb{C}, \;\; \chi(\lambda) := \det\left(\mathrm{Id} - \operatorname{diag}\left(e^{-\frac{\lambda}{c_j}}\right)\mathbb{B}\right)$$

associated to the matrix $\mathbb{B}$, Liapunov's theorem follows for difference semigroups.
Corollary. *$(T(t))_{t\geq 0}$ is exp. stable on $\mathrm{L}^2([0,1],\mathbb{C}^m) \Leftrightarrow \exists \delta > 0$ s.t. $\chi(\lambda) = 0 \Rightarrow \Re(\lambda) \leq -\delta$.*
Unfortunately exponential stability is hard to find in many applications. In fact, it frequently occurs that A has imaginary eigenvalues. To examine this problem, one can reduce the stability concept and switch to a smaller space. The first step is to apply the Jacobs–DeLeeuw–Glicksberg splitting.
Proposition. *Let $(A, D(A))$ be a dissipative difference operator on $\mathrm{L}^2([0,1],\mathbb{C}^m)$. Then we have the orthogonal splitting*

$$\mathrm{L}^2([0,1],\mathbb{C}^m) = X_r \oplus X_s. \tag{4}$$

where X_s denotes the part of all states x, where $\|T(t)x\| \to 0$.
So the only space where one can expect good stability properties is X_s. However, in the case of finitely many imaginary eigenvalues, exponential stability fails even on $X_s \cap D(A)$.
Theorem. *Let $(A, D(A))$ be a difference operator on $\mathrm{L}^2([0,1],\mathbb{C}^m)$ generating the bounded difference semigroup $(T(t))_{t\geq 0}$. If*

1. *there exists $\alpha \in \mathbb{R}$ such that $\chi(i\alpha) = 0$ and*
2. *there exists $\lambda_0 > 0$ such that $\chi(\lambda) \neq 0$ for all $\lambda \in i\mathbb{R}\backslash i(-\lambda_0, \lambda_0)$,*

then $(T(t))_{t\geq 0}$ is not exponentially stable on $X_s \cap D(A)$.

To overcome this problem one examines the intermediate concept of polynomial stability.
Theorem. *Let $(A, D(A))$ be a difference operator on $\mathrm{L}^2([0,1],\mathbb{C}^m)$ generating the bounded difference semigroup $(T(t))_{t\geq 0}$. If there exists $C, \lambda_0 > 0$ and $N \in \mathbb{N}$ such that*

$$|\chi(\lambda)| \geq \frac{C}{|\lambda|^N} \text{ for } \lambda \in i\mathbb{R}\backslash i(-\lambda_0, \lambda_0),$$

then $(T(t))_{t\geq 0}$ *is polynomially stable on* $X_s \cap D(A)$*. More precisely, for every* $f \in X_s \cap D(A)$ *and* $\varepsilon > 0$ *there exists a constant* M *such that*

$$\|T(t)f\| \leq \frac{M}{t^{\frac{1}{N}}}\|f\|_A, \ \ t > 0.$$

References

[1] A. Borichev and Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. **347**, (2010), 455-478.
[2] K.-J. Engel, B. Klöss, M. Kramar Fijavvz, R. Nagel and E. Sikolya, *Maximal controllability for boundary control problems*, Appl. Math. and Opt. **62** (2010), 205-227.
[3] G. Greiner, *Perturbing the boundary conditions of a generator*, Houston J. Math. **13**, (1987), 213-229.
[4] B. Klöss, *Difference operators as semigroup generators*, Semigroup Forum **81**, (2010), 461-482.
[5] B. Klöss, *Difference Semigroups for Vibrating Networks*, PhD-thesis published by Dr. Hut Verlag, 2010.
[6] B. Klöss, *The Flow Approach to Waves in Networks*, submitted to Operators and Matrices (2010).

Fast control cost for heat-like semigroups: Lebeau-Robbiano strategy and Hautus test

Luc Miller

(joint work with Thomas Duykaerts)

Since the seminal work of Russell and Weiss in [7], resolvent conditions for various notions of admissibility, observability and controllability, and for various notions of linear evolution equations have been studied intensively, sometimes under the name of infinite-dimensional Hautus test, cf. [8, 2]. This talk based on [1] investigates resolvent conditions for null-controllability in arbitrary time: necessary conditions for general semigroups, and sufficient conditions for analytic normal semigroups and semigroups with negative self-adjoint generators.

1. Introduction

Let $-A$ be the generator of a strongly continuous semigroup on a Hilbert space $\mathcal{E}$. Let C be a bounded operator from the domain $\mathrm{D}(A)$ with the graph norm to another Hilbert space $\mathcal{F}$. The norms in $\mathcal{E}$ and $\mathcal{F}$ are denoted $\|\cdot\|$. We refer to the monograph [8] for a full account of the control theory of semigroups.

Recall the usual *admissibility* condition (for some time $T > 0$ hence all $T > 0$),

$$\exists K_T > 0, \ \forall v \in \mathrm{D}(A), \quad \int_0^T \|Ce^{-tA}v\|^2 dt \leq K_T\|v\|^2. \tag{1}$$

If C is admissible for A then null-controllability at time T is equivalent to *final-observability* at time T (cf. [8], i.e.

$$\exists \kappa_T > 0, \ \forall v \in \mathcal{E}, \quad \|e^{-TA}v\|^2 \leq \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt. \tag{2}$$

The control property investigated here is (2) *for all* $T > 0$.

1.1. **Control cost.** The coefficient κ_T in (2) is the *control cost*: it is the ratio of the size of the input over the size of the initial state which the input steers to the zero final state in a lapse of time T. It blows up as $T \to 0$. E.g. for the heat semigroup on a compact manifold M with Dirichlet boundary conditions observed from a subset Ω: $\kappa_T \leq c_0 \exp(2c/T)$, $T \in (0,1)$, where c_0 is a positive constant, implies $c \geq d^2/4$ where d is the furthest a point of M can be from Ω, and is implied by $c > 3L^2/4$ where L is the length of the longest generalized geodesic in M which does not intersect Ω ($L < +\infty$ is known as the condition of Bardos-Lebeau-Rauch).

For many evolutions of parabolic type, κ_T is bounded by $c_0 \exp(2c/T^\beta)$ where c, c_0 and β are positive constants. E.g. thermoelastic plates without rotatory inertia, the plate equation with square root damping, diffusions in discontinuous media or in a potential well, diffusions generated by the fractional Laplacian or non-selfadjoint elliptic generators, cf. references in [5].

1.2. **Resolvent conditions.** The resolvent condition: $\exists M > 0$,

$$\|v\|^2 \leq \frac{M}{(\operatorname{Re}\lambda)^2}\|(A-\lambda)v\|^2 + \frac{M}{\operatorname{Re}\lambda}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0. \tag{3}$$

was introduced in [7] as a necessary condition for exact observability in infinite time of exponentially stable semigroups.

When A is skew-adjoint (equivalently when the semigroup is a unitary group), it was proved in [3] that the following resolvent condition is necessary and sufficient for final-observability (hence exact observability) in some time $T > 0$: $\exists M > 0$,

$$\|v\|^2 \leq M\|(iA-\lambda)v\|^2 + M\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \in \mathbb{R}.$$

We refer to [6] for more background and references. This result was extended to some more general groups in [2, theorem 1.2].

When $-A$ generates an exponentially stable normal semigroup, [2, theorem 1.3] proves that the resolvent condition (3) is sufficient for the weaker notion:

$$\exists T > 0, \exists \kappa_T > 0, \forall v \in \mathcal{E}, \|e^{-TA}v\|^2 \leq \kappa_T \int_0^\infty \|Ce^{-tA}v\|^2 dt. \tag{4}$$

In this framework (4) implies (2) for *some* time T.

But it seems that resolvent conditions for final-observability for *any* $T > 0$ in (2) has not been investigated yet, although it is quite natural for heat-like semigroups.

2. Results

2.1. **Necessary resolvent conditions for semigroups.** The proof mainly consists in changing i into -1 in [3, lemma 5.2]. Cf. also the proof of [7, theorem 1.2].

Theorem 1. *Let* $B_T = \sup_{t\in[0,T]} \|e^{-tA}\|$ *be the semigroup bound up to time* T. *If* (1) *and* (2) *hold then :* $\forall v \in \mathrm{D}(A)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$,

$$\|v\|^2 \leq 2e^{2T\operatorname{Re}\lambda}\left((B_T^2 + 2\kappa_T K_T)\frac{\|(A-\lambda)v\|^2}{(\operatorname{Re}\lambda)^2} + \kappa_T\frac{\|Cv\|^2}{\operatorname{Re}\lambda}\right),$$

Theorem 2. *If final-observability* (2) *holds for all* $T \in (0, T_0]$ *with the control cost* $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$ *for some positive* β, c *and* c_0 *then the resolvent condition*

$$\|v\|^2 \leq a_0 e^{2a(\operatorname{Re}\lambda)^\alpha}\left(\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0,$$

holds with power $\alpha = \frac{\beta}{\beta+1}$ *and rate* $a = c^{\frac{1}{\beta+1}}\frac{\beta+1}{\beta^\alpha}$.

It still holds for $\lambda \in \mathbb{C}$ *if* $\operatorname{Re}\lambda$ *is replaced by* $\operatorname{Re}_+\lambda := \max\{\operatorname{Re}\lambda, 0\}$.

2.2. **Sufficient resolvent conditions for an analytic normal semigroup.** The proof is based on the Lebeau-Robbiano strategy of [5]. N.b. (1) is not assumed.

Theorem 3. *Assume that* $-A$ *generates an analytic normal semigroup, hence there exists* $\omega \in \mathbb{R}$ *and* $\theta \in [0, \frac{\pi}{2})$ *such that* $\sigma(A) \subset \{z \in \mathbb{C} : \arg(z-\omega) \leq \theta\}$.

The resolvent condition with $\alpha \in (0,1)$, $\omega_0 < \omega$, $\lambda_0 > \omega_0$, *positive* a_0 *and* a,

$$\|v\|^2 \leq \frac{\cos^2\theta}{(\lambda-\omega_0)^2}\|(A-\lambda)v\|^2 + a_0 e^{2a\lambda^\alpha}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \geq \lambda_0,$$

implies final-observability (2) *for all time* $T > 0$ *with the control cost estimate*

$$\limsup_{T\to 0} T^\beta \ln \kappa_T \leq 2^{1-\beta} a^{\beta+1} (\beta+1)^{\beta(\beta+1)} \beta^{-\beta^2}, \quad \textit{where } \beta = \frac{\alpha}{1-\alpha}.$$

2.3. **Sufficient resolvent condition for a negative self-adjoint generator.** The proof combines the Lebeau-Robbiano strategy of [5], the control transmutation method of [4] (which deduces the final-observability of the heat-like equation $\dot{v} + Av = 0$ from the exact observability of the wave-like equation $\ddot{w} + Aw = 0$) and results on resolvent conditions from [6].

Theorem 4. *Assume that the positive self-adjoint operator* A *and the operator* C *bounded from* $\mathrm{D}(\sqrt{A})$ *with the graph norm to* $\mathcal{F}$ *satisfy the admissibility and observability conditions with nonegative powers* γ *and* δ, *positive* L_* *and* M_*:

$$\|Cv\|^2 \leq L_*\lambda^\gamma\left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|v\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \geq \inf A,$$

$$\|v\|^2 \leq M_*\lambda^\delta\left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \geq \inf A.$$

If $\gamma + \delta < 1$ *then final-observability* (2) *holds for all* $T > 0$ *with the cost estimate*

$$\limsup_{T\to 0} T^\beta \ln \kappa_T < +\infty, \quad \textit{where } \beta = \frac{1+\gamma+\delta}{1-\gamma-\delta}.$$

The assumption of the control transmutation method corresponds to $\gamma = \delta = 0$. The Russell-Weiss condition (3) corresponds to $\delta = -1$.

A logarithmic improvement of this theorem is also proved in [1] thanks to this new variant of the Lebeau-Robbiano strategy of [5]:

Theorem 5. *Assume the admissibility condition* (1) *and that* $-A$ *generates a normal semigroup. If the logarithmic observability condition on spectral subspaces*

$$\|v\|^2 \leq a_0 e^{2a\lambda/(\log(\log\lambda))^\alpha \log\lambda)} \|Cv\|^2, \quad v \in \mathbf{1}_{\operatorname{Re} A<\lambda}\mathcal{E}, \quad \lambda \geq \lambda_0.$$

holds with $\alpha > 2$, λ_0, a_0, *a positive then final-observability* (2) *holds for all* $T > 0$.

References

[1] T. Duyckaerts and L. Miller, *Resolvent conditions for the control of parabolic equations*, manuscript, 2010, soon available online at `http://hal.archives-ouvertes.fr`.

[2] B. Jacob and H. Zwart, *On the Hautus test for exponentially stable C_0-groups*, SIAM J. Control Optim. **48** (2009), no. 3, 1275–1288.

[3] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444.

[4] L. Miller, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim. **45** (2006), no. 2, 762–772.

[5] L. Miller, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Contin. Dyn. Syst. B **14** (2010), no. 4, 1465–1485.

[6] L. Miller, *Resolvent conditions for the control of unitary groups*, manuscript, 2010.

[7] D. L. Russell and G. Weiss, *A general necessary condition for exact observability*, SIAM J. Control Optim. **32** (1994), no. 1, 1–23.

[8] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser Verlag, Basel, 2009.

C_0–Semigroups: my view

Rainer Nagel

I. Basic Theory

Every C_0-**semigroup** $(T(t))_{t\geq 0}$ has a **generator** A whose **resolvent** $R(\lambda, A)$ exists at least for λ in some right halfplane. The basic theory deals with the relations between these three objects and shows that each one determines the other two uniquely. So the motivation for this theory can come from dynamical systems ($\widehat{=}$ C_0-semigroups), differential equations ($\widehat{=}$ generators) or holomorphic functions ($\widehat{=}$ resolvents). While this theory has reached a certain completion, I want to state what I consider the major open problem.

Problem 1.
Let A and B be two (unbounded) linear operators on a Banach space. Under which conditions and for which choice of domains is the "sum" $A + B$ the generator of a C_0-semigroup?

Clearly, the answer should include all / many of the known perturbation results (see [5] and [3, Chap. VI.9]).

Special Case.
Let $A(t), t \in \mathbb{R}$, be a family of generators on a Banach space X. Consider

$$\mathcal{X} := C_0(\mathbb{R}, X)$$

and

$$\mathcal{A}f(s) := -f'(s) + A(s)f(s)\ ,\ s \in \mathbb{R},$$

on some appropriate domain. Under which conditions on $A(\cdot)$ and for which domain is $\mathcal{A}$ the generator of a C_0-semigroup?

Clearly, the answer should include the so-called Kato conditions for the wellposedness of nonautonomous abstract Cauchy problems (see e.g. [5]).

II. Asymptotic Behavior

The most fascinating aspect (to me) of this theory concerns the behavior of the semigroup $(T(t))_{t\geq 0}$ as $t \to \infty$. For a systematic treatment see the recent monograph by Eisner [2]. Quite standard are by now the various characterizations (and counterexamples) of exponential stability through the location of the spectrum of the generator. For strong stability, i.e., $\lim_{t\to\infty} \|T(t)x\| = 0$ for each $x \in X$, the Arendt-Batty-Lyubich-Vu-Theorem yields a nice and useful sufficient spectral condition (see [1, Chap.5.5]). A satisfactory characterization (or even a useful sufficient spectral condition) for weak stability, i.e., $\lim_{t\to\infty} \langle T(t)x, x'\rangle = 0$ for all $x \in X$, $x' \in X'$ is still missing.

The following recent theorem by T. Eisner shows that C_0-semigroups can have a quite irregular behavior (as opposed to the above stability concepts).

Theorem 2. [see [2], Theorem IV.3.20]
In the set of all unitary C_0-semigroups on a separable infinite-dimensional Hilbert space ***most*** *(in the sense of category) semigroups have the following properties.*

(1) *there exists a set $M \subset \mathbb{R}_+$ with density 1 such that*

$$\lim_{t\to\infty, t\in M} T(t) = 0 \text{ weakly, and}$$

(2) *for every $\lambda \in \{z \in \mathbb{C} \mid |z| = 1\}$ there exists $\{t_j^{(\lambda)}\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} t_j^{(\lambda)} = \infty$ such that*

$$\lim_{j\to\infty} T(t_j^{(\lambda)}) = 0 \text{ strongly.}$$

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems* , Birkhäuser Verlag, 2001.
[2] T. Eisner, *Stability of Operators and C_0-Semigroups*, Birkhäuser Verlag, 2010.
[3] K.-J. Engel, R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.
[4] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, 1957.

[5] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1995.
[6] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

Well-posedness, controllability and model reduction

Mark R. Opmeer

The first part of the talk considered distributional control systems [1] and (distributional) resolvent linear systems [2]. Some open problems were mentioned. In particular, whether every analytic function that is bounded in norm by a polynomial on an exponential region is the transfer function of such a system (if 'exponential region' is replaced here by right half-plane, then this is known to be true [1]) and whether a (generalized) Lax-Phillips semigroup formulation of these systems is possible.

The second part of the talk considered model reduction. It was argued that the Hankel operator of the system being in Schatten classes is important for the error analysis in model reduction. It was shown [3] that for the usual class of systems with an analytic semigroup and with at least one of the input or the output space finite-dimensional, the Hankel operator in fact belongs to all the Schatten classes S_p, $p > 0$. It was further shown that certain coupled hyperbolic-parabolic partial differential equations have a property that could be described as generating a partial analytic semigroup. This again has applications in model reduction. The proof of this semigroup property used well-posedness of input-state-output systems where a mixture of L^1 and L^2 well-posedness was needed.

The third part of the talk considered systems that are exactly controllable and exactly observable on the same state space and how these systems from a model reduction point of view are somewhat problematic.

References

[1] Mark R. Opmeer, *Distribution semigroups and control systems*, J. Evol. Equ., **6** (2006), 145–159.
[2] Mark R. Opmeer, *Infinite-dimensional linear systems: a distributional approach*, Proc. London Math. Soc., **91** (2005) 738–760.
[3] Mark R. Opmeer, *Decay of Hankel singular values of analytic control systems*, Systems Control Lett., **59** (2010), 635–638.

Interpolation and finite-time controllability

Jonathan R. Partington
(joint work with Birgit Jacob, Sandra Pott)

We present results from [4]. Consider systems of the form

(S) $\dot{x}(t) = Ax(t) + Bu(t), \quad (t \geq 0), \qquad \text{with} \quad x(0) = x_0.$

We suppose that A is the generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space H and that the eigenvectors of $-A$ form a Riesz basis $(\phi_n)_{n\in\mathbb{N}}$ of H with the corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ forming a Blaschke sequence in the open right half-plane $\mathbb{C}_+$. The eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{C}_+$ are uniformly bounded away from the imaginary axis. Moreover, we assume a finite-dimensional input space, with the operator B given by

$$Bv = \sum_{n=1}^{\infty} \langle v, b_n \rangle \phi_n, \qquad v \in \mathbb{C}^N,$$

for a sequence $(b_n)_n \subset \mathbb{C}^N$.

In [2], infinite-time exact controllability of the system (S) in the case $N = 1$ was shown to be equivalent to solution of the following interpolation problem in the Hardy space $H^2(\mathbb{C}_+)$:

- For every $(c_n) \in \ell^2$ there is a function $g \in H^2(\mathbb{C}_+)$ with $b_n g(\lambda_n) = c_n$ for each n.

Results of McPhail [6] were used to solve this problem in terms of Carleson measures [2]; the case $N > 1$ was analysed in [3]. From now on, for convenience of exposition, we sometimes state results just for the case $N = 1$.

For finite-time controllability, inputs lie in $L^2(0,\tau)$ for some $\tau > 0$ and we are led to consider interpolation in the model space $K_{\Theta_\tau} = H^2(\mathbb{C}_+) \ominus \Theta_\tau H^2(\mathbb{C}_+)$, where $\Theta_\tau(s) = e^{-s\tau}$. Previous work has concentrated on the case when the sequence (λ_n) is Carleson (an admissibility assumption), but this is not necessary here. The following condition on the Blaschke product β with zeroes (λ_n) is central to our considerations: it was introduced in an entirely different context in [5].

(JZ) There are constants a, $\delta > 0$ such that $\|(\beta(s))^{-1}\| \leq \frac{1}{a}$ on the strip $S_\delta = \{s \in \mathbb{C} : 0 < \operatorname{Re} s < \delta\}$.

The following lemma is easily deduced from [7, Lemma D.4.4.4, Cor. D.4.4.5].

Lemma 1. The following conditions are equivalent:

(1) the Hankel operator $\Gamma_{\overline{\beta}\Theta_\tau}$ has norm strictly less than 1, i.e., $\operatorname{dist}(\overline{\beta}\Theta_\tau, H^\infty(\mathbb{C}_+)) < 1$;

(2) given $F \in H^2(\mathbb{C}_+)$ there exists a function $G \in K_{\Theta_\tau}$ such that $F(\lambda_n) = G(\lambda_n)$ for all n.

For $\lambda \in \mathbb{D}$ we write K_λ for the normalized reproducing kernel for $H^2 = H^2(\mathbb{D})$, that is,

$$K_\lambda(z) = \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda}z}, \qquad (z \in \mathbb{D}).$$

For $b \in H^2_0$, we write $\Gamma_{\overline{b}}$ for the Hankel operator $\Gamma : H^2 \to \overline{H^2_0}$, defined by $\Gamma_b f = P_-(\overline{b}f)$. Then we have the following quantitative form of Bonsall's reproducing kernel thesis [1], which we deduce from an analogous result for Carleson embeddings given in [8].

Theorem 2. For a Hankel operator $\Gamma = \Gamma_{\overline{b}}$ we have

$$\|\Gamma\| \le M \sup_{\lambda\in\mathbb{D}} \|\Gamma K_\lambda\|,$$

where M can be taken to be $4\sqrt{2e}$.

This result is used in deriving an interpolation theorem, as follows:

Theorem 3. Condition (JZ) holds if and only if if the equivalent conditions of Lemma 1 hold for some $\tau > 0$. More precisely, there exists a constant $m > 0$ such that given a, δ from Condition (JZ), the equivalent conditions of Lemma 1 hold for each $\tau > \frac{2}{\delta} \log\left(\frac{m}{\sqrt{\pi}}(a^{-1}+1)\log(N+1)\right)$. In the case $N = 1$, m may be chosen as $\frac{4\sqrt{2e}}{\log 2}$.

Finally, we deduce a result on finite-time controllability:

Theorem 4. Suppose that Condition (JZ) holds and that the system (S) is exactly controllable in infinite time. Then (S) is exactly controllable in any time τ satisfying

$$\tau > \frac{2}{\delta} \log\left(\frac{m}{\sqrt{\pi}}(a^{-1}+1)\log(N+1)\right)$$

for some constant $m > 0$. For $N = 1$, m may be chosen as $\frac{4\sqrt{2e}}{\log 2}$.

References

[1] F. F. Bonsall, *Boundedness of Hankel matrices*, J. London Math. Soc. (2) **29** (1984), no. 2, 289–300.

[2] B. Jacob and J.R. Partington, *On controllability of diagonal systems with one-dimensional input space*, Systems Control Lett. **55** (2006), 321–328.

[3] B. Jacob, J.R. Partington and S. Pott, *Interpolation by vector-valued analytic functions, with applications to controllability*, J. Funct. Anal. **252** (2007), 517–549.

[4] B. Jacob, J.R. Partington and S. Pott, *Weighted interpolation in Paley-Wiener spaces and finite-time controllability*, J. Funct. Anal. **259** (2010), 2424–2436.

[5] B. Jacob and H. Zwart, *Properties of the realization of inner functions*, Math. Control Signals Systems **15** (2002), no. 4, 356–379.

[6] J. D. McPhail, *A weighted interpolation problem for analytic functions*, Studia Math. **96** (1990), no. 2, 105–116.

[7] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz. Vol. 2. Model operators and systems.* Translated from the French by Andreas Hartmann. American Mathematical Society, Providence, RI, 2002.
[8] S. Petermichl, S. Treil and B. D. Wick, *Carleson potentials and the reproducing kernel thesis for embedding theorems*, Illinois J. Math. **51** (2007), no. 4, 1249–1263.

Scattering and Impedance Passive and Conservative Systems

Olof J. Staffans
(joint work with George Weiss)

In my talk I discussed systems which are either scattering or impedance passive or conservative. There are several similarities between scattering and impedance systems, but there are also significant differences.

In both cases we are talking about input/state/output systems, which have an input $u(t)$ in a Hilbert input space $\mathcal{U}$, a state $x(t)$ in a Hilbert state space $\mathcal{X}$, and an output $y(t)$ in a Hilbert output space $\mathcal{Y}$. The dynamics of the system is defined by an equation of the type

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \qquad t \geq 0, \tag{1}$$

where S is a closed densely defined operator from $\mathcal{X} \oplus \mathcal{U}$ to $\mathcal{X} \oplus \mathcal{Y}$ with domain $\operatorname{dom}(S)$. We say that (x, u, y) is a classical trajectory of (1) if x is continuously differentiable, u and y are continuous and, (1) holds.

In the case of a scattering passive system the classical trajectories satisfy the power inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \leq \|u(t)\|^2_{\mathcal{U}} - \|y(t)\|^2_{\mathcal{Y}}, \qquad t \geq 0, \tag{2}$$

whereas the corresponding inequality for an impedance passive system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2_{\mathcal{X}} \leq (y(t), u(t))_{\mathcal{U}}, \qquad t \geq 0; \tag{3}$$

in the impedance case we assume, in addition, that $\mathcal{Y} = \mathcal{U}$. In the case of conservative systems the two inequalities above are replaced by equalities, and they are required to hold both for the original systems, and for the dual systems that one gets by replacing S by its adjoint S^*.

A scattering passive system is always well-posed, and in the case of a scattering passive system the operator S is a so called *system node.* On the contrary, an impedance passive system need not be well-posed, and it need not even be a system node. However, impedance systems have a very simple characterization of a different type. We can always split the operator S into two parts $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, where $A\&B$ maps $\operatorname{dom}(S)$ into the state space $\mathcal{X}$ and $C\&D$ maps $\operatorname{dom}(S)$ into Y. Impedance passivity is characterized by the fact that the operator $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is maximal dissipative.

There is a simple method that can be used to convert impedance passive systems into scattering passive systems, which in the case of an impedance conservative system results in a scattering conservative system. The idea is the following. If we denote the impedance input by e (for "effort") and the impedance output by f (for "flow"), and if we map each trajectory $(x(t), e(t), f(t))$ of the impedance system $\begin{bmatrix} \dot{x}(t) \\ f(t) \end{bmatrix} = S_{\rm imp} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ into a new family of functions $(x(t), u(t), y(t))$ by taking

$$u(t) = \tfrac{1}{\sqrt{2}}[e(t) + f(t)],$$
$$y(t) = \tfrac{1}{\sqrt{2}}[e(t) - f(t)],$$

then this family is the set of classical trajectories of a scattering passive system $\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_{\rm sca} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$. The above mapping is called the *external Cayley transform.* The above idea leads to the following theorem, which is proved in [WS10]:

Theorem 1. *Let* $S_{\rm imp} = \begin{bmatrix} [A\&B]_{\rm imp} \\ [C\&D]_{\rm imp} \end{bmatrix}$ *be an operator on* $X \oplus U$ *with domain* $\operatorname{dom}(S_{\rm imp}) \subset X \oplus U$ *such that* $T := \begin{bmatrix} [A\&B]_{\rm imp} \\ -[C\&D]_{\rm imp} \end{bmatrix}$ *(with the same domain) is maximal dissipative. Then the operator*

$$E_{\rm imp} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\rm imp} \end{bmatrix} \right) \tag{4}$$

is injective on $\operatorname{dom}(S_{\rm imp})$*. We denote its range by* $\operatorname{dom}(S_{\rm sca})$ *and we define* $S_{\rm sca}$ *(with domain* $\operatorname{dom}(S_{\rm sca})$*) by*

$$S_{\rm sca} = \begin{bmatrix} [A\&B]_{\rm sca} \\ [C\&D]_{\rm sca} \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\rm imp} \\ 0 \quad 0 \end{bmatrix} \right) E_{\rm imp}^{-1}. \tag{5}$$

Then $S_{\rm sca}$ *is a scattering passive system node and* $E_{\rm imp}^{-1} = E_{\rm sca}$ *from* (7).

We denote by $A_{\rm sca}$*,* $B_{\rm sca}$ *and* $C_{\rm sca}$ *the semigroup generator, the control operator and the observation operator of* $S_{\rm sca}$*, and we denote by* $\widehat{\mathfrak{D}}_{\rm sca}$ *its transfer function. Then, for all* $s \in \mathbb{C}_+$*,*

(6)
$$\begin{bmatrix} (sI - A_{\rm sca})^{-1} & \frac{1}{\sqrt{2}}(sI - A_{\rm sca})^{-1}B_{\rm sca} \\ \frac{1}{\sqrt{2}}C_{\rm sca}(sI - A_{\rm sca})^{-1} & \frac{1}{2}(I + \widehat{\mathfrak{D}}_{\rm sca}(s)) \end{bmatrix} = \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\rm imp} \\ -[C\&D]_{\rm imp} \end{bmatrix} \right)^{-1}.$$

The operator $S_{\rm imp}$ *can be recovered from* $S_{\rm sca}$ *via the formulas*

$$E_{\rm sca} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\rm sca} \end{bmatrix} \right), \tag{7}$$

$$S_{\rm imp} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\rm sca} \\ 0 \quad 0 \end{bmatrix} \right) E_{\rm sca}^{-1}. \tag{8}$$

The system node $S_{\rm sca}$ *is scattering conservative if and only if* T *is skew-adjoint.*

The above theorem can be used to construct many interesting scattering passive or conservative system nodes by applying the external Cayley transform to an impedance passive or conservative system. The proofs of the result mentioned in the abstract by George Weiss in this mini workshop are partially based on the above theorem.

Generally speaking, many equations from mathematical physics come naturally in an impedance formulation. For example, in an electrical system there is a natural division of signals into pairs of voltages and currents, the product of which gives the power. The name "impedance" is, in fact, taken from circuit theory, where it is used for the transfer function from current inputs to voltage outputs. A similar situation occurs in many partial differential equations, where the boundary conditions naturally split into conditions of Dirichlet and Neumann types.

Often the impedance formulation is algebraically simpler to work with than the scattering formulation, but on the other hand, the scattering version of a system has better well-posedness properties.

References

[KS09] Mikael Kurula and Olof J. Staffans, *Well-posed state/signal systems in continuous time*, Complex Anal. Oper. Theory **4** (2009), 319–390.

[Kur10] Mikael Kurula, *On passive and conservative state/signal systems in continuous time*, Integral Equations Operator Theory **67** (2010), 377–424, 449.

[WS10] George Weiss and Olof J. Staffans, *A class of scattering passive linear systems comprising the Maxwell equations*, in preparation, 2010.

Some perturbation results in exact observability theory

Marius Tucsnak

(joint work with Nicolae Cindea, George Weiss)

The study of the robustness of the exact observability property of infinite dimensional systems with respect to perturbations of the generator is a relatively recent subject which has been initiated, at least in an abstract setting, in Hadd [3]. On the other hand, Bardos, Lebeau and Rauch [2] introduced, on particular systems governed by partial differential equations a technique which is by now designated as the *compactness uniqueness method.* This method can be seen as a perturbation argument, tackling the perturbation of the generator by compact operators.

The aim of this talk is to present the compactness uniqueness method in an abstract framework, connecting this type of argument with some recent perturbation results, based on a simultaneous observability theorem proved in Tucsnak and Weiss [4]. We illustrate the abstract results by a detailed study of a perturbed Euler-Bernoulli plate equation.

To state our main abstract result, let X and Y be complex Hilbert spaces which are identified with their duals. $\mathbb{T}$ is a strongly continuous semigroup on X, with generator $A : \mathcal{D}(A) \to X$ and growth bound $\omega_0(\mathbb{T})$. The space X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed.

Theorem 1. *Suppose that $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$. Assume that (A, C) is exactly observable in time $\tau > 0$, i.e., there exists $k_\tau > 0$ such that*

$$\left(\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \, \mathrm{d}t\right)^{1/2} \geq k_\tau \|z_0\| \qquad (z_0 \in \mathcal{D}(A)).$$

Let $P \in \mathcal{L}(X)$ and let $\mathbb{T}^{cl}$ be the strongly continuous semigroup on X generated by $A+P$. Let V be a closed invariant subspace of $\mathbb{T}^{cl}$ and let $P_V \in \mathcal{L}(V,X)$ be the restriction of P to V. Denote

$$M_V = \sup\left\{ \|\mathbb{T}^{cl}_t z_0\| \;\middle|\; t \in [0,\tau],\ z_0 \in V\ ,\ \|z_0\| \leq 1 \right\}.$$

If

$$\|P_V\| \leq \frac{k_\tau}{\tau M_V \|C\|_\tau},$$

then $(A+P, C)$ is exactly observable in time τ on V, i.e., there exists $k^V_\tau > 0$ such that

$$\left(\int_0^\tau \|C\mathbb{T}^{cl}_t z_0\|^2 \,\mathrm{d}t \right)^{1/2} \geq k^V_\tau \|z_0\| \qquad (z_0 \in V \cap \mathcal{D}(A)).$$

For a proof of the above theorem we refer to [5, Section 6.3].

To describe an application of the above result to systems described by partial differential equations, let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N}^*)$ be an open and nonempty set with a C^2 boundary or let Ω be a rectangle. We consider the following initial and boundary value problem :

$$\ddot{w}(x,t) + \Delta^2 w(x,t) - a\Delta w(x,t) + b(x)w(x,t) = 0, \quad \text{for } (x,t) \in \Omega \times (0,\infty) \tag{1}$$

$$w(x,t) = \Delta w(x,t) = 0, \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty) \tag{2}$$

$$w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), \quad \text{for } x \in \Omega, \tag{3}$$

where $a > 0$, $b \in L^\infty(\Omega)$, $w_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $w_1 \in L^2(\Omega)$. We consider the output given by

$$y(t) = \dot{w}(\cdot,t)|_{\mathcal{O}}, \tag{4}$$

where $\mathcal{O}$ is an open and nonempty subset of Ω and a dot denotes differentiation with respect to the time t:

$$\dot{w} = \frac{\partial w}{\partial t}, \quad \ddot{w} = \frac{\partial^2 w}{\partial t^2}.$$

For $n = 2$ the equations (1)-(3) model the vibration of a perturbed Euler-Bernoulli plate with a hinged boundary.

We have the following result, proved in [1] :

Theorem 2. *Let $\mathcal{O} \in \Omega$ be an open and nonempty subset of Ω such that (1)-(4), with $a = 0$, $b = 0$, is exactly observable. Thenthe system (1)-(4) is exactly observable for every $a \geq 0$, $b \in L^\infty(\Omega)$.*

References

[1] N. Cindea and M. Tucsnak, *Internal exact observability of a perturbed Euler-Bernoulli equation*, Annals of the Academy of Romanian Scientists, Series on Mathematics and its Applications **2** (2010), 205–221.
[2] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control. and Optim., **30** (1992), 1024–1065.
[3] S. Hadd, *Exact controllability of infinite dimensional systems persists under small perturbations*, Journal of Evolution Equations **5** (2005), 545–555.
[4] M. Tucsnak and G. Weiss, *Simultaneous exact controllability and some applications*, SIAM J. Control Optim., **38** (2000), 1408–1427.
[5] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkäuser Advanced Texts, Birkäuser, Basel, 2009.

Stochastic PDEs such as Zakai's equation and $\mathbf{H}^\infty$-calculus

Lutz Weis

(joint work with Jan van Neerven and Mark Veraar)

An important equation arising in non-linear filter theory is Zakai's equation

$$dX_t = \Delta X_t + \alpha \nabla X_t \, dW(t) \quad , \quad X(0) = x_0 \tag{1}$$

where $W(t)$ is a cylindrical Wiener process and the solution is expected to be (at least) a $L_p(\mathbb{R}^n)$-valued process for $p \geq 2$. The principal difficulty with this equation is that the noise term depends on the gradient of the solution and this defines precisely the border line of "loss of regularity" one can hope to handle in the stochastic perturbation term.

For equations of the type

$$\partial X_t \; = \; -AX_t + \alpha A^{1/2} \, dW(t) \;, \tag{2}$$

there are, so far, two quite different theories that can show the existence and uniqueness of solutions.

- Da Prato [1] treats generators $-A$ of bounded contractive analytic semigroups on a Hilbert space H. He uses that, in this case, the norm in H is equivalent to the square function norm

$$\|x\|_H \; \approx \; \left(\int_0^\infty \left\| A^{1/2} e^{-tA} x \right\|^2 dt \right)^{1/2} \; . \tag{3}$$

- Krylov [6] established a theory for a large class of elliptic differential operators on $L_p(\mathbb{R}^n)$, where $p \geq 2$, including not only equations of the type (2) but also non-linear equations. As a starting point, he found solutions for equation (1) leaning strongly on methods from harmonic analysis such as Paley-Littlewood theory and BMO-spaces.

We have developed a new approach which extends both of these theories. Our two main tools are:

- an integration theory for UMD-spaces, a class of Banach spaces which includes L_p-spaces for $1 < p < \infty$, worked out in [11],
- the boundedness of the H^∞ functional calculus of the generator A.

The H^∞-calculus takes the role of contractivity in Da Prato's Hilbert space theory (indeed, an analytic generator on a Hilbert space has a bounded H^∞-calculus if and only if e^{-tA} is contractive in an equivalent Hilbert space norm) as well as the role of the harmonic analysis techniques employed by Krylov. In essence this is possible since the boundedness of the H^∞-calculus of A for an analytic generator $-A$ on L_p, for $1 < p < \infty$, can be characterized by the square-function estimate (cf. [5, 8])

$$\left\| \left(\int_0^\infty \left| A^{1/2} e^{-tA} x \right|^2 \, dt \right)^{1/2} \right\|_{L_p} \approx \|x\|_{L_p} , \tag{4}$$

which reduces to the classical Paley-Littlewood estimates if $A = -\Delta$ and to (3) if we consider the Hilbert space case. As far as Paley-Littlewood theory goes, one could say that the H^∞-calculus provides a "custom-made Fourier analysis" for the operator A and a "guideline" on how to extend known results for the Laplace operator Δ to generators of analytic semigroups.

The usefulness of the H^∞-calculus in stochastic analysis is highlighted by the fact that the same square function estimate (4) that characterizes its boundedness also appears in the Ito isomorphism for stochastic integrals in L_p-spaces, e.g. for a deterministic L_p-valued function f, for $1 < p, q < \infty$, we have

$$\left(\mathbb{E} \left\| \int_0^T f(t) \, d\beta(t) \right\|_{L_p}^q \right)^{1/q} \approx \left\| \left(\int_0^T |f(t)|^2 \, dt \right)^{1/2} \right\|_{L_p}$$

where $\beta(\cdot)$ is a Brownian motion (cf. [11]).

Here is a sample result we obtain from our methods. Consider the equation

$$dX(t) = -A(t) X(t) \, dt + \sum_{j=1}^\infty B_j(t) X(t) \, d\beta_j(t) , \tag{5}$$

where:

(i) the operators $A(t)$, $t \in [0, T]$, have uniformly bounded H^∞ functional calculi of the same angle $\sigma < \frac{\pi}{2}$ on a Banach space E that is a closed subspace of a space $L_p(\mu)$ with $2 \le p < \infty$

(ii) $B_j(t) : D(A) \to D\left(A^{1/2}\right)$, for $t \in [0, T]$ and $j \in \mathbb{N}$, is a family of linear operators satisfying a uniform (in t) estimate

$$\mathbb{E} \left\| \sum_{j=1}^\infty \gamma_j B_j(t) x \right\|_{D\left(A^{1/2}\right)} \le L \|x\|_{D(A)}$$

for a sequence $\{\gamma_j\}$ of independent $N(0, 1)$-distributed Gaussian variables

(iii) $\beta_j(\cdot)$, for $j \in \mathbb{N}$, is a sequence of independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}_t, P)$.

Theorem. *Assume (i), (ii), and (iii) from just above hold. Then, for L small enough and $x_0 \in D\left(A^{1/2}\right)$, equation* (5) *has a unique strong solution*

$$X\colon\ [0,T] \times \Omega \to D(A)\ ,$$

specifically, for each $t \in [0,T]$, as functions in E,

$$X(t) = x_0 - \int_0^t AX(s)\ ds + \sum_{j=0}^{\infty} \int_0^T B_j(s)\, X(s)\ d\beta_j(s) \tag{6}$$

P-almost everywhere. Furthermore:

- *$X(\cdot)$ has almost surely continuous paths in E*
- *$X(t)$ has almost surely paths in $L_q([0,T], D(A))$ and $X \in L_q([0,T] \times \Omega, D(A))$.*

Also, the sum in (6) *converges in $L_q([0,T] \times \Omega, D(A))$.*

This linear result, essentially contained in [9], will be used in [10] in order to prove the corresponding theorem for equations with nonlinear $B_j(\cdot)$ and an additional nonlinear term "$F(t, X(t))\ dt$".

The boundedness of the H$^\infty$-calculus can also be expressed in purely operator theoretic terms: it is required that, for each bounded analytic function f on $\Sigma_\sigma := \{\lambda\colon\ |\arg \lambda| < \sigma\}$, the Dunford integral

$$f(A)\, x\ =\ \int_{\partial \Sigma_\sigma} f(\lambda)\, R(\lambda, A)\ d\lambda\ \ ,\quad x \in D(A)$$

can be extended to a bounded operator on E. A further advantage of using the H$^\infty$-calculus in our approach is that we can point to a large literature on this subject (see, e.g., [2, 3, 4, 7] and the literature quoted therein). It is known that operators defined by systems of elliptic operators with rather general coefficient and boundary conditions on a domain in $\mathbb{R}^n$ or a manifold admit a bounded H$^\infty$-calculus, as do Schrödinger operators with rather general potentials and Stokes operators on Helmholtz spaces. Our method allows us to use results, motivated by regularity questions for deterministic evolution equations, as building blocks in our study of regularity results for stochastic evolution equations.

References

[1] Giuseppe Da Prato and Jerzy Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[2] Robert Denk, Matthias Hieber, and Jan Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **166** (2003), no. 788.

[3] Markus Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006.

[4] Nigel Kalton, Peer Kunstmann, and Lutz Weis, *Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators*, Math. Ann. **336** (2006), no. 4, 747–801.

[5] N.J. Kalton and L. Weis, *The H^∞-calculus and square function estimates*, (2004), (preprint).
[6] N. V. Krylov, *An analytic approach to SPDEs*, Stochastic partial differential equations: six perspectives, Math. Surveys Monogr., vol. 64, Amer. Math. Soc., Providence, RI, 1999, pp. 185–242.
[7] Peer C. Kunstmann and Lutz Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations, Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311.
[8] Christian Le Merdy, *On square functions associated to sectorial operators*, Bull. Soc. Math. France **132** (2004), no. 1, 137–156.
[9] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis, *Stochastic evolution equations and maximal L_p regularity*, (in preparation).
[10] ———, *Stochastic maximal L_p regularity*, Ann. Probab., (to appear).
[11] ———, *Stochastic integration in UMD Banach spaces*, Ann. Probab. **35** (2007), no. 4, 1438–1478.
[12] ———, *Stochastic evolution equations in UMD Banach spaces*, J. Funct. Anal. **255** (2008), no. 4, 940–993.

A special class of scattering passive linear systems

George Weiss
(joint work with Olof Staffans)

Given four operators A, B, C, D on appropriate Hilbert spaces, a natural question is whether they determine a scattering passive or conservative (in particular, well-posed) linear system via the equations $\dot{x} = Ax + Bu$, $y = Cx + Du$. This has been studied for the first time in Arov and Nudelman [1], using earlier results about discrete-time scattering passive systems and translating those results using the internal Cayley transform. More results about scattering passive systems were derived in Staffans and Weiss [7] (where they were called dissipative systems) and relatively simple necessary and sufficient conditions for a system node to be scattering conservative were provided in Malinen *et al* [2]. A good overview of these results can be found in the book Staffans [6], and the connection with impedance passive and conservative systems is studied in Staffans [3, 4, 5].

It is of interest to identify large classes of systems where the operators A, B, C, D have a special structure observed in models of mathematical physics, which implies that the system is scattering passive or conservative. Indeed, if we then find a system with this special structure, then we do not have to take the trouble of checking the conditions for scattering passivity or conservativity given in the papers listed earlier (this kind of checking need not be straightforward). Such a special class of conservative systems ("from thin air") has been introduced in Weiss and Tucsnak [9] and further studied in Tucsnak and Weiss [8] and in Staffans [5]. In this paper we give a larger special class, which includes the systems introduced in [9] and also others. We were led to introduce this class by our failure to fit the Maxwell equations into the framework of [9].

In this paper we consider a linear system Σ whose state space X can be decomposed as $X = H \oplus E$, where H and E are Hilbert spaces. The Hilbert space U is both the input space and the output space of Σ. We identify H, E and U with their duals H', E' and U'. The Hilbert space E_0 is a dense subspace of E and the embedding $E_0 \hookrightarrow E$ is continuous. We denote by E_0' the dual of E_0 with respect to the pivot space E, so that

$$E_0 \subset E \subset E_0',$$

densely and with continuous embeddings. Such triples of Hilbert spaces are often encountered in the abstract treatment of partial differential equations. We denote $X_0 = H + E_0$, so that $X_0' = H + E_0'$. We decompose the state of Σ as follows:

$$x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \qquad z_0 \in H, \quad w_0 \in E.$$

We assume that

$$L \in \mathcal{L}(E_0, H), \qquad K \in \mathcal{L}(E_0, U), \qquad G \in \mathcal{L}(E_0, E_0') \tag{1}$$

$$\Re\langle G w_{_0}, w_{_0}\rangle \leq 0 \qquad \forall\ w_{_0} \in E_0, \tag{2}$$

and we define $\overline{A} \in \mathcal{L}(X_0, X_0')$, $B \in \mathcal{L}(U, X_0')$ and $\overline{C} \in \mathcal{L}(X_0, U)$ by

$$\overline{A} = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ K^* \end{bmatrix}, \qquad \overline{C} = \begin{bmatrix} 0 & -K \end{bmatrix}. \tag{3}$$

The equations of the system are

$$\dot{x}(t) = \overline{A}x(t) + Bu(t), \qquad y(t) = \overline{C}x(t) + u(t), \tag{4}$$

where x is the state trajectory, u is the input function and y is the output function. Note that the differential equation above is an equation in X_0'.

We define the domain $\mathcal{D}(A)$ by

$$\mathcal{D}(A) = \{x_0 \in X_0 \mid \overline{A}x_0 \in X\} \tag{5}$$

and we denote by A and C the restrictions of $\overline{A}$ and $\overline{C}$ to $\mathcal{D}(A)$. More explicitly,

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X_0 \;\middle|\; L^* z_0 + \left(G - \frac{1}{2}K^*K\right) w_0 \in E \right\}. \tag{6}$$

Under the assumptions made so far, A is not necessarily closed. But for (4) to define a scattering passive system, we need A to be the generator of a strongly continuous semigroup of operators on X. One way to overcome this problem would be to assume that L is closed. This would indeed work, but it would be too restrictive: it would eliminate the Maxwell equations which we would like to fit into this abstract framework. A better alternative is to assume the following:

(7) $\begin{bmatrix} L \\ K \end{bmatrix}$ (with domain E_0) is closed as an unbounded operator $E \to H + U$.

As we shall see later, this assumption implies that A is maximal dissipative and hence it generates a semigroup of contractions.

Informal statement of the main result. *The equations* (4) *determine a scattering passive system with state space X. This system is scattering conservative if and only if G is skew-adjoint.*

References

[1] D.Z. Arov and M.A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral Eq. & Operator Theory*, 24:1–45, 1996.
[2] J. Malinen, O.J. Staffans and G. Weiss. When is a linear system conservative? *Quarterly of Applied Mathematics*, 64:61–91, 2006.
[3] O.J. Staffans. Passive and conservative continuous-time impedance and scattering systems. Part I: well-posed systems. *Mathematics of Control, Signals and Systems*, 15:291–315, 2002.
[4] O.J. Staffans. Passive and conservative infinite-dimensional impedance and scattering systems (from a personal point of view). In *Mathematical Systems Theory in Biology, Communication, Computation, and Finance*, volume 134 of *IMA Volumes in Mathematics and its Applications*, pages 375–414. Springer-Verlag, New York, 2002.
[5] O.J. Staffans. Stabilization by collocated feedback. pp. 261–278 in *Directions in Mathematical Systems Theory and Optimization*. A. Rantzer and C.I. Byrnes, eds. Springer-Verlag, Berlin, LNCIS vol. 286, 2002.
[6] O.J. Staffans. *Well-Posed Linear Systems*. Cambridge University Press, UK, 2004.
[7] O.J. Staffans and G. Weiss. Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup. *Trans. American Math. Soc.*, 354:3229–3262, 2002.
[8] M. Tucsnak and G. Weiss. How to get a conservative well-posed linear system out of thin air, Part II: Controllability and stability. *SIAM J. Control Optim.*, 42:907–935, 2003.
[9] G. Weiss and M. Tucsnak. How to get a conservative well-posed linear system out of thin air, Part I: Well-posedness and energy balance. *ESAIM: Control, Optim. Calc. Var.* 9:247–274, 2003.

Bilinear Forms on the Dirichlet Space

Brett D. Wick

(joint work with Nicola Arcozzi, Richard Rochberg, and Eric T. Sawyer)

1. Overview

Let $\mathcal{D}$ be the classical Dirichlet space, the Hilbert space of holomorphic functions on the disk with inner product

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA$$

and normed by $\|f\|_{\mathcal{D}}^2 = \langle f, f\rangle_{\mathcal{D}}$. Given a holomorphic *symbol function* b we define the associated Hankel type bilinear form, initially for $f, g \in \mathcal{P}(\mathbb{D})$, the space of polynomials, by

$$T_b\,(f, g) := \langle fg, b\rangle_{\mathcal{D}}\,.$$

The norm of T_b is

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} := \sup\left\{|T_b\,(f, g)| : \|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1\right\}.$$

We say a positive measure μ on the disk is a *Carleson measure for* $\mathcal{D}$ if

$$\|\mu\|_{CM(\mathcal{D})} := \sup\left\{\int_{\mathbb{D}} |f|^2\, d\mu : \|f\|_{\mathcal{D}} = 1\right\} < \infty,$$

and that a function b is in the space $\mathcal{X}$ if the measure $d\mu_b := |b'(z)|^2\, dA$ is a Carleson measure. We norm $\mathcal{X}$ by

$$\|b\|_{\mathcal{X}} := |b(0)| + \left\| |b'(z)|^2\, dA \right\|_{CM(\mathcal{D})}^{1/2}$$

and denote by $\mathcal{X}_0$ the norm closure in $\mathcal{X}$ of the space of polynomials. Our main result is

Theorem 1.

(1) *T_b is bounded if and only if $b \in \mathcal{X}$. In that case*

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\mathcal{X}}\,.$$

(2) *T_b is compact if and only if $b \in \mathcal{X}_0$.*

This result, which had been conjectured by Rochberg for some time, is part of an intriguing pattern of results involving boundedness of Hankel forms on Hardy spaces in one and several variables and boundedness of Schrödinger operators on the Sobolev space.

It is easy to see that $\|T_b\|_{\mathcal{D}\times\mathcal{D}} \leq C\,\|b\|_{\mathcal{X}}$. To obtain the other inequality we must use the boundedness of T_b to show $|b'|^2\, dA$ is a Carleson measure. Analysis of the capacity theoretic characterization of Carleson measures due to Stegenga allows us to focus attention on a certain set V in $\mathbb{D}$ and the relative sizes of $\int_V |b'|^2$ and the capacity of the set $\bar{V} \cap \partial\bar{\mathbb{D}}$. To compare these quantities we construct $V_{\exp}$, an expanded version of the set V which satisfies two conflicting conditions. First, $V_{\exp}$ is not much larger than V, either when measured by $\int_{V_{\exp}} |b'|^2$ or by the capacity of the $\overline{V_{\exp}} \cap \partial\bar{\mathbb{D}}$. Second, $\mathbb{D} \setminus V_{\exp}$ is well separated from V in a way that allows the interaction of quantities supported on the two sets to be controlled. Once this is done we can construct a function $\Phi_V \in \mathcal{D}$ which is approximately one on V and which has Φ_V' approximately supported on $\mathbb{D} \setminus V_{\exp}$. Using Φ_V we build functions f and g with the property that

$$|T_b(f,g)| = \int_V |b'|^2 + \text{ error.}$$

The technical estimates on Φ_V allow us to show that the error term is small and the boundedness of T_b then gives the required control of $\int_V |b'|^2$.

Once the first part of the theorem is established, the second follows rather directly.

2. Reformulation in Terms of Weak Factorization

In his proof Nehari used the fact that any function $f \in H^1(\mathbb{D})$ could be factored as $f = gh$ with $g, h \in H^2(\mathbb{D})$, $\|f\|_{H^1(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})} \|h\|_{H^2(\mathbb{D})}$. In [3] the authors develop a weak substitute for this. For two Banach spaces of functions, $\mathcal{A}$ and $\mathcal{B}$, defined on the same domain, define the weakly factored space $\mathcal{A} \odot \mathcal{B}$ to be the completion of finite sums $f = \sum a_i b_i$; $\{a_i\} \subset \mathcal{A}$, $\{b_i\} \subset \mathcal{B}$ using the norm

$$\|f\|_{\mathcal{A}\odot\mathcal{B}} = \inf\left\{\sum \|a_i\|_{\mathcal{A}} \|b_i\|_{\mathcal{B}} : f = \sum a_i b_i\right\}.$$

It is shown in [3] that $H^2(\partial\mathbb{B}^n) \odot H^2(\partial\mathbb{B}^n) = H^1(\partial\mathbb{B}^n)$ and consequentially

$$\left(H^2(\partial\mathbb{B}^n) \odot H^2(\partial\mathbb{B}^n)\right)^* = BMO(\partial\mathbb{B}^n). \tag{1}$$

(In this context, by "=" we mean equality of the function spaces and equivalence of the norms.) We think of $\mathcal{D} \odot \mathcal{D}$ as a type of "H^1" space and of $\mathcal{X}$ as a type of "BMO" space. That viewpoint is developed further in [2].

Corollary 1. *For $b \in \mathcal{X}$ set $\Lambda_b h = T_b(h,1)$, then $\Lambda_b \in (\mathcal{D} \odot \mathcal{D})^*$. Conversely, if $\Lambda \in (\mathcal{D} \odot \mathcal{D})^*$ there is a unique $b \in \mathcal{X}$ so that for all $h \in \mathcal{P}(\mathbb{D})$ we have $\Lambda h = T_b(h,1) = \Lambda_b h$. In both cases $\|\Lambda_b\|_{(\mathcal{D}\odot\mathcal{D})^*} \approx \|b\|_{\mathcal{X}}$. Namely,*

$$(\mathcal{D} \odot \mathcal{D})^* = \mathcal{X}. \tag{2}$$

Proof. If $b \in \mathcal{X}$ and $f \in \mathcal{D}\odot\mathcal{D}$, say $f = \sum g_i h_i$ with $\sum \|g_i\|_{\mathcal{D}} \|h_i\|_{\mathcal{D}} \le \|f\|_{\mathcal{D}\odot\mathcal{D}} + \varepsilon$, then

$$\begin{aligned} |\Lambda_b f| &= \left|\sum_{i=1}^{\infty} \langle g_i h_i, b\rangle_{\mathcal{D}}\right| = \left|\sum_{i=1}^{\infty} T_b(g_i, h_i)\right| \\ &\le \|T_b\| \sum_{i=1}^{\infty} \|g_i\|_{\mathcal{D}} \|h_i\|_{\mathcal{D}} \le \|T_b\| \left(\|f\|_{\mathcal{D}\odot\mathcal{D}} + \varepsilon\right). \end{aligned}$$

It follows that $\Lambda_b f = \langle f, b\rangle_{\mathcal{D}}$ defines a continuous linear functional on $\mathcal{D} \odot \mathcal{D}$ with $\|\Lambda_b\| \le \|T_b\|$.

Conversely, if $\Lambda \in (\mathcal{D} \odot \mathcal{D})^*$with norm $\|\Lambda\|$, then for all $f \in \mathcal{D}$

$$|\Lambda f| = |\Lambda(f \cdot 1)| \le \|\Lambda\| \|f\|_{\mathcal{D}} \|1\|_{\mathcal{D}} = \|\Lambda\| \|f\|_{\mathcal{D}}.$$

Hence there is a unique $b \in \mathcal{D}$ such that $\Lambda f = \Lambda_b f$ for $f \in \mathcal{D}$. Finally, if $f = gh$ with $g, h \in \mathcal{D}$ we have

$$\begin{aligned} |T_b(g,h)| &= |\langle gh, b\rangle_{\mathcal{D}}| = |\Lambda_b f| = |\Lambda f| \\ &\le \|\Lambda\| \|f\|_{\mathcal{D}\odot\mathcal{D}} \le \|\Lambda\| \|g\|_{\mathcal{D}} \|h\|_{\mathcal{D}}, \end{aligned}$$

which shows that T_b extends to a continuous bilinear form on $\mathcal{D} \odot \mathcal{D}$ with $\|T_b\| \le \|\Lambda\|$. By Theorem 1 we conclude $b \in \mathcal{X}$ and collecting the estimates that $\|\Lambda\| = \|\Lambda_b\|_{(\mathcal{D}\odot\mathcal{D})^*} \approx \|T_b\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\mathcal{X}}$. □

References

[1] N. Arcozzi, R. Rochberg and E. Sawyer, *Carleson measures for analytic Besov spaces*, Rev. Mat. Iberoamericana **18** (2002), 443-510.

[2] N. Arcozzi, R. Rochberg, E. Sawyer and B. Wick. Function Spaces Related to the Dirichlet Space, J. London Math. Soc., to appear.

[3] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*. Ann. of Math. (2) 103 (1976), no. 3, 611—635.

[4] D. Stegenga, *Multipliers of the Dirichlet space*. Illinois J. Math. 24 (1980), no. 1, 113–139.

Hamiltonians and Riccati equations for admissible control operators

Christian Wyss
(joint work with Birgit Jacob, Hans Zwart)

We consider the algebraic Riccati equation from the problem of linear quadratic optimal control,

$$A^*X + XA - XBB^*X + C^*C = 0,$$

and the connected Hamiltonian operator matrix

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

An operator X is a solution of the Riccati equation if and only if its graph subspace $\Gamma(X)$ is invariant under the Hamiltonian. We use this well-known fact to construct infinitely many solutions for the case that the control and observation operators B and C are admissible.

In the finite-dimensional setting, the connection between the Riccati equation and the Hamiltonian was extensively studied and led to a complete description of all solutions, see e.g. [7, 9, 10]. In the infinite-dimensional setting, only some results in this direction are known: Kuiper and Zwart [6] studied the case where B and C are bounded and T is a Riesz-spectral operator. They obtained a characterisation of all bounded solutions in terms of the eigenvectors of T. Langer, Ran and van de Rotten [8] considered T as an operator in an indefinite inner product space to prove the existence of nonnegative and nonpositive solutions, also for bounded B, C. In [11, 12] these two results were extended to the case that T has a Riesz basis of invariant subspaces and BB^* and C^*C are unbounded closed operators on the state space.

Here we consider the following setting:

(a) A is a normal operator with compact resolvent on a Hilbert space H;
(b) $B \in L(U, H_{-s})$, $C \in L(H_s, Y)$ with $0 \le s \le 1/2$, where $H_s = \mathcal{D}(|A|^s)$ and H_{-s} is the dual of H_s with respect to the pivot space H;
(c) T considered as an operator on $H \times H$ with $\mathcal{D}(T) = \{x \in H_{1-s} \times H_{1-s} \,|\, Tx \in H \times H\}$ has a compact resolvent and a Riesz basis of generalised eigenvectors.

One of our main tools is the indefinite inner product on $H \times H$ given by

$$\langle x|y\rangle = (J_1 x|y), \qquad J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix},$$

where $(\cdot|\cdot)$ denotes the usual inner product on $H \times H$. The Hamiltonian is skew-symmetric with respect to $\langle\cdot|\cdot\rangle$, i.e. $\langle Tx|y\rangle = -\langle x|Ty\rangle$ for all $x, y \in \mathcal{D}(T)$. As a consequence of the theory of indefinite inner product spaces, see e.g. [1, 3], the spectrum $\sigma(T)$ is symmetric with respect to the imaginary axis. In particular, if $\sigma(T) \cap i\mathbb{R} = \emptyset$, then $\sigma(T)$ consists of skew-conjugate pairs of eigenvalues $(\lambda, -\overline{\lambda})$. In this case, we say that a subset $\sigma \subset \sigma(T)$ is *skew-conjugate* if it contains exactly one eigenvalue from each pair. In [12] it was shown that then the spectral subspace

W_σ corresponding to σ is equal to its own J_1-orthogonal complement, $W_\sigma = W_\sigma^{\langle\perp\rangle}$. Our first main result is now the following:

Theorem 1. *Suppose that (A, B) is approximately controllable, and that A has no non-observable eigenvalues on the imaginary axis. Then:*

(i) $\sigma(T) \cap i\mathbb{R} = \emptyset$.

(ii) *For every skew-conjugate $\sigma \subset \sigma(T)$ there exists a selfadjoint operator X on H and a dense subspace $D \subset \mathcal{D}(X)$ such that $W_\sigma = \Gamma(X)$ and X is a solution of the Riccati equation*

$$X(Av - BB^*Xv) = -C^*Cv - A^*Xv, \quad v \in D.$$

Since there are infinitely many possible choices for σ, we obtain infinitely many solutions X. In addition, the solution corresponding to the spectrum in the left half-plane is nonnegative, the one corresponding to the right half-plane is nonpositive. This is a consequence of the dissipativity of T with respect to the indefinite inner product

$$[x|y] = (J_2 x|y), \qquad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Our second result is a sufficient condition for assumption (c) for the case of bounded C:

Theorem 2. *Suppose that*

(i) $B \in L(U, H_{-s})$, $s < 1/2$, $C \in L(H, Y)$,

(ii) *A is selfadjoint, negative, with simple eigenvalues $0 > \lambda_0 > \lambda_1 > \dots$, $\lambda_k - \lambda_{k+1} \to \infty$, and*

$$\sum_{k=0}^{\infty} \frac{1}{|\lambda_k|^{2(1-2s)}} < \infty. \tag{1}$$

Then T has a compact resolvent and a Riesz basis of generalised eigenvectors.

For the proof of this result, we decompose T as

$$T = S + R, \qquad S = \begin{pmatrix} A & -BB^* \\ 0 & -A^* \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ -C^*C & 0 \end{pmatrix}.$$

The eigenvectors of S are given by an explicit formula and, using a theorem of Bari [2, 4], we show that they form a Riesz basis. Since $R \in L(H \times H)$ is bounded, a standard perturbation result, see e.g. [5], then yields a Riesz basis of eigenvectors and at most finitely many generalised eigenvectors of T.

As an example, we apply our theory to the heat equation on the unit interval with boundary control:

$$\begin{aligned} &H = L^2([0,1]), \\ &Av = v'', \quad \mathcal{D}(A) = \{v \in H^2([0,1]) \,|\, v'(0) = v(1) = 0\}, \\ &B^*v = -v(0). \end{aligned}$$

Theorems 1 and 2 apply since $B \in L(\mathbb{C}, H_{-s})$ for $s > 1/4$ and condition (1) is satisfied for $s < 3/8$.

References

[1] T. Y. Azizov, I. S. Iokhvidov. *Linear Operators in Spaces With an Indefinite Metric.* John Wiley & Sons, Chichester, 1989.
[2] N. K. Bari. *Biorthogonal systems and bases in Hilbert space.* Moskov. Gos. Univ. Učenye Zapispki Mathematika, **148(4)** (1951), 69–107. (Russian).
[3] J. Bognar. *Indefinite Inner Product Spaces.* Springer, Berlin, 1974.
[4] I. C. Gohberg, M. G. Krein. *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space.* American Mathematical Society, Providence, 1969.
[5] T. Kato. *Perturbation Theory for Linear Operators.* Springer, Berlin, 1980.
[6] C. R. Kuiper, H. J. Zwart. *Connections between the algebraic Riccati equation and the Hamiltonian for Riesz-spectral systems.* J. Math. Systems Estim. Control, **6(4)** (1996), 1–48.
[7] P. Lancaster, L. Rodman. *Algebraic Riccati Equations.* Oxford University Press, Oxford, 1995.
[8] H. Langer, A. C. M. Ran, B. A. van de Rotten. *Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations.* In *Linear Operators and Matrices*, volume 130 of Oper. Theory Adv. Appl., pages 235–254. Birkhäuser, Basel, 2002.
[9] K. Mårtensson. *On the matrix Riccati equation.* Inform. Sci., **3** (1971), 17–49.
[10] J. E. Potter. *Matrix quadratic solutions.* SIAM J. Appl. Math., **14(3)** (1966), 496–501.
[11] C. Wyss. *Perturbation theory for Hamiltonian operator matrices and Riccati equations.* PhD thesis, University of Bern, 2008.
[12] C. Wyss. *Hamiltonians with Riesz bases of generalised eigenvectors and Riccati equations.* Indiana Univ. Math. J., to appear.

Spectral conditions implied by observability

Hans Zwart

(joint work with F-Z. El Alauoi)

Observability and its dual notion controllability are important system theoretic properties. However, showing that a given system processes these properties can be a non-trivial task, see for instance the books of [5, 6, 13], and the references therein. Hence it can be very useful to have simple tests for (lack of) observability/controllability. Under the assumption that the output operator is relatively compact we derive necessary conditions for exact and final state observability. These results fits in a long tradition of necessary conditions for exact observability/controllability. In 1975, Triggiani [11] showed that exact controllability is not possible when the input operator is compact. Two years later he proved the same result for compact semigroups, [12]. Thus if the range of the input operator is finite-dimensional, then the system will never be exactly controllable provided the input operator is bounded. Since the one-dimensional wave equation is exactly controllable by boundary control, [8], it is clear that this theorem does not hold for unbounded input operators with finite-dimensional range. However, having a finite-dimensional range, exact controllability gives conditions on the system operator, see [2, 3, 9]. Using Weyl characterization of the essential spectral, in [1] it was showed that exact controllability is impossible when the input operator is relatively compact, and the self-adjoint system operator has essential spectrum.

For the dual notion of exact observability, we extend these results in two ways. To explain and to formulate this, we first have to introduce some notation.

A denotes the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X. The domain of A is denoted by $D(A)$. By C we denote the linear operator from the domain of A to the Hilbert space Y. The operator C is assumed to be relatively compact with respect to A, which is equivalent to assuming that $C(rI-A)^{-1}$ is a compact operator from X to Y for some (or any) r in the resolvent set of A.

We associate to the operators A and C the system $\Sigma(A,-,C)$ as

$$\dot{x}(t) = Ax(t) \qquad x(0)=x_0 \tag{1}$$

$$y(t) = Cx(t). \tag{2}$$

Since A generates the strongly continuous semigroup $(T(t))_{t\geq 0}$, we have that the first equation possesses the unique solution $x(t) = T(t)x_0$. For $x_0 \in D(A)$, the output $y(t)$ is given by $CT(t)x_0$. If this map can be extended to a bounded map from X to $L^2((0,t_f);Y)$, then C is said to be an *admissible* output operator for the semigroup $(T(t))_{t\geq 0}$. We denote this extended map by $\mathcal{O}$. Thus if C is admissible, then there exists an $M_f > 0$ such that for all $x_0 \in X$

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt =: \|\mathcal{O}x_0\|^2_{L^2(0,t_f)} \leq M_f \|x_0\|^2. \tag{3}$$

Using the semigroup property it is easy to show that the boundedness of the observability map is independent of t_f, i.e., if (3) holds, then for any $t_f > 0$, there exists an $\tilde{M}_f > 0$ such that

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt \leq \tilde{M}_f \|x_0\|^2.$$

The system $\Sigma(A,-,C)$ is *exactly observable in finite-time* if there exists a $t_f > 0$ and an $m_f > 0$ such that

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt \geq m_f \|x_0\|^2, \qquad x_0 \in D(A). \tag{4}$$

A weaker definition of observability is final state observability. The system $\Sigma(A,-,C)$ is *final state observable in finite-time* if there exists a $t_f > 0$ and an $m_f > 0$ such that

$$\int_0^{t_f} \|CT(t)x_0\|^2 dt \geq m_f \|T(t_f)x_0\|^2, \qquad x_0 \in D(A). \tag{5}$$

Note that for these definitions we did not assumed that C is admissible.

We prove the following assertions;

- If $\Sigma(A,-,C)$ is exactly observable in finite-time and C is admissible, then the relative compactness of C implies the compactness of the resolvent operator of A;

- If $\Sigma(A, -, C)$ is final state observable in finite-time and C is relatively compact, then the approximate point spectrum of A consists of point spectrum only. Furthermore, for all $s \in \mathbb{C}$ the kernel of $(sI - A)$ is finite-dimensional, and the range of $sI - A$ is closed.

The proof of the second assertion goes via the Hautus test, which was introduced in [10] and further studied in [4, 7].

References

[1] G. Geymonat and V. Valente. A noncontrollability result for systems of mixed order. *SIAM J. Control Optim.*, 39(3):661–672 (electronic), 2000.

[2] B. Jacob and J.R. Partington. On controllability of diagonal systems with one-dimensional input space. *Systems Control Lett.*, 55(4):321–328, 2006.

[3] B. Jacob and H. Zwart. Exact observability of diagonal systems with a finite-dimensional output operator. *Systems Control Lett.*, 43(2):101–109, 2001.

[4] B. Jacob and H. Zwart. On the Hautus test for exponentially stable C_0-groups. *SIAM J. Control Optim.*, 48(3):1275–1288, 2009.

[5] V. Komornik. *Exact Controllability and Stabilizability. The Multiplier Method.* Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris, 1994.

[6] J.-L. Lions. *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1988. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[7] L. Miller. Controllability cost of conservative systems: resolvent condition and transmutation. *J. Funct. Anal.*, 218(2):425–444, 2005.

[8] P.C. Parks. On how to shake a piece of string to a standstill. In *Recent Mathematical Devolopments in Control, (Proc. Conf., Bath, 1972)*, pages 267–287. Academic Press, London and New York, 1973.

[9] R. Rebarber and G. Weiss. Necessary conditions for exact controllability with a finite-dimensional input space. *Systems Control Lett.*, 40(3):217–227, 2000.

[10] D.L. Russell and G. Weiss. A general necessary condition for exact observability. *SIAM J. Control Optim.*, 32(1):1–23, 1994.

[11] R. Triggiani. On the lack of exact controllability for mild solutions in Banach spaces. *J. Math. Anal. Appl.*, 50:438–446, 1975.

[12] R. Triggiani. A note on the lack of exact controllability for mild solutions in Banach spaces. *SIAM J. Control Optimization*, 15(3):407–411, 1977.

[13] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

Reporter: Sven-Ake Wegner

Participants

Dr. Bernhard H. Haak
Mathematiques et Informatique
Universite Bordeaux I
351, cours de la Liberation
F-33405 Talence Cedex

Dr. Tuomas Hytönen
Dept. of Mathematics & Statistics
University of Helsinki
P.O.Box 68
FIN-00014 University of Helsinki

Prof. Dr. Birgit Jacob
Fachbereich Mathematik
Bergische Universität Wuppertal
Gaußstr. 20
42097 Wuppertal

Dr. Bernd Klöss
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Prof. Dr. Luc Miller
U. F. R. de Sciences Economiques
Universite de Paris X - Nanterre
200, Avenue de la Republique
F-92001 Nanterre Cedex

Prof. Dr. Rainer Nagel
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Dr. Mark Opmeer
Department of Mathematical Sciences
University of Bath
Claverton Down
GB-Bath BA2 7AY

Prof. Dr. Jonathan R. Partington
School of Mathematics
University of Leeds
GB-Leeds LS2 9JT

Prof. Dr. Olof Johan Staffans
Matematiska Institutionen
Abo Akademi
Fänriksgatan 3
FIN-20500 Abo

Prof. Dr. Marius Tucsnak
Institut Elie Cartan
-Mathematiques-
Universite Henri Poincare, Nancy I
Boite Postale 239
F-54506 Vandoeuvre les Nancy Cedex

Dr. Sven-Ake Wegner
FB C: Mathematik u. Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
42119 Wuppertal

Prof. Dr. Lutz Weis
Mathematisches Institut I
Universität Karlsruhe
76128 Karlsruhe

Dr. George Weiss
Department of Electrical
Engineering Systems
Tel Aviv University
Ramat Aviv 69978
ISRAEL

Prof. Dr. Brett D. Wick
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta , GA 30332-0160
USA

Dr. Christian Wyss
FB C: Mathematik u. Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
42119 Wuppertal

Prof. Dr. Hans Zwart
Department of Applied Mathematics
University of Twente
P.O.Box 217
NL-7500 AE Enschede