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Set Theory

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Abstract. This stimulating workshop exposed some of the most exciting recent develops in set theory, including major new results about the proper forcing axiom, stationary reflection, gaps in $P(\omega)/\mathrm{Fin}$, iterated forcing, the tree property, ideals and colouring numbers, as well as important new applications of set theory to C^*-algebras, Ramsey theory, measure theory, representation theory, group theory and Banach spaces.

Mathematics Subject Classification (2000): 03E02, 03E04, 03E10, 03E15, 03E17, 03E35, 03E45, 03E55, 03E60, 03E75.

Introduction by the Organisers

This was an exciting workshop which divided almost evenly between pure set theory and applications of set theory to other fields. There were 52 postdoctoral and 4 doctoral participants with a high percentage of young people, making for a lively atmosphere. We scheduled only 11 long (50-minute) talks; the remaining 18 talks were short (30-minutes), allowing for ample time for informal discussion and collaboration.

Among the highlights of the workshop were the following: Viale presented exciting work showing that any standard approach to proving the consistency of PFA requires a supercompact. Todorcevic presented deep work on the study of higher-order gaps in $P(\omega)/\mathrm{Fin}$, while Farah and Törnquist presented talks establishing the unclassifiability of separable C^*-algebras in the sense o descriptive set theory. Jensen described his ultimate generalisation of Namba forcing and Neeman worked miracles with forcings built from finite conditions. Zapletal launched a new program mixing ideals with equivalence relations, Gitik solved the normality problem for precipitous ideals (negatively), Simon Thomas connected large cardinals with representation theory and Zdomskyy presented a new approach to preserving large cardinals after applying a wide variety of iterations with fusion. Louveau presented a major new result in dual Ramsey theory, Sinapova explained her deep work on the tree property at the successor of a singular and Sargsyan brought us up-to-date on the influence of large cardinals on the structure of HOD.

The number of new results connecting set theory with other fields of mathematics was a striking feature of this workshop, and points toward an even richer future for an already dynamic subject.

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Workshop: Set Theory

Table of Contents

Abstracts

On the notion of guessing model

Matteo Viale

We present a brief account on the notion of guessing model which is analyzed and introduced in [1] The ultimate and most likely out of reach ambition in this work is to provide by means of guessing models useful tools to show that for a given model W of MM, $(\aleph_2)^W$ has an arbitrarily high degree of supercompactness in some simply definable inner model V.

A guessing model come in pair with an infinite cardinal δ:

- $\aleph_0$-guessing models provide an interesting characterization of all large cardinal axioms which can be described in terms of elementary embedding $j : V_\gamma \to V_\lambda$. In particular supercompactness, hugeness , and the axioms I_1 and I_3 can be characterized in terms of the existence of appropriate $\aleph_0$-guessing models.
- In a paper with Weiss [2] we showed that PFA implies that there are $\aleph_1$-guessing models, and that in many interesting models W of PFA such $\aleph_1$-guessing models M can be used to show that in some inner model V of W, $M \cap V$ is an $\aleph_0$-guessing models belonging to V and witnessing that $\aleph_2$ is supercompact in V.
- In [1] I also outline some interesting properties guessing models have in models of MM. For example assume θ is inaccessible in W, then:
  (1) If W models PFA, then for a stationary set G of $\aleph_1$-guessing models $M \prec H_\theta$ the isomorphism-type of M is uniquely determined by the ordinal $M \cap \aleph_2$ and the order type of $M \cap Card$ where $Card$ is the set of cardinals in H_θ.
  (2) In the seminal paper of Foreman Magidor and Shelah [4] on Martin's maximum and in a recent work by Sean Cox [3] several strong forms of diagonal reflections are obtained, for example Cox shows:
  > Assume MM holds in V. Then for every regular θ there is S stationary set of models $M \prec H_\theta$ such that every $M \in T$ computes correctly stationarity in the following sense:
  > *For every $X \in M$ and every set $R \in M$ subset of $[X]^{\aleph_0}$ if R is projectively stationary in V then R reflects on $[M \cap X]^{\aleph_0}$.*

  (3) We can improve (1) and (2) above to further argue that in a model V of MM, $G \cap S$ is stationary.

Such results even if rather technical are attributing to $\aleph_2$ properties shared by supercompact cardinals in the sense that $\aleph_0$-guessing models M are characterized by property (1) when $\aleph_2$ is replaced by some suitable inaccessible cardinal $\kappa \in M$ and satisfy many strenghtenings of property (2).

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Stationary and semi stationary reflection principles

Boban Veličković
(joint work with Hiroshi Sakai)

We study consequences of stationary and semi stationary set reflection. We show that the semi-stationary reflection principle implies the Singular Cardinal Hypothesis, the failure of weak square principles, etc. We also consider two cardinal tree properties introduced recently by Weiss and prove that they follow from stationary and semi stationary set reflection augmented with a weak form of Martin's Axiom. We also show that there are some differences between the two reflection principles which suggest that stationary set reflection is analogous to supercompactness whereas semi-stationary set reflection is analogous to strong compactness.

k-gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$

Stevo Todorcevic
(joint work with Antonio Aviles)

This will be an overview of the joint work with Antonio Aviles over the last few years on a higher-dimensional theory of gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$. This new theory originally motivated by an application to the theory of function spaces is a natural extension of the classical one-dimensional theory of gaps developed by Hausdorff a century ago. We shall examine both the combinatorial and the descriptive set-theoretic side. For example, we shall identify a finite basis of analytic k-gaps for each finite dimension k.

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$\aleph_1$-perfect mad families

Jörg Brendle

The starting point of our considerations is a classical result of Mathias [Ma] which says that an infinite maximal almost disjoint (*mad*, for short) family on the natural numbers ω cannot be an analytic subset of the Baire space $[\omega]^\omega$.

Let $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega$ is an infinite mad family$\}$, the classical almost disjointness number. Similarly define $\mathfrak{a}_{\mathrm{Borel}} = \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite family of Borel a.d. families such that $\bigcup \mathcal{A}$ is mad$\}$, as well as $\mathfrak{a}_{\mathrm{closed}} = \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite family of closed a.d. families such that $\bigcup \mathcal{A}$ is mad$\}$. Then $\aleph_1 \leq \mathfrak{a}_{\mathrm{Borel}} \leq \mathfrak{a}_{\mathrm{closed}} \leq \mathfrak{a}$ where the first inequality follows from Mathias' Theorem. By an unpublished result of Raghavan (personal communication), this inequality can be improved to $\mathfrak{t} \leq \mathfrak{a}_{\mathrm{Borel}}$ where $\mathfrak{t} = \min\{|\mathcal{A}| : \mathcal{A}$ is a tower, that is, $\mathcal{A}$ is a $\subseteq^*$-decreasing sequence of infinite subsets of ω without a lower bound$\}$ is the tower number. A better lower bound for $\mathfrak{a}$ is the unbounding number $\mathfrak{b}$, and it is natural to ask whether one even has $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{Borel}}$ or $\mathfrak{b} \leq \mathfrak{a}_{\mathrm{closed}}$. In joint work with Yurii Khomskii (Amsterdam), we proved that this is not the case.

Theorem 1. (Brendle, Khomskii) *$\mathfrak{a}_{\mathrm{closed}} < \mathfrak{b}$ is consistent.*

In particular $\mathfrak{a}_{\mathrm{closed}} < \mathfrak{a}$ is consistent.

Let us briefly sketch the framework of this proof. Let $\bar{A} = (A_\sigma : \sigma \in \omega^{<\omega})$ be a partition of ω (or of a subset of ω) into infinite sets. Put $A_\sigma = \{a_\sigma^n : n \in \omega\}$. With $\bar{A}$, we naturally associate a perfect almost disjoint (*a.d.*, for short) family $\mathcal{X}^{\bar{A}}$ as follows: for $f \in \omega^\omega$, let $X_f^{\bar{A}} = \{a_{f\restriction n}^{f(n)} : n \in \omega\}$ and let $\mathcal{X}^{\bar{A}} = \{X_f^{\bar{A}} : f \in \omega^\omega\}$.

If $\mathcal{A}$ is a family of such partitions such that $\bar{A} \neq \bar{B} \in \mathcal{A}$ and $f, g \in \omega^\omega$ implies $|X_f^{\bar{A}} \cap X_g^{\bar{B}}| < \aleph_0$, then $\mathcal{X}^{\mathcal{A}} = \bigcup\{\mathcal{X}^{\bar{A}} : \bar{A} \in \mathcal{A}\}$ is an a.d. family on ω.

Now assume $M \subseteq V$ is a countable model of a large enough fragment of ZFC, and let $\mathcal{A} = \{\bar{A}_i : i \in \omega\} \subseteq M$.

Main Lemma. *There is $\bar{C} = (C_\sigma : \sigma \in \omega^{<\omega})$ in V such that:*

(1) *Letting $\mathcal{C} = \mathcal{A} \cup \{\bar{C}\}$, $\mathcal{X}^{\mathcal{C}}$ is still a.d.,*

(2) *Assume $V' \supseteq V$, $M' \supseteq M$, $M' \subseteq V'$ is a countable model, and all reals in V which are splitting over M are still splitting over M'. Then the following holds in V': If $Y \in [\omega]^\omega$ belongs to M' and $|Y \cap X_g^{\bar{A}_i}| < \aleph_0$ for all i and all $g \in \omega^\omega$, then there is $f \in \omega^\omega$ such that $X_f^{\bar{C}} \subseteq Y$.*

The proof is rather technical.

To prove the Theorem using the Main Lemma, perform a finite support iteration of Hechler forcing of length κ, where $\kappa = \kappa^\omega \geq \aleph_2$ is regular, over a model of CH. The generic extension satisfies $\mathfrak{b} = \mathfrak{c} = \kappa$. The Main Lemma, in conjunction with the fact that iterated Hechler forcing preserves reals which are splitting, allows to build up a family $\mathcal{A}$ of $\aleph_1$ many partitions in the ground model such that $\mathcal{X}^{\mathcal{A}}$ is mad even in the generic extension. We call such a mad family an $\aleph_1$-perfect mad family. Thus $\mathfrak{a}_{\text{closed}} = \aleph_1$ in the extension.

The following problems remain open:

Problem 1. *Is $\mathfrak{a}_{\text{Borel}} = \mathfrak{a}_{\text{closed}}$?*

Problem 2. *Is $\mathfrak{a}_{\text{Borel}} > \mathfrak{b}$ consistent?*

Problem 3. (Raghavan) *Is $\mathfrak{h} \leq \mathfrak{a}_{\text{Borel}}$ where $\mathfrak{h}$ denotes the distributivity number?*

We now turn to definability of mad families.

By a result of Miller [Mi], there is a coanalytic mad family in the constructible universe L. This is still true in many forcing extensions of $V = L$, e.g. the Cohen, random, and Sacks extensions. Friedman and Zdomskyy [FZ] proved that $\mathfrak{b} = \aleph_2$ is consistent with the existence of a $\mathbf{\Pi}^1_2$ mad family, and Fischer, Friedman, and Zdomskyy [FFZ] showed the same for arbitrarily large $\mathfrak{b}$. Friedman and Zdomskyy [FZ] asked whether one can also have simultaneously $\mathfrak{b} > \aleph_1$ and a $\mathbf{\Sigma}^1_2$ mad family. By Raghavan's result mentioned above, $\mathfrak{t} > \aleph_1$ implies that there are no $\mathbf{\Sigma}^1_2$ mad families. By the theorem sketched above, adding Hechler reals over L preserves an $\aleph_1$-perfect mad family which is $\mathbf{\Sigma}^1_2$:

Corollary 1. (Brendle, Khomskii) *It is consistent that $\mathfrak{b}$ is arbitrarily large and there exists a $\mathbf{\Sigma}^1_2$ mad family.*

Further problems are:

Problem 4. *Is it consistent that $\mathfrak{b} > \aleph_1$ and there exists a $\mathbf{\Pi}^1_1$ mad family?*

Problem 5. *Is the existence of a $\mathbf{\Pi}^1_1$ mad family equivalent to the existence of a $\mathbf{\Sigma}^1_2$ mad family?*

Problem 6. *Does the Ramsey property for $\mathbf{\Sigma}^1_2$ sets imply that there is no $\mathbf{\Sigma}^1_2$ mad family?*

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Descriptive set theory and the classification of separable C*-algebras, Part I

Ilijas Farah
(joint work with Andrew Toms, Asger Törnquist)

In this talk I described an ongoing joint work with Andrew Toms and Asger Törnquist ([2]). The abstract classification program initiated by Kechris and his collaborators (e.g., [5]) was one of the main themes in set theory over the last two decades. Similarly, Elliott's classification program for nuclear C*-algebras played a central role in the study of C*-algebras over a similar time span ([1]). We apply methods of the abstract classification, in particular Greg Hjorth's results and concept of turbulence ([4]), to analyze the classification problem for separable C*-algebras as follows.

The first classification result in the field of C*-algebras was Glimm's classification of unital UHF algebras. In 1960 Glimm has proved that all UHF algebras are inductive limits of full matrix algebras, $M_n(\mathbb{C})$ for $n \in \mathbb{N}$, and that they are classified by a 'supernatural' number $\prod_{p \text{ prime}} p^{n(p)}$. Building on the work of Brattelli, in 1976 George Elliott proved a classification result for the larger class of AF algebras. AF algebras are inductive limits of finite-dimensional C*-algebras. Elliott has shown that a complete invariant for AF algebras is the ordered abelian group K_0. This classification has a remarkable additional feature of being *functorial*: every homomorphism between K_0-groups lifts to a *-homomorphism of the corresponding algebras. If homomorphism is an isomorphism, then it lifts to a *-isomorphism of C*-algebras. An even finer result is true: the automorphism group of $K_0(A)$ is in exact correspondence to the outer automorphism group of A.

Elliott invariant $\mathrm{Ell}(A)$ of a C*-algebra A is obtained by expanding $K_0(A)$ by adding another abelian group, $K_1(A)$, as well as tracial simplex $T(A)$ of A and the pairing map ρ that associates states on $K_0(A)$ to traces in $T(A)$. By a remarkable result known as the Bott periodicity, all the higher order K-groups are are isomorphic to K_0 or K_1 and $K_1(A)$ is isomorphic to $K_0(C([0,1)) \otimes A)$ (e.g., [7]). Elliott's program postulated that all infinite-dimensional, separable, simple, unital, and nuclear C*-algebras A are classified by the invariant $\mathrm{Ell}(A)$. Moreover, the classification was purported to be functorial.

Elliott program has enjoyed a series of remarkable successes (see e.g., [9]). Nuclear, simple, purely infinite C*-algebras (modulo a technical assumption known as the Universal Coefficient Theorem, UCT) were classified by the ordered groups K_0 and K_1 alone (hence by countable structures) in a sweeping work of Kirchberg and Phillips. This essentially reduced Elliott's program to (i) classification of finite C*-algebras and (ii) showing that infinite C*-algebras cannot have finite projections.

In 1998 Villadsen constructed a separable, simple, unital, and nuclear C*-algebra A whose K_0 group is not *weakly unperforated.* This means that some $n \cdot x > 0$ does not imply $x > 0$ for some $x \in K_0(A)$. Villadsen's algebra was an inductive limit of subalgebras of algebras of continuous functions from a compact, finite dimensional, metric space into a finite-dimensional C*-algebra. Its properties depended on the existence of the so-called *Bott projection* in the algebra of continuous functions from S^2 into $M_2(\mathbb{C})$).

In 1999, Jiang and Su constructed an infinite-dimensional separable, simple, unital, and nuclear C*-algebra $\mathcal{Z}$ such that $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$. This was not considered to be a counterexample to Elliott's program since $\mathbb{C}$ is not an infinite-dimensional C*-algebra. Jiang and Su also proved that $\mathrm{Ell}(A \otimes \mathcal{Z}) = \mathrm{Ell}(A)$ for every separable, simple, unital, and nuclear C*-algebra A, thus raising the question whether all such A are $\mathcal{Z}$-stable, i.e., isomorphic to $A \otimes \mathcal{Z}$.

In 2002 Rørdam constructed an algebra which had a remarkable feature of having both infinite and finite projections, thus showing the answer to be negative. In 2003 Toms independently constructed a stably finite counterexample to Elliott's conjecture. These results were rapidly followed by Toms's construction of a separable, simple, unital, and nuclear C*-algebra A that was not $\mathcal{Z}$-stable and yet had the property that $F(A) = F(A \otimes \mathcal{Z})$ for every continuous, homotopy invariant functor. Both Rørdam and Toms constructed their counterexamples using refinements of Villadsen's techniques.

The invariant $W(A)$ used by Toms to distinguish A and $A \otimes \mathcal{Z}$ has attracted major attention, overtaking the central stage in the updated version of Elliott's program. Remarkably, for simple $\mathcal{Z}$-stable C*-algebras $W(A)$ amplified with $K_0(A)$ provides exactly the same information as the original Elliott invariant (N. Brown, Perera and Toms, 2007).

As remarkable as they are, Rørdam and Toms non-classification results were not nonclassification results from the point of view of abstract classification theory as presented in e.g. [4]. The starting point of [2] was a desire to show that separable, simple, unital, and nuclear C*-algebras cannot be effectively classified by simple invariants.

Let us now describe the abstract classification theory standpoint. Recall that if (X, E) and (Y, F) are analytic equivalence relations on standard Borel spaces, E is *Borel-reducible* to F, in symbols $E \leq_B F$, if there is a Borel-measurable map $f\colon X \to Y$ such that $x \mathrel{\mathrm{E}} y \Leftrightarrow f(x) \mathrel{\mathrm{E}} f(y)$.

Model results are given in [10] and [3] where anti-classification results were proved for von Neumann factors with separable predual and separable Banach spaces, respectively. In the former work it was shown, using Hjorth's [4], that no major class of factors that has not been already classified in the work of Connes and others can be effectively classified by countable structures. In the latter work a bit more was proved: every analytic equivalence relation is Borel-reducible to the isomorphism relation E_b of separable Banach spaces. In particular, E_b is not even Borel-reducible to an orbit equivalence relation of a continuous Polish group action on a Polish space. (The isomorphism of factors with separable predual is known to be induced by an action of the unitary group.)

A Borel space of separable C*-algebras was introduced in [6] and the Elliott invariant takes values in a naturally defined standard Borel space. Using Kechris's results, in [2] we proved that the computation of the Elliott invariant is Borel. This set the stage for our non-classification results, described in [8].

I will finish by saying that even the Cuntz semigroup $W(A)$ ranges over a standard Borel space, and that the map $A \mapsto W(A)$ is given by a Borel function. This suggests that every natural and absolute classification of separable structures can be given a Borel model, and raises the question whether all such models are necessarily equivalent?

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Descriptive set theory and the classification of separable C*-algebras, Part II

Asger Törnquist

(joint work with Andrew Toms, Ilijas Farah)

In the first part of this talk, Ilijas Farah set the stage for a systematic investigation of the classification of nuclear simple separable C*-algebras from the point of view of descriptive set theory. This part is dedicated to providing some more details of the results that have been achieved in a joint effort of Ilijas Farah, Andrew Toms and the speaker.

A C*-algebra is nothing but a norm-closed $*$-subalgebra of $\mathcal{B}(H)$, the algebra of all bounded operators on a complex Hilbert space H. Thus, if we fix a separable complex Hilbert space H, then every sequence in $\mathcal{B}(H)$ generates a separable C*-algebra, and conversely, all isomorphism types of separable C*-algebras appear in this way. Moreover, if we give $\mathcal{B}(H)$ the Borel structure induced by the weak operator topology, then $\mathcal{B}(H)$ becomes a standard Borel space. It follows that the elements of

$$\Gamma(H) = \mathcal{B}(H)^{\mathbb{N}}$$

parametrize all separable C*-algebras. For $\gamma \in \Gamma$, we let $C^*(\gamma)$ be the C*-algebra generated by $\{\gamma(n) : n \in \mathbb{N}\}$. We let $\simeq^\Gamma$ be the relation

$$\gamma \simeq^\Gamma \gamma' \iff C^*(\gamma) \text{ is isomorphic to } C^*(\gamma').$$

This can be shown to be an analytic equivalence relation, and so Γ provides with a reasonable setting to study the descriptive set theory of the isomorphism relation of separable C*-algebras. The space Γ is what we call a *standard Borel parametrization.* It is by no means the only reasonable parametrization of the class of separable C*-algebras one can dream up, though the other parametrizations that we have considered turn out to be identical to this one from the descriptive set-theoretic point of view. It does raise the following question:

Question 1. Are all reasonable parametrizations by elements in a standard Borel space of the class of separable C*-algebras equivalent?

We leave this question for now, but return to the theme at the end of the talk.

The Elliott programme targets a special class of C*-algebras, namely those separable C*-algebras that are *nuclear* and *simple*. Simple is easy to explain: It just means that there are no proper, non-trivial closed two-sided ideals. What this immediately does is that it rules out the class of algebras of the form $C(K)$, the continuous complex-valued functions on a compact Polish space from being considered, which are not simple unless $\mathrm{card}(K) = 1$.

Nuclearity is a more delicate matter. I will not discuss it in this talk, but simply say that it is an amenability condition. For those familiar with Banach algebras it suffices to say that a C*-algebra is nuclear precisely when it is amenable as a Banach algebra (in the sense of B.E. Johnson, say.) Further, the class of nuclear C*-algebras has good permanence properties: It is preserved through inductive limits, formation of matrix algebra, etc.

A large part of our work on the descriptive set theory of C*-algebras was motivated by the classification problem for nuclear simple separable C*-algebras. Where does this classification problem belong in the Borel reducibility hierarchy? The following two results give upper and lower bounds:

Theorem 1. (Farah-Toms-Törnquist.) *The homeomorphism relation for compact Polish spaces is Borel reducible to the isomorphism relation for (unital[1]) nuclear simple separable C*-algebras. In particular, isomorphism in this class is not classifiable by countable structures.*

[1]An algebra is unital if it has a multiplicative unit, usually denoted 1.

Theorem 2. (F-T-T.) *The isomorphism relation for* unital *nuclear simple separable C*-algebras is Borel reducible to an orbit equivalence relation induced by a Borel action of a Polish group on a Polish space. In particular, it is not the* $\leq_B$*-maximal analytic equivalence relation.*

In the talk I will focus on Theorem 2, and I will give a rough sketch of the proof. The proof necessitates proving "Borel versions" of two important theorems of Kirchberg about nuclear simple separable C*-algebras, and the more general class of *exact* C*-algebras.

The first of these is the "$A \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$ Theorem": it states that a C*-algbera A is nuclear, unital and simple precisely when $A \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$. This theorem has an effective counterpart: Given (a code for) a nuclear simple separable unital C*-algebra A, an isomorphism between $A \otimes O_2$ $\mathcal{O}_2$ can be computed in a Borel way.

The 2nd is Kirchberg's exact embedding theorem, which states that any exact separable C*-algebra can be embedded into $\mathcal{O}_2$, and thus $\mathcal{O}_2$ provides a "universal" space for exact (and therefore nuclear) separable C*-algebras. Again, an effective version of this is possible: Given (the code for) an exact C*-algebra A, (the code for) an embedding into $\mathcal{O}_2$ can be computed in a Borel way.

As a corollary, we obtain that the parametrization of exact (and nuclear) C*-algebras given by the class of sub-C*-algebras of $\mathcal{O}_2$ is equivalent to the parametrization we obtain from considering the $\gamma \in \Gamma$ such that $C^*(\gamma)$ is exact (nuclear). Thus we eventually come back to the theme introduced by Question 1 above.

Universal flows of closed subgroups of S_∞

Lionel Nguyen Van Thé

This work is related to a question asked by Kechris, Pestov and Todorcevic in [1]. Recall that for a topological group G, a G-flow is a topological Hausdorff space X together with a continuous action of G on X. A G-flow is *compact* when X is; it is *minimal* when the orbit of every point $x \in X$ is dense; it is *universal* when it can be mapped homomorphically onto any compact minimal G-flow. It is a general result in topological dynamics that every Hausdorff topological group G admits a unique compact minimal G-flow which is also universal. In [1], it is shown that for some closed subgroups of S_∞, two combinatorial properties are relevant in order to compute the universal minimal flow. Those are respectively called *Ramsey property* and *ordering property* (or, more generally, *expansion property*). In particular, it is shown that the ordering property is equivalent to minimality of a certain flow, while the conjunction of the ordering property and of the Ramsey property is equivalent to the conjunction of minimality and universality of the same flow. A natural question is therefore: is Ramsey property alone equivalent to universality?

It turns out that the answer is: no in general. In particular, it is shown that while the Ramsey property is linked to the extreme amenability of a certain group (a topological group is *extremely amenable* iff all of its continuous actions on compact spaces admit a fixed point), universality is linked to the *relative extreme amenability* of a certain pair of groups (a pair of topological groups (G, H) is *relatively extremely amenable*, where H is a subgroup of G, when every continuous action of G on any compact space has an H-fixed point). This result is used to produce a negative answer to the question above. It is also used to provide a combinatorial reformulation of universality, which turns out to be weaker than the Ramsey property.

It could be however that in some particular remarkable cases (which are the ones originally described in [1]), Ramsey property and universality coincide. This is the subject of some work in progress with Yonatan Gutman.

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Subcomplete forcing

Ronald Jensen

We discuss the class of subcomplete forcings and their iterations. We show, in particular, that if κ is inaccessible, $A \subset \kappa$ and GCH holds below κ, then there is a subcomplete P, such that whenever G is P-generic, then in $V[G]$ we have:

- $\kappa = \aleph_2$
- if $\tau \in (\omega_1, \kappa)$ is regular in V, then
$$cf(\tau) = \begin{cases} \omega & \text{if } \tau \in A \\ \omega_1 & \text{if } \tau \notin A \end{cases}$$
- all stationary subsets of κ remain stationary.

It follows that if κ is strongly inaccessible, then by subcomplete forcing we can achieve: $\kappa = \omega_2$ and $cf(\tau) = \omega$ for all regular $\tau \in (\omega_1, \kappa)$.

Forcing with side conditions

Itay Neeman

Forcing axioms have been central in set theory over the last four decades, serving both as axiomatic center points for consistency proofs, and as objects of study in their own right. The forcing axiom associated to a cardinal κ and a class of posets $\mathcal{F}$ states that for any $\mathbb{P} \in \mathcal{F}$ and any family A of κ dense sets of $\mathbb{P}$, there is a filter over $\mathbb{P}$ which meets every dense set in A. The most common axioms involve the class of c.c.c. posets (Martin's Axiom), the class of proper posets (Proper Forcing Axiom) and the class of semi-proper posets (Semi-Proper Forcing Axiom). In the case of c.c.c. posets, the forcing axiom associated to any κ is consistent. For proper and semi-proper posets, only $\kappa = \omega_1$ is possible. It was expected initially that there would be analogs of the axioms with greater κ. But already in the case of proper forcing, the naive approach to arrange a higher analog fails for reasons involving preservation (or rather lack of preservation) of any high analog of properness under iterations. Moreover the proper forcing axiom, and even some fragments of the axiom, was seen to imply that the continuum is ω_2. In particular then a high analog cannot be an actual strengthening of the axiom or even the fragments.

In my talk I introduced an alternative proof of the consistency of the proper forcing axiom, avoiding the countable support that is normally used in connection with iterations of proper posets, and using instead finite support with the addition of models as side conditions. Side conditions are needed that allow proving the preservations of two cardinals, ω_1 and a supercompact cardinal θ. The existence of a poset of side conditions preserving two cardinals is a recent development due independently to Friedman and Mitchell, and in my talk I presented a substantial simplification of their approach.

Most importantly, the use of side conditions frees the consistency proof of the proper forcing axiom from the need to use countable supports for preservation of properness, and paves the way for a high analog of the axiom, for simultaneously meeting ω_2 dense sets, rather than ω_1.

In the talk, I presented a poset of side conditions that preserves three cardinals, ω_1, ω_2, and a supercompact θ. I then used it to obtain a high analog of the proper forcing axiom, for simultaneously meeting ω_2 dense sets, in posets that satisfy a combined clause on existence of master conditions for both countable models and models of size ω_1.

Canonical Ramsey Theory on Polish Spaces

Jindřich Zapletal
(joint work with Vladimir Kanovei, Marcin Sabok)

The talk serves as an advertisement for a book of the same title written jointly with Vladimir Kanovei and Marcin Sabok. It deals primarily with the following problem:

Question 1. *Let E be a Borel equivalence relation on a Polish space X and let I be a σ-ideal on the same space. Is there a Borel I-positive set $B \subset X$ such that the equivalence $E \restriction B$ is significantly simpler than E on the whole space?*

Here, the phrase "significantly simpler" can be interpreted in several ways. In a particularly favorable cases, we may hope for $E \restriction B$ to be one of finitely many or countably many forms presribed beforehand. In other cases, we can only get $E \restriction B$ to be strictly below E in the Borel reducibility ordering. In other cases still, we get a strong negative type result: the Borel reducibility complexity of the equivalence relation E does not change by restricting to an I-positive Borel set. The results change according to the ideal and equivalence in question. We pay attention mainly to the σ-ideals associated with standard forcing notions, and equivalences in well known Borel bireducibility classes. The main hope is that we will discover new ways of proving irreducibility results. The following definitions are central:

Definition 1. Let I be a σ-ideal on a Polish space X. A Borel equivalence E is in the *spectrum* of the ideal I if there is a Borel I-positive set $B \subset X$ and a Borel equivalence relation F on B bireducible with E, such that for every Borel I-positive set $C \subset B$, the equivalence relation $F \upharpoonright C$ is still bireducible with E.

For every forcing P, it makes sense to evaluate the spectrum of the ideal I naturally associated with the poset P. Thus, E_0 and the identity belongs to the spectrum of Silver forcing, but no other equivalence relations classifiable by countable structures do. E_{K_σ} belongs to the spectrum of the Laver forcing. On the other hand, E_{K_σ} does not belong to the spectrum of the ideal generated by closed measure zero sets.

Definition 2. The ideal I has the *Silver property* if for every Borel equivalence relation E, either there is a Borel I-positive set of pairwise inequivalent elements, or the underlying space decomposes into countably many equivalence classes and a set in I.

So for example, the ideal of countable sets has the Silver property as per the Silver dichotomy. However, other σ-ideal have this property, for example the ideal of σ-compact sets in ω^ω or the ideal of σ-porous sets on $\mathbb{R}$. The nondominating ideal on ω^ω has the Silver property for the equivalence relations classifiable by countable structures, but not in general.

Definition 3. Let E be a Borel equivalence relation on a Polish space X and $x \in X$ be a set generic point. The model $V[x]_E$ is the model of all sets hereditarily definable from parameters in the ground model and from $[x]_E$ in some large collapse forcing extension.

This is a model of ZFC that depends only on the E-equivalence class of x and not on x itself. It allows to connect equivalence relations with models of set theory. The computations of intermediate models are often known, and they can be brought to bear on equivalence relations via this definition.

There are numerous open questions and promisin lines of research in the subject. We will include one typical representative:

Question 2. *Let E be a Borel equivalence relation on $\mathbb{R}^\omega$ reducible to an orbit equivalence. Is there a product of perfect sets on which E is smooth?*

To explain, clearly E_1 does not change its Borel complexity on any product of perfect sets, and it is widely believed to be the simplest equivalence relation not reducible to an orbit equivalence. The answer to the question is positive for orbit equivalences of the permutation group. The question is clearly connected to the computation of the spectrum of the product of countably many copies of Sacks forcing.

Borel ideals as tools

Michael Hrušák

The talk concerns the use of the combinatorics of Borel ideals on ω in perhaps unexpected areas.

Here, an *ideal* is a subset of $\mathcal{P}(\omega)$ closed under subsets and unions. The space $\mathcal{P}(\omega)$ is equipped with the usual Polish topology, and therefore it makes sense to speak about descriptive set theoretic complexity of ideals on ω.

We will use Borel ideals to solve two rather old questions.

The first one, due to Canjar, concerns the existence of ultrafilters $\mathcal{U}$ such that the corresponding Mathias-Prikry forcing $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real.

Let $\mathcal{F}$ be a filter on ω. The *Mathias-Prikry forcing* associated with $\mathcal{F}$ is the partial order

$$\mathbb{M}_{\mathcal{F}} = \{\langle s, F\rangle : s \in [\omega]^{<\omega}, F \in \mathcal{F}\}$$

ordered by

$$\langle s, F\rangle \leq \langle s', F'\rangle \text{ iff } s \supseteq s', F \subseteq F' \text{ and } s \setminus s' \subseteq F'.$$

Canjar himself has shown that such ultrafilters exist assuming $\mathfrak{d} = \mathfrak{c}$ and that any such ultrafilter has to be a P-point without rapid Rudin-Keisler predecessors and asked:

Question 1. *(Canjar 1980) Is every P-point $\mathcal{U}$ without rapid Rudin-Keisler predecessors such that the corresponding Mathias-Prikry forcing $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real?*

Soon after that C. Laflamme introduced the notion of a strong P-point:

An ultrafilter is a *strong P-point* if for any sequence $\langle C_n : n < \omega\rangle$ of compact subsets of $\mathcal{U}$ (considering $\mathcal{U}$ as a subset of 2^ω with the product topology), there is an interval partition $\langle I_n : n < \omega\rangle$ such that $\bigcup_{n<\omega} I_n \cap X_n \in \mathcal{U}$ for each choice of $X_n \in C_n$.

Laflamme noticed that: The forcing $\mathbb{M}_{\mathcal{U}}$ does not add a dominating rea$\Rightarrow \mathcal{U}$ is a strong P-point $\Rightarrow \mathcal{U}$ is a P-point without rapid RK-predecessors anad asked whether eithr of the implications can be reversed.

Question 2. *(Laflamme 1981) Is every strong P-point $\mathcal{U}$ such that the corresponding Mathias-Prikry forcing $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real? Is every P-point without rapid RK-predecessors a strong P-point?*

In a joint work with A. Blass, H. Minami and J. Verner we were able to answer all of these questions by proving:

Theorem 1. *Let $\mathcal{U}$ be a free filter on ω. Then $\mathbb{M}_{\mathcal{F}}$ does not add a dominating real if and only if $\mathcal{U}$ is a strong P-point if and only if the filter $\mathcal{U}^{<\omega}$ is a P^+-filter.*

Here,given a filter $\mathcal{F}$ on ω let

$$\mathcal{F}^{<\omega} = \{A \subseteq [\omega]^{<\omega} \setminus \{\emptyset\} :\ \exists F \in \mathcal{F}\ [F]^{<\omega} \setminus \{\emptyset\} \subseteq A\},$$

and a filter $\mathcal{F}$ on ω is a *P^+-filter* if every decreasing sequence of $\mathcal{F}$-positive sets has a positive pseudo-intersection (mod *fin*).

Theorem 2 (H.-Verner). *The generic ultrafilter added by $\mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}}$ is a P-point without rapid RK-predecessors which is not a strong P-point. In fact, the same is true for any quotient over a tall F_σ P-ideal.*

Here $\mathcal{I}_{\frac{1}{n}}$ denotes the *summable ideal*:

$$\mathcal{I}_{\frac{1}{n}} = \{A \subset \omega : \Sigma_{n \in A} \frac{1}{n} < \infty\}$$

The second application of Borel ideals we will talk about deals with continuous selections and uses combinatorics of an ideal closely realted to Rado's Random graph.

Definition 1 (Michael 1951). A *weak selection* on a topological space X is a continuous function $\varphi : X^2 \to X$ such that $\varphi((x, y)) = \varphi((y, x)) \in \{x, y\}$ for every $x, y \in X$.

A topological space X is *weakly orderable* if it admits a weaker linearly ordered topology or, equivalently, if it admits a continuous one-to-one function to a linearly orderable space.

Proposition 1. *(Michael 1951)*

- *Every weakly orderable space admits a weak selection.*
- *Every connected space which admits a weak selection is weakly orderable.*

The question, implicit in Michaels paper whether the first implication can be reversed was asked explicitly in a 1981 paper by J. van Mill and E. Wattel

Question 3. *(van Mill-Wattel 1981) Is every space which admits a weak selection weakly orderable?*

They gave a positive answer for compact spaces.

Theorem 3. *(van Mill-Wattel, 1981) Every compact space which admits a weak selection is orderable, in particular, weakly orderable.*

With I. Martinez Ruiz we have shown that the answer is negative (even for separable locally compact spaces).

Theorem 4 (H.-Martinez Ruiz). *There is a separable locally compact space admitting a weak selection which is not weakly orderable.*

Some further uses of Borel ideals as tools will be briefly mentioned.

On the complexity of the isomorphism problem for measures on Boolean algebras

Mirna Džamonja
(joint work with Piotr Borodulin–Nadzieja)

A celebrated result in measure theory is the theorem of Maharam in 1942 which states that if μ is a homogeneous σ-additive measure on a σ-complete Boolean algebra $\mathfrak{B}$, then the measure algebra of $(\mathfrak{B}, \mu)$ is isomorphic to the measure algebra of some 2^κ with the natural product measure. Moreover, every measure algebra can be decomposed into a countable sum of such algebras where the measure is homogeneous. This provides us with a very beautiful classification of σ-complete measure algebras and it is natural to ask if a similar characterisation can be obtained under weaker assumptions. In particular, a natural class to consider is formed by pairs $(\mathfrak{B}, \mu)$ where $\mathfrak{B}$ is any Boolean algebra, not necessarily σ-complete, and μ is a *strictly positive finitely additive* measure on $\mathfrak{B}$, where μ without loss of generality assigns measure 1 to the unit element of $\mathfrak{B}$.

A closely connected, but not the same problem, is that of obtaining a combinatorial characterisation of Boolean algebras $\mathfrak{B}$ which *support a measure*, that is, for which there is a finitely additive μ which is strictly positive on $\mathfrak{B}$. It is easy to see that such an algebra must satisfy the countable chain condition ccc and the question of the sufficiency of this condition was raised by Tarski. Horn and Tarski in 1948 suggested various other chain conditions and Gaifman in showed that in fact a rather strong condition of being a union $\mathfrak{B} = \bigcup_{n<\omega} \mathfrak{B}_n$ where each B_{n+1} is n-linked, does not suffice for $\mathfrak{B}$ to support a measure. Kelley in 1951 gave an exact combinatorial characterisation of Boolean algebras that support a measure, which we therefore call Kelley algebras. This characterisation is that $\mathfrak{B} \setminus \{0\} = \bigcup_{n<\omega} \mathfrak{B}_n$, where each B_n has a positive *Kelley intersection number*. The Kelley intersection number for a family $\mathcal{F}$ of sets is said to be $\geq \alpha$ if for every $n < \omega$ and every sequence of n elements of $\mathcal{F}$ (possibly with repetitions), there is a subsequence of length at least $\alpha \cdot n$ which has a nonempty intersection. Then the Kelley intersection number is the sup of all α such that the intersection number of is $\geq \alpha$. This characterisation is unfortunately not very useful in practice, as it is hard to check, but nevertheless, it sheds light on our initial problem of classification. Namely, it shows that every σ-centred Boolean algebra does support a measure. The σ-centred Boolean algebras are exactly the subalgebras of $\mathcal{P}(\omega)$, which for various good reasons are considered to be unclassifiable.

This detour shows that if we hope to have a classification of Kelley algebras, we should better first restrict to some reasonable subclass. Maharam's theorem suggests that there should be some cardinal invariant at least as a first dividing line, so the natural first reduction is to consider only those Boolean algebras that support a *separable* measure. This can be easily defined by noticing that a strictly positive measure μ on a Boolean algebra $\mathfrak{B}$ induces a metric d given by $d(a, b) = \mu(a \Delta b)$. This gives rise to the cardinal characteristics given by the density character of this metric space, which is exactly the Maharam type of a measure algebra if the algebra is σ-closed and the measure σ-additive and homogeneous. The measure μ is said to be *separable* if the density character defined above is equal to ω. In this paper we shall only consider separable measures. The question of a combinatorial characterisation of Boolean algebras that support a separable strictly positive measure has already been considered by many authors, including Talagrand in 1981, who proposed a plausible candidate characterisation. In 2008 Džamonja and Plebanek have shown that there is a ZFC counterexample to this characterisation, therefore putting the characterisation programme back to zero.

Going back to the fact that all subalgebras of $\mathcal{P}(\omega)$ are Kelley algebras, we note that it is rather easy to construct atomic separable measure, counting measures. Many subalgebras of $\mathcal{P}(\omega)$ only support such a measure, so it is more natural to restrict our attention to the non-atomic case. For a Boolean algebra to support a non-atomic measure it is of course necessary that the algebra itself be non-atomic, so we shall mostly consider such algebras. Džamonja and Plebanek showed that Martin's axiom (for cardinals $< \mathfrak{c}$) implies that every non-atomic Boolean algebra of size less than $\mathfrak{c}$ supports a non-atomic separable measure, which shows on the one hand that the characterisation of algebras supporting a non-atomic separable measure is really about algebras of size $\mathfrak{c}$ (it is easy to see that such a measure cannot exists on an algebra of size $> \mathfrak{c}$), and on the other hand that a classification is difficult since it potentially includes all small enough non-atomic algebras.

Since the work of Harrington and Louveau, through much recent work in descriptive set theory, there has emerged a powerful machinery for showing the complexity of various classification problems. Namely, suppose that we wish to classify the objects in a certain class, say we are in a Polish space and we wish to classify definable subspaces of it according to some equivalence relation. The equivalence relation may be understood as having the same invariant. If this classification is useful, then the invariant should be definable and checking if two objects are in the same class should be doable in a definable way. If we show that such a definable classification is not possible, then we have shown that the class we started is unclassifiable in reasonable terms. In this paper our aim is to show that the class of Boolean algebras supporting a separable measure is unclassifiable. We cannot approach this problem directly using the classification techniques of equivalence classes in Polish spaces, since the Boolean algebras in question cannot be coded as elements of a Polish space. However, since we are after a non-classification result, we may restrict to a subclass of our initial class, which may be seen as coming from a Polish space and show a non-classification of even that smaller class. This is exactly what we do, by showing that even the (separable) measures on the Cantor algebra are nonclassifiable.

That result shows that the density of the metric space induced by a measure is not a good invariant for classification. We are therefore led to search for a different invariant, which we find by reconsidering the definition of separability. Namely a measure μ on $\mathcal{B}$ is separable iff there is a countable $\mathfrak{A} \subseteq \mathfrak{B}$ which is μ-*dense* in the sense that for every $b \in \mathcal{B}$ and $\varepsilon > 0$ there is $a \in \mathfrak{A}$ such that $\mu(a \Delta b) < \varepsilon$. Let us say that $\mathfrak{A} \subseteq \mathfrak{B}$ is *μ-uniformly dense* if for every $b \in \mathcal{B}$ and $\varepsilon > 0$ there is $a \in \mathfrak{A}$ such that $a \leq b$ and $\mu(b \setminus a) < \varepsilon$. The *uniform density* of $(\mathfrak{B}, \mu)$ can be defined as the smallest cardinal κ such that there is a μ-uniformly dense $\mathfrak{A} \subseteq \mathfrak{B}$ of size κ. Kelley algebras whose uniform density is $\aleph_0$ are called *uniformly regular* and they have been considered in the literature, for example in by Mercourakis, and it was shown that uniform regularity is quite different than separability. We show that in fact for the purposes of characterisation, uniformly regular measures are much superior to the separable ones, since we prove that a Boolean algebra supports a non-atomic uniformly regular measure iff it is a subalgebra of the so called Jordan algebra, and algebra which is well known in the literature.

We recall in that it is also known that algebras supporting a non-atomic separable measure are subalgebras of a fixed algebra (namely the Cohen one) but that there is no iff characterisation known. Our results suggest that the classification of non-atomic Kelley algebras should proceed with the invariant being the uniform density rather than the Maharam type.

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A model with a precipitous ideal but without normal one

Moti Gitik

T. Jech and K. Prikry asked if it is possible to have a model with a precipitous ideal but without normal one.

We show the following:

Theorem 1. *$Con(\exists \kappa o(\kappa) = \kappa^{++}) \rightarrow Con(\exists$ a precipitous ideal on $\aleph_1$ but there is no normal precipitous ideal$)$.*

The next result follows from the proof:

Theorem 2. *There is a model with a supercompact in which there is no precipitous filters on $\aleph_1$ extending $Cub_{\aleph_1} \restriction S$, for some fixed stationary $S \subseteq \aleph_1$.*

The following related problems remain open.

Question 1. *Is the assumption $o(\kappa) = \kappa^{++}$ needed for a model with a precipitous ideal on $\aleph_1$ but without a normal one?*

We think that it is likely to be possible to show that if $\aleph_1$ is ∞–semi precipitous with a witnessed forcing satisfying $\aleph_3$–c.c. and with image of $\aleph_1$ under the corresponding generic embedding is at least $\aleph_3$, then $o(\kappa) = \kappa^{++}$ in an inner model. But probably there is no need to go via a construction of such ∞–semi precipitous.

Question 2. *Is it possible to have a GCH model with a precipitous ideal on $\aleph_1$ but without a normal one?*

By [7] large cardinals not far from $o(\kappa) = \kappa^{++}$ are needed for such a model.

Question 3. *Is it possible to generalize the present result to cardinals bigger than $\aleph_1$?*
Simplest case: Is there a model with a precipitous ideal on $\aleph_2$ but without a normal one?

The next question is well known with partial answers given by Schimmerling, Velickovic [13], Woodin [14] (8.1 Condensation Principles) and recently by Wu.

Question 4. *Is it consistent that there is a supercompact cardinal and $\aleph_1$ does not carries a precipitous ideal?*

Question 5. *Is it consistent that there is a supercompact cardinal and $\aleph_1$ does not carries a precipitous filters that are Q–points, i.e. isomorphic to filters which extend $Cub_{\aleph_1}$?*

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On partial square sequences

John Krueger

A famous theorem in set theory is the result that the failure of the square principle $\Box_\kappa$, for a regular uncountable cardinal κ, is equiconsistent with a Mahlo cardinal. Solovay proved that if $\lambda > \kappa$ is a Mahlo cardinal, then in any generic extension by the Lévy collapse $\textsc{Coll}(\kappa, <\lambda)$, $\lambda = \kappa^+$ and $\neg\Box_\kappa$. On the other hand, Jensen proved that $\neg\Box_\kappa$ implies that κ^+ is Mahlo in L.

Partial square sequences were introduced by Shelah as a weakening of the square principle. Let $\nu < \kappa^+$ be regular, and let $A \subseteq \kappa^+ \cap \mathrm{cof}(\nu)$. We say that A *carries a partial square* if there exists a sequence $\langle c_\alpha : \alpha \in A\rangle$ satisfying: (a) c_α is a club subset of α; (b) $\mathrm{ot}(c_\alpha) = \nu$; (c) if γ is a limit point of c_α and c_β, then $c_\alpha \cap \gamma = c_\beta \cap \gamma$.

A significant difference between the square principle and partial squares is that, while $\square_\kappa$ is independent of ZFC, the existence of partial squares is provable in ZFC. For example, Shelah proved that if κ is a regular uncountable cardinal, then $\kappa^+ \cap \mathrm{cof}(<\kappa)$ splits into κ many pairwise disjoint subsets each of which carries a partial square.

Magidor constructed a model of set theory which satisfies a strong form of stationary set reflection, using a weakly compact cardinal. In this model there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ which carries a partial square.

We define a forcing iteration which destroys the stationarity of any subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$ which carries a partial square, using a greatly Mahlo cardinal. We also obtain the lower bound, by showing that if no stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$ carries a partial square, then κ^+ is greatly Mahlo in L. Thus the statement that there exists a regular uncountable cardinal κ such that no stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$ carries a partial square is equiconsistent with a greatly Mahlo cardinal.

Unitary representations of oligomorphic groups

Todor Tsankov

Traditionally, representation theory is restricted to studying representations of locally compact groups, and for a good reason: the Haar measure provides an invaluable tool for constructing and analyzing representations. It gives rise to the left-regular representation (so that every locally compact group has at least one faithful representation) but also allows to define convolution of functions and various useful topologies on function spaces on the group. And indeed, many standard theorems of representation theory break down for non-locally compact groups: for example, the group of homeomorphisms of the reals has no non-trivial unitary representations (Megrelishvili [6]), while the group of all measurable maps from $[0,1]$ to the circle has a faithful unitary representation (by multiplication on $L^2([0,1])$) but has no irreducible representations (this example is due to Pestov; see [2, Example C.5.10]). Nevertheless, some non-locally compact groups do have a nice representation theory: for example, the infinite symmetric group S_∞ and the unitary group of a separable, infinite-dimensional Hilbert space both have only countably many irreducible representations that separate points and every representation splits as a sum of irreducibles (Lieberman [5] and Kirillov [4]). In both situations, the representation theory is quite similar to the one for compact groups. We present a similar classification result for the representations of oligomorphic permutation groups.

Let S_∞ be the group of *all* permutations (not necessarily of finite support) of a countable infinite set $\mathbf{X}$. It becomes naturally a topological group if equipped with the pointwise convergence topology (where $\mathbf{X}$ is taken to be discrete). For us, a *permutation group* will be a *closed* subgroup of S_∞ equipped with its natural action on the set $\mathbf{X}$. It is well known that the topological groups that can be realized as permutation groups are exactly the *Polish* (separable, completely metrizable) groups that admit a basis at the identity consisting of open subgroups. The basis is given by the stabilizers of finite sets.

A natural way in which permutation groups arise is as *automorphism groups of countable structures* in model theory, i.e. one fixes some relations and functions on the set $\mathbf{X}$ and considers the group of all permutations that preserve them. We will restrict our attention to *oligomorphic* permutation groups which are defined as follows.

Definition 1. A permutation group $G \curvearrowright \mathbf{X}$ is called *oligomorphic* if the induced action $G \curvearrowright \mathbf{X}^n$ has only finitely many orbits for each n.

The following property of topological groups will also be relevant for us.

Definition 2. A topological group G is called *Roelcke precompact* if for every open neighborhood of the identity U, there exists a finite set F such that $G = UFU$.

In the above definition, one can obviously restrict U to belong to a basis at the identity, so for closed subgroups of S_∞, the definition has the following equivalent form: $G \leq S_\infty$ is Roelcke precompact iff for every open subgroup $V \leq G$, the set of double cosets $\{VxV : x \in G\}$ is finite. One has the following basic characterization.

Theorem 1. *For a closed subgroup $G \leq S_\infty$, the following are equivalent:*

(i) *G is Roelcke precompact;*
(ii) *for every continuous action $G \curvearrowright \mathbf{X}$ on a countable set $\mathbf{X}$ with finitely many orbits, the induced action $G \curvearrowright \mathbf{X}^n$ has finitely many orbits for each n;*
(iii) *G can be written as an inverse limit of oligomorphic groups.*

Note that an oligomorphic group can never be locally compact: if $V \leq G$ is a compact open subgroup, then the union of finitely many V double cosets will be compact.

A standard way to produce structures with oligomorphic automorphism groups is the so-called Fraïssé construction: given a class of finite structures satisfying a certain amalgamation property, there is a way to build an infinite structure that contains all structures in the class as substructures and is moreover homogeneous. We refer the reader to [3] for the general theory and just present a few examples.

- The Fraïssé limit of all finite sets without structure is a countably infinite set $\mathbf{X}$. The corresponding group is S_∞, the group of all permutations of $\mathbf{X}$.
- The Fraïssé limit of all finite linear orders is the countable dense linear order without endpoints $(\mathbf{Q}, <)$.
- The Fraïssé limit of all finite boolean algebras is the countable atomless boolean algebra which is isomorphic to the algebra of all clopen subsets of the Cantor space $2^{\mathbb{N}}$. The corresponding automorphism group is $\mathrm{Homeo}(2^{\mathbb{N}})$, the group of all homeomorphisms of $2^{\mathbb{N}}$.
- The Fraïssé limit of all finite vector spaces over a fixed finite field $\mathbf{F}_q$ is the countable-dimensional vector space over $\mathbf{F}_q$.

- The Fraïssé limit of all finite graphs is the *random graph*, the unique countable graph R such that for every two finite disjoint sets of vertices U, V, there exists a vertex x which is connected by an edge to all vertices in U and to no vertices in V.

The main theorem, proved in [7], describes all representations of an oligomorphic group as direct sums of irreducible representations which are induced from irreducible representations of finite quotients of open subgroups. More precisely, the following holds.

Theorem 2. *Let G be an oligomorphic group. Then every irreducible unitary representation of G is of the form $\mathrm{Ind}_{C(V)}^{G}(\sigma)$, where $V \leq G$ is an open subgroup, $C(V)$ is the commensurator of V, $V \trianglelefteq C(V)$, and σ is an irreducible representation of the* finite *group $C(V)/V$. Moreover, every unitary representation of G is a sum of irreducibles.*

We also provide a complete description of when two representations of the form $\mathrm{Ind}_{C(V)}^{G}(\sigma)$ are equivalent.

As every oligomorphic group has only countably many distinct open subgroups, this means that every oligomorphic group has only countably many irreducible representations.

If one is given a realization of an oligomorphic group as the automorphism group of a countable structure, it is usually possible to give a more concrete description of the representations in terms of the structure. For example, for the random graph, one can take a finite (induced) subgraph $A \subseteq R$ and set V to be the pointwise stabilizer of A. Then $C(V)$ is the setwise stabilizer of A and $C(V)/V \cong \mathrm{Aut}(A)$. As a result, one obtains that irreducible representations of the automorphism group of the random graph are obtained by induction from irreducible representations of automorphism groups of finite graphs (and in fact, this correspondence is one-to-one if one takes care of the obvious identifications).

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An analysis of the amenablity problem for Thompson's group F

Justin Tatch Moore

Thompson's group F is a well studied object which has been rediscovered several times in different contexts. Abstractly, it is the group generated by x_n $(n \in \omega)$ subject to the relations $x_i^{-1} x_n x_i = x_{n+1}$ for each $i < n$. The generators x_n for $n > 1$ are redundant and in fact F has a finite presentation (although this is not obvious). F also can be defined as a group of automorphisms of $[0, 1]$ as follows. Let $\mathcal{T}$ denote the collection of all subsets T of $[0, 1]$ which contain $\{0, 1\}$ and satisfy that if $s < t$ are consecutive elements of T, then there are natural numbers p and q such that $s = p/2^q$ and $t = (p + 1)/2^q$. If $S, T \in \mathcal{T}$ have an equal number of elements, then the unique order preserving map from S to T extends piecewise linearly to a map of $[0, 1]$ to $[0, 1]$. The collection of all such maps with composition forms a group isomorphic to F. F acts on $\mathcal{T}$ by set-wise application. There are several other natural ways of viewing $\mathcal{T}$. One which we will need below is that $\mathcal{T}$ is the collection of all variable free terms in a binary operation $\frown$ and a constant symbol 1.

A basic question about F which still remains open is whether it is *amenable*: does it support a finitely additive translation invariant probability measure? Part of the interest in this question stems from the fact that it is known that F does not contain a copy of the free group on two generators and therefore a negative answer would give a finitely presented counterexample to the so called von Neumann-Day Conjecture. This conjecture has been resolved, but only recently for finitely presented groups and by a counterexample which is both complex and *ad hoc* [4].

My approach to this problem has been to determine what can be said about invariant measures on F and the subsets of F which must be assigned measure 0 by any invariant measure. The most quotable results along these lines were obtained in [3]. First recall that an action of a finitely generated group is amenable if and only if for every $\epsilon > 0$ there is an *ϵ-Følner set* for the action with respect to a fixed finite generating set — that is a finite set A such that $\sum_\gamma |A \triangle A \cdot \gamma| < \epsilon |A|$, where γ ranges over the finite set of generators. If an action is amenable, we can therefore ask how fast the minimum cardinality of a $1/n$-Følner set grows as n increases. In [3] it is shown that, if F is amenable, then there is a constant C such that the minimum cardinality of a C^{-n}-Følner set is at least $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (this is true for both the action of F on itself and on $\mathcal{T}$). This is derived from the following qualitative result which is of independent interest: if μ is an F-invariant measure on $\mathcal{T}$ and I_i $(i \leq l)$ is a sequence of intervals such that $0 < \min I_i \leq \max I_{i+1} < 1$ for all $i < l$, then μ-a.e. T satisfy that $|T \cap I_i|$ $(i \leq l)$ is strictly increasing or strictly decreasing.

This last qualitative property of F-invariant measures on $\mathcal{T}$ places limitations on the methods of construction which might be successful in constructing an invariant measure. The main purpose of this talk is to present two candidates for how to build such invariant measures which naturally build in this qualitative feature.

The first is non-constructive and is motivated by a well known result of Ellis: if $(S, *)$ is a compact left topological semigroup, then S has an idempotent. Typically this result is applied by starting with a discrete semigroup S and then using iterated integration to extend the binary operation to βS — the Cech-Stone compactification of S. In fact the operation also extends to the finitely additive probability measures on S. One can ask whether there is an analog of Ellis's theorem for arbitrary binary operations. While Ellis's theorem yields an idempotent ultrafilter on S when S is a semigroup, there are no idempotent ultrafilters on the free binary system on one generator. It seems plausible, however, that there would be an idempotent probability measure. In fact one can prove that if μ is an idempotent measure on $(\mathcal{T}, \hat{})$, then μ is invariant with respect to the action of F.

Conjecture 1. *If $(S, *)$ is any binary system, then there is an idempotent probability measure on S.*

It is interesting to note that the process of extending the operation to the space of probability measures requires a choice in the order of integration and this choice would account for the asymmetry noted above which ergodic invariant measures must exhibit.

The second approach to proving that F is amenable involves attempts to build explicit Følner sets for F. The main ingredient is the assumption of the existence of a system of functions f_T $(T \in \mathcal{T})$ satisfying the following properties:

- Each f_T is a non-identity increasing function from $\mathbb{N}$ to $\mathbb{N}$ and $f_S = f_T$ if S is equivalent to T by the repeated applications of the *left self distributive law* $a\hat{}(b\hat{}c) = (a\hat{}b)\hat{}(a\hat{}c)$.
- $\mathrm{crit}(f_{A\hat{}B}) = f_A(\mathrm{crit}(f_B))$ where $\mathrm{crit}(f_T) = \min\{n : f_T(n) \neq n\}$.
- $\{\mathrm{crit}(f_T) : T \in \mathcal{T}\} = \mathbb{N}$.

Such a family of functions is unique if it exists, in which case the functions f_T are all recursive. The existence of such a system is known to follow form the existence of a rank-to-rank elementary embedding but it is not known to follow from ZFC [2]. Such functions allow for the definition of a number of finite-to-one functions from $\mathcal{T}$ into $\mathbb{N}$ which seem relevant to building Følner sets for F. The main template is:

$$\phi(T) = \begin{cases} 0 & \text{if } T = 1 \\ \max(\phi(A), f_A(\phi(B))) & \text{if } T = A\hat{}B \end{cases}$$

This function is finite-to-one and satisfies that $\phi(T) \leq \phi(T \cdot x_k)$ whenever $T \cdot x_k$ is defined. It can be shown that $\mathcal{A}_n = \{T \in \mathcal{T} : \phi(T) \leq n\}$ does not contain a Følner sequence, but I conjecture that if these sets are additionally constrained by some condition which only concerns the LD-class of $T \in \mathcal{T}$, then one obtains sets which contain a Følner sequence. A test conjecture is the following. Let $\mathcal{A}_{n,p}$ denote those T which are iterated right divisors of $1^{[p+1]}$ (the $p+1$st right associated power of the generator 1) and which satisfy $\phi(T) \leq n$.

Conjecture 2. *For every $\epsilon > 0$, there is a p_0 such that if $p > p_0$ then for all but finitely many n, $\mathcal{A}_{n,p}$ is ϵ-Følner.*

Moreover, it should be possibly to take p_0 to be $O(1/\epsilon^k)$ for some k.

Additionally, I conjecture that the amenability of F is equivalent to the existence of the system of functions f_T $(T \in \mathcal{T})$.

Conjecture 3. *The Følner function of F is bi-primitive recursive in any of the functions f_T.*

Dougherty has shown that none of the functions f_T are primitive recursive but that each is primitive recursive in any other, assuming that the system exists [1].

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On the Bergman and Steinhaus properties for infinite products of finite groups

Simon Thomas
(joint work with Jindřich Zapletal)

In this talk, we discussed the Bergman and Steinhaus properties for infinite products of finite groups.

Definition 1. Suppose that G is a non-finitely generated group.

(a) G has *countable cofinality* if $G = \bigcup_{n\in\omega} G_n$ can be expressed as the union of a countable increasing chain of proper subgroups. Otherwise, G has *uncountable cofinality*.
(b) G is *Cayley bounded* if for every symmetric generating set S, there exists an integer $n \geq 1$ such that every element $g \in G$ can be expressed as a product $g = s_1 \cdots s_n$, where each $s_i \in S \cup \{ 1 \}$.
(c) G has the *Bergman property* if G has uncountable cofinality and is Cayley bounded.

Definition 2. Let G be a topological group. Then G has the *Steinhaus property* if there exists a *fixed* integer $k \geq 1$ such that for every symmetric countably syndetic subset $W \subseteq G$, the k-fold product W^k contains an open neighborhood of the identity element 1_G.

The talk focused on groups of the form $\prod SL(2, p_n)$, where $(p_n \mid n \in \omega)$ is an increasing sequence of primes. This may seem a strange choice, given that the results of Saxl-Shelah-Thomas [1] and Thomas [2] imply that the Bergman and Steinhaus properties always fail for such groups.

Theorem 1 (ZFC). *If $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes, then:*

(a) $\prod SL(2, p_n)$ *has countable cofinality;*
(b) $\prod SL(2, p_n)$ *is not Cayley bounded; and*
(c) $\prod SL(2, p_n)$ *does not have the Steinhaus property.*

However, this is not the end of the story. The arguments in both [1] and [2] make use of an ultraproduct $\prod_{\mathcal{U}} SL(2, p_n)$, where $\mathcal{U}$ is a nonprincipal ultrafilter over ω; and it is natural to ask whether the existence of such an ultrafilter is either necessary or sufficient in order to establish the failure of the Bergman and Steinhaus properties for $\prod SL(2, p_n)$. Of course, when considering this kind of question, we cannot work with the usual ZFC axioms of set theory since these already imply the existence of nonprincipal ultrafilters over arbitrary infinite sets. Instead we will work with the axiom system $ZF + DC$, where DC is the following weak form of the Axiom of Choice.

Axiom of Dependent Choice (DC). *Suppose that X is a nonempty set and that R is a binary relation on X such that for all $x \in X$, there exists $y \in X$ with $x\, R\, y$. Then there exists a function $f : \omega \to X$ such that $f(n)\, R\, f(n+1)$ for all $n \in \omega$.*

The following result shows that the existence of a nonprincipal ultrafilter over ω is indeed necessary in order to prove the failure of the Bergman property. It is currently not known whether the failure of the Steinhaus property implies the existence of a nonprincipal ultrafilter over ω.

Theorem 2 ($ZF+DC$). *Let $(\, p_n \mid n \in \omega\,)$ be an increasing sequence of primes. If $\prod SL(2, p_n)$ does not have the Bergman property, then there exists a nonprincipal ultrafilter over ω.*

On the other hand the following result shows that the existence of a nonprincipal ultrafilter over ω is not sufficient to prove the failure of either the Bergman property or the Steinhaus property. (Here LC indicates that the proof makes use of a suitable large cardinal hypothesis.)

Theorem 3 (LC). *If $(\, p_n \mid n \in \omega\,)$ is a sufficiently fast growing sequence of primes, then $\prod SL(2, p_n)$ has both the Bergman property and the Steinhaus property in $L(\mathbb{R})[\mathcal{U}]$.*

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Fusion and large cardinals

Lyubomyr Zdomskyy

(joint work with Radek Honzik, Sy-D. Friedman)

An important technique in large cardinal set theory is that of extending an elementary embedding $j : M \to N$ of inner models to an elementary embedding $j^* : M[G] \to N[G^*]$ of generic extensions of them. For example, using a reverse Easton iteration of forcings adding α^{++} many Cohen subsets to every inaccessible cardinal α below a strong cardinal κ, Woodin produced a model where κ remains measurable and GCH fails at κ. As a complementary technique to the above mentioned proof of Woodin, Friedman and Thompson suggested in [2] to use perfect trees, using *fusion* as a substitute for distributivity. This allowed them to provide, among other results, a new proof of Woodin's theorem. Another example of the use of fusion for extending elementary embeddings is given in our joint work with Friedman [3], where we used an iteration with supports of size $\leq \kappa$ of a suitably defined uncountable version of the Miller and Sacks forcings (the latter is an alternative name for the perfect tree forcing).

These results suggest to isolate some property (or properties) $\mathcal{B}$ such that whenever κ is a *large* cardinal (e.g., strong cardinal) and $\langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ is an iteration with supports of size $\leq \kappa$ such that $\Vdash_{\mathbb{P}_\xi}$ "$\underset{\sim}{\mathbb{Q}}_\xi$ has the property $\mathcal{B}$" for all $\xi < \gamma$, then $(\kappa^+)^V$ is a cardinal in $V^{\mathbb{P}_\gamma}$ and κ remains *large* in this forcing extnsion. One of such properties is a suitable modification of the reasonable B-boundedness of Roslanowski and Shelah from [5] given by the definition below.

In what follows $\bar{\mu}$ denotes an increasing sequence $\langle \mu_\alpha : \alpha < \kappa \rangle$ of regular cardinals below an inaccessible cardinal κ such that $|\prod_{\xi<\alpha} f(\xi)| < \mu_\alpha$ for every $f : \alpha \to \mu_\alpha$. Let $\mathcal{U}$ be a family of unbounded subsets of κ which is closed under diagonal intersections. For a poset $\mathbb{Q}$ we denote by $\mathcal{U}^{\mathbb{Q}}$ the closure of $\mathcal{U}$ under diagonal intersections in $V^{\mathbb{Q}}$.

Definition 1. Let $\mathbb{Q}$ be a forcing notion.

For a condition $p \in \mathbb{Q}$ we define a game $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ lasts κ steps and results in a sequence $\left\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \kappa \right\rangle$ constructed by the players. The αth round is played as follows:

(1) First, Generic chooses a non-empty set I_α of cardinality $< \mu_\alpha$ and a collection $\langle p^\alpha_t : t \in I_\alpha \rangle$ of pairwise incompatible elements of $\mathbb{Q}$ such that
   (a) for any $J \subset \alpha$ and $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} I_\xi$, if there exists a lower bound for the set $\{q^\xi_{t_\xi} : \xi \in J\} \cup \{p^\alpha_t\}$ for some $t \in I_\alpha$, then p^α_t is such a lower bound;
   (b) for any limit $\alpha < \kappa$, a cofinal subset J of α, and a sequence $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} I_\xi$, the set $\{t \in I_\alpha : \forall\, \xi \in \alpha\, (p^\alpha_t \leq q^\xi_{t_\xi})\}$ has size at most $|\alpha|$.

(2) Antigeneric answers by picking a collection $\langle q^\alpha_t : t \in I_\alpha \rangle$ such that $q^\alpha_t \leq p^\alpha_t$ for all $t \in I_\alpha$.

Generic wins this play $\langle I_\alpha, \langle p_t^\alpha, q_t^\alpha : t \in I_\alpha\rangle : \alpha < \kappa\rangle$ if there exists $p^* \leq p$ such that $p^* \Vdash \{\alpha < \kappa : \exists\, t \in I_\alpha\ (q_t^\alpha \in \Gamma_{\mathbb{Q}})\} \in \mathcal{U}^{\mathbb{Q}}$.

We say that $\mathbb{Q}$ is *reasonably B_e-bounding*[1] *over* $\mathcal{U}, \bar{\mu}$ if $\mathbb{Q}$ is $< \kappa$ strategically closed and the Generic has a winning strategy in $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ for all $p \in \mathbb{Q}$. If $\mathcal{U} = \{\kappa\}$, then forcing notions which are reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$ will be called *reasonably A_e-bounding over $\bar{\mu}$*. $\square$

If we remove items $(a), (b)$ (or just item (b)) from Definition 1(1), we get the definition of the game $\mathfrak{D}^{rc\mathbf{B}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ and of *reasonably B-bounding over $\mathcal{U}, \bar{\mu}$* forcing notions introduced in [5], respectively. If $\mathcal{U} = \{\kappa\}$, then forcing notions which are reasonably B-bounding over $\mathcal{U}, \bar{\mu}$ are called [5] *reasonably A-bounding over $\bar{\mu}$*.

Assume GCH in V and let κ be an inaccessible limit of inaccessible cardinals in V. We define in V a poset $\mathbb{R}_\kappa$ as follows. Let $\mathbb{R}_0$ be the trivial forcing. For $i < \kappa$ let $\underset{\sim}{\mathbb{S}}_i$ be a $\mathbb{R}_i$-name for the lottery sum of all $< \rho_i$-directed closed posets whose underlying set is a subset of $H(\rho_i^{++})^{V^{\mathbb{R}_i}}$, where ρ_i is the ith inaccessible cardinal below κ. In other words, let $\mathbb{S}_i$ be a $\mathbb{R}_i$-name for the poset $\{\langle \mathbb{S}, s\rangle : \mathbb{S}$ is a $< \rho_i$-directed closed posets whose underlying set is a subset of $H(\rho_i^{++})^{V^{\mathbb{R}_i}}$ and $s \in \mathbb{S}\} \cup \{1\}$, ordered with 1 above everything else and $\langle \mathbb{S}, s\rangle \leq \langle \mathbb{S}', s'\rangle$ when $\mathbb{S} = \mathbb{S}'$ and $s \leq s'$. Let $\mathbb{R}_\kappa$ be the iteration $\langle \mathbb{R}_\xi, \underset{\sim}{\mathbb{S}}_\xi : \xi < \kappa\rangle$ with Easton support.

The following theorem is the main result we are going to present.

Theorem 1. *Suppose GCH holds and $j : V \to M$ is a (κ, κ^{++})-extender ultrapower*[2] *and $H(\kappa^{++})^V = H(\kappa^{++})^M$. Let $\mathcal{U}$ be a normal filter on κ contained in the measure derived from j. Let also $\mathbb{R}_\kappa$ be the poset defined above and in $V^{\mathbb{R}_\kappa}$ let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma\rangle$ be a κ-support iteration such that $\gamma \leq \kappa^{++}$ and $\Vdash_{\mathbb{R}*\mathbb{P}_\xi}$ "$\underset{\sim}{\mathbb{Q}}_\xi$ is a $< \kappa$ directed closed reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$ poset of size $\leq \kappa^+$" for all $\xi < \gamma$. Then j can be extended to an elementary embedding $j^{**} : V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma} \to M^{j(\mathbb{R}_\kappa * \mathbb{P}_\gamma)}$ so that $H(\kappa^{++})$ of $V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma}$ and $M^{j(\mathbb{R}_\kappa * \mathbb{P}_\gamma)}$ coincide. In particular, κ remains measurable in $V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma}$.*

Theorem 1 covers iterations of the "plain" Sacks forcing at κ introduced in [4] and also of some posets from [5]. In particular, we have the following

Corollary 1. *Suppose GCH holds and $j : V \to M$ is an (κ, κ^{++})-extender ultrapower such that $H(\kappa^{++})^V = H(\kappa^{++})^M$. Let $\mathcal{U}$ be the measure derived from j. Then there exists a κ^{++}-c.c. κ-proper poset $\mathbb{P}$ of size κ^{++} such that*

(1) *j can be extended to an elementary embedding $j^* : V^{\mathbb{P}} \to M^{j(\mathbb{P})}$;*
(2) *If $\mathcal{W} \in V^{\mathbb{P}}$ is a normal measure on κ such that $\mathcal{U} \subset \mathcal{W}$, then $\mathfrak{d}_{\mathcal{W}} = \kappa^+$;*
(3) *If $\mathcal{W} \in V^{\mathbb{P}}$ is a normal measure on κ such that $\mathcal{U} \not\subset \mathcal{W}$, then $\mathfrak{d}_{\mathcal{W}} = \kappa^{++}$.*

However, Theorem 1 does not cover *arbitrary* Sacks posets at κ (e.g., those where a node on αth splitting level of a tree has $> |\alpha|^+$ immediate successors, see [1] where such posets were used to force a prescribed number of normal measures on a measurable cardinal.) Iterations with support $\leq \kappa$ of the poset Miller(κ)

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[1] The letter "e" comes from "extending embeddings"

[2] I.e., $M = \{j(f)(a) : f \in V, f : H(\kappa) \to V$, and $a \in H(\kappa^{++})\}$.

introduced in [3] are also not in the scope of applications of Theorem 1. We plan to discuss possible strategies how to overcome these difficulties.

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Bounding coloring numbers by powers of choice numbers in all infinite graphs

Menachem Kojman

The *choice number* or *list-chromatic number* $\chi_\ell(G)$ of a graph $G = (V, E)$ is the least cardinal κ such that for every assignment of a list $L(v)$ of κ colors to each $v \in V$ there exists a valid coloring c of V such that $c(v) \in L(V)$ for each vertex $v \in V$. The *coloring number* $Col(G)$ of G is the least κ such that there is a well-ordering $\prec$ of V satisfying that each vertex $v \in V$ has fewer than κ neighbors in $\{u : u \prec v\}$. For every G it holds that $\chi(G) \leq \chi_\ell(G) \leq Col(G)$ (where $\chi(G)$ is the usual *chromatic number* of G.)

N. Alon proved that $Col(G) \leq c2^{\chi_\ell(G)}$ for some constant c for every finite graph G and asked if some analogous bound holds in the infinite case. Using Shelah's *revised GCH Theorem* we prove that for every graph G with infinite $\chi_\ell(G)$ it holds that

$$Col(G) \leq \beth_\omega(\chi_\ell(G)).$$

Better upper bounds hold for proper initial segments of the cardinals or by assuming very weak forms of the Singular Cardinals Hypothesis. The main point, though, is that countably many power-set operations suffice to bound the coloring number in *all* graphs in ZFC.

More on continuous pair colorings on Polish spaces

Stefan Geschke

1. Introduction

Let X be a Polish space. A coloring $c : [X]^2 \to 2$ of the unordered pairs of X is *continuous* if for all $x, y \in X$ with $x \neq y$ there are disjoint open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and for all $a \in U$ and all $b \in V$, $c(x, y) = c(a, b)$.

Continuous colorings made an appearance in the theory of planar convexity in [5] and have been studied more systematically in [4]. Some more information was obtained in [1] and [2]. The new results presented here without citation will appear in [3].

In the light of the importance of the $\mathcal{G}_0$-dichotomy for analytic graphs [6], it seems to be natural to consider instead of a continuous coloring $c : [X]^2 \to 2$ the clopen graph

$$(X, \{(x, y) \in X^2 : x \neq y \wedge c(x, y) = 1\})$$

on X associated to c. Several graph-theoretic cardinal invariants are degenerate for clopen graphs in the sense that they are either countable or $2^{\aleph_0}$. Examples include the clique number, the chromatic number, and the Borel chromatic number. A cardinal invariant that is not degenerate in this sense is the *cochromatic number* which we call *homogeneity number* in the context of continuous colorings.

Definition 1. Let $c : [X]^2 \to 2$ be a continuous coloring on a Polish space X. The *homogeneity number* $\mathfrak{hm}(c)$ is the least size of a family of c-homogeneous subsets of X that covers X.

The following facts were already proved in [5]:

Theorem 1. *a) There is a continuous coloring $c_{\min} : [2^\omega]^2 \to 2$ such that for every uncountably homogeneous continuous coloring c on a Polish space we have $\mathfrak{hm}(c_{\min}) \leq \mathfrak{hm}(c)$.*

b) $2^{\aleph_0} \leq (\mathfrak{hm}(c_{\min}))^+$

c) It is consistent that for every continuous coloring c on a Polish space we have $\mathfrak{hm}(c) < 2^{\aleph_0}$.

In [4], the following result was shown:

Theorem 2. *a) There is a continuous coloring $c_{\max} : [2^\omega]^2 \to 2$ such that for every continuous coloring c on a Polish space, $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\max})$.*

b) It is consistent that $\mathfrak{hm}(c_{\min}) < \mathfrak{hm}(c_{\max})$.

Finally, after some initial results in [5] and [4], in [1] it was determined that $\mathfrak{hm}(c_{\min})$ is large compared to other cardinal characteristics of the continuum.

Theorem 3. $\mathrm{cof}(\mathfrak{t}\mathit{null}) \leq \mathfrak{hm}(c_{\min})$

In particular, $\mathfrak{hm}(c_{\min})$ is larger than all cardinal characteristics in Cichoń's diagram.

2. Depth and width

Definition 2. Let $A \subseteq \omega^\omega$ be a closed set. A continuous coloring $c : [A]^2 \to 2$ is of *width* $m \in \omega$ if no node in the tree

$$T(A) = \{s \in \omega^{<\omega} : \exists x \in A(s \subseteq x)\}$$

has more than m immediate successors.

For $\{x, y\} \in [\omega^\omega]^2$ let

$$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}.$$

The coloring c is of depth $k \in \omega$ if for all $\{x, y\} \in [A]^2$ the color $c(x, y)$ only depends on $x \restriction \Delta(x, y) + k$ and $y \restriction \Delta(x, y) + k$.

Note that if we speak of depth and width of a coloring this implicitly means that the coloring is defined on a closed subset of ω^ω. The methods developed in [4] can be used to show the following:

Lemma 1. *a) Every continuous coloring on a zero-dimensional, compact metric space X is isomorphic to a coloring of depth 2.*

b) For every continuous coloring c on a Polish space there is a continuous coloring d of depth 1 such that $\mathfrak{hm}(c) = \mathfrak{hm}(d)$.

Using a construction similar to the construction of $c_{\max}$ in [4] together with Lemma 1 a) we obtain

Theorem 4. *There is a continuous coloring $c_{\text{universal}} : [\omega^\omega]^2 \to 2$ of depth 2 such that every continuous coloring on a zero-dimensional, compact metric space embeds into $c_{\text{universal}}$ in the natural sense.*

Using an entirely new argument, we show that many colorings have the same homogeneity number as $c_{\min}$.

Theorem 5. *Let c be a continuous coloring of finite width and depth. If $\mathfrak{hm}(c)$ is uncountable, then $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min})$.*

3. Analysis of continuous colorings in terms of finite induced subgraphs

We assume all classes of graphs to be closed under isomorphism. A class of finite graphs is *nontrivial* if it contains a graph that has an edge and a graph with two points that are not connected by an edge. A class of finite graphs is *closed* if it is closed under taking induced subgraphs and under *substitution*.

Given two graphs G and F with disjoint sets $V(G)$ and $V(F)$ of vertices and a vertex $v \in V(G)$, H is obtained by *substituting F for v in G* if $V(H)$ is the disjoint union of the sets $V(G) \setminus \{v\}$ and $V(F)$, any two vertices in $V(G) \setminus \{v\}$ are connected in H if they are connected in G, any two vertices in $V(F)$ are connected in H if they are connected in F, and a vertex $u \in V(G) \setminus \{v\}$ is connected in H to a vertex $w \in V(F)$ if u and v form an edge in G.

Definition 3. Let c be a continuous coloring on a Polish space X. Let $\mathrm{age}(c)$ denote the class of all finite graphs G that are isomorphic to an induced subgraph of the clopen graph associated with c.

The coloring c is *self-similar* if for every nonempty open set $O \subseteq X$ we have $\mathrm{age}(c \restriction O) = \mathrm{age}(c)$.

Theorem 6. *a) If c is a self-similar continuous coloring on a Polish space and $\mathfrak{hm}(c)$ is uncountable, then $\mathrm{age}(c)$ is a nontrivial closed class of finite graphs.*

b) If $\mathcal{A}$ is a closed class of finite graphs then there is a selfsimilar continuous coloring $c_{\mathcal{A}}$ of depth 1 on a compact subset of ω^ω such that $\mathrm{age}(c_{\mathcal{A}}) = \mathcal{A}$. With these properties, the coloring $c_{\mathcal{A}}$ is unique up to bi-embeddability.

c) For each continuous coloring c on a Polish space we have $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\mathrm{age}(c)})$. If c is self-similar, then $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\mathrm{age}(c)})$.

The smallest nontrivial closed class of finite graphs is the class generated by a single edge and its complement. This is the class of finite P_4*-free graphs*, i.e., the class of finite graphs that do not contain induced subgraphs isomorphic to P_4, the path on four vertices. Let $\mathcal{A}$ be the class of finite P_4-free graphs and let $\mathcal{B}$ be the class of all finite graphs. Then $c_{\min}$ is bi-embeddable with $c_{\mathcal{A}}$ and $c_{\max}$ is bi-embeddable with $c_{\mathcal{B}}$.

In [2] it was shown that every uncountably homogeneous, continuous coloring c on a Polish space whose associated graph is P_4-free satisfies $\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min})$. Generalizing this result, we obtain the following corollary of Theorem 5:

Theorem 7. *If $\mathcal{D}$ is a nontrivial closed class of finite graphs that is generated by finitely many graphs, then $\mathfrak{hm}(c_{\mathcal{D}}) = \mathfrak{hm}(c_{\min})$.*

Another important closed class of finite graphs is the class of perfect graphs. That this class is closed is just Lovász' Substition Theorem [7]. If $\mathcal{C}$ denotes the class of finite perfect graphs, then an analysis of the proof of Theorem 2 b) shows that actually $\mathfrak{hm}(c_{\mathcal{C}}) < \mathfrak{hm}(c_{\mathcal{B}})$ is consistent. It is possible to give a purely combinatorial condition on a pair $(\mathcal{E}, \mathcal{F})$ of closed classes of finite graphs that is sufficient for the consistency of $\mathfrak{hm}(c_{\mathcal{E}}) < \mathfrak{hm}(c_{\mathcal{F}})$. Unfortunately, it is currently not known whether there is any such pair of classes that satisfies the condition and is substantially different from the pair $(\mathcal{C}, \mathcal{B})$.

In particular, we do not know whether there are uncountably homogeneous continuous colorings c_1, c_2, and c_3 on Polish spaces such that for any two of them the homogeneity numbers can be separated in a generic extension of the set-theoretic universe.

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On isometric representations and maximal symmetry

Christian Rosendal
(joint work with Valentin Ferenczi)

0.1. **Existence of non-trivial isometries.** It has been a longstanding question whether a Banach space can admit only trivial continuous linear operators. A major breakthrough was due to W.T. Gowers and B. Maurey in the 1990's, with the definition of the hereditarily indecomposable space GM, which has few operators in the sense that every operator on it is a strictly singular perturbation of a multiple of the identity [7]. Finally, a stricter question was recently solved by S. Argyros and R. Haydon [1], with the construction of a space on which every operator is a compact perturbation of a multiple of the identity map. The space AH of Argyros and Haydon has a Schauder basis, and so all compact operators are limits of finite-rank operators. Therefore the space AH has no other operators than those that necessarily arise on any Banach space from the existence of the identity map and the finite dimensional maps.

The presented results address another similar issue, namely:

Question 1. *Does every Banach space contain a non trivial isometry?*

Now, K. Jarosz [8] showed that any Banach space may be equivalently renormed to have only the trivial isometries $\lambda\mathrm{Id}$, where λ is a scalar of modulus 1. But of course, this does not prevent the group of isometries to be extremely non-trivial for some other norm and is not an *isomorphic* property of the space. So we seek a more structural result on the isometry groups $(\mathrm{Isom}(X), \|\cdot\|)$, where $\|\cdot\|$ varies over all equivalent norms on X.

Note that an infinite-dimensional Banach space X may always be equipped with an equivalent norm associated to an isomorphic representation of X as the ℓ_1-sum $F \oplus_1 H$, where F is any choice of a finite dimensional space, in which case $\mathrm{Isom}(X)$ will at least contain a subgroup isomorphic to $\mathrm{Isom}(F)$. In other words for any Banach space X and any group G of isometries on a finite dimensional space, G will appear as a subgroup of $\mathrm{Isom}(X)$ in some equivalent norm on X.

Therefore we should have a less restrictive concept of what we mean by "trivial isometries" if we look for results about all possible equivalent renormings. One of our main results is that there do exist spaces with as few isometries as possible, in the sense that under any equivalent norm the only isometries are those which arise naturally from a decomposition as above:

Theorem 1. *Let X be a separable, complex, reflexive, hereditarily indecomposable Banach space without a Schauder basis. Then for any equivalent norm on X, there exists an isometry invariant decomposition $F \oplus H$ of X such that F is finite dimensional and all isometries act trivially on H.*

0.2. **Transitivity of norms.** The previous result is closely related to a classical open problem in Banach space theory.

A norm on a Banach space X is said to be *transitive* or *isotropic* if for any two points x, y of the unit sphere $\mathcal{S}_X$ of X, there exists a surjective linear isometry T on X such that $Tx = y$. The following conjecture was stated by S. Banach in [2] and attributed to S. Mazur.

Conjecture 1 (Mazur's rotation problem)**.** *Any separable Banach space with a transitive norm is isometric to ℓ_2.*

Weaker notions of transitivity were put forth by A. Pełczyński and S. Rolewicz in 1962 at the International Mathematical Congress in Stokholm [9] and also published by S. Rolewicz in 1972 [10].

A norm on a Banach space is said to be *almost transitive* if for any point x of the unit sphere $\mathcal{S}_X$ of X, the Isom(X)-orbit of x is dense in $\mathcal{S}_X$. Almost transitive norms are not too difficult to obtain. For example the classical norm on $L_p([0,1])$, $1 \leqslant p < +\infty$, is almost transitive, but the norm on ℓ_p, $1 \leqslant p < +\infty, p \neq 2$ is not [10]. It is also known that the non-trivial ultrapower of a space with an almost transitive norm will have a transitive norm, see [4]. Therefore there are many examples of non-separable, non-Hilbert spaces with a transitive norm.

Finally, a formally weaker notion is that of a *convex transitive* norm on a Banach space X, which means that for any x of $\mathcal{S}_X$, the convex hull of the Isom(X)-orbit of x is dense in the closed unit ball B_X of X.

More interesting than asking whether a specific norm on a Banach space X is transitive, almost transitive, or convex transitive, is the question whether X admits an equivalent norm with one of these forms of transitivity. For example, $C([0,1], \mathbb{R})$ has an equivalent almost transitive norm [4], although the supremum norm itself is not, [6] p. 195. There also are separable spaces with no equivalent almost transitive norm. Actually, any non superreflexive space, which is reflexive, has the Radon-Nikodym property, or is an Asplund space, fails to admit an equivalent almost transitive norm [4]. This gives a long list of spaces with no equivalent almost transitive norm: c_0, ℓ_1, Tsirelson's space T, Schlumprecht's space S, Gowers-Maurey's space GM, for example.

However the question of whether any superreflexive space admited an equivalent almost transitive norm remained open. Even the possibility that every superreflexive space could have a transitive norm was not disconsidered, see [5, 4].

Theorem 2. *There exists a separable superreflexive complex Banach space X such that for any equivalent norm on X, there exists an isometry invariant decomposition*

$$X = F \oplus H,$$

where F is finite-dimensional and $T|_H = \lambda \mathrm{Id}$ for any isometry T on X. It follows that X has no equivalent almost transitive or even convex transitive norm.

0.3. **Maximal norms.** Our results are even more strongly related to the notion of a *maximal norm* on a Banach space [9], which is a weaker notion than any of the three forms of transitivity considered in the previous section. While by the result of Jarosz it is always possible to renorm a given Banach space to obtain only trivial isometries, a best norm on a Banach space is one displaying as much symmetry as possible. For example, a space with an unconditional basis always has an equivalent norm for which all maps acting by change of signs of the coordinates are isometries, with similar results holding for spaces with symmetric or subsymmetric bases.

A norm $\|\cdot\|$ on a Banach space X is said to be *maximal* if whenever $|||\cdot|||$ is an equivalent norm on X such that

$$\mathrm{Isom}(X, \|\cdot\|) \leqslant \mathrm{Isom}(X, |||\cdot|||),$$

then actually

$$\mathrm{Isom}(X, \|\cdot\|) = \mathrm{Isom}(X, |||\cdot|||).$$

Observe that if G is a bounded group of isomorphisms on a Banach space $(X, \|\cdot\|)$, then the renorming $|||x||| = \sup_{g\in G} \|gx\|$ turns G into a group of isometries of X. So equivalently, a norm $\|\cdot\|$ on X is maximal if $\mathrm{Isom}(X, \|\cdot\|)$ is a maximal bounded subgroup of the general linear group $GL(X)$ of X.

Once one has obtained a maximal norm $\|\cdot\|$ on a Banach space, one may claim to have found a norm which is optimal for the problem of preserving the symmetries of the space, and in that sense is an optimal choice of norm on the space. Of course not every norm on a Banach space is maximal, but many spaces admit natural maximal norms. Furthermore, Rolewicz [10] proved that any convex transitive norm must be maximal. Actually, a norm $\|\cdot\|$ is *uniquely maximal* if whenever $|||\cdot|||$ is an equivalent norm such that

$$\mathrm{Isom}(X, \|\cdot\|) \leqslant \mathrm{Isom}(X, |||\cdot|||),$$

then $|||\cdot|||$ is a scalar multiple of $\|\cdot\|$. E. Cowie [3] proved the reverse direction of Rolewicz's result by showing that a norm on a Banach space is convex transitive if and only if it is uniquely maximal.

Therefore all three notions of norm transitivity considered earlier imply maximality, and in particular the norm on $L_p([0,1])$ is maximal. Rolewicz also proved that if a space has a 1-symmetric basic sequence, then the norm is maximal. Therefore the usual norms on the spaces c_0 and ℓ_p are maximal, though not uniquely maximal.

It was a longstanding question, formulated by Wood in [11], whether any Banach space must admit an equivalent maximal norm. Our results allow us to answer this question in the negative.

Theorem 3. *Let X be a separable, complex, reflexive hereditarily indecomposable Banach space without a Schauder basis. Then X admits no equivalent maximal norm and hence $GL(X)$ contains no maximal bounded subgroup.*

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A Dual Ramsey result for finite co-structures with forbidden configurations

Alain Louveau

Many Ramsey-type results for finite structures have been obtained in the 70's and 80's, including Dual Ramsey results, see e.g. [2]. These results assert, for a given class of structures, that for all elements A and B of the class, there is an element C in the class with the property that for any partition of the substructures of C of type A in two pieces, one of the pieces contains all substructures of type A of a substructure of type B of C. In the direct case, structures and substructures have their usual (model-theoretic) meaning, while in the dual case, one considers co-structures (relations are now sets of labelled partitions), and "substructures" are now understood as quotients.

Recently, an important link has been discovered between these Ramsey results for classes of finite structures, and the dynamical properties of the automorphism group of their infinite (direct or inverse) limit, see [1].

This has prompted new investigations about the extent of the Ramsey property among classes of finite structures. In [4], S. Solecki has proposed a new approach to dual Ramsey results, proving in particular that the Ramsey property above holds for the class of all finite co-structures, with co-relations and functions, of any given finite type.

At the Oberwolfach meeting, I presented a similar result, for relational co-structures which avoid certain types of forbidden configurations. The proof relies heavily on Solecki's technique and results. Earlier results of this form had been obtained by Nesetril and Rödl [3], and our result generalize them, by weakening the conditions put on the forbidden configurations.

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Prikry type forcing and stationary reflection at $\aleph_{\omega+1}$

Martin Zeman

Given a regular cardinal λ, a stationary set $S \subseteq \lambda$ and an ordinal $\nu < \lambda$ of uncountable cofinality, we say that S reflects at ν if and only if $S \cap \nu$ is stationary in ν. In this situation ν is called a reflection point of S. If $\mathcal{S}$ is a family of stationary subsets of λ and ν is a reflection point of every $S \in \mathcal{S}$ we say that the sets from $\mathcal{S}$ simultaneously reflect at ν. The requirement that "sufficiently many" stationary subsets of λ have reflection points implies the consistency of large cardinals relative to ZFC; this is a fact known for a long time. For cardinal successors of small regular cardinals equiconsistency results were established by Jensen, Harrington and Shelah in the case of simple reflection and by Baumgartner and Magidor in the case of simultaneous reflection. In either case the consistency strength is quite low.

The situation at successors of of singular cardinals is quite different, as the consistency strength is known to be very high, although the gap between the known lower and upper bounds is immense. The statement "Every stationary subset of $\aleph_{\omega+1}$ has a reflection point" is consistent relative to the existence of infinitely many supercompact cardinals by a result of Magidor. The best known lower bound, on the other hand, is merely at the level of many Woodin cardinals, due to Steel and later improved by Sargsyan.

The inner model theoretic considerations in [6] indicate that the consistency strength of stationary reflection should be in the region still consistent with extender models for short extenders, hence should be significantly below any nontrivial instance of supercompactness, and the large cardinal axioms emerging from these considerations seem to be natural candidates for the consistency strength. Since such large cardinal axioms do not have much influence on the universe beyond the cardinal successor of the cardinals in question, the only tool currently available for turning large cardinals into $\aleph_\omega$ without changing combinatorics at $\aleph_{\omega+1}$ too much seems to by a Prikry type forcing of the kind used for obtaining failure of the Singular Cardinal Hypothesis at $\aleph_\omega$ and variations thereof; the forcing $\mathbb{P}$ used in our construction is essentially a variation due to Woodin; it is described for instance in [3].

Let us say that $\mathsf{SC}(\kappa)$ holds if and only if κ is strongly inaccessible and the set

$$\mathfrak{S}_\kappa = \{x \in [\kappa^+]^{<\kappa} \mid \mathsf{otp}(x) \text{ is a cardinal}\}$$

is stationary. Burke and Jensen proved independently that $\mathsf{SC}(\kappa)$ implies the failure of $\square_\kappa$ and by the results in [6], $\mathsf{SC}(\kappa)$ is equivalent to the failure of $\square_\kappa$ in extender models. It is easy to see that $\mathsf{SC}(\kappa)$ is much weaker than the κ^+-supercompactness of κ.

The intuition that the consistency strength of the failure of $\square_{\aleph_\omega}$ should be relatively low is supported by the following result.

Theorem 1. *[Zeman 2002] Assume* $\mathsf{GCH} + \mathsf{SC}(\kappa)$ *hold and there is a normal measure on* κ. *Then* $\neg\square_{\aleph_\omega,<\omega}$ *holds in the generic extension via* $\mathbb{P}$. *The analogous conclusion is also true for uncountable cofinalities where the forcing* $\mathbb{P}$ *is replaced by the obvious modification of Magidor forcing.*

The failure of the finite family square principle seems to be the best possible here: Since κ^+ is not collapsed, the forcing $\mathbb{P}$ necessarily adds a $\square_{\aleph_\omega,\omega}$-sequence, due to a result of Cummings-Schimmerling [2]. In general, various results from singular cardinal combinatorics indicate that both squares and stationary reflection principles change their behavior dramatically once we begin considering families of size equal to the cofinality of the singular cardinal in question. It is believed that the failure of $\square_{\aleph_\omega,\omega}$ has significantly higher consistency strength, possibly at the level of supercompactness. There is one more issue in connection with the above theorem: It is assumed that κ is measurable, although it feels like this assumption is unnecessary. Indeed, the least κ that satisfies $\mathsf{SC}(\kappa)$ is not measurable. Moreover, the measure on κ in Theorem 1 is only needed to make the use of a Prikry type forcing possible. This leads to a natural question:

Question 1. *Is it possible to eliminate the assumption on measurability of* κ *from Theorem 1?*

The above theorem on the failure of the square principle motivates the result on stationary reflection. The general intuition is that stationary reflection should be of higher consistency strength than that of failure of square. In any case, it is not clear how to obtain a model for stationary reflection starting from the large cardinal axiom $\mathsf{SC}(\kappa)$ alone. Here we use a stronger axiom which we denote by $\mathsf{QC}(\kappa)$. This axiom was introduced by Jensen in [5] and it is easy to verify that it is still much weaker than any nontrivial version of supercompactness. Unfortunately, we do not have a combinatorial formulation of the axiom. $\mathsf{QC}(\kappa)$ asserts that for every $A \subseteq \kappa^+$ there are $\lambda > \kappa$, a set $A' \subseteq \lambda^+$ and an elementary embedding $\sigma : (H_{\kappa^+}, A) \to (H_{\lambda^+}, A')$ with critical point κ.

Theorem 2. *[Faubion,Zeman 2010] Assume* $\mathsf{GCH} + \mathsf{QC}(\kappa)$. *In the generic extension via* $\mathbb{P}$ *we have* $\kappa = \aleph_\omega$, $\kappa^+ = \aleph_{\omega+1}$ *and the following kind of stationary* *reflection principle holds. If $S_1, \dots, S_n$ is a finite family of stationary subsets of $\aleph_{\omega+1}$ disjoint from $S^* = \{\xi < \kappa^+ \mid \mathsf{cf}^{\mathbf{V}}(\xi) = \kappa\}$ then $S_1, \dots, S_n$ reflect simultaneously at some $\nu < \aleph_{\omega+1}$.*

Our goal is to obtain the full reflection at $\aleph_{\omega+1}$. It is easy to see that the set S^* is non-reflecting both in the ground model and the generic extension and it is possible to add a club subset of $\aleph_{\omega+1}$ via the usual distributive forcing that is disjoint with S^*. This of course may add new non-reflecting stationary sets, but it is plausible that the presence of the embeddings guaranteed by $\mathsf{QC}(\kappa)$ may make it possible to iterate adding club subsets of $\aleph_{\omega+1}$ in the Harrington-Shelah style [4] to obtain the full reflection. Modifying the Harrington-Shelah construction to the current context is the focus of our current research on this topic.

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The tree property and not SCH for small cardinals

Dima Sinapova

The tree property at κ^+ states that every tree with height κ^+ and levels of size at most κ has an unbounded branch. Equivalently, there are no Aronszajn trees on κ^+. In 1980's Woodin and others asked if the failure of the Singular Cardinal Hypothesis (SCH) at a singular κ implies the existence of an Aronszajn tree at κ^+. To motivate the question we note a few facts about Aronszajn trees that illustrate the tension between the tree property and the failure of SCH. Results of Shelah showed that the tree property holds at successors of limits of strongly compact cardinals. On the other hand, Solovay showed that SCH holds above a strongly compact.

With regard to SCH, in order to violate SCH at κ, one has to use Prikry type forcing. For a while all such constructions preserved κ^+. Prikry forcing at κ that preserves κ^+ adds a weak square sequence to κ, which is equivalent to the existence of a special Aronszajn tree at κ^+. Then in 2008 Gitik-Sharon [1] showed that the failure of SCH is consistent with the negation of weak square. This result suggested that the failure of SCH may also be consistent with the tree property. And indeed, recently Neeman [2] showed that that the failure of SCH at κ is consistent with the tree property at κ^+ for a singular κ of cofinality ω. The next question is whether his result can be pushed down to small cardinals. It turns out that we can. We show the following theorem [3]:

Theorem 1. *Suppose that in V, $\langle\kappa_n \mid n < \omega\rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension in which:*

(1) $\kappa = \kappa_0 = \aleph_{\omega^2}$,
(2) *the tree property holds at* $\aleph_{\omega^2+1}$,
(3) *SCH fails at* $\aleph_{\omega^2}$.

The reason for the choice of $\aleph_{\omega^2}$ is as follows. The most direct way of obtaining Neeman's result for small cardinals is to combine the Prikry type forcing from Gitik-Sharon [1] with collapses. Prior to forcing with that poset, we have added many subsets to κ in order to violate SCH at κ in the final model. As this reflects down, when collapsing we have to leave some space in between elements of the Prikry sequence below κ. This yields a model where κ becomes $\aleph_{\omega^2}$. It is open whether ω^2 can be changed to ω. We conclude with the following open question:

Question 1. *Is it consistent to have the tree property at $\aleph_{\omega+1}$ and not SCH at $\aleph_\omega$?*

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Fodor-type Reflection Principle and very weak square principles

Sakae Fuchino

Fodor-type Reflection Principle (FRP) is the principle which asserts that the following FRP(κ) holds for all regular $\kappa > \aleph_1$:

For any stationary $S \subseteq E^\kappa_\omega$ and $g\colon S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(1) $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$.
(2) for any $f\colon S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

FRP(κ) for regular κ follows from the reflection principle RP($[\kappa]^{\aleph_0}$) of stationary subsets S of $[\kappa]^{\aleph_0}$ to a subset of κ of cardinality and cofinality ω_1.

FRP is weaker than most of the other known reflection principles in that it is preserved under c.c.c. generic extensions. In particular, the size of the continuum is not bounded under FRP. In [9], [4], [5] and [6], it is proved that FRP is equivalent to many "mathematical" reflection theorems over ZFC.

FRP is inconsistent with the Weak Square Pinciple $\square^*_\kappa$ for cardinals $\kappa > \omega$ of cofinality ω since FRP implies the failure of ADS$_\kappa$ of [2] ([7]).

On the other hand it is consistent with some very weak versions of the square principle at such cardinals.

For example, starting from a supercompact cardinal, we can easily show the consistency of

(3) FRP + the very weak square of Foreman and Magidor [3] for all uncountable cardinals.

This is in strong contrast with the situation under MM, where the very weak square does no hold at cardinals of countable cofinality ([1]).

Starting from a model of (3) and forcing with an appropriate c.c.c. poset, we obtain e.g. a model of

(4) FRP + $\square^{***}_{\aleph_1,\kappa}$ of [10] for all successor cardinals $\kappa > \aleph_1$ + CH + MA.

H. Sakai proved recently that under GCH and $\mathrm{MA}^+(\sigma\text{-closed})$ there is a poset for any regular cardinal $\lambda > \aleph_1$ forcing $\mathrm{FRP}(\lambda)+\square(\lambda)$ ([8]). In contrast, B. Velickovic's proof of $\neg = \square(\lambda)$ from PFA can be modified to show the same consequence from the Weak Reflection Principle (WRP).

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Pointwise definable models of set theory

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(joint work with David Linetsky, Jonas Reitz)

This extended abstract is based on joint work [3] of myself with Linetsky and Reitz. The theme of the work arises with what we call the "math tea" argument, perhaps heard at some of the better math teas, asserting:

> "There must be real numbers we cannot describe or define, because there are uncountably many real numbers, but only countably many definitions."

Does it withstand scrutiny?[1] An object is *definable* in a structure $\mathcal{M}$ if it is the unique object r satisfying an assertion $\mathcal{M} \models \varphi[r]$. For example, no reals are definable in the real line $\langle \mathbb{R}, < \rangle$, but exactly the algebraic reals are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$. Additional reals become definable as we add structure or move to higher orders, such as $\langle H_{\omega_2}, \in \rangle$ or $\langle V_{\omega+\omega}, \in \rangle$. One naturally considers only structures that are themselves definable with respect to the set-theoretic background, but this gives rise to subtle meta-mathematical issues, for the notion of being definable in $\langle V, \in \rangle$ is not expressible in set theory.

Definition 1. A structure $\mathcal{M}$ is *pointwise definable* if every element of $\mathcal{M}$ is definable without parameters in $\mathcal{M}$.

There are a number of easy folklore observations. If ZFC is consistent, then there are continuum many non-isomorphic pointwise definable models of ZFC, by considering the collection of definable elements of any model of $V = \mathrm{HOD}$. Pointwise definable models with the same theory are isomorphic, and indeed, the pointwise definable models of ZFC are exactly the prime models of the theory $\mathrm{ZFC} + V = \mathrm{HOD}$. Pointwise definability is a strong form of $V = \mathrm{HOD}$, since the ordinal parameters are not needed. If there is a transitive model of ZFC, then there are continuum many transitive pointwise-definable models of ZFC, essentially because any countable transitive model of ZFC has a perfect set of forcing extensions. The minimal transitive model of ZFC is pointwise definable by a simple condensation argument, and this generalizes to higher levels of the constructible hierarchy. For example, if there is an uncountable transitive model of ZF, then there are arbitrarily large $\alpha < \omega_1^L$ for which L_α is a pointwise definable model of ZFC. This fact implies that if there is an uncountable transitive model of ZF, then every real is an element of a pointwise definable ω-standard model of $\mathrm{ZFC}+V = L$. The reason is that the conclusion is true in L by the previous observation, but the statement itself is Π^1_2, hence absolute to V. Indeed, every countable transitive model M of set theory has an end-extension to a (possibly nonstandard) model $M^+ \models \mathrm{ZFC}+V = L$, such that M^+ is pointwise definable. This latter fact admits curious instances, for instance when M has many large cardinals or true $0^\sharp$ is in M, but still M is extended to a pointwise definable model of $V = L$.

There can be no uniform definition of the class of definable elements, although in some models of ZFC, it can happen that the definable elements form a definable class. In others, such as in any nonprincipal ultrapower of a pointwise definable model, the definable elements do not form a class. There are models in which the definable elements form a class, but there is no definability map $r \mapsto \psi_r$ mapping each definable element to a definition of it. But other models have both the

[1]We leave aside the remark of eight-year-old Horatio, who announced, "Sure, papa, I can describe any number. Let me show you: you tell me a number, and I'll tell you a description of it!".

definable elements and a definability map existing as sets, although in no model can a definability map be definable.

Theorem 1. *Every countable model of* ZFC *has a pointwise definable class forcing extension.*

An earlier independent version of this theorem was mentioned by Ali Enayat [1], who was concerned with the *Paris* models, models of ZF in which every ordinal is definable without parameters.

Theorem 2 (Enayat [1])**.** *If L has an uncountable transitive model of ZF, then there are Paris models of arbitrarily large cardinality.*

These models are very large, but have only countably many ordinals. The proof uses model-theoretic methods and $\mathcal{L}_{\omega_1,\omega}$ logic, such as Morley's two-cardinal theorem, and a result of Harvey Friedman showing that every model $M \models$ ZF has extensions with same ordinals of size $\beth_\alpha$, where $\alpha = \mathrm{ORD}^M$.

The proof of theorem 1 makes use of the following result of Simpson, which he proved first for PA.

Theorem 3 (Simpson [4])**.** *Let $\langle M, \in\rangle$ be a countable model of* ZFC*. Then, there is an M-generic class $U \subseteq M$ such that $\langle M, \in, U\rangle \models \mathrm{ZFC}(U)$ and every element of M is definable in $\langle M, \in, U\rangle$.*

Consider now the extension of theorem 1 to Gödel-Bernays set theory, also known as von Neumann-Gödel-Bernays set theory, a second-order set theory that is conservative over ZFC. Models have form $\langle M, S, \in\rangle$, where $\langle M, \in\rangle \models$ ZFC and $S \subset P(M)$ is a family of *classes*, such that instances of Replacement and Separation are allowed to use finitely many class parameters (but not to quantify over classes), and there is a global choice class. GBC is conservative over ZFC since every ZFC model $\langle M, \in\rangle$ can be extended to a GBC model $\langle M, S, \in\rangle$ by adding a generic global well-ordering and letting S consist of the definable (with set parameters) classes of M relative to it.

Theorem 4. *Every countable model of Gödel-Bernays set theory has a pointwise definable extension, where every set and class is first-order definable without parameters.*

Theorem 4 is proved first in the case of the *principal* GBC models $\langle M, S, \in\rangle$, which have some $X \in S$ such that every class in S is definable in $\langle M, \in, X\rangle$. Natural examples of principal models include the ZFC definability extensions; and the principal GBC models are closed under class forcing. A non-example is obtained by successive forcing extensions $M, M[G_0], M[G_0, G_1], \cdots$, whose union is non-principal. No model $\langle M, S, \in\rangle$ of Kelly-Morse set theory is principal as a GBC model, since KM proves the existence of a truth predicate relative to any one class. For example, if κ is inaccessible, then $\langle V_\kappa, V_{\kappa+1}, \in\rangle$ is easily seen to be non-principal since $V_{\kappa+1}$ has size 2^κ.

In order to achieve the full GBC theorem, we show that every GBC model $\langle M, S, \in\rangle$ can be extended to a principal model. The initial idea was to use meta-class forcing to code up all the classes in one class. For example, if κ is inaccessible, then $\langle V_\kappa, V_{\kappa+1}, \in\rangle$ is extended to a principal GBC model $V_\kappa[G]$ by forcing with $\mathrm{Coll}(\kappa, 2^\kappa)$, and something similar works with KM models. The general GBC case is treated by the following.

Theorem 5. *(S. Friedman) Every countable* GBC *model* $\mathcal{M} = \langle M, S, \in\rangle$ *has an extension to a principal* GBC *model* $\mathcal{M}[Y] = \langle M[Y], S[Y], \in\rangle$.

The extension $\mathcal{M}[Y]$ is not a forcing extension, but is built as an increasingly partial generic extension by a descending sequence of class partial orders $\mathbb{Q}_n$, with $Y \subset \mathbb{Q}_n$ increasingly but only partially generic for each n. The classes of the original model are hidden away, coded into increasingly difficult subclasses of Y.

Combining all the arguments, the final conclusion is that every countable model of Gödel-Bernays set theory has a pointwise definable extension, where every set and class is first-order definable without parameters, establishing Theorem 4.

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Many countable support iterations of proper forcings preserve Souslin trees

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(joint work with Saharon Shelah)

We show [3] that there are many models of $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{cof}(\mathcal{M}) = \aleph_2$ in which the club principle holds and there are Souslin trees. The proof consists of the following main steps:

(1) We give some conditions on a forcing in terms of games that imply that the forcing is $(T, Y, \mathcal{S})$-preserving. A special case of $(T, Y, \mathcal{S})$-preserving is preserving the Souslinity of an ω_1-tree.
(2) We show that some tree-creature forcings from [5] satisfy the sufficient condition for one of the strongest games.
(3) Without the games, we show that some linear creature forcings from [5] are $(T, Y, \mathcal{S})$-preserving. There are non-Cohen preserving examples.
(4) For the wider class of non-elementary proper forcings we show that ω-Cohen preserving for certain candidates implies $(T, Y, \mathcal{S})$-preserving.
(5) We give a less general but hopefully more easily readable presentation of a result from [6, Chapter 18, §3]: If all iterands in a countable support iteration are proper and $(T, Y, \mathcal{S})$-preserving, then also the iteration is $(T, Y, \mathcal{S})$-preserving. This is a presentation of the so-called case A in which a division in forcings that add reals and those who do not is not needed.

In [2] we showed: Many proper forcings from [5] with finite or countable H(n) (see Section 2.1) force over a ground model with $\diamondsuit_{\omega_1}$ in a countable support iteration the club principle. After ω_1 iteration steps the diamond holds anyway.

This work is related to Juhasz' question [4]: "Does Ostaszewski's club principle imply the existence of a Souslin tree?"

Partial positive answers are known: In a model of the club principle and $\mathrm{cov}(\mathcal{M}) > \aleph_1$ by Miyamoto [1, Section 4] there are Souslin trees. Brendle showed [1, Theorem 6]: In a model of the club principle and $\mathrm{cof}(\mathcal{M}) = \aleph_1$ there are Souslin trees. Now in this work we add examples of ZFC models that witness there can be a Souslin tree and in the examples from [5] the club principle holds and neither sufficient conditions holds, i.e., in some of our examples we have $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{cof}(\mathcal{M}) = \aleph_2$.

In particular, items 2 and 4 apply to Miller forcing. So we have two proofs that in the Miller model there is a Souslin tree and we have that the club principle holds. It is known that in the Miller model $\mathfrak{d} = \aleph_2$ and $\mathrm{cov}(\mathcal{M}) = \aleph_1$. Item 3 applies to the Blass-Shelah forcing and gives another model of this kind, that is, in contrast to the Miller forcing, not ω-Cohen preserving and increases the splitting number. Besides these particular examples the main technical work in [3] is an investigation of preserving Souslin trees.

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HOD mice

Grigor Sargsyan

We give the definition of a HOD mouse below $AD_{\mathbb{R}}$ + "Θ is regular" and its generalization to higher levels. In particular, we outline a comparison argument for HOD mice at a level of overlapped Woodins.

Forcings constructed along morasses

Bernhard Irrgang

I used the opportunity to talk on my project to construct forcings along morasses which was motivated by the following theorem and (still open) problem of Todorcevic's.

Theorem 1 (Todorcevic). *If $\square_{\omega_1}$ holds, then there exists a ccc forcing that adds a function $f : \omega_2 \times \omega_2 \to \omega$ which is not constant on any $A \times B \subseteq \omega_2 \times \omega_2$ with $otp(A) = otp(B) = \omega$.*

Question 1 (Todorcevic). *Is it consistent that there exists a function $f : \omega_3 \times \omega_3 \to \omega$ which is not constant on any $A \times B \subseteq \omega_3 \times \omega_3$ with $otp(A) = otp(B) = \omega$?*

My **naive idea** to answer this question was as follows:

First step: Replace $\square_{\omega_1}$ by the existence of a simplified $(\omega_1, 1)$-morass. Note that the existence of a simplified $(\omega_1, 1)$-morass implies $\square_{\omega_1}$.

Second step: Reformulate the construction of the forcing as a typical morass construction. That is, use the morass as an index set for a recursive construction of a system of embeddings between forcings and take its direct limit.

Third step: Carry out the same construction along a simplified $(\omega_1, 2)$-morass. This yields a forcing of size ω_3 which might do the right thing.

My first attempt to construct a forcing along a simplified $(\omega_1, 1)$-morass used Tennenbaum's forcing to add a Suslin tree. As it turned out, this kind of construction leads to a ccc forcing which can be densely embedded into a forcing of size ω_1.

Theorem 2. *If there is a simplified $(\omega_1, 1)$-morass, then there exists a ccc forcing of size ω_1 that adds an ω_2-Suslin tree.*

Since ccc forcings of size ω_1 preserve GCH, the method cannot be used to answer Todorcevic's question. But with a slight modification it is possible to reprove some known results:

Theorem 3 (Galvin). *It is consistent that there exists a function $f : [\omega_2]^2 \to \omega$ such that $\{\xi < \alpha \mid f(\xi, \alpha) = f(\xi, \beta)\}$ is finite for all $\alpha < \beta < \omega_2$.*

Theorem 4 (Todorcevic). *It is consistent that there exists a function $f : \omega_2 \times \omega_2 \to \omega$ which is not constant on any $A \times B \subseteq \omega_2 \times \omega_2$ with $otp(A) = otp(B) = \omega$.*

Theorem 5 (Koszmider). *It is consistent that there exists a sequence $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ such that $X_\alpha \subseteq \omega_1$, $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_2$.*

Theorem 6 (Baumgartner, Shelah). *It is consistent that there exists an (ω, ω_2)-superatomic Boolean algebra.*

I also have an example of a construction along a simplified $(\omega_1, 2)$-morass. It is motivated by a theorem of Juhasz. Let X be a topological space. Its spread is defined by

$$spread(X) = sup\{card(D) \mid D \text{ discrete subspace of } X\}.$$

Theorem 7 (Hajnal, Juhasz - 1967)**.** *If X is a Hausdorff space, then $card(X) \leq 2^{2^{spread(X)}}$.*

In his book "Cardinal functions in topology" (1971), Juhasz explicitly asks if the second exponentiation is really necessary. This was answered by Fedorcuk (1975).

Theorem 8 (Fedorcuk)**.** *In L, there exists a 0-dimensional Hausdorff (and hence regular) space with spread ω of size $\omega_2 = 2^{2^{spread(X)}}$.*

This is a consequence of $\diamondsuit$ (and GCH). There was no such example for the case $spread(X) = \omega_1$. By thinning-out Cohen forcing along a simplified gap-2 morass, one obtains:

Theorem 9. *If there is a simplified $(\omega_1, 2)$-morass, then there exists a ccc forcing of size ω_1 which adds a 0-dimensional Hausdorff space X of size ω_3 with spread ω_1.*

Hence there exists such a forcing in L. By the usual argument for Cohen forcing, it preserves GCH. So the existence of a 0-dimensional Hausdorff space with spread ω_1 and size $2^{2^{spread(X)}}$ is consistent.

Gap-1 morasses can be pictured as two-dimensional structures, gap-2 morasses as three-dimensional structures. In this sense my results can be summarized by

	two-dimensional	three-dimensional
preserving GCH	Suslin tree	topological space
not preserving GCH	coloring and similar examples	???

.

It would be interesting to find a three-dimensional construction of a forcing which destroys CH. If my original approach to Todorcevic's question works, the necessary forcing would be of this form.

More on my approach can be found in my papers [1, 2, 3]. An excellent source for most of the other topics mentioned is [4].

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