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# Geometric Quantization in the Non-compact Setting

Organised by
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February 13th – February 19th, 2011

Abstract. The purpose of the workshop was to bring together mathematicians interested in "quantization of manifolds" in a broad sense: given classical data, such as a Lie group $G$ acting on a symplectic manifold $M$, construct a quantum version, that is a representation of $G$ in a vector space $Q(M)$ reflecting the classical properties of $M$.

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## Introduction by the Organisers

The workshop, *Geometric Quantization in the Non-compact Setting*, organized by Lisa Jeffrey (Toronto), Xiaonan Ma (Paris) and Michèle Vergne (Paris) was held February 13th - February 19th, 2011.

The meeting was attended by 48 participants, representing researchers from many European countries, and Australia, Canada, China, Japan, USA. Unfortunately, Lisa Jeffrey could not be present because a minor injury before the meeting made it impossible for her to travel.

The meeting was devoted to the following theme (and adjacent themes). Let $G$ be a Lie group acting on a manifold $M$. Assume that $G$ preserves some data $(D)$, such as a symplectic structure, or a differential operator on $M$, or a fibration, ..., then the aim of Geometric Quantization is to associate to these data a representation of $G$ in a vector space $Q(M, D)$, and to analyze the relations of the quantum space $Q(M, D)$ with the classical data $(M, D)$. There are diverse constructions of the quantum space $Q(M, D)$. They should all have some functorial properties, summarized in the maxim (only a hope or guiding principle rather than an established fact in this very general setting): quantization commutes with reduction.

To report on the recent progress overcoming a certain number of difficulties, arising in the case of a noncompact manifold, or a noncompact group $G$, was an important goal for this meeting. However, important results on the quantization in the case of compact manifolds were also reported in this meeting.

Thus the topic of our meeting included deformation quantization of functions on a symplectic manifold via Toeplitz operators, branching rules for unitary representations of real Lie groups, equivariant index of transversally elliptic operators, quantization of Hamiltonian manifolds with proper moment maps, group valued moment maps, Lagrangian fibrations.

It was not clear to the organizers that our choice of participants working on these many diverse topics and with many different techniques (topological $K$-theory, analytic estimates, $C^*$-algebras, representation theory) could lead to anything other than a series of talks with disjoint attendance. However, we think that the meeting was very successful in making bridges between the many different approaches towards a general common goal. This is certainly due to the very unique atmosphere of the Oberwolfach setting.

There were 22 talks of approximately 50 minutes, 7 talks of 30 minutes, and a session of short talks by young postdocs. All the speakers presented interesting new results, and they were concerned with clearly communicating the results of their research to an audience, possibly not familiar with techniques used, although interested in same themes. Thus our meeting was successful due to the efforts of the speakers. Certainly, this meeting will produce new ideas in the future in the participants' research, generated by attending a live presentation of new points of view.

Let us give some details on the topics of the conference:

• Quantization $Q(M, L)$ of a noncompact symplectic manifold $M$ provided with a line bundle $L$.

Methods via $C^*$-algebras, or transversally elliptic operators or cutting methods were presented.

Hamiltonian manifolds such as the cotangent bundle of a manifold are a classical topic in mechanics. The list of other interesting Hamiltonian manifolds include the coadjoint orbits of real reductive Lie groups, representation spaces of fundamental groups of a surface of genus $g$ with value in a compact Lie group (moduli spaces of flat bundles) or in a complex lie group (Hitchin moduli spaces). Results on the quantisation of those manifolds were discussed.

• Toeplitz algebras: this leads to quantisation of the algebra of functions on a symplectic manifold by studying asymptotic $k$ estimates of the quantisation $Q(M, L^k)$ when the line bundle $L$ is raised to its $k$-th power.

• The equivariant index of elliptic operators or of transversally elliptic operators.

Methods via $C^*$-algebras, the heat kernel or topological $K$-theory were presented.

• Quantisation of integrable systems. This includes the theory of Lagrangian fibrations (possibly singular) and Bohr-Sommerfeld orbits.

• Spectrum of the Laplacian or the hypoelliptic Laplacian. Zeta functions of the Laplacian, analytic torsion.

Detailed information on the topics presented are given in the abstracts.

We had asked several young researchers to prepare a poster on their research before coming to Oberwolfach. Wednesday evening was then devoted to a special session of short talks and posters. Talks given by the younger researchers were dynamic and very well prepared. Furthermore, although the talks were necessarily very short due to the lack of time, we had a poster session just after the introductory talks, and scientific informal discussions. This was the "must-see event" of the workshop, and it went very well.

On behalf of all participants, we would like to thank the staff for their concern in providing the best material conditions for our stay. The setting of Oberwolfach is beautiful, the food of excellent quality, the library full of resources, and the staff extremely helpful.

Thanks to Oberwolfach grants, four young researchers: Solha (Barcelona), Deltour (Montpellier), Hochs (Utrecht), Szilagyi (Geneve) could participate to our workshop.

Finally, as organizers, we would like to thank the director and his staff for their great help in the scientific organization. In particular, the director explained the general policy of the Oberwolfach meeting to us, which was very helpful.

## 1. Program of the conference

Monday, 14/02/2011

9h00-10h00 G. Marinescu
Toeplitz operators and geometric quantization

10h10–11h10 P. Ramacher
Singular equivariant heat asymptotics and Lefchetz formulas

11h20 -12h20 M. Duflo
Kirillov's formula and Box splines

14h30–15h30 P-E. Paradan
Spin quantization in the compact and non-compact setting

16h00-17h00 T. Kobayashi
Geometric quantization, limits and restrictions–some examples for elliptic and minimal orbits

17h10-18h10 B. Ørsted

Deformation of Fourier transformation

Tuesday, 15/02/2011

9h00-10h00 J. Brüning
Formulas for the multiplicities of the equivariant index

10h10–11h10 W. Zhang
Geometric quantisation for proper moment maps

11h20 -12h20 L. Boutet de Monvel
Asymptotic equivariant index of Toeplitz operators on spheres

16h00-17h00 W. Müller
Dynamical zeta functions and analytic torsion

17h10-18h10 K-I. Yoshikawa
Singularities and Analytic Torsion

Evening talk: 20h00-21h00 J.M. Bismut
Hypoelliptic Laplacian

Wednesday, 16/02/2011

8h50-9h40 G. Kasparov
K-theoretic index theorems for transversally elliptic operators

9h50-10h40 V. Mathai
Geometric quantization commutes with reduction

11h00–11h50 P. Piazza
Eta cocycles, relative pairings and Godbillon-Vey index theorem

12h00 -12h50 T. Schick
$L^2$-Betti numbers and their values

20h00 –21h30 Short talks and Poster session
M. Hamilton
Real and complex quantization of flag manifolds
P. Hochs
Quantization commutes with reduction at non-trivial representation
S. Fitzpatrick
Quantization of manifolds with f-structure
R. Solha

Geometric quantization of integrable systems with singularities
Z. Szilagyi
An equivariant Jeffrey-Kirwan formula in non-compact case

Thursday, 17/02/2011

9h00-10h00 H. Fujita
Localization of Riemann-Roch numbers via Torus fibrations

10h10–11h10 A. Szenes
Enumerative topology of quotients

11h20 -12h20 E. Meinrenken
Verlinde formulas for non-simply connected groups

16h00-17h00 G. Heckman
On the regularization of the Kepler problem

17h10- 17h40 G. Deltour
Symplectic and Hamiltonian properties of holomorphic co-adjoint orbits

17h40– 18h10 A-L. Mare
On the image of real loci of symplectic manifolds under moment maps

Friday, 18/02/2011

9h00-9h30 E. Miranda
From action-angle coordinates to geometric quantization: a 30-minute round trip

9h30-10h00 C. Procesi (presented by M. Vergne)
Multiplicities formulas for transversally elliptic operators

10h20-10h50 S. Wu
Quantization of the cotangent bundle of Lie groups

10h50–11h20 J. Huebschmann
Singular Kähler quantization on the moduli space of semi- stable holomorphic vector bundles on a curve

11h40– 12h10 S. Goette
Perturbative analysis of the $L^2$ heat kernel for large times

13h50–14h50 A. Alekseev
Tropical avatar of the Gelfand-Zeitlin integrable system

## Workshop: Geometric Quantization in the Non-compact Setting

## Table of Contents

# Abstracts

## The hypoelliptic Laplacian

Jean-Michel Bismut

The purpose of the talk was to given an elementary introduction to aspects of the hypoelliptic Laplacian connected with the object of the conference.

The following 3 questions were given as a motivation for the talk, to which the hypoelliptic Laplacian provides the proper answer:

(1) A trace formula can be viewed as a Lefschetz formula. Indeed, let $X$ be a compact Riemannian manifold. The trace of the heat kernel $\mathrm{Tr}^{L_2^X}\left[e^{t\Delta^X}\right]$ is the trace of the 'group element' $g = e^{t\Delta^X}$ on the vector space $L_2^X$. If we consider the Hilbert space $L_2^X$ as the cohomology of some complex, then $\mathrm{Tr}^{L_2^X}\left[e^{t\Delta^X}\right]$ is the evaluation of a Lefschetz trace. Index theoretic methods teach us that this trace should be equal to the supertrace of some new 'heat operator' also depending on a parameter $b > 0$ on an adequate resolution of $L_2^X$. For a complete development of this line of thought, we refer to [6].

(2) Let $G$ be compact connected Lie group, let $\mathfrak{g}$ be its Lie algebra equipped with an invariant scalar product. Let $p_t(g)$ be the heat kernel on $G$. In [1], Atiyah raised the question of relating the evaluation of $p_t(g)$ as a sum over a lattice in the Lie algebra $\mathfrak{t}$ of a maximal torus to the localization formulas of Duistermaat-Heckman [9], Berline-Vergne [2]. Arguments by Frenkel [10] have shown that when certain smooth paths in $G$ are identified with coadjoint orbits of the loop group, such formulas are formal consequences of Kirillov versus Lefschetz formulas for the characters of the loop group. On the other hand, in finite dimensions, there is an efficient Gaussian proof of localization formulas [3]. In [5], it is shown that the hypoelliptic Laplacian provides the analytic counterpart to this Gaussian proof, which is now extended to infinite dimensions.

(3) Witten's deformation of the Hodge-de Rham Laplacian [13] provides an interpolation between classical Hodge theory on a manifold $X$ and the Morse theory associated with a Morse function $f$. If $X$ is replaced by its loop space $LX$, and if $f$ is now any of the classical Lagrangian function on $LX$ like the energy $E$, can one perform a similar interpolation? The problem with the question is that there is no Hodge theory on $LX$. The critical points of the $E$ are the closed geodesics. In [4], we have provided the proper construction of an existing object, the hypoelliptic Laplacian in de Rham theory, which interpolates between the standard Hodge-de Rham Laplacian of $X$ and the Lie derivative associated with the generator of the geodesic flow.

The analytic theory of the hypoelliptic Laplacian has been developed by Lebeau and ourselves [8].

If $X$ is a Riemannian manifold, and if $\pi : \mathcal{X} \to X$ is the total space of its tangent bundle (or some bigger bundle), the hypoelliptic Laplacian $\mathcal{L}_b, b > 0$ is an operator of the form

$$\mathcal{L}_b = \frac{1}{2b^2}\left(-\Delta^V + |Y|^2 - n\right) - \frac{\nabla_Y}{b} + \dots \tag{1}$$

In (1), the first operator is the harmonic oscillator along the fibre, the second operator is the generator of the geodesic flow. The remaining terms, in general nonscalar, are related to the specific geometrical data one is trying to deform. If one ignores these terms, as $b \to 0$, $\mathcal{L}_b$ tends in the proper sense to $-\Delta^X/2$.

In the talk, the case of $\mathbf{R}$ and its compact quotient $S^1$ was extensively reviewed. In this case, the relevant operator is the operator of Kolmogorov [11],

$$L_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{b} y \frac{\partial}{\partial x}. \tag{2}$$

The nonelliptic, non self-adjoint operator $L_b$ can be shown to be conjugate by an unbounded conjugation to the self-adjoint elliptic operator

$$M_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{3}$$

The conjugation, a bizarre version of Egorov's theorem, takes the form

$$\exp\left(b\frac{\partial^2}{\partial x \partial y}\right) L_b \exp\left(-b\frac{\partial^2}{\partial x \partial y}\right) = M_b. \tag{4}$$

Let $N^{\Lambda^\cdot(\mathbf{R})}$ be the number operator of $\Lambda^\cdot(\mathbf{R})$. Set

$$\mathcal{L}_b = L_b + \frac{N^{\Lambda^\cdot(\mathbf{R})}}{b^2}. \tag{5}$$

In the talk, for $t > 0$, I established the identity of operators acting on $C^{\infty,c}(\mathbf{R}, \mathbf{R})$,

$$\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right) = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b\right)\right]. \tag{6}$$

In the right-hand side, the supertrace is taken with respect to the variable $y$, and also with respect to $\Lambda^\cdot(\mathbf{R})$. By introducing the proper index theoretic formalism, equation (6) can be viewed as an operator valued index formula, which explains the independence on $b > 0$ of the right-hand side. Making $b \to 0$ exactly gives the left-hand side. Making $b \to +\infty$ gives the classical Gaussian formula for the heat kernel on $\mathbf{R}$.

In our proof of Selberg's trace formula [6], the above formalism was extended to general symmetric spaces of noncompact type. The Dirac operator of Kostant [12] plays a crucial role in the constructions.

For more details on various aspects of this talk, we refer to the survey [7].

## References

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## Asymptotic Equivariant Index of Toeplitz Operators on the Sphere

Louis Boutet de Monvel

Toeplitz operators, in the sense of my book with V. Guillemin, live on arbitrary contact manifolds, generalising pseudodifferential operators. We recall that an oriented contact manifold $X$ is a smooth manifold of odd dimension $2n-1$ equipped with a contact form $\lambda$ (two forms define the same structure if they are positive multiples of each other). Here $X$ will always be supposed to be compact. $X$ is the basis of a symplectic cone $\Sigma$, e.g. the set of positive multiples of $\lambda$ in $T^*X$ (the correspondence between symplectic cones and contact manifolds is an equivalence of categories). In pseudodifferential theory the relevant symplectic cone is the cotangent bundle $T^*V$ of a smooth manifold $V$, deprived of its zero section, and the contact manifold is the cotangent sphere $S^*V$.

Toeplitz operators behave much in the same way as pseudodifferential operators. They act on a scale of Hilbert spaces $\mathbb{H}^s$ (mimicking the Sobolev spaces), and they give rise to the same symbolic calculus, which lives on $\Sigma$: if $P$ is of degree $p$ (acting continuously $\mathbb{H}^s \to \mathbb{H}^{s-p}$), its symbol $\sigma_p(P)$ (or $\sigma_P$) is a smooth function on $\Sigma$, homogeneous of degree $p$; for two operators we have

$$\sigma_{p+q}(PQ) = \sigma_p(P)\sigma_q(Q), \qquad \sigma_{p+q-1}([P,Q]) = \frac{1}{i}\{\sigma_p(P), \sigma_q(Q)\}$$

where $[P, Q]$ denotes the commutator and $\{\ ,\ \}$ the Poisson bracket of $\Sigma$.

$\mathbb{H} : \bigcup \mathbb{H}^s$ is the range of a generalised Szegö projector (see [2, 3]) mimicking the Szegö projector of a boundary complex structure. If $\Sigma$ is embedded in a cotangent bundle $T^*V$, this is a Fourier integral projector in $L^2(V)$. Toeplitz operators mod $C^\infty$ acting on $\mathbb{H}$ mod $C^\infty$ define a sheaf, locally isomorphic to the sheaf of pseudodifferential operators acting on microfunctions; they form a sheaf of algebras which can be represented by a star-product (not canonically, as for pseudodifferential operators).

More generally one can define Toeplitz operators acting on sections of vector bundles (generalised Szegö projectors acting on sections of vector bundles exist).

There is an obvious notion of elliptic operator (the principal symbol is elliptic). Such an operator has an index. However this index in general cannot give rise to a useful topological formula. In fact it can be shown that the Toeplitz space $\mathbb{H}$ is essentially well defined, independently of the choice of a Szegö projector, or of the manner in which $\Sigma$ is embedded: for any two choices $\mathbb{H}_1, \mathbb{H}_2$ there exists a Fourier integral operator $F : \mathbb{H}_1 \to \mathbb{H}_2$ which behaves as an elliptic operator - in particular it is Fredholm and preserves Toeplitz operators. But it is not better than Fredholm and one has no control on its index; only the Toeplitz algebra mod $C^\infty$ is well defined, and unique (up to non unique isomorphism). A comparable pseudodifferential situation is the following: let $E$ be a vector bundle on the cotangent sphere $S^*V$ ($V$ compact). Then $E$ can be realised as a direct summand of some trivial bundle $L$, with projector $p_E$. It is immediate that $p_E$ is the symbol of a pseudodifferential projector $P_E$, with range $\mathbb{H}_E$ and one can then define pseudodifferential operators $\mathbb{H}_E \to \mathbb{H}_F$; if such an operator is elliptic it has an index; but this cannot be intelligently computed because the projectors $P_E, P_F$ are by no means unique and without further data the range is at best defined up to a finite dimensional space.

All the constructions and statements above allow a compact group action; in particular if $G$ is a compact group acting on a compact contact manifold $X$, there exists an invariant generalised Szegö projector, and this is "essentially unique" as above. The equivariant asymptotic index theory is an outgrowth of Atiyah's equivariant index theory [1] for transversally elliptic operators in presence of a compact group action, but still meaningful for Toeplitz operators; we used it in [4] to prove the Atiyah-Weinstein conjecture about the relative index of CR structures.

Associated to the group action is the characteristic set $\mathrm{char}\,\mathfrak{g} \subset \Sigma$, which is the set where the symbols of all infinitesimal generators vanish (if $u \in \mathfrak{g}$ - the Lie algebra - it defines a vector field $L_u$ on $X$ which preserves the Szegö projector, and a Toeplitz operator $T_u$ (the restriction of $L_u$ to $\mathbb{H}$); $\mathrm{char}\,\mathfrak{g}$ is the set where all symbols $\sigma(T_u)$ vanish; its basis $Z \subset X$ is the set where all vectors $L_u$ are orthogonal to the contact form). A Toeplitz operator (or system of such) is $G$-elliptic if its principal symbol is invertible on $\mathrm{char}\,\mathfrak{g}$ (transversally elliptic in [1] - but there is nothing much to be transversal to in the Toeplitz context). If an equivariant Toeplitz system $A$ is $G$-elliptic it can be shown, as in [1], that although it may not be Fredholm, its restriction to all isotypic components are. Its equivariant index is then defined as a virtual infinite representation $\mathrm{Ind}^G(A)$ whose character is a central distribution on $G$:

$$\sum_\alpha \frac{1}{d_\alpha}\mathrm{Ind}(A_\alpha)\ \chi_\alpha$$

($\alpha$ ranges among all classes of irreducible representations, with character $\chi_\alpha$ and degree $d_\alpha$; $A_\alpha$ denotes the component ot type $\alpha$ of $A$).

As above, although the equivariant index exists, it is not computable because the Toeplitz spaces are at best defined up to finite dimensional spaces. The asymptotic index was introduced to palliate this: it is the equivariant index mod finite representations; its character is the singularity of a central distribution on $G$ (a distribution mod $C^\infty$).

If a Toeplitz system $A$ is $G$-elliptic, its symbol defines an element $[A]$ of the equivariant K-theory (with compact support) $K^G(X - Z)$. It is immediate that the asymptotic index is additive and deformation invariant, so it defines an index map: $K^G(X - Z) \to \widehat{R}_G/R_G$ .

It is also shown (cf [3, 4]) that, for index computation purpose, any $G$-elliptic Toeplitz system can be equivariantly embedded in a larger and simpler $G$-contact manifold, just as in the 1968 proof of the Atiyah-Singer index theorem. The embedding preserves the asymptotic index (not the absolute); the K-theoretic counterpart is the Bott periodicity homomorphism, which is well defined since the normal bundle of a contact embedding is symplectic. In particular one can embed in the sphere of a numeric space $\mathbb{C}^N$ with a unitary action of $G$.

To understand what the index looks like, as indicated by Atiyah [1], the first case to examine is the case where $G$ is a torus acting diagonally on a sphere. So let now $X = \mathbb{S}^{2N-1} \subset V = \mathbb{C}^N$ be a sphere, equipped with a unitary action of a torus $G = \mathbb{R}^m/\mathbb{Z}^m$; the contact form is $\lambda = \mathrm{Im}\,(\bar{z}\cdot dz)$, the Szegö projector is the orthogonal projector on the space of boundary values of holomorphic functions:

$$Sf(z) = \frac{1}{\mathrm{vol}(X)}\int_X \frac{f(w)d\sigma(w)}{(1 - z\cdot\bar{w})^N}\ .$$

The group action is given by $g\cdot z = (\chi_k(g)z_k)$ where $\chi_k = e^{2i\pi\xi_k}$ ($k = 1\cdots N$) are characters of $G$ ($\xi_k \in \mathfrak{g}^*$ integral linear forms). If $u \in \mathfrak{g}$ the symbol of $\frac{1}{i}T_u$ is $\sum \xi_k(u)z_k\bar{z}_k$ (on unit covectors - this is also the moment of $L_u$).

We will from now on suppose that there is no fixed point: $\xi_k \neq 0$ (otherwise if $X_1$ is the sphere orthogonal to the fixed sphere, it is immediate that $X_1 - Z$ is a deformation retract of $X - Z$ so we are reduced to the former case).

A first simple case is when the action of $G$ is elliptic, i.e. $Z = \emptyset$, equivalently the infinitesimal characters $\xi_k$ generate a strictly convex cone in $\mathfrak{g}^*$. In this case all equivariant Toeplitz system are $G$-elliptic. The base space $\mathbb{H}$ is the space of holomorphic functions; the element of $\widehat{R}_G$ it defines is $\beta^{-1}$, with $\beta = \Pi(1 - \chi_k)$ (the convention is that the inverse is expanded as a series of positive powers of the $\chi_k$); it is the index of the $G$-elliptic operator $\mathbb{H} \to 0$, whose asymptotic index thus is $\beta^{-1}$ mod $R_G$. All other indices are multiples of this by finite elements in $R_G$.

In this case (no fixed point) we have $K^G(X) = R_G/R_G\beta$ so the asymptotic index map is an isomorphism of $K^G(X)$ to its image $R_G\beta^{-1}/R_G \subset \widehat{R}_G/R_G$.

In general, let $X'$ be an elliptic subsphere of $X$ in a coordinate subspace $V'$. The embedding procedure mentioned above takes the following form: let $k_V$" be the Koszul complex of the orthogonal subspace $V$" to $X'$ (in negative degrees):

$$(\mathbb{E}^k, d'') \quad \text{with } \mathbb{E}^k = \mathbb{H} \otimes \Lambda^k V''^* \quad d''\omega(z', z'') = \sum dz''_j \frac{\partial \omega}{\partial z''_j}$$

This is an equivariant resolution of $\mathbb{H}_{V'}$. If $A$ is a Toeplitz system on $X'$, $A$ on $X'$ and $k_{V"} \otimes A$ on $X$ have the same equivariant index. The K-theoretic element defined by the symbol of $k_W$ is $\beta_W$, so we have $[k_W \otimes A] = \beta_W \cdot [A]$, the image by the Bott periodicity homomorphism.

It is natural to conjecture that all elliptic systems from $X$ are obtained from elliptic subspheres in that manner, i.e. that the $\beta_{V"}$ generate $K^G(X - Z)$, when $V$" describes the set of all orthogonal subspaces to elliptic subspheres. This is easily seen to be true if $G$ is the circle group. It is also true if the characters go by opposite pairs (that case can be reduced to the case studied by Atiyah in [1]; the 'maximal complex structures', which provide the K-theoretical generators there, exactly correspond to the maximal elliptic subspheres). It is also true in various other examples, but I have no proof for the general case.

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## Multiplicities and the equivariant index


Jochen Brüning

(joint work with Franz Kamber and Ken Richardson)


Let $M$ be a smooth manifold which is closed, oriented, and connected, and let $E^{\pm} \to M$ be two smooth complex vector bundles over $M$. Consider moreover a first order differential operator $D^+ : C^1(M, E^+) \to C^1(M, E^-)$ which we assume to be elliptic. Finally, let a compact Lie group, $G$, act effectively on $M$ and $E^{\pm}$, hence on the smooth sections of $E^{\pm}$, such that $G$ commutes with $D^+$. We pick $G$-invariant metrics $g^{TM}$ on $M$ and $h^{E^{\pm}}$ on $E^{\pm}$ and we put $E := E^+ \oplus E^-$. Then we can form the adjoint operator $D^- := (D^+)^\dagger : C^1(M, E^-) \to C^1(M, E^+)$ such that $D := D^+ \oplus D^- : C^1(M, E) \to C^1(M, E)$ is symmetric and essentially self-adjoint in the Hilbert space $L^2(M, E)$, and its self-adjoint closure is discrete.

The equivariant index of $D$ is by definition the virtual character

$$\operatorname{ind} D(g) = \operatorname{tr}^{\ker D^+}(g) - \operatorname{tr}^{\ker D^-}(g), \tag{1}$$

which, by the argument of McKean-Singer, can also be written as

$$\operatorname{ind} D(g) = \operatorname{tr}_s^{L^2(M,E)}\left(ge^{-tD^2}\right),\ t > 0, \tag{2}$$

where $\operatorname{tr}_s$ denotes the supertrace with respect to the grading of $E$. Now let $\rho : G \to \operatorname{Aut} V_\rho$ be an irreducible representation of $G$; we are interested in the multiplicitiy of $\rho$ in the virtual representation $\operatorname{ind} D$, i.e. in the integer

$$\operatorname{ind}_\rho D := \dim \operatorname{Hom}_G(V_\rho, \ker D^+) - \dim \operatorname{Hom}_G(V_\rho, \ker D^-). \tag{3}$$

The purpose of this work is to present an explicit formula for $\operatorname{ind}_\rho D$ in terms of the geometric data. We start with several ways to express $\operatorname{ind}_\rho D$ which have been used in the study of this problem.

$$\operatorname{ind}_\rho D = \int_G \operatorname{ind}_\rho D(g)\overline{\chi_\rho(g)}dg \tag{4}$$

$$= (\dim V_\rho)^{-1} \operatorname{tr}_s^{L^2(M,E)}\left(P_\rho e^{-tD^2}\right) \tag{5}$$

$$= (\dim V_\rho)^{-1} \operatorname{ind} D_\rho^+ \tag{6}$$

$$= \int_{M\times G} \overline{\chi_\rho(g)} \operatorname{tr}_s^{E_p}\left(ge^{-tD^2}(g^{-1}p, p)\right) dpdg \tag{7}$$

$$=: \int_M k_\rho^D(t, p)dp. \tag{8}$$

Here $P_\rho$ denotes the orthogonal projection in $L^2(M, E)$ onto the $\rho$-isotypical subspace, explicitly

$$P_\rho s = \dim V_\rho \int_G \overline{\chi_\rho(g)}\, gs\, dg,$$

where $s \in L^2(M, E)$ and we use the normalized biinvariant integral on $G$. Also, the operator $D_\rho$ is a self-adjoint Fredholm operator which we will describe below. Atiyah, Segal, and Singer [1] have generalized the index theorem for elliptic operators to a formula for $\operatorname{ind} D(g)$ which localizes on the fixed point set of $g$ and which gives a formula of the kind we are looking for in the case of finite groups, via (4), which can be extended to orbifolds [6]. Berline and Vergne [2] have generalized this further to a full asymptotic expansion of the equivariant heat kernel $k_\rho^D(t, p)$ on the diagonal; however, it seems quite difficult to evaluate the integral over $G$ in (4) or (7) if the $G$-action has isotropy groups of varying dimensions.

In our approach, we make crucial use of the stratification of $M$ defined by the $G$-action. To describe it, we denote by $([G_j])_{j=1}^r$ the finitely many orbit types of the $G$-action. For subgroups $H, K$ of $G$ we write $[H] \leq [K]$ if $K$ is conjugate to a subgroup of $H$; then we may arrange the labeling in such a way that $[G_i] \leq$

$[G_j]$ implies $i \leq j$. The strata of the $G$-action are then the submanifolds $M_i := M_{[G_i]}, i = 0 \dots r$ which form a decomposition of $M$. $M_0$ is the top stratum of $M$ which is an open and dense subset, while we call

$$\Sigma := \cup_{i=1}^{r} M_i$$

the singular set or the union of the singular strata; the connected components of $\Sigma$ will be labeled as $(\Sigma_j)_{j=1}^{\bar{r}}$. Note that a minimal stratum is closed in $M$.

For our final result, we have to introduce an assumption. We note that the operator $D_\rho^+$ in (6) is unitarily equivalent to an elliptic first order differential operator induced by $D$ in $L^2(M_0, E_\rho)$ where $E_{\rho,p} := E_{p,\rho}$ (with respect to the action of $G_p$), as explained in [3]; this operator is discrete and hence Fredholm. Thus it follows from (6) that $\operatorname{ind}_\rho D$ is invariant under $G$-equivariant deformations which preserve ellipticity, a fact we use repeatedly. We then *assume* that near any minimal stratum, $\Sigma$, the operator $D$ can be deformed equivariantly to an operator product $D_\Sigma * D_{N\Sigma}$ acting on a vector bundle $E = E_\Sigma \boxtimes E_{N\Sigma}$ with both factors $G$-bundles, cf. [5, Ch.19.2]. In order to restrict our final computations to the $\rho$-isotypical parts, we will have to decompose further

$$E_{N\Sigma_j} = \oplus_{k\geq 1} E_{N\Sigma_j,k},$$

into a finite sum of isotypical $G$-bundles. This will induce operators $D_{\Sigma_j,k} = D_{\Sigma_j} \otimes I_{E_{N\Sigma_j,k}}$ acting on the smooth sections of $E_{\Sigma_j} \boxtimes E_{N\Sigma_j,k}$ over $\Sigma_j$.

Now we can formulate our result.

**Theorem 1.** *There is data $\tilde{M}, \tilde{E}$, and $\tilde{D}$ canonically constructed from $M, E, G$ and $D$ which enjoy analogous properties, in particular an effective $G$-action which commutes with $\tilde{D}$. In addition, the following assertions hold.*
*1. $G$ acts on $\tilde{M}$ with one orbit type.*
*2. There is a smooth surjective map $\tilde{\pi} : \tilde{M} \to M$ which is a $2^r$-sheeted covering outside $(\tilde{\pi})^{-1}(\Sigma)$.*
*3. For each $G$-invariant submanifold $U$ of $M$ there is a distinguished preimage $\tilde{U} \subset \tilde{M}$ which is a $G$-invariant submanifold of $\tilde{M}$.*
*4. The submanifolds $\tilde{\Sigma}_j$ are closed.*
*5. We obtain the formula*

$$\operatorname{ind}_\rho D = \int_{\tilde{M}_0} k_\rho^{\tilde{D}}(t) + \sum_{j,k} c_{j,k} \int_{\tilde{\Sigma}_j} k_\rho^{\tilde{D}_{\Sigma_j,k}}(t); \tag{9}$$

*the constants $c_{j,k}$ can be computed explicitly*

We add some comments on the proof of the theorem. The manifold $\tilde{M}$ is called the *desingularization* of $M$ and is, in principle, well known. Its construction proceeds inductively, starting each time with a (closed) minimal stratum. This can be viewed as an inverse to Seeley's conic degeneration [7] which needs to be applied here to Euclidean balls in the normal bundle of a singular stratum. Therefore, the change in index from the ball to the cylinder, by blowing up the origin, can be computed explicitly which leads eventually to the formula of the theorem. Note that the integrals in (9) can be rewritten as integrals over the respective $G$-orbit spaces of the Atiyah-Singer integrand computed for the quotient operators. As a consequence, we have $\operatorname{ind}_\rho D = 0$ if all singular strata have odd dimensional orbit spaces.

Of course, applying the theorem is technically demanding but doable. As one of the most interesting applications, we obtain a basic index theorem for Riemannian foliations, a problem that has been open for some time, cf. [4].

Finally, it should be noted that the theorem above extends to transversally elliptic operators without modifications.

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## Symplectic and Hamiltonian properties of holomorphic coadjoint orbits

Guillaume Deltour

Let $G$ be a connected Lie group, and $K$ a compact connected Lie subgroup of $G$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ their Lie algebras. Any coadjoint orbit $\mathcal{O}$ of $G$ carries a natural $G$-invariant symplectic structure, the Kirillov-Kostant-Souriau form $\Omega_\mathcal{O}$ of $\mathcal{O}$. It is a well-known fact that the symplectic $G$-manifold $(\mathcal{O}, \Omega_\mathcal{O})$ is actually a Hamiltonian $K$-manifold, for the induced action of $K$, with moment map $\Phi_K : \mathcal{O} \subset \mathfrak{g}^* \to \mathfrak{k}^*$ obtained by composing the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ with the canonical projection $\mathfrak{g}^* \to \mathfrak{k}^*$. The map $\Phi_K$ is called the *orbit projection of $\mathcal{O}$ relatively to the subgroup $K$*. When the moment map $\Phi_K$ is proper, the Kirwan's Hamiltonian Convexity Theorem [4, 7, 12] asserts that the image of the orbit projection $\Phi_K$ intersects a fixed Weyl chamber of some maximal torus of $K$ in a locally polyhedral convex set $\Delta_K(\mathcal{O})$, called *moment polyhedron* of the orbit projection of $\mathcal{O}$.

**Question 1.** *Can we determine the equations of the moment polyhedron $\Delta_K(\mathcal{O})$?*

So far, this question has been mainly studied for $G$ compact. With this assumption, any orbit projection is proper, and the moment polyhedron $\Delta_K(\mathcal{O})$ is actually a convex polytope. The study of this polytope's equations culminated in the 2000's, starting with the complete resolution of the famous Horn's Eigenvalue

Problem by Klyachko [5] and Knutson-Toa-Woodward [6]. This Eigenvalue Problem corresponds to the diagonal injection of $K = U(n)$ into $G = U(n) \times U(n)$. In the general compact case $K \subset G$, Berenstein-Sjamaar [1] gave a set of equations determining the polytope $\Delta_K(\mathcal{O})$, using Geometric Invariant Theory (GIT).

In the compact setting, the moment polytope, associated to the orbit projection, has a strong relation with the restriction of irreducible representations of $G$ relatively to the subgroup $K$. Indeed, if $\widehat{G}$ (resp. $\widehat{K}$) denotes the set of dominant weights of $G$ (resp. $K$), then, for all $\lambda \in \widehat{G}$ and $\mu \in \widehat{K}$, we have the following equivalence :

$$\mu \in \Delta_K(G \cdot \lambda) \quad \Longleftrightarrow \quad \exists N \geq 1,\ V^K_{N\mu} \subseteq V^G_{N\lambda},$$

where $V^G_\nu$ (resp. $V^K_\nu$) denotes the irreducible representation of $G$ (resp. $K$) with highest weight $\nu \in \widehat{G}$ (resp. $\nu \in \widehat{K}$).

In algebraic geometry, one studies the following set,

$$C^+_{\mathbb{Q}} := \{(\mu, \nu) \text{ dominant rational weight of } K \times G \mid \exists N \geq 1,\ (V^K_{N\mu} \otimes V^G_{N\nu})^K \neq 0\},$$

which is a polyhedral convex cone, called the *semiample cone of the complete flag manifold of* $K \times G$. This GIT object has been defined more generally for any projective variety by Dolgachev-Hu [3]. The important fact is that the equations of the polytope $\Delta_K(G \cdot \lambda)$ are determined by the ones of $C^+_{\mathbb{Q}}$, since $\Delta_K(G \cdot \lambda)$ is a rational polytope and the set of its rational points is equal to an affine section of $C^+_{\mathbb{Q}}$. Then, it suffices to compute the equations of $C^+_{\mathbb{Q}}$, which is done by applying GIT techniques based on Hilbert-Mumford's criterion.

Unfortunately, in the non-compact setting, we can't apply these GIT techniques directly. However, in some special cases of non-compact coadjoint orbits, one can obtain the equations of $\Delta_K(\mathcal{O})$ by computing the ones of the semiample cone related to a particular compactification of $\mathcal{O}$. In this note, we will study the orbit projection of *holomorphic coadjoint orbits.*

Let $G$ be a connected, non-compact, semisimple, real Lie group with finite center, such that $G/K$ (where $K$ is a maximal compact subgroup of $G$) is Hermitian. For instance, the classical simple groups of such type are $Sp(\mathbb{R}, 2n)$, $SO^*(2n)$, $SU(p, q)$ et $SO(p, 2)$. The holomorphic coadjoint orbits are the elliptic coadjoint orbits of $G$ which are endowed with a natural invariant Kählerian structure compatible with the K-K-S symplectic form. These orbits corresponds to Harish-Chandra's holomorphic discrete series.

First, we have to study the symplectic structure of holomorphic coadjoint orbits. McDuff [8] proved that the Hermitian symmetric space $G/K$, which can be thought of as the coadjoint orbit $\mathcal{O}_{\lambda_0}$ of some element $\lambda_0 \in \mathfrak{k}^*$ with stabilizer $K$, endowed with the K-K-S symplectic form, is symplectomorphic to a symplectic vector space. The first main result of my Ph.D. thesis proves a generalization of McDuff's symplectomorphism for holomorphic coadjoint orbits: it shows that there exists a $K$-equivariant symplectomorphism between any holomorphic coadjoint orbit $\mathcal{O}_\lambda$ and the product of symplectic manifolds $K \cdot \lambda \times \mathfrak{p}$, where $K \cdot \lambda$ is the associated compact coadjoint orbit (endowed with its K-K-S symplectic form), and $\mathfrak{p}$ is a symplectic vector space given by the Cartan decomposition of $\mathfrak{g}$. As a corollary, this gives another proof of the equality of the two moment polyhedra $\Delta_K(\mathcal{O}_\lambda)$ and $\Delta_K(K \cdot \lambda \times \mathfrak{p})$ proved by Paradan [10], and Nasrin [9] when $\lambda$ is central in $\mathfrak{k}^*$.

Enventually, it boils down to compute the equations of the polyhedron $\Delta_K(K \cdot \lambda \times \mathfrak{p})$. The idea is to apply GIT methods not directly on $K \cdot \lambda \times \mathfrak{p}$, but on the semiample cone of the projective variety $K/T \times K/T \times \mathbb{P}(\mathfrak{p} \oplus \mathbb{C})$, where $T$ is a maximal torus of $K$. This semiample cone is the following set,

$$C_{\mathbb{Q}}(\mathfrak{p})^+ = \{(\mu, \nu, r) \mid \exists N \geq 1,\ (V^K_{N\mu} \otimes V^K_{N\nu} \otimes \mathbb{C}_{\leq Nr}[\mathfrak{p}])^K \neq 0\},$$

where $\mu$ and $\nu$ are in the set of rational dominant weights of $K$, and $r$ is in the set of non-negative rational numbers. The equations of $C_{\mathbb{Q}}(\mathfrak{p})^+$ are obtained by using the notion of well covering pairs introduced by Ressayre [11].

Then, we obtain the equations of $\Delta_K(T^*K \times \mathfrak{p})$ by linear projection of the ones of $C_{\mathbb{Q}}(\mathfrak{p})^+$ (where we forget the last rational variable $r$). Indeed, one can see that the set of rational points of the polyhedral convex cone $\Delta_K(T^*K \times \mathfrak{p})$ is

$$\{(\mu, \nu, r) \mid \exists N \geq 1,\ (V^K_{N\mu} \otimes V^K_{N\nu} \otimes \mathbb{C}[E])^K \neq 0\}.$$

Then, by refining Ressayre's criterion in this setting, we note that the projection of the equations of $C_{\mathbb{Q}}(\mathfrak{p})^+$ has a good behavior, since the equations of $\Delta_K(T^*K \times \mathfrak{p})$ is induced by a particular subset of equations of $C_{\mathbb{Q}}(\mathfrak{p})^+$.

Finally, since $\Delta_K(K \cdot \Lambda \times \mathfrak{p})$ is an affine section of $\Delta_K(T^*K \times \mathfrak{p})$, the equations of $\Delta_K(K \cdot \Lambda \times \mathfrak{p})$ are obvioulsy induced by the ones of the polyhedral convex cone $\Delta_K(T^*K \times \mathfrak{p})$.

These results concern a substantial part of elliptic coadjoint orbits of such group $G$. So, now, the question that remains is: is it possible to extend them to all elliptic coadjoint orbits of $G$? In particular, the generalization of McDuff's theorem would prove that the symplectic slice at $\lambda$ of the Hamiltonian action of $K$ on $\mathcal{O}_\lambda$ is actually a global symplectic slice.

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## Kirillov's formula and Box splines

Michel Duflo
(joint work with Michèle Vergne)

I explained how the two topics in the title are used to prove some cases of "$[Q, R] = 0$" in the context of real reductive groups, not necessarily compact.

Let $G$ be a connected real reductive group with a compact Cartan subgroup $T \subset G$. Let $\mathfrak{g}$ be the Lie algebra of $G$, $\mathfrak{g}^*$ its dual space. Harish-Chandra associated to a coadjoint orbit $M \subset \mathfrak{g}^*$, which is elliptic, regular, and admits an equivariant *Spin*$^c$-structure, an irreducible unitary representation $\pi_M$ of $G$. The representations of $G$ obtained in this manner are exactly the *discrete series*, that is the representations which can be realized as an irreducible subrepresentation of the left regular representation of $G$ in $L^2(G)$. We consider $\pi_M$ as the *Spin*$^c$-quantization of $M$. This claim may be supported in various ways, in particular by the work of W. Schmid realizing concretely $\pi_M$ as a space of $L^2$-solutions of a suitable Dirac operator.

Kirillov's formula provides a direct link between $M$ and $\pi_M$, through the character of $\pi_M$.

Recall that the character of an irreducible unitary representation $\pi$ of $G$ is a generalized function on $G$, informally denoted by $tr(\pi(g))$. We need the following notations : for $x \in \mathfrak{g}$,

$$j_{\mathfrak{g}}(x) = det(\frac{e^{ad(x)/2} - e^{ad(-x)/2}}{ad(x)}),$$

and $\beta_M$ is the Liouville measure of the symplectic manifold $M$. Kirillov's formula says that for $x$ in a suitable subset of $\mathfrak{g}$, the character $tr(\pi_M(e^x))$ and the Fourier transform of $\beta_M$ are related by the following formula :

$$(1) \qquad j_{\mathfrak{g}}(x)^{1/2} tr(\pi_M(e^x)) = \int_M e^{im(x)} d\beta_M(m).$$

In this case, formula (1) is due to Rossmann [3]. Note that $\pi_M$ may be sometimes also defined for non regular coadjoint orbits $M$; but formula (1) holds usually only for regular orbits.

We suppose that $G'$ is another connected reductive group with a compact Cartan subgroup. We suppose that $G$ is a closed subgroup of $G'$. Let $\pi' := \pi'_{M'}$ be a discrete series of $G'$ associated as above to a coadjoint orbit $M' \subset \mathfrak{g}'^*$ of $G'$. We denote by $p : M' \subset \mathfrak{g}'^* \to \mathfrak{g}^*$ the moment map (the restriction map) for the action of $G$ in $M'$. We consider the restriction $\pi'|_G$ of $\pi'$ to $G$ as the $Spin^c$-quantization of the $G$-manifold-$M'$.

**We assume that $p$ is proper**. It is known that in this case $\pi'|_G$ is an Hilbertian direct sum of discrete series $\pi_M$ of $G$ occurring with finite multiplicities $m(M)$. On the other hand, the direct image $p_*(\beta_{M'})$ is a $G$-invariant measure on $\mathfrak{g}^*$. A variant of the problem "$[Q, R] = 0$" could be : is it possible to compute the multiplicities $m(M)$ from the measure $p_*(\beta_{M'})$? This is maybe too optimistic. To explain what is going on, we need more notations.

Let $V := \mathfrak{t}^*$. We denote by $P \subset V$ the set of $\lambda$ such that $e^{i\lambda}$ is a character of $T$, $\Delta(\mathfrak{g}')$, $\Delta(\mathfrak{g})$, $\Delta(\mathfrak{g}'/\mathfrak{g})$ the multisets of non zero roots of $\mathfrak{t}$ in the corresponding spaces. We denote by $\Phi$ a choice of positive roots for $\Delta(\mathfrak{g}'/\mathfrak{g})$. We may describe $\Phi$ as a list $\Phi = [\alpha_1, \ldots, \alpha_N]$ of non zero vectors in $V$, with repetitions allowed.

The (centered) *box spline* $B_\Phi$ is the probability measure on $V$ defined by the formula

(2) $$B_\Phi(f) = \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} f(t_1\alpha_1 + \cdots + t_N\alpha_N)\, dt_1 \ldots dt_N.$$

The relevance of $B_\Phi$ to our problem comes from the fact that the Fourier transform of $B_\Phi$ is equal to $j_{\mathfrak{g}'}(x)^{1/2}/j_{\mathfrak{g}}(x)^{1/2}$ for $x \in \mathfrak{t}$. Thus the box spline $B_\Phi$ takes care of the difference between the Kirillov formula for the discrete series of $G$ and of $G'$.

The properties of the box spline we need depend strongly on its vertex set $Ver(\Phi) \subset T$, that is the set of $s \in T$ such that there exists a basis $\sigma \subset \Phi$ of $\mathfrak{t}^*$ such that, for all $\alpha \in \sigma$, the value of $e^{i\alpha}$ at $s$ is 1.

When $\Phi$ is unimodular (i. e. $Ver(\Phi) = \{1\}$), one can recover the multiplicities $m(M)$ from the measure $p_*(\beta_{M'})$ "by inverting the box spline", see the report of M. Vergne in the same workshop. The formula involves the differential operator of Khovanskii-Pukhlikov type

(3) $$\hat{A}_\Phi = \Pi_{\phi\in\Phi} \frac{\partial_\phi}{e^{\partial_\phi/2} - e^{-\partial_\phi/2}}.$$

In general, Kirillov's formula (1) does not provide a sufficient information on the representation $\pi_M$ to determine it. However, building on the Harish-Chandra method of descent, we gave in [2] a formula relating, for $s \in T$ and $x \in \mathfrak{g}$ commuting with $s$, $tr(\pi(se^x))$ and the manifold $M^s$ of fixed points of $s$ in $M$. The final result is that the multiplicities $m(M)$ can be recovered from the measures $p_*(\beta_{M'^s})$, $s \in Ver(\Phi)$. The formula involves the differential operators, generalizing the Khovanskii-Pukhlikov operators, introduced in [1].

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## Quantization of manifolds with $f$-structure

Sean Fitzpatrick

Let $M$ be a smooth, compact manifold, and let $\varphi \in \Gamma(M, \mathrm{End}(TM))$ be an $f$-structure on $M$; that is, $\varphi$ is an endomorphism field satisfying [12]

$$\varphi^3 + \varphi = 0.$$

The complementary projection operators $l = -\varphi^2$ and $m = \varphi^2 + \mathrm{Id}_{TM}$ determine a splitting $TM = \ker\varphi \oplus \operatorname{im}\varphi$ of the tangent bundle of $M$. The restriction of $\varphi$ to $\operatorname{im}\varphi$ squares to $-\mathrm{Id}_{\operatorname{im}\varphi}$, and thus an $f$-structure is equivalent to an almost CR structure together with a choice of complement to the Levi distribution.

Given $(M, \varphi)$ it is always possible to find a compatible metric $g$ and connection $\nabla$ such that [11]

$$g(\varphi X, Y) + g(X, \varphi Y) = 0 \quad \text{and} \quad \nabla\varphi = \nabla g = 0.$$

Using the data $(\varphi, g, \nabla)$, we can construct a differential operator whose principal symbol is of the type we studied in [6]. In certain settings with additional structure, this operator may be interesting from the point of view of CR geometry. In both symplectic and contact geometry, there is a notion of compatible $f$-structure. In this symplectic case this is of course an almost complex structure, while in the contact case, it is an almost contact structure. We can describe two "quantization" procedures for manifolds with $f$-structure that reduce to familiar methods in symplectic geometry when our $f$-structure is a compatible almost complex structure, as well as to the geometric quantization of contact manifolds described in [7] in the almost contact case.

Let $E_{1,0} \subset T_{\mathbb{C}}M$ denote the $+i$-eigenbundle of $\varphi$ which, as noted above, defines an almost CR-structure on $M$. We use the data $(\varphi, g, \nabla)$ to construct an odd first-order differential operator $\mathsf{D}$ acting on sections of $\mathcal{S} = \Lambda E_{0,1}^*$, where $E_{0,1} = \overline{E_{1,0}}$. The construction is based on the usual construction of a Dirac operator on a complex manifold (see for example [2, Section 3.6]): the metric $g$ allows us to construct the bundle of Clifford algebras $\mathbb{C}l(E)$, whose fibre over $x \in M$ is the complexified Clifford algebra of $E_x^*$ with respect to the inner product induced by $g$. The Clifford bundle then acts on $\mathcal{S}$ via the Clifford action $\mathbf{c}$ defined for $\alpha \in \Gamma(M, E^*)$ by $\mathbf{c}(\alpha)\gamma = \sqrt{2}\left(\alpha^{0,1} \wedge \gamma - \iota(\alpha^{1,0})\gamma\right)$, where the contraction is defined using $g$. The bundles $E^*$ and $T^*$ are orthogonal with respect to $g$, and we let

$\pi_{E^*} : T^*M \to E^*$ denote the orthogonal projection. We then define $\mathsf{D}$ by the composition

$$\Gamma(M,\mathcal{S}) \xrightarrow{\nabla} \Gamma(M,T^*M\otimes\mathcal{S}) \xrightarrow{\pi_{E^*}} \Gamma(M,E^*\otimes\mathcal{S}) \xrightarrow{\mathbf{c}} \Gamma(M,\mathcal{S}).$$

When the almost CR structure determined by $\varphi$ is CR-integrable, we can also define the $\overline{\partial}_b$ operator of the resulting tangential CR complex, and construct another odd first-order differential operator acting on sections of $\mathcal{S}$, given in this case by $\mathsf{D}_b = \sqrt{2}(\overline{\partial}_b + \overline{\partial}_b^*)$, where $\overline{\partial}_b^*$ denotes the formal adjoint of $\overline{\partial}_b$, with respect to the pairing induced by the metric $g$. This operator satisfies $\mathsf{D}_b^2 = 2\Box_b$, where $\Box_b$ denotes the Kohn-Rossi Laplacian [9, 5]. When $M$ is equipped with the additional structure of an *almost $\mathcal{S}$-manifold*, as defined in [4], there exists a canonical connection $\nabla^{LP}$ analogous to the Tanaka-Webster connection of a strongly pseudoconvex CR manifold of hypersurface type [10]. Although $\nabla^{LP}$ necessarily has torsion, we can show that if we take $\nabla = \nabla^{LP}$ in the definition of $\mathsf{D}$ given above, then $\mathsf{D} = \mathsf{D}_b$.

We can also consider the case of a compact Lie group $G$ acting smoothly on $M$ such that $\varphi$, $g$ and $\nabla$ (and hence $\mathsf{D}$) are $G$-invariant. Such group action preserves the splitting $T_{\mathbb{C}}M = E_{1,0}\oplus E_{0,1}\oplus(T\otimes\mathbb{C})$, where $T = \ker\varphi$. The principal symbol of $\mathsf{D}$ is given by $\sigma_P(\mathsf{D})(x,\zeta) = i\mathbf{c}(\pi_{E^*}(\zeta_x))$ for $(x,\zeta)\in T^*M$, so that the results of [6] apply whenever the vector fields generated by the infinitesimal action span $T$. When this is the case, the operator $\mathsf{D}$ is $G$-transversally elliptic, since $\sigma_P(\mathsf{D})$ is invertible for all nonzero $\alpha\in E_x^*$. The equivariant index of $\mathsf{D}$ can therefore be defined as a distribution (i.e. generalized function) on $G$. In the almost $\mathcal{S}$ case, where the subbundle $T$ is trivial (and assuming we have twisted by a complex line bundle $\mathcal{V}$), the germ of the equivariant index of $\mathsf{D}$ near the identity element in $G$ is given (for $X\in\mathfrak{g}$ sufficiently small) by the formula

$$\operatorname{index}^G(\mathsf{D})(e^X) = \frac{1}{(2\pi i)^n}\int_M \operatorname{Td}(E,X)\operatorname{Ch}(\mathcal{V},X)\mathcal{J}(E,X),$$

where $n = \operatorname{rank}E/2$. The formulas for the equivariant index near other elements of $G$, and for the case when $T$ is not trivial, are similar. The term $\mathcal{J}(E,X)$ is an equivariant differential form with generalized coefficients defined as follows: Let $\theta\in\mathcal{A}^1(T^*M)$ denote the Liouville 1-form on $T^*M$, let $\iota : E^0 \hookrightarrow T^*M$ denote the inclusion of the annihilator of $E$ (which may be identified with $T^*$), and let $p : E^0\to M$ denote the projection mapping. The equivariant differential of $\theta$ is given by $D\theta(X) = d\theta - \theta(X_M)$, where $X_M$ denotes the vector field generated by $X\in\mathfrak{g}$, and $\mathcal{J}(E,X)$ is defined by

$$\mathcal{J}(E,X) = (2\pi i)^{-\operatorname{rank}T} p_*\iota^* e^{iD\theta(X)}.$$

The form $\mathcal{J}(E,X)$ was studied carefully in [6] and shown to be well-defined whenever the group action is transverse to the subbundle $E$.

By analogy with the symplectic case, we interpret the above as an index-theoretic description of a "quantization" $Q(M)$ given by the virtual $G$-representation $Q(M) = \ker\mathsf{D} - \ker\mathsf{D}^*$; when $\mathsf{D}$ is $G$-transversally elliptic, this representation has a well-defined (distributional) virtual character given by the equivariant index of $\mathsf{D}$. When the fundamental 2-form $\Phi$ given by

$$\Phi(X,Y) = g(X, \varphi Y)$$

is closed, it defines a symplectic structure on the fibres of $E$. A Hermitian line bundle $\mathbb{L}$ equipped with a connection $\nabla^{\mathbb{L}}$ whose curvature is equal to $\Phi$, is a *quantum bundle* [5], and we can take the $L^2$ sections of this bundle as a "prequantization" of $M$. If our $f$-structure is CR-integrable, the resulting CR structure is a natural analogue of a complex polarization; if in addition $\mathbb{L}$ is *CR holomorphic*, we can identify the space of polarized sections with the CR holomorphic $L^2$ sections of $\mathbb{L}$, and take this to be an alternative definition of $Q(M)$. In the almost $\mathcal{S}$ case, a suitable example is given by the trivial bundle $M \times \mathbb{C}$. Associated to the 2-form $\Phi$ is the set

$$\mathcal{P}(M,\Phi) = \{(f,X) \in C^\infty(M) \times \Gamma(M,TM) : df = -\iota(X)\Phi\},$$

which has a natural Poisson structure [8] such that the assignment

$$(f,X) \mapsto -i\nabla^{\mathbb{L}}_X + f$$

defines a Lie algebra homomorphism from $\mathcal{P}(M,\Phi)$ to the space of Hermitian operators on $Q(M)$. We note that given an "observable" $f \in C^\infty(M)$, the corresponding vector field $X$ is only defined up to sections of $T$, and that not every $f \in C^\infty(M)$ can be considered an "observable": if $(f,X) \in \mathcal{P}(M,\Phi)$, then $Yf = 0$ for any section $Y$ of $T$.

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## Localization of Riemann-Roch numbers via torus fibrations

Hajime Fujita

(joint work with Mikio Furuta, Takahiko Yoshida)

**1. Riemann-Roch number and its localization.** Let $(M, \omega)$ be a symplectic manifold. In this report we assume that $(M, \omega)$ is prequantized. Namely there exists a Hermitian line bundle with connection $(L, \nabla)$ whose curvature form is equal to $-\sqrt{-1}\omega$ and we fix it. We take and fix an $\omega$-compatible almost complex structure. Then we have a Dirac type operator $D = \bar{\partial}_L + \bar{\partial}_L^*$ defined by the Dolbeault operator $\bar{\partial}_L$ with values in $L$ and its formal adjoint $\bar{\partial}_L^*$. If $M$ is closed, then the analytic index of $D$ can be defined as an invariant of $(M, \omega)$ and it is called the *Riemann-Roch number*. We denote it by $RR(M)$ for short. In the context of spin$^c$-quantization $RR(M)$ is nothing other than the dimension of the quantization. There are several facts concerning localizations of $RR(M)$.

(i) If $M$ has a structure of a Lagrangian fibration without singular fibers, then $RR(M)$ is equal to the number of *Bohr-Sommerfeld fibers* ([1]). It is a localization of $RR(M)$ to BS fibers.

(ii) If $M$ has a structure of a toric manifold, then $RR(M)$ is equal to the number of the integral lattice points in the momentum polytope ([2]). It is a localization of $RR(M)$ to integral lattice points.

(iii) If a compact Lie group acts on $(M, \omega, L, \nabla)$, then the "quantization commutes with reduction ($[Q, R] = 0$)" holds. There are several proofs using localization of equivariant integrals ([8], [13]), symplectic cutting ([10],[11]), cobordism invariance ([6]), or analytic method ([12]).

Since toric manifold is a Lagrangian fibration with singular fibers, (ii) is a generalization of (i) to singular fibration cases. Similar results for the moduli space of flat $SU(2)$-connections over a Riemann surface and the Gelfand-Cetlin system are known ([9], [7]). All these facts can be understood in a formal framework of localization. The point is to define a local invariant for any neighborhood of each singular locus so that the resulting invariant has several topological properties. In joint works [3, 4, 5] with M. Furuta and T. Yoshida, we introduced a geometric structure which gives a sufficient condition to define such an invariant and a unified approach to the localizations (i), (ii) and (iii)(for torus actions). Our theorem can be stated as follows.

**Theorem** (Fujita-Furuta-Yoshida). *Let $(X, \omega, L, \nabla, J)$ be a prequantized symplectic manifold with an $\omega$-compatible almost complex structure $J$. Let $V$ be an open subset of $X$ whose complement $X \setminus V$ is compact. Suppose that $V$ is equipped with an "acyclic compatible system". Then we can define an integer $RR(X, V)$ which satisfies the following properties.*

(P1) *$RR(X, V) = RR(X)$ for a closed manifold $X$.*

(P2) *Deformation invariance.*
(P3) *Exciton property.*
(P4) *Gluing formula.*
(P5) *Product formula.*

By (P1) and (P3) we have a localization formula of the Riemann-Roch number.

**Corollary**. *Let $(X, V, \omega, L, \nabla, J)$ be the data as in Theorem. Suppose that $X$ is a closed manifold. Then we have the following equality.*

$$RR(X) = \sum_i RR(X_i, X_i \cap V),$$

*where $X_i$ is any small open neighborhood of a connected component of $X \setminus V$.*

For a Lagrangian fibration we can take $V$ to be the complement of BS-fibers and singular fibers. In particular we have a localization of the Riemann-Roch number to any neighborhood of BS fibers and singular fibers ([3]). In [3] we showed that if $X_i$ is a neighborhood of a BS-fiber, then $RR(X_i, X_i \cap V_i) = 1$, and hence, we have a generalization of Andersen's theorem (i). For a toric manifold we can take $V$ to be the complement of the inverse images of lattice points, and we have a proof of Danilov's theorem (ii) from the view point of localization of index. Using the equivariant version, we also have a proof of $[Q, R] = 0$ for Hamiltonian torus actions ([5]). The points in our proof of $[Q, R] = 0$ are the product formula (P5) and a variant of the equivariant version which we call the *G-acyclicity*.

**2. Acyclic compatible system.** In this report we explain our geometric structure the "acyclic compatible system" in a simple setting, $S^1$-bundle arising from an $S^1$-action. See [4, 5] for the full version.

Let $(X, \omega, L, \nabla, J)$ be a prequantized symplectic manifold with an $\omega$-compatible almost complex structure $J$. Suppose that there is an open subset $V$ of $X$ with the following properties.

(1) $X \setminus V$ is compact.
(2) $S^1$ acts on $V$ without any fixed points and preserving $\omega$ and $J$ on $V$.
(3) All orbits are acyclic, i.e., for all $x \in V$, the restriction $(L, \nabla)|_{S^1 \cdot x}$ does not have any non-trivial global parallel sections.

By (2) the natural projection to the orbit space $\pi : V \to V/S^1$ has a structure of an $S^1$-bundle (in the orbifold category). In our setting we do not use the non-degeneracy of the symplectic form $\omega$.

**3. Definition of $RR(X, V)$.** We define $RR(X, V)$ by using a variant of the Witten deformation. Let $T\pi \to V$ be the tangent bundle along $S^1$-orbits. Let $D_\pi : \Gamma(\wedge^\bullet T^*\pi \otimes L|_V) \to \Gamma(\wedge^\bullet T^*\pi \otimes L|_V)$ be the differential operator along orbits whose restriction to each orbit is the deRham operator on the orbits with values in $L|_{\mathrm{orbit}}$. Take and fix a smooth function $\rho$ on $X$ such that supp$\rho$ is contained in $V$ and it is identically 1 on the complement of a compact set of $X$. For any Dirac type operator $D$ on $X$ and $t \geq 0$ we put $D_t := D + t\rho D_\pi : \Gamma(\wedge^\bullet T^*X^{0,1} \otimes L) \to \Gamma(\wedge^\bullet T^*X^{0,1} \otimes L)$. Note that we may think $\rho D_\pi = \rho D_\pi \otimes \mathrm{id}$ as an operator acting on $\Gamma(\wedge^\bullet T^*X^{0,1}\otimes L)=\Gamma(\wedge^\bullet(T\pi\otimes\mathbb{C}\oplus(T\pi\otimes\mathbb{C})^\perp)\otimes L)$. A key to define $RR(X,V)$ is the following vanishing theorem.

**Proposition.** *Suppose that $X$ is closed or $V$ has a cylindrical end and all the data are translationally invariant on the end. If $X=V$, then there do not exist any non-trivial $L^2$-solutions of $D_tf=0$ for any $t\gg 0$.*

By the above vanishing theorem, we have that in the cylindrical end case the space of $L^2$-solutions of $D_tf=0$ is finite dimensional for any $t\gg 0$. In this case we can define $RR(X,V)$ as the index of $D_t(t\gg 0)$ using the APS-type boundary condition. For more general case we can deform all the data so that they have cylindrical end structure and are translationally invariant on the end. Then we define $RR(X,V)$ to be the index of the deformed data. This index satisfies (P1), (P2), (P3) and (P4). The product formula (P5) can be verified in the generalized version in [4].

There are two points in the proof of the vanishing theorem. (a) Using the $S^1$-invariant canonical flat structure on each orbit we can check that the anti-commutator $DD_\pi+D_\pi D$ is a differential operator along the orbit. (b) The acyclicity for $(L,\nabla)|_{S^1\cdot x}$ is equivalent to the vanishing of the cohomology of the flat line bundle, $H^\bullet(S^1\cdot x,(L,\nabla)|_{S^1\cdot x})=\{0\}$, and hence it is also equivalent to $\mathrm{Ker}(D_\pi|_{S^1\cdot x})=\{0\}$. Using these two points and the a priori estimate for the positive operator $D_\pi^2$, we can estimate the norms of sections in the kernel of $D_t^2$ as in the similar way for the Witten deformation.

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## The perturbative expansion of the $L^2$-heat kernel for large times

Sebastian Goette
(joint work with Sara Azzali, Thomas Schick)

In this talk, we sketch an important technical step in the generalisation of the signature theorem for families and the Bismut-Lott theorem [2] to an $L^2$-setting. Our proof is based on the McKean-Singer formula for the heat kernels associated to the fibrewise de Rham operator. As a byproduct, we can define $L^2$-versions of the family signature $\eta$- and $\rho$-invariants, and of the Bismut-Lott analytic torsion. Similar problems have also been considered by Heitsch-Lazarov [4], [5] and Benameur-Heitsch [1] for foliations. Whereas Benameur and Heitsch have to assume that the Novikov-Shubin invariants of the leafwise operators are larger than $\frac{1}{2}\dim B$, our method works if the Novikov-Shubin invariants are merely positive. However, we can only work with the fibrewise de Rham operator, which we may regard either as Euler operator or as signature operator.

Let $p\colon M \to B$ be a proper submersion, let $\mathcal{A}$ be a von Neumann algebra with a faithful finite normal trace, and let $\mathcal{F} \to M$ be a family of finite dimensional $\mathcal{A}$-modules with a flat $\mathcal{A}$-linear connection $\nabla^{\mathcal{F}}$ and a family of admissible scalar products $g^{\mathcal{F}}$. For the signature operator, we either demand that $g^{\mathcal{F}}$ is parallel, or we make the more general assumptions of [6] involving a real von Neumann algebra with a family of real $\mathcal{A}$-modules, a parallel nondegenerate bilinear form $(\,\cdot\,,\,\cdot\,)$ and an endomorphism $J^{\mathcal{F}}$ such that $g^{\mathcal{F}} = (\,\cdot\,, J^{\mathcal{F}}\,\cdot\,)$.

As a simple example, let a group $\Gamma$ act freely and cocompactly on the fibres of a fibration $\tilde{p}\colon \tilde{M} \to B$, and consider

$$\tilde{p}\colon \tilde{M} \longrightarrow \tilde{M}/\Gamma = M \stackrel{p}{\longrightarrow} B\ .$$

Let $\mathcal{A}$ be the group von Neumann algebra of $\Gamma$, then

$$\mathcal{F} = \tilde{M} \times_\Gamma \ell^2(M)$$

is a family of finite-dimensional $\mathcal{A}$-modules, and there is an $\mathcal{A}$-linear isomorphism of infinite-dimensional bundles of vertical differential forms,

$$\Omega^\bullet(M/B;\mathcal{F}) = \Omega^\bullet(\tilde{M}/B) \to B\ .$$

For the proof of local index theorems, one regards the curvature of the Bismut superconnection $\mathbb{A}$ acting on $\Omega^\bullet(M/B;\mathcal{F})$. Because the leading term of $\mathbb{A}^2$ is a Laplacian, the operator $e^{-t\mathbb{A}^2}$ is a generalisation of the fibrewise heat operator. We use a trick due to Bismut and Lott [2] and Lott [6], which allows us to write the curvature of the Bismut superconnection as

$$\mathbb{A}^2 = -\mathbb{X}^2\ .$$

Here $\mathbb{X}$ is an operator that differentiates only along the fibres, and hence an endomorphism of $\Omega^\bullet(M/B;\mathcal{F})$.

We perform the usual rescaling

$$\mathbb{X}_t = t^{\frac{1}{2}} X_0 + X_1 + t^{-\frac{1}{2}} X_2$$

with $\mathbb{X}_1 = \mathbb{X}$ and with $X_i \in \Omega^i\big(B; \operatorname{End}\Omega^\bullet(M/B)\big)$. The operator $X_0 = d^* - d$ is a skewadjoint elliptic first order differential operator along the fibres, whereas $X_1$ and $X_2$ are bounded operators. Then we are interested in the von Neumann trace of the fibrewise heat kernel $e^{\mathbb{X}_t^2}$ for large $t$.

We use Duhamel's formula, therefore let

$$\Delta^k = \Big\{\, \bar{s} = (s_0, \dots, s_k) \subset [0,1]^k \;\Big|\; s_0 + \dots + s_k = 1 \,\Big\}$$

denote the standard simplex, and write $\mathbb{X}_t = \sqrt{t} X_0 + R_t$. Then

$$(1) \quad \mathbb{X}_t^a\, e^{\mathbb{X}_t^2} = \sum_{k=0}^{\dim B} \int_{\Delta^k} \big(\sqrt{t}\, X_0 + R_t\big)^a\, e^{s_0 t X_0^2}\, \big(\sqrt{t}\,(X_0 R_t + R_t X_0) + R_t^2\big) \\ e^{s_1 t X_0^2} \cdots e^{s_k t X_0^2}\, d^k(s_0, \dots, s_k)\;.$$

Note that even though $t$ may be very large, we also encounter instances of the heat kernel for small times whenever at least one of the simplex coordinates $s_i$ is small. Thus, it is difficult to give a uniform estimate of the trace of the integrand over the whole $k$-simplex.

Let $P \in \operatorname{End}\Omega(M/B;\mathcal{F})$ denote the fibrewise $L^2$-projection onto the reduced $L^2$-cohomology, represented by the bundle of fibrewise harmonic forms

$$\mathcal{H} = \ker X_0 \subset \Omega^\bullet(M/B;\mathcal{F})\;.$$

Let $\tau$ denote the von Neumann trace on $\operatorname{End}\Omega(M/B;\mathcal{F})$, and let $\|\cdot\|_\tau$ denote the induced norm on the trace class operators.

**Lemma.** *There exists a function $\vartheta\colon (0,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty}\vartheta(t) = 0$ and a constant $C$ such that for all $s$, $t > 0$,*

$$(1) \qquad \Big\| t^{\frac{c}{2}} X_0^c\, e^{stX_0^2} \Big\|_\infty \le C\, s^{-\frac{c}{2}}\;,$$

$$(2) \qquad \Big\| e^{stX_0^2} - P \Big\|_\tau = \vartheta(st)\;,$$

$$(3) \qquad \Big\| t^{\frac{c}{2}} X_0^c\, e^{stX_0^2} \Big\|_\tau \le C\, s^{-\frac{c}{2}}\, \vartheta\Big(\frac{st}{2}\Big) \qquad \textit{for } c = 1,\, 2\;,$$

$$(4) \qquad \Big\| \int_0^{\bar{s}} t^{\frac{c}{2}} X_0^c\, e^{stX_0^2}\, ds \Big\|_\infty \le C\, \bar{s}^{c-1} \qquad \textit{for } c = 0,\, 1\;,$$

$$(5) \qquad \int_0^{\bar{s}} t X_0^2\, e^{stX_0^2}\, ds = e^{\bar{s}tX_0^2} - \mathrm{id}\;.$$

One can now fix $\bar{s}(t) > 0$ and decompose the simplex $\Delta^k$ in regions where certain $s_i$ are smaller than $\bar{s}(t)$ and the remaining are larger. Then one integrates over the small simplex coordinates before considering the limit $t \to \infty$. Let

$$R = X_1 = \lim_{t\to\infty} R_t\;.$$

With a careful choice of $\bar{s}\colon (0,\infty) \to (0,1)$, one obtains

$$\dots R_t\, t^{\frac{c}{2}} X_0^c\, e^{s_i t X_0^2}\, R_t \dots \quad \rightsquigarrow \quad \begin{cases} R\,(P-\mathrm{id})\,R & \text{if } s_i \text{ is small and } c=2, \\ R\,P\,R & \text{if } s_i \text{ is large and } c=0, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

if $t$ is large. Integration over the remaining simplex coordinates finally proves that

$$\lim_{t\to\infty} \mathbb{X}_t^a\, e^{\mathbb{X}_t^2} = (PRP)^a\, e^{(PRP)^2} \tag{2}$$

in the trace norm, as in the classical case, see [2].

The bundle $\mathcal{H} \to B$ carries the flat Gauß-Manin connection $\nabla^{\mathcal{H}}$ and an $L^2$-metric $g^{\mathcal{H}}_{L^2}$ induced by the inclusion $\mathcal{H} \subset \Omega^\bullet(M/B;\mathcal{F})$. If the fibres are oriented and $4k$-dimensional, there exists a natural splitting $\mathcal{H} = \mathcal{H}^+ - \mathcal{H}^-$. More generally, in the context of [6], the real bundle $\mathcal{H}_\mathbb{R}$ carries an automorphism $J^{\mathcal{H}}$ if the fibres are even-dimensional. Using the McKean-Singer trick and equation (2), one can prove cohomological versions of the index theorems in [2] and [6].

By adapting the method of Cheeger and Gromov [3], one defines $L^2$-signature $\eta$-invariants $\eta_\tau(T^HM, g^{TX}, J^{\mathcal{F}})$ and $L^2$-analytic torsion forms $\mathcal{T}_\tau(T^HM, g^{TX}, g^{\mathcal{F}}) \in \Omega^\bullet(B)$. In the following theorems, the exterior derivative $d$ on $C^0$-forms on $B$ is understood in a weak sense.

**Theorem.** *Assume that the Novikov-Shubin invariant of $X_0$ is positive and that the fibres are even-dimensional and oriented. Then*

$$d\eta_\tau(T^HM, g^{TX}, J^{\mathcal{F}}) = \int_{M/B} \hat{L}(TX, \nabla^{TX})\mathrm{ch}_\tau(\mathcal{F}, J^{\mathcal{F}}) - \mathrm{ch}_\tau(\mathcal{H}, J^{\mathcal{H}})\ .$$

*If the fibres are oriented and odd-dimensional, then*

$$d\eta_\tau(T^HM, g^{TX}, J^{\mathcal{F}}) = \int_{M/B} \hat{L}(TX, \nabla^{TX})\mathrm{ch}_\tau(\mathcal{F}, J^{\mathcal{F}})\ .$$

**Theorem.** *Assume that the Novikov-Shubin invariant of $X_0$ is positive. Then*

$$d\mathcal{T}_{L^2}(T^HM, g^{TX}, g^F) = \int_{M/B} e(TX, \nabla^{TX})\, \mathrm{ch}^\circ_\tau(\mathcal{F}, g^{\mathcal{F}}) - \mathrm{ch}^\circ_\tau(\mathcal{H}, g^{\mathcal{H}}_{L^2})\ .$$

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## Real and complex quantization of flag manifolds

Mark Hamilton
(joint work with Hiroshi Konno)

**Background.** Given a symplectic manifold $(M, \omega)$, the geometric quantization is defined in terms of sections of a *prequantum line bundle,* which is a complex line bundle $L$ over $M$ with a connection whose curvature is $\omega$. Naïvely, we want the quantization to be the space of sections of $L$, but this space is "too big." We restrict by choosing certain sections using a *polarization.*

The most common type of polarization is a *Kähler polarization* or *complex polarization,* which is given by a compatible complex structure on $M$; in this case, the quantization is the space of holomorphic sections of $L$.

Another type of polarization is a *real polarization,* given by a foliation of $M$ into Lagrangian submanifolds. In this case the sections considered are those that are "flat along the leaves," i.e. covariant constant (with respect to the connection on $L$) in directions tangent to the leaves of the polarization. There are some subtleties involved in the definition, since such sections cannot be defined on all of $M$ but only on certain leaves of the polarization called *Bohr-Sommerfeld leaves,* but we do not address those here. The main result about quantization using real polarizations is a theorem of Śniatycki, who proved in [S] that, under certain nice conditions, the dimention of the real quantization is equal to the number of Bohr-Sommerfeld leaves of the polarization.

We call the quantizations coming from these two types of polarizations the "complex quantization" and "real quantization" of $M$. If a manifold admits both a real and Kähler polarization, it is natural to ask if the resulting quantizations are the same.

**Gelfand-Cetlin and flag manifolds.** Let $\mathcal{F}$ be the complex flag manifold $\mathcal{F} = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n | \dim V_k = k\}$. Since $\mathcal{F}$ is a complex manifold (of dimension $d = \frac{n(n-1)}{2}$), it has a natural complex polarization.

There is also a completely integrable system on $\mathcal{F}$, called the *Gelfand-Cetlin system,* which we denote by $\lambda \colon \mathcal{F} \to \mathbb{R}^d$. It was introduced and studied by Guillemin and Sternberg in [GS]. By the Arnol'd-Liouville theorem, a completely integrable system gives a structure very much like a real polarization, except that it has singularities, and so we can think of the Gelfand-Cetlin system as giving a singular real polarization on $\mathcal{F}$. This was observed by Guillemin and Sternberg, who showed that the number of Bohr-Sommerfeld fibres for this system equals the dimension of the space of holomorphic sections, and so the real and complex quantizations of $\mathcal{F}$ are "the same." However, this was expressed in terms of the equality of two numbers which were computed by other means (combinatorial and representation-theoretic), and did not give any direct relationship between the quantizations.

We give a direct relationship between the real and complex quantizations of $\mathcal{F}$, by deforming the complex structure on $\mathcal{F}$ in such a way that holomorphic sections converge to distributional sections supported on the Bohr-Sommerfeld fibres, which can be seen as elements of the real quantization. Our construction is based on two techniques.

**Toric degeneration of flag manifolds.** There exists a family of Kähler manifolds $X_t$, $t \in \mathbb{C}$, with constant dimension, such that $X_1 = \mathcal{F}$ and $X_0$ is a toric variety.

Toric degenerations of flag manifolds were studied by various authors over the past decade or two. Most recently, Nishinou, Nohara, and Ueda in [NNU] constructed a toric degeneration of $\mathcal{F}$ that carries the Gelfand-Cetlin integrable system on $\mathcal{F}$ to the integrable system on $X_0$ coming from the torus action. (In their construction, they use a "degeneration in stages," where the parameter $t$ is not in $\mathbb{C}$ but in $\mathbb{C}^{n-1}$.) More history of toric degenerations is given in [NNU].

**Deformation of complex structure of a toric manifold.** Recently, Baier, Florentino, Mourao, and Nunes [BFMN] constructed a deformation of the complex structure on a symplectic toric manifold, such that an element of the canonical basis of holomorphic sections (corresponding to an integer lattice point in the moment polytope) converges to a distributional section supported on the Bohr-Sommerfeld fibre over the same integer lattice point.

**Our result.** Our construction is a combination of these two techniques, applied to the flag manifold case. Naïvely, we degenerate $\mathcal{F}$ to $X_0$ and then apply BFMN's deformation to the $X_0$, which is toric; the actual process is slightly more complicated.

First of all, BFMN's deformation does not apply directly to $X_0$, because $X_0$ is (almost always) singular. However, it sits inside $\mathcal{P} = \prod_{k=1}^{n-1} \mathbb{P}(\wedge^k \mathbb{C}^n)$ which is a smooth toric manifold, in such a way that "everything" is compatible: the real polarization, the complex structure, and the actions of the respective tori on $X_O$ and $\mathcal{P}$. We apply the deformation to $\mathcal{P}$ and keep track of what happens on $X_0$.

Second, we cannot directly apply the toric degeneration either, because the manifolds $X_t$ are no longer diffeomorphic to $\mathcal{F}$ after the first "stage" of the degeneration. Instead, we use an approximation to NNU's degeneration in stages so that the degenerating manifolds remain diffeomorphic to $\mathcal{F}$. By a delicate limiting argument, we prove convergence of the holomorphic sections.

In the end, we obtain:

**Theorem.** *View the complex flag manifold $\mathcal{F}$ as a symplectic manifold, with symplectic form $\omega$, complex structure $J_{\mathcal{F}}$, and prequantum line bundle $L$. Denote by $\Delta \subset \mathbb{R}^d$ the convex polytope that is the image of the Gelfand-Cetlin system $\lambda$. Then there exists a one-parameter family $\{J_s\}_{s\in[0,\infty)}$ of complex structures on $\mathcal{F}$, compatible with $\omega$, such that $J_0 = J_{\mathcal{F}}$ and the Kähler polarizations defined by $J_s$ converge to the real polarization defined by $\lambda$, in the following sense: For each $s \in [0, \infty)$, there exists a basis $\{\sigma_s^m \mid m \in \Delta \cap \mathbb{Z}^d\}$ of the space of holomorphic sections $H^0(L)$ such that, for each $m \in \mathrm{Int}\Delta \cap \mathbb{Z}^d$, the section $\frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1}}$ converges* *as $s \to \infty$ to a delta-function section $\delta_m$ supported on the Bohr-Sommerfeld fiber $\lambda^{-1}(m)$.*

Thus, we have a direct relationship between the holomorphic sections $\sigma_0^m$, which are elements of the complex quantization of $\mathcal{F}$, and the distributional sections $\delta_m$, which can be seen as elements of the real quantization.

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## On the Regualirization of the Kepler Problem

Gert Heckman

In the lecture I discuss four proofs of Kepler's ellipse law. The first one (hard to imagine, but maybe new) is geometric in origin and was obtained by Van Haandel and Heckman, while teaching a master class for high school students in their final grade. This is the proof I like best, and would describe as a proof from "The Book". The second proof is the original proof by Newton, rephrased in modern terminology, and explaining his beautiful comparison of the inverse square force (Kepler problem) with the harmonic oscillator (Hooke law). The third proof is the one of Bernoulli and Hermann, that is found in most physics text books. Rewrite the equation of motion for position vector as function of time via polar coordinates as a second order equation of the inverse distance as a function of the angle. This equation is easy to solve, but the proof is black magic. The final proof is the construction of a canonical bijection between the part of phase space of the Kepler problem with negative energy to the punctured cotangent bundle of the sphere (with the cotangent space above the north pole deleted) which intertwines the Kepler Hamiltonian with the Delaunay Hamiltonian for the sphere. The essential step is due to Moser: stereographic projection after geometric Fourier transform. However under this geometric transformation Kepler and Delaunay flow are intertwined with different times. The map can be adjusted according to a formula by Ligon and Schaaf (1976) with improvements by Cushman and Duistermaat (1997) so that the two times also match. Our main point (collaboration with Tim de Laat) is that the Ligon-Schaaf formula can be understood in an almost trivial way from the Moser regularization paper (1970).

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## Quantisation commutes with reduction at nontrivial representations

Peter Hochs

### 1. Quantisation and noncommutative geometry

Landsman [6] formulated a version of the quantisation commutes with reduction problem that is meaningful for noncompact groups acting on noncompact manifolds, as long the orbit space of the action is compact. Consider a prequantisable Hamiltonian action by a Lie group $G$ on a symplectic manifold $(M, \omega)$, such that $M/G$ is compact. Landsman defines the geometric quantisation of this action using the *analytic assembly map* from the Baum–Connes conjecture [1, 8] in noncommutative geometry

$$\mu_M^G : K_0^G(M) \to K_0(C^*G).$$

Here $K_0^G(M)$ is the equivariant $K$-homology of $M$ [2], in which the (Spin$^c$- or Dolbeault-)Dirac operator $\not{D}_M$ associated to all the data given above naturally defines a class $[\not{D}_M] \in K_0^G(M)$. Furthermore, $K_0(C^*G)$ is the $K$-theory of the full group $C^*$-algebra of $G$. We will in fact use a version for the reduced group $C^*$-algebra (see below), but this is not suitable for Landsman's purposes. The geometric quantisation of the action is then defined as

$$Q_G(M, \omega) := \mu_M^G[\not{D}_M] \qquad \in K_0(C^*G).$$

If $M$ and $G$ are compact, there is a natural isomorphism

$$K_0(C^*G) \cong R(G),$$

with $R(G)$ the representation ring, which maps the geometric quantisation defined in this way to the standard equivariant index of the Dirac operator $\not{D}_M$.

Landsman proceeds to define a reduction map

$$R_G^0 : K_0(C^*G) \to \mathbb{Z},$$

that coresponds to taking the multiplicity of the trivial representation in the compact case. It is functorially induced by the continuous map

$$C^*G \to \mathbb{C},$$

which on the dense subset $C_c(G) \subset C^*G$ is given by

$$\varphi \mapsto \int_G \varphi(g)\, dg.$$

His generalised quantisation commutes with reduction conjecture then reads

$$R_G^0 \circ Q_G(M, \omega) = \mathrm{index}(\not{D}_{M_0}),$$

where $\not{D}_{M_0}$ is the Dirac operator on the symplectic quotient $M_0$ of $M$ at $0 \in \mathfrak{g}^*$. This conjecture was proved under additional assumptions in [3]. A complete proof was given by Mathai, Zhang and Bunke in [7].

## 2. Discrete series representations

For applications in representation theory, it is useful to have a version of quantisation commutes with reduction that is valid for reduction at other representations that the trivial one. In the compact situation, the case for reduction at the trivial representation implies the case for reduction at any irreducible representation. In the noncompact case, a similar principle does not hold (or is not known yet).

We have obtained a result [4] for reduction at discrete series representations of semisimple groups. Therefore, suppose from now on that $G$ is a semisimple Lie group with finite centre, that has discrete series representations. We also assume $G$ to be connected to avoid some technical difficulties, although this assumption is not essential.

A key assumption is that the momentum map associated to the action,

$$\Phi : M \to \mathfrak{g}^*,$$

takes values in the *strongly elliptic set* $\mathfrak{g}^*_{\mathrm{se}} \subset \mathfrak{g}^*$, which is the set of all elements of $\mathfrak{g}^*$ with compact stabilisers under the coadjoint action. Heuristically, the coadjoint orbits inside this set are associated with discrete sreies representations (see Schmid [11], Parthasarathy [10] and also Paradan [9]). In [13], Proposition 2.6, Weinstein proves that $\mathfrak{g}^*_{\mathrm{se}}$ is nonempty if and only if $\operatorname{rank} G = \operatorname{rank} K$, which is Harish-Chandra's criterion for the existence of discrete series representations of $G$.

We define geometric quantisation as

$$Q_G(M, \omega) = \mu_M^G[\not{D}_M] \in K_0(C^*_r G),$$

where now $\mu_M^G$ is the version of the analytic assemmply map with values in the $K$-theory of the *reduced* group $C^*$-algebra $C^*_r G$ of $G$, and $\not{D}_M$ is the Spin$^c$-Dirac operator on $M$ associated to all the data. Following Lafforgue [5], we define a reduction map at a discrete series representation $(\mathcal{H}, \pi)$ of $G$, as a map

$$R_G^{\mathcal{H}} : K_0(C^*_r(G)) \to \mathbb{Z},$$

in the following way. Consider the continuous map

$$C^*_r(G) \to \mathcal{K}(\mathcal{H})$$

(the $C^*$-algebra of compact operators on $\mathcal{H}$), given on $C_c(G) \subset C^*_r(G)$ by

$$\varphi \mapsto \int_G \varphi(g)\, \pi(g)\, dg.$$

Since $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$, this map induces a map $K_0(C^*_r(G)) \to \mathbb{Z}$ on $K$-theory, which by definition is $R_G^{\mathcal{H}}$. The reason why we now use the reduced group $C^*$-algebra whereas Landsman used the full one, is that his reduction map $R_G^0$ is defined for the full group $C^*$-algebra, while the map $R_G^{\mathcal{H}}$ is defined for the reduced one.

With these preparations in place, the result can be stated as follows. Fix a maximal compact subgroup $K < G$, a maximal torus $T < K$, and a positive Weyl chamber $\mathfrak{t}^*_+$. Let $\mathcal{H}$ be an irreducible discrete series representation. Let $\lambda \in i\mathfrak{t}^*$ be its Harish-Chandra parameter such that $(\alpha, \lambda) > 0$ for all compact positive roots $\alpha$. We will write $(M_\lambda, \omega_\lambda) := (M_{-i\lambda}, \omega_{-i\lambda})$ for the symplectic reduction of $(M, \omega)$ at $-i\lambda \in \mathfrak{t}^*_+ \cap \mathfrak{g}^*_{\mathrm{se}}$.

**Theorem 1** (Quantisation commutes with reduction at discrete series representations)**.** *If $-i\lambda$ is in the image of $\Phi$, then*

$$\boxed{R^{\mathcal{H}}_G \circ Q_G(M, \omega) = (-1)^{\frac{\dim G/K}{2}} Q(M_\lambda, \omega_\lambda).}$$

*If $-i\lambda$ does not lie in the image of $\Phi$, then the integer on the left hand side equals zero.*

The quantisation $Q(M_\lambda, \omega_\lambda)$ of the (compact) reduced space is the usual Spin$^c$-quantisation in the compact setting.

## 3. Quantisation commutes with induction

Our proof of this result is based on a reduction to the compact case. Crucial roles are played by the *Hamiltonian induction* and *Dirac induction* maps. Dirac induction is the map

$$\text{D-Ind}^G_K : R(K) \to K_0(C^*_r G)$$

used in the Connes–Kasparov conjecture [5, 12]. Hamiltonian induction maps a compact Hamilton $K$-manifold $(N, \nu)$ to a Hamiltonian $G$-manifold

$$\text{H-Ind}^G_K(N, \nu) = \big(M := G \times_K N, \omega\big),$$

with $\omega$ some symplectic form on the fibred product $G \times_K N$. Under the assumptions we have made (the main one being that $\Phi(M) \subset \mathfrak{g}^*_{\mathrm{se}}$), the Hamiltonian induction map is invertible. Its inverse is called taking *Hamiltonian cross-sections*, and is defined by

$$\text{H-Cross}^G_K(M, \omega) = \big(N := \Phi^{-1}(\mathfrak{k}^*), \omega|_N\big).$$

Hamiltonian induction extends to all relevant data, such as prequantisations and equivariant Spin$^c$-structures.

Our central result is the fact that *quantisation commutes with induction*:

**Theorem 2.** *Let $(N, \nu)$ be a compact Hamiltonian $K$-manifold. Suppose $(N, \nu)$ is equivariantly* Spin$^c$*-prequantisable, and let $Q_K(N, \nu) \in R(K)$ be its usual* Spin$^c$*-quantisation. Then*

$$\text{D-Ind}^G_K \circ Q_K(N, \nu) = Q_G \circ \text{H-Ind}^G_K(N, \nu).$$

The quantisation commutes with reduction result for discrete series representations follows readily from this theorem.

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## Singular Kähler quantization on the moduli space of semi-stable holomorphic vector bundles on a curve

Johannes Huebschmann

A well-known construction of Seshadri [20] involving Grothendieck's quot-scheme [4] yields the moduli space of semi-stable holomorphic vector bundles on a curve (equivalently: Riemann surface) of fixed rank and degree as a normal projective variety.

In the coprime case, as a projective variety, the moduli space is non-singular and acquires a Kähler structure. In the general case, an infinite-dimensional approach due to Atiyah and Bott yields the moduli space by a version of infinite-dimensional Kähler reduction [1]. A construction worked out by L. Jeffrey and me [7], [12] that involves a suitable extended moduli space establishes the moduli space by symplectic reduction in finite dimensions. The moduli space thus acquires the structure of a stratified symplectic space in the sense of [21].

Our aim is to construct the moduli space as a purely finite dimensional Kähler quotient and to develop singular quantization on that space via singular Kähler reduction. To this end, relative to the appropriate symmetry group, we endow an open and dense non-singular stratum of the space of holomorphic maps from the curve to a suitable Grassmannian with an invariant Kähler structure together with a momentum mapping; that open and dense stratum consists of those holomorphic maps that parametrize semi-stable holomorphic vector bundles on the given curve, cf. e. g. [17]. Kähler reduction in finite dimensions [15], [16] then yields the moduli space. This construction complements that of Seshadri. Among our tools are Chen's theory of "differentiable spaces" [3] or, equivalently, Souriau's theory of "diffeological spaces" [22] and the equivariant Chern-Weil construction à la Berline and Vergne [2]; the latter is here a crucial ingredient for the construction of the requisite momentum mapping, a momentum mapping associated with a closed equivariant 2-form being an equivariantly closed extension of that 2-form. The resulting Kähler quotient construction endows the moduli space with a stratified Kähler structure [9] and explains in particular the in general singular structure of the moduli space, that singular structure being finer than the ordinary complex analytic singularity structure. Suffice it to mention here the following: a *stratified symplectic space* is a stratified space whose strata are symplectic manifolds together with a Poisson algebra of continuous functions such that, for each stratum, the restriction mapping on that algebra of continuous functions goes into the smooth functions on that stratum and is actually a Poisson mapping relative to the ordinary smooth symplectic Poisson algebra on that stratum; a *stratified Kähler space* is a complex analytic space that is, furthermore, endowed with a complex analytic stratification together with a compatible stratified symplectic structure such that, on each stratum, the pieces of structure combine to a Kähler structure; this kind of structure is considerably more subtle than just that of a stratified space whose strata are Kähler manifolds and, in particular, is not equivalent to Grauert's notion of stratified Kählerian space. As for the moduli spaces of semi-stable holomorphic vector bundles on a curve under discussion, in the special case of genus two, rank two, degree zero, and trivial determinant—in the literature, this case is considered exceptional—the moduli space is complex projective 3-space, the semi-stable points that are not stable constitute a Kummer surface [18], and this surface arises here as the singular locus of an exotic stratified Kähler structure on complex projective 3-space even though, with the standard structure, complex projective 3-space is non-singular. Here the term "exotic" is intended to refer to the fact that that structure on projective 3-space is essentially different from the standard (non-singular Fubini-Study) Kähler structure [9].

Equivariant Kähler quantization of the data then proceeds via an appropriate equivariant holomorphic line bundle (a determinant bundle) on that open and dense stratum of the space of holomorphic maps, and Kähler reduction yields a holomorphic line bundle or, more generally, coherent sheaf, on the moduli space. Taking holomorphic sections, we obtain a costratified Hilbert space [10], a quantum structure that has the classical singularities as its shadow; we recall that a costratified Hilbert space consists of a system of Hilbert spaces, one for each stratum, each stratum, apart from those at the "bottom", being a non-compact Kähler manifold (unless there is a single stratum), together with bounded operators which reflect the stratification on the classical level.

The constructions admit generalizations to other principal bundles on a curve (a general compact Lie group being substituted for the unitary group) and, even more generally, to the situation of Hermite-Einstein theory.

The construction of the moduli space as a stratified Kähler space and the subsequent quantization procedure is part of a research program that addresses the issue of quantization in the presence of classical phase space singularities [8]-[11], [13], [14].

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## K-theoretic index theorems for transversally elliptic operators

Gennadi Kasparov

Let $X$ be a complete Riemannian manifold (in general, non-compact), equipped with a proper isometric action of a Lie group $G$. To simplify things, we will assume that $X/G$ is compact. Let $T(X)$ be the tangent bundle of $X$, $T^*(X)$ the cotangent bundle and $p : T(X) \to X$ the projection. We identify $T(X)$ with $T^*(X)$ via the Riemannian metric. We will denote covectors on $X$ by $\xi$. The Lie algebra of the group $G$ will be denoted $\mathfrak{g}$. We will consider properly supported pseudo-differential operators on $X$ of the Hörmander class $\rho = 1, \delta = 0$.

Let $x \in X$ and $f_x : G \to X$ be the map defined by $g \mapsto g(x)$. We denote by $f'_x : \mathfrak{g} \to T_x(X)$ the tangent (first derivative) map of $f_x$ at the identity of $G$, and by $f'^*_x : T^*_x(X) \to \mathfrak{g}^*$ the dual map. It is easy to see that for any $x \in X,\ g \in G,\ v \in \mathfrak{g}$, one has: $g(f'_x(v)) = f'_{g(x)}(Ad(g)(v))$.

Let us consider a trivial vector bundle $\mathfrak{g}_X = X \times \mathfrak{g}$ over $X$ with the $G$-action given by $(x, v) \mapsto (g(x), Ad(g)(v))$. Because the $G$-action on $X$ is proper, there exists a $G$-invariant Riemannian metric on $\mathfrak{g}_X$. Equivalently, one can say that there exists a smooth map from $X$ to the space of Euclidean norms on $\mathfrak{g}$: $x \mapsto ||\cdot||_x$, such that for any $x \in X,\ g \in G,\ v \in \mathfrak{g}$, one has: $||Ad(g)(v)||_{g(x)} = ||v||_x$.

It is clear that the map $f' : \mathfrak{g}_X \to T(X)$ is $G$-equivariant. Note that by multiplying our Riemannian metric on $\mathfrak{g}_X$ by a certain strictly positive function lifted from $X/G$, we can also arrange that the following condition is satisfied: for any $v \in \mathfrak{g}$, $||f'_x(v)|| \leq ||v||_x$. We will assume this, and so $||f'_x|| \leq 1$ for any $x \in X$.

We identify $\mathfrak{g}_X$ with its dual bundle via the Riemannian metric and define a $G$-invariant quadratic form $q$ on cotangent vectors $\xi \in T^*_x(X)$ by $q_x(\xi) = |(f'_x f'^*_x(\xi), \xi)| = ||f'^*_x(\xi)||^2_x$. It follows from the above that $q_x(\xi) \leq ||\xi||^2$ for any covector $\xi$.

*Note that a covector* $\xi \in T^*_x(X)$ *is orthogonal to the orbit passing through* $x$ *if and only if* $q_x(\xi) = 0$.

In the definition of the symbol, as well as in the definition of the $K$-theoretic index class of a transversally elliptic operator, we will use the following convention: we replace the operator $A$ with $\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$ on $E = E^0 \oplus E^1$ acting on the vector bundle $E = E^0 \oplus E^1$. It means that we consider $E$ as a $\mathbf{Z}_2$-graded vector bundle, and $A$ is a selfadjoint operator on $E$ of grading degree 1.

**Definition.** A properly supported $G$-invariant pseudo-differential operator $A$ of order 0 on a vector bundle $E$ over $X$ will be called transversally elliptic if its symbol $\sigma_A$ satisfies the following condition: *for any compact subset $K \subset X$ and any $\epsilon > 0$, there exists $c > 0$ such that for any $x \in K$,*

$$||\sigma_A^2(x,\xi) - 1||_{(x,\xi)} \leq c \cdot \frac{1 + q_x(\xi)}{1 + ||\xi||^2} + \epsilon.$$

The notation $|| \cdot ||_{(x,\xi)}$ means the norm of an endomorphism of the vector bundle $p^*(E)$ on $T^*(X)$ at the point $(x,\xi)$.

In order to define the $K$-theory symbol class, we first need to introduce an analog of the algebra of scalar symbols of negative order. We define this algebra of symbols as the following commutative $C^*$-algebra $\mathfrak{S}_G(X) \subset C_b(T^*(X))$.

**Definition.** The symbol algebra $\mathfrak{S}_G(X)$ is the norm-closure in $C_b(T^*(X))$ of the set of all smooth, bounded, compactly supported in the $x$-variable functions $b(x,\xi)$ on $T^*(X)$ which satisfy the following conditions 1), 2):

1) The exterior derivative $d_x b(x,\xi)$ is norm bounded uniformly in $\xi$, and for the exterior derivative $d_\xi$ there is an estimate:

$$||d_\xi b(x,\xi)|| \leq C \cdot (1 + ||\xi||)^{-1}$$

where the constant $C$ depends only on $b$ and not on $(x,\xi)$;

2) for any $\epsilon > 0$, there exists $c > 0$ such that for any $x \in X$,

$$|b(x,\xi)| \leq c \cdot \frac{1 + q_x(\xi)}{1 + ||\xi||^2} + \epsilon.$$

Note that condition 1) is just a weak version of the Hörmander property of the $\rho = 1, \delta = 0$ class of symbols for order 0 operators. Under the assumption that the condition 1) is satisfied, the condition 2) is equivalent to the usual condition of transverse ellipticity of Atyiah-Singer.

Let $E$ be a vector bundle on $X$, and consider the norm-closure of the set of all smooth bounded sections of $p^*(E)$ over $T^*(X)$ which satisfy the conditions of the definition of the algebra $\mathfrak{S}_G(X)$ (with $|b(x,\xi)|$ replaced with $||b(x,\xi)||_{(x,\xi)}$). Call it $\mathfrak{S}_G(E)$. Then $\mathfrak{S}_G(E)$ is a Hilbert module over $\mathfrak{S}_G(X)$, with the obvious multiplication and the pointwise inner product given by the Hermitian metric of $p^*(E)$. The algebra $C_0(X)$ acts on $\mathfrak{S}_G(E)$ by multiplication.

Let $A$ be a formally self-adjoint, properly supported, $G$-invariant pseudo-differential operator $A$ of order 0 on $X$ in the Hörmander class $\rho = 1, \delta = 0$. We assume that its symbol $\sigma_A$ satisfies the conditions of the definition of transverse ellipticity. Then this symbol is a bounded operator on the Hilbert module $\mathfrak{S}_G(E)$ and $f(\sigma_A^2 - 1) \in \mathfrak{S}_G(E)$ for any $f \in C_0(X)$. It allows to define an element $[\sigma_A] \in K_0(C^*(G, \mathfrak{S}_G(X)))$.

On the other hand, as noticed by P. Julg in the case of compact $G$ and $X$, the natural covariant representation of the algebra $C^*(G, C_0(X))$ on $L^2(E)$, together with the transversally elliptic operator $A$ of order 0, define an element of the dual $K$-homology group $K^0(C^*(G, C_0(X)))$. This remains true for the non-compact $G$ and $X$.

In order to state the index theorem we need certain $K$-theory elements. We will use $KK$-theory and its small modification suggested by G. Skandalis, $KK_{sep}$, in order to deal with non-separable algebras. Note that the algebra $\mathfrak{S}_G(X)$ is not separable. For simplicity we omit the subscrip *sep*.

We will use two canonical elements: the Dirac element

$$[d_{X,G}] \in K^0(C^*(G, \mathfrak{S}(X)))$$

and the local dual Dirac element

$$[\Theta_{X,G}] \in K_0(C^*(G, C_0(X)) \otimes C^*(G, \mathfrak{S}(X))).$$

We can now state the

**Inverse Index Theorem.**

$$[\sigma_A] = [\Theta_{X,G}] \otimes_{C^*(G,C_0(X))} [A] \in K_0(C^*(G, \mathfrak{S}(X))).$$

Finally, to state the index theorem, we transform the symbol $[\sigma_A]$ into an element $[\tilde{\sigma}_A] \in KK_0(C^*(G, C_0(X)), C^*(G, \mathfrak{S}(X)))$ by a certain natural homomorphism.

**Index Theorem.** Let $X$ be a complete Riemannian manifold and $G$ a Lie group which acts on $X$ properly and isometrically. Let $A$ be a properly supported $G$-invariant transversally elliptic operator on $X$ of order 0. Then

$$[A] = [\tilde{\sigma}_A] \otimes_{C^*(G,\mathfrak{S}(X))} [d_{X,G}] \in K^0(C^*(G, C_0(X))).$$

The proof of the Inverse Index Theorem uses a certain rotation homotopy in the neighborhood of the diagonal of $X \times X$. The Index Theorem follows from the Inverse one.

It is important to note that there is Poincare duality between the symbol and the index (such duality also exists in the elliptic case). Poincare duality here is the isomorphism between the groups $K_0(C^*(G, \mathfrak{S}(X)))$ and $K^0(C^*(G, C_0(X)))$. It maps the symbol into the index. So if the symbol and the index elements are defined as above, they carry the same information.

Note also that one can define a Clifford symbol of a transversally elliptic operator. This can be done by replacing the algebra $\mathfrak{S}_G(X)$ with a much smaller (and separable) algebra $C_{\tau\oplus\mathfrak{g}}(X)$ which is $KK$-equivalent to $\mathfrak{S}_G(X)$. The above index theorems and the Poincare duality remain true after this replacement.

## Geometric quantization, limits, and restrictions– some examples for elliptic and nilpotent orbits

Toshiyuki Kobayashi

The Kirillov–Kostant–Duflo orbit philosophy relates the set of equivalence classes of irreducible unitary representations of a Lie group $G$ with the set of coadjoint orbits. Our expectation is that this correspondence is given by a "geometric quantization":

$$Q : \mathfrak{g}^* / \operatorname{Ad}^*(G) \to \widehat{G}, \tag{1}$$

satisfying functorial properties (e.g. $[Q,R]=0$, $[Q,\mathrm{Limit}]=0$). This works perfectly for simply connected nilpotent $G$. However, for reductive $G$, there is no reasonable bijection between $\widehat{G}$ and $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ (or its subset requiring some integrality conditions). Nevertheless we know more or less what $Q$ should be for semisimple orbits. For example, $Q(\mathcal{O}^G)$ is realized in a certain Dolbeault cohomology group on $\mathcal{O}^G$ for an integral elliptic orbit $\mathcal{O}^G$, and $Q(\mathcal{O}^G)$ is given by a (classical) parabolic induction for a hyperbolic orbit $\mathcal{O}^G$.

Let $H$ be a subgroup of $G$, $\mathfrak{h}\subset\mathfrak{g}$ their Lie algebras, and $\mathrm{pr}:\mathfrak{g}^*\to\mathfrak{h}^*$ the restriction map. Take any coadjoint orbit $\mathcal{O}^G\subset\mathfrak{g}^*$. Then the natural inclusion $\iota:\mathcal{O}^G\hookrightarrow\mathfrak{g}^*$ gives the momentum map of the Hamiltonian action of $G$ on $\mathcal{O}^G$ endowed with the Kirillov–Kostant–Souriau symplectic form, and the composition $\mu:=\mathrm{pr}\cdot\iota:\mathcal{O}^G\to\mathfrak{h}^*$ gives that for $H$.

For a coadjoint orbit $\mathcal{O}^H\subset\mathfrak{h}^*$, we set

$$n(\mathcal{O}^G,\mathcal{O}^H):=\#(\mu^{-1}(\mathcal{O}^H)/H)=(\mathcal{O}^G\cap\mathrm{pr}^{-1}(\mathcal{O}^H))/H.$$

Our concern is with the case where $G$ and $H$ are non-compact reductive groups. For $\mathcal{O}^G$ such that $Q(\mathcal{O}^G)\in\widehat{G}$ is well-defined, we raise:

**Conjecture 1.** (1) *The restriction of the unitary representation* $Q(\mathcal{O}^G)|_H$ *is multiplicity-free, namely, the ring* $\operatorname{End}_H(Q(\mathcal{O}^G))$ *is commutative if*

$$n(\mathcal{O}^G,\mathcal{O}^H)\le 1 \quad \textit{for any } \mathcal{O}^H\in\mathfrak{h}^*/\operatorname{Ad}^*(H). \tag{2}$$

(2) *If* $\mathcal{O}^G_\lambda$ *is a family of coadjoint orbits with parameter* $\lambda$ *such that the restrictions* $Q(\mathcal{O}^G_\lambda)|_H$ *are multiplicity-free, then* (2) *holds for all* $\mathcal{O}^G_\lambda$.

We present some non-compact settings for Conjecture 1 (2), and show some evidence of the Conjecture. For a simple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, we set

$$\mathcal{C}^*_{\mathfrak{k}}:=([\mathfrak{k},\mathfrak{k}]+\mathfrak{p})^\perp\subset\mathfrak{g}^*.$$

We note $\mathcal{C}^*_{\mathfrak{k}}\neq 0$ iff $G/K$ is a Hermitian symmetric space. Assume that a coadjoint orbit $\mathcal{O}^G$ satisfies

$$\mathcal{O}^G\cap\mathcal{C}^*_{\mathfrak{k}}\neq\emptyset. \tag{3}$$

Let $\{\nu_1,\dots,\nu_k\}$ be the maximal set of strongly orthogonal set in $\Delta(\mathfrak{p}_+^{-\tau},\mathfrak{t}^\tau)$ (see [2] for more details). For $\mathbb{A}=\mathbb{Z}$ or $\mathbb{R}$, we define

$$\mathcal{C}^+_{\mathbb{A}}:=\{\sum_{j=1}^k a_j\nu_j : a_1\ge\cdots\ge a_k\ge 0,\ a_j\in\mathbb{A}\ (1\le j\le k)\}.$$

**Theorem $\mathbf{B_{hol}}$ and $\mathbf{B^Q_{hol}}$** ([2, 4])**.** *Suppose* $(G,H)$ *is a symmetric pair of holomorphic type. For any* $\mathcal{O}^G_\lambda$ *satisfying the condition* (3)*, we have:*

(1) $\mu:\mathcal{O}^G_\lambda\to\mathfrak{h}^*$ *is proper, and* $n(\mathcal{O}^G_\lambda,\mathcal{O}^H)\le 1$ *for any* $H$*-coadjoint orbit* $\mathcal{O}^H$ *in* $\mathfrak{h}^*$*. Further,* $n(\mathcal{O}^G_\lambda,\mathcal{O}^H)\neq 0$ *only if* $O^H$ *is elliptic. More precisely,*

$$\mu(\mathcal{O}^G_\lambda)=\coprod_{\mu\in\lambda+\mathcal{C}^+_{\mathbb{R}}}\mathcal{O}^H_\mu.$$

(2) *The restriction of the unitary representation* $Q(\mathcal{O}^G_\lambda)|_H$ *is discretely decomposable and multiplicity-free. More precisely,*

$$Q(\mathcal{O}^G_\lambda)|_H \simeq \sum^{\oplus}_{\mu\in\lambda|_{\mathfrak{t}^\tau}+\rho(\mathfrak{p}_+^{-\tau})+C_{\mathbb{Z}}^+} Q(\mathcal{O}^H_\mu) \qquad \text{(discrete direct sum).}$$

**Theorem $\mathbf{B_{anti}}$ and $\mathbf{B^Q_{anti}}$** ([2, 4])**.** *Suppose* $(G,H)$ *is a symmetric pair of anti-holomorphic type. For any* $\mathcal{O}^G_\lambda$ *satisfying the condition* (3)*, we have:*

(1) *The momentum map* $\mu : \mathcal{O}^G \to \mathfrak{h}^*$ *is not proper. Further,* $n(\mathcal{O}^G_\lambda, \mathcal{O}^H) \leq 1$ *for any* $H$*-coadjoint orbit* $\mathcal{O}^H$ *in* $\mathfrak{h}^*$*. More precisely,* $n(\mathcal{O}^G_\lambda, \mathcal{O}^H) \neq 0$ *if and only if* $\mathcal{O}^H$ *is hyperbolic. Hence,*

$$\mu(\mathcal{O}^G_\lambda) = \coprod_{\mu\in(\mathfrak{a}_\mathfrak{h})^*_+} \mathcal{O}^H_\mu.$$

(2) *The restriction* $Q(\mathcal{O}^G_\lambda)|_H$ *is decomposed only by continuous spectrum:*

$$Q(\mathcal{O}^G_\lambda)|_H \simeq \int_{(\mathfrak{a}_\mathfrak{h})^*_+} Q(\mathcal{O}^H_\mu) d\mu \qquad \text{(direct integral).}$$

A remarkable feature of Theorem $\mathrm{B_{anti}}$ is that the image $\mu(\mathcal{O}^G_\lambda)$ is independent of $\lambda$ in contrast to Theorem $\mathrm{B_{hol}}$.

The geometric quantization of nilpotent orbits is non-trivial. Observing that any nilpotent orbit $\mathcal{O}_{\mathrm{nilp}}$ can be approximated by semisimple orbits $\mathcal{O}_\nu$, we propose:

**Problem 1.** *Construct a representation* $Q(\mathcal{O}_{\mathrm{nilp}})$ *from the knowledge of geometric quantizations* $Q(\mathcal{O}_\nu)$ *for semisimple orbits that approach to* $\mathcal{O}_{\mathrm{nilp}}$*.*

Here is an example for which the idea works. Let $G = O(p,q)$, and set

$$f := E_{12} - E_{21},\ h := E_{1,p+q} + E_{p+q,1} \in \mathfrak{g}.$$

For a parameter $\nu > 0$, we introduce a family of minimal elliptic and hyperbolic orbits

$$\mathcal{O}^{\mathrm{ell}}_\nu := \mathrm{Ad}^*(G)(\nu f), \quad \mathcal{O}^{\mathrm{hyp}}_\nu := \mathrm{Ad}^*(G)(\nu h).$$

**Theorem C** ([6])**.**

$$\lim_{\nu\downarrow 0} \mathcal{O}^{\mathrm{hyp}}_\nu = \lim_{\nu\downarrow 0} \mathcal{O}^{\mathrm{ell}}_\nu = \mathcal{O}^{\mathrm{nilp}}_0 \cup \mathcal{O}_{\mathrm{min}} \cup \{0\}.$$

Here $\mathcal{O}^{\mathrm{hyp}}_\nu$, $\mathcal{O}^{\mathrm{ell}}_\nu$, and $\mathcal{O}^{\mathrm{nilp}}_0$ are hyperbolic, elliptic, and nilpotent orbits of dimension $2(p+q-2)$, and $\mathcal{O}_{\mathrm{min}}$ is the minimal nilpotent orbit. Then, we can construct $Q(\mathcal{O}_{\mathrm{min}})$ from the knowledge of $Q(\mathcal{O}^{\mathrm{hyp}}_\nu)$ or $Q(\mathcal{O}^{\mathrm{ell}}_\nu)$ as follows:

**Theorem $\mathbf{C}^Q$** ([6, 7])**.** *For* $p+q$ *even and* $p, q \geq 2$*, there exists the following two non-splitting exact sequences of* $G$*-modules:*

$$0 \to \varpi_{\mathrm{min}} \to Q(\mathcal{O}^{\mathrm{hyp}}_{-1}) \overset{\widetilde{\Delta}}{\to} Q(\mathcal{O}^{\mathrm{hyp}}_1) \to 0,$$

$$0 \to \varpi_{\mathrm{min}} \to Q(\mathcal{O}^{\mathrm{ell}}_{-1}) \to Q(\mathcal{O}^{\mathrm{ell}}_1) \to 0.$$

*Remark.* (1) The same representation $\varpi_{\min}$ appears as a subrepresentation of the two completely different representations $Q(\mathcal{O}_{-1}^{\mathrm{hyp}})$ and $Q(\mathcal{O}_{1}^{\mathrm{ell}})$.

(2) We have used $Q$ by a little abuse of notation, namely, as an "analytic continuation" of $Q$. We note that neither $Q(\mathcal{O}_{\pm 1}^{\mathrm{hyp}})$ nor $Q(\mathcal{O}_{-1}^{\mathrm{ell}})$ is unitarizable.

(3) The intertwining operator $\widetilde{\Delta}$ is given by the Yamabe operator in the conformal geometry (see [7]) for the pseudo-Riemannian manifold $\mathcal{O}_{1}^{\mathrm{hyp}} \simeq (S^{p-1} \times S^{q-1})/\mathbb{Z}_2$.

Finally we discuss a direct approach to get a quantization $Q(\mathcal{O}_{\min}^{G})$, namely, to construct an irreducible unitary representation from a real minimal nilpotent orbit $\mathcal{O}_{\min}^{G}$. Here is an optimistic approach:

**Approach.** *Find an appropriate Lagrangian submanifold $C$ of $\mathcal{O}_{\min}^{G}$, and construct an irreducible unitary representation $Q(\mathcal{O}_{\min}^{G})$ of $G$ on $L^2(C)$.*

We list some difficulties:

- The group $G$ cannot act geometrically on any such $C$.
- There does not exist any invariant polarization on $\mathcal{O}_{\min}^{G}$.
- For some group $G$, there is no candidate for $Q(\mathcal{O}_{\min}^{G})$.

However, we can give some affirmative results in the following setting:

**Theorem D and D$^{Q}$** ([1, 3])**.** *Suppose $G$ is the conformal group of any real simple Jordan algebra $V$. Then $C := \mathcal{O}_{\min}^{G} \cap V$ is Lagrangian in $\mathcal{O}_{\min}^{G}$, and the above approach works for an appropriate covering of $G$ except for $\mathfrak{g} \simeq \mathfrak{so}(p,q)$ ($p+q$ odd).*

A generalized Fourier transform is studied in details in [3].

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## Quantization of the moduli space of flat $G$-bundles using $G$-valued moment maps

Derek Krepski

Let $G$ be a compact, connected Lie group and let $\Sigma$ be a compact oriented surface of genus $g$ with $r$ boundary components. Recall from [AMM] that the moduli space $M_G(\Sigma; \mathbb{C}_1, \ldots, \mathbb{C}_r)$ of flat $G$ bundles over $\Sigma$ with boundary holonomies in prescribed conjugacy classes $\mathbb{C}_j \subset G$ can be realized as a symplectic quotient of a quasi-Hamiltonian $G$-space:

$$M_G(\Sigma; \mathbb{C}_1, \ldots, \mathbb{C}_r) = (G^{2g} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_r)//G.$$

That is, $M_G(\Sigma; \mathbb{C}_1, \ldots, \mathbb{C}_r) = \Phi^{-1}(e)/G$ where $\Phi : X = G^{2g} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_r \to G$ is the *group-valued moment map* for the *quasi-Hamiltonian G-space X* given by

$$\Phi(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots c_r) = \prod_i a_i b_i a_i^{-1} b_i^{-1} \prod_j c_j.$$

Important examples of quasi-Hamiltonian $G$-spaces include conjugacy classes $\mathbb{C} \hookrightarrow G$ with moment map the inclusion (the quasi-Hamiltonian counterpart to coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ in the ordinary setting of Hamiltonian group actions), and the double $D(G) = G \times G$ with moment map $(a, b) \mapsto aba^{-1}b^{-1}$.

In a recent paper [M2], E. Meinrenken has defined the quantization $\mathbb{Q}(M) \in R_k(G)$, the level $k$ fusion ring (or Verlinde algebra), for (pre-quantized) quasi-Hamiltonian $G$-spaces $M$. Several basic properties are established, including

(1) $\mathbb{Q}(M_1 \times M_2) = \mathbb{Q}(M_1)\mathbb{Q}(M_2)$ (compatibility with fusion products), and
(2) $\mathbb{Q}(M//G) = \mathbb{Q}(M)^G$ (quantization commutes with reduction).

**Problem**: In the framework of quasi-Hamiltonian $G$-spaces, compute $\mathbb{Q}(M_G(\Sigma; \mathbb{C}_1, \ldots, \mathbb{C}_r))$.

When $G$ is simply connected, such a computation is carried out in [M2]. For $G$ non-simply connected, there are some complications.

For example, there are obstructions to pre-quantization if $\pi_1(G)$ is non-trivial. Indeed, suppose $G$ is simple. Then in [K1] it is shown that the obstruction to level $k$ pre-quantization is a cohomology class $\tilde{\phi}^*(kx) \in H^3(G \times G; \mathbb{Z})$, where $\tilde{\phi} : G \times G \to \tilde{G}$ is the canonical lift of the group commutator $\phi : G \times G \to G$ to the universal covering group $\tilde{G}$, and $x \in H^3(\tilde{G}; \mathbb{Z}) \cong \mathbb{Z}$ represents a generator. This obstruction vanishes precisely when $k$ is a multiple of $l_0$, displayed in Table 1 for every *non-simply connected* compact simple Lie group $G$.

| $G$ | $SU(n)/\mathbb{Z}_k$ $n \geq 2$ | $PSp(n)$ $n \geq 1$ | $SO(n)$ $n \geq 7$ | $PO(2n)$ $n \geq 4$ | $Ss(4n)$ $n \geq 2$ | $PE_6$ | $PE_7$ |
|---|---|---|---|---|---|---|---|
| $l_0$ | $\mathrm{ord}_k(\frac{n}{k})$ | 1, $n$ even<br>2, $n$ odd | 1 | 2, $n$ even<br>4, $n$ odd | 1, $n$ even<br>2, $n$ odd | 3 | 2 |

TABLE 1. The integer $l_0$. Notation: $\mathrm{ord}_k(x)$ denotes the order of $x$ mod $k$ in $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

*Remark*: The numbers appearing in Table 1 also appear in the paper of Toledano-Laredo [TL] but in a different context. Specifically, there is an integer $l_b$ defined in [TL] ($l_b$ = smallest integer $l$ for which the restriction to the integer lattice of $G$

of $lB(-,-)$, where $B$ is the basic inner product on $\mathfrak{g}$, is integral.) The integer $l_b$ is then used to help classify central extensions

$$1 \to S^1 \to \widehat{LG} \to LG \to 1.$$

It was observed *by inspection* that $l_b = l_0$. A partial explanation for the coincidence can be found in [K2], where an explicit pre-quantum line bundle is constructed using $l_b$ in the context of Hamiltonian loop group actions.

Specializing to the case $G = SO(3)$, explicit computations for $\mathbb{Q}(D(SO(3))) \in R_k(SU(2))$ can be found in [M1]. Further calculations involving conjugacy classes $\mathbb{C} \subset SO(3)$ have also been obtained [MK]. Similar computations for other non-simply connected $G$ are the subject of future work.

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## On the image of real loci of symplectic manifolds under moment maps


Liviu Mare

(joint work with Lisa Jeffrey)


Let $(M, \omega)$ be a connected symplectic manifold equipped with a Hamiltonian action of a torus $T$ and let $\Phi : M \to \mathfrak{t}^*$ be a moment map, where $\mathfrak{t} := \mathrm{Lie}(T)$. Let also $\tau$ be an automorphism of $M$, which is involutive, i.e. it satisfies $\tau \circ \tau = \mathrm{id}_M$. We also assume that $\tau$ is anti-symplectic and compatible with the $T$-action. That is, we have

$$\tau^*\omega = -\omega$$

$$\tau(g.m) = g^{-1}.\tau(m), \text{ for all } g \in T, m \in M.$$

We denote by $M^\tau$ the fixed point set of $\tau$. The following result has been proved by Duistermaat, see [2, Theorem 2.5]:

**Theorem 1.** (Duistermaat) *If $M$ is compact and $M^\tau \neq \emptyset$ then we have*

$$\Phi(M^\tau) = \Phi(M).$$

This result has been generalized by Hilgert, Neeb, and Plank [3], [4] and O'Shea and Sjamaar [6] to the case when the moment map $\Phi$ is proper. Our goal is to present a generalization of Theorem 1 to the context of non-proper moment maps. We are motivated by the following example.

**Example.** (see [7, Counterexample 0.1]) We consider the space $M := \mathbb{C}^2 \setminus (\mathbb{D} \times \mathbb{D})$, where $\mathbb{D}$ is the closed unit disk in $\mathbb{C}$. It is an open subset of $\mathbb{C}^2$, invariant under the canonical action of the torus $T^2$ on $\mathbb{C}^2$. The corresponding moment map is $\Phi : M \to \mathbb{R}^2$, $\Phi(z,w) = \frac{1}{2}(|z|^2, |w|^2)$, where we have identified the dual vector space of $\mathrm{Lie}(T^2)$ with $\mathbb{R}^2$. The main observation of [7] is that

$$\Phi(M) = \{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0 \text{ and } y \geq 0\} \setminus \{(x,y) \in \mathbb{R}^2 \ : \ x \leq \frac{1}{2} \text{ and } y \leq \frac{1}{2}\},$$

which is not a convex subset of $\mathbb{R}^2$. The reason is that $\Phi$ is not a proper map: indeed, the image of $\Phi$ is not a closed subset of $\mathbb{R}^2$. However, if $\tau : M \to M$ is given by $\tau(z,w) = (\bar{z}, \bar{w})$, then the assumptions above on $M, \omega, \tau$, and the $T^2$-action are satisfied and we have $\Phi(M^\tau) = \Phi(M)$. Let us finally observe that the map $\Phi : M \to \Phi(M)$ is a closed map (in fact, it is a proper map); moreover, all fibers of $\Phi$ are connected subspaces of $M$.

This is just a particular case of the following theorem, which is our main result.

**Theorem 2.** *Let $\tau$ be an anti-symplectic involutive automorphism of the symplectic manifold $(M, \omega)$ which is compatible with the $T$-action. Assume that the map $\Phi : M \to \Phi(M)$ is a closed map and all of its fibers are connected. If $M^\tau \neq \emptyset$, then $\Phi(M^\tau) = \Phi(M)$.*

To prove the theorem we will use the following result of Birtea, Ortega, and Ratiu, see [1, Corollary 2.18].

**Proposition 3.** (Birtea, Ortega, and Ratiu) *If $\Phi : M \to \Phi(M)$ is a closed map and all of its fibers are connected then the map $\Phi : M \to \Phi(M)$ is an open map.*

*Proof of Theorem 2.* Let us first note that $M^\tau$ is a closed subspace of $M$. Thus $\Phi(M^\tau)$ is a closed subspace of $\Phi(M)$. We now show that $\Phi(M^\tau)$ is an open subspace of $\Phi(M)$. Indeed, take $m \in M^\tau$. By [3, Proof of Theorem 2.3], there exists a $\tau$-invariant neighborhood $U$ of $m$ in $M$ such that $\Phi(U \cap M^\tau) = \Phi(U)$. By Proposition 3, the map $\Phi : M \to \Phi(M)$ is open. Thus, $\Phi(U \cap M^\tau)$ is open in $\Phi(M)$, i.e. it is an open neighborhood of $\Phi(m)$ in $\Phi(M)$. This implies that $\Phi(M^\tau)$ is open in $\Phi(M)$, as desired. Consequently, $\Phi(M^\tau)$ must be equal to $\Phi(M)$. $\square$

From Theorem 2 we can now deduce the following result, which seems to be new even in the case when $M$ is compact.

**Corollary 4.** *Under the hypotheses of Theorem 2, the map $\Phi|_{M^\tau} : M^\tau \to \Phi(M^\tau)$ is an open map.*

*Proof.* Let $V$ be an open subset of $M$ such that $V \cap M^\tau$ is non-empty. We will show that $\Phi(V \cap M^\tau)$ is open in $\Phi(M^\tau)$. Indeed, take $m \in V \cap M^\tau$. Let $U$ be a neighborhood of $m$ in $M$ which is $\tau$-invariant and satisfies $\Phi(U \cap M^\tau) = \Phi(U)$.

As before, such a $U$ exists by an argument which can be found in [3, Proof of Theorem 2.3]; moreover, we can assume that $U \subset V$. We deduce that

$$\Phi(U) = \Phi(U \cap M^\tau) \subset \Phi(V \cap M^\tau).$$

Since $\Phi(U)$ is open in $\Phi(M)$ and $\Phi(M) = \Phi(M^\tau)$ (see Theorem 2), we deduce that $\Phi(V \cap M^\tau)$ is a neighborhood of $\Phi(m)$ in $\Phi(M^\tau)$. Since $m$ was chosen arbitrarily, we deduce that $\Phi(V \cap M^\tau)$ is open in $\Phi(M^\tau)$, QED □

**Application.** Let $N$ be a symplectic manifold with a Hamiltonian action of a compact non-abelian Lie group $G$ and moment map $\Psi : N \to \mathfrak{g}^*$, where $\mathfrak{g} := \mathrm{Lie}(G)$. Assume that the map $\Psi$ is proper. Let $T \subset G$ be a maximal torus and denote its Lie algebra by $\mathfrak{t}$. We also consider a closed Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^* \subset \mathfrak{g}^*$ and denote its interior by $\mathrm{Int}(\mathfrak{t}^*_+)$. Recall that the principal face $\sigma_{\mathrm{prin}}$ of the latter polyhedral cone is the minimal face $\sigma$ with the property that the Kirwan polytope $\Psi(N) \cap \mathfrak{t}^*_+$ is contained in the closure of $\sigma$. Assume that $\sigma_{\mathrm{prin}} = \mathrm{Int}(\mathfrak{t}^*_+)$ (for a concrete situation when this assumption is fulfilled we refer to the example described in [8, Section 2.4]). By [5, Theorem 3.7], the pre-image $M := \Psi^{-1}(\mathrm{Int}(\mathfrak{t}^*_+))$ is a symplectic cross-section of the $G$-action, hence a symplectic submanifold, which is $T$-invariant. The moment map of the $T$-action on $M$ is $\Phi = \Psi|_M : M \to \mathfrak{t}^*$. The image of this map is $\mathrm{Int}(\mathfrak{t}^*_+) \cap \Psi(N)$, which is in general not a closed subset of $\mathfrak{t}^*$ (see again [8, Section 2.4]). Thus the map $\Phi : M \to \mathfrak{t}^*$ is in general not proper. However, the hypotheses in Theorem 2 are satisfied. Indeed, the map $\Phi : M \to \Phi(M)$ is proper, since for any $K \subset \Phi(M)$ which is compact, we have $\Phi^{-1}(K) = \Psi^{-1}(K)$. Any fiber of the map $\Phi : M \to \Phi(M)$ is also a fiber of $\Psi$, hence it is connected. Let us now consider an involution $\tau$ of $N$ which is antisymplectic and compatible with the $T$-action on $N$. The last condition implies that $\Psi \circ \tau = \Psi$ (cf. [2]). Thus $M$ is $\tau$-invariant. Theorem 2 implies that $\Phi(M^\tau) = \Phi(M) = \Psi(N) \cap \mathrm{Int}(\mathfrak{t}^*_+)$.

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## Toeplitz operators and geometric quantization

George Marinescu
(joint work with Xiaonan Ma)

The goal of this talk is to review the Berezin-Toeplitz quantization in the framework of spin$^c$ quantization for symplectic manifolds.

The aim of the geometric quantization theory of Kostant and Souriau is to relate the classical observables (smooth functions) on a phase space (a symplectic manifold) to the quantum observables (bounded linear operators) on the quantum space (sections of a line bundle). Toeplitz operators and more generally Toeplitz structures were introduced in geometric quantization by Berezin [1] and Boutet de Monvel-Guillemin [3]. Using the analysis of Toeplitz structures [3], Bordemann-Meinrenken-Schlichenmaier [2] and Schlichenmaier [10] gave asymptotic expansion for the composition of Toeplitz operators in the Kähler case.

We will first show that in the general symplectic case the kernel of the spin$^c$ operator is a good substitute for the space of holomorphic sections used in Kähler quantization. Then we present an approach to the asymptotic expansion of Toeplitz operators using kernel calculus and the off-diagonal asymptotic expansion of the Bergman kernel of Dai-Liu-Ma (based in turn on analytic localization techniques of Bismut-Lebeau). For more details and a global picture see the article [5].

### 1. Spectral gap of the spin$^c$ Dirac operator

Let $(X,\omega)$ be a compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$, with compatible almost complex structure $J : TX \to TX$. Let $g^{TX}$ be the associated Riemannian metric, $g^{TX}(u,v) = \omega(u,Jv)$. Let $(L,h^L,\nabla^L) \to X$ be Hermitian line bundle, endowed with a Hermitian metric $h^L$ and a Hermitian connection $\nabla^L$, whose curvature is $R^L = (\nabla^L)^2$. We assume that the *prequantization condition* is fulfilled:

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega\,. \tag{1}$$

Let $(E,h^E,\nabla^E) \to X$ be a Hermitian vector bundle. We will be concerned with asymptotics in terms of high tensor powers $L^p \otimes E$, when $p \to \infty$, that is, we consider the semi-classical limit $\hbar = 1/p \to 0$.

Let us consider the Clifford connection $\nabla^{\mathrm{Cliff}}$ on $\Lambda^{0,\bullet}T^*X$ (see e.g. [8, §. 1.3]). The connections $\nabla^L$, $\nabla^E$ and $\nabla^{\mathrm{Cliff}}$ induce the connection

$$\nabla_p = \nabla^{\mathrm{Cliff}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{L^p\otimes E} \quad \text{on } \Lambda^{0,\bullet}T^*X \otimes L^p \otimes E.$$

The *spin$^c$ Dirac operator* is defined by

$$D_p = \sum_{j=1}^{2n} \mathbf{c}(e_j)\nabla_{p,e_j} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E)\,. \tag{2}$$

where $(e_j)_{j=1}^{2n}$ local orthonormal frame of $TX$ and $\mathbf{c}(v)$ is the Clifford action of $v \in TX$.

If $(X, J, \omega)$ is Kähler then $D_p = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ so $\ker(D_p) = H^0(X, L^p \otimes E)$ for $p \gg 1$. The following result shows that $\ker(D_p)$ has all semiclassical properties of $H^0(X, L^p \otimes E)$. The proof is based on a direct application of the Lichnerowicz formula for $D_p^2$. Note that the metrics $g^{TX}$, $h^L$ and $h^E$ induce an $L^2$-scalar product on $\Omega^{0,\bullet}(X, L^p \otimes E)$, whose completion is denoted $(\Omega^{0,\bullet}_{(2)}(X, L^p \otimes E), \|\cdot\|_{L^2})$.

**Theorem 1.1** ([6, Th. 1.1, 2.5], [8, Th. 1.5.5])**.** *There exists $C > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \bigoplus_{k>0} \Omega^{0,k}(X, L^p \otimes E)$ we have*

$$\|D_p s\|^2_{L^2} \geq (2p - C)\|s\|^2_{L^2} \,. \tag{3}$$

*Moreover, the spectrum of $D_p^2$ verifies*

$$\operatorname{spec}(D_p^2) \subset \{0\} \cup [2p - C, +\infty[\,. \tag{4}$$

By the Atiyah-Singer index theorem we have for $p \gg 1$

$$\dim\ker(D_p) = \int_X \operatorname{Td}(T^{(1,0)}X)\operatorname{ch}(L^p \otimes E) = \operatorname{rank}(E)\,\frac{p^n}{n!}\int_X \omega^n + O(p^{n-1})\,. \tag{5}$$

Theorem 1.1 shows the forms in $\ker(D_p)$ concentrate asymptotically in the $L^2$ sense on their zero-degree component and (5) shows that $\dim\ker(D_p)$ is a polynomial in $p$ of degree $n$, as in the holomorphic case.

## 2. Toeplitz operators in spin$^c$ quantization

Let us introduce the orthogonal projection $P_p : \Omega^{0,\bullet}_{(2)}(X, L^p \otimes E) \longrightarrow \ker(D_p)$, called the Bergman projection in analogy to the Kähler case. Its integral kernel is called *Bergman kernel*. The *Toeplitz operator* with symbol $f \in \mathcal{C}^\infty(X, \operatorname{End}(E))$ is

$$T_{f,p} : \Omega^{0,\bullet}_{(2)}(X, L^p \otimes E) \to \Omega^{0,\bullet}_{(2)}(X, L^p \otimes E)\,, \quad T_{f,p} = P_p f P_p$$

A draw-back is that $T_{f,p} \circ T_{g,p} \neq T_{fg,p}$. However, equality holds in the asymptotic sense. To make this precise we introduce the following definition. A (generalized) *Toeplitz operator* is a sequence $(T_p)$ of linear operators $T_p \in \operatorname{End}(\Omega^{0,\bullet}_{(2)}(X, L^p \otimes E))$ verifying $T_p = P_p\, T_p\, P_p$, such that there exist a sequence $g_l \in \mathcal{C}^\infty(X, \operatorname{End}(E))$ with the property that for all $k \geq 0$, there exists $C_k > 0$ so that

$$\Big\| T_p - \sum_{l=0}^{k} T_{g_l,p}\, p^{-l} \Big\| \leq C_k\; p^{-k-1} \qquad \text{for any } p \in \mathbb{N}^*, \tag{6}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators.

We express (6) symbolically by $T_p = P_p\big(\sum_{l=0}^k p^{-l} g_l\big)P_p + O(p^{-k-1})$. If this holds for any $k \in \mathbb{N}$, then we write $T_p = P_p\big(\sum_{l=0}^\infty p^{-l} g_l\big)P_p + O(p^{-\infty})$.

By the Bergman kernel expansion of Dai-Liu-Ma we obtain the expansion of the integral kernels of $T_{f,p}$, $T_{g,p}$ and hence of $T_{f,p} \circ T_{g,p}$. We check then that $T_p := T_{f,p} \circ T_{g,p}$ satisfies the characterization of Toeplitz operators [7, Th. 4.9], [8, Lemmas 7.2.2, 7.2.4, Th. 7.3.1] in terms of the off-diagonal asymptotic expansion of their integral kernels. We obtain thus the following.

**Theorem 2.1** ([7, Th. 1.1], [8, Th. 7.4.1]). *Let $f, g \in \mathcal{C}^\infty(X, \mathrm{End}(E))$. The composition $(T_{f,\,p} \circ T_{g,\,p})$ is a Toeplitz operator, i.e.,*

$$T_{f,\,p} \circ T_{g,\,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),\,p} + O(p^{-\infty}), \tag{7}$$

*where $C_r$ are bidifferential operators, $C_0(f,g) = fg$ and $C_r(f,g) \in \mathcal{C}^\infty(X, \mathrm{End}(E))$. Let $f, g \in \mathcal{C}^\infty(X)$ and let $\{\cdot,\cdot\}$ be the Poisson bracket on $(X, \omega)$. Then*

$$\big[T_{f,\,p}\,, T_{g,\,p}\big] = \frac{\sqrt{-1}}{p} T_{\{f,g\},\,p} + O(p^{-2}). \tag{8}$$

In view of Theorem 2.1 we define an associative star-product on $\mathcal{C}^\infty(X, \mathrm{End}(E))$

$$f * g := \sum_{k=0}^{\infty} C_k(f,g) \hbar^k \in \mathcal{C}^\infty(X, \mathrm{End}(E))[[\hbar]]\,, \quad f, g \in \mathcal{C}^\infty(X, \mathrm{End}(E))\,, \tag{9}$$

where $C_r(f,g)$ are determined by (7). This is the *Berezin-Toeplitz star product*. We denote by Ric the Ricci curvature of $(X, g^{TX})$ and set $\mathrm{Ric}_\omega = \mathrm{Ric}(J\cdot,\cdot)$.

**Theorem 2.2** ([9]). *Assume that $(X, \omega)$ is Kähler and let $f, g \in \mathcal{C}^\infty(X, \mathrm{End}(E))$. Then $C_1(f,g) = -\frac{1}{2\pi}\langle \nabla^{1,0} f, \overline{\partial}^E g\rangle_\omega$. If $f, g \in \mathcal{C}^\infty(X)$, then*

$$C_2(f,g) = \frac{1}{8\pi^2}\langle D^{1,0}\partial f, D^{0,1}\overline{\partial} g\rangle + \frac{\sqrt{-1}}{4\pi^2}\langle \mathrm{Ric}_\omega, \partial f \wedge \overline{\partial} g\rangle - \frac{1}{4\pi^2}\langle \partial f \wedge \overline{\partial} g, R^E\rangle_\omega\,. \tag{10}$$

This result has been used by J. Fine [4] for the quantization of Mabuchi energy.

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## Geometric quantisation commutes with reduction

Varghese Mathai
(joint work with Weiping Zhang)

In 2005, Hochs and Landsman [3] proposed a generalization of the Guillemin–Sternberg [2] conjecture to the case of locally compact symplectic manifolds and locally compact Lie groups, using some of the ideas surrounding the Baum–Connes conjecture. Indeed, Landsman viewed this as a consequence of the functoriality of quantization. Let $(M,\omega)$ be a symplectic manifold with a Hamiltonian action of a Lie group $G$. That is, for all $V \in \mathfrak{g}$, there is a smooth function $f_V$ satisfying $i_V\omega = df_V$. Then $(M,\omega)$ has a moment map, $\Phi : M \longrightarrow \mathfrak{g}^*$ defined as $\Phi(m)(V) = f_V(m)$. The Marsden–Weinstein quotient $(M_G, \omega_G)$, i.e.

$$M_G = \Phi^{-1}(0)/G,$$

turns out to be a symplectic manifold if 0 is a regular value of $\Phi$. Suppose that $[\omega] \in H^2(M,\mathbb{Z})$. Recall the quantisation $Q$ commutes with reduction diagram, where $R_C$ denotes classical reduction and $R_Q$ quantum reduction:

$$\begin{array}{ccc} (G \circlearrowleft M,\omega) & \xrightarrow{Q} & G \circlearrowleft Q(M,\omega) \\ {\scriptstyle R_C}\big\downarrow & & \big\downarrow{\scriptstyle R_Q} \\ (M_G,\omega_G) & \xrightarrow{Q} & Q(M_G,\omega_G) \end{array}$$

The commutativity of the diagram boils down to the equality,

$$\dim(Q(M,\omega)^G) = \dim(Q(M_G,\omega_G)).$$

To define the quantum spaces, first note that by assumption, there is a prequantum complex line bundle $L \to M$ whose first Chern class is $[\omega]$. In fact, $[\omega + \Phi] \in H^2_G(M,\mathbb{Z})$, so that $L$ is a $G$-equivariant line bundle over $M$. Then $M$ has a G-invariant $Spin^{\mathbb{C}}$-structure and an equivariant $Spin^{\mathbb{C}}$-Dirac operator denoted $\eth^L_M$, with equivariant index,

$$\text{index}_G(\eth^L_M) = [\ker(\eth^{L+}_M)] - [\ker(\eth^{L-}_M)] \in R(G),$$

where $R(G)$ denotes the representation ring of $G$. Define the quantum space,

$$Q(M,\omega) = \text{index}_G(\eth^L_M) \in R(G).$$

The equivariant line bundle $L$ descends to a line bundle $L_G$ over $M_G$, which inherits a $Spin^{\mathbb{C}}$-structure and so an $Spin^{\mathbb{C}}$-Dirac operator denoted $\eth^{L_G}_{M_G}$. The index is,

$$\text{index}(\eth^{L_G}_{M_G}) = \ker(\eth^{L_G+}_{M_G}) - \ker(\eth^{L_G-}_{M_G})$$

Define the quantum space,

$$Q(M_G,\omega_G) = \text{index}(\eth^{L_G}_{M_G}).$$

Then the *Guillemin-Sternberg conjecture* is:-

$$\dim\left(\text{index}_G(\eth^L_M)^G\right) = \dim(\text{index}(\eth^{L_G}_{M_G}))$$

Equivalently, the following diagram commutes:-

$$\begin{array}{ccc}
(G \circlearrowleft M, \omega) & \xrightarrow{Q} & G \circlearrowleft Q(M,\omega) \in R(G) \\
{\scriptstyle R_C}\downarrow & & \downarrow{\scriptstyle R_Q} \\
(M_G, \omega_G) & \xrightarrow{Q} & Q(M_G, \omega_G) \in \mathbb{Z}
\end{array}$$

This was first proved by Meinrenken, with partial results by others earlier, including by Guillemin-Stenberg who established the Kähler case. Alternate proofs were later given by Vergne, Paradan and also by Tian-Zhang [5]. Using standard ideas from the context of the Baum-Connes conjecture, Hochs-Landsman defined a generalization of the Guillemin-Sternberg conjecture for noncompact groups $G$ and noncompact manifolds $M$. Additionally, one assumes that the Hamiltonian action $G \circlearrowleft (M, \omega)$ is *proper* and *cocompact*, (i.e. $M/G$ is compact). The only changes are:-

- The representation ring $R(G)$ is replaced by $K_0(C^*(G))$, i.e. the usual $K_0$-group of the group $C^*$-algebra of $G$.
- The equivariant index, $\text{index}_G(\eth_M^L) \in R(G)$ is replaced by, $\mu_M^G([\eth_M^L]) \in K_0(C^*(G))$, where

  $$\mu_M^G : K_0^G(M) \to K_0(C^*(G))$$

  is the *analytic assembly map*, $K_0^G(M)$ is the *equivariant analytical $K$-homology group* defined by $G \circlearrowleft M$, and $[\eth_M^L]$ is the class in $K_0^G(M)$ of the $Spin^{\mathbb{C}}$ Dirac operator $\eth_M^L$.
- Therefore, the quantisation of the unreduced space $(G \circlearrowleft M, \omega)$ is now given by

$$Q(M,\omega) = \text{index}_G(\eth_M^L) \in K_0(C^*(G)), \tag{1}$$

  where

  $$\text{index}_G(\eth_M^L) := \mu_M^G([\eth_M^L])$$

  purely as a matter of notation.
- The map $R_Q : R(G) \to \mathbb{Z}$ is replaced by the map

  $$R_Q = \left(\textstyle\int_G\right)_* : K_0(C^*(G)) \to \mathbb{Z}$$

  functorially induced by map

  $$\int_G : C^*(G) \to \mathbb{C}$$

  given by

  $$f \mapsto \int_G f(g)\, dg$$

  (defined on $f \in L^1(G)$ or $f \in C_c(G)$ and extended to $f \in C^*(G)$ by continuity). Here we make the usual identification of $K_0(\mathbb{C})$ with $\mathbb{Z}$.

Let $G$ be a unimodular Lie group, let $(M, \omega)$ be a symplectic manifold, and let $G \circlearrowright M$ be a proper strongly Hamiltonian action. Suppose 0 is a regular value of the momentum map $\Phi$. Suppose that the action is cocompact and admits an equivariant prequantum line bundle $L$. Assume there is an almost complex structure $J$ on $M$ compatible with $\omega$. Let $\eth_M^L$ be the Dirac operator on $M$ associated to $J$ and coupled to $L$, and let $\eth_{M_G}^{L_G}$ be the Dirac operator on the reduced space $M_G$, coupled to the reduced line bundle $L_G$. Then

$$\left(\textstyle\int_G\right)_* \circ \mu_M^G \left[\eth_M^L\right] = \mathrm{index}\left(\eth_{M_G}^{L_G}\right).$$

Equivalently, the following diagram commutes:-

$$\begin{array}{ccc} (G \circlearrowright M, \omega) & \xrightarrow{\;Q\;} & G \circlearrowright Q(M, \omega) \in K_0(C^*(G)) \\ {\scriptstyle R_C}\big\downarrow & & \big\downarrow{\scriptstyle R_Q} \\ (M_G, \omega_G) & \xrightarrow{\;Q\;} & Q(M_G, \omega_G) \in \mathbb{Z} \end{array}$$

**Theorem** (Mathai-Zhang [4], quantization commutes with reduction)**.** *Under the hypotheses of the Hochs-Landsman conjecture, there is a positive integer $p_0$ such that for all integers $p$ such that $p \geq p_0$, one has*

$$\left(\textstyle\int_G\right)_* \circ \mu_M^G \left[\eth_M^{L^p}\right] = \mathrm{index}\left(\eth_{M_G}^{L_G^p}\right).$$

*When $G$ has an Ad-invariant metric, then we can take $p_0 = 1$.*

This essentially proves the Hochs-Landsman conjecture. In the case when $G$ is a finitely generated discrete group, this theorem gives an extremely general index theorem for orbifolds. The idea of the proof is to use a variant of Witten deformation, adapting the technique in [5]. Due to lack of space, we will not give a sketch of proof, but we mention here that a key step is an interesting new index theorem that we prove in our context. Roughly speaking, in a special case it says:-

**Theorem** (Mathai-Zhang [4])**.** *Under the hypotheses of the Hochs-Landsman conjecture, for any $G$-equivariant first order elliptic differential operator $D$ acting on $\Omega^{0,\bullet}(M, L^p)$, then the induced operator*

$$D^G : \Omega^{0,\bullet}(M, L^p)^G \longrightarrow \Omega^{0,\bullet}(M, L^p)^G$$

*is a Fredholm operator.*

For a precise statement, we refer to [4]

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## Verlinde Formulas for non-simply connected groups

Eckhard Meinrenken

The purpose of this talk is to describe an application of the recently developed *quantization of group-valued moment maps* [6] to Verlinde-type formulas for the quantization of moduli spaces. The talk is related to, and in part based on, joint work with A. Alekseev and C. Woodward from over 10 years ago [2], and with D. Krepski from less than 10 months ago [5].

### 1. Notation

Let $G$ be a compact, simple, simply connected Lie group, with maximal torus $T$. The Lie algebras are denoted $\mathfrak{g}, \mathfrak{t}$. We denote by $Q \subset P \subset \mathfrak{t}^*$ the root lattice and the weight lattice, and by $Q^\vee \subset P^\vee \subset \mathfrak{t}$ the co-root lattice and the co-weight lattice. (We work with *real* weights and co-weights; these differ from the complex weights by factors of $2\pi i$.) Then

$$Z(G) = P^\vee/Q^\vee \subset T = \mathfrak{t}/Q^\vee.$$

Let $\mathfrak{t}_+$ be the choice of a fundamental Weyl chamber, $\alpha_{\max} \in Q$ the highest root, and

$$\Delta = \{\xi \in \mathfrak{t}_+ |\ \langle \alpha_{\max}, \xi \rangle \leq 1\}$$

the fundamental alcove. The alcove parametrizes conjugacy classes in $G$, in the sense that for any conjugacy class $\mathcal{C}$ there is a unique $\xi \in \Delta$ such that $\exp(\xi) \in \mathcal{C}$. The center $Z(G)$ acts on the set of conjugacy classes by translation, this defines an action on $\Delta$. The basic inner product on $\mathfrak{g}$ is the unique invariant inner product such that $||\alpha_{\max}^\vee|| = \sqrt{2}$. Use the inner product to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, hence $\mathfrak{t}$ with $\mathfrak{t}^*$. One knows that $Q^\vee \subset P$ under this identification. Let

$$P_+ = P \cap \mathfrak{t}_+$$

be the dominant weights. The representation ring $R(G)$, viewed as a ring of characters, has basis the characters $\chi_\lambda$ of irreducible representations of highest weight $\lambda \in P_+$. The intersection

$$P_k = P \cap k\Delta$$

are the *level $k$ weights.* For all $\lambda \in P_k$, the points $(\lambda + \rho)/(k + h^\vee)$, where $\rho$ is the half-sum of positive roots and $h^\vee = 1 + \langle \rho, \alpha_{\max}^\vee \rangle$ is the dual Coxeter number, lie in the interior of $\Delta$, hence

$$t_\lambda = \exp(\frac{\lambda + \rho}{k + h^\vee}) \in T$$

are regular elements of the maximal torus. The *level $k$ fusion ring* (Verlinde algebra) is the quotient

$$R_k(G) = R(G)/I_k$$

where $I_k$ is the ideal of characters vanishing at all points $t_\lambda, \lambda \in P_k$. The images $\tau_\mu$, $\mu \in P_k$ of the characters $\chi_\mu$ form an additive basis of $R_k(G)$. The maps $R(G) \to \mathbb{C}$ given by evaluation at points $t_\lambda$ descend to the fusion ring, and any element of $R_k(G)$ is determined by these values. Hence the complexified ring $R_k(G) \otimes \mathbb{C}$ has another basis $\tilde{\tau}_\mu$, given by the conditions

$$\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}.$$

In the new basis, the fusion product is diagonalized: $\tilde{\tau}_\lambda \tilde{\tau}_{\lambda'} = \delta_{\lambda,\lambda'}\tilde{\tau}_\lambda$. The two basis are related by the *S-matrix*:

$$S_{\lambda\mu} = c \sum_{w \in W} (-1)^{l(w)} e^{-\frac{2\pi i}{k+h^\vee} w(\lambda+\rho)\cdot(\mu+\rho)}.$$

Here $c$ is a scalar, which can be chosen such that $S$ is a unitary matrix, and $S_{0,\lambda} > 0$. One has

$$\tilde{\tau}_\lambda = S_{\lambda 0} \sum_{\mu \in P_k} \overline{S}_{\lambda\mu} \tau_\mu.$$

## 2. Verlinde Formulas

Let $\Sigma$ be a surface of genus $g$ without boundary, and

$$\mathcal{M}_G(\Sigma) = \frac{\{\text{flat connections on } \Sigma \times G\}}{\text{gauge equivalence}}$$

the moduli space of flat $G$-connections. In terms of holonomies, it has a description $\mathcal{M}_G(\Sigma) = \Phi^{-1}(e)/G$ where $\Phi\colon G^{2g} \to G$ is the map

$$\Phi(a_1, b_1, \ldots, a_g, b_g) = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}.$$

The space $\Omega^1(\Sigma, \mathfrak{g})$ of connections on $\Sigma \times G$ carries a symplectic structure, given by pointwise inner product $\cdot$ followed by integration over $\Sigma$. As observed by Atiyah-Bott, this induces a symplectic structure on $\mathcal{M}_G(\Sigma)$, turning it into a singular symplectic space. Furthermore, there exists a pre-quantization at any integer level $k \in \mathbb{N}$, i.e. a pre-quantum line bundle $L$ with curvature the $k$-th multiple of the symplectic form. (For the purposes of this talk, we ignore the singularities - in reality we work with appropriate desingularizations.) We take the index of the Spin$_c$-Dirac operator on $M$ with coefficients in $L$ to be the quantization

$$\mathcal{Q}(M) = \text{index}(\partial\!\!\!/_L) \in \mathbb{Z}$$

for the given level $k$. According to the Verlinde formula [7],

$$\mathcal{Q}(M) = \sum_{\lambda \in P_k} S_{0,\lambda}^{2-2g}.$$

Our approach to the Verlinde formula uses the theory of group-valued moment maps [1]. A *q-Hamiltonian G-space* $(M, \omega, \Phi)$ is a $G$-manifold $M$ with an invariant 2-form $\omega$ and an equivariant map $\Phi\colon M \to G$ such that

(1) $\mathrm{d}\omega = -\Phi^*\eta, \quad \eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G),$

(2) $\iota(\xi_M)\omega = -\frac{1}{2}\xi \cdot \Phi^*(\theta^L + \theta^R)$,
(3) $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$,

with $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan forms. The first condition says that the pair $(\omega, \eta)$ defines a cocycle in relative cohomology for the map $\Phi$, and we define a *level $k$ pre-quantization* to be an integral lift of $k[(\omega, \eta)] \in H^3(\Phi, \mathbb{R})$. Given such a pre-quantization, the construction in [6] produces a push-forward map $\Phi_*\colon K^0_G(M) \to R_k(G)$. (This uses the interpretation of $R_k(G)$ in terms of twisted $K$-theory, due to Freed-Hopkins-Teleman.) We then define the quantization as

$$\mathcal{Q}(M) = \Phi_*(1) \in R_k(G).$$

Some properties are $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$ and the *quantization commutes with reduction* theorem $\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G)$, where $M/\!\!/G = \Phi^{-1}(e)/G$ is the symplectic quotient and the superscript $G$ signifies the coefficient of $\tau_0$. Furthermore, there is a localization formula

$$\mathcal{Q}(M)(t_\lambda) = \sum_{F \subset M^{t_\lambda}} \int_F \frac{\hat{A}(F)\,\mathrm{Ch}(\mathcal{L}_F, t_\lambda)^{1/2}}{D_\mathbb{R}(\nu_F, t_\lambda)}.$$

Without going into details, we remark that these are the usual Atiyah-Singer fixed point contributions for a certain $\mathrm{Spin}_c$-structure on $TM|_F$, with associated line bundle $\mathcal{L}_F$. As described in [6], these arise along fixed point manifolds even though $M$ itself need not carry a natural $\mathrm{Spin}_c$-structure, in general. As a special case, this applies to the *double* $D(G) = G \times G$ with $G$-action the conjugation action on each factor, and moment map $(a, b) = aba^{-1}b^{-1}$. The fixed point set of $t_\lambda$ is simply $F = T \times T$, since $t_\lambda$) is regular. It becomes straightforward to evaluate the fixed point contribution of $F$, and one obtains $\mathcal{Q}(D(G))(t_\lambda) = S_{0,\lambda}^{-2}$. Hence

$$\mathcal{Q}(D(G)) = \sum_{\lambda \in P_k} S_{0,\lambda}^{-2}\tilde{\tau}_\lambda \Rightarrow \mathcal{Q}(D(G)^g) = \sum_{\lambda \in P_k} S_{0,\lambda}^{-2g}\tilde{\tau}_\lambda$$
$$\Rightarrow \mathcal{Q}(D(G)^g/\!\!/G) = \sum_{\lambda \in P_k} S_{0,\lambda}^{2-2g}$$

where we used $\tilde{\tau}_\lambda^G = S_{0,\lambda}^2$. But $D(G)^g/\!\!/G = \mathcal{M}_G(\Sigma)$.

## 3. Fuchs-Schweigert formulas

In their 1999 paper [3], Fuchs and Schweigert considered generalizations of the Verlinde formulas to non-simply connected groups. As in our discussion of the Verlinde formulas, we will only consider the case without markings. Consider

$$G' = G/Z$$

where $Z$ is a subgroup of the center $Z(G)$. To simplify notation, we assume $Z = Z(G)$. The space $D(G')^g/\!\!/G'$ is the space of flat $G$-bundles, which is a disconnected space with components labeled by topological types of $G'$-bundles. The space of flat connections on the *trivial* bundle is, instead,

$$\mathcal{M}_{G'}(\Sigma) = D(G')^g/\!\!/G$$

where $D(G') = D(G)/(Z \times Z)$ is viewed as a q-Hamiltonian $G$-space (note that the $G$-moment map descends). By a theorem of D. Krepski [4], the space $D(G')$ is pre-quantizable at level $k$ if and only if $P^\vee \cdot P^\vee \subset \frac{1}{k}\mathbb{Z}$. The various inequivalent pre-quantizations are related by homomorphims $\psi \in \mathrm{Hom}(Z\times Z, \mathrm{U}(1))$. We found the following formula for the quantization of $D(G')$:

$$\mathcal{Q}(D(G')) = \frac{1}{|Z|^2} \sum_{\gamma=(\gamma_1,\gamma_2)\in Z^2} \phi(\gamma) \sum_{\lambda \in P_k^\gamma} S_{0,\lambda}^{-2} \tilde{\tau}_\lambda$$

Here $P_k^\gamma$ are the level $k$ weights that are fixed under the action of both $\gamma_1, \gamma_2 \in Z$ on $P_k$. (The action of $Z$ on $\Delta$ induces an action on level $k$ weights.) The most subtle point is the computation of the phase factor $\phi(\gamma) \in \mathrm{U}(1)$. It depends on the choice of pre-quantization, and is explictly given by

$$\phi(\gamma) = \psi(\gamma) e^{-2\pi i k\, ((1-w_*)^{-1}\zeta_1)\cdot\zeta_2},$$

where $\psi \in \mathrm{Hom}(Z^2, \mathrm{U}(1))$, the $\zeta_i \in \Delta$ exponentiate to $c_i$, and $w_* \in W$ is the Coxeter transformation. Following the argument for $\mathcal{M}_G(\Sigma)$, we now obtain

$$\mathcal{Q}(\mathcal{M}_{G'}(\Sigma)) = \frac{1}{|Z|^{2g}} \sum_{\gamma=(\gamma_1,\ldots,\gamma_{2g})\in Z^{2g}} \phi(\gamma_1, \ldots, \gamma_{2g}) \sum_{\lambda \in P_k^\gamma} S_{0,\lambda}^{2-2g}$$

where the phase factor is the product of the phase factors $\phi(\gamma_{2i-1}, \gamma_{2i})$ defined above. This is the formula conjectured by Fuchs-Schweigert. The more general Fuchs-Schweigert formulas with markings can be addressed similarly; for the case of $G' = \mathrm{SO}(3)$ this is fully worked out in [5].

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**From action-angle coordinates to geometric quantization: a round trip**

Eva Miranda

The philosophy of geometric quantization is to find and understand a "(one-way) dictionary" that "translates" classical systems into quantum systems . In this way, a quantum system is associated to a classical system in which observables (smooth functions) become operators of a Hilbert space and the classical Poisson bracket becomes the commutator of operators. In this process, the choice of additional geometric structures (polarizations) plays an important rôle. A desired property is that the quantization obtained does not depend on the polarization. Another rule in the game is that of keeping track of the symmetries on both sides. This is the deep link of geometric quantization with representation theory. The quantization commutes with reduction "principle" becomes realistic in some geometric quantization set-ups.

Our point of view in this big endeavour is very modest. We plan to construct a "representation space" in the case the polarization is given by a real polarization. For this, we follow the definition of Kostant of the representation spaces via higher cohomology groups with coefficients in the sheaf of flat sections. In this short note, we will not discuss either the (pre)Hilbert structure of this space nor the quantization rules.

## 1. Quantization via real polarizations

Let $(M^{2n}, \omega)$ be a symplectic manifold such that $[\omega]$ is integral. Under these circumstances (see for instance [14] or [6]), there exists a complex line bundle $\mathbb{L}$ with a connection $\nabla$ over $M$ such that $curv(\nabla) = \omega$. The symplectic manifold $(M^{2n}, \omega)$ is called prequantizable and the pair $(\mathbb{L}, \nabla)$ is called *a prequantum line bundle* of $(M^{2n}, \omega)$. In order to construct the representation space we need to restrict the space of sections to a subspace of sections which are flat in "priviledged" directions given by a polarization. In this note we will just consider a real polarization. A real polarization $\mathcal{P}$ is a foliation whose leaves are Lagrangian submanifolds. Integrable systems provide natural examples of real polarizations. If the manifold $M$ is compact the "moment map": $F : M^{2n} \to \mathbb{R}^n$ has singularities that correspond to *equilibria*. Consider the following:

**Example 1.1.** *Consider $M = S^1 \times \mathbb{R}$ and $\omega = dt \wedge d\theta$. Take as $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = td\theta$. Now, let $\mathcal{P} =< \frac{\partial}{\partial\theta} >$ then flat sections satisfy, $\nabla_X \sigma = X(\sigma) - i < \theta, X > \sigma$. Thus $\sigma(t, \theta) = a(t).e^{it\theta}$ and Bohr-Sommerfeld leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.*

This example shows that flat sections are not globally defined but they exist along a subset of leaves of the polarization. These are called Bohr-Sommerfeld leaves. The characterization of Bohr-Sommerfeld leaves for regular fibrations under some conditions is a well-known result by Guillemin and Sternberg ([4]). In particular the set of Bohr-Sommerfeld leaves is discrete and is given by "action" coordinates.

**Theorem 1.1** (Guillemin-Sternberg)**.** *If the polarization is a regular fibration with compact leaves over a simply connected base $B$, then the Bohr-Sommerfeld set is discrete and assuming that the zero-fiber is a Bohr-Sommerfeld leaf, the Bohr-Sommerfeld set is given by, $BS = \{p \in M, (f_1(p), \dots, f_n(p)) \in \mathbb{Z}^n\}$ where $f_1, \dots, f_n$ are global action coordinates on $B$.*

This result connects with Arnold-Liouville-Mineur theorem for action-angle coordinates for integrable systems. When we consider a toric manifolds the base $B$ may be identified with the image of the moment map by the toric action (Delzant polytope).

In view of the previous theorem, it would make sense to "quantize" these systems counting Bohr-Sommerfeld leaves. When the polarization is an integrable system with global action-angle coordinates, Bohr-Sommerfeld leaves are just *"integral"* Liouville tori. But why? Following the idea of Kostant [7], in the case there are no global sections denote by $\mathcal{J}$ the sheaf of flat sections along the polarization, we can then define the quantization as $\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M, \mathcal{J})$. Then quantization is given by precisely the following theorem of Sniatycki [13]:

**Theorem 1.2** (Sniatycki)**.** *If the leaf space $B^n$ is a Hausdorff manifold and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibres, then all the cohomology groups vanish except for degree half of the dimension of the manifold. Furthermore, $\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$, and the dimension of $H^n(M^{2n}, \mathcal{J})$ is the number of Bohr-Sommerfeld leaves.*

There are two different approaches to compute this sheaf cohomology:

(1) Using a fine resolution of the complex: Namely, we can define the sheaf: $\Omega^i_{\mathcal{P}}(U) = \Gamma(U, \wedge^i \mathcal{P})$. and $\mathcal{C}$ to be the sheaf of complex-valued functions that are locally constant along $\mathcal{P}$. Consider the natural (fine) resolution

$$0 \to \mathcal{C} \xrightarrow{i} \Omega^0_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^2_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \cdots$$

The differential operator $d_{\mathcal{P}}$ is the restriction of the exterior differential to the directions of the distributions (as in foliated cohomology). We can use this resolution to obtain a fine resolution of $\mathcal{J}$ by twisting the previous resolution with the sheaf $\mathcal{J}$.

(2) A different approach used in [2] and [5] is the one of Čech cohomology which turns out to be useful when we consider integrable systems with singularities.

1.1. **Applications to the general case of Lagrangian foliations.** This *fine resolution approach* can be useful to compute this geometric quantization for regular foliations (including those not coming from integrable systems like irrational slope on the torus).

In [9] we use the classification of foliations on the torus (Kneser-Denjoy-Schwartz theorem) together with basic properties of this sheaf cohomology to compute the geometric quantization of a torus. In the case of irrational slope we can compute the quantization (see [9]) and we obtain that the quantization space is always infinite dimensional. However, if we compute the limit case of the foliated cohomology ($\omega = 0$), we obtain that this foliated cohomology is finite dimensional if the irrationality measure of $\eta$ and is infinite dimensional if the irrationality measure of $\eta$ is infinite. The results contained in [9] seem to generalize a result of El Kacimi [8] for foliated cohomology.

Most computations in [9] rely on what we call *"geometric quantization computation kit"* (essentially a Künneth formula and a Mayer-Vietoris theorem in this context). This Künneth formula is very helpful to extend results to higher dimension by reduction to the 2-dimensional case (whenever the corresponding theorem for reduction also holds within the category of foliations considered).

## 2. Quantization using singular action-angle coordinates

Consider the case of rotations of the sphere. There are two leaves of the polarization which are singular and correspond to fixed points of the action. What happens if we go to the edges and vertexes of Delzant's polytope? This case and, more generally, that of toric manifolds was considered by Mark Hamilton in [2].

**Theorem 2.1** (Hamilton)**.** *For a $2n$-dimensional compact toric manifold and let $BS_r$ be the set of regular Bohr-Sommerfeld leaves, $\mathcal{Q}(M) = H^n(M;\mathcal{J}) \cong \bigoplus_{l\in BS_r}\mathbb{C}$*

Then this geometric quantization does not see the singular elliptic points. In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

In order to consider more general singularities, we need to review some results for normal forms of integrable system. The theorem of Guillemin-Marle-Sternberg gives normal forms in a neighbourhood of fixed points of a toric action. This can be generalized to normal forms of integrable systems (not always toric) that we call non-degenerate. A proof of this theorem in the elliptic case can be found in [1]. For the other cases see the author's thesis [10] where the idea of symplectic orthogonal decomposition is used and the paper [12].

**Theorem 2.2** (Eliasson-Miranda)**.** *There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.*

The local model is given by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega = \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$. and the components of the moment map are:

(1) Regular $f_i = p_i$ for $i = 1, ..., k$;
(2) Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;
(3) Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;
(4) focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1$, $j = 1, ..., k_f$.

We can use these models to compute geometric quantization in these cases. In the case of non-degenerate singularities in dimension 2 (only elliptic and hyperbolic singularities), we [5] obtain the following:

**Theorem 2.3** (Hamilton and Miranda)**.** *The quantization of a compact surface endowed with an integrable system with non-degenerate singularities is given by,*

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l \in BS_r} \mathbb{C} \ ,$$

*where $\mathcal{H}$ is the set of hyperbolic singularities.*

In particular, this theorem shows that this quantization depends strongly on the polarization (for more details see [5]).

2.1. **New directions.** The case of general non-degenerate singularities in higher dimensions is a joint work of the author with Romero Solha and uses the above-mentioned "geometric quantization computation kit" together with the results in [11] and [10]. For these singular real polarizations a "quantization commutes with reduction" principle seems to hold.

Finally, we have learned from the symplectic case that action-angle coordinates are useful to compute geometric quantization. We can use the existence of (partial) action-angle coordinates for Poisson manifolds (recently explored in [3]) to compute geometric quantization in the Poisson context. This is a joint project with Mark Hamilton.

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## Dynamical zeta functions and the spectrum of hyperbolic manifolds

Werner Müller

It is a basic problem of quantum mechanics to understand the relation between a classical Hamiltonian system and its quantization. In particular, one wants to understand the semi-classical limit as $\hbar \to 0$ for the quantized system. An example are the Bohr-Sommerfeld quantization rules, which apply to integrable systems. The opposite of integrable systems are Hamiltonian systems whose dynamics is "chaotic", which means that the flow when restricted to a fixed energy surface is ergodic and has almost everywhere positive Ljapunov exponents. The study of quantum systems with chaotic behavior of the underlying classical Hamiltonian system is the content of "quantum chaos". An example of a chaotic dynamical system is the geodesic flow on a manifold of negative curvature. In this talk we will discuss some aspects of these questions for the geodesic flow.

Let $(X, g)$ be a compact Riemannian manifold. Let $H(x, \xi)$ be the Hamiltonian on the cotangent bundle $T^*(X)$ which is defined by

$$H(x,\xi) = \frac{1}{2} \parallel \xi \parallel_x^2 .$$

It gives rise to the geodesic flow $\phi^t$ on the unit-cotangent bundle $S^*(X)$. Let $\Delta = d^* d$ denote the Laplace operator on $X$. Recall that in local coordinates it is given by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right),$$

where $g = \sum_{i,j=1}^n g_{ij} dx_i \otimes dx_j$, $g = \det(g_{ij})$, and $g^{kl}$ are the components of the inverse of the matrix $(g_{ij})$. By the usual quantization procedure (see [LL]), $\hbar^2 \Delta$ is the Hamilton operator of the quantized geodesic flow. It is well known that $\Delta$ is an essentially self-adjoint operator in $L^2(X)$ and its spectrum $\operatorname{Spec}(\Delta)$ consists of a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$$

of finite multiplicities with corresponding eigenstates $\phi_i$, $i \in \mathbb{N}_0$, which can be chosen to form an orthonormal basis of $L^2(X)$. On the classical side we have the length spectrum of closed geodesics

$$\mathcal{L}(X, g) := \{\ell(\gamma) \colon \gamma \text{ closed geodesic of } X\},$$

where $\ell(\gamma)$ denotes the length of $\gamma$. Part of the problem described above is to understand the relation between $\mathcal{L}(X, g)$ and $\operatorname{Spec}(\Delta)$. In this respect Chazarain [1] proved the following result. Let

$$u(t) = \operatorname{Tr}\left(e^{it\sqrt{\Delta}}\right),$$

where the trace is understood in the distributional sense. Then one has

$$\text{sing supp}(u) \subseteqq \mathcal{L}(X, g).$$

This was generalized by Duistermaat and Guillemin [2]. Moreover, they showed if there is only a finite number of periodic geodesics of period $T$: $\gamma_1, \cdots, \gamma_N$ and that for each $\gamma_i$ the Poincaré map $P_{\gamma_i}$ satisfies the Lefschetz condition $\det(I - P_\gamma) \neq 0$, then $u(t)$ is smooth in an interval $0 < |t - T| < a$ and

$$\lim_{t \to T} (t - T)u(t) = \sum_{i=1}^{N} \mathrm{Tr}(H_{\gamma_i}) \frac{|T|}{2\pi} (\sqrt{-1})^{\sigma_i} \left| \det(I - P_{\gamma_i}) \right|^{-1/2},$$

where $\sigma_i$ is the Maslov index of $\gamma_i$. In [5] Guillemin improved this result to a full asymptotic expansion. The coefficients are called wave trace invariants. They play an important role in spectral geometry.

For compact locally symmetric spaces the Selberg trace formula (STF) provides a much more precise relation between $\mathcal{L}(X, g)$ and $\mathrm{Spec}(\Delta)$. For example, consider a compact, oriented hyperbolic $n$-manifold $X = \Gamma \backslash \mathbb{H}^n$. Let $\varphi \in C_c^\infty(\mathbb{R})$ be even and $f = \hat{\varphi}$. Then the STF is the following identity

$$\sum_{j=0}^{\infty} f\left(\sqrt{\lambda_j - (n-1)^2/4}\right) = \int_{\mathbb{R}} f(\lambda)\, d\mu_{\mathrm{PL}}(\lambda) + \sum_{[\gamma] \neq e} \frac{\ell(\gamma_0)}{D(\gamma)} \varphi(\ell(\gamma)).$$

Here $d\mu_{\mathrm{PL}}$ denotes the Plancherel measure, $\ell(\gamma_0)$ is the length of the primitive closed geodesic associated to $\gamma$, $D(\gamma)$ is the discriminant of $\gamma$, and $[\gamma]$ runs over the non-trivial $\Gamma$-conjugacy classes (see [12] for details).

In order to study the relation between $\mathcal{L}(X, g)$ and $\mathrm{Spec}(\Delta)$ it is convenient to introduce zeta functions associated to these spectra. On the geometric side these are *dynamical zeta functions*. The first example of such a zeta function was first introduced by Selberg [11] – the Selberg zeta function of a closed hyperbolic surface $\Gamma \backslash \mathbb{H}^2$. Ruelle [10] introduced dynamical zeta functions in a more general context. The *Ruelle zeta function* $R(s)$ associated to the geodesic flow on a compact hyperbolic manifold $X = \Gamma \backslash \mathbb{H}^n$ is defined as

$$R(s) := \prod_{\substack{[\gamma] \neq e \\ \text{prime}}} (1 - e^{-s\ell(\gamma)}), \quad \mathrm{Re}(s) > n - 1.$$

Using the Selberg trace formula, one can show that $R(s)$ admits a meromorphic extension to the whole complex plane and satisfies a functional equation.

We study a more general type of zeta functions, called *twisted Ruelle zeta functions*. They arise as follows. Let $\rho \colon \Gamma \to \mathrm{GL}(V)$ be a finite-dimensional complex representation of $\Gamma$. Put

$$R(s; \rho) := \prod_{\substack{[\gamma] \neq e \\ \text{prime}}} \det\left(\mathrm{I} - \rho(\gamma) e^{-s\ell(\gamma)}\right)^{(-1)^{n-1}}.$$

The infinite product converges absolutely in some half-plane $\mathrm{Re}(s) > c$. Note that there is some similarity with the Artin $L$-function. Let $K/L$ be a Galois extension of number fields with Galois group $G = \mathrm{Gal}(K/L)$ and let $(V, \rho)$ be a representation of $G$. Then the Artin $L$-function is defined to be

$$L(s;\rho) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s;\rho).$$

Here $\mathfrak{p}$ runs over the prime ideals of $L$ and for an unramified $\mathfrak{p}$, the Euler factor is given by

$$L_{\mathfrak{p}}(s;\rho) := \det\left(\mathrm{I} - \rho(\varphi_{\mathfrak{P}/\mathfrak{p}})N(\mathfrak{p})^{-s}\right)^{-1},$$

where $\mathfrak{P}$ is any prime of $K$ which lies over $\mathfrak{p}$ and $\varphi_{\mathfrak{P}/\mathfrak{p}} \in G$ denotes the corresponding Frobenius element.

Our main result concerning $R(s;\rho)$ is the following theorem.

**Theorem 1.** *$R(s;\rho)$ admits a meromorphic extension to $\mathbb{C}$ and satisfies a functional equation.*

The existence of a meromorphic extension was proved by Fried [4] by different methods. Our method relies on an extension of the Selberg trace formula [7], by which we can handle non-unitary representations of the lattice $\Gamma$. This method also implies the functional equation.

As in the arithmetic case, it is interesting to study the behavior of $R(s;\rho)$ at special points, from which one expects to extract interesting identities relating $\mathcal{L}(X,g)$ and $\mathrm{Spec}(\Delta)$. In this respect, a point of special interest is $s = 0$. The behavior of $R(s;\rho)$ at $s = 0$ is related to the analytic torsion, whose definition we recall next. Given a representation $\rho\colon \Gamma \to \mathrm{GL}(V)$, let $E_\rho$ denote the associated flat vector bundle over $X = \Gamma\backslash\mathbb{H}^n$. Pick a Hermitian fiber metric in $E_\rho$. Let $\Delta_{\rho,p}\colon \Lambda^p(X,E_\rho) \to \Lambda^p(X,E_\rho)$ be the Laplace operator on $E_\rho$-valued $p$-forms on $X$. Let

$$\zeta_p(s;\rho) := \sum_{\lambda_i} \lambda_i^{-s},$$

where $\lambda_i$ runs over the nonzero eigenvalues of $\Delta_{\rho,p}$, counted with multiplicities. The series converges absolutely and locally uniformly in the half-plane $\mathrm{Re}(s) > n/2$ and admits a meromorphic extention to $\mathbb{C}$ which is regular at $s = 0$. Then the analytic torsion $T_X(\rho) \in \mathbb{R}^+$ is defined by

$$\log T_X(\rho) := \frac{1}{2}\sum_{p=1}^{n}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}.$$

If $n$ is odd and $H^*(X,E_\rho) = 0$, then $T_X(\rho)$ is independent of the choice of the fiber metric in $E_\rho$. With these notations we have

**Theorem 2.** *Let $n$ be odd. Let $\rho$ be a representation of $\Gamma$ which is either unitary or is given by restriction to $\Gamma$ of a representation of $\mathrm{SO}_0(n,1)$. Assume that $H^*(X;E_\rho) = 0$. Then $R(s;\rho)$ is regular at $s = 0$ and we have*

$$|R(0;\rho)| = T_X(\rho)^2.$$

For $\rho$ unitary this result is due to Fried [3]. If $\rho$ is obtained by restriction to $\Gamma$ of a representation of $\mathrm{SO}_0(n,1)$, Theorem 2 was proved by Wotzke in his thesis

[13]. A different proof is based on the extended Selberg trace formula [7]. This method can be used to extend Theorem 2 to general representations $\rho$.

In [8] we consider hyperbolic 3-manifolds $X$ and use Theorem 2 to study the behavior of $T_X(\rho)$ as $\rho$ grows. More precisely, let $X = \Gamma\backslash\mathbb{H}^3$ be a compact. oriented hyperbolic 3-manifold. Then we may regard $\Gamma$ as a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. Given $m \in \mathbb{N}$, let $\tau_m = \mathrm{Sym}^m$ be the $m$-th symmetric power representation of the standard representation of $\mathrm{SL}(2,\mathbb{C})$. Then we have

$$-\log T_X(\tau_m) = \frac{\mathrm{Vol}(X)}{4\pi} m^2 + O(m)$$

as $m \to \infty$. This formula has applications to the cohomology of arithmetic subgroups of $\mathrm{SL}(2,\mathbb{C})$ (see [9]).

It is interesting to see how these results can be extended to the non-compact case. Let $G = \mathrm{SO}_0(n,1)$ and let $\Gamma \subset G$ be a lattice, i.e., $\Gamma$ is a discrete subgroup such that $\mathrm{Vol}(\Gamma\backslash G) < \infty$, where the volume is computed with respect to any Haar measure. We assume that $\Gamma$ is torsion free. Then $X = \Gamma\backslash\mathbb{H}^n$ is a complete hyperbolic manifold of finite volume. Assume that $X$ is non-compact. Then the Laplace operator has a non-empty continuous spectrum which is equal to $[(n-1)^2/4, \infty)$. Besides of the continuous spectrum, $\Delta$ has a pure point spectrum which consists of a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. In general, the point spectrum may be finite. The only eigenvalue which we know to exist for sure is $\lambda_0 = 0$. So the eigenvalues are certainly not the right spectral parameters associated to the quantized system. Instead resonances come into play. This means the following. It is known that the resolvent, regarded as operator

$$(\Delta - s(n-1-s))^{-1} \colon L^2_{\mathrm{cpt}}(X) \to H^2_{\mathrm{loc}}(X)$$

has a meromorphic extension from the half-plane $\mathrm{Re}(s) > n-1$ to the whole complex plane. Poles of the analytic continuation of the resolvent in the half-plane $\mathrm{Re} < (n-1)/2$ are called scattering poles, because they coincide with the poles of the scattering matrix. Let $\sigma(X)$ be the union of the eigenvalues and the set of scattering resonances. The point $(n-1)/2$ has also to be included. This is the *resonance set*. Then the Selberg trace formula sets up correspondence

$$\mathcal{L}(X) \leftrightarrow \sigma(X).$$

Now one can start to investigate the corresponding zeta functions on the geometric and spectral side.

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## Deformations of the Fourier transform

Bent Ørsted
(joint work with Salem Ben Said, Toshiyuki Kobayashi)

The Fourier transform and its basic properties may be understood in many different ways; one is the relation to the representation theory of the double cover of the real symplectic group, namely as a special element in the metaplectic representation. From this point of view we shall introduce a natural two-parameter family of deformations of the Fourier transform, in effect interpolating between two minimal representations of two different semisimple Lie groups; at the same time the deformation allows introducing the differential-difference operators for Coxeter groups found by C. Dunkl [3]. The two minimal representations in question have the physical interpretations as the quantum harmonic oscillator resp. Kepler problem - curiously, for the classical systems with two degrees of freedom, such a connection was discovered by Newton (in modern terms by considering the square map of the complex numbers).

We construct a two-parameter family of actions $\omega_{k,a}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by differential-difference operators on $\mathbb{R}^N \setminus \{0\}$. Here, $k$ is a multiplicity-function for the Dunkl operators, and $a > 0$ arises from the interpolation of the two $\mathfrak{sl}(2,\mathbb{R})$ actions on the Weil representation of $Mp(N,\mathbb{R})$ and the minimal unitary representation of $O(N+1,2)$. We prove that this action $\omega_{k,a}$ lifts to a unitary representation of the universal covering of $SL(2,\mathbb{R})$, and can even be extended to a holomorphic semigroup $\Omega_{k,a}$. In the $k \equiv 0$ case, our semigroup generalizes the Hermite semigroup studied by R. Howe ($a = 2$) [4] and the Laguerre semigroup by T. Kobayashi with G. Mano ($a = 1$) [7]. One boundary value of our semigroup $\Omega_{k,a}$ provides us with $(k,a)$-*generalized Fourier transforms* $\mathcal{F}_{k,a}$, which includes the Dunkl transform $\mathcal{D}_k$ ($a = 2$) [5] and a new unitary operator $\mathcal{H}_k$ ($a = 1$), namely a Dunkl–Hankel transform. We establish the inversion formula, and a generalization of the Plancherel theorem, the Hecke identity, the Bochner identity, and a Heisenberg uncertainty relation for $\mathcal{F}_{k,a}$. We also find kernel functions for

$\Omega_{k,a}$ and $\mathcal{F}_{k,a}$ for $a = 1, 2$ in terms of Bessel functions and the Dunkl intertwining operator. See [1] and the preprint [2] for more details and proofs.

First we find a holomorphic semigroup $\mathcal{I}_{k,a}(z)$ with two parameters $k$ and $a$. Dunkl operators are differential-difference operators associated to a finite reflection group on the Euclidean space. They were introduced by C. Dunkl [3]. This subject was motivated partly from harmonic analysis on the tangent space of the Riemannian symmetric spaces, and resulted in a new theory of non-commutative harmonic analysis 'without Lie groups'. The Dunkl operators are also used as a tool for investigating an algebraic integrability property for the Calogero–Moser quantum problem related to root systems.

Our holomorphic semigroup $\mathcal{I}_{k,a}(z)$ is built on Dunkl operators. To fix notation, let $\mathfrak{C}$ be the Coxeter group associated with a root system $\mathcal{R}$ in $\mathbb{R}^N$. For a $\mathfrak{C}$-invariant real function $k \equiv (k_\alpha)$ (*multiplicity function*) on $\mathcal{R}$, we write $\Delta_k$ for the Dunkl Laplacian on $\mathbb{R}^N$ (see [3]).

We take $a > 0$ to be a deformation parameter, and introduce the following differential-difference operator

$$\Delta_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a. \tag{1}$$

Here, $\|x\|$ is the norm of the coordinate $x \in \mathbb{R}^N$, and $\|x\|^a$ in the right-hand side of the formula stands for the multiplication operator by $\|x\|^a$. Then, $\Delta_{k,a}$ is a symmetric operator on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ consisting of square integrable functions on $\mathbb{R}^N$ against the measure $\vartheta_{k,a}(x)dx$, where the density function $\vartheta_{k,a}(x)$ on $\mathbb{R}^N$ is given by

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha\in\mathcal{R}} |\langle\alpha, x\rangle|^{k_\alpha}. \tag{2}$$

Then $\vartheta_{k,a}(x)$ has a degree of homogeneity $a - 2 + 2\langle k\rangle$, where $\langle k\rangle := \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k_\alpha$ is the index of $k = (k_\alpha)$.

The $(k, a)$-*generalized Laguerre semigroup* $\mathcal{I}_{k,a}(z)$ is defined to be the semigroup with infinitesimal generator $\frac{1}{a}\Delta_{k,a}$, that is,

$$\mathcal{I}_{k,a}(z) := \exp\Big(\frac{z}{a}\Delta_{k,a}\Big), \tag{3}$$

for $z \in \mathbb{C}$ such that $\operatorname{Re} z \geq 0$.

**Theorem 1.** *With notation as above,*

*(1)* $\mathcal{I}_{k,a}(z)$ *is a holomorphic semigroup in the complex right-half plane* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ *in the sense that* $\mathcal{I}_{k,a}(z)$ *is a Hilbert–Schmidt operator on* $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ *satisfying*

$$\mathcal{I}_{k,a}(z_1) \circ \mathcal{I}_{k,a}(z_2) = \mathcal{I}_{k,a}(z_1 + z_2), \quad (\operatorname{Re} z_1, \operatorname{Re} z_2 > 0),$$

*and that the scalar product* $(\mathcal{I}_{k,a}(z)f, g)$ *is a holomorphic function of* $z$ *for* $\operatorname{Re} z > 0$, *for any* $f, g \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

*(2)* $\mathcal{I}_{k,a}(z)$ *is a one-parameter group of unitary operators on the imaginary axis* $\operatorname{Re} z = 0$.

The 'boundary value' of the $(k,a)$-generalized Laguerre semigroup $\mathcal{I}_{k,a}(z)$ on the imaginary axis gives a one-parameter family of unitary operators. The case $z=0$ gives the identity operator, namely, $\mathcal{I}_{k,a}(0)=\mathrm{id}$. The particularly interesting case is when $z=\frac{\pi i}{2}$, and we set

$$\mathcal{F}_{k,a} := c\,\mathcal{I}_{k,a}\Big(\frac{\pi i}{2}\Big) = c\exp\Big(\frac{\pi i}{2a}(\|x\|^{2-a}\Delta_k - \|x\|^a)\Big)$$

where the phase factor $c = e^{i\frac{\pi}{2}(\frac{2\langle k\rangle+N+a-2}{a})}$. Then, the unitary operator $\mathcal{F}_{k,a}$ for general $a$ and $k$ satisfies the following significant properties:

**Theorem 2.** *Suppose $a>0$ and $a+2\langle k\rangle+N-2>0$.*

*(1) $\mathcal{F}_{k,a}$ is a unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*
*(2) $\mathcal{F}_{k,a}\circ E = -(E+N+2\langle k\rangle+a-2)\circ\mathcal{F}_{k,a}$.*
*Here, $E=\sum_{j=1}^N x_j\partial_j$.*
*(3) $\mathcal{F}_{k,a}\circ\|x\|^a = -\|x\|^{2-a}\Delta_k\circ\mathcal{F}_{k,a}$,*
*$\mathcal{F}_{k,a}\circ(\|x\|^{2-a}\Delta_k) = -\|x\|^a\circ\mathcal{F}_{k,a}$.*
*(4) $\mathcal{F}_{k,a}$ is of finite order if and only if $a\in\mathbb{Q}$. Its order is $2p$ if $a$ is of the form $a=\frac{p}{q}$, where $p$ and $q$ are positive integers that are relatively prime.*

We call $\mathcal{F}_{k,a}$ a $(k,a)$-*generalized Fourier transform* on $\mathbb{R}^N$. The representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ generated by $\frac{i}{a}\|x\|^a$ and $\frac{i}{a}\|x\|^{2-a}\Delta_k$ lifts to the universal covering group $\widetilde{SL(2,\mathbb{R})}$:

**Theorem 3.** *If $a>0$ and $a+2\langle k\rangle+N-2>0$, then $\omega_{k,a}$ lifts to a unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$.*

The Hilbert space $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$ decomposes discretely as a direct sum of unitary representations of the direct product group $\mathfrak{C}\times\widetilde{SL(2,\mathbb{R})}$:

$$L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx) \simeq \sum_{m=0}^{\infty}{}^{\oplus}\, \mathcal{H}_k^m(\mathbb{R}^N)\big|_{S^{N-1}} \otimes \pi\Big(\frac{2m+2\langle k\rangle+N-2}{a}\Big), \tag{4}$$

where $\mathcal{H}_k^m(\mathbb{R}^N)$ stands for the representation of the Coxeter group $\mathfrak{C}$ on the eigenspace of the Dunkl Laplacian (the space of spherical $k$-harmonics of degree $m$) and $\pi(\nu)$ is an irreducible unitary lowest weight representation of $\widetilde{SL(2,\mathbb{R})}$ of weight $\nu+1$. The unitary isomorphism (4) is constructed explicitly by using Laguerre polynomials.

The unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$ extends furthermore to a holomorphic semigroup of a complex three dimensional semigroup. Basic properties of the holomorphic semigroup $\mathcal{I}_{k,a}(z)$ defined in (3) and the unitary operator $\mathcal{F}_{k,a}$ can be read from the 'dictionary' of $\mathfrak{sl}(2,\mathbb{R})$ as follows:

$$i\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \longleftrightarrow \frac{1}{a}\Delta_{k,a}$$

$$\exp iz\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \longleftrightarrow \mathcal{I}_{k,a}(z) = \exp(\frac{z}{a}\Delta_{k,a})$$

$$w_0 = \exp\frac{\pi}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \longleftrightarrow \mathcal{F}_{k,a} \text{ (up to the phase factor)}$$

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## Spin-quantization in the compact and non-compact setting

Paul-Emile Paradan

We explain how the "quantization commutes with reduction" phenomenon of Guillemin-Sternberg [3] holds in the context of the metaplectic correction. In the first part the talk, we expose the main result of the preprint arXiv:0911.1067 that will appear in Journal of Symplectic Geometry. In the second part of this talk we explain how we can extend this result in the non-compact setting.

Let $K$ be a compact connected Lie group with Lie algebra $\mathfrak{k}$. An Hamiltonian K-manifold $(M,\omega,\Phi)$ is Spin-prequantized if $M$ carries an equivariant Spin$^c$ structure $P$ with determinant line bundle being a Kostant-Souriau line bundle over $(M,2\omega,2\Phi)$. Let $\mathcal{D}_P$ be the Spin$^c$ Dirac operator attached to $P$, where $M$ is oriented by its symplectic form. When $M$ is compact the Spin-quantization of $(M,\omega,\Phi)$ corresponds to the equivariant index of the elliptic operator $\mathcal{D}_P$, and is denoted

$$\mathcal{Q}^K_{\text{spin}}(M) \in R(K).$$

Let $\widehat{A}(M)(X)$ be the equivariant A-genus class: it is an equivariant analytic function from a neigborhood of $0 \in \mathfrak{k}$ with value in the algebra of differential forms on $M$. The Atiyah-Segal-Singer index theorem [1] tell us that

$$\mathcal{Q}^K_{\rm spin}(M)(e^X) := (\frac{i}{2\pi})^{\dim M/2} \int_M e^{i(\omega + \langle \Phi, X\rangle)} \widehat{A}(M)(X) \tag{1}$$

for $X \in \mathfrak{k}$ small enough. It shows in particular that $\mathcal{Q}^K_{\rm spin}(M) \in R(K)$ does not depend of the choice of the Spin-prequantum data.

This notion of Spin-quantization is closely related to the notion of *metaplectic correction.* Suppose that $(M, \omega, \Phi)$ carries a Kostant-Souriau line bundle $L_\omega$, and that the bundle of half-forms $\kappa_J^{1/2}$ associated to an invariant almost complex structure $J$ is well defined. In this case, $(M, \omega, \Phi)$ is Spin-prequantized by the Spin$^c$-structure defined by $J$ and twisted by the line bundle $L_\omega \otimes \kappa_J^{1/2}$. The crucial point here is that the corresponding Spin-quantization of $(M, \omega, \Phi)$ does not depend of the choice of the almost complex structure.

One wants to compute geometrically the multiplicities of $\mathcal{Q}^K_{\rm spin}(M) \in R(K)$ in a way similar to the famous "quantization commutes with reduction" phenomenon of Guillemin-Sternberg [3, 7, 8, 16, 9, 17, 4, 18, 14]. This question was first addressed in the work of Cannas-Karshon-Tolman [2] and Vergne [17] in the case of a circle action. The non-abelian group action case was first studied by Jeffrey-Kirwan [4] and by the author [10], but both papers made fairly strong assumptions: in [4] they suppose that $0 \in \mathfrak{k}^*$ has a big enough neighborhood of regular values of the moment map, and in [10] one asks that the infinitesimal stabilizers of the $K$-action are abelian. We will now explain how a "quantization commutes with reduction" theorem holds in the general case. Note that C. Teleman also obtained some results [15][Proposition 3.10] in the algebraic setting.

The striking difference with the standard Guillemin-Sternberg phenomenon is the *rho shift* that we explain now. Let $T$ be a maximal torus of $K$ with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be the closed Weyl chamber. We will look at $\mathfrak{t}^*_+$ as a disjoint union of its open faces, the maximal one being its interior $(\mathfrak{t}^*_+)^o$. Let $\rho \in (\mathfrak{t}^*_+)^o$ be the half sum of the positive roots. At each open face $\tau$ of $\mathfrak{t}^*_+$, we associate the term $\rho_\tau$ which is the half sum of the positive roots which are orthogonal to $\tau$. We note that $\rho - \rho_\tau \in \tau$ is to the orthogonal projection of $\rho$ on $\tau$.

For any $\xi \in \mathfrak{t}^*_+$ and any face $\tau$ containing $\xi$ in its closure, we consider the *shifted symplectic reduction*

$$M^\tau_\xi := \Phi^{-1}(\xi + \rho - \rho_\tau)/K_\tau$$

where $K_\tau$ is the common stabiliser of points in $\tau$. Note that $\xi + \rho - \rho_\tau \in \tau$ when $\xi \in \overline{\tau}$.

We are particularly interested to the smallest face $\sigma$ of the Weyl chamber so that the Kirwan polytope $\Delta_K(M) := \Phi(M) \cap \mathfrak{t}^*_+$ is contained in the closure of $\sigma$. It is not hard to see that the Spin-prequantum data on $(M, \omega, \Phi)$ descents to the shifted symplectic reduction $M^\sigma_\mu$ when $\mu$ is a dominant weight belonging to $\overline{\sigma}$. Then $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) \in \mathbb{Z}$ is naturally defined when $\mu + \rho - \rho_\sigma$ is a regular value of the moment map. In general, the number $\mathcal{Q}_{\rm spin}(M^\sigma_\mu)$ is defined by shift-desingularization.

By definition $\mathcal{Q}_{\rm spin}(M^\sigma_\mu)$ vanishes when $\mu+\rho-\rho_\sigma \notin \Delta_K(M)$, but in fact we can strengthen this vanishing property: $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) = 0$ if $\mu+\rho-\rho_\sigma$ does not belong to the *relative interior* of the Kirwan polytope $\Delta_K(M)$.

Recall that the irreducible representation $V^K_\mu$ of $K$ are parametrized by their highest weight $\mu \in \widehat{K} \subset \mathfrak{t}^*_+$.

Our "Spin-quantization commutes with reduction" theorem is the following.

**Theorem** [13] *Let $(M,\omega,\Phi)$ be a compact Spin-prequantized Hamiltonian $K$-manifold. Let $\sigma$ be the smallest face of the Weyl chamber so that $\Delta_K(M) \subset \overline{\sigma}$. We have*

$$\mathcal{Q}^K_{\rm spin}(M) = \sum_{\mu\in\widehat{K}\cap\overline{\sigma}} \mathcal{Q}_{\rm spin}(M^\sigma_\mu)\, V^K_\mu.$$

Suppose now that the manifold $M$ is (possibly) non-compact but with a *proper* moment map $\Phi$. Here the Spin-quantization of $(M,\omega,\Phi)$ is an admissible representation

$$\mathcal{Q}^{-\infty,K}_{\rm spin}(M) \in R^{-\infty}(K) := \hom(R(K),\mathbb{Z})$$

which is defined as an index of a transversally elliptic operator on $M$ [5, 6, 12] .

Note that in this context, the reduced spaces $M^\sigma_\mu$ are compact, and one can define their Spin-quantization $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) \in \mathbb{Z}$.

Using the technique of symplectic cutting developed in [11, 12], we are able to prove that the multiplicity of $V^K_\mu$ in $\mathcal{Q}^{-\infty,K}_{\rm spin}(M)$ is equal to $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) \in \mathbb{Z}$. This result will appear in a subsequent paper.

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## Eta cocycles, relative pairings and the Godbillon-Vey index theorem

Paolo Piazza
(joint work with Hitoshi Moriyoshi)

Connes' index theorem for $G$-proper manifolds [3], with $G$ an étale groupoid, unifies under a single statement most of the existing (longitudinal) index theorems. We shall focus on a particular case of such a theorem, that of foliated bundles. Thus, let $N$ be a closed compact manifold. Let $\Gamma \to \tilde{N} \to N$ be a Galois $\Gamma$-cover. Let $T$ be a smooth oriented compact manifold with an action of $\Gamma$ which is assumed to be by diffeomorphisms, orientation preserving and locally faithful, as in [7]. Let $Y = \tilde{N} \times_\Gamma T$ and let $(Y, \mathcal{F})$ be the associated foliation. (This is an example of $G$-proper manifold with $G$ equal to the groupoid $T \rtimes \Gamma$.) Let $D$ be a $\Gamma$-equivariant family of Dirac operators on the fibration $\tilde{N} \times T \to T$; such a family induces a longitudinal Dirac operator on $(Y, \mathcal{F})$.

If $T =$ point and $\Gamma = \{1\}$ we have a compact manifold and Connes' index theorem reduces to the Atiyah-Singer index theorem. If $\Gamma = \{1\}$ we simply have a fibration and the theorem reduces to the Atiyah-Singer family index theorem. If $T =$ point then we have a Galois covering and Connes' index theorem reduces to the Connes-Moscovici higher index theorem. If $\dim T > 0$ and $\Gamma \neq \{1\}$, then Connes' index theorem is a higher foliation index theorem on the foliated manifold $(Y, \mathcal{F})$.

One fascinating higher index is the so-called Godbillon-Vey index on a codimension 1 foliation (thus we take $T = S^1$ in this case). Following the treatment of Moriyoshi-Natsume in [7] the corresponding higher index formula can be stated in the following way: there is a cyclic 2-cocycle $\tau_{GV}$ on $C^\infty_c(Y, \mathcal{F}) := C^\infty((\tilde{N} \times \tilde{N} \times S^1)/\Gamma)$ which can be paired with the (compactly supported) index class $\mathrm{Ind}^c(D) \in K_0(C^\infty_c(Y, \mathcal{F}))$; there is a holomorphically closed subalgebra $\mathfrak{A}$, $C^\infty_c(Y, \mathcal{F}) \subset \mathfrak{A} \subset C^*(Y, \mathcal{F})$, containing the $C^*$-index class $\mathrm{Ind}(D)$ and such that $\tau_{GV}$ extends to $\mathfrak{A}$; the pairing $\langle \mathrm{Ind}(D), [\tau_{GV}] \rangle$ can be written down explicitly and it involves the Godbillon-Vey class of the foliation, $GV \in H^3(Y)$. As an example, let $\Sigma_g$ be a closed compact riemann surface of genus $g \geq 2$ and let $\Gamma \to H^2 \to \Sigma_g$

be the associated universal covering; for the particular 3-dimensional example given by $Y = H^2 \times_\Gamma S^1$, with $\Gamma < PSL(2,\mathbb{R})$ acting on $S^1$ by fractional linear trasformation, the higher index formula reads

$$\langle \mathrm{Ind}(D), [\tau_{\mathrm{GV}}] \rangle = \int_Y \omega_{GV} \tag{1}$$

with $\omega_{GV}$ an explicit closed 3-form on $Y$ such that $[\omega_{GV}] = GV \in H^3(Y)$. In particular, we find that $\langle \mathrm{Ind}(D), [\tau_{\mathrm{GV}}] \rangle = \mathfrak{gv}(Y,\mathcal{F})$, with $\mathfrak{gv}(Y,\mathcal{F})$ the Godbillon-Vey invariant of the foliation $(Y,\mathcal{F})$. Thus, a purely geometric invariant of the foliation $(Y,\mathcal{F})$, $\mathfrak{gv}(Y,\mathcal{F})$, is in fact a higher index.

One might wonder if Connes' general index theorem on $G$-proper manifolds can be extended to foliated bundles with boundary, in the spirit of the seminal work of Atiyah-Patodi-Singer [1]. For simplicity, let us concentrate on foliated bundles. Then, under a polynomial growth assumption on the group $\Gamma$ and requiring, as usual, invertibility of the boundary operator, such an extension was proved by Leichtnam and Piazza in [4]. The structure of the (higher) index formula in [4] is precisely the one displayed by the classic Atiyah-Patodi-Singer index formula. Thus there is a local contribution, which is the one appearing in the corresponding higher index formula in the closed case, and there is a boundary-correction term, which is a *higher eta invariant.* This higher eta invariant should be thought of as a secondary higher invariant of the operator on the boundary (indeed, the index class for the boundary operator is always zero). We remark that some of the interesting geometric applications of the theory do employ this secondary invariant in order to tackle classification problems that cannot be treated by ordinary higher indices.

We now make the crucial observation that the polynomial growth assumption in [4] excludes many interesting (typically type III) examples and higher indices; in particular it excludes the possibility of proving a Atiyah-Patodi-Singer formula for the Godbillon-Vey higher index.

*The main goal of my talk was to explain recent results, in collaboration with Hitoshi Moriyoshi, establishing such a formula.* See the announcement [5] and the complete paper [6]. Notice that this formula constitutes the first instance of a higher APS index theorem on type III foliations. Notice also that, consequently, we define *a Godbillon-Vey eta invariant* on the boundary-foliation; this is a *type III eta invariant.* In tackling this specific index problem we develop what we believe is a new approach to index theory on geometric structures with boundary, heavily based on the interplay between absolute and relative pairings. We think that this new method can be applied to a variety of situations.

Let us give a brief description of our main results.

It is clear from the structure of the classic Atiyah-Patodi-Singer index formula that one of the basic tasks in the theory is to split in a precise way the interior contribution from the boundary contribution in the higher index formula. We look at operators on the boundary through the translation invariant operators on the associated infinite cyclinder; by Fourier transform these two pictures are equivalent. We solve the Atiyah-Patodi-Singer higher index problem on a foliated bundle with boundary $(X_0, \mathcal{F}_0)$, $X_0 = \tilde{M} \times_\Gamma T$, by solving the associated $L^2$-problem on the associated foliation with cylindrical ends $(X, \mathcal{F})$. With the goal of splitting the interior contribution from the boundary contribution in mind, we define a short exact sequence of $C^*$-algebras

$$0 \to C^*(X, \mathcal{F}) \to A^*(X, \mathcal{F}) \to B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0\,.$$

This is an extension by the foliation $C^*$-algebra $C^*(X, \mathcal{F})$ of a suitable algebra of *translation invariant operators* on the cylinder; we call it the Wiener-Hopf extension. We briefly denote the Wiener-Hopf extension as $0 \to C^*(X, \mathcal{F}) \to A^* \to B^* \to 0$. These $C^*$-algebras are the receptacle for the two $C^*$-index classes we will be working with. Thus, given a $\Gamma$-equivariant family of Dirac operators $(D_\theta)_{\theta \in T}$ with invertible boundary family $(D^\partial_\theta)_{\theta \in T}$ we prove that there exist an index class $\mathrm{Ind}(D) \in K_*(C^*(X, \mathcal{F}))$ and a relative index class $\mathrm{Ind}(D, D^\partial) \in K_*(A^*, B^*)$ . The higher Atiyah-Patodi-Singer index problem for the Godbillon-Vey cocycle consists in proving that there is a well defined paring $\langle \mathrm{Ind}(D), [\tau_{GV}] \rangle$ and giving a formula for it, with a structure similar to the one displayed by the Atiyah-Patodi-Singer index formula. Now, as in the case of Moriyoshi-Natsume, $\tau_{GV}$ is initially defined on the small algebra $J_c(X, \mathcal{F})$ of $\Gamma$-equivariant smoothing kernels of $\Gamma$-compact support; however, because of the structure of the parametrix on manifolds with cylindrical ends, there does *not* exist an index class in $K_*(J_c(X, \mathcal{F}))$. Hence, even *defining* the index pairing is not obvious. We solve this problem by showing that there exists a holomorphically closed intermediate subalgebra $\mathfrak{J}$ containing the index class $\mathrm{Ind}(D)$ but such that $\tau_{GV}$ extends. More on this in a moment. This point involves elliptic theory on manifolds with cylindrical ends in an essential way.

Once the higher Godbillon-Vey index is defined, we search for an index formula for it. Our main idea is to show that such a formula is a direct consequence of the equality

$$\langle \mathrm{Ind}(D), [\tau_{GV}] \rangle = \langle \mathrm{Ind}(D, D^\partial), [(\tau^r_{GV}, \sigma_{GV})] \rangle \tag{2}$$

where on the right hand side a new mathematical object, the *relative* Godbillon-Vey cocycle, appears. The relative Godbillon-Vey cocycle is built out of the usual Godbillon-Vey cocycle by means of a very natural procedure. First, we proceed algebraically. Thus we first look at a subsequence of $0 \to C^*(X, \mathcal{F}) \to A^* \to B^* \to 0$ made of small algebras, call it $0 \to J_c(X, \mathcal{F}) \to A_c \to B_c \to 0$; $J_c(X, F)$ are, as above, the $\Gamma$-equivariant smoothing kernels of $\Gamma$-compact support; $B_c$ is made of $\Gamma \times \mathbb{R}$-equivariant smoothing kernels on the cylinder of $\Gamma \times \mathbb{R}$-compact support. The $A_c$ cyclic 2-cochain $\tau^r_{GV}$ is obtained from $\tau_{GV}$ through a regularization à la Melrose. The $B_c$ cyclic 3-cocycle $\sigma_{GV}$ is obtained by *suspending* $\tau_{GV}$ on the cylinder with Roe's 1-cocycle. We call this $\sigma_{GV}$ the *eta cocycle* associated to $\tau_{GV}$. One proves, but it is not quite obvious, that $(\tau^r_{GV}, \sigma_{GV})$ is a relative cyclic 2-cocycle for $A_c \to B_c$. We obtain in this way a relative cyclic cohomology class $[\tau^r_{GV}, \sigma_{GV}] \in HC^2(A_c, B_c)$.

We remark here that for technical reasons having to do with the extension of these cocycles to suitable smooth subalgebras, see below, we shall have to consider the cyclic cocycle and the relative cyclic cocycle obtained from $\tau_{GV}$ and $(\tau^r_{GV}, \sigma_{GV})$ through the $S$ operation in cyclic cohomology, see [2]: thus we consider $S^{p-1}\tau_{GV}$ and $(S^{p-1}\tau^r_{GV}, \frac{3}{2p+1}S^{p-1}\sigma_{GV})$ obtaining in this way a class in $HC^{2p}(J_c)$ and a relative class in $HC^{2p}(A_c, B_c)$. With a small abuse of notation we still denote these cyclic $2p$-cocycles by $\tau_{GV}$ and $(\tau^r_{GV}, \sigma_{GV})$.

Once the algebraic theory is clarified, we need to pair the class $[\tau_{GV}] \in H^{2p}(J_c)$ and the relative class $[\tau^r_{GV}, \sigma_{GV}] \in HC^{2p}(A_c, B_c)$ with the corresponding index classes $\mathrm{Ind}(D) \in K_*(C^*(X, \mathcal{F}))$ and $\mathrm{Ind}(D, D^\partial) \in K_*(A^*, B^*)$. To this end we construct an *intermediate* short exact subsequence $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ of Banach algebras, sitting half-way between $0 \to C^*(X, \mathcal{F}) \to A^* \to B^* \to 0$ and $0 \to J_c(X, \mathcal{F}) \to A_c \to B_c \to 0$. Much work is needed in order to define such a subsequence and prove that

$$\mathrm{Ind}(D) \in K_*(\mathfrak{J}) \cong K_*(C^*(X, \mathcal{F}))\,, \quad \mathrm{Ind}(D, D^\partial) \in K_*(\mathfrak{A}, \mathfrak{B}) \cong K_*(A^*, B^*)\,.$$

Even more work is needed in order to establish that the Godbillon-Vey cyclic $2p$-cocycle $\tau_{GV}$ and the relative cyclic $2p$-cocycle $(\tau^r_{GV}, \sigma_{GV})$ extend for $p$ large enough from $J_c$ and $A_c \to B_c$ to $\mathfrak{J}$ and $\mathfrak{A} \to \mathfrak{B}$, thus defining elements

$$[\tau_{GV}] \in HC^{2p}(\mathfrak{J}) \quad \text{and} \quad [\tau^r_{GV}, \sigma_{GV}] \in HC^{2p}(\mathfrak{A}, \mathfrak{B}).$$

We have now made sense of both sides of the equality (2) $\langle \mathrm{Ind}(D), [\tau_{GV}]\rangle = \langle \mathrm{Ind}(D, D^\partial), [(\tau^r_{GV}, \sigma_{GV})]\rangle$. The equality itself is proved by establishing and using the excision formula: if $\alpha_{\mathrm{ex}} : K_*(\mathfrak{J}) \to K_*(\mathfrak{A}, \mathfrak{B})$ is the excision isomorphism, then

$$\alpha_{\mathrm{ex}}(\mathrm{Ind}(D)) = \mathrm{Ind}(D, D^\partial) \quad \text{in} \quad K_*(\mathfrak{A}, \mathfrak{B})\,.$$

The index formula is obtained by explicitly writing the relative pairing

$$\langle \mathrm{Ind}(D, D^\partial), [(\tau^r_{GV}, \sigma_{GV})]\rangle$$

in terms of the graph projection $e_D$, multiplying the operator $D$ by $s > 0$ and taking the limit as $s \downarrow 0$. The final formula in the 3-dimensional case (always with an invertibility assumption on the boundary family) reads:

$$\langle \mathrm{Ind}(D), [\tau_{GV}]\rangle = \int_{X_0} \omega_{GV} - \eta_{GV}\,, \tag{3}$$

with $\omega_{GV}$ equal, as in the closed case, to (a representative of) the Godbillon-Vey class $GV$ and

$$\eta_{GV} := \frac{(2p+1)}{p!} \int_0^\infty \sigma_{GV}([\dot{p_t}, p_t], p_t, \ldots, p_t, p_t)dt\,, \tag{4}$$

with $p_t := e_{tD^{\mathrm{cyl}}}$ the graph projection associated to the cylindrical Dirac family $tD^{\mathrm{cyl}}$. Observe that by Fourier transform *the Godbillon-Vey eta invariant* $\eta_{GV}$ only depends on the boundary family $D^\partial \equiv (D^\partial_\theta)_{\theta \in T}$.

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## Singular equivariant heat asymptotics

Pablo Ramacher

Let $M$ be a closed $n$-dimensional Riemannian manifold, and $E$ a complex $C^\infty$-vector bundle over $M$. Consider an elliptic differential operator

$$P : C^\infty(M,E) \longrightarrow L^2(M,E)$$

of order $d$ on $E$, regarded as an operator in the Hilbert space $L^2(M,E)$ of square integrable sections of $E$, its domain being the space $C^\infty(M,E)$ of smooth sections. In addition, assume that $P$ is symmetric and positive. Under these assumptions, $P$ has discrete spectrum, and there exists an orthonormal basis of $L^2(M,E)$ consisting of smooth sections $\{e_j\}$ such that $Pe_j = \lambda_j e_j$, $|\lambda_j| \to \infty$. Consider now the heat equation

$$(\partial_t + P)h(x,t) = 0, \qquad \lim_{t\to\infty} h(x,t) = f(x), \qquad t > 0,$$

with initial condition $f \in C^\infty(M,E)$. Its solution is given by $h(x,t) = e^{-tP}f(x)$, where $e^{-tP}$ denotes the heat operator. It has a smooth kernel $K(t,x,y,P) \in \mathrm{Hom}(E_y,E_x)$, and is of $L^2$-trace class, its trace being given by

$$\mathrm{tr}_{L^2}(e^{-tP}) = \sum_{j\geq 0}(e^{-tP}e_j,e_j)_{L^2} = \sum_{j\geq 0} e^{-t\lambda_j} = \int_M \mathrm{tr}\, K(t,x,x,P)\, dM(x).$$

Approximating the heat operator with pseudodifferential operators, it can be shown that the heat trace has the asymptotic expansion

$$\mathrm{tr}_{L^2}(e^{-tP}) \sim \sum_{j\geq 0} a_j(P) t^{(j-n)/d}, \qquad t \to 0,$$

where the coefficients $a_j(P)$ are invariantly defined. Let now $G$ be a compact Lie group acting isometrically and effectively on $M$, and $(\pi_\chi, V_\chi)$, an unitary irreducible representation of $G$ corresponding to the character $\chi \in \hat{G}$. We are interested in an asymptotic expansion of

$$\mathcal{L}_\chi(t) = \int_G \mathrm{tr}_{L^2}(T_g \circ e^{-tP})\overline{\chi(g)}\, dg = \int_{G\times M} \mathrm{tr}\,[T_g \circ K(t,gx,x,P)]\overline{\chi(g)}\, dM(x)\, dg$$

as $t \to 0$, where $T_g : E_{gx} \to E_x$ is a fiber linear map depending smoothly on $x \in M$. Using an approximation of $e^{-tP}$ by pseudodifferential operators, this reduces to the problem of finding asymptotics for oscillatory integrals of the form

$$I(\mu) = \int_{T^*Y} \int_G e^{i\mu\Phi(x,\xi,g)} a(gx, x, \xi, g)\, dg\, d(T^*Y)(x, \xi), \qquad \mu \to +\infty,$$

via the generalized stationary phase theorem, where $(\kappa, Y)$ are local coordinates on $M$, $d(T^*Y)$ is the canonical volume density on the cotangent bundle $T^*Y$, and $dg$ is the volume density of a left invariant metric on $G$, while $a$ is a compactly supported amplitude, and $\Phi(x, \xi, g) = \langle \kappa(x) - \kappa(gx), \xi\rangle$. While for free group actions the critical set of the phase function $\Phi$ is a smooth manifold, this is no longer the case for general non-transitive group actions, so that, a priori, the principle of the stationary phase can not be applied. Nevertheless, this obstacle can be circumvented by partially resolving the singularities of the critical set, yielding asymptotics for $\mathcal{L}_\chi(t)$ in the case of singular group actions. The existence of such expansions could probably lead to equivariant Lefschetz formulae for arbitrary compact group actions.

## Atiyah's question about possible values of $L^2$-Betti numbers

Thomas Schick
(joint work with Mikaël Pichot, Andrzej Zuk)

Atiyah defined $L^2$-Betti numbers of a compact manifold $M$ in terms of harmonic $L^2$-forms on the universal covering. A priori, these are arbitrary non-negative real numbers. However, their alternating sum is the Euler characteristic. This lead Atiyah to the question about the possible values these $L^2$-Betti numbers can assume, in particular whether they always have to be rational. Varies conjectures in this direction have been popularized as the "strong Atiyah conjecture". This conjecture predicts in particular that the numbers are integers if the fundamental group $\Gamma$ of $M$ is torsion-free.

Indeed, the $L^2$-Betti numbers can (by Dodziuk's $L^2$-Hodge de Rham theorem), also be computed from the cellular chain complex of the universal covering, which is a chain complex of free $\mathbb{Z}[\Gamma]$-modules and it turns out that the strong Atiyah conjecture is really (equivalent to) a purely algebraic statement about elements of the integral group ring $\mathbb{Z}[\Gamma]$, as following.

Let $A$ be a $d \times d$-matrix over $\mathbb{Z}\Gamma$. It acts by left convolution multiplication as bounded operator on the Hilbert space $l^2(\Gamma)^n$. Let $p_A$ be the orthogonal projection onto the null space of this operator. Let $\delta_e \in l^2(\Gamma)$ be the characteristic function of the identity element and $\delta_e^i \in l^2(\Gamma)^n$ the vector with entry $\delta_e$ at position $i$ and 0 at all other positions. Then $b^{(2)}(A) := \sum_{i=1}^d \langle p_A \delta_e^i, \delta_e^i \rangle_{l^2(\Gamma)^n}$. This is a normalized trace of the projector onto the kernel of $A$. The possible values of $L^2$-Betti numbers of manifolds with fundamental group $\Gamma$ coincides with the possible values of $b^{(2)}(A)$ where $A$ varies over matrices over $\mathbb{Z}\Gamma$. Note that $\Gamma$ must be finitely presented to be the fundamental group of a compact manifold.

The task now is to find groups $\Gamma$ and elements $A \in \mathbb{Z}[\Gamma]$ where $\ker(A)$ and $b^{(2)}(A)$ are explicitly calculable and have unexpected, namely transcendental values.

First calculations in this direction for the random walk operator on the lamplighter group have been carried out in [2], where a complete eigenspace decomposition is derived. This has been taken up for free lamplighter groups, i.e. the restricted wreath product of $\mathbb{Z}/2\mathbb{Z}$ by non-abelian free groups. Recently an explicit irrational $L^2$-Betti number for a random walk operator on a free lamplighter group has been computed by Lehner and Wagner [4]. The main point here is to get a hold on the combinatorial difficulties of understanding all finite connected subgraphs in the Cayley graph of a free group (a regular tree) and the kernel of the graph Laplacian on these.

In a different direction and slightly earlier, Austin [1] uses suitable quotients of the free lamplighter group and tailor-made (very much generalized) relatives of the random walk operator to produce an uncountable set of $L^2$-Betti numbers, so that transcendental ones have to exist. The drawback, however, is that this method is not explicit and does not give finitely presented groups as examples.

The main point of the work [5] reported on in the talk is, to refine the work of Austin in such a way as to arrive at explicit calculations. For this, a different class of operators is used.

We arrive at the following results:

- there are explicit finitely presented groups and elements in their group ring with transcendental $L^2$-Betti numbers; therefore also closed manifolds with such $L^2$-Betti numbers.
- every algebraic number is an $L^2$-Betti number of a closed manifold, moreover every real number which admits a Turing machine producing its decimal expansion (in correct order), e.g. $\pi$.
- purely algebraically, every element of $\mathbb{R}_{\geq 0}$ is a $b^{(2)}(A)$ for a suitable matrix over $A$ the integral group ring of a suitable discrete group.

Very similar results have been obtained independently, using a way to implement Turing machines into groups and elements of their group ring, by Lukasz Grabowski in [3].

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# Geometric quantisation of integrable systems with nondegenerate singularities

Romero Solha
(joint work with Eva Miranda)

## 1. abstract

This talk shows an attempt to extend some results by Snyaticki, Guillemin and Sternberg in geometric quantisation considering regular fibrations as real polarisations to the singular setting. The generic real polarisations concerned here are given by integrable systems with nondegenerate singularities (in the Morse-Bott sense). And the definition of geometric quantisation used is the one suggested by Kostant; via higher cohomology groups. The case of nondegenerate singularities was obtained in dimension 2 by Hamilton and Miranda [1] and the completely elliptic case was considered by Hamilton [2] in any dimension. The approach is to combine previous results of Miranda and Presas [3] on a Künneth formula to reduce to the 2-dimensional case.

## 2. The regular case

Let $(M^{2n}, \omega)$ be a symplectic manifold such that $[\omega]$ is integral. In [4] is proved the existence of $(L, \nabla)$ a complex line bundle with connection over $M$ that satisfies $curv(\nabla) = -i\omega$. Under these conditions the symplectic manifold $(M^{2n}, \omega)$ is called prequantisable and the pair $(L, \nabla)$ a prequantum line bundle of $(M^{2n}, \omega)$.

The definition of geometric quantisation used here is the one suggested by Kostant via higher cohomology groups. A real polarisation $P$ is a integrable subbundle of $TM$ whose leaves are lagrangian submanifolds. And the quantisation of $(M^{2n}, \omega, L, \nabla, P)$ is

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M; \mathcal{J})$$

where $\mathcal{J}$ is the sheaf of flat sections, i.e.: the space of local sections $s$ of $L$ such that $\nabla_X s = 0$, for all sections $X$ of $P$.

**Theorem:** (Sniatycki) *If the leaf space $B^n$ is a Hausdorff manifold and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibres, then $\mathcal{Q}(M^{2n}) = H^n(M^{2n}; \mathcal{J})$, and the dimension of $H^n(M^{2n}; \mathcal{J})$ is the number of Bohr-Sommerfeld leaves.*

A leaf $\ell$ of $P$ is a Bohr-Sommerfeld leaf if there is a nonzero flat section $s : \ell \to L$, and a theorem due to Guillemin and Sternberg [6] guarantees that the set of (Liouville tori) Bohr-Sommerfeld leaves is discrete.

The main problem of Sniatycki's result is that the topological condition on the leaf space is too strong. For the simplest example of lagrangian fibration, toric manifold, Delzant's result [7] implies that the leaf space is a polytope. Thus instead of regular lagrangian fibrations is not unnatural to consider singular foliations.

## 3. The nondegenerate singular case

For an integrable system $F : M^{2n} \to \mathbb{R}^n$ on a symplectic manifold the Liouville integrability condition implies that the distribution of the hamiltonian vector fields of the components of the moment map generates a lagrangian foliation (possible) with singularities; which is just the level sets of the moment map. This is an example of a generalised real polarisation, i.e.: a integrable subbundle of $TM$ whose leaves are generically lagrangian submanifolds (except for some singular isotropic leaves).

There is a classification of the kinds of singularities that can appear in integrable systems (see e.g.: [9] and references therein). Here it will be considered just the nondegenerate singularities (in the Morse-Bott sense). For them there exists a normal form [8, 9, 10] and the results of [2, 1] rely on it.

For toric manifolds Hamilton [2] has shown that Sniatycki's theorem holds and the elliptic singularities give no contribution to the quantisation (just the trivial vector space $\{0\}$). Also Sniatycki's theorem holds for 2-dimensional compact integrable systems, whose momentum map has only nondegenerate singularities, as has been show by Hamilton and Miranda [1]. In that case elliptic singularities give no contribution and each hyperbolic singularity gives a $\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}$ contribution. And they conjecture the following:

**Conjecture:** (Miranda and Hamilton) *For a $2n$-dimensional compact integrable system, whose momentum map has only nondegenerate singularities, $\mathcal{Q}(M) = H^n(M;\mathcal{J})$. Moreover, the cohomology $H^n(M;\mathcal{J})$ has contributions of the form $\mathbb{C}$ for each (Liouville tori) Bohr-Sommerfeld leaf, $\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}$ for the hyperbolic singularities and the elliptic points give no contribution (just the trivial vector space $\{0\}$).*

The idea to prove this is to use a singular Poincaré lemma [11] which gives:

**Theorem:** (Miranda and Solha)

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\mathrm{d}_\nabla} \mathcal{S}^1_P(L) \xrightarrow{\mathrm{d}_\nabla} \cdots \xrightarrow{\mathrm{d}_\nabla} \mathcal{S}^n_P(L) \xrightarrow{\mathrm{d}_\nabla} 0$$

*is a fine resolution for $\mathcal{J}$. where $\mathcal{S}^k_P(L)$ is the associated sheaf of $\Gamma(\bigwedge^k P^* \otimes L)$ and $\mathrm{d}_\nabla$ is the exterior derivative obtained by twisting the foliated cohomology exterior derivative $\mathrm{d}_P$ with the connection $\nabla$ of the prequantum line bundle. Therefore its cohomology computes geometric quantisation ($H^k(M;\mathcal{J}) \cong H^k(S_P{}^\bullet(L))$).*

With it is possible to mimic the proof of a Künneth formula which exists for the regular lagrangian case [3] to reduce to the 2-dimensional case.

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## Equivariant Intergation and moduli spaces

András Szenes

We give a review of localization principles and their application to the cohomology of moduli spaces.

Localization is a major tool in calculating classical enumerative quatities. As a basic example, consider the problem of finding the number of lines in 3-space intersecting 4 given lines. This number may be expressed as the intersection number on the Grassmannian: $\int_{\mathrm{Gr}(2,4)} c_1(E^*)^4$, where $E$ is the tautological bundle. Bott's localization theorem form the 60's gives a rather complicated formula for this quantity: a sum of 6 rational functions in 4 variables, which, after being brought to common denominators, reduces to an integer.

A more efficient method is the Witten-Jeffrey-Kirwan reduction principle from the early 90s. It uses the fact that $\mathrm{Gr}(2,4)$ may be obtained as a (GIT) quotient of the linear space of 2-by-4 matrices by the group $GL(2)$. One needs to take the compact diagonal torus $U(1)^2 \subset GL(2)$; denote the linear weights of this torus by $a$ and $b$. The fixed point set of this commutative subgroup consists of a single point: the zero matrix. The contribution at this point is given by a residue:

$$\int_{\mathrm{Gr}(2,4)} c_1(E^*)^4 = \operatorname*{Res}_{a=0} \operatorname*{Res}_{b=0} \frac{-(a-b)^2(a+b)^4 \, da\, db}{2a^4b^4}. \tag{1}$$

This residue may be easily calculated by hand: the result is 2.

Under conditions that the Chern classes generate cohomology, these intersection numbers give a complete description of the cohomology ring of a space $M$ endowed with a torus action with isolated fixed points. Moreover, this is done in the most natural manner, since residues have an internal duality, thus these cohomology rings end up being represented by residue cycles. This idea was developed for toric varieties in my joint work with Michele Vergne on the cohomology ring of toric varieties and mirror symmetry.

Remarkably, this approach may also be used for calculating the cohomology rings of non-compact varieties, which do not satisfy Poincaré duality. This may be done by formal Berline-Vergne localization. Assume that $M$ is a non-compact $T = U(1)$-manifold. Then, under appropriate conditions, calculating the fixed point contributions, one obtains a functional on the equivariant Chern ring of $M$ with values in meromorphic functions in the variable $u$, where the equivariant cohomology of a point is $H_T^*(\mathrm{pt}) = \mathbb{C}[u]$. Again, under certain conditions, this functional satisfies Poincaré duality, which allows us to calculate $H_T^*(M)$. Then one has $H^*(M) = H_T^*(M)/uH_T^*(M)$. Using this method, I, jointly with T. Hausel, obtained results on the cohomology ring of the moduli spaces of Higgs bundles in rank 2. This method corresponds to the Bott localization formula in the compact case described above.

Recently, my student Zs. Szilagyi, wrote down a formula for these equivariant intersection numbers in the case when we obtain our non-compact $T$-manifold $M$ as a $G$-quotient of a $T \times G$ manifold $\tilde{M}$. This formula is thus the equivariant version of the method represented by the formula (1) above. Using this new formula, one can attack a number of cohomology rings in an efficient manner. These calculations are joint work in progress.

## An equivariant Jeffrey-Kirwan formula in non-compact case

Zsolt Szilágyi

**Introduction.** We consider (non-compact) symplectic manifolds which admit Hamiltonian action of $S = S^1$ with compact $S$-fixed point sets.

Let $(M, \omega)$ be a symplectic manifold with commuting Hamiltonian actions of a compact group $G$ and the circle $S$ with moment maps $\mu : M \to \mathfrak{g}^*$ and $\varphi : M \to \mathfrak{s}^* \simeq \mathbb{R}$, respectively. We suppose that $\varphi$ is proper and bounded below. Moreover suppose that $0 \in \mathfrak{g}^*$ is a regular value of $\mu$ and consider the symplectic quotient $M_0 = M /\!\!/ G$. Define the integration on $M$ and $M_0$ by Atiyah-Bott-Berline-Vergne formula

$$\oint_M \alpha := \sum_{F \subset M^{T\times S}} \int_F \frac{i_F^* \alpha}{e_{T\times S}(\mathcal{N}_F)}, \qquad \alpha \in H_{G\times S}(M),$$

$$\oint_{M_0} \beta := \sum_{F_0 \subset M_0^S} \int_{F_0} \frac{i_{F_0}^* \beta}{e_S(\mathcal{N}_{F_0})}, \qquad \beta \in H_S(M_0),$$

where $T \subset G$ is a maximal torus of rank $r$.

**An equivariant version of the Jeffrey-Kirwan formula.** For $\alpha \in H_{G\times S}(M) \simeq H_{T\times S}(M)^W$ we have

$$\oint_{M_0} \kappa_S(\alpha e^{\omega-\mu-\varphi}) = \frac{1}{|W| vol(T)} \mathrm{JKRes}_u^{s>u} \left( \oint_M \Delta(u)\alpha(u,s) e^{\omega - \langle \mu_T, u\rangle - \varphi\cdot s} \right) du,$$

where $W$ is the Weyl group, $\Delta$ is the product of roots, $\kappa_S : H_{G\times S}(M) \to H_S(M_0)$ is the Kirwan map.

**Definition of** $\mathrm{JKRes}_u^{s>u}$**.** Let $\{u^1, \ldots, u^r\}$ be a basis of $\mathfrak{t}^*$ and $\{s\}$ be a basis of $\mathfrak{s}^* \simeq \mathbb{R}$. Consider linear forms $\lambda, \gamma_1, \ldots, \gamma_r$ on $\mathfrak{t} \oplus \mathfrak{s}$. The polarization of linear forms on $\mathfrak{t} \oplus \mathfrak{s}$ is induced by the ordered basis $\{s, u^1, \ldots, u^r\}$ as follows: $\gamma_i = \gamma_0 s + \gamma_1 u^1 + \ldots + \gamma_r u^r$ is polarized if the first non-zero coefficient is positive. (We suppose that all $\gamma_i$'s are polarized.) If $pr_1(\gamma_1), \ldots, pr_1(\gamma_r)$ form a basis of $\mathfrak{t}^*$ then we can write $\lambda = \lambda_0 s + \lambda_1\gamma_1 + \ldots + \lambda_r\gamma_r$ (suppose $\lambda_i \neq 0$) and

$$\mathrm{JKRes}_u^{s>u}\left(\frac{e^{\lambda}du}{\gamma_1^{n_1+1}\cdots\gamma_r^{n_r+1}}\right) = \begin{cases} \dfrac{1}{|\det[\gamma_{ij}]_{i,j=1}^r|}\dfrac{\lambda_1^{n_1}\cdots\lambda_r^{n_r}e^{\lambda_0 s}}{n_1!\cdots n_r!} & \lambda_1, \ldots, \lambda_r > 0, \\ 0 & \text{otherwise.} \end{cases}$$

**HyperKähler version.** Now let $(M, \omega_1, \omega_2, \omega_3)$ be a hyperKähler manifold. We suppose that $S$ is Hamiltonian with respect to the symplectic form $\omega_{\mathbb{R}} := \omega_1$ and the complex moment map $\mu_{\mathbb{C}} := \mu_2 + \sqrt{-1}\mu_3 : M \to \mathfrak{g}^* \otimes \mathbb{C}$ is $S$-equivariant: $\mu_{\mathbb{C}}(\sigma \cdot m) = \sigma^k \mu_{\mathbb{C}}(m)$, $\sigma \in S$. Consider the hyperKähler reduction $M/\!/\!/\!/G = \mu_{\mathbb{R}}^{-1}(\xi) \cap \mu_{\mathbb{C}}^{-1}(0)/G$. Then

$$\oint_{M/\!/\!/\!/G} \kappa(\alpha e^{\omega_{\mathbb{R}} - \mu + \xi - \varphi}) = \frac{1}{|W| vol(T)} \mathrm{JKRes}_u^{s>u}\left(\oint_M \Delta\Delta_{\mathbb{C}}\alpha e^{\omega_{\mathbb{R}} - \mu + \xi - \varphi}\right) du,$$

where $\Delta_{\mathbb{C}} = (ks)^r \prod_w (ks + w)$, (product over all roots).

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## Multiplicities of the equivariant index of a transversally elliptic operator


Michèle Vergne

(joint work with Claudio de Concini, Claudio Procesi)


Let $G$ be a compact Lie group acting on a $G$-manifold $M$. Let $\mathfrak{g}$ be the Lie algebra of $G$ with dual vector space $\mathfrak{g}^*$. Then $T^*M$ is a $G$-Hamiltonian manifold, with moment map $\mu : T^*M \to \mathfrak{g}^*$ given by $\langle \mu([x, \xi]), X \rangle = \langle \xi, X_x \rangle$. Here $x \in M, \xi \in T_x^*M$, $X \in \mathfrak{g}$ and $X_x$ is the tangent vector at $x$ produced by the infinitesimal action of $X \in \mathfrak{g}$. By definition $T_G^*M = \mu^{-1}(0)$.

An element $\sigma \in K_G^0(T_G^*M)$ will be called a transversally elliptic symbol. It can be represented as an equivariant morphism $\sigma([x, \xi]) : E_x^+ \to E_x^-$ between two equivariant bundles $E^{\pm}$ on $M$ (lifted to $T_G^*M$) invertible outside a compact subset of $T_G^*M$.

Atiyah-Singer ([1]) have associated to $\sigma$ a (virtual) trace class representation index($\sigma$) of $G$:

$$\operatorname{index}(\sigma) = \sum_{\lambda \in \hat{G}} m(\lambda)\lambda$$

where $m(\lambda) \in \mathbb{Z}$ is the multiplicity of the representation $\lambda$ in the representation index($\sigma$).

Our aim is to produce a cohomological formula for $m(\lambda)$. Remark that the computation of $m(\lambda)$ is already problematic, in the case of an elliptic symbol where Atiyah-Bott-Segal-Singer fixed point formula is available as a sum of meromorphic functions. This fixed point formula does not lead to an intrinsic formula for $m(\lambda)$. When $\sigma$ is not transversally elliptic, no fixed point formula is available.

Our idea is that $m(\lambda)$ is the quantum analogue of the Duistermaat-Heckman measure, so that the function $m(\lambda)$ should be a piecewise polynomial function on $\hat{G}$, computable in function of the equivariant Chern character of $\sigma$ and of the equivariant $\hat{A}$ genus of $T^*M$. We give a meaning to this statement in our work [4].

It is possible to reduce the computations to the case where $G$ is a torus. To simplify the statements of our results, in this abstract, we assume that the torus $G$ acts effectively on $M$ and that every stabilizer of a point $m \in M$ is connected.

Consider the equivariant cohomology with compact supports $H^*_{G,c}(T^*_G M)$. In [3], we have defined a map:

$$\text{infdex} : H^*_{G,c}(T^*_G M) \to \mathcal{D}'(\mathfrak{g}^*) \tag{1}$$

associating to an element $\alpha$ of $H^*_{G,c}(T^*_G M)$ a distribution infdex($\alpha$) on $\mathfrak{g}^*$.

For any $\alpha \in H^*_{G,c}(T^*_G M)$, the distribution infdex($\alpha$) is piecewise polynomial: there exists a finite union of affine hyperplanes $\mathcal{H} = \cup_i H_i$, and, on each connected component of the complement of $\mathcal{H}$, infdex($\alpha$) is given by a polynomial function.

We consider a piecewise polynomial function on $\mathfrak{g}^*$ as a function $f = (f_\tau)$ on the open set $\mathfrak{g}^* \setminus \mathcal{H}$. Here $\tau$ is a connected component (a tope) and $f_\tau$ a polynomial function on $\tau$. We can apply to a piecewise polynomial function $(f_\tau)$ a constant coefficient differential operator of infinite order $P(\partial)$. The formula is $P(\partial)_{pw}(f_\tau) = (P(\partial)f_\tau)$. We say that a piecewise polynomial function $f = (f_\tau)$ is continuous at a point $\lambda \in \mathfrak{g}^*$ if all the values $f_\tau(\lambda)$ are equal whenever the connected component $\tau$ contains $\lambda$ in its closure. We can then define $f(\lambda)$.

Assume that the tangent bundle to $M$ is stably trivial: there exists a real representation space $R$ of $G$ such that $TM$ is stably equivalent to $M \times R$ as a $G$-manifold. Consider the list $L$ of weights $\pm a \in \mathfrak{g}^*$ of the action of $G$ in $R_{\mathbb{C}}$ and the infinite order differential operator on $\mathfrak{g}^*$ defined by

$$\hat{A}(\partial) = \prod_{a \in L} \frac{\partial_a}{(1 - e^{-\partial_a})}.$$

We recall that the Chern character associate to a transversally elliptic symbol $\sigma$ an element ch($\sigma$) in (a completion of) $H^*_{G,c}(T^*_G M)$ (see [6]).

**Theorem**

Let $\sigma \in K^0_G(T^*_G M)$ be a transversally elliptic symbol. If the tangent bundle to $M$ is stably equivalent to $M \times R$, then

(i) the piecewise polynomial function $\hat{A}(\partial)_{pw}\text{infdex}(\text{ch}(\sigma))$ is continuous at any point $\lambda$ of the lattice $\hat{G} \subset \mathfrak{g}^*$.

(ii) we have

$$\text{index}(\sigma) = \sum_\lambda (\hat{A}(R)_{pw}\text{infdex}(\text{ch}(\sigma)))(\lambda)e^\lambda.$$

The idea of this formula comes by taking the Fourier transform of the cohomological index formula given by Berline-Vergne, [2] and by Paradan-Vergne [6]. The result follows from applying the miraculous deconvolution formula of Dahmen-Micchelli in approximation theory [5].

Finally, we can reduce the computation to the case treated above of stably trivial tangent bundle by embedding $M$ in a vector space $V$ and taking direct images.

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## Quantisation of the cotangent bundle of Lie groups

Siye Wu

(joint work with William D. Kirwin)

We consider a family of adapted complex structures on the cotangent bundle of a Lie group and find the BKS pairing relating the corresponding half-form quantisation. We show that the resulting bundle of quantum Hilbert spaces over the space of polarisations is flat. The vertical polarisation as a limit of complex polarisations yields the coherent state transform (or the Segal-Bargmann-Hall transform). We show that there is another limit of the complex polarisations that corresponds to the Peter-Weyl theorem.

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## Singularities and analytic torsion

Ken-Ichi Yoshikawa

### 1. Introduction

Let $M$ be a compact Kähler manifold and let $F$ be a holomorphic Hermitian vector bundle on $M$. Let $\square_{n,q} = 2(\bar{\partial}+\bar{\partial}^*)^2$ be the Laplacian acting on $(n,q)$-forms on $M$ with values in $F$ and let $\zeta_{n,q}(s)$ be its zeta function:

$$\zeta_{n,q}(s) := \sum_{\lambda\in\sigma(\square_{n,q})\setminus\{0\}} \lambda^{-s} \dim E(\lambda, \square_{n,q}),$$

where $\sigma(\square_{n,q})$ denotes the set of eigenvalues of $\square_{n,q}$ and $E(\lambda, \square_{n,q})$ denotes the eigenspace of $\square_{n,q}$ with eigenvalue $\lambda$. Let $\omega_M$ be the canonical bundle of $M$ and write $\omega_M(F) := \omega_M \otimes F$. Then the analytic torsion of $(M, \omega_M(F))$ is defined as

$$\tau(M, \omega_M(E)) := \exp[-\sum_{q\geq 0}(-1)^q q\, \zeta'_{n,q}(0)].$$

In this note, we fix the following notation. Let $X$ be a connected Kähler manifold of dimension $n+1$ and let $S \subset \mathbf{C}$ be the unit disc. Let $\pi\colon X \to S$ be a proper surjective holomorphic map with connected fibers. Let $\Sigma_\pi \subset X$ be the critical locus of $\pi$. We assume $\pi(\Sigma_\pi) = \{0\}$. Then $\pi\colon X \setminus \pi^{-1}(0) \to S \setminus \{0\}$ is a family of compact Kähler manifolds of dimension $n$. For $s \in S$, we set $X_s := \pi^{-1}(s)$. Then $X_s$ is equipped with the Kähler metric induced from the Kähler metric on $X$.

Let $\xi$ be a holomorphic Hermitian vector bundle on $X$. Set $\xi_s := \xi|_{X_s}$ for $s \in S$.

**Question 1.** *Determine the behavior of* $\tau(X_s, \omega_{X_s}(\xi_s))$ *as* $s \to 0$.

Under certain assumptions, we settle this question.

### 2. Nakano semi-positive vector bundles

Let $R^\xi$ be the curvature of $\xi$ with respect to the Chern connection and write $h_\xi(\sqrt{-1}\, R^\xi(\cdot), \cdot) = \sum_{i,j,\alpha,\beta} R_{\alpha\bar{\beta}i\bar{j}}\, (e^\vee_\alpha \otimes \bar{e}^\vee_\beta) \otimes (\theta_i \wedge \bar{\theta}_j)$, where $h_\xi$ is the Hermitian metric on $\xi$ and $\{e^\vee_\alpha\}$ (resp. $\{\theta_i\}$) is a local *unitary* frame of $\xi^\vee$ (resp. $\Omega^1_X$). Then $\xi$ is said to be *Nakano semi-positive* if for all $(\eta^\alpha_i) \in \mathbf{C}^{r(n+1)}$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}}\, \eta^\alpha_i \bar{\eta}^\beta_j \geq 0.$$

If $\xi$ is Nakano semi-positive on $X$, then the direct image sheaf $R^q\pi_*\omega_{X/S}(\xi)$ is a torsion-free sheaf on $S$ by [5], so that $R^q\pi_*\omega_{X/S}(\xi)$ has the structure of a holomorphic vector bundle on $S$ for all $q \geq 0$. Set $\ell_q := \operatorname{rk} R^q\pi_*\omega_{X/S}(\xi) \in \mathbf{Z}_{\geq 0}$.

## 3. Semistable reduction and comparison of direct images

Let $(T, 0)$ be another unit disc of $\mathbf{C}$. By the semi-stable reduction theorem [3], there exist an integer $\nu > 0$ and a diagram

$$\begin{array}{ccc} F\colon (Y, Y_0) & \to & (X, X_0) \\ f\downarrow & & \downarrow\pi \\ \mu\colon (T, 0) & \to & (S, 0) \end{array}$$

where $\mu(t) = t^\nu$, $Y_0$ is a *reduced*, normal crossing divisor of $Y$, and $Y$ is smooth.

Since $F^*\xi$ is Nakano semi-positive, $R^q f_*\omega_{Y/T}(F^*\xi)$ is a holomorphic vector bundle on $T$. By [4], there is a natural injective homomorphism of holomorphic vector bundles of equal rank $\ell_q$

$$\varphi_q\colon R^q f_*\omega_{Y/T}(F^*\xi) \hookrightarrow \mu^* R^q\pi_*\omega_{X/S}(\xi).$$

Hence we may regard $R^q f_*\omega_{Y/T}(F^*\xi) \subset \mu^* R^q\pi_*\omega_{X/S}(\xi)$ via $\varphi_q$ and we can write $(\mu^* R^q\pi_*\omega_{X/S}(\xi)/R^q f_*\omega_{Y/T}(F^*\xi))_0 \cong \bigoplus_{1\leq\alpha\leq\ell_q} \mathbb{C}\{t\}/(t^{e_\alpha^{(q)}})$, where $e_\alpha^{(q)} \in \mathbf{Z}_{\geq 0}$.

## 4. The Gauss map

Let $\mathbf{P}(TX)^\vee$ be the projective bundle such that $\mathbf{P}(TX)^\vee_x$ is the set of hyperplanes of $T_xX$ for $x \in X$. Define the Gauss map $\gamma\colon X \setminus \Sigma_\pi \to \mathbf{P}(TX)^\vee$ by

$$\gamma(x) = [\ker(\pi_*)_x] = [T_x X_{\pi(x)}] \in \mathbf{P}(T_xX)^\vee, \qquad x \in X \setminus \Sigma_\pi.$$

By [2], there is a resolution $q\colon (\widetilde{X}, E) \to (X, \Sigma_\pi)$ of the indeterminacy of $\gamma$. Namely,

- $q|_{\widetilde{X}\setminus E}\colon \widetilde{X} \setminus E \cong X \setminus \Sigma_\pi$ is an isomorphism;
- $\widetilde{\gamma} := \gamma \circ q$ extends to a holomorphic map from $\widetilde{X}$ to $\mathbf{P}(TX)^\vee$.

Let $\mathcal{H}$ be the tautological quotient bundle on $\mathbf{P}(TX)^\vee$; we get an exact sequence $0 \to \mathcal{U} \to \Pi^*TX \to \mathcal{H} \to 0$ of holomorphic vector bundles on $\mathbf{P}(TX)^\vee$, where $\Pi\colon \mathbf{P}(TX)^\vee \to X$ is the projection and $\mathcal{U} \to \mathbf{P}(TX)^\vee$ is the universal bundle.

## 5. The main theorems

**Theorem 1.** [7] *If $\xi$ is Nakano semi-positive on $X$ and if there is a projective algebraic manifold containing $X$ as an open subset, then as $s \to 0$*

$$\log\tau(X_s, \omega_{X_s}(\xi_s)) = \left(\alpha + \frac{\chi}{\deg\mu}\right)\log|s|^2 + \rho\,\log(-\log|s|^2) + c + O\left(\frac{1}{\log|s|}\right),$$

*where $\rho \in \mathbf{Z}$ (see the end of this note for its formula), $c \in \mathbf{R}$ and*

$$\alpha := \int_{(\pi\circ q)^{-1}(0)\cap E} \widetilde{\gamma}^*\left\{\frac{\operatorname{Td}(\mathcal{H}^\vee)^{-1} - 1}{c_1(\mathcal{H}^\vee)}\right\} q^*\{\operatorname{Td}(TX)\operatorname{ch}(\xi)\} \in \mathbf{Q},$$

$$\chi := \sum_{q\geq 0}(-1)^q \dim_{\mathcal{O}_T/\mathfrak{m}_0}\left(\frac{\mu^* R^q\pi_*\omega_{X/S}(\xi)}{R^q f_*\omega_{Y/T}(F^*\xi)}\right)_0 = \sum_{q\geq 0}(-1)^q\sum_\lambda e_\lambda^{(q)} \in \mathbf{Z}.$$

**Theorem 2.** [7] *Assume that $\xi$ is Nakano semi-positive on $X$. If $X_0$ is reduced, normal and has only canonical (equivalently rational) singularities, then there exist $c \in \mathbf{C}$, $r \in \mathbf{Q}_{>0}$, $l \in \mathbf{Z}_{\geq 0}$ such that as $s \to 0$*

$$\log\tau(X_s, \omega_{X_s}(\xi_s)) = \alpha\,\log|s|^2 + c + O\left(|s|^r(\log|s|)^l\right).$$

**Theorem 3.** [8] *The complex Hessian $\partial_{s\bar{s}}\log\tau(X_s,\omega_{X_s}(\xi_s))$ has Poincaré growth:*

$$\partial_{s\bar{s}}\log\tau(X_s,\omega_{X_s}(\xi_s)) = \frac{\rho}{|s|^2(\log|s|)^2} + O\left(\frac{1}{|s|^2(\log|s|)^3}\right) \qquad (s\to 0).$$

The key to the proofs of these theorems is the following structure theorem for the singularity of $L^2$-metric on $R^q\pi_*\omega_{X/S}(\xi)$, as well as the structure theorem for the singularity of Quillen metric on $\det R\pi_*\omega_{X/S}(\xi)$ [1], [6].

**Theorem 4.** [7] *Endow $R^q\pi_*\omega_{X/S}(\xi)$ with the $L^2$-metric $h_{L^2}$.*

(1) *By a suitable choice of a basis $\{\Psi_\alpha^{(q)}\}$ of $R^q\pi_*\omega_{X/S}(\xi)$, the Hermitian matrix $G(s) := \left(h_{L^2}(\Psi_\alpha^{(q)}(s), \Psi_\beta^{(q)}(s))\right) \in \mathrm{Herm}(\ell_q)$ admits the expression*

$$G(t^\nu) = D(t)\cdot H(t)\cdot\overline{D(t)}, \qquad D(t) = \mathrm{diag}(t^{-e_1^{(q)}},\dots,t^{-e_{\ell_q}^{(q)}}).$$

(2) *One has the expression on $T\setminus\{0\}$*

$$H(t) = \sum_{m=0}^{n} A_m(t)\,(\log|t|^2)^m, \quad A_m(t)\in C^\infty(S,\mathrm{Herm}(\ell_q)).$$

(3) *Defining the real-valued functions $a_k(t)\in C^\infty(S)$, $0\leq k\leq n\ell_q$, by*

$$\det H(t) = \sum_{k=0}^{n\ell_q} a_k(t)\,(\log|t|^2)^k,$$

*one has $a_k(0)\neq 0$ for some $0\leq k\leq n\ell_q$. Set $\rho_q := \max\{k;\, a_k(0)\neq 0\}$. Then*

$$\det H(t) = (\log|t|^2)^{\rho_q}\,\{a_{\rho_q}(0) + O(1/\log|t|)\}.$$

By this last estimate, we get $\rho = \sum_q(-1)^q\rho_q$.

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## Quantization on noncompact symplectic manifolds

Weiping Zhang
(joint work with Xiaonan Ma)

In this talk we report our joint work with Xiaonan Ma on the resolution of a conjecture due to Michèle Vergne [13] concerning the geometric quantization on noncompact symplectic manifolds.

To be more precise, let $(M, \omega)$ be a (not necessarily) compact symplectic manifold with symplectic form $\omega$. We assume that $(M, \omega)$ is prequantizable, that is, there exists a complex line bundle $L$ (called a prequantized line bundle) carrying a Hermitian metric $h^L$ and a Hermitian connection $\nabla^L$ such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega. \tag{1}$$

Let $J$ be an almost complex structure on $TM$ such that

$$g^{TM}(u, v) = \omega(u, Jv), \quad u,\ v \in TM \tag{2}$$

defines a $J$-invariant Riemannian metric on $TM$.

Let $G$ be a compact connected Lie group with Lie algebra denoted by $\mathbf{g}$. We assume that the compact connected Lie group $G$ acts on $M$ and that this action lifts to an action on $L$. Moreover, we assume that $G$ preserves $g^{TM}$, $J$, $h^L$ and $\nabla^L$.

For any $K \in \mathbf{g}$, let $K^M$ be the vector field generated by $K$ over $M$.

Let $\mu : M \to \mathbf{g}^*$ be the moment map defined by the Kostant formula

$$2\pi\sqrt{-1}\mu(K) = \nabla^L_{K^M} - L_K,\ K \in \mathbf{g}. \tag{3}$$

Then $\mu$ verifies the Hamiltonian action condition that for any $K \in \mathbf{g}$,

$$d\mu(K) = i_{K^M}\omega. \tag{4}$$

From now on, we assume that the following fundamental assumption holds.

**Fundamental Assumption**. The moment map $\mu : M \to \mathbf{g}^*$ is proper, in the sense that the inverse image of a compact subset is compact.

Fix a maximal torus of $G$ and let $\Lambda^*_+ \subset \mathbf{g}^*$ be the corresponding set of dominant weights of irreducible representations of $G$.

Take any $\gamma \in \Lambda^*_+$. If $\gamma$ is a regular value of the moment map $\mu$, then one can construct the Marsden-Weinstein symplectic reduction $(M_\gamma, \omega_\gamma)$, where $M_\gamma = \mu^{-1}(G\cdot\gamma)/G$ is a *compact* orbifold (since $\mu$ is proper). Moreover, the line bundle $L$ (resp. the almost complex structure $J$) induces a prequantized line bundle $L_\gamma$ (resp. an almost complex structure $J_\gamma$) over $(M_\gamma, \omega_\gamma)$. One can then construct the associated Spin$^c$-Dirac operator (twisted by $L_\gamma$), $D_+^{L_\gamma} : \Omega^{0,\mathrm{even}}(M_\gamma, L_\gamma) \to \Omega^{0,\mathrm{odd}}(M_\gamma, L_\gamma)$ on $M_\gamma$, of which the index

$$(5) \qquad Q(L_\gamma) := \mathrm{Ind}\left(D_+^{L_\gamma}\right) := \dim \mathrm{Ker}\left(D_+^{L_\gamma}\right) - \dim \mathrm{Coker}\left(D_+^{L_\gamma}\right) \in \mathbf{Z},$$

is well-defined. If $\gamma \in \Lambda_+^*$ is not a regular value of $\mu$, then by proceeding as in [8] (cf. [9, §7.4]), one still gets a well-defined quantization number $Q(L_\gamma)$ extending the above definition.

On the other hand, let $\mathbf{g}^*$ be equipped with an $\mathrm{Ad}_G$-invariant metric. Then $\mathcal{H} = |\mu|^2$ is $G$-invariant. Let $X^{\mathcal{H}} = -J(d\mathcal{H})^*$ be the Hamiltonian vector field associated to $\mathcal{H}$.

Since $\mu$ is proper, for any $a > 0$, $M_a := \mathcal{H}^{-1}([0,a]) = \{x \in M : \mathcal{H}(x) \leq a\}$ is a compact subset of $M$. Recall that by Sard's theorem, the set of critical values of the function $\mathcal{H} : M \to \mathbf{R}$ is of measure zero.

For any regular value $a > 0$ of $\mathcal{H}$, it is clear that $X^{\mathcal{H}}$ is nowhere zero on $\partial M_a = \mathcal{H}^{-1}(a)$. Thus the triple $(M_a, X^{\mathcal{H}}, L)$ defines a transversally elliptic symbol

$$(6) \qquad \sigma_{L,X^{\mathcal{H}}}^{M_a} = \sqrt{-1}c(\cdot + X^{\mathcal{H}}) \otimes \mathrm{Id}_L,$$

where $c(\cdot)$ is the Clifford action on $\Lambda(T^{*(0,1)}M)$, in the sense of Atiyah [1] and Paradan [9]. Let $\mathrm{Ind}(\sigma_{L,X^{\mathcal{H}}}^{M_a}) \in R[G]$ be the corresponding transversal index in the sense of [9].

For any $\gamma \in \Lambda_+^*$, let $V_\gamma^G$ denote the corresponding irreducible representation of $G$, let $Q(L)_a^\gamma \in \mathbf{Z}$ denote the multiplicity of $V_\gamma^G$ of $\mathrm{Ind}(\sigma_{L,X^{\mathcal{H}}}^{M_a}) \in R[G]$.

**Theorem 1.** *a) For any $\gamma \in \Lambda_+^*$, there exists $a_\gamma > 0$ such that $Q(L)_a^\gamma \in \mathbf{Z}$ does not depend on $a \geq a_\gamma$, with $a$ a regular value of $\mathcal{H}$.*

*b) $Q(L)_a^{\gamma=0} \in \mathbf{Z}$ does not depend on $a > 0$, with $a$ a regular value of $\mathcal{H}$.*

According to Theorem 1, for any $\gamma \in \Lambda_+^*$, we have a well-defined integer $Q(L)_a^\gamma$ not depending on the large enough regular value $a > 0$. From now on we denote it by $Q(L)^\gamma$.

**Theorem 2.** *For any $\gamma \in \Lambda_+^*$, the following identity holds,*

$$(7) \qquad Q(L)^\gamma = Q(L_\gamma).$$

*Remark 3.* If the zero set of $X^{\mathcal{H}}$ is compact, then Theorem 1 was already known, while Theorem 2 was conjectured by Michèle Vergne in [13, §4.3]. Thus Theorem 2 provides a solution of Vergne's conjecture even when the zero set of $X^{\mathcal{H}}$ is non-compact. If $M$ is compact, then Theorem 2 reduces to the famous Guillemin-Sternberg geometric quantization conjecture [4] first proved in [7] and [8].

*Outline of Proof.* Our proof of Theorems 1 and 2 is analytic. We first interpret the transversal indices appearing in the context through the analytic indices of Atiyah-Patodi-Singer [2] type, by making use of a result of Braverman [3]. We then prove Theorems 1 and 2 by analyzing the corresponding APS type indices, by adapting the analytic methods developed in [11] and [12]. Extra difficulties appear in dealing with the $\gamma \neq 0$ case. For more details, see [5] and [6].

*Remark 4.* For an alternate proof of Theorems 1 and 2, see [10].

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