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Automorphic Forms: New Directions

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Abstract. The workshop on Automorphic Forms: New Directions provided a nice glimpse of the many streams of current research activity in this very active area. Topics included the relative trace formula and periods of automorphic forms, Arthur packets and locally/globally generic representations, Eisenstein cohomology, special values of L-functions, algebraic modular forms, p-adic modular forms, arithmetic theta functions, endoscopy and CAP representations, and proofs of the Gross-Prasad conjecture and the local Langlands conjecture for GL(n). The group of participants was notably broad in terms of nationality and age and the meeting confirmed the continued vigor of research in the field of automorphic representations.

Mathematics Subject Classification (2000): 11Fxx.

Introduction by the Organisers

The theory of automorphic representations has been an extremely active area of research over the past four decades since the introduction of the powerful tools of representation theory into the classical theory of automorphic forms by Langlands, Harish-Chandra, Piatetski-Shapiro and others. The subject already had very deep roots in number theory and geometry and there is now a vast program of conjectures encompassing, on the one hand, the theory of automorphic representations per se and, on the other, Grothendieck's theory of motives. Much progress has been made in recent years. The main goal of this meeting was to survey the most recent developments and to provide a glimpse of the new directions that are opening up, where one might imagine important future growth will take place. The wide range of current research was evident as topics included: periods of automorphic representations and the relative trace formula (Lapid, Feigon, Sakellaridis), Arthur packets, locally and globally generic representations and the Ramanujan conjecture (Shahidi), Eisenstein cohomology and applications to special values of L-functions (Harder, Grbac), algebraic modular forms (Buzzard), p-adic modular forms (Mahnkopf) and automorphic forms valued in arithmetic Chow groups (Liu), endoscopic transfer and CAP representations (Soudry, Jiang), automorphic forms on covering groups (Ikeda, Savin), and existence questions (Muic). Two highlights were the lecture by Waldspurger detailing his proof of the local Gross-Prasad conjecture for orthogonal groups and the lecture by Scholze explaining his new proof of the local Langlands conjecture for GL(n). The group of participants was notably broad in terms of nationality and age, and the meeting confirmed the continued vigor of research in the theory of automorphic representations.

There were 44 participants, coming mainly from Europe, North America and Asia, among them 4 young researchers who participated as Oberwolfach Leibniz Graduate Students and 2 US Junior Oberwolfach Fellows. The organizers are very grateful to the Leibniz-Gemeinschaft and the NSF for this support. The staff of the Mathematische Forschungsinstitut Oberwolfach was - as always - extremely supportive and helpful. We thank them for providing excellent working conditions.

Workshop: Automorphic Forms: New Directions

Table of Contents

Abstracts

Algebraic automorphic representations

Kevin Buzzard

Weil observed that if χ was an algebraic grössencharacter then one can attach a 1-dimensional p-adic Galois representation to χ. One can even construct a family of representations—one for each prime number p (or more generally for each finite place of the coefficient field).

Taniyama, in his work on Tate modules of CM abelian varieties, was led to the more general notion of a compatible family of Galois representations of higher dimension. Eichler and Shimura attached a compatible family of 2-dimensional Galois representations to a holomorphic modular eigenform of weight 2. Serre quickly noticed that various congruences for the coefficients of the Δ function noticed by Ramanujan and others could be explained by a generalisation of this picture to higher weight modular forms, and shortly afterwards these representations were constructed by Deligne.

The notion of what it meant for an automorphic representation for a general connected reductive G to be "of type A_0", or "algebraic", was formulated in the 1970s, but perhaps not much was done with it at the time, in that generality. It was Clozel who specialised to the case of $\mathrm{GL}(n)$ over a number field, and conjectured that given an algebraic automorphic representation π of $\mathrm{GL}(n)$ there should be a compatible family of n-dimensional p-adic Galois representations attached to π. Clozel also formulated a notion of "arithmetic" for automorphic representations of $GL(n)$ and conjectured that π was arithmetic if and only if it was algebraic.

Inspired by work of Toby Gee on the emerging p-adic and mod p Langlands philosophy, Gee and I set out to generalise Clozel's conjectures to an arbitrary connected reductive group over a number field. This was not supposed to be a big project—the idea was just that we should formulate some notion of what it means for an automorphic representation of $G(\mathbf{A}_K)$ to be arithmetic/algebraic ($\mathbf{A}_K$ the adeles of a number field K, G now an arbitrary connected reductive group), and to an algebraic automorphic representation there should be an associated compatible family of p-adic Galois representations, presumably (given what we know about the Satake isomorphism) taking values in the ($\overline{\mathbf{Q}}_p$-valued points of the) L-group of G.

Here is what went wrong, and which turned this idle question into a paper.

1) Given an elliptic curve $E/\mathbf{Q}$ there's an associated automorphic representation of $\mathrm{GL}(2,\mathbf{A}_\mathbf{Q})$, with trivial central character. This automorphic representation is "clearly" algebraic (in the sense that it is cohomological, so any theory that suggests that such a gadget is not algebraic is probably not the right theory!). It descends to an automorphic representation of $\mathrm{PGL}(2,\mathbf{A}_\mathbf{Q})$ which is also "clearly" algebraic. And yet the p-adic Galois representation attached to this automorphic representation does not take values in $\mathrm{SL}(2,\overline{\mathbf{Q}}_p)$, not even after twisting (because

the cyclotomic character has no square root; think about complex conjugation). Yet SL(2) is the L-group of PGL(2).

2) However, given a representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/K) \to \mathrm{GL}(2, \mathbf{C})$ with $\det(\rho(c)) = 1$ (c a complex conjugation), there is conjectured to be, and if the image of ρ is solvable then there really is, an associated automorphic representation on $\mathrm{GL}(2, \mathbf{A_Q})$, which is also "clearly" algebraic. Furthermore, this automorphic reprsesentation has trivial central character (and hence descends to $\mathrm{PGL}(2, \mathbf{A_Q})$ if, and only if, the image of ρ lands in $\mathrm{SL}(2, \mathbf{C})$. This and the previous observation seem to almost lead to a contradiction!

3) Clozel, Harris and Taylor used the cohomology of Shimura varieties to attach Galois representations to certain cohomological automorphic representations of certain unitary groups U. One would perhaps expect that the associated Galois representations were taking values in the L-group of U. But this is easily checked not to be the case in general—indeed the Galois representations are taking values in a group whose dimension is one bigger than that of the L-group of U (rather analogous to 1) above). It was suggested to us that perhaps the group recieving the Galois representations was the L-group of the associated general unitary group—but we could prove that this was not in general the case. So whatever is the *recipe* being used to predict where an Frobenius element should be sent, if it is not the element of the L-group given by Langlands' interpretation of the Satake isomorphism?

What we thought would be an elementary exercise turned into quite an interesting puzzle! Now Gee and I have a very good understanding of what we believe is the conjectural solution to this puzzle. we can formulate not one but two notions of "algebraic" in the theory of automorphic representations (we called them "L-algebraic" and "C-algebraic"), and not one but two notions of "arithmetic" (we called them "L-arithmetic" and "C-arithmetic"; L is ambiguous and either stands for "Langlands" or "L-function"; C is also ambiguous and either stands for "Clozel" or "cohomological"). One might conjecture that π is L-algebraic iff it is L-arithmetic, and C-algebraic iff it is C-arithmetic. Analysing the difference between the L- and the C- notions, and how and when it is possible to move by twisting from one notion to the other, cleared up the issues numbered 1) to 3) above completely, in the sense that we now have a complete conjectural understanding of what is going on.

Progrès récents sur la conjecture locale de Gross-Prasad

Jean-Loup Waldspurger

1. Enoncé de la conjecture

Soient F une extension finie d'un corps $\mathbb{Q}_p$, V un espace vectoriel sur F de dimension finie d, q une forme quadratique non dégénérée sur V, $V = W \oplus D$ une décomposition orthogonale, où D est une droite. On note G et H les groupes spéciaux orthogonaux de V et W (H s'identifie au sous-groupe des éléments de G qui agissent par l'identité sur D). Soient π et σ des représentations lisses irréductibles de $G(F)$, resp. $H(F)$, dans un espace complexe E_π, resp. E_σ. On note $Hom_{H(F)}(E_\pi, E_\sigma)$ l'espace des homomorphismes de $H(F)$-modules de E_π dans E_σ et on note $m(\pi,\sigma)$ la dimension de cet espace. D'après [1], on a $m(\pi,\sigma) \leq 1$. La conjecture locale de Gross-Prasad calcule cette multiplicité $m(\pi,\sigma)$ sous certaines hypothèses.

La conjecture s'énonce plus simplement si, au lieu d'un espace V, on en considère deux. Supposons pour fixer les idées d impair et $d \geq 5$. On sait que, pour un discriminant fixé, il y a deux classes d'équivalence d'espaces quadratiques (c'est-à-dire munis d'une forme quadratique non dégénérée) de dimension d et du discriminant fixé. Leurs groupes spéciaux orthogonaux sont formes intérieures l'un de l'autre, l'un est déployé et l'autre n'est pas quasi-déployé. On introduit, pour un discriminant fixé qui importe peu, ces deux espaces que l'on note V_i et V_a, de groupes spéciaux orthogonaux G_i et G_a, et l'on suppose G_i déployé. On suppose donnée une décomposition orthogonale $V_i = W_i \oplus D$. Il y a alors aussi une décomposition orthogonale $V_a = W_a \oplus D$, avec la même droite quadratique D. Les espaces quadratiques W_i et W_a ont mêmes dimensions et discriminants, mais ne sont pas équivalents. On note H_i et H_a leurs groupes spéciaux orthogonaux.

Notons W_F le groupe de Weil de F et $W_{DF} = W_F \times SL(2,\mathbb{C})$ le groupe de Weil-Deligne. Considérons les homomorphismes $\varphi : W_{DF} \to Sp(d-1,\mathbb{C})$ tels que la restriction de φ à $SL(2,\mathbb{C})$ soit algébrique et le composé de φ avec l'injection $Sp(d-1,\mathbb{C}) \to GL(d-1,\mathbb{C})$ soit semi-simple. Le groupe $Sp(d-1,\mathbb{C})$ agit par conjugaison sur cet ensemble d'homomorphismes. Notons $\Phi^{orth}(d)$ l'ensemble des classes de conjugaison. A tout $\varphi \in \Phi^{orth}(d)$, la conjecture de Langlands associe un L-paquet fini $\Pi^{G_i}(\varphi)$, resp. $\Pi^{G_a}(\varphi)$, de représentations lisses irréductibles de $G_i(F)$, resp. $G_a(F)$.

Le discriminant commun de W_i et W_a définit un caractère quadratique de W_F, notons-le δ. On considère les homomorphismes $\varphi' : W_{DF} \to O(d-1,\mathbb{C})$ vérifiant les mêmes conditions que ci-dessus et tels que $det \circ \varphi'_{|W_F} = \delta$. Le groupe $SO(d-1,\mathbb{C})$ agit par conjugaison sur l'ensemble de ces homomorphismes. Notons $\Phi^{orth}(d-1,\delta)$ l'ensemble des classes de conjugaison. A tout $\varphi' \in \Phi^{orth}(d-1,\delta)$, la conjecture de Langlands associe un L-paquet fini $\Pi^{H_i}(\varphi')$, resp. $\Pi^{H_a}(\varphi')$, de représentations lisses irréductibles de $H_i(F)$, resp. $H_a(F)$.

Pour $\varphi \in \Phi^{orth}(d)$, on dit que φ est générique si et seulement si $\Pi^{G_i}(\varphi)$ contient une représentation admettant un modèle de Whittaker. Une définition analogue vaut pour $\varphi' \in \Phi^{orth}(d-1,\delta)$.

Conjecture. *Soient $\varphi \in \Phi^{orth}(d)$ et $\varphi' \in \Phi^{orth}(d-1,\delta)$ deux éléments génériques. Alors il y a un unique couple $(\pi,\sigma) \in (\Pi^{G_i}(\varphi) \times \Pi^{H_i}(\varphi')) \sqcup (\Pi^{G_a}(\varphi) \times \Pi^{H_a}(\varphi'))$ tel que $m(\pi,\sigma) = 1$. Pour les autres couples, cette multiplicité est nulle.*

Cf. [3] conjecture 6.9. Soient φ et φ' comme dans cet énoncé. Notons $S(\varphi)$ le centralisateur dans $Sp(d-1,\mathbb{C})$ de l'image de φ et $S(\varphi')$ le centralisateur dans $SO(d-1,\mathbb{C})$ de celle de φ'. Ce sont des groupes algébriques. Leurs groupes de composantes $S(\varphi)/S(\varphi)^0$ et $S(\varphi')/S(\varphi')^0$ sont abéliens, isomorphes à des produits finis de copies de $\mathbb{Z}/2\mathbb{Z}$. On note $(S(\varphi)/S(\varphi)^0)^\vee$ et $(S(\varphi')/S(\varphi')^0)^\vee$ leurs duaux. Une deuxième assertion de la conjecture de Langlands est qu'il y a des bijections

$$\begin{array}{ccc} (S(\varphi)/S(\varphi)^0)^\vee & \to & \Pi^{G_i}(\varphi) \sqcup \Pi^{G_a}(\varphi) \\ \epsilon & \mapsto & \pi(\varphi, \epsilon) \end{array}$$

et

$$\begin{array}{ccc} (S(\varphi')/S(\varphi')^0)^\vee & \to & \Pi^{H_i}(\varphi') \sqcup \Pi^{H_a}(\varphi') \\ \epsilon' & \mapsto & \pi(\varphi', \epsilon'). \end{array}$$

Ces bijections doivent satisfaire des propriétés très contraignantes liées à la théorie de l'endoscopie. Gross et Prasad définissent deux éléments $\epsilon_{\varphi,\varphi'} \in (S(\varphi)/S(\varphi)^0)^\vee$ et $\epsilon'_{\varphi,\varphi'} \in (S(\varphi')/S(\varphi')^0)^\vee$, en utilisant des valeurs de facteurs ϵ.

Conjecture. *L'unique couple* (π, σ) *de la conjecture précédente est le couple* $(\pi(\varphi, \epsilon_{\varphi,\varphi'}), \pi(\varphi', \epsilon'_{\varphi,\varphi'}))$.

Cf. [3] conjecture 6.9.

2. Généralisations

Dans [4], Gross et Prasad généralisent les définitions et conjectures ci-dessus en remplaçant la décomposition $V = W \oplus D$ par $V = W \oplus Z$, où Z est un espace quadratique de dimension impaire dont le groupe spécial orthogonal est déployé.

Ces conjectures concernent des couples (G, H) de groupes spéciaux orthogonaux en dimensions de parités distinctes. Dans [2], Gan, Gross et Prasad posent des conjectures similaires pour d'autres couples de groupes (G, H):

- les couples de groupes unitaires en dimensions de parités distinctes;
- les couples formés d'un groupe symplectique et d'un groupe métaplectique (revêtement de degré 2 d'un groupe symplectique);
- les couples de groupes unitaires en dimensions de même parité.

Dans les deux derniers cas, la représentation de Weil est utilisée dans la définition des multiplicités.

3. Résultats dans le cas spécial orthogonal

Les résultats valent pour des couples de groupes spéciaux orthogonaux en dimensions de parités distinctes, c'est-à-dire dans la situation de la section 1 ou la généralisation du premier paragraphe de la section 2. Pour les groupes classiques, les conjectures de paramétrage sont en passe d'être démontrées par Arthur, nous les admettons. On a alors le théorème suivant.

Théorème. *Les conjectures de la section 1 sont vérifiées.*

Cf. [6], [5]. La preuve repose sur trois ingrédients:

- des formules intégrales calculant les multiplicités, ou des valeurs de facteurs ϵ, en termes de caractères de représentations, dans le cas où ces représentations sont tempérées;
- la théorie de l'endoscopie et de l'endoscopie tordue pour les groupes spéciaux orthogonaux et les groupes linéaires tordus;

- le théorème suivant, dû à Moeglin; on le formule pour les groupes spéciaux orthogonaux en dimension impaire, mais il vaut aussi en dimension paire ou pour les groupes symplectiques.

Théorème. *Soit $\varphi \in \Phi^{orth}(d)$. Alors φ est générique si et seulement si tout élément de $\Pi^{G_i}(\varphi) \sqcup \Pi^{G_a}(\varphi)$ est l'induite irréductible d'une représentation tempérée d'un sous-groupe de Levi.*

Cf. [5], corollaire 2.14.

References

[1] A. Aizenbud, D. Gourevitch, S. Rallis, G. Schiffmann, *Multiplicity one theorems*, Annals of Math. **172** (2010), 1407–1434.

[2] W. T. Gan, B. Gross, D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, prépublication (2009)

[3] B. Gross, D. Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}*, Can. J. of Math. **44** (1992), 974-1002.

[4] B. Gross, D. Prasad, *On irreducible representations of $SO_{2n+1} \times SO_{2m}$*, Can. J. of Math. **46** (1994), 930-950.

[5] C. Moeglin, J.-L. Waldspurger, *La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général*, prépublication (2009)

[6] J.-L. Waldspurger, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spéciaux orthogonaux*, prépublication (2009)

Unitary periods and distinction; representation-theoretic aspects

Erez Lapid

(joint work with Brooke Feigon, Omer Offen)

Let G be a reductive group defined over a local field F and let H be a Zariski closed subgroup of G defined over F. We also write $G = G(F)$, $H = H(F)$. For a representation (π, V) of G consider

$$\operatorname{Hom}_H(\pi, \mathbf{C}) = \{\ell \in V^* : \ell(\pi(h)v) = \ell(v) \quad \forall h \in H, v \in V\}.$$

One says that (G, H) is a *Gelfand pair* if for every irreducible representation π of G we have $\dim \operatorname{Hom}_H(\pi, \mathbf{C}) \leq 1$. Equivalently, $\dim \operatorname{Hom}_G(\pi, C^\infty(H\backslash G)) \leq 1$. Suppose for simplicity that G is quasi-split and let B be a Borel subgroup defined over F. A necessary condition (even for finite multiplicity) is that H has an open orbit $\mathcal{O}$ on $B\backslash G$, so that H is a *spherical subgroup*. An important special case of spherical subgroups is the fixed points of an involution (defined over F) – the *symmetric case*. Not every spherical subgroup gives rise to a Gelfand pair. There is (at least) a cohomological obstruction: $H(F)$ should act transitively on $\mathcal{O}(F)$. Indeed, for a principal series $\pi = \operatorname{Ind}_B^G(\chi)$ "in general position" (suitably defined) we have

$$\dim \operatorname{Hom}_H(\pi, \mathbf{C}) \geq \#[H(F)\text{-orbits in } \mathcal{O}(F)]$$

(with equality in many cases – cf. [40]). The notion of Gelfand pairs has been studied a lot in the literature since the pioneering work of Gelfand, Kazhdan and

Bernstein [17, 6, 8]. In recent years there have been spectacular developments in showing that certain pairs (G, H) are Gelfand pairs, and even stronger results on multiplicity one taking into account L-packets and forms of (G, H). (Aizenbud-Gourevitch [3, 2], Waldspurger [47, 46, 32] and others).

Sakellaridis and Sakellaridis-Venkatesh study many aspects of spherical subgroups (both local and global) [39, 41]. For instance they show that $\dim \mathrm{Hom}_H(\pi, \mathbf{C})$ is always finite in that case. In the Archimedean case, harmonic analysis on $H \backslash G$ was studied extensively in the symmetric case by van den Ban, Delorme, Schlichtkrull and many others.

1. Distinguished representations

If $\mathrm{Hom}_H(\pi, \mathbf{C}) \neq 0$, we say that π is *H-distinguished*. One motivation for this notion is global: a non-zero period integral

$$\int_{H(F)\backslash H(\mathbf{A})} \varphi(h)\, dh \quad \varphi \in \pi$$

on (say) a cuspidal representation $\pi = \otimes \pi_v$ of $G(\mathbf{A})$ (where now G is defined over a global field F) gives rise to locally distinguished representations π_v of $G(F_v)$. In this case we say that π is (globally) *H-distinguished*.

What characterizes H-distinguished representations (in the case $H = G^\theta$)? On first approximation (and in some cases, precisely), there should exist another (explicit) involution $\tilde{\theta}$ such that π is H-distinguished $\implies \tilde{\theta}(\pi) \simeq \pi$. (In general, we cannot expect this condition to be sufficient, e.g. for $\theta = \mathrm{id}$.) This is indeed the case whenever one can prove multiplicity one (for instance using the Gelfand-Kazhdan method, or its refinements). In general, we can only expect the *L-packet* of π to be $\tilde{\theta}$-stable. (Otherwise put, π is a functorial lift from some G'.) Also, we had better consider certain inner forms of G and H together. From now on we discuss a particular case where E/F is a quadratic extension, $G = \mathrm{GL}_n(E)$ and θ is a Galois involution defining a unitary group.

I will summarize recent joint work with Brooke Feigon and Omer Offen [16], following up on work of Jacquet [27, 28].

1.1. Finite field case.

Let τ be the non-trivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. The group $G = \mathrm{GL}_n(\mathbb{F}_{q^2})$ acts transitively on the set

$$X = \{\Phi \in G : \tau(\Phi^t) = \Phi\}$$

of non-degenerate hermitian forms by $\Phi \bullet g := \tau(g^t)\Phi g$. The stabilizer of Φ under this action is the unitary group $H = U_n(\mathbb{F}_q)$. We have the following result due to Gow.

Theorem 1 ([18]). *For any $\pi \in \hat{G}$*

- $\dim \mathrm{Hom}_H(\pi, \mathbf{C}) \leq 1$.
- $\mathrm{Hom}_H(\pi, \mathbf{C}) \neq 0 \iff \pi$ *is* $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$*-invariant, i.e.* $\pi^\tau \simeq \pi$.

1.2. **Base change.** There is yet another interesting characterization of Galois-invariant representations. For $m \geq 1$ let $\widehat{\mathrm{GL}_n(\mathbb{F}_{q^m})}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)}$ be the set of $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$-invariant irreducible representations of $\mathrm{GL}_n(\mathbb{F}_{q^m})$.

Theorem 2 (Shintani [44]). *There is a bijective correspondence*

$$\widehat{\mathrm{GL}_n(\mathbb{F}_{q^m})}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)} \longleftrightarrow \widehat{\mathrm{GL}_n(\mathbb{F}_q)}$$

characterized by certain character identities.

To put things in perspective we recall the so-called Glauberman correspondence. If σ is an automorphism of a finite group G such that $\gcd(\mathrm{ord}(\sigma), |G|) = 1$ then

$$\hat{G}^{\sigma} \longleftrightarrow \widehat{G^{\sigma}}$$

2. Local fields

To what extent do these results carry over to local fields? Namely,

- multiplicity one for unitary periods,
- characterization of distinction by Galois invariance,
- relation to base change.

By the cohomological obstruction mentioned before, we do not have multiplicity one. In fact, for unramified principle series the multiplicity is $\geq 2^{n-1}$.

Conjecture 1 (Jacquet). *Let E/F be a quadratic extension of local fields. Then a representation π of $\mathrm{GL}_n(E)$ is distinguished by the quasi-split unitary group if and only if π is Galois invariant.*

2.1. **Cyclic base change.** Let E be a cyclic (Galois) extension of degree m of a local field F and let τ be a generator of $\mathrm{Gal}(E/F)$. Let

$$\mathrm{Nm} : E^* \to F^*$$

be the norm map homomorphism given by

$$\mathrm{Nm}\, x = x x^{\tau} \dots x^{\tau^{m-1}}.$$

By class field theory $F^*/\mathrm{Nm}\, E^* \simeq \mathrm{Gal}(E/F)$.

Theorem 3 (Arthur-Clozel [4]). *To any $\pi \in \widehat{\mathrm{GL}_n(F)}$ there exists $\mathrm{bc}(\pi) \in \widehat{\mathrm{GL}_n(E)}^{\mathrm{Gal}(E/F)}$ characterized by character identities. Moreover if $\mathrm{bc}(\pi_1) = \mathrm{bc}(\pi_2)$ and π_1 is square-integrable then $\pi_2 \simeq \pi_1 \otimes \omega \circ \det$ for some character ω of $F^*/\mathrm{Nm}\, E^*$.*

From now on let E/F be a quadratic extension of either local or global fields. Let τ be the non-trivial Galois involution and let ω be the corresponding quadratic character of either F^* or $F^*\backslash\mathbb{I}_F$ by class field theory. Set $G' = G'_n = \mathrm{GL}_n/F$ and $G = G_n = \mathrm{Res}_{E/F}\mathrm{GL}_n$, i.e. $G(F) = \mathrm{GL}_n(E)$, $X = X_n$ – the space of hermitian non-singular matrices, as a variety over F with right action by G. In the p-adic case there are exactly two orbits of $G(F)$ on $X(F)$ according to discriminant. In the Archimedean case there are $n+1$ orbits (according to signature). Denote by G^x the stabilizer (the unitary group defined by x).

2.2. **The supercuspidal case.**

Theorem 4 (Jacquet). *Suppose that F is p-adic and π is supercuspidal and G^x-distinguished. Then π is τ-invariant.*

The Theorem is proved by a global argument due to Harder-Langlands-Rapoport [22] and independently by Oda [35], which shows that any globally distinguished cuspidal representation is Galois invariant. In order to apply the global argument one embeds π as a local component of a globally distinguished cuspidal representation. (See [20] for a general result.)

In many cases one can give a purely local proof (which also gives multiplicity one) based on the Bushnell-Kutzko classification of supercuspidal representations [11] as induced from compact open subgroups modulo the center [19, 38, 21]. However, I am not aware of a purely local argument which avoids classification. Using the geometric Lemma of Bernstein-Zelevinsky [7] we can deduce (purely locally)

Theorem 5 ([16]). *Let π be any irreducible representation of G (F p-adic). Suppose that π is distinguished by a unitary group. Then $\pi^\tau \simeq \pi$.*

In the same token, we can reduce all questions about distinction and multiplicity to the case where the supercuspidal support of π is contained in $\sigma \otimes |\det|^{\mathbf{Z}}$ for some supercuspidal σ, which is τ-invariant. (We call these π's pure of type σ.) For the non-quasi-split unitary group the condition for distinction is conjecturally $\#\{\pi' : \mathrm{bc}(\pi') = \pi\} > 1$.

2.3. **Archimedean situation.** A more elaborate filtration in the archimedean case yields

Theorem 6 (Aizenbud+Lapid). *Let π be a representation of $\mathrm{GL}_n(\mathbf{C})$. Suppose that π is the Langlands quotient of $\chi_1, \dots, \chi_n$ (characters of $\mathbf{C}^*$). Then π is distinguished by $U_{p,q} \implies \pi$ is τ-invariant (i.e., $\mathrm{Gal}(\mathbf{C}/\mathbb{R})$ stabilizes the multiset $\{\chi_1, \dots, \chi_n\}$) and the number of $\mathrm{Gal}(\mathbf{C}/\mathbb{R})$-orbits of size two does not exceed $\min(p, q) = \operatorname{rank} U_{p,q}$. The converse holds if π is generic.*

Conjecture 2. *The converse holds in general.*

In order to attack the other direction of Jacquet's conjecture, as well as to obtain results about multiplicity we need to use the full force of Jacquet's relative trace formula.

2.4. **Bessel distributions.** Let (π, V), $(\hat{\pi}, \hat{V})$ be two representations and $(\cdot,\cdot) : V \times \hat{V} \to \mathbf{C}$ a non-degenerate G-invariant bilinear form. This gives rise to an isomorphism $\iota : \pi^\vee \to \hat{\pi}$ where $\pi^\vee$ is the contragredient. For $\ell \in V^*$ and $\hat{\ell} \in \hat{V}^*$ we define

$$\mathcal{B}^{\ell,\hat{\ell}}_{(\pi,\hat{\pi},(\cdot,\cdot))}(f) = \hat{\ell} \circ \iota(\ell \circ \pi(f)).$$

(The map $\ell \mapsto \ell \circ \pi(f)$ defines an isomorphism $V^* \simeq \mathrm{Hom}_{G\text{-right}}(\mathcal{S}(G), \pi^\vee)$.) For example, for $\pi = \sigma \otimes \sigma^\vee$, $\hat{\pi} = \sigma^\vee \otimes \sigma$ on $G = H \times H$ with the standard pairing (doubled) and $\ell =$ standard pairing, $\hat{\ell} =$ standard pairing we get

$$\mathcal{B}(f_1 \otimes f_2) = \operatorname{tr} \sigma(f_1 * f_2^\vee).$$

The following example will be important for us. Let π be non-degenerate representation of $\mathrm{GL}_n(F)$ with Whittaker model $\mathcal{W}^\psi(\pi)$. Consider the pairing

$$(W, \hat{W}) = \int_{N_n \backslash P_n} W(p)\hat{W}(p)\, dp, \quad W \in \mathcal{W}^\psi(\pi), \hat{W} \in \mathcal{W}^{\psi^{-1}}(\pi^\vee)$$

where P_n is the mirabolic subgroup consisting of matrices of GL_n whose last row is $(0, \dots .0, 1)$. We write $\mathbb{W}(\pi) = (\mathcal{W}^\psi(\pi), \mathcal{W}^{\psi^{-1}}(\pi^\vee), (\cdot, \cdot))$ and consider

$$\mathbb{B}_\pi(f) = \mathcal{B}^{\delta_{w_0}, \delta_e}_{\mathbb{W}(\pi)}(f)$$

where $w_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. (For $\mathrm{GL}_2(\mathbb{R})$ this distribution is essentially represented by the usual Bessel function [15, 5].) The distribution $\mathbb{B}_\pi$ is left and right ψ-equivariant. Hence (at least in the p-adic case) it depends only on the ψ-orbital integrals of f. There is also a relative analogue of the construction of Bessel distributions. Let π be a representation of $G = \mathrm{GL}_n(E)$. Observe that

$$\begin{aligned}\mathrm{Hom}_G(X, \pi^*) &:= \{\alpha : X \to \pi^* | \alpha_{x \bullet g} = \alpha_x \circ \pi(g)\ \forall x \in X, g \in G\} \\ &\simeq \oplus_{x \in X/G} \mathrm{Hom}_{G^x}(\pi, \mathbf{C}) \simeq \mathrm{Hom}_G(\mathcal{S}(X), \pi^\vee).\end{aligned}$$

For $\alpha \in \mathrm{Hom}_G(X, \pi^*)$ and $\hat{\ell} \in \hat{\pi}^*$ we consider the distribution

$$\tilde{\mathcal{B}}^{\alpha, \ell}_{(\pi, \hat{\pi}, (\cdot, \cdot))}(\Phi) = \hat{\ell} \circ \iota(\alpha(\Phi)), \ \Phi \in \mathcal{S}(X)$$

on X. If π is generic then for any $\alpha \in \mathrm{Hom}_G(X, \mathcal{W}(\pi)^*)$, $\tilde{\mathcal{B}}^{\alpha, \delta_e}_{\mathbb{W}(\pi)}$ is $(N_n(E), \psi)$-equivariant and hence depends only on the ψ-orbital integrals of Φ.

2.5. **Matching functions and distributions.** The regular double cosets ${}^tN' \backslash G'/N' = \overline{N_n}(F) \backslash GL_n(F)/N_n(F)$ are parameterized by diagonal elements. Similarly, for $X/N_n(E)$. We write $\Phi \longleftrightarrow f'$ if

$$\omega(a_1 a_3 \dots) \int_{N_n'(F)^2} f'({}^t u_1 a u_2)\psi'(u_1 u_2)\, du_1\, du_2 = \int_{N_n(E)} \Phi(a \bullet u)\psi(u)\, du$$

for any $a = \mathrm{diag}(a_1, \dots, a_n) \in G'(F)$.

Theorem 7 (Jacquet – p-adic case [25], Aizenbud-Gourevitch – archimedean case [1])**.** *The condition*

$$D(\Phi) = D'(f') \ \ \forall \Phi \longleftrightarrow f'$$

defines an isomorphism $D \leftrightarrow D'$ of vector spaces between the $(\overline{N'} \times N', \psi' \times \psi')$-equivariant distributions o G' and the (N, ψ)-equivariant distributions on X.

Theorem 8 ([16]). *Let π' be a non-degenerate representation of $G'_n(F)$ and $\pi = \mathrm{bc}(\pi')$ (assumed generic). Then $\exists!\ \alpha^{\pi'} \in \mathrm{Hom}_G(X, \mathcal{W}(\pi)^*)$ such that $\tilde{\mathcal{B}}^{\alpha^{\pi'},\delta_e}_{\mathbb{W}(\pi)} \leftrightarrow \mathbb{B}_{\pi'}$. Moreover, in the p-adic case $\alpha^{\pi'}_x \not\equiv 0$ unless $\pi' = \pi' \otimes \omega$ and G^x is not quasi-split.*

Corollary 1. *Jacquet's conjecture holds for non-degenerate representations.*

3. Global periods

It is a remarkable fact that albeit lack of local uniqueness, global periods factorize nevertheless. More precisely

Theorem 9 (Jacquet's factorization Theorem [24]). *Suppose that $\pi = \mathrm{bc}(\pi')$ is cuspidal. Then for any factorizable φ in the space of π we have*

$$\int_{G^x(F)\backslash G^x(\mathbf{A})} \varphi(h)\ dh = 2L^S(1, \pi' \times \tilde{\pi}' \otimes \omega) \prod_{v \in S} \alpha_x^{\pi'_v}(W_v)$$

where $W(\varphi) = \int_{N_n(E)\backslash N_n(\mathbf{A}_E)} \varphi(u)\psi(u)^{-1}\ du = \prod_v W_v$ and S is a sufficiently large finite set of places outside of which W_v is the standard spherical vector.

This theorem is a consequence of the relative trace formula comparison. On top of smooth matching it requires the *fundamental lemma* (proved in Ngô's thesis in the positive characteristic case [33, 13, 34] and later on by Jacquet in general [26, 27]) and the spectral expansion [31].

Let us now consider induced representations. Let $P = MU$ be a standard parabolic subgroup, $M = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k}$, $X^M = X \cap M = X_{n_1} \times \cdots \times X_{n_k}$.

Given a representation σ of M, for each $x \in X^M$ and $\alpha \in \mathrm{Hom}_{M^x}(\sigma, \mathbf{C})$ we can define a family of elements of $\mathrm{Hom}_{G^x}(I(\sigma, \lambda), \mathbf{C})$, $\lambda \in \mathbf{C}^k$ by meromorphic continuation. (In a more general context, cf. [9] – p-adic case, [10, 12] – archimedean case.

A variant of this construction gives a map

$$\alpha \in \mathrm{Hom}_M(X^M, \sigma^*) \to J(\alpha, \lambda) \in \mathrm{Hom}_G(X, I(\sigma, \lambda)^*),\ \lambda \in \mathbf{C}^k.$$

In the case of unramified principle series these are essentially the *spherical functions* which were computed explicitly by Y. Hironaka (for E/F unramified) [23].

4. Local functional equations

4.1. **Local coefficients.** Consider the intertwining operator

$$W(\sigma, \lambda) : I(\mathcal{W}(\sigma), \lambda) \to \mathcal{W}(I(\sigma, \lambda))$$

given by the (analytic continuation of the) Jacquet integral

$$W(g, \varphi, \lambda) = \int_{w_0 \bar{U} w_0^{-1}} \varphi(w_0^{-1} ug)\psi(u)^{-1}\ du.$$

We have

$$W(w\sigma, w\lambda) \circ M(w, \sigma, \lambda) = C_M(w, \sigma, \lambda) W(\sigma, \lambda)$$

where $C_M(w, \sigma, \lambda)$ are Shahidi's local coefficients, [42, 43] expressible in terms of γ-factors for Rankin-Selberg integrals (whose theory was developed by Jacquet, Piatetski-Shapiro and Shalika).

4.2. **The functional equations.**

Theorem 10 (FLO). *Let σ' be a generic representation of M', $\sigma = \mathrm{bc}(\sigma')$. Then for $\lambda \in \mathbf{C}^k$ we have*

$$\alpha^{I(\sigma',\lambda)} \circ W(\cdot, \lambda) = \pm \frac{C_M(w_0, \sigma, \lambda)}{C_{M'}(w_0, \sigma', \lambda)} J(\alpha^{\sigma'}, \lambda)$$

and (consequently)

$$J(\alpha^{w\sigma'}, w\lambda) \circ M(w, \lambda) = \textit{explicit scalar function} \times J(\alpha, \lambda).$$

These functional equations are reminiscent to those of Shahidi. One difference is that in our case there is no uniqueness.

The Theorem is proved by analyzing the continuous part of the relative trace formula using earlier work by Lapid-Rogawski and Offen [30, 36, 37].

5. Applications

Theorem 11 (FLO). *Suppose that π is square integrable and τ-invariant. Then* $\dim \mathrm{Hom}_{G^x}(\pi, \mathbf{C}) = 1$ *for all $x \in X$.*

More generally, suppose that $\pi = \delta_1 \times \cdots \times \delta_k$ where δ_i are essentially square integrable and distinct. Then $\{\alpha^{\pi'} : \mathrm{bc}(\pi') = \pi\}$ forms a basis for $\mathrm{Hom}_G(X, \pi^*)$.

The Theorem is no longer true in the singular case (where δ_i are not distinct).

In fact, $\dim \mathrm{Hom}_G(X, \pi^*)$ is upper semi-continuous in the parameters while $\#\{\pi' : \mathrm{bc}(\pi') = \pi\}$ is not.

We expect the following

Conjecture 3. *If $\pi = \delta_1 \times \cdots \times \delta_k$ with δ_i essentially square-integrable and τ-invariant then* $\dim \mathrm{Hom}_G(X, \pi^*) = 2^k$.

As explained to me by Yiannis Sakellaridis, Hironaka's results imply that at least for E/F unramified we have $\dim \mathrm{Hom}_G(X, \pi^*) = 2^n$ for any generic unramified representation.

5.1. **Ladder representations.** An interesting class of pure representations is obtained as Langlands quotients of $\delta_1 \times \cdots \times \delta_k$ where $\delta_i \times \delta_{i+1}$ is reducible for all $i = 1, \ldots, k-1$. We call them ladder representations. (We can parameterize them as intervals $[a_1, b_1], \ldots, [a_k, b_k]$ such that $a_i, b_i \in \mathbf{Z}$, $a_i - 1 \le b_{i+1} < b_i$ for all i.)

For instance any *Speh representation* (the Langlands quotient of $\delta \otimes |\det \cdot|^{(m-1)/2} \times \cdots \times \delta \otimes |\det \cdot|^{(1-m)/2}$ where δ is square integrable) is ladder.

Theorem 12 ([16]). *For any τ-invariant ladder representation π and $x \in X$ we have* $\dim \mathrm{Hom}_{G^x}(\pi, \mathbf{C}) = 1$.

Corollary 2. *Jacquet's conjecture holds for unitarizable representations.*

5.2. **Tadić's determinantal formula.** For the existence of local invariant functionals for (Galois-invariant) Speh representations we use the following property of the ladder representations.

Proposition 1 ([29]). *The kernel of the projection $\delta_1 \times \cdots \times \delta_k \to \pi$, i.e., the kernel of the longest intertwining operator, is spanned by the kernels of the co-rank one intertwining operators (where we switch the order of δ_i and δ_{i+1}), $i = 1, \ldots, k-1$.*

Fix a supercuspidal σ and consider a ladder representation

$$\pi = \mathrm{LQ}(\Delta(\sigma, [a_1, b_1]) \times \cdots \times \Delta(\sigma, [a_k, b_k])).$$

The proposition is closely related to the following character formula, originally proved by Tadić for Speh representations [45].

Theorem 13 ([29]). *In the Grothendieck ring $\oplus_{n\geq 0}\mathcal{R}(\mathrm{GL}_n(F))$ we have*

$$\pi = \det(\Delta(\sigma, [a_i, b_j])_{i,j=1,\ldots,k}$$
$$= \sum_{w \in S_k : \forall i\, b_{w(i)} \geq a_i - 1} \operatorname{sgn} w\ \Delta(\sigma, [a_1, b_{w(1)}]) \times \cdots \times \Delta(\sigma, [a_k, b_{w(k)}]).$$

In the case of Speh representations a different proof was given by Chenevier-Renard [14]. There is yet another approach, which works in general, due to Badulescu.

Suppose that we have an arbitrary irreducible representation $\pi = \mathrm{LQ}(\delta_1, \ldots, \delta_k)$. Consider the kernel $\mathcal{K}$ of the longest intertwining operator $M(w_0)$, i.e., the kernel of the projection $\delta_1 \times \cdots \times \delta_k \to \pi$.

In order to analyze Jacquet's conjecture in general it is necessary to analyze $\mathcal{K}$. Clearly, $\mathcal{K}$ contains the kernels of any intertwining operator. It seems decisive to know whether in general

$$\mathcal{K} = \sum_{w = w_1 w_2 : w_1 \text{ simple reflection}, M_{w_2} \text{ isomorphism}} \operatorname{Ker} M_w.$$

Note that by Proposition 1 this holds for ladder representations (where the sum is over simple w's).

5.3. **Imprimitive representations.** We call a representation (parabolically) *imprimitive* if it is not parabolically induced from any proper parabolic subgroup.

Any irreducible representation can be written uniquely (up to permutation) as $\pi_1 \times \cdots \times \pi_k$ where π_i are imprimitive. Thus, the imprimitive representations are the prime elements of the representations theory of $\mathrm{GL}(n)$. Any ladder representation is imprimitive, but not conversely. For instance $\mathrm{LQ}(\Delta(\sigma, [1, 2]) \times \Delta(\sigma, [-1, 1]) \times \sigma)$ is imprimitive. (This example was shown to me by Minguez.)

Conjecture 4. *If $\pi_1, \ldots, \pi_k$ are imprimitive, τ-invariant and $\pi = \pi_1 \times \cdots \times \pi_k$ is irreducible then* $\dim \mathrm{Hom}_G(X, \pi^*) = 2^k$.

References

[1] Avraham Aizenbud and Dmitry Gourevitch. Smooth transfer of Kloosterman integrals (the archimedean case). preprint.

[2] Avraham Aizenbud and Dmitry Gourevitch. Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis's theorem. *Duke Math. J.*, 149(3):509–567, 2009. With an appendix by the authors and Eitan Sayag.

[3] Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann. Multiplicity one theorems. *Ann. of Math. (2)*, 172(2):1407–1434, 2010.

[4] James Arthur and Laurent Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[5] Ehud Moshe Baruch and Zhengyu Mao. Bessel identities in the Waldspurger correspondence over the real numbers. *Israel J. Math.*, 145:1–81, 2005.

[6] I. N. Bernšteĭn and A. V. Zelevinskiĭ. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.

[7] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.

[8] Joseph N. Bernstein. P-invariant distributions on $\mathrm{GL}(N)$ and the classification of unitary representations of $\mathrm{GL}(N)$ (non-Archimedean case). In *Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 of *Lecture Notes in Math.*, pages 50–102. Springer, Berlin, 1984.

[9] Philippe Blanc and Patrick Delorme. Vecteurs distributions H-invariants de représentations induites, pour un espace symétrique réductif p-adique G/H. *Ann. Inst. Fourier (Grenoble)*, 58(1):213–261, 2008.

[10] Jean-Luc Brylinski and Patrick Delorme. Vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégrales d'Eisenstein. *Invent. Math.*, 109(3):619–664, 1992.

[11] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[12] Jacques Carmona and Patrick Delorme. Base méromorphe de vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs: equation fonctionnelle. *J. Funct. Anal.*, 122(1):152–221, 1994.

[13] Ngô Báo Châu. Le lemme fondamental de Jacquet et Ye en caractéristique positive. *Duke Math. J.*, 96(3):473–520, 1999.

[14] Gaëtan Chenevier and David Renard. Characters of Speh representations and Lewis Caroll identity. *Represent. Theory*, 12:447–452, 2008.

[15] James W. Cogdell and Ilya Piatetski-Shapiro. *The arithmetic and spectral analysis of Poincaré series*, volume 13 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1990.

[16] Brooke Feigon, Erez Lapid, and Omer Offen. On representations distinguished by unitary groups. *preprint.*

[17] I. M. Gel'fand and D. A. Kajdan. Representations of the group $\mathrm{GL}(n, K)$ where K is a local field.
In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 95–118. Halsted, New York, 1975.

[18] Roderick Gow. Two multiplicity-free permutation representations of the general linear group $\mathrm{GL}(n, q^2)$. *Math. Z.*, 188(1):45–54, 1984.

[19] Jeffrey Hakim and Zhengyu Mao. Supercuspidal representations of $\mathrm{GL}(n)$ distinguished by a unitary subgroup. *Pacific J. Math.*, 185(1):149–162, 1998.

[20] Jeffrey Hakim and Fiona Murnaghan. Globalization of distinguished supercuspidal representations of $\mathrm{GL}(n)$. *Canad. Math. Bull.*, 45(2):220–230, 2002.

[21] Jeffrey Hakim and Fiona Murnaghan. Tame supercuspidal representations of GL(n) distinguished by a unitary group. *Compositio Math.*, 133(2):199–244, 2002.
[22] G. Harder, R. P. Langlands, and M. Rapoport. Algebraische Zyklen auf Hilbert-Blumenthal-Flächen. *J. Reine Angew. Math.*, 366:53–120, 1986.
[23] Yumiko Hironaka. Spherical functions and local densities on Hermitian forms. *J. Math. Soc. Japan*, 51(3):553–581, 1999.
[24] Hervé Jacquet. Factorization of period integrals. *J. Number Theory*, 87(1):109–143, 2001.
[25] Hervé Jacquet. Smooth transfer of Kloosterman integrals. *Duke Math. J.*, 120(1):121–152, 2003.
[26] Hervé Jacquet. Kloosterman identities over a quadratic extension. *Ann. of Math. (2)*, 160(2):755–779, 2004.
[27] Hervé Jacquet. Kloosterman identities over a quadratic extension. II. *Ann. Sci. École Norm. Sup. (4)*, 38(4):609–669, 2005.
[28] Hervé Jacquet. Distinction by the quasi-split unitary group. *Israel J. Math.*, 178:269–324, 2010.
[29] Erez Lapid and Alberto Mínguez. On a determinantal formula of Tadić. *preprint*.
[30] Erez Lapid and Jonathan Rogawski. Stabilization of periods of Eisenstein series and Bessel distributions on GL(3) relative to U(3). *Doc. Math.*, 5:317–350 (electronic), 2000.
[31] Erez M. Lapid. On the fine spectral expansion of Jacquet's relative trace formula. *J. Inst. Math. Jussieu*, 5(2):263–308, 2006.
[32] Colette Mœglin and Jean-Loup Waldspurger. Conjecture de Gross-Prasad, pour les groupes spéciaux orthogonaux: le cas général. *preprint*.
[33] Bao Châu Ngô. Le lemme fondamental de Jacquet et Ye en caractéristiques égales. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(3):307–312, 1997.
[34] Báo Châu Ngô. Faisceaux pervers, homomorphisme de changement de base et lemme fondamental de Jacquet et Ye. *Ann. Sci. École Norm. Sup. (4)*, 32(5):619–679, 1999.
[35] Takayuki Oda. *Periods of Hilbert modular surfaces*, volume 19 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1982.
[36] Omer Offen. Stable relative Bessel distributions on GL(n) over a quadratic extension. *Amer. J. Math.*, 129(5):1183–1226, 2007.
[37] Omer Offen. Unitary periods and Jacquet's relative trace formula. In *Automorphic forms and L-functions I. Global aspects*, volume 488 of *Contemp. Math.*, pages 183–236. Amer. Math. Soc., Providence, RI, 2009.
[38] Dipendra Prasad. On a conjecture of Jacquet about distinguished representations of GL(n). *Duke Math. J.*, 109(1):67–78, 2001.
[39] Yiannis Sakellaridis. Spherical functions on spherical varieties. *preprint*.
[40] Yiannis Sakellaridis. On the unramified spectrum of spherical varieties over p-adic fields. *Compos. Math.*, 144(4):978–1016, 2008.
[41] Yiannis Sakellaridis and Akshay Venkatesh. Periods and harmonic analysis on spherical varieties. *preprint*.
[42] Freydoon Shahidi. Fourier transforms of intertwining operators and Plancherel measures for GL(n). *Amer. J. Math.*, 106(1):67–111, 1984.
[43] Freydoon Shahidi. A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. *Ann. of Math. (2)*, 132(2):273–330, 1990.
[44] Takuro Shintani. Two remarks on irreducible characters of finite general linear groups. *J. Math. Soc. Japan*, 28(2):396–414, 1976.
[45] M. Tadić. On characters of irreducible unitary representations of general linear groups. *Abh. Math. Sem. Univ. Hamburg*, 65:341–363, 1995.
[46] J.-L. Waldspurger. Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2ème partie: extension aux représentations tempérées. *preprint*.
[47] J.-L. Waldspurger. Une formule intégrale reliée à la conjecture locale de Gross-Prasad. *Compos. Math.*, 146(5):1180–1290, 2010.

Arthur Packets and the Ramanujan Conjecture

Freydoon Shahidi

Let G be a quasisplit connected reductive group defined over a local field k (real, complex or p–adic). Let ψ be an Arthur parameter for G, i.e., $\psi \in \Psi(G/k)$, the set of equivalence classes of homomorphism from $W'_k \times SL_2\mathbb{C})$ into ${}^LG$, under ${}^LG^0$–conjugacy (cf [A]). Let $\phi_\psi \colon W'_k \to {}^LG$ be defined by $\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix})$. Fix a Borel subgroup B of G over k. Let $P = MN$ be the standard parabolic sub–group of G for which ϕ_ψ factors through ${}^LM$ and ${}^LM$ is smallest such. Let $\pi^M(\phi_\psi)$ be the L–packet of $M(k)$ attached to ϕ_ψ. Let $\sigma \in \pi^M(\phi_\psi)$. Then $\pi^G(\phi_\psi)$ will consist of Langlands quotients $J(\sigma)$, $\forall \sigma \in \pi^M(\phi_\psi)$. Let r be the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$, the Lie algebra of ${}^LN$. Assume there exists a σ such that $J(\sigma)$ is generic (thus so is σ), i.e., having a Whittaker model. Assume

$$\text{(a)} \qquad L(s, \tilde{r} \cdot \phi_\psi) = L(s, \sigma, r),$$

where the L–function on the left are those of Artin, while the right is those attached by Langlands–Shahidi method to σ and r (cf. [Sh1]). In this lecture, we prove that under validity of (a), if $\pi^G(\phi_\psi)$ has a generic member π, then ϕ_ψ is tempered. The convers is a conjecture in [Sh1] which is now proved in many cases. The result is valid with no assumptions if $k = \mathbb{R}$ or $\mathbb{C}$, or π is $G(O_k)$–unramified when k is p–adic. In the cases of classical groups this should also be immediate since LLC (local Langlands conjecture) is valid for them by the work of Arthur and Jiang–Soudry (proved by D. Ban and B. Liu by other techniques including LLC, at least in part). The proof relies on representation theoretic techniques from our method developed in [CSh].

There are important global consequences of this result. For example, under a certain conjecture of Clozel (and Arthur), this proves that locally generic cuspidal representations of $G(\mathbb{A}_k)$, $\mathbb{A}_k$ adeles of a number field, are always tempered at almost all places, i.e., are of Ramanujan type. This also gives strong evidence that the isotypic cuspidal component of a locally generic representation always contains a globally generic one and thus there should be no L–function obstruction for this equivalency up to isomorphisms. We refer to [Sh2] for further details and discussions.

References

[A] J. Arthur, *On some problems suggested by the trace formula.* In: Lie Group Representations, II, College Park, Md, 1982/1983; Lecture Notes in Math. vol. 1041, Springer, Berlin (1984), 1–49.

[CSh] W. Casselman and F. Shahidi, *On irreducibility of standard modules for generic representations.* Ann. Sc. École Norm. Sup. (4) vol. 31 (1998), 561–589

[Sh1] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures: Complementary series for p–adic groups.* Ann. of Math. (2) vol. 132 (1990), 273–330

[Sh2] F. Shahidi, *Arthur Packets and the Ramanujan Conjecture.* Kyoto J. Math Memorial Issue for M. Nagata II, Vol. 51, Number 1, (2011), 1–23

Periods over spherical subgroups: an extension of some of the Langlands conjectures.

Yiannis Sakellaridis

1. Summary

Periods of automorphic forms over spherical subgroups tend to: (1) distinguish images of functorial lifts and (2) give information about L-functions.

This raises the following questions, given a spherical variety $X = H\backslash G$: Locally, which irreducible representations admit a non-zero H-invariant functional or, equivalently, appear in the space of functions on X? Globally, can the period over H of an automorphic form on G be related to some L-value?

The conjectural answer involves a "dual group" associated to X and can be seen as a generalization of part of the Langlands conjectures for the case $X =$a group under left and right multiplication by itself. The purpose of this talk is to describe the dual group and discuss evidence suggesting that the relative trace formula of Jacquet is the correct framework for a more precise formulation of the conjectures.

2. Spherical varieties

A homogeneous variety X for a reductive algebraic group G is called "spherical" if (over the algebraic closure) a Borel subgroup of G acts with a dense orbit on X. This includes symmetric spaces, flag varieties, and other interesting spaces such as the "Gross-Prasad" variety $\mathrm{SO}_n \backslash (\mathrm{SO}_n \times \mathrm{SO}_{n+1})$. If $X = H\backslash G$ is spherical, then H is called a spherical subgroup of G.

Let G be defined over a global field k. Then spherical varieties for G (or, more precisely, affine embeddings thereof) give rise to interesting distributions on the automorphic quotient $[G] := G(k)\backslash G(\mathbb{A})$, cf. [Sa]. At present there are comparatively few spherical varieties for which we can prove that the associated distributions have good analytic properties, and those include:

- The cases where H is reductive. The associated distribution is then the "period integral" over H:

$$\mathcal{P}_H(\phi) := \int_{[H]} \phi(h)dh. \tag{1}$$

- When H is the semidirect product of a reductive subgroup of a Levi subgroup with the unipotent radical of an associated parabolic, the latter endowed with an idele class character normalized by the former. This gives rise to "Whittaker-type" periods.
- When X happens, by coincidence, to have an open embedding into a flag variety (for some other group). This gives rise to Rankin-Selberg integrals.

Although only periods appear explicitly in this talk, the local discussion holds for (almost) all spherical varieties and the global discussion holds whenever we can make sense of the associated distribution on the automorphic quotient.

The word "almost" was inserted to exclude certain spherical varieties whose "rank one degenerations" contain, in a suitable sense, the space $\mathrm{PO}_2 \backslash \mathrm{PGL}_2$. These varieties, and the associated periods, seem to be sharing certain features of metaplectic groups, cf. [Ja91], and the notion of a dual group is not suitable to describe their spectrum.

3. Local conjecture

We switch to local notation: k is a p-adic or archimedean field, $H = H(k)$ etc.

We ask ourselves which irreducible representations π admit a non-zero:

$$\pi \underset{\neq 0}{\overset{H}{\longrightarrow}} \mathbb{C} \Leftrightarrow \pi \hookrightarrow C^\infty(H\backslash G) \subset C^\infty(X).$$

A related question is, which

$$\pi \in L^2(X)$$

in the sense of the Plancherel formula (Fell topology)?

For π a G-discrete series, the two questions are equivalent, as a consequence of the theory of asymptotics of such embeddings.

For simplicity, assume from now on that G is split. Recall the following (weak) formulation of the Local Langlands Conjecture: $L^2(G)$ admits a direct integral decomposition:

$$L^2(G) = \int_{\{\phi\}} \mathcal{H}_\phi \mu_G(\phi)$$

where ϕ ranges over the set of conjugacy classes of tempered Langlands parameters: $\phi : WD_k \to {}^L G$, which has natural orbifold structure;
μ_G is the canonical Plancherel measure, and lives in the class of Lebesgue measure for this orbifold;
$\mathcal{H}_\phi = \bigoplus_{\pi\in\Pi_\phi} \mathcal{H}_\pi$, where $\mathcal{H}_\pi$ is the completion of $C_c^\infty(G)$ with respect to the norm $\| \bullet \|_\pi : C_c^\infty(G) \overset{\text{matrix coeff.}}{\longrightarrow} \pi \otimes \bar{\pi} \overset{\|\bullet\|\cdot\|\bullet\|}{\longrightarrow} \mathbb{R}_+$.

Conjecture 5 (S.-Venkatesh, [SV]). *Let X be a spherical variety over a local field. Then $L^2(X)$ admits a direct integral decomposition:*

$$L^2(X) = \int_\psi \mathcal{H}_\psi \mu(\psi),$$

where:

- *ψ varies over X-*distinguished Arthur parameters *modulo conjugacy;*
- *$\mu_X \in$ class of Lebesgue measure on such parameters.*

Here $\mu_X, \| \bullet \|_\psi^2$ are not canonical, but their *product* is.

Theorem 14 (S.-Venkatesh, [SV]). *Description of* $\|\bullet\|_\psi^2 \cdot \mu_X$ up to discrete spectra.

Here "discrete spectra" means the discrete-modulo-center part of $L^2(X_\Theta)$, where Θ varies over certain degenerations of X. Unlike the case of symmetric varieties (such as the group variety), in the general case the variation of discrete series with central character cannot be obtained by "twisting", and one needs to use a technique called "unfolding".

Finally, the "X-distinguished Arthur parameters" of the Conjecture are those which factor as:

$$\begin{array}{ccc} WD_k \times \mathrm{SL}_2 & \overset{\phi\times\mathrm{Id}}{\dashrightarrow} & \check{G}_X \times \mathrm{SL}_2 \\ & \searrow^{\psi} & \downarrow \\ & & \check{G} \end{array}$$

taken modulo $\check{G}_X$-conjugacy.

Here $\check{G}_X$ is the "dual group" of the spherical variety, defined in [SV]. It is closely related, but not identical to, the dual group constructed by Gaitsgory and Nadler [GN10]. It has the same Weyl group, but not always the same roots, as two root systems associated to the spherical variety by Brion and Knop [Br90, Kn96]. The map $\check{G}_X \times \mathrm{SL}_2 \to \check{G}$ is a canonical one.

4. Global conjecture

A global conjecture about the value of $|\mathcal{P}_H(\phi)|^2$ was described in the talk, assuming that the functional $\mathcal{P}_H$ is factorizable (i.e. a pure tensor). It is a generalization of the Ichino-Ikeda conjecture for the Gross-Prasad case [II10]; rougly speaking, it expresses the local Euler factors for $|\mathcal{P}_H(\phi)|^2$ in terms of generalized characters (also called spherical characters or Bessel distributions) appearing in a suitably normalized Plancherel formula for $X = H\backslash G$. The conjecture can be proven for period integrals which "unfold" to known cases of the conjecture, such as: the Whittaker period in GL_n, the space $\mathrm{GL}_n \backslash(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$, the Rankin-Selberg integral for $\mathrm{GL}_n \times \mathrm{GL}_n$ and (partially) the cases $\mathrm{Sp}_{2n} \backslash \mathrm{GL}_{2n}$ and $\mathrm{GL}_n \times \mathrm{GL}_n \backslash \mathrm{GL}_{2n}$.

References

[Br90] M. Brion, *Vers une généralisation des espaces symétriques.* J. Algebra 134 (1990), no. 1, 115–143.

[GN10] D. Gaitsgory and D. Nadler, *Spherical varieties and Langlands duality.* Mosc. Math. J. 10 (2010), no. 1, 65–137.

[II10] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture.* Geom. Funct. Anal. 19 (2010), no. 5, 1378–1425.

[Ja91] H. Jacquet, Représentations distinguées pour le groupe orthogonal. C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 13, 957–961.

[Kn96] F. Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties.* J. Amer. Math. Soc. 9 (1996), no. 1, 153–174.

[Sa] Y. Sakellaridis, *Spherical varieties and integral representations of L-functions.* arXiv:0905.4245, submitted for publication.

[SV] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties.* In preparation, partial draft available at: `http://math.newark.rutgers.edu/~sakellar/harmonic-draft.pdf`.

On special values of automorphic L–functions and Eisenstein cohomology

Neven Grbac
(joint work with Joachim Schwermer)

We present the results of a joint work with Joachim Schwermer regarding the automorphic cohomology of a connected classical algebraic group G defined over $\mathbb{Q}$. The presented results are the content of recent papers [2], [3], [4].

The automorphic cohomology captures essential information on the cohomology of arithmetic congruence subgroups of $G(\mathbb{R})$. More precisely, the automorphic cohomology of G is, by definition, the direct limit over open compact subgroups C of $G(\mathbb{A}_f)$ of the cohomology of double coset spaces

$$X_C = G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{R}}C,$$

where $\mathbb{A}$ is the ring of adèles of $\mathbb{Q}$, $\mathbb{A}_f$ the ring of finite adèles, and $K_{\mathbb{R}}$ is a fixed maximal compact subgroup of $G(\mathbb{R})$. As proved in [5], it can be computed as the relative Lie algebra cohomology of the space of automorphic forms on $G(\mathbb{A})$.

The space of automorphic forms on $G(\mathbb{A})$ admits a direct sum decomposition according to their cuspidal support (see [7], [8], [6]). The summands are indexed by associate classes of parabolic $\mathbb{Q}$-subgroups of G, and associate classes of cuspidal automorphic representations of their Levi factors. The summand indexed by the associate class of a proper parabolic $\mathbb{Q}$-subgroup P, and the associate class of a cuspidal automorphic representation π of the Levi factor $L_P(\mathbb{A})$ of P, is spanned by all possible residues and principal values of derivatives of Eisenstein series attached to π at certain values of their complex parameter.

This decomposition gives rise to the corresponding decomposition of automorphic cohomology of G. The summands indexed by the full group G form the so–called cuspidal cohomology. The natural complement to cuspidal cohomology, formed by the summands indexed by associate classes of proper parabolic $\mathbb{Q}$-subgroups, is called Eisenstein cohomology. We study necessary conditions for non–vanishing of individual summands in Eisenstein cohomology.

The main result presented in this talk is that the Eisenstein series attached to π can possibly give rise to a non–trivial cohomology class in the Eisenstein cohomology of G only if the evaluation point satisfies a certain "half–integral" property. In order to explain what this property means, we restrict our attention to one of the $\mathbb{Q}$-split classical groups SO_{2n+1}, Sp_n, and SO_{2n} defined over $\mathbb{Q}$ of $\mathbb{Q}$-rank n.

For such G, the Levi factor L_P of any proper parabolic $\mathbb{Q}$-subgroup P of G is of the form

$$L_P \cong GL_{r_1} \times \ldots \times GL_{r_d} \times G',$$

where $r_1, \ldots, r_d$ are positive integers such that their sum is not greater than the rank n of G, and G' is the (possibly trivial) classical group of the same type as G, but of smaller rank equal to $n - r_1 - \ldots - r_d$. Hence, the space of complex parameters for the Eisenstein series attached to a cuspidal automorphic representation π of $L_P(\mathbb{A})$ is isomorphic to $\mathbb{C}^d$, and we can take as the basis the determinant on each of the general linear factors of L_P. Then, the d–tuple $(s_1, \ldots, s_d) \in \mathbb{C}^d$ corresponds to the character of $L_P(\mathbb{A})$ given by

$$(g_1, \ldots, g_d, h) \mapsto |\det(g_1)|^{s_1} \cdot \ldots \cdot |\det(g_d)|^{s_d},$$

for $g_i \in GL_{r_i}(\mathbb{A})$ and $h \in G'(\mathbb{A})$.

In this natural basis for the space of complex parameters, our "half–integral" property of the evaluation point shows that the Eisenstein series attached to π can possibly give rise to a non–trivial cohomology class in the Eisenstein cohomology of one of the groups SO_{2n+1}, Sp_n, and SO_{2n}, only if

$$s_1, s_2, \ldots, s_d \in \frac{1}{2}\mathbb{Z},$$

that is, all coordinates of the evaluation point are half–integers.

We observe that if one only considers the cohomology classes represented by square–integrable automorphic forms, this result is in accordance with Arthur's conjectural description of the discrete spectrum of the classical group $G(\mathbb{A})$ presented in [1].

This result also shows that, if we assume that π is globally generic (with respect to a fixed non–trivial additive character ψ of $k\backslash\mathbb{A}$), and apply the Langlands–Shahidi method, developed in [9], to study the poles of Eisenstein series, then with regard to cohomological questions we only need to understand the analytic properties of the automorphic L–functions at certain special values of their complex argument. For example, in the case of the $\mathbb{Q}$-split classical groups SO_{2n+1}, Sp_n and SO_{2n} already considered above, the symmetric and exterior square automorphic L–functions appear in the Langlands–Shahidi normalizing factors of intertwining operators with the argument of the form $2s_i$. Since our "half–integral" property implies that only the evaluation points with $2s_i \in \mathbb{Z}$ matter, the unknown analytic properties of symmetric and exterior square L–functions inside the critical strip $0 < Re(s) < 1$ play no role in computing the cohomological contribution.

References

[1] J. Arthur, An introduction to the trace formula, in: J. Arthur, D. Ellwood, R. Kottwitz, eds., *Harmonic analysis, the trace formula, and Shimura varieties*, *Clay Math. Proc.* **4**, Amer. Math. Soc., Providence, RI, 2005, pp. 1–263.

[2] N. Grbac, J. Schwermer, On Eisenstein series and the cohomology of arithmetic groups, *C. R. Math. Acad. Sci. Paris* **348** (2010), 597–600.

[3] N. Grbac, J. Schwermer, On residual cohomology classes attached to relative rank one Eisenstein series for the symplectic group, *Int. Math. Res. Not. IMRN*, to appear.

[4] N. Grbac, J. Schwermer, Eisenstein series, cohomology of arithmetic groups, and automorhic L–functions at half–integral arguments, preprint.

[5] J. Franke, Harmonic analysis in weighted L_2-spaces, *Ann. Sci. École Norm. Sup. (4)* **31** (1998), 181–279.

[6] J. Franke, J. Schwermer, A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups, *Math. Ann.* **311** (1998), 765–790.
[7] R.P. Langlands, *On the Functional Equations Satisfied by Eisenstein Series*, *Lect. Notes Math.*, Vol. **544**, Springer–Verlag, Berlin–New York, 1976.
[8] C. Mœglin, J.–L. Waldspurger, *Décomposition spectrale et séries d'Eisenstein*, Progress in Math. **113**, Birkhäuser Verlag, Basel, 1994.
[9] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math. (2)* **132** (1990), 273–330.

Special values of certain cohomological L-functions

Günter Harder

This is a report on joint work with A. Raghuram ([4]. We generalize the method for proving rationality results of special values of $L-$ functions from my talk in OWR 2008/5 to the group $\tilde{G} = \mathbf{Gl}_N/\mathbb{Z}$, Nodd. (see also [2]) For the following notations and general results I refer to [3].

For any reductive group scheme $G/\mathbb{Z}$ and any highest weight module $\mathcal{M}_{\lambda,\mathbb{Z}}$ we consider the long exact sequence

$$\to H^\bullet_c(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}}) \to H^\bullet(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}}) \to H^\bullet(\partial S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}}) \to H^{\bullet+1}_c(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}}) \to$$

as a sequence of modules under the integral Hecke algebra, this is our basic object of interest.

We still have the inner cohomology $H^\bullet_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}})$ and we know that after a suitable finite extension $F/\mathbb{Q}$ it decomposes into a direct sum of isotypical components absolutely irreducible modules for the Hecke-algebra

$$H^\bullet_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F) = \bigoplus_{\pi_f} H^\bullet_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(\pi_f).$$

In [3] 3.1 I explain how to attach a cohomological (motivic) L-function $L^{\mathrm{coh}}(\pi_f, r, s)$ to π_f and a representation r of the dual Langlands group. This L-function is equal to the usual L-function in the theory of automorphic forms up to a shift in the argument s. It is defined in terms of the integral structure of the Hecke-module π_f. and it has the following property: If we twist our coefficient system $\mathcal{M}_{\lambda,\mathbb{Z}}$ by a character $\delta : G \to \mathbf{G_m}$ then $\pi_f \otimes |\delta_f|^{-1}$ occurs in $H^\bullet_!(S^G_{K_f}, \mathcal{M}_{\lambda+\delta,\mathbb{Z}} \otimes F)$ and we have

$$L^{\mathrm{coh}}(\pi_f, r, s) = L^{\mathrm{coh}}(\pi_f \otimes |\delta_f|^{-1} r, s).$$

It is part of the general philosophy of Langlands that there should exist motives $\mathbb{M}(\pi_f, r)$ such that we have an equality

$$L^{\mathrm{coh}}(\pi_f, r, s) = L(\mathbb{M}(\pi_f, r), s) = \prod_p L(\pi_p, r, s)$$

where for almost all primes $L(\pi_p, r, s) = \det(\mathrm{Id} - r(\Phi_p^{-1})p^{-s})^{-1}$ Consider the group $G = \mathbf{Gl}_n/\mathbb{Z}$ and a regular highest weight λ. Then a π_f which occurs as isotypical module in the inner cohomology occurs in lowest degree $b_n = n^2/4$ (resp. $(n^2-1)/4$) with multiplicity 2 (resp. 1) if n is even (resp. odd). We have an action of $\pi_0(G(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ on $H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(\pi_f))$, which commutes with the Hecke algebra. In the even case

$$H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(\pi_f)) = H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)_+(\pi_f)) \oplus H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)_-(\pi_f))$$

and in the odd case $\pi_0(G(\mathbb{R}))$ acts by a sign character $\epsilon(\lambda)$ on $H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(\pi_f))$.

In the even case we can choose an Hecke module isomorphism

$$T^{\text{arith}}(\pi_f) : H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)_+(\pi_f)) \xrightarrow{\sim} H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)_-(\pi_f))$$

which is unique up to an element in $F^\times$. (A more meticulous argumentation shows that we can pin down $T^{\text{arith}}(\pi_f)$ up to an element in $\mathcal{O}^\times_{F,S}$, where S is a computable finite set of primes which has to be inverted). If we choose an embedding $\iota : F \hookrightarrow \mathbb{C}$, then we can construct a canonical isomorphism

$$T^{\text{trans}}(\pi_f, \iota) : H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F \otimes_{F,\iota} \mathbb{C})_+(\pi_f)) \xrightarrow{\sim} H^{b_n}_!(S^G_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F \otimes_{F,\iota} \mathbb{C})_-(\pi_f))$$

and this provides an array of periods $\Omega(\pi_f) = \{\dots, \Omega(\pi_f, \iota), \dots\}_{\iota:F\hookrightarrow\mathbb{C}}$, which is defined by

$$\Omega(\pi_f, \iota) T^{\text{trans}}(\pi_f, \iota) = T^{\text{arith}}(\pi_f) \otimes_{F,\iota} \mathbb{C}.$$

These periods have the following property: If we modify λ to $\lambda + d\det$ then $\Omega(\pi_f \otimes |\det_f|^{-d}) = (-1)^d \Omega(\pi_f)$. We start from the group $\tilde{G}$, a highest weight module $\mathcal{M}_{\lambda,\mathbb{Z}}$ and a finite extension $F/\mathbb{Q}$ and consider the restriction map

$$r : H^\bullet(S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F) \to H^\bullet(\partial S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F).$$

Now I refer to the description of the cohomology of the boundary in terms of the cohomology of the boundary strata of the Borel-Serre compactification. The strata are labelled by the conjugacy classes of parabolic subgroups over $\mathbb{Q}$. For any decomposition $N = n + n'$ we have a pair of opposing conjugacy classes P, Q whose Levi quotient is $\mathbf{Gl}_n \times \mathbf{Gl}_{n'} = M$. We realize them as a pair of opposing parabolic subgroups containing the standard maximal split torus T. They provide a submodule

$$H^\bullet_!(\partial_P S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Q}) \oplus H^\bullet_!(\partial_Q S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Q}) \subset H^\bullet(\partial S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Q}).$$

which satisfies Manin-Drinfeld with respect to $H^\bullet(\partial S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Q})$. ([3] , 2.4.1).) Then it becomes clear that the image of r intersected with this direct summand is a $\mathbb{Q}$ -sub-vector space. We apply the principles from [3]. Kostants theorem combined with the decomposition into isotypical subspaces yields

$$H^\bullet_!(\partial_P S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F) = \bigoplus_{w \in W^P} \bigoplus_{\sigma_f} I^G_P H^\bullet_!(S^M_{K_f}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(w \cdot \lambda)(\sigma_f)$$

$$H^\bullet_!(\partial_Q S^{\tilde{G}}_{K_f}, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F) = \bigoplus_{w' \in W^Q} \bigoplus_{\sigma'_f} I^G_Q H^\bullet_!(S^M_{K_f}, H^{l(w')}(\mathfrak{u}_Q, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(w' \cdot \lambda)(\sigma'_f).$$

In [2] 1.2.1 we establish a one-to-one correspondences $w \leftrightarrow w'$, $l(w) + l(w') = \dim(U_P)$, and $\sigma_f \leftrightarrow \sigma'_f, \sigma_f = \sigma'_f \otimes |\rho^{-2}_{U,f}|$. (See [2] 1.2.3.

Under these special circumstances the theory Eisenstein cohomology tells us that

$$\mathrm{Eis}(\sigma_f) \subset I_P^G H_!^\bullet(S_{K_f}^M, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(\sigma_f) \oplus I_Q^G H_!^\bullet(S_{K_f}^M, H^{l(w')}(\mathfrak{u}_Q, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(\sigma_f')$$

is the graph of homomorphism between the two summands as Hecke modules. Hence we know how to identify the two summands we find

$$\mathrm{Eis}(\sigma_f) = \{\psi_f + c(\sigma_f) T^{\mathrm{arith}}(\psi_f)\}$$

where $c(\sigma_f) \in F$ (it may happen that $c(\sigma_f) = \infty$).

The Langlands-Shahidi method yields a formula for the term $c(\sigma_f)$. Recall that $M = \mathbf{Gl}_n \times \mathbf{Gl}_{n'} = G_1 \times G_2$ is a product and accordingly

$$H_!^\bullet(S_{K_f}^M, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_{\lambda,\mathbb{Z}} \otimes F)(w \cdot \lambda)(\sigma_f) = H_!^\bullet(S_{K_f^{(1)}}^{G_1}, \mathcal{M}_{\lambda_1})(\sigma_{1,f}) \otimes H_!^\bullet(S_{K_f^{(2)}}^{G_2}, \mathcal{M}_{\lambda_2})(\sigma_{2,f})$$

The $c(\sigma_f)$ can be expressed in terms of special values of the completed cohomological L function

$$\Lambda^{\mathrm{coh}}(\sigma_{1,f} \times \sigma_{2,f}, \tau_1 \times \tau_2^\vee, s) = L_\infty(w, \lambda, s) L^{\mathrm{coh}}(\sigma_{1,f} \times \sigma_{2,f}, \tau_1 \times \tau_2^\vee, s).$$

Here the τ_i are the tautological representations of the Langlands dual of G_i and $\tau_2^\vee$ is the dual. The factor $L_\infty(w, \lambda, s)$ is product of Γ factors depending on w, λ.

To give the formula we look again at the embeddings $\iota : F \hookrightarrow \mathbb{C}$, then $\Lambda^{\mathrm{coh}}(\sigma_{1,f} \times \sigma_{2,f} \circ \iota, \tau_1 \times \tau_2^\vee, s)$ becomes an honest function in the complex variable s. If n is even then

$$\iota(c(\sigma_f)) = c(\sigma_f \circ \iota) = \Big(\frac{C_\infty(w,\lambda)}{\Omega(\sigma_{1,f}, \iota))}\Big)^{e(w,\lambda)} \frac{\Lambda^{\mathrm{coh}}(\sigma_{1,f} \times \sigma_{2,f} \circ \iota, \tau_1 \times \tau_2^\vee, a(w,\lambda))}{\Lambda^{\mathrm{coh}}(\sigma_{1,f} \times \sigma_{2,f} \circ \iota, \tau_1 \times \tau_2^\vee, a(w,\lambda)+1))},$$

where $a(w, \lambda)$ is an integer (depending explicitly on w, λ, I do not give the formula) and $e(w, \lambda)$ is ± 1, the value depends on the parity of $a(w, \lambda)$. The number $C_\infty(w, \lambda)$ is a non zero rational number, which we have not yet computed (see [2],2.2).

The question arises what are the values of $a(w, \lambda)$ which occur here. Recall that the cohomological L does not change if we twist $\sigma_{1,f} \times \sigma_{2,f}$ by $|\det_{1,f}|^a \times |\det_{2,f}|^b$, then $a(w, \lambda)$ changes into $a(w, \lambda) + a - b$, this means that the same L function is evaluated at different arguments. To answer to this question we need a combinatorial lemma concerning the symmetric group S_N, which we believe is true and which we have checked in the cases $n = 1, 2$ and some other cases.

Assuming the combinatorial lemma *the numbers* $a(w, \lambda)$, *which occur are exactly those numbers* ν *for which* ν *and* $\nu + 1$ *are critical (see* [1]*)for the hypothetical motive* $\mathbb{M}(\sigma_{1,f}, \tau_1) \times \mathbb{M}(\sigma_{2,f}, \tau_2^\vee)$. *This set of critical values can also be read off from the factor* $L_\infty(w, \lambda, s)$.

Hence we can say: If we accept the existence of the the motive $\mathbb{M}(\sigma_{1,f}, \tau_1) \times \mathbb{M}(\sigma_{2,f}, \tau_2^\vee)$ then we have proved a rationality statement for the special values of its L-function, which is very close to Delignes conjecture in [1]. If we do not believe in this motive, then we still have a theorem on the special values of certain automorphic L-functions.

References

[1] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales (French),* Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions , Part 2, pp. 313–346, Amer. Math. Soc., Providence, R.I., 1979.

[2] G. Harder, *Arithmetic aspects of Rank one Eisenstein cohomology, with an appendix by Don Zagier* in Cycles, Motives, Shimura Varieties (ed. by V. Srinivas), Studies in Mathematics, TIFR.

[3] G. Harder, *Five Chapters on the cohomology of arithmetic groups*: http://www.math.uni-bonn.de/people/harder/Manuscripts/buch

[4] G. Harder, A. Raghuram *Eisenstein Cohomology and ratios of critical values of Rankin–Selberg L-functions*, Submitted to Comptes Rendus

Sato-Tate conjecture for families of automorphic representations

Sug Woo Shin

(joint work with Nicolas Templier)

Mathematics is concerned with not only exact formulas but also asymptotic formulas. The question is rich when it comes to the asymptotic behaviors and formulas of *arithmetic* origin. Many sophisticated questions can be asked via the means of "equidistribution".

A general setup is the following. Let X be a nice topological space (which hosts arithmetic invariants of interest). Denote by $C(X)$ the space of complex-valued continuous functions equipped with sup norm. Let μ be a measure on X defining a linear functional on $C(X)$. Let $\{\mathcal{F}_k\}_{k\geq 1}$ be a sequence of finite subsets of X such that $|\mathcal{F}_k| \to \infty$ as $k \to \infty$. The sequence $\{\mathcal{F}_k\}_{k\geq 1}$ is said to be μ-*equidistributed* if μ is the limit of the averaged counting measure $\frac{1}{|\mathcal{F}_k|}\sum_{x\in\mathcal{F}_k}\delta_x$ as $k \to \infty$.

In number theory we wish to detect the equidistribution property when $\mathcal{F}_k$ are formed from arithmetic invariants, for instance rational points (or cycles) on algebraic varieties, Fourier coefficients of modular forms, or Frobenius eigenvalues on the Tate modules of abelian varieties, just to name a few. A general philosophy is that when arithmetic invariants do not obey exact formulas, they tend to be random. For a simplest example, the fact that primes are equidistributed between 1 mod 4 and 3 mod 4 means that choosing a prime mod 4 is the same as flipping a coin. To put it another way, equidistribution in number theory shows that arithmetic invariants are *predictably* unpredictable.

In my talk I considered (the analogue of) the Sato-Tate conjecture for families of automorphic representations. Roughly speaking, the aim is to show that the Satake parameters at p of the given family of automorphic representations are equidistributed as p tends to infinity according to a natural measure (the so-called Sato-Tate measure), perhaps under a suitable condition. Although the Sato-Tate conjecture can be formulated for a single automorphic representation, it is not easy to cleanly state it in general and turns out to be extremely difficult already for $GL(2)$. (It should be a consequence of the Langlands functoriality conjecture, but the latter is far from fully established.) Our observation is that the analogue for families can still be attacked with the trace formula and various other techniques.

Now let us be more precise. Let G be a connected semisimple group over $\mathbb{Q}$ with trivial center. (This amounts to working with automorphic representations of a reductive group with fixed central character.) Although it is unnecessary, G is assumed to be split to simplify the exposition. Let $\mathcal{A}_{\mathrm{disc}(G)}$ denote the set of isomorphism classes of discrete automorphic representations of $G(\mathbb{A})$. Let T be a maximal torus of G. Let $U \subset G(\mathbb{A}^\infty)$ be an open compact subgroup and $\xi \in X_*(T)$ be a highest weight parameter for an irreducible algebraic representation of G over $\mathbb{C}$. A sequence of such objects is denoted by U_k and ξ_k, respectively. A $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ is said to have level U if π^∞ has a nonzero U-fixed vector, and weight ξ if $\pi \otimes \xi^\vee$ has nontrivial Lie algebra cohomology in some degree. Two types of families $\{\mathcal{F}_k\}$ are going to be considered where $\mathcal{F}_k$ in each case consists of all $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ with

(1) level U_k and weight ξ, as $U_k \to \{1\}$ (level goes to infinity), or
(2) level U and weight ξ_k, as $\xi_k \to \infty$.

The precise definition for $U_k \to \{1\}$ and $\xi_k \to \infty$ is technical and not to be written out in this report. By Harish-Chandra's finiteness theorem, each $\mathcal{F}_k$ is a finite set. For a reason coming from the trace formula, $\mathcal{F}_k$ is allowed to be a multi-set; the same π may appear with multiplicity $a(\pi) \in \mathbb{Z}_{>0}$.

We introduce the habitat for Satake parameters. Let $\widehat{T}$ be the (complex) dual torus of T, and $\widehat{T}_c$ the maximal compact subtorus of $\widehat{T}$. Denote by Ω the Weyl group. Let p be a prime and assume that π_p is unramified. The Satake isomorphism canonically associates to π_p a point $s_{\pi_p} \in \widehat{T}/\Omega$. Moreover π_p is tempered if and only if $s_{\pi_p} \in \widehat{T}_c/\Omega$. The locus $(\widehat{T}/\Omega)^{\mathrm{unit}}$ for unitary π_p contains $\widehat{T}_c/\Omega$. There are two natural measures to consider in this context. The Plancherel measure $\widehat{\mu}_p^{\mathrm{pl}}$ is defined on $(\widehat{T}/\Omega)^{\mathrm{unit}}$ and supported on $\widehat{T}_c/\Omega$. The Sato-Tate measure $\widehat{\mu}_p^{\mathrm{ST}}$ on $\widehat{T}_c/\Omega$ is defined as follows: Let $\widehat{G}_c$ be a maximal compact subgroup of $\widehat{G}$. Any $g \in \widehat{G}_c$ can be pulled into $\widehat{T}_c$ by conjugation, and this defines a map $\widehat{G}_c \to \widehat{T}_c/\Omega$. The pushforward of the Haar measure on $\widehat{G}_c$ under this map is $\widehat{\mu}_p^{\mathrm{ST}}$.

Let $\mathcal{H}(G(\mathbb{Q}_p))$ denote the unramified Hecke algebra for $G(\mathbb{Q}_p)$. Note that $\phi_p \in \mathcal{H}(G(\mathbb{Q}_p))$ naturally defines a function $\widehat{\phi}_p$ on $\widehat{T}/\Omega$ by $s_{\pi_p} \mapsto \mathrm{tr}\pi_p(\phi_p)$. One can define a truncated Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p))^{\leq\kappa}$ for each $\kappa \in \mathbb{Z}_{\geq 1}$ which is a non-canonical increasing exhaustive filtration of $\mathcal{H}(G(\mathbb{Q}_p))$ by complex vector spaces. To keep this article to a reasonable length we skip the definition of truncation.

To formulate equidistribution we need to define an averaged counting measure on $\widehat{T}/\Omega$ (or its subspace). For each $\mathcal{F}_k$ as above, let $a_{\mathcal{F}}(\pi)$ denote the multiplicity of π in the multi-set $\mathcal{F}_k$. (Set $a_{\mathcal{F}_k}(\pi) = 0$ if $a_{\mathcal{F}_k}(\pi) \notin \mathcal{F}_k$.) Define a measure on $(\widehat{T}/\Omega)^{\mathrm{unit}}$ (which is often restricted to $\widehat{T}_c/\Omega$)

$$\widehat{\mu}_{\mathcal{F}_k,p} := \frac{1}{|\mathcal{F}_k|} \sum_{\pi \in \mathcal{F}_k} a_{\mathcal{F}_k}(\pi)\delta_{\pi_p}.$$

We are ready to state the main result.

Theorem. Let $\{\mathcal{F}_k\}$ be a family of automorphic representations of $G(\mathbb{A})$ as above. Then there exists a polynomial $P(\kappa) \in \mathbb{R}[x]$ (with nonnegative coefficients) and a constant $C > 0$ such that for any prime p, any $\kappa \geq 1$ and any $\phi_p \in \mathcal{H}(G(\mathbb{Q}_p))$,

- $\widehat{\mu}_{\mathcal{F}_k,p}(\widehat{\phi}_p) - \widehat{\mu}^{\mathrm{pl}}(\widehat{\phi}_p) = O(N(U_k)^{-C_1}p^{P(\kappa)})$ for a family of type 1.
- $\widehat{\mu}_{\mathcal{F}_k,p}(\widehat{\phi}_p) - \widehat{\mu}^{\mathrm{pl}}(\widehat{\phi}_p) = O(m(\xi_k)^{-C_1}p^{P(\kappa)})$ for a family of type 2.

Here $N(U_k), m(\xi_k) \in \mathbb{Z}_{\geq 1}$ are integers measuring the "sizes" of U_k and ξ_k. (If $G = PGL(2)$, these may be thought of as the usual level and weight for modular forms.)

Two interesting consequences immediately follow. First if p is fixed and $k \to \infty$ then the right hand side tends to zero. This means that the "Satake parameters" s_{π_p} are equidistributed on $\widehat{T}_c/\Omega$ according to the Plancherel measure. Second, let $\{p_k\}$ be a sequence of primes such that $p_k \to \infty$. Suppose that $\lim_{k\to\infty} p_k^m/N(U_k) = 0$ (resp. $\lim_{k\to\infty} p_k^m/m(\xi_k) = 0$) for all $m \geq 1$. Then for any "polynomial function" $\widehat{\phi}$ on $\widehat{T}_c/\Omega$ we obtain

$$\lim_{k\to\infty} \widehat{\mu}_{\mathcal{F}_k,p_k}(\widehat{\phi}) = \widehat{\mu}^{\mathrm{ST}}(\widehat{\phi}). \tag{1}$$

This follows from the above theorem and the relatively easy fact that $\widehat{\mu}^{\mathrm{ST}}$ is the limit of $\widehat{\mu}^{\mathrm{pl}}$ as primes tend to ∞. Formula (1) is exactly the Sato-Tate equidistribution for families as alluded to in the title.

A few words should be said about the proof of the main theorem. The first step, considered standard, is to interpret $\widehat{\mu}_{\mathcal{F}_k,p}$ as the spectral side of Arthur's trace formula with Euler-Poincaré functions at infinity. (This can be done for various choices of the multiplicity $a_{\mathcal{F}_k}(\pi)$.) Thus one has a geometric expansion for $\widehat{\mu}_{\mathcal{F}_k,p}$ in terms of orbital integrals on G and its Levi subgroups. The orbital integral at 1 on G turns out to yield $\widehat{\mu}^{\mathrm{pl}}(\widehat{\phi})$ by the Plancherel formula. Therefore the main problem is to bound the remaining terms in the geometric expansion as in the right hand side of the theorem. This requires various techniques to estimate the number of rational conjugacy classes, volumes of locally symmetric spaces (coming from centralizers of semisimple elements), orbital integrals and stable discrete series characters at infinity.

Periods and central values of quadratic base change L-functions for GL$(2n)$

Brooke Feigon
(joint work with Kimball Martin, David Whitehouse)

Let E/F be a quadratic extension of number fields, π a cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbb{A}_F)$ and π_E the base change of π. Let $X(E,\pi)$ denote the set of isomorphism classes of quaternion algebras D/F such that there exists an embedding of E into D and an automorphic representation π^D on $D^\times$ such that the Jacquet-Langlands transfer of π^D is π. Then Waldspurger proved in [5] that

$L(1/2,\pi_E)=0$ if and only if $\int_{E^\times \mathbb{A}_F^\times\backslash \mathbb{A}_E^\times}\phi(t)dt=0$ for all $\phi\in\pi^D, D\in X(E,\pi)$. This result was reproved by Jacquet using the relative trace formula [3].

In this talk we explain some progress on extending Jacquet's approach to Waldspurger's result to higher rank. That is, let π be a cuspidal automorphic representation of $\mathrm{PGL}(2n,\mathbb{A}_F)$ that transfers to an automorphic representation π^D of $\mathrm{PGL}(n,D(\mathbb{A}_F))$, where we choose D such that there exists an embedding E into D. We show that, under certain conditions, if there exists $\phi\in\pi^D$ such that

$$\int_{\mathrm{PGL}(n,E)\backslash \mathrm{PGL}(n,\mathbb{A}_E)}\phi(h)dh\neq 0 \tag{0.1}$$

then $L(1/2,\pi_E)\neq 0$ and $L(s,\pi,\Lambda^2)$ has a pole at $s=1$.

The method we use is a comparison of two "simple" relative trace formulas: one on $(\mathrm{GL}(n,F)\times\mathrm{GL}(n,F))\backslash\mathrm{GL}(2n,F)/(\mathrm{GL}(n,F)\times\mathrm{GL}(n,F))$ and one on $\mathrm{GL}(n,E)\backslash\mathrm{GL}(n,D)/\mathrm{GL}(n,E)$. We avoid convergence issues on the geometric and spectral sides by choosing our test function to be a matrix coefficient of a supercuspidal representation at one place and to have support on the elliptic set at another place. We are then able to compare the two relative trace formulas on the geometric side by using work of Guo's [2] and reducing matching functions to the known case of quadratic base change. The full fundamental lemma at the split places and a recent result of Ramakrishan's [4], allow us to reduce the spectral sides to a single representation. By work of Friedberg-Jacquet [1] the periods occurring in the spectral expansion of the relative trace formula for $\mathrm{GL}(2n,F)$ are related to $L(1/2,\pi_E)$. The result now follows as (0.1) occurs in the spectral expansion of the relative trace formula for $\mathrm{GL}(n,D)$.

References

[1] S. Friedberg and H. Jacquet, *Linear periods*, J. Reine Angew. Math., **443**, (1993), 91–139.
[2] J. Guo, *On a generalization of a result of Waldspurger*, Canad. J. Math., **48**, (1996), no. 1, 105–142.
[3] H. Jacquet, *Sur un résultat de Waldspurger*, Ann. Sci. Éc. Norm. Supér., IV., **19** (1986), no. 2, 185–229.
[4] D. Ramakrishnan, *A mild Tchebotarev theorem for* GL(n), preprint, 2011.
[5] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), no. 2, 173–242.

On the Existence of Cuspidal Automorphic Forms

Goran Muić

Existence and construction of cusp forms is a fundamental problem in the modern theory of automorphic forms ([1], [11], [3]). In this talk we address the issue of existence of cusp forms using an extension and refinement of a classical method of (adelic) compactly supported Poincaré series. Our approach is based on the spectral decomposition of compactly supported Poincaré series. This method was successfully applied in the case of a cocompact discrete subgroup of a semisimple Lie group [6] to give some quantitative information on the decomposition of the corresponding L^2–space. It is extended in the non–cocompact settings and adélic settings in [7]. Some examples related to the existence of cusps forms for $SL_2(\mathbb{R})$ can be found in [8]. The relation to the action of unramified Hecke algebra in [9]. The joint work with Allen Moy [10] contains some further refining of results of [7] using Moy–Prasad filtration [5], [4].

We introduce some notation. Let G be a semisimple algebraic group defined over a number field k. We write V_f (resp., V_∞) for the set of finite (resp., Archimedean) places. For $v \in V_\infty \cup V_f$, we write k_v for the completion of k at v; if $v \in V_f$, then we let $\mathcal{O}_v$ be the ring of integers of k_v. The group G is unramified over k_v for almost all $v \in V_f$. In this case G is defined over $\mathcal{O}_v$ and $K_v \overset{def}{=} G(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup of $G(k_v)$. Let $G_\infty = \prod_{v \in V_\infty} G(k_v)$. This is a semisimple Lie group with finite center; let K_∞ and $\mathfrak{g}_\infty$ be a maximal compact subgroup and the (real) Lie algebra of G_∞, respectively. Let $G(\mathbb{A}_f)$ be the restricted product of all $G(k_v)$, $v \in V_f$, w.r. to the groups K_v defined above. The group $G(k)$ is diagonally embedded into $G(\mathbb{A})$ which is the restricted product of all $G(k_v)$, $v \in V_\infty \cup V_f$, w.r. to the groups K_v defined above. We assume that

$$G_\infty \text{ is not compact.}$$

Let $L \subset G(\mathbb{A}_f)$ be an open–compact subgroup. Then the intersection $\Gamma = \Gamma_L = G(k) \cap L \subset G(\mathbb{A}_f)$, which is taken in $G(\mathbb{A}_f)$ where we consider $G(k)$ diagonally embedded in $G(\mathbb{A}_f)$. The group Γ can be identified with a discrete subgroup of G_∞. It is called a congruence subgroup.

Motivated by [3], where they considered the existence of cuspidal automorphic forms in $L^2_{cusp}(\Gamma\backslash G_\infty)^{K_\infty}$ for level zero congruence subgroup Γ, we consider the following problem. Let $\hat{K}_\infty$ be the set of equivalence classes of irreducible representations of K_∞. Let $\delta \in \hat{K}_\infty$. Then, we want to prove that there exists infinitely many automorphic cuspidal representations in $L^2_{cusp}(\Gamma\backslash G_\infty)$ which contain δ.

We recall that by a well–known theorem of Gelfand, Piatetski Shapiro, Graev and Langlands (if not trivial) $L^2_{cusp}(\Gamma \setminus G_\infty)$ decomposes into a Hilbert direct sum of irreducible subspaces of G_∞:

$$L^2_{cusp}(\Gamma \setminus G_\infty) = \hat{\oplus}_j \mathfrak{H}^j, \tag{0.1}$$

where each irreducible unitary representation of G_∞ appears with a finite multiplicity.

The first observation ([6], Theorem 2.1) shows that not all K_∞–types δ are necessary to detect the representations $\mathfrak{H}^j$.

Theorem 0.2. *Let ψ be a non–zero K_∞–finite square–integrable automorphic form (for example, ψ could belong to the space of K_∞–finite vectors for some $\mathfrak{H}^j$). Then, the $(\mathfrak{g}_\infty, K_\infty)$–module of generated by ψ contains a non–trivial isotypic component for some $\delta \in \hat{K}_\infty$ such that there is a non–zero $K_\infty \cap \Gamma$–invariant vector in the space of δ.*

This result is a sort of Frobenius reciprocity for the restriction of "the induced representation" $L^2(\Gamma \setminus G_\infty)$ to K_∞. In Section 3 of [6] we explain the method of approach to the problem assuming that Γ is cocompact in G_∞. The main point is that given $\delta \in \hat{K}_\infty$ such that there is a non–zero $K_\infty \cap \Gamma$–invariant vector in the space of δ, we can construct $\psi(g) = \sum_{\gamma \in \Gamma} \varphi(\gamma g)$, for appropriate $\varphi \in C_c^\infty(G_\infty)$, such that it is non–zero, transforms on the right as δ and its support which is invariant on the right under Γ does not contain a connected component of the Lie group G_∞. Then, since we assume that Γ is cocompact in G_∞, $L^2(\Gamma \setminus G_\infty) = L^2_{cusp}(\Gamma \setminus G_\infty)$ and we can decompose ψ according to (0.1): $\psi = \sum_j \psi_j$. The requirements on ψ implies that all ψ_j transform on the right as δ and for infinitely many j's $\psi_j \neq 0$. This obviously implies that there exists infinitely many automorphic cuspidal representations in $L^2_{cusp}(\Gamma \backslash G_\infty)$ which contain δ. When Γ is not cocompact in G_∞, then $L^2(\Gamma \setminus G_\infty) \neq L^2_{cusp}(\Gamma \setminus G_\infty)$, and we need an additional requirement

$$\int_{\Gamma \cap U_{P,\infty} \backslash U_{P,\infty}} \psi(u g_\infty) du = 0, \quad g_\infty \in G_\infty,$$

for all proper k–parabolic subgroups P of G. (We write U_P for the unipotent radical.) Here $U_{P,\infty} = \prod_{v \in V_\infty} U_P(k_v)$.

This additional requirement is very difficult to understand without working in adelic settings. Now, we present approach from [7]. Let $\mathcal{A}_{cusp}(G(k) \backslash G(\mathbb{A}))$ be the space of K_∞–finite cuspidal automorphic forms for $G(\mathbb{A})$. This is a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module. In particular, it is a smooth $G(k_v)$–module for $v \in V_f$. This fact enables us to apply the local Bernstein's theory and decompose according to the Bernstein classes $\mathfrak{M}_v$ the smooth module

$$\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A})) = \oplus_{\mathfrak{M}_v} \mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v).$$

If $\mathfrak{M}_v$ is a Bernstein's class of (M_v, ρ_v), where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is an (irreducible) supercuspidal representation of M_v, then, by definition, $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v)$ is the largest $G(k_v)$–submodule of $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))$ such that its every irreducible subquotient is a subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v \rho_v)$, for some unramified character χ_v of M_v. Here P_v is an arbitrary parabolic subgroup of $G(k_v)$ containing M_v as a Levi subgroup. Obviously, this is also a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module decomposition. Further, we can iterate this for v ranging over a finite set of places, and as a result, we arrive at the question of non–triviality of a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v \in T)$, where $T \subset V_f$ is a finite and non–empty set of places. The following theorem gives a rather precise information on the structure of $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v \in T)$.

Theorem 0.3. *Let T be a finite set of places of k such that G is unramified over k_v for $v \in V_f - T$. For $v \in T$, let $\mathfrak{M}_v$ be a Bernstein's class of $G(k_v)$ determined by (M_v, ρ_v). We define $\mathfrak{P}$ to be the set of all k–parabolic subgroups P such that a Levi factor of $P(k_v)$ contains a $G(k_v)$–conjugate of M_v for all $v \in T$. Then we have the following:*

(i) $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v \in T) \neq 0$.

(ii) *Assume that $\mathfrak{P} = \{G\}$. Then for a sufficiently small open–compact subgroup $L \subset G(\mathbb{A}_f)$ of the form $L = \prod_{v \in T} L_v \times \prod_{v \in V_f - T} G(\mathcal{O}_v)$, there exist* *infinitely many K_∞–types δ which depend on L such that a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module $\mathcal{A}_{cusp}(G(k) \setminus G(\mathbb{A}))(\mathfrak{M}_v;\ v \in T)$ contains infinitely many irreducible representations of the form $\pi^j_\infty \otimes_{v \in V_f} \pi^j_v$, where π^j_v is unramified for $v \in V_f - T$, π^j_v belongs to the class $\mathfrak{M}_v$ and it contains a non–trivial vector invariant under L_v for $v \in T$, and the irreducible unitarizable $(\mathfrak{g}_\infty, K_\infty)$–module π^j_∞ contains δ. The set of equivalence classes of representations in $\{\pi^j_\infty\}$ which contribute to $L^2_{cusp}(\Gamma_L \setminus G_\infty)$ is infinite.*

We remark that an optimal choice (see [10]) of L_v may be in future obtained along the lines and definitions of [2] and [5].

References

[1] J. Arthur, *An introduction to the trace formula. Harmonic analysis, the trace formula, and Shimura varieties,* 1–263, Clay Math. Proc., **4**, Amer. Math. Soc., Providence, RI, 2005.

[2] C. Bushnell and P. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types,* Proc. London Math. Soc. (3) 77 (1998), no. 3, 582–634.

[3] E. Lindenstrauss and A. Venkatesh, *Existence and Weyl's law for spherical cusps forms,* Geom. Funct. Anal. **1**7 (2007), no. 1, 220251.

[4] A. Moy, *Lecture notes delivered in Beijing: http://www.math.ust.hk/ amoy/beirun.pdf*

[5] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal K-types,* Comment. Math. Helv. **7**1 (1996), 98–121.

[6] G. Muić, *On the decomposition of $L^2(\Gamma\backslash G)$ in the cocompact case.* J. Lie Theory **1**8 (2008), no. 4, 937–949.

[7] G. Muić, *Spectral Decomposition of Compactly Supported Poincaré Series and Existence of Cusp Forms,* Compositio Math. **1**46, No. 1 (2010) 1-20.

[8] G. Muić, *On the Cusp Forms for the Congruence Subgroups of $SL_2(\mathbb{R})$,* Ramanujan J. **2**1 No. 2 (2010), 223-239.

[9] G. Muić, *On the Structure of the Space of Cusp Forms For a Semisimple Group Over a Number Field,* Proc. Amer. Math. Soc. **138** (2010), 3147-3158.

[10] G. Muić and A. Moy, *On the Cusp Forms for the Congruence Subgroups of a Semisimple Lie group*, in preparation.

[11] J.-L. Li and J. Schwermer, *On the cuspidal cohomology of arithmetic groups,* Amer. J. Math. **1**31 (2009), no. 5, 1431–1464.

Parametrizing tempered near equivalence classes of cuspidal representations of $\widetilde{\mathbf{Sp}}_{2n}(\mathbb{A})$ with CAP representations of $\mathbf{Sp}_{4n}(\mathbb{A})$

David Soudry

(joint work with David Ginzburg, Dihua Jiang)

1. Functorial lift and descent $\widetilde{\mathrm{Sp}}_{2n} \leftrightarrow \mathrm{GL}_{2n}$

Let F be a number field, and denote by $\mathbb{A}$ its ring of Adeles. We fix a nontrivial character ψ of $F\backslash\mathbb{A}$. Let $\tilde{\pi}$ be an irreducible, automorphic, genuine, cuspidal representation of the Adelic metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, acting in a given subspace of cusp forms, which we keep denoting by $\tilde{\pi}$. Assume that $\tilde{\pi}$ is ψ–generic, that is the Fourier coefficient, with respect to the standard Whittaker character determined by ψ is not trivial on (the space of) $\tilde{\pi}$. By use of the ψ– theta correspondence to

$SO_{2n+1}(\mathbb{A})$, and [1], we can lift $\tilde{\pi}$, almost everywhere, with respect to ψ, to an irreducible automorphic, cuspidal representation τ of $GL_{2n}(\mathbb{A})$. We will assume that τ is cuspidal. Then, using Rankin-Selberg integrals which represent the standard (partial) L-function $L^S_\psi(\tilde{\pi}\times\tau,s)$, we know that $L(\tau,\wedge^2,s)$ has a pole at $s=1$, and that $L(\tau,\frac{1}{2})\neq 0$. The point is that

$$L^S_\psi(\tilde{\pi}\times\tau,s)=L^S(\tau\times\tau,s)=L^S(\tau\times\hat{\tau},s)$$

has a pole at $s=1$, and since the Rankin-Selberg integrals above depend on an Eisenstein series $E_{\tau,s}$ on $Sp_{4n}(\mathbb{A})$, corresponding to the parabolic induction

$$\mathrm{Ind}_{P^{4n}_{2n}(\mathbb{A})}^{Sp_{4n}(\mathbb{A})}\tau|\det\cdot|^{s-\frac{1}{2}},$$

we see that $E_{\tau,s}$ has a pole at $s=1$, and this implies the two conditions above on the pole of the exterior square L-function of τ and on the central value on the standard L-function of τ. See [3], [11]. In general, we denote by P^{2m}_k the standard parabolic subgroup of Sp_{2m}, whose Levi part is isomorphic to $GL_k\times Sp_{2(m-k)}$; we will denote its unipotent radical by N^{2m}_k.
Conversely, starting with such a representation τ, consider the Eisenstein series $E_{\tau,s}$ on $Sp_{4n}(\mathbb{A})$, as above. It has a pole at $s=1$ (for suitable sections). Denote by $\mathcal{E}_\tau$ the corresponding residual representation of $Sp_{4n}(\mathbb{A})$. Now we apply to $\mathcal{E}_\tau$ Fourier-Jacobi coefficients attached to the symplectic partition $[(2n),1^{2n}]$:

$$\widetilde{D}^{4n}_{2n,\psi}(\mathcal{E}_\tau)=\tilde{\pi}_\psi(\tau)=\tilde{\pi}. \tag{1}$$

This is the descent of $\mathcal{E}_\tau$ (we also say "the descent of τ") to $\widetilde{Sp}_{2n}(\mathbb{A})$.

Theorem 15. *$\tilde{\pi}_\psi(\tau)$ is an irreducible, automorphic, cuspidal and ψ-generic representation of $\widetilde{Sp}_{2n}(\mathbb{A})$; it lifts almost everywhere, with respect to ψ, to the representation τ.*

This theorem, except the assertion about irreducibility, was proved in [4]–[7]. The irreducibility of $\tilde{\pi}_\psi(\tau)$ follows from [9], [10].
In general, Fourier–Jacobi coefficients, as above, are defined, as follows. Consider, in Sp_{2k}, the following subgroup, which is the unipotent radical of the standard parabolic subgroup, whose Levi part is isomorphic to $GL_1^r\times Sp_{2(k-r)}$.

$$U^{2k}_r=\left\{u(z,v,y)=\begin{pmatrix} z & v & y\\ & I_{2(k-r)} & v'\\ & & z^*\end{pmatrix}\in Sp_{2k}: z\in Z_r\right\},$$

where Z_r is the group of upper unipotent $r\times r$ matrices. Consider the following character of $U^{2k}_r(\mathbb{A})$,

$$\psi_{U^{2k}_r}(u(z,v,y))=\psi(z_{1,2}+z_{2,3}+\cdots+z_{r-1,r}).$$

It is trivial on $U^{2k}_r(F)$. The map

$$\ell_{k-r}: u(z,v,y)\mapsto(v_r;y_{r,1})$$

projects U^{2k}_r onto the Heisenberg group in $2(k-r)+1$ variables, $\mathcal{H}_{2(k-r)+1}$.
Let π be an automorphic representation of $Sp_{2k}(\mathbb{A})$. Consider, for $\phi\in\mathcal{S}(\mathbb{A}^{k-r})$,

the theta series $\theta^{\phi}_{\psi^{-1},k-r}$ on $\mathcal{H}_{2(k-r)+1}(\mathbb{A})\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$. Embed $\mathrm{Sp}_{2(k-r)}$ inside Sp_{2k} as $\mathrm{diag}(I_r, \mathrm{Sp}_{2(k-r)}, I_r)$. Now, define, for $\varphi_\pi \in \pi$ and $\tilde{g} \in \widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$, projecting to $g \in \mathrm{Sp}_{2(k-r)}(\mathbb{A})$,

$$FJ^{\phi}_{\psi,k-r}(\varphi_\pi)(\tilde{g}) = \int\limits_{U_r^{2k}(F)\backslash U_r^{2k}(\mathbb{A})} \varphi_\pi(ug)\theta^{\phi}_{\psi^{-1},k-r}(\ell_{k-r}(u)\tilde{g})\psi^{-1}_{U_r^{2k}}(u)du.$$

This is a Fourier-Jacobi coefficient of φ_π corresponding to the symplectic partition $[(2r), 1^{2(k-r)}]$ of $2k$. It is a smooth automorphic function on $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$ and has a uniform moderate growth. We denote

$$\widetilde{D}^{2k}_{2(k-r),\psi}(\pi) = Span\{FJ^{\phi}_{\psi,k-r}(\varphi_\pi) : \varphi_\pi \in \pi,\ \phi \in \mathcal{S}(\mathbb{A}^{k-r})\}.$$

The metaplectic group $\widetilde{\mathrm{Sp}}_{2(k-r)}(\mathbb{A})$ acts on $\widetilde{D}^{2k}_{2(k-r),\psi}(\pi)$ by right translations. Similarly, if $\tilde{\pi}$ is an automorphic representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$, we define, for $\varphi_{\tilde{\pi}} \in \tilde{\pi}$, the Fourier-Jacobi coefficient $FJ^{\phi}_{\psi,k-r}(\varphi_{\tilde{\pi}})(g)$, on $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$, and, similarly, we define the $\mathrm{Sp}_{2(k-r)}(\mathbb{A})$–module $D^{2k}_{2(k-r),\psi}(\tilde{\pi})$. In (1), we took $\mathrm{Sp}_{2k} = \mathrm{Sp}_{4n}$, $\pi = \mathcal{E}_\tau$, $r = n$.

2. CAP representations of $\mathrm{Sp}_{4n}(\mathbb{A})$

Let us relax the conditions on τ. Instead of the condition $L(\tau, \frac{1}{2}) \neq 0$, let us require that there is $a \in F^*$, such that $L(\tau \otimes \chi_a, \frac{1}{2}) \neq 0$, where χ_a is the quadratic character of $F^*\backslash\mathbb{A}^*$, determined by a (the Hilbert symbol $(\cdot, a)$). Assume that $L(\tau, \frac{1}{2}) = 0$, so that $a \notin (F^*)^2$. By Theorem 15,

$$\widetilde{D}^{4n}_{2n,\psi^a}(\mathcal{E}_{\tau\otimes\chi_a}) = \tilde{\pi}$$

is an irreducible, automorphic, cuspidal, ψ^a–generic representation of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which lifts, almost everywhere, with respect to ψ^a to $\tau \otimes \chi_a$, and, hence, it lifts, almost everywhere, with respect to ψ, to τ. The representation $\tilde{\pi}$ is not ψ–generic, since if it where, then, as before, we would get that $L(\tau, \frac{1}{2}) \neq 0$, contrary to our present assumption. As a result of our main theorems (to follow), we will obtain

Theorem 16. *There is a unique CAP representation π of $\mathrm{Sp}_{4n}(\mathbb{A})$, which is isomorphic, at almost all finite places v, to the unramified constituent of*

$$\mathrm{Ind}^{\mathrm{Sp}_{4n}(F_v}_{P^{4n}_{2n}(F_v)} \tau_v|\det\cdot|^{\frac{1}{2}},$$

such that

$$\widetilde{D}^{4n}_{2n,\psi}(\pi) = \tilde{\pi}.$$

The existence of CAP representations with respect to $\mathrm{Ind}^{\mathrm{Sp}_{4n}(\mathbb{A}}_{P^{4n}_{2n}(\mathbb{A})}\tau|\det\cdot|^{\frac{1}{2}}$ was proved in [8]. We followed the same ideas and discovered a lot more.

3. Near equivalence classes determined by τ

Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L(\tau, \wedge^2, s)$ has a pole at $s = 1$, and there is a quadratic character χ of $F^*\backslash\mathbb{A}^*$, such that $L(\tau \otimes \chi, \frac{1}{2}) \neq 0$ (we allow $\chi = 1$). We define the following sets of nearly equivalent automorphic representations.

Let $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ denote the set of irreducible, automorphic, cuspidal (genuine) representations $\tilde{\pi}$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which lift almost everywhere, with respect to ψ, to τ.

Let $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$ denote the set of irreducible, automorphic representations π of $\mathrm{Sp}_{4n}(\mathbb{A})$, in the discrete automorphic spectrum, such that π_v is isomorphic, at almost all finite places v, to the unramified constituent of $\mathrm{Ind}_{P^{4n}_{2n}(F_v)}^{\mathrm{Sp}_{4n}(F_v}\tau_v|\det\cdot|^{\frac{1}{2}}$, and such that $\widetilde{D}^{4n}_{2n,\psi}(\pi) \neq 0$. In both sets the representations are counted with multiplicities.

From Theorem 15 and from Section 2, we know that $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$ is not empty, and even contains generic elements. Our goal is to set a bijection between the two sets above. More precisely, we want to show that the descent map

$$\Psi(\pi) = \widetilde{D}^{4n}_{2n,\psi}(\pi)$$

is bijective from $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$ to $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$. We can prove this, modulo a result on irreducibility of residual Eisenstein series, which should follow from Arthur's work and from the work of Moeglin. What we can prove, without any assumption is the following. Let $\mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau, \psi)$ be the subset of cuspidal elements in $\mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$. Define

$$\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi) = \begin{cases} \mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau, \psi) & , \quad L(\tau, \frac{1}{2}) = 0 \\ \mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau, \psi) \cup \{\mathcal{E}_\tau\} & , \quad L(\tau, \frac{1}{2}) \neq 0 \end{cases}$$

From Arthur's work, we should get that $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi) = \mathcal{N}_{\mathrm{Sp}_{4n}}(\tau, \psi)$. We can prove

Theorem 17. *For each $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$, $\Psi(\pi)$ is an irreducible representation which lies in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$. Moreover, the restriction of Ψ to $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau, \psi)$ is surjective on $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau, \psi)$.*

In [8], the goal was simply to construct elements in $\mathcal{N}^0_{\mathrm{Sp}_{4n}}(\tau, \psi)$. What brought us back to this was the following observation.

Let π be a CAP representation, with respect to $\mathrm{Ind}_{P^{4n}_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A}}\tau|\det\cdot|^{\frac{1}{2}}$. The proof that the constant terms, along unipotent radicals of parabolic subgroups, of $\Psi(\pi)$, are all zero, is the same as we already had for the case $\pi = \mathcal{E}_\tau$, because the proof here was to take an unramified finite place v, and show that the local version, given through Jacquet modules, $\widetilde{D}^{4n}_{2n,\psi_v}(\pi_v)$, has trivial Jacquet modules, along unipotent radicals of parabolic subgroups of $\widetilde{\mathrm{Sp}}_{2n}(F_v)$. For such v, π_v is the unramified constituent of $\mathrm{Ind}_{P^{4n}_{2n}(F_v)}^{\mathrm{Sp}_{4n}(F_v}\tau_v|\det\cdot|^{\frac{1}{2}}$, and all we needed to know was that $\tau_v \cong \hat{\tau}_v$ and that the central character of τ_v is trivial. Similarly, the proof that an irreducible subrepresentation of $\Psi(\pi)$ lifts almost everywhere, with respect to ψ, to τ, was obtained by computation of Jacquet modules at each unramified place. Thus, the fact that each irreducible subrepresentation of $\Psi(\pi)$ lies in $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ is straightforward.

4. Constructing elements in $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$ and the map Ψ

Starting with $\tilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$, we will construct elements $\pi \in \mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$, such that $\Psi(\pi) = \tilde{\pi}$. We consider the Eisenstein series of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$ attached to

$$\mathrm{Ind}_{\tilde{P}^{6n}_{2n}(\mathbb{A})}^{\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})} \gamma_\psi \tau|\det\cdot|^s \otimes \tilde{\pi}.$$

It has a pole at $s = 1$ (since $L^S_\psi(\tilde{\pi}\times\tau, s)$ has a pole at $s = 1$). Denote the residual representation by $\mathcal{E}_{\tau,\tilde{\pi},\psi}$. Now, we apply descent to $\mathrm{Sp}_{4n}(\mathbb{A})$:

$$\Phi(\tilde{\pi}) = D^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\tau,\tilde{\pi},\psi}).$$

We prove

Theorem 18. *1.* $\Phi(\tilde{\pi}) \neq 0$.
2. $\Phi(\tilde{\pi})$ *is square-integrable. Moreover, it is cuspidal if and only if* $\tilde{\pi}$ *is not* ψ-*generic. If* $\tilde{\pi}$ *is* ψ*-generic, then (*$\mathcal{E}_\tau$ *exists and)*

$$\Phi(\tilde{\pi}) = \mathcal{E}_\tau \oplus \textit{cuspidal representation.}$$

3. Each irreducible, cuspidal subrepresentation π *of* $\Phi(\tilde{\pi})$ *is CAP with respect to*

$$\mathrm{Ind}_{P^{4n}_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A}} \tau|\det\cdot|^{\frac{1}{2}}.$$

4. Each irreducible, subrepresentation π *of* $\Phi(\tilde{\pi})$ *satisfies*

$$\widetilde{D}^{4n}_{2n,\psi}(\pi) \neq 0.$$

To prove (1), we prove much more, namely

Theorem 19. *We have*

$$\widetilde{D}^{4n}_{2n,\psi}(D^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\tau,\tilde{\pi},\psi})) = \tilde{\pi},$$

that is

$$\Psi(\Phi(\tilde{\pi})) = \tilde{\pi}$$

(equality of spaces).

This theorem results from a precise identity that we obtain for the double descent above. The proof of (2) in Theorem 18 relies on proving that

$$D^{6n}_{4(n-\ell),\psi^{-1}}(\mathcal{E}_{\tau,\tilde{\pi},\psi}) = 0,$$

for $1 \leq \ell \leq n$. This is done by computing analogous Jacquet modules at one unramified place. Similarly with (3). For (4), let π be an irreducible, cuspidal subrepresentation of $\Phi(\tilde{\pi})$. Then the L^2-product, along $\mathrm{Sp}_{4n}(F)\backslash\mathrm{Sp}_{4n}(\mathbb{A})$,

$$< \pi, D^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\tau,\tilde{\pi},\psi}) >_{L^2(\mathrm{Sp}_{4n})}$$

is not identically zero. When we replace the residue by the Eisenstein series at s, the last pairing can be unfolded, for $Re(s)$ large, to an Eulerian integral representing $L^S(\pi\times\tau, s+\frac{1}{2})$. It contains, as an inner integral, the pairing

$$< \widetilde{D}^{4n}_{2n,\psi}(\pi), \tilde{\pi} >_{L^2(\mathrm{Sp}_{2n})} .$$

See [2]. In particular, $\Psi(\pi)=\widetilde{D}^{4n}_{2n,\psi}(\pi)\neq 0$, and hence $\pi\in\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Finally, if $\pi=\mathcal{E}_\tau$, then we already know that $\Psi(\mathcal{E}_\tau)\neq 0$.

5. The composition $\Phi(\Psi(\pi))$.

Let $\pi\in\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$. Consider the residual Eisenstein series $\mathcal{E}_{\tau,\pi}$ on $\mathrm{Sp}_{8n}(\mathbb{A})$, relative to $\mathrm{Ind}^{\mathrm{Sp}_{8n}(\mathbb{A})}_{P^{8n}_{2n}(\mathbb{A})}\tau|\det\cdot|^s\otimes\pi$, at $s=\frac{3}{2}$. We prove a similar identity, as the one for Theorem 19.

Theorem 20.

$$D^{6n}_{4n,\psi^{-1}}(\widetilde{D}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi}))=\pi.$$

We also prove

Theorem 21. *$\widetilde{D}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ is a square-integrable representation of $\widetilde{\mathrm{Sp}}_{6n}(\mathbb{A})$. Moreover, it has a unique nontrivial constant term, namely the one along the unipotent radical N^{6n}_{2n}. Denote it by $\mathcal{C}_{N^{6n}_{2n}}$. Then*

$$\mathcal{C}_{N^{6n}_{2n}}(\widetilde{D}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi}))=\gamma_\psi\delta^{\frac{1}{2}}_{P^{6n}_{2n}}|\det\cdot|^{-1}\tau\otimes\Psi(\pi).$$

Let $\tilde{\pi}$ be an irreducible subrepresentation of $\Psi(\pi)$. By Theorem 21, $\widetilde{D}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi})$ contains an irreducible subrepresentation σ of $\mathcal{E}_{\tau,\tilde{\pi},\psi}$. As in Theorem 18(4), we prove that $D^{6n}_{4n,\psi^{-1}}(\sigma)\neq 0$. By Theorem 20,

$$D^{6n}_{4n,\psi^{-1}}(\sigma)\subset D^{6n}_{4n,\psi^{-1}}(\widetilde{D}^{8n}_{6n,\psi}(\mathcal{E}_{\tau,\pi}))=\pi.$$

Since π is irreducible,, we get

$$\pi=D^{6n}_{4n,\psi^{-1}}(\sigma).$$

Hence

$$\pi\subset D^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\tau,\tilde{\pi},\psi})=\Phi(\tilde{\pi}).$$

Apply Ψ. Then by Theorem 19,

$$\Psi(\pi)\subset\Psi(\Phi(\tilde{\pi}))=\tilde{\pi},$$

and hence $\Psi(\pi)=\tilde{\pi}$ is irreducible. This proves that $\Psi(\pi)$ is irreducible, and that Ψ (when restricted to $\mathcal{N}'_{\mathrm{Sp}_{4n}}(\tau,\psi)$) is surjective. Moreover, we also proved

Theorem 22.

$$\pi\subset\Phi(\Psi(\pi)).$$

If we know the irreducibility of $\mathcal{E}_{\tau,\tilde{\pi},\psi}$ (and this should follow from the works of Arthur and the works of Moeglin), then we get

$$\pi = D^{6n}_{4n,\psi^{-1}}(\mathcal{E}_{\tau,\tilde{\pi},\psi}) = \Phi(\tilde{\pi}) = \Phi(\Psi(\pi)),$$

and hence Ψ is bijective, with

$$\Psi^{-1} = \Phi.$$

When $\tilde{\pi}$ is generic we can prove that $\mathcal{E}_{\tau,\tilde{\pi},\psi}$ is irreducible, and then we conclude that $\Phi(\tilde{\pi})$ is irreducible. Finally, we have the following application.

Theorem 23. *Assume that* $L(\tau\otimes\chi_a, \frac{1}{2}) \neq 0$. *Then* $\mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ *contains a unique* ψ^a*-generic representation. This is* $\tilde{\pi}_{\psi^a}(\tau\otimes\chi_a) = \widetilde{D}^{4n}_{2n,\psi}(\mathcal{E}_{\tau\otimes\chi_a})$. *In particular,* $\tilde{\pi}_{\psi^a}(\tau\otimes\chi_a)$ *occurs with multiplicity one within the* ψ^a*-generic representations.*

It is easy to reduce to the case $\chi_a = 1$ $(a \in (F^*)^2)$. If $\tilde{\pi} \in \mathcal{N}_{\widetilde{\mathrm{Sp}}_{2n}}(\tau,\psi)$ is ψ-generic, then, by what we explained before, $\Phi(\tilde{\pi})$ is irreducible. By Theorem 18(2), we conclude that $\Phi(\tilde{\pi}) = \mathcal{E}_\tau$, and so,

$$\tilde{\pi} = \Psi(\Phi(\tilde{\pi})) = \Psi(\mathcal{E}_\tau) = \tilde{\pi}_\psi(\tau).$$

References

[1] J. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi, *Functoriality for the classical groups*, Publ. Math. IHES **99**, (2004), 163-233.

[2] D. Ginzburg, D. Jiang, S. Rallis and D. Soudry, *L-functions for symplectic groups using Fourier-Jacobi models*, to appear in a volume in honor of Steve Kudla.

[3] D. Ginzburg, S. Rallis and D. Soudry, *L-functions for symplectic groups*, Bull. Soc. Math. France **126**, (1998), 181-244.

[4] D. Ginzburg, S. Rallis and D. Soudry, *On explicit lifts of cusp forms from* GL_m *to classical groups*, Annals of Math. **150** (1999), 807-866.

[5] D. Ginzburg, S. Rallis and D. Soudry, *On a correspondence between cuspidal representations* GL_{2n} *and* $\widetilde{\mathrm{Sp}}_{2n}$, JAMS **12** No. 3 (1999), 849-907.

[6] D. Ginzburg, S. Rallis and D. Soudry, *Lifting cusp forms on* GL_{2n} *to* $\widetilde{\mathrm{Sp}}_{2n}$*: the unramified correspondence*, Duke Math. J. **100**, No. 2, (1999), 243-266.

[7] D. Ginzburg, S. Rallis and D. Soudry, *Endoscopic representations of* $\widetilde{\mathrm{Sp}}_{2n}$, JIMJ**1**, No. 1, (2002), 77-123.

[8] D. Ginzburg, S. Rallis and D. Soudry, *Construction of CAP representations for symplectic groups using the descent method*, in Automorphic representations, L-functions and applications: progress and prospects, a volume in honor of Steve Rallis, Ohio State Univ. Math. Res. Inst. Publ., **11**, de Gruyter, Berlin, (2005), 193224.

[9] D. Jiang and D. Soudry, *The local converse theorem for* $\mathrm{SO}(2n+1)$ *and applications*, Annals of Math. **157**, (2003), 743-806.

[10] D. Jiang and D. Soudry, *Generic representations and local Langlands reciprocity law for p-adic* $\mathrm{SO}(2n+1)$, in Contributions to automorphic forms, geometry and number theory, a volume in honor of J. Shalika, The Johns Hopkins University Press, Baltimore (2004), 457-519.

[11] D. Soudry, *On langlands functoriality from classical groups to* GL_n, in Automorphic Forms I, Asterisque **298**, (2005), 335-390.

Stabilization of the trace formula for the covering groups of SL_2 and some applications

Tamotsu Ikeda
(joint work with K.Hiraga)

In this talk, we gave a partial stabilization of the trace formula for the covering groups over SL_2 of even degree. Local theory for the transfer was developed by Adams, Schultz, and Trehan for covering groups of SL_r. As for the global theory, Wen-Wei Li gave a stabilization of the trace formula for the metaplectic double covering groups of Sp_r. In this talk, we discuss a stabilization of elliptic terms of the trace formula for the covering groups of SL_2 with even degree.

Let F be a local field. We assume n is even and $\sharp\mu_n(F) = n$, where μ is the group of n-th roots of unity. The n-fold covering $\widetilde{\mathrm{SL}_2(F)}$ of $\mathrm{SL}_2(F)$ defined by the Kubota 2-cocycle $c(g_1, g_2)$. An element of $\widetilde{\mathrm{SL}_2(F)}$ is denoted by $[g, \zeta]$, $g \in \mathrm{SL}_2(F)$, $\zeta \in \mu_n$. $[g, 1]$ is simply denoted by $[g]$ or g.

We define two maps $\tau^+ : \mathrm{GL}_2(F) \to \mathrm{SL}_2(F)$ and $\tau^- : \mathrm{GL}_2(F) \to \mathrm{SL}_2(F)$ by $\tau^+(g) = (\det g)^{-n/2} g^n$ and $\tau^-(g) = -(\det g)^{-n/2} g^n$. Then τ^+ and τ^- factors through $\tau^+ : \mathrm{PGL}_2(F) \to \mathrm{SL}_2(F)$ and $\tau^- : \mathrm{PGL}_2(F) \to \mathrm{SL}_2(F)$.

We fix an additive character $\psi : F \to \mathbb{C}^\times$ once and for all. Let $\alpha_\psi(x)$ be the Weil constant of $x \in F^\times$ with respect to an additive character ψ. For $[h, \zeta] \in \widetilde{\mathrm{SL}_2(F)}$ and $g \in \mathrm{PGL}_2(F)$, we define the transfer factors $\delta^+_\psi([h, \zeta], g)$ and $\delta^-_\psi([h, \zeta], g)$ as follows (cf. [1], [8]).

If $h \in \mathrm{SL}_2(F)$ is stably conjugate to $\tau^+(g)$, then we set

$$\delta^+_\psi([h, \zeta], g) = \begin{cases} \zeta \dfrac{\alpha_\psi(1)}{\alpha_\psi(\det g)} c((\det g)^{n/2}, h) & \text{if } n \equiv 2 \bmod 4, \\ \zeta c((\det g)^{n/2}, h) & \text{if } n \equiv 0 \bmod 4. \end{cases}$$

If $h \in \mathrm{SL}_2(F)$ is not stably conjugate to $\tau^+(g)$, then we set $\delta^+_\psi([h, \zeta], g) = 0$. We also set $\delta^-_\psi([h, \zeta], g) := \alpha_\psi(1)^2 c(-1, h) \delta^+_\psi([-h, \zeta], g)$. Then these transfer factors $\delta^+_\psi([h, \zeta], g)$ and $\delta^-_\psi([h, \zeta], g)$ have the following properties.

Lemma. Let $g, g' \in \mathrm{PGL}_2(F)$ and $\tilde{h}, \tilde{h}' \in \widetilde{\mathrm{SL}_2(F)}$.

(1) If g and g' are conjugate in $\mathrm{PGL}_2(F)$, then $\delta^\pm_\psi(\tilde{h}, g) = \delta^\pm_\psi(\tilde{h}, g')$.
(2) If $\tilde{h}$ and $\tilde{h}'$ are conjugate in $\widetilde{\mathrm{SL}_2(F)}$, then $\delta^\pm_\psi(\tilde{h}, g) = \delta^\pm_\psi(\tilde{h}', g)$.

Here, the sign changes simultaneously.

Lemma. Let $g, g' \in \mathrm{PGL}_2(F)$, $h, h' \in \mathrm{SL}_2(F)$, and $\varepsilon, \varepsilon' \in \{+, -\}$. Assume that the following conditions (1), (2), (3), and (4) hold:

(1) $\tau^\varepsilon(g) = h$, and $\tau^{\varepsilon'}(g') = h'$.
(2) h and h' are elliptic.
(3) h and h' are stably conjugate.
(4) g and g' are not conjugate.

Then we have

$$\frac{\delta_\psi^{\varepsilon'}([h],g')}{\delta_\psi^{\varepsilon}([h],g)} = -\frac{\delta_\psi^{\varepsilon'}([h'],g')}{\delta_\psi^{\varepsilon}([h'],g)}.$$

Let $I([h],\tilde{\varphi})$ and $I(g,\varphi)$ be normalized orbital integrals for $h \in \widetilde{\mathrm{SL}_2(F)}$, $g \in \mathrm{PGL}_2(F)$, $\tilde{\varphi} \in \widetilde{C}_0(\widetilde{\mathrm{SL}_2(F)})$ and $\varphi \in C_0(\mathrm{PGL}_2(F))$. Here, $\widetilde{C}_0(\widetilde{\mathrm{SL}_2(F)})$ (resp. $C_0(\mathrm{PGL}_2(\mathrm{F}))$) is the space of anti-genuine locally constant compactly supported functions on $\widetilde{\mathrm{SL}_2(F)}$ (resp. the space of locally constant compactly supported functions on $\mathrm{PGL}_2(F)$).

For $\varepsilon \in \{+,-\}$, $\varphi^\varepsilon \in C_0(\mathrm{PGL}_2(F))$ is a transfer of $\tilde{\varphi} \in \widetilde{C}_0(\widetilde{\mathrm{SL}_2(F)})$ with respect to δ_ψ^ε, if

$$\sum_h \delta_\psi^\varepsilon([h],g) I([h],\tilde{\varphi}) = I(g,\varphi^\varepsilon)$$

for any semi-simple element $g \in \mathrm{PGL}_2(F)$ such that $\tau^\varepsilon(g)$ is regular. Here, h extends over a set of representatives for conjugacy classes of $\mathrm{SL}_2(F)$. For any anti-genuine function $\tilde{\varphi} \in \widetilde{C}_0^\infty(\widetilde{\mathrm{SL}_2(F)})$, there exists a transfer $\varphi^\varepsilon \in C_0^\infty(\mathrm{PGL}_2(F))$ of $\tilde{\varphi}$ with respect to δ_ψ^ε.

Assume F is non-archimedean, $n \in \mathfrak{o}^\times$, and ψ is of order 0. In this case, there is a canonical splitting for the covering $\widetilde{\mathrm{SL}_2(F)} \to \mathrm{SL}_2(F)$ over the maximal compact subgroup $\mathrm{SL}_2(\mathfrak{o})$. By this splitting, we regard $\mathrm{SL}_2(\mathfrak{o})$ as a subgroup of $\widetilde{\mathrm{SL}_2(F)}$. Then, there exists a canonical isomorphism

$$\mathcal{H}(\mathrm{PGL}_2(F)//\mathrm{PGL}_2(\mathfrak{o})) \simeq \widetilde{\mathcal{H}}(\widetilde{\mathrm{SL}_2(F)}//\mathrm{SL}_2(\mathfrak{o})),$$

which preserves the transfers. Here, $\mathcal{H}(\mathrm{PGL}_2(F)//\mathrm{PGL}_2(\mathfrak{o}))$ is the Hecke algebra for $(\mathrm{PGL}_2(F),\mathrm{PGL}_2(\mathfrak{o}))$, and $\widetilde{\mathcal{H}}(\widetilde{\mathrm{SL}_2(F)}//\mathrm{SL}_2(\mathfrak{o}))$ is the anti-gemuine Hecke algebra for $(\widetilde{\mathrm{SL}_2(F)},\mathrm{SL}_2(\mathfrak{o}))$.

Now let F be an algebraic number field such that $\sharp\mu(F) = n$, Let $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ be the n-fold metaplectic covering of $\mathrm{SL}_2(\mathbb{A})$. Let $C_0(\mathrm{PGL}_2(\mathbb{A}))$ be the space of smooth functions on $\mathrm{PGL}_2(\mathbb{A})$ with compact support. For $g = (g_v) \in \mathrm{PGL}_2(\mathbb{A})$ and $\varphi = \prod_v \varphi_v \in C_0(\mathrm{PGL}_2(\mathbb{A}))$, set

$$I(g,\varphi) := \prod_v I(g_v,\varphi_v).$$

Similarly, let $\widetilde{C}_0(\widetilde{\mathrm{SL}_2(\mathbb{A})})$ be the space of anti-genuine smooth functions on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ with compact support. For $h = (h_v) \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $\tilde{\varphi} = \prod_v \tilde{\varphi}_v \in \widetilde{C}_0(\widetilde{\mathrm{SL}_2(\mathbb{A})})$,

$$I(h,\tilde{\varphi}) := \prod_v I(h_v,\tilde{\varphi}_v).$$

Theorem. We have

$$2 \sum_{\substack{h \in \mathrm{SL}_2(F)/\sim \\ h:\ \text{ell. reg.}}} I(h, \tilde{\varphi}) = \sum_{\substack{g \in \mathrm{PGL}_2(F)/\sim \\ \tau^{\pm}(g):\ \text{ell. reg.}}} \left(I(g, \varphi^+) + I(g, \varphi^-) \right).$$

Here, $/\sim$ means the conjugacy.

As an application of the trace formula, we can construct an generalization of the theory of Kohnen plus subspace for Hilbert modular forms of half-integral weight.

References

[1] J. Adams, *Characters of covering groups of SL(n)*, Journal of the Inst. of Math. Jussieu, **2** (2003), 1–21.

[2] Y. Flicker, *Automorphic forms on covering groups of* GL(2), Inv. Math. **57** (1980), 119–182.

[3] S. Gelbart, I. I. Piatetski-Shapiro, *Some Remarks on Metaplectic Cusp Forms and the Correspondence of Shimura and Waldspurger*, Israel J. Math. **44** (1983), 97–126.

[4] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249–266.

[5] H. Y. Loke and G. Savin, *Representations of the two-fold central extension of* $\mathrm{SL}_2(\mathbb{Q}_2)$, Pacific J. Math. **247** (2010), 435–454.

[6] J. Schultz, *Lifting of Characters on* $\widetilde{\mathrm{SL}}_2(F)$ *and* $\mathrm{SO}_{1,2}(F)$ *for* F *a non-archimedean local field*, dissertation, University of Maryland, 1998.

[7] G. Shimura, *On modular forms of half integral weight*, Ann. Math. **97** (1973) 440–481.

[8] A. Trehan, *Lifting of characters and functions on metaplectic groups*, Dissertation, University of Maryland, 2004.

[9] J.-L. Waldspurger. *Correspondances de Shimura et quaternions*, Forum Math. **3** (1991), 219–307.

[10] Wen-Wei Li, *Transfert d'intégrales orbitales pour le groupe métaplectique*, preprint. arXiv:0906.4053

[11] Wen-Wei Li, *La formule des traces pour les revêtements de groupes réductifs connexes. I. Le développement géométrique fin*, preprint, arXiv:1004.4011

[12] Wen-Wei Li, *Le lemme fondamental pondéré pour le groupe métaplectique*, preprint, arXiv:1006.4780

Generating series and arithmetic theta lifting

Yifeng Liu

We formulate a general arithmetic inner product formula as the arithmetic counterpart of the classical Rallis inner product formula [12], by constructing so-called arithmetic theta lifting using Kudla's special cycles and generating series [7], [5], [6]. The arithmetic inner product formula has been studied in the book [8] for certain cases for the reductive pair $(\widetilde{\mathrm{SL}}(2), \mathrm{O}(3))$. We will focus on the case of unitary groups as developed in the upcoming paper [11].

Let us first introduce some notations and briefly recall the Rallis inner product formula in a special case. Let F be a number field, E/F a quadratic extension with the nontrivial Galois involution τ. We let $\mathbb{A}$ (resp. $\mathbb{A}_f$) and $\mathcal{M}$ (resp. $\mathcal{M}_f$) be the ring of (resp. finite) adèles and the set of all (resp. finite) places of F. Let $\eta_{E/F} : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be the associated quadratic character and $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ a nontrivial additive character. For $r \geq 1$, denote W_r the skew-hermitian space over

E (w.r.t. τ) of rank $2r$ defined by the matrix $w_r = \begin{pmatrix} & \mathbf{1}_r \\ \mathbf{1}_r & \end{pmatrix}$, where $\mathbf{1}_r$ is the identity matrix of rank r. Let $G = \mathrm{U}(W_r)$ be the corresponding unitary group. Given any (non-degenerate) hermitian space V over E of rank $m \geq 1$ and $H = \mathrm{U}(V)$, we obtain a reductive pair (G, H) in the sense of Howe inside the symplectic group $\mathrm{Sp}(\mathbf{W})$ for some symplectic space $\mathbf{W}$ over F of rank $4rm$. By [3], for any character $\eta : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ extending $\eta_{E/F}^m$, we can use η to lift $G(\mathbb{A}) \times H(\mathbb{A})$ to the ($\mathbb{C}^\times$-) metaplectic cover of $\mathrm{Sp}(\mathbf{W})(\mathbb{A})$ and hence have the Weil representation $\omega_{\eta,\psi}$ of $G(\mathbb{A}) \times H(\mathbb{A})$ on the space of Schwartz functions $\mathcal{S}(V^r(\mathbb{A}))$. For any $\phi \in \mathcal{S}(V^r(\mathbb{A}))$, we form the classical theta series $\theta(g, h; \phi) = \sum_{x \in V^r(E)} \omega_{\eta,\psi}(g, h)\phi(x)$, which is a smooth, slowly increasing function on $G(F)\backslash G(\mathbb{A}) \times H(F)\backslash H(\mathbb{A})$. Now let $\pi \subset \mathcal{A}_0(G)$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ inside the space of cuspidal automorphic forms of G and $\pi^\vee$ the contragredient representation realizing on the complex conjugation of π. For $f \in \pi$ and $\phi \in \mathcal{S}(V^r(\mathbb{A}))$, define the theta lifting

$$\theta_\phi^f(h) = \int_{G(F)\backslash G(\mathbb{A})} f(g)\theta(g, h; \phi)\mathrm{d}g$$

which is an automorphic form of H. Similarly, we define $\theta_{\phi^\vee}^{f^\vee}$ for $f^\vee \in \pi^\vee$ and $\phi^\vee \in \mathcal{S}(V^r(\mathbb{A}))^\vee$. One can ask if the theta lifting θ_ϕ^f is nontrivial for some f and ϕ. The answer will relate to the central value of the L-function $L(\frac{1}{2}, \pi)$ defined in [2] when $r = n, m = 2n$ and assuming $\eta = 1$ for simplicity. In fact, by [4] and [10], we have the following Rallis inner product formula (assuming V is anisotropic for simplicity):

$$\langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H := \int_{H(F)\backslash H(\mathbb{A})} \theta_\phi^f(h)\theta_{\phi^\vee}^{f^\vee}(h)\mathrm{d}h = \frac{L(\frac{1}{2}, \pi)}{2\prod_{i=1}^{2n} L(i, \eta_{E/F}^i)} \prod_{v \in S} Z_v^*(0)$$

where $S \subset \mathcal{M}$ is a finite subset and $Z_v^*(s) = Z_v^*(s; ; f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$ is certain normalized zeta integral as considered in [3]. In fact, $Z_v^*(0)$ can be viewed as a nonzero invariant functional in $\mathrm{Hom}_{G_v \times G_v}(I_{2n,v}(0), \pi_v^\vee \times \pi_v)$, where $I_{2n,v}(0)$ is the space of certain degenerate principal series of the doubling unitary group. By [9], we have $I_{2n,v}(0) = \bigoplus R(V_v)$ where the direct sum is taken over all isomorphic classes of hermitian spaces V_v over F_v of rank $2n$ and $R(V_v)$ is the space of Siegel-Weil sections obtained from V_v. By the theta dichotomy [3], [13], [1], we know that $\mathcal{I}(V_v, \pi_v) := \mathrm{Hom}_{G_v \times G_v}(R(V_v), \pi_v^\vee \times \pi_v) \neq 0$ for exactly one V_v. Moreover, it is conjectured that, for that V_v, $Z_v^*(0)$ is the unique element, up to constant, in $\mathcal{I}(V_v, \pi_v)$, which is true when $n = 1$ [11]. Define $\epsilon(\pi_v) = \eta_{E/F,v}((-1)^n \mathrm{disc}(V_v)) \in \{\pm 1\}$ for the unique V_v such that $\mathcal{I}(V_v, \pi_v) \neq 0$ and $\epsilon(\pi) = \prod_{v \in \mathcal{M}} \epsilon(\pi_v)$. There are two cases: (a) $\epsilon(\pi) = 1$; (b) $\epsilon(\pi) = -1$.

For (a), there is V/F, unique up to isomorphism, such that $Z_v^*(0)|_{R(V_v)} \neq 0$ for all $v \in \mathcal{M}$ and then the obstruction for some theta lifting of π to $H = \mathrm{U}(V)$ being nontrivial is the vanishing of $L(\frac{1}{2}, \pi)$. For (b), there is no chance to get any nontrivial theta lifting and moreover we prove that $L(\frac{1}{2}, \pi) = 0$. In this sense, the classical theory misses the second half. The arithmetic theory comes in when we try to fill up the case (b). For this purpose, we need more assumptions. Let we assume that: (1) F is totally real and E/F is totally imaginary; (2) π_v is the discrete series representation whose minimal K-type is $\det^n \times \det^{-n}$ for v archimedean. It is easy to see that $\mathcal{I}(V_v, \pi_v) \neq 0$ for V_v of signature $(2n, 0)$ when v is archimedean.

We let $\mathbb{V}$ to be the (unique up to isomorphism) hermitian space over $\mathbb{A}$ of rank $2n$ such that $\mathcal{I}(\mathbb{V}_v, \pi_v) \neq 0$, then $\mathbb{V} \ncong V \otimes_F \mathbb{A}$ for any V/F. Instead of the reductive group H over F, we have Shimura varieties Sh_K for $K \subset \mathbb{H}(\mathbb{A}_f)$ (sufficiently small) open compact subgroups of $\mathbb{H}(\mathbb{A}_f)$ where $\mathbb{H} = \mathrm{U}(\mathbb{V})$. They are smooth and quasi-projective over E of dimension $2n-1$. Instead of the theta series, we construct a generating series $Z_\phi(g)$ using Kudla's special cycles on Sh_K, for $\phi = \phi_\infty^0 \phi_f$ whose archimedean components are the Gaussian. By construction, $Z_\phi(g)$ is a function on $G(\mathbb{A})$ whose values are formal series in $\mathrm{Ch}^n := \varinjlim_K \mathrm{Ch}^n(\mathrm{Sh}_K) \otimes \mathbb{C}$. We have

Theorem 24 (Modularity of generating series, [11])**.** *For any linear functional ℓ of Ch^n, denote $\ell(Z_\phi)(g) = \ell(Z_\phi(g))$,*
(1) If $\ell(Z_\phi)(g)$ is absolutely convergent, then it is an automorphic form of G;
(2) If $n = 1$, $\ell(Z_\phi)(g)$ is always absolutely convergent.

We remark that: (1) The generating series can be defined for general (r, m) with $r \leq m - 1$ and the above theorem still holds. In particular, the second part is true only assuming $r = 1$. (2) There is a similar result in the context of symplectic-orthogonal pair which is proved in [14] and our proof is based on the one there. In what follows, we assume that $F \neq \mathbb{Q}$ when $n > 1$ to avoid the problem of compactification (see [11] for the general discussion). Parallel to the definition of theta lifting, we define the following *arithmetic theta lifting*

$$\Theta_\phi^f = \int_{G(F)\backslash G(\mathbb{A})} f(g) Z_\phi(g) \mathrm{d}g$$

which is in Ch^n. Moreover, we prove that it is cohomologically trivial. Hence we can talk about the (conjectural) Beilinson-Bloch height pairing $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}}$, which is just the Néron-Tate height pairing when $n = 1$. We have the following.

Conjecture 6 (Arithmetic inner product formula, [11])**.** *Let π, $\mathbb{V}$, f, $f^\vee$, ϕ and $\phi^\vee$ be as above, then*

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} = \frac{L'(\frac{1}{2}, \pi)}{2 \prod_{i=1}^{2n} L(i, \eta_{E/F}^i)} \prod_{v \in S} Z_v^*(0).$$

Using some ideas in [15] to our situation, we are able to prove the following.

Theorem 25 ([11])**.** *The above conjecture holds when $n = 1$.*

References

[1] Z. Gong and L. Grenié, *An inequality for local unitary theta correspondence*, preprint *http://dspace-unibg.cilea.it/handle/10446/506* (2009).

[2] S. Gelbart, I. Piatetski-Shapiro, S. Rallis, *Explicit Constructions of Automorphic L-Functions*, Lecture Notes in Mathematics **1254**, Springer-Verlag Berlin Heidelberg (1987).
[3] M. Harris, S. Kudla and W. Sweet, *Theta dichotomy for unitary group*, J. Amer. Math. Soc. **9** (1996), 941–1004.
[4] A. Ichino,*A regularized Siegel-Weil formula for unitary groups*, Math. Z. **247** (2004), 241–277.
[5] S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. **146** (1997), 545–646.
[6] S. Kudla, *Derivatives of Eisenstein series and generating functions for arithmetic cycles*, Séminaire Boubaki **876**, Astérisque **276**, Société Mathématique de France, Paris (2002), 341–368.
[7] S. Kudla and J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 121–172.
[8] S. Kudla, M. Rapoport and T. Yang, *Modular Forms and Special Cycles on Shimura Curves*, Annals of Mathematics Studies **161**, Princeton University Press, Princeton and Oxford (2006).
[9] S. Kudla, W. Sweet, *Degenerate principal series representations for $U(n,n)$*, Isr. J. Math **98** (1997), 253–306.
[10] J.-S. Li, *Non-vanishing theorems for the cohomology of certain arithemetic quotients*, J. Reine Angew. Math **428** (1992), 177–217.
[11] Y. Liu, *Arithmetic theta lifting and L-derivatives for unitary groups I. & II.*, to appear in Algebra and Number Theory.
[12] S. Rallis, *Injectivity properties of liftings associated to Weil representations*, Compositio Math. **52** (1984), 136–169.
[13] A. Paul, *Howe correspondence for real unitary groups*, J. Funct. Anal. **159**(1998), 384–431.
[14] X. Yuan, S.-W. Zhang and W. Zhang, *The Gross-Kohnen-Zagier theorem over totally real fields*, Compositio Math. **145** (2009), 1147–1162.
[15] X. Yuan, S.-W. Zhang and W. Zhang, *Gross-Zagier formula*, preprint *http://www.math.columbia.edu/~szhang/papers/Preprints.htm*, to appear in Annals of Mathematics Studies.

A new approach to the Local Langlands Correspondence for $\mathbf{GL}_n$ over p-adic fields

Peter Scholze

Fix a p-adic field F, i.e., a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$. Recall that the Local Langlands Correspondence, which is now a theorem due to Harris-Taylor, [3], and Henniart, [4], asserts that there should be a canonical bijection between the set of isomorphism classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of irreducible n-dimensional representations of the Weil group W_F of F, denoted $\pi \longmapsto \sigma(\pi)$. One possible local characterization of this bijection was given by Henniart, showing that there is at most one family of bijections defined for all $n \geq 1$ preserving L- and ϵ-factors of pairs, and also compatible with some basic operations on both sides such as twisting with characters.

We have a new local characterization of the Local Langlands Correspondence. Assume that $F = \mathbb{Q}_p$ for simplicity. Recall that it is known that the cohomology of the Lubin-Tate tower realizes the Local Langlands Correspondence, cf. e.g. [3], Theorem C, but only for supercuspidal representations. The idea is that replacing the Lubin-Tate space, i.e. the moduli space of one-dimensional formal groups of height n, by the moduli space of one-dimensional p-divisible groups of height n, one adds exactly the extra amount of information necessary to get the Langlands Correspondence for all irreducible smooth representations.

With this in mind, we just repeat the construction of the Lubin-Tate tower, except that we start with objects defined over a finite field: Take some integer $r \geq 1$ and a one-dimensional p-divisible group $\overline{H}$ of height n over $\mathbb{F}_{p^r}$. Looking at its Dieudonné module, this is equivalent to giving an element

$$\delta \in \mathrm{GL}_n(\mathbb{Z}_{p^r})\mathrm{diag}(p, 1, \ldots, 1)\mathrm{GL}_n(\mathbb{Z}_{p^r})$$

up to σ-conjugation by an element of $\mathrm{GL}_n(\mathbb{Z}_{p^r})$, where σ is the absolute Frobenius of $\mathbb{Z}_{p^r}$. Let R_δ be the deformation space of $\overline{H}$, with universal deformation H, and let $R_{\delta,m}/R_\delta$ be the covering parametrizing Drinfeld-level-m-structures on H. Then $\mathrm{GL}_n(\mathbb{Z}/p^m)$ acts on $R_{\delta,m}$. We choose $\ell \neq p$ and take the global sections of the nearby-cycle sheaves in the sense of Berkovich:

$$R\psi_\delta = \varinjlim H^0(R\psi_{\mathrm{Spf}\ R_{\delta,m}} \bar{\mathbb{Q}}_\ell) \ ,$$

and their alternating sum $[R\psi_\beta]$. These objects carry an action of $W_{\mathbb{Q}_{p^r}} \times \mathrm{GL}_n(\mathbb{Z}_p)$. Now take an element $\tau \in W_{\mathbb{Q}_p}$ projecting to the r-th power of geometric Frobenius, and let $h \in C_c^\infty(\mathrm{GL}_n(\mathbb{Z}_p))$ have values in $\mathbb{Q}$. Define a new function $h^\vee \in C_c^\infty(\mathrm{GL}_n(\mathbb{Z}_p))$ by $h^\vee(g) = h((g^{-1})^t)$.

Theorem 1 Define a function $\phi_{\tau,h}$ on $\mathrm{GL}_n(\mathbb{Q}_{p^r})$ by

$$\phi_{\tau,h}(\delta) = \mathrm{tr}(\tau \times h^\vee | [R\psi_\delta]) \ ,$$

if δ is as above, and by 0 else. Then $\phi_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_{p^r}))$, with values in $\mathbb{Q}$ independent of ℓ.

This allows us to define a function $f_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_p))$ by requiring that it has matching (twisted) orbital integrals. We use the normalization of Haar measures that gives maximal compact subgroups volume 1. Note that this function $f_{\tau,h}$ itself is not well-defined, but e.g. its orbital integrals and its traces on representations are.

For general p-adic fields F, we have an analogous definition of $f_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F))$ depending on $\tau \in W_F$ projecting to a positive power of geometric Frobenius and $h \in C_c^\infty(\mathrm{GL}_n(\mathcal{O}))$.

Theorem 2

(a) For any irreducible smooth representation π of $\mathrm{GL}_n(F)$ there is a unique semisimple n-dimensional representation $\mathrm{rec}(\pi)$ of W_F such that for all τ and h as above,

$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\mathrm{rec}(\pi))\mathrm{tr}(h|\pi) \ .$$

Write $\sigma(\pi) = \mathrm{rec}(\pi)(\frac{1-n}{2})$.

(b) If π is a subquotient of the normalized parabolic induction of the irreducible representation $\pi_1 \otimes \cdots \otimes \pi_t$ of $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_t}(F)$, then $\sigma(\pi) = \sigma(\pi_1) \oplus \ldots \oplus \sigma(\pi_t)$.

(c) The map $\pi \longmapsto \sigma(\pi)$ induces a bijection between the set of isomorphism classes of supercuspidal irreducible smooth representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of irreducible n-dimensional representations of W_F.

(d) The bijection defined in (c) is compatible with twists, central characters, duals, and L- and ϵ-factors of pairs, hence is the standard correspondence.

One main difference to the known proofs is that a previous result from [1] allows us to give a direct proof of part (c), i.e. the bijectivity of the correspondence. We do so without making use of the numerical Local Langlands Correspondence of Henniart, [5]. This argument relies on a geometric result from [1] describing the inertia-invariant nearby cycles in certain regular situations. This determines the inertia invariants $\sigma(\pi)^{I_F}$ for all irreducible smooth representations π, and implies that there are no supercuspidal representations that stay supercuspidal after any series of base-changes.

Moreover, we have a different proof of the local-global-compatibility result at all places due to Harris-Taylor, [3], avoiding the use of Igusa varieties and instead reducing all the necessary counting of points directly to the classical work of Kottwitz, [6], in the unramified case.

References

[1] P. Scholze, *The Langlands-Kottwitz approach for some simple Shimura varieties*, 2010, arXiv:1003.2451.

[2] P. Scholze, *The Local Langlands Correspondence for* GL_n *over p-adic fields*, 2010, arXiv:1010.1540.

[3] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies 151.

[4] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. 139 (2), 2000, 439–455.

[5] G. Henniart, *La conjecture de Langlands locale numérique pour* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) 21, 1988, 497–544.

[6] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. 5 (2), 1992, 373–444.

P-adic families of modular forms

Joachim Mahnkopf

1. Introduction

Let $G, G'/\mathbb{Q}$ be reductive algebraic groups and let

$$\varphi : {}^LG \to {}^LG'$$

be a morphism of the corresponding L-groups. According to the (conjectural) functoriality principle there should be a map

$$\tilde{\varphi} : \text{Automorphic representations on } G \rightsquigarrow \text{Automorphic representations on } G'$$

such that $t_{\pi',\ell} = \varphi(t_{\pi,\ell})$ for all unramified primes ℓ; here, $t_{\pi,\ell}$ is the Langlands parameter attached to the local component π_ℓ of π. One possible method to prove such a statement is the comparison of trace formulas. Roughly, this means the following. We denote by $\mathcal{H}_G$ resp. $\mathcal{H}_{G'}$ the Hecke algebra attached to $G(\mathbb{A})$ resp. $G'(\mathbb{A})$.

1. Step: Find an identity between traces of Hecke operators $T \in \mathcal{H}_G$ acting on $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and traces of Hecke operators $T' \in \mathcal{H}_{G'}$ acting on $L^2_0(G'(\mathbb{Q})\backslash G'(\mathbb{A}))$, which implies the existence of the mapping φ.

2. Step: Verify this identity by comparing trace formulas on G and on G'.

Roughly, the functoriality principle relates the existence of Automorphic representations on different groups. Somewhat similar, the theory of p-adic families of modular forms relates the existence of modular forms in different weights. It was our motivation to find out whether the comparison of trace formulas can also be applied to prove the existence of the families of modular forms, which are predicted by the theory of p-adic families. In the following we describe such a comparison of trace formulas following [3].

2. P-adic families of modular forms

We fix some notation. We fix a prime $p \in \mathbb{N}$ an integer $N \in \mathbb{N}$, which is relatively prime to p, and a Dirichlet character $\chi : \mathbb{Z}/(pN)^* \to \mathbb{C}^*$. We denote by $\Gamma = \Gamma_1(pN) \leq \mathrm{SL}_2(\mathbb{Z})$ the Hecke subgroup of level pN, $\mathcal{M}_k = \mathcal{M}_k(\Gamma, \chi\omega^{-k})$ is the space of holomorphic modular forms of level Γ, weight k and nebentype $\chi\omega^{-k}$ (ω is the Teichmuller character) and $\mathcal{S}_k = \mathcal{S}_k(\Gamma, \chi\omega^{-k})$ is the subspace of cusp forms. We denote by $T_\ell = \Gamma \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} \Gamma$ the classical Hecke operator attached to the prime $\ell \in \mathbb{N}$ and we denote by

$$\mathcal{H} = \langle T_\ell,\ \ell\, \text{prime} \rangle$$

the $\mathbb{Z}$-algebra generated by the Hecke operators. We fix a p-adic valuation v_p on $\bar{\mathbb{Q}}_p$ and for any $\alpha \in \mathbb{Q}_{\geq 0}$ we denote by $\mathcal{S}_k^\alpha \leq \mathcal{S}_k$ the slope α subspace. According to the *Conjecture of Mazur-Gouvea* (cf. [2]) any eigenform $f_0 \in \mathcal{S}_{k_0}^\alpha$ fits in a p-adic analytic family of eigenforms of varying weight k, i.e. there are • a finite flat $\mathbb{Z}_p[[X]]$-algebra R • morphisms $\eta_k : R \to \bar{\mathbb{Q}}_p$ • a formal q -expansion $F = \sum_{n>0} A_n q^n \in R[[q]]$ such that the following holds: for any k the specialization $f_k = \eta_k(F) := \sum_{n>0} \eta_k(F) q^n$ is an eigenform in $\mathcal{S}_k^\alpha$ and $f_{k_0} = f_0$. Hence, $(f_k)_k$ is a family of eigenforms, which depends p-adic analytically on the weight k and which passes through the given eigenform f_0. Thus, the theory of p-adic families relates the existence of modular forms in different weights.

3. Comparison of trace formulas

We want to establish the existence of a family of eigenforms passing through a given eigenform by a comparison of trace formulas. To this end instead of eigenforms f satisfying $T_\ell f = \lambda_\ell f$ for all primes ℓ we consider the (corresponding) systems of eigenvalues $\lambda = (\lambda_\ell)_\ell$. For any sequence $\lambda = (\lambda_\ell)_\ell$ of algebraic numbers we denote by

$$\mathcal{M}_k(\lambda) = \{f \in \mathcal{M}_k : \text{ for all } \ell \text{ there is } n_\ell \in \mathbb{N} : (T_\ell - f)^{n_\ell} = 0\}$$

the generalized eigenspace attached to $\lambda = (\lambda_\ell)_\ell$ and

$$\Phi_k^\alpha = \{\lambda = (\lambda_\ell)_\ell : \mathcal{M}_k(\lambda) \neq 0 \text{ and } v_p(\lambda_p) = \alpha\}$$

is the set of systems of eigenvalues appearing in slope α. We then want to show that any $\lambda_0 \in \Phi_{k_0}^\alpha$ fits in a p-adic analytic family of systems of eigenvalues $(\lambda_k)_k$, where $\lambda_k \in \Phi_k^\alpha$. We proceed in 2 steps.

1. Step. Any λ_0 fits in a *p-adic continuous* family of eigenvalues $(\lambda_k)_k$. Here, continuous means that there is $a \in \mathbb{Q}_{\geq 0}$ such that $k \equiv k' \pmod{p^m}$ implies $v_p(\lambda_{k,\ell} - \lambda_{k',\ell}) > am$ for all primes ℓ.

2. Step. Any continuous family is analytic. Here we assume $\alpha = 0$, i.e. we restrict to the *ordinary case*.

As in the talk we will only report on the first step. If any λ_0 fits in a continuous family as in step 1, then there are mappings

$$\varphi_k : \Phi_{k_0}^\alpha \to \Phi_k^\alpha$$

such that $v_p(\varphi(\lambda) - \lambda) > am$ if $k \equiv k_0 \pmod{p^m}$. We want to establish the existence of the mappings φ_k by comparing trace formulas in weight k_0 and k. To this end we introduce certain mod p^c multiplicities: for any $\lambda = (\lambda_\ell)_\ell$ and any $c \in \mathbb{N}$ we set

$$\mathbf{m}_k^\alpha(\lambda, c) = \sum_{\substack{\mu \in \Phi_k^\alpha \\ v_p(\mu - \lambda) > c}} \dim \mathcal{M}_k(\mu).$$

Thus, $\mathbf{m}_k^\alpha(\lambda, c)$ counts the number of eigenforms, which are congruent to a given eigenform modulo p^c. The map φ_k now exists if there is $a \in \mathbb{Q}_{\geq 0}$ with the following property: for any λ there is $c = c(\lambda) > am$ such that $\mathbf{m}_{k_0}^\alpha(\lambda, c) \neq 0$ implies $\mathbf{m}_k^\alpha(\lambda, c) \neq 0$. In fact we have:

Proposition. *Assume that* $\mathrm{tr}T|_{\mathcal{M}_k^\alpha}$ *is a continuous function of the weight* k *for all* $T \in \mathcal{H}$, *i.e. there is* $A \in \mathbb{Q}_{\geq 0}$ *such that* $k \equiv k' \pmod{p^m}$ *implies that* $v_p(\mathrm{tr}T|_{\mathcal{M}_k^\alpha} - \mathrm{tr}T|_{\mathcal{M}_{k'}^\alpha}) \geq Am$. *Then there is* $a \in \mathbb{Q}_{\geq 0}$ *with the following property: for all* $k \equiv k_0 \pmod{p^m}$ *and all* $\lambda \in \Phi_{k_0}^\alpha$ *there is* $c = c(\lambda) > am$ *such that*

$$\mathbf{m}_{k_0}^\alpha(\lambda, c) = \mathbf{m}_k^\alpha(\lambda, c).$$

To prove the Proposition we first define the number a (a will depend on A). According to a Theorem of Hida/Buzzard (cf. [1]) there is $L = L(\alpha) \in \mathbb{N}$ such $\dim \mathcal{M}_k^\alpha < p^L$ for all weights $k \geq 2$ and we then construct an element $e \in \mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Q}$ and specify a natural number $c > am$ such that

- $\mathrm{tr}\, e|_{\mathcal{M}_{k_0}^\alpha} \equiv \mathbf{m}_{k_0}^\alpha(\lambda, c) \pmod{p^L}$,
- $\mathrm{tr}\, e|_{\mathcal{M}_k^\alpha} \equiv \mathbf{m}_k^\alpha(\lambda, c) \pmod{p^L}$,
- the p-adic value of the denominator of e is bounded by $Am - L$.

Using the continuity of $\mathrm{tr}e|_{\mathcal{M}_k^\alpha}$ and the bound of Buzzard/Hida we see that the above two equations imply that $\mathbf{m}_{k_0}^\alpha(\lambda, c) = \mathbf{m}_k^\alpha(\lambda, c)$.

Using the topological trace formula we verify the continuity of $\mathrm{tr}T|_{\mathcal{M}_k^\alpha}$, which together with the above Proposition yields the existence of continuous p-adic families of eigenforms passing through a given eigenform f_0.

4. Final Remarks

1.) In [3] we also verified analyticity of the trace in the slope 0 case and used this to show that continuous families of slope 0 are analytic. Hence, we obtain the existence of analytic slope 0 families of eigenforms passing through a given eigenform of slope 0.

2.) Using the Topological Trace Formula we compute in [3] the Lefschetz number of Hecke operators acting on the cohomology of quotients of the upper half plane with coefficients in finite dimensional representations of GL_2. From this we deduce continuity of the trace in the finite slope case and analyticity in the slope 0 case. We hope that by computing Lefschetz numbers of Hecke operators acting on cohomology with coefficients in some Verma type modules we can even conclude analyticity of the trace in the finite slope case and, hence, the existence of p-adic analytic families of finite slope.

3.) The main obstacle to generalizing the approach to higher rank is the use of the Theorem of Hida/Buzzard on the boundedness of the slope subspace of the space of modular forms. Therefore, in [4] we generalized their result to arbitrary Chevalley groups, hence, our approach should carry over to higher rank groups having discrete series.

4.) The proofs in [3] do not make use of rigid analytic methods or p-adic Fredholm theory and are elementary in a sense comparable to [1].

References

[1] Buzzard, K., *Families of modular forms* (21st Journées arithmétique) (Rome 2001), J. Théor. Nombres Bordeaux **13** (2001), no. 1, 43-52

[2] Mazur, B., Gouvea, F., *Families of modular forms*, Math. of Computation **58** 1992

[3] Mahnkopf, J., *Trace identities and p-adic families of modular forms* http://www.mat.univie.ac.at/algebra/people/jmahnkopf.shtml

[4] Mahnkopf, J., *On the Cohomology of irreducible representations of Chevalley groups*, preprint, 20 pp.,

Construction of Endoscopy Transfers for Classical Groups

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(joint work with David Ginzburg, David Soudry)

This is a report of our work in progress on construction of endoscopy transfers for quasi-split classical groups.

This construction uses certain Fourier coefficients of residues of certain Eisenstein series as kernels of integral transforms. The formulation works for all quasi-split classical groups. However, we discuss in this notes a particular case, which explains the idea and the method behind the construction.

Let F be a number field, and $\mathbb{A}$ be the ring of adeles of F. Let τ be an irreducible unitary cuspidal automorphic representation of $GL_{2a}(\mathbb{A})$, with a positive integer a. Assume that τ is self-dual and with trivial central character. It is well-known that τ is either of symplectic type (i.e. the partial exterior square L-function $L^S(s,\tau,\wedge^2)$ has a pole at $s=1$) or of orthogonal type (i.e. the partial symmetric square L-function $L^S(s,\tau,\vee^2)$ has a pole at $s=1$).

For a positive integer m, consider the F-split even special orthogonal group SO_{2am}. Assume that if τ is of symplectic type, take m to be even; and if τ is of orthogonal type, take m to be odd. When m is odd and τ is of orthogonal type, we denote by ϵ the automorphic descent of τ from GL_{2a} to SO_{2a}, following the work of Ginzburg, Rallis, and Soudry [7].

When $m=2n$, take the standard parabolic subgroup $P_{(2a)^n}$ with the Levi part being $GL_{2a}^{\times(n)}$; and when $m=2n+1$, take the standard parabolic subgroup $P_{(2a)^n}$ with the Levi part being $GL_{2a}^{\times(n)}\times SO_{2a}$. Consider the cuspidal data $(P_{(2a)^n},\tau^{\otimes(n)})$ when $m=2n$, and $(P_{(2a)^n},\tau^{\otimes(n)}\otimes\epsilon)$ when $m=2n+1$, respectively. Let $\underline{s}:=(s_1,s_2,\cdots,s_n)\in\mathbb{C}^n$. Following Langlands, there are Eisenstein series $E(g,\Phi_{\tau^{\otimes(n)}},\underline{s})$ when $m=2n$, and $E(g,\Phi_{\tau^{\otimes(n)}\otimes\epsilon},\underline{s})$ when $m=2n+1$, attached to the above cuspidal data. It is not hard to check that $E(g,\Phi_{\tau^{\otimes(n)}}$ has a pole at $\underline{s}=(n-\frac{1}{2},n-\frac{3}{2},\cdots,\frac{1}{2})$ when $m=2n$, and $E(g,\Phi_{\tau^{\otimes(n)}\otimes\epsilon},\underline{s})$ has a pole at $\underline{s}=(n,n-1,\cdots,1)$ when $m=2n+1$ ([8]).

We denote by $\mathcal{E}_{(\tau,m)}(g)$ the iterated residue of this Eisenstein series at the given point, which is a square-integrable automorphic form on $SO_{2am}(\mathbb{A})$.

In the following, we construct kernel functions for the integral transform, which gives certain types of endoscopy transfers ([1]). In order to explain the idea and the method of the construction, we consider the following type of endoscopy transfer

$$SO_{2ab}\times SO_{2k}\to SO_{2k+2ab}, \tag{1}$$

where b is a positive integer.

The kernel functions for such a construction are given by certain Fourier coefficients of the residues $\mathcal{E}_{(\tau,2k+b)}(g)$ on $SO_{2a(2k+b)}(\mathbb{A})$. Consider the standard parabolic subgroup $P_{(2k)^{a-1}}$ of $SO_{2a(2k+b)}$ with the Levi part being $GL_{2k}^{\times(a-1)}\times SO_{2ab+4k}$. The unipotent radical is denoted by $V_{a,k,l}$. It is clear that

$$V_{a,k,l}/[V_{a,k,l},V_{a,k,l}]\cong M_{(2k)\times(2k)}^{\oplus(a-2)}\oplus M_{(2k)\times(ab+k)}\oplus M_{(2k)\times(2k)}\oplus M_{(2k)\times(ab+k)} \tag{2}$$

where $M_{m\times n}$ denotes the matrix of size $m\times n$. The elements on the right hand side is denoted by $(X_1,\cdots,X_{a-2};Y_1,Y_2,Y_3)$. Take a non-trivial additive character ψ of $F\backslash\mathbb{A}$, and define a character of $V_{a,k,l}(\mathbb{A})$ by

$$\psi_{a,k,l}(v):=\psi(trace(X_1+\cdots X_{a-2}+Y_2)). \tag{3}$$

It is left $V_{a,k,l}(F)$-invariant. It is easy to check that the stabilizer of the character $\psi_{a,k,l}$ in the Levi subgroup $GL_{2k}^{\times(a-1)}\times SO_{2ab+4k}$ is $SO_{2k}^{\Delta(a)}\times SO_{2ab+2k}$. The elements

$$(g,\begin{pmatrix}h_1 & h_2\\ h_3 & h_4\end{pmatrix})\in SO_{2k}\times SO_{2ab+2k}$$

correspond to the elements

$$(g^{\Delta(a-1)},\begin{pmatrix}h_1 & & h_2\\ & g & \\ h_3 & & h_4\end{pmatrix})\in GL_{2k}^{\times(a-1)}\times SO_{2ab+4k}.$$

The Fourier coefficient defined by

$$\Theta^{\tau,\psi}_{a,k,l}(g,h):=\int_{V_{a,k,l}(F)\backslash V_{a,k,l}(\mathbb{A})}\mathcal{E}_{(\tau,2k+b)}(v(g,h))\overline{\psi}_{a,k,l}(v)dv \tag{4}$$

is automorphic over $SO_{2k}(\mathbb{A})\times SO_{2ab+2k}(\mathbb{A})$. Let σ and π be irreducible cuspidal automorphic representations of $SO_{2k}(\mathbb{A})$ and $SO_{2ab+2k}(\mathbb{A})$, respectively. Then the main integral is defined as follows:

$$I^{\psi}_{a,k,l}(\tau,\sigma;\pi):=\int_{[SO_{2k}]}\int_{[SO_{2ab+2k}]}\Theta^{\tau,\psi}_{a,k,l}(g,h)\varphi_\sigma(g)\varphi_{\pi^\vee}(h)dhdg, \tag{5}$$

where $[SO_{2k}]:=SO_{2k}(F)\backslash SO_{2k}(\mathbb{A})$ and $[SO_{2ab+2k}]:=SO_{2ab+2k}(F)\backslash SO_{2ab+2k}(\mathbb{A})$. We make the following conjecture.

Conjecture: *Let σ and π be irreducible cuspidal automorphic representations of $SO_{2k}(\mathbb{A})$ and $SO_{2ab+2k}(\mathbb{A})$, respectively. Assume that if τ is of symplectic type, take b to be even; and if τ is of orthogonal type, take b to be odd, and assume that the integral $I^{\psi}_{a,k,l}(\tau,\sigma;\pi)$ is nonzero for some choice of $\varphi_\sigma\in V_\sigma$ and $\varphi_{\pi^\vee}\in V_{\pi^\vee}$. Then σ has a global Arthur parameter $\psi_{SO_{2k}}$ if and only if π has a global Arthur parameter $\psi_{SO_{2k}}\oplus(\tau,b)$.*

Note that the global Arthur parameters are referred to [1]. The condition that if τ is of symplectic type, take b to be even; and if τ is of orthogonal type, take b to be odd is to make (τ,b) a global Arthur parameter for SO_{2ab}.

The following special cases of our conjecture are proved in [4] and [5].

Theorem: *The above conjecture holds for tempered representation σ and for either $b=1$ if τ is orthogonal or $b=2$ if τ is symplectic.*

In [6], we explore the problem on cuspidality of the above construction for a fixed σ with b varying. This leads to the problem on the first occurrence in such endoscopy tower, generalizing the classical theory on first occurrence in theta correspondences.

References

[1] Arthur, James, *An introduction to the trace formula.* Harmonic analysis, the trace formula, and Shimura varieties, 1–263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.
[2] Ginzburg, David, *A Construction of CAP Representations for Classical Groups.* International Math. Research Notices No. 20 (2003), 1123-1140.
[3] Ginzburg, David, *Endoscopic Lifting in Classical Groups and Poles of Tensor L Functions.* Duke Math. Journal Vol. 141, No. 3 (2008), 447-503.
[4] Ginzburg, David; Jiang, Dihua; Soudry, David, , (tempered)
[5] Ginzburg, David; Jiang, Dihua; Soudry, David, *Endoscopy Correspondences for Orthogonal Groups: unramified correspondence*, in preparation, 2011.
[6] Ginzburg, David; Jiang, Dihua; Soudry, David, *Endoscopy Correspondences for Orthogonal Groups: cuspidality*, in preparation, 2011
[7] Ginzburg, David; Rallis, Stephen; Soudry, David, *On explicit lifts of cusp forms from GL_m to classical groups*, Annals of Math 150 (1999), 807-866.
[8] Jiang, Dihua; Liu, Baiying; Zhang, Lei, *Residuea of certain Eisenstein series for classical groups*, in preparation, 2011.

Representations of Metaplectic Groups

Gordan Savin

(joint work with Wee Teck Gan)

Let k be a non-archimedean local field of characteristic zero and residual characteristic p. Let $(W, \langle -, - \rangle)$ be a symplectic vector space of dimension $2n$ over k, with associated symplectic group $\mathrm{Sp}(W)$. The group $\mathrm{Sp}(W)$ has a unique two-fold central extension $\mathrm{Mp}(W)$ which is called the metaplectic group:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

The purpose of this work is to investigate the (genuine) representation theory of $\mathrm{Mp}(W)$. The prototype of our results is the work of Waldspurger who considered the case $\dim W = 2$. We obtain extensions of essentially all of Waldspurger's results mentioned above to the case of general W's. First, one has the following theorem, whose proof was sketched in [GGP], based on a key result of Kudla-Rallis [KR].

Theorem 1. *For each non-trivial additive character $\psi : k \to \mathcal{C}^\times$, there is a bijection*

$$\Theta_\psi : Irr(\mathrm{Mp}(W)) \longleftrightarrow Irr(\mathrm{SO}(V^+)) \sqcup Irr(\mathrm{SO}(V^-)),$$

where V^+ (respectively V^-) is the split (resp. non-split) quadratic space of discriminant 1 and dimension $2n+1$. This bijection is given by the theta correspondence (with respect to ψ) for the group $\mathrm{Mp}(W) \times \mathrm{SO}(V^\pm)$.

Corollary 2. *Assume the local Langlands correspondence for $\mathrm{SO}(V^\pm)$. Then one obtains a local Langlands correspondence for $\mathrm{Mp}(W)$, i.e. a bijection (depending on ψ)*

$$\mathcal{L}_\psi : Irr(\mathrm{Mp}(W)) \longleftrightarrow \Phi(\mathrm{Mp}(W))$$

where $\Phi(\mathrm{Mp}(W))$ is the set of pairs (ϕ, η) such that

- *$\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathcal{C})$ is a $2n$-dimensional symplectic representation of the Weil-Deligne group WD_k of k;*
- *η is an irreducible representation of the (finite) component group $A_\phi = \pi_0(Z_{\mathrm{Sp}_{2n}(\mathbb{C})}(\phi))$.*

Since the local Langlands correspondence for $\mathrm{SO}(V^\pm)$ is known for $\dim V = 5$ (by [GT] and [GTW]), the statement of the corollary is unconditional in this case. The general case has been announced by Arthur and will appear in his forthcoming book [A]. One may ask if the local Langlands correspondence given in Corollary 2 satisfies certain typical properties. For example, for a representation σ of $\mathrm{Mp}(W)$ with L-parameter ϕ, one would expect that σ is a discrete series representation if and only if ϕ does not factor through any proper Levi subgroup. To a large extent, such questions amount to whether the bijection Θ_ψ satisfies the analogous properties. More precisely, we have:

Theorem 3. *Suppose that $\pi \in Irr(\mathrm{SO}(V))$ and $\sigma \in Irr(\mathrm{Mp}(W))$ correspond under Θ_ψ. Then we have:*

(i) π is a discrete series representation if and only if σ is a discrete series representation.

(ii) π is tempered if and only if σ is tempered.

(iii) In general, suppose that

$$\pi = J_Q(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \pi_0), \quad s_1 > s_2 > .... > s_r > 0$$

is a Langlands quotient of $\mathrm{SO}(V)$, where Q is a parabolic subgroup of $\mathrm{SO}(V)$ with Levi subgroup $\mathrm{GL}_{n_1} \times ... \times \mathrm{GL}_{n_r} \times \mathrm{SO}(V_0)$, the τ_i's are unitary tempered representations of GL_{n_i}, and π_0 is a tempered representation of $\mathrm{SO}(V_0)$. Then

$$\sigma = J_{\tilde{P}}(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \Theta_\psi(\pi_0))$$

where $\tilde{P}$ is the parabolic subgroup of $\mathrm{Mp}(W)$ with Levi subgroup $\tilde{\mathrm{GL}}_{n_1} \times_{\mu_2} ... \times_{\mu_2} \tilde{\mathrm{GL}}_{n_r} \times \mathrm{Mp}(W_0)$.

Given an irreducible representation π of $\mathrm{SO}(V)$, exactly one extension of π to $\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm 1\}$ has nonzero theta lift to $\mathrm{Mp}(W)$. Let π^ϵ denote the extension of π where -1 acts as ϵ. We have:

Theorem 4. *Let π be an irreducible representation of $\mathrm{SO}(V)$. Then π^ϵ participates in theta correspondence (with respect to ψ) with $\mathrm{Mp}(W)$ if and only if*

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi).$$

Here $\epsilon(s, \pi, \psi)$ is the standard epsilon factor defined by Lapid-Rallis [LR] using the doubling method; its value at $s = 1/2$ is independent of ψ, and $\epsilon(V)$ is the Witt invariant of V.

Finally, we investigate how the local Langlands correspondence $\mathcal{L}_\psi$ depends on ψ. For this, we shall of course assume the local Langlands correspondence for $\mathrm{SO}(V^{\pm})$ so that Corollary 2 makes sense. To state the result, we recall that $\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$ is a symplectic representation of WD_k, and if we write $\phi = \bigoplus_i n_i \cdot \phi_i$ as a direct sum of irreducible representations ϕ_i with some multiplicities n_i, then the component group A_ϕ is given by

$$A_\phi = \prod_{i:\phi_i \text{ symplectic}} \mathbb{Z}/2\mathbb{Z}a_i,$$

so that A_ϕ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with a canonical basis. Now we have:

Theorem 5. *For $\sigma \in Irr(\mathrm{Mp}(W))$ and $c \in k^\times$, let*

$$\mathcal{L}_\psi(\sigma) = (\phi, \eta) \text{ and } \mathcal{L}_{\psi_c}(\sigma) = (\phi_c, \eta_c).$$

Then:

(i) $\phi_c = \phi \otimes \chi_c$, where χ_c is the quadratic character associated to $c \in k^\times/k^{\times 2}$.

It follows by (i) that we have canonical identification of component groups:

$$A_\phi = A_{\phi_c} = \oplus_i \mathbb{Z}/2\mathbb{Z}a_i,$$

so that it makes sense to compare η and η_c.

(ii) the characters η and η_c are related by:

$$\eta_c(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i) \cdot \epsilon(1/2, \phi_i \otimes \chi_c) \cdot \chi_c(-1)^{\frac{1}{2}\dim \phi_i} \in \{\pm\}.$$

It is interesting to note that the proof of this last theorem makes use of the Gross-Prasad conjecture for tempered representations of special orthogonal groups, which is recently demonstrated by Waldspurger.

References

[A] J. Arthur, *Automorphic representations of classical groups*, in preparation.

[GGP] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups*, to appear in Asterisque Vol. 118.

[GT] W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp(4), to appear in Annals of Math.

[GTW] W. T. Gan and W. Tantono, *The local Langlands conjecture for* GSp(4) *II: the case of inner forms*, preprint.

[KR] S. Kudla and S. Rallis, *On first occurrence in the local theta correspondence*, in *Automorphic representations, L-functions and applications: progress and prospects*, 273–308, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[LR] E. Lapid and S. Rallis, *On the local factors of representations of classical groups*, in *Automorphic representations, L-functions and applications: progress and prospects*, 309–359, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[M1] G. Muić, *On the structure of theta lifts of discrete series for dual pairs* $(\mathrm{Sp}(n), \mathrm{O}(V))$, Israel J. Math. 164 (2008), 87–124

[M2] G. Muić, *Theta lifts of tempered representations for dual pairs* $(\mathrm{Sp}_{2n}, \varnothing(V))$, Canadian J. Math. 60 No. 6 (2008), 1306-1335.

[PSR] I. Piatetski-Shapiro and S. Rallis, *ϵ factor of representations of classical groups*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), no. 13, 4589–4593.

[W1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures Appl. (9) 59 (1980), no. 1, 1–132.
[W2] J.-L. Waldspurger, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spciaux orthogonaux*, preprint.

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