Mathematisches Forschungsinstitut Oberwolfach

Report No. 19/2011

DOI: 10.4171/OWR/2011/19

Geometric Methods of Complex Analysis

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April 10th – April 16th, 2011

Abstract. The purpose of this workshop was to discuss recent results in Several Complex Variables, Complex Geometry and Symplectic/Contact Geometry with a special focus on the interplay and exchange of ideas among these areas, which proved to be very fruitful in the last years. The main topics of the workshop included Symplectic and Contact Geometry, Almost Complex Geometry, Pluripotential Theory and the Monge-Ampère equation, Complex Dynamics, Geometric Questions of Complex Analysis (including Theory of Foliations) and Applications, the $\bar{\partial}$-equation and Geometry.

Mathematics Subject Classification (2000): 32xx, 53xx, 14xx, 37Fxx.

Introduction by the Organisers

The workshop *Geometric Methods of Complex Analysis* attracted 53 researchers from 16 countries. Both, leading experts in the field and young researchers (including five Ph. D. students) were well represented in the meeting and gave talks. Rather wide spectrum of topics related to Complex Analysis (and this was one of the aims of the workshop) was covered by the talks and unformal discussions. All 24 lectures presented on the meeting can be conditionally divided into the following groups.

Symplectic and Contact Geometry was represented by talks of H. Geiges and V. Shevchishin. Geiges explained Eliashberg's idea for proving Cerf's theorem which is based on the method of filling with holomorphic discs. Shevchishin gave a description of the diffeotopy group of a rational or ruled complex surface.

Almost Complex Geometry was represented by the talks of B. Saleur and A. Gournay (both being young researchers). Saleur described a generalization of the classical Borel's and Bloch's theorems to the case of a smooth almost complex structure on $\mathbb{P}^2$ which is tamed by the Fubini-Study form. Gournay presented a generalization of the Runge approximation theorem for the case of (pseudo-) holomorphic maps from a compact Riemann surface to a compact (almost-) complex manifold.

Pluripotential Theory and the Monge-Ampère equation is an important topic which was well represented on the meeting. Four lectures in this area were given by V. Guedj, S. Boucksom, L. Lempert and D. Coman. Guedj presented a solution of the analogue of the Calabi conjecture in a big cohomology class inspired by viscosity techniques. Boucksom explained a variational approach to complex Monge-Ampère equations which gives characterization of Kähler-Einstein metrics and has applications to the Kähler-Ricci flow. Lempert presented a result on nonexistence (in general) of geodesics connecting two given points in the space of Kähler metrics. This solves a long standing open problem connected to extremal metrics on Kähler manifolds. Coman described a result on extension of plurisubharmonic functions from analytic subvarieties with sharp growth control.

Complex Dynamics was represented by the talks of N. Sibony, H. Peters and E. Bedford. Sibony presented results on finiteness of entropy of a meromorphic map of a compact Kähler manifold and of foliations by Riemann surfaces. In the first case he has also provided a bound for entropy by the maximum of the logarithm of the dynamical degrees. Peters explained the role of limit varieties for a Fatou component for selfmaps in two complex variables. For holomorphic endomorphisms he gave a classification of the Fatou components under the assumption of uniqueness. Bedford discussed periodicities and positivity of entropy for linear fractional recurrences in 3-space.

Geometric Questions of Complex Analysis including Theory of Foliations and Applications were represented by the talks of S. Ivashkovich, H. Samuelsson, F. Kutzschebauch, D. Popovici, B. Jöricke, J. Globevnik, G. Bharali, G. Henkin, E. Rousseau and S. Orevkov. Ivashkovich presented (for any given integer $d \geq 1$) an example of a rational self-map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree d without holomorphic fixed points. He also described different topologies on the space of meromorphic maps. Samuelsson described a generalization of classical theorems by Čirka and Axler-Shields to the multidimensional case. Kutzschebauch gave a complete positive solution of Gromov's Vaserstein problem. Namely, he proved the existence of holomorphic factorization of null-homotopic holomorphic mappings from a reduced Stein space into $SL_n(\mathbb{C})$ in a product of upper and lower diagonal matrices. Popovici presented the new concept of "strongly Gauduchon manifold" and explained how using this concept one can prove a long-standing conjecture: if all the fibres, except of one, of a holomorphic family of compact complex manifolds are projective, then the remaining fibre is Moishezon. Jöricke gave a sharp lower bound of the 4-ball genus of an arbitrary analytic knot L contained in a small tubular neighborhood of a given smoothly analytic knot K in terms of the 4-ball genus of K and the "Umlaufszahl" of L with respect to K. Globevnik characterized pairs of points a, b in $\mathbb{C}^2$ having the property that if a function $f \in \mathcal{C}^\infty(b\mathbb{B})$, where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^2$, extends holomorphically inside $\mathbb{B}$ along each complex line passing either through a or through b, then f extends holomorphically to the whole of $\mathbb{B}$. Bharali explained how to achieve pseudoconvex bumping near a weakly pseudoconvex boundary point of some finite-type pseudoconvex domains in $\mathbb{C}^3$. Henkin presented recent results in the theory of complex Radon transforms and their applications. Rousseau described his work on holomorphic mappings $f : \mathbb{C}^p \to X$ of generic maximal rank into a projective manifold of dimension n, such that the image of f is tangent to a foliation $\mathcal{F}$ on X. He also discussed a generalized Green-Griffiths-Lang conjecture and presented several results of algebraic degeneracy in the strong sense. Orevkov explained how using the projective duality of plane projective complex curves one can give a complete solution to the problem of classification of systems of orthogonal polynomials in two variables.

The $\bar\partial$-equation and Geometry were represented by the talks of T. Ohsawa, J. Ruppenthal and M. Andersson. Ohsawa presented his results related to pseudoconvexity, the variation of the Bergman kernels and Levi flat manifolds which were based on refined L^2-theorems. Ruppenthal explained how L^2-Dolbeault cohomology groups $H^{0,q}_{(2)}(X - \mathrm{Sing}X)$ can be described by the cohomology of the sheaf of germs of meromorphic functions with poles according to a certain effective divisor on a resolution $\pi : N \to X$ of a singular space. Andersson discussed global division problems on algebraic varieties and presented generalizations to singular varieties of various results previously known for smooth varieties. He also gave an analytic proof of the Briançon-Skoda-Huneke theorem and combining the residue theory with integral formulas he obtained semiglobal Koppelman formulas for $\bar\partial$ on analytic spaces.

Workshop: Geometric Methods of Complex Analysis

Table of Contents

Abstracts

Viscosity solutions to degenerate complex Monge-Ampère equations

Vincent Guedj
(joint work with P. Eyssidieux and A. Zeriahi)

Pluripotential theory lies at the fundation of the approach to degenerate complex Monge-Ampère equations on compact Kähler manifolds as developed in [11], [9], [15], [5] and many others. This method is global in nature, since it relies on some delicate integrations by parts.

On the other hand, a standard PDE approach to second-order degenerate elliptic equations is the method of viscosity solutions, see [6] for a survey. This method is local in nature - and solves existence and unicity problems for weak solutions very efficiently. Our main goal in this article is to develop the viscosity approach for complex Monge-Ampère equations on compact complex manifolds.

Whereas the viscosity theory for real Monge-Ampère equations has been developed by P.L. Lions and others (see e.g.[13]), the complex case has not been studied until very recently. There is a viscosity approach to the Dirichlet problem for the complex Monge-Ampère equation on a smooth hyperconvex domain in a Stein manifold in [12]. This recent article does not however prove any new results for complex Monge-Ampère equations since this case serves there as a motivation to develop a deep generalization of plurisubharmonic functions to Riemannian manifolds with some special geometric structure (e.g. exceptional holonomy group). To the best of our knowledge, there is no reference on viscosity solutions to complex Monge-Ampère equations on compact Kähler manifolds.

There has been some recent interest in adapting viscosity methods to solve degenerate elliptic equations on compact or complete Riemannian manifolds [2]. This theory can be applied to complex Monge-Ampère equations only in very restricted cases since it requires the Riemann curvature tensor to be nonnegative. Using [14], a compact Kähler manifold with a non-negative Riemannian curvature tensor has an étale cover which is a product of a symmetric space of compact type (e.g.: $\mathbb{P}^n(\mathbb{C})$, Grassmannians) and a compact complex torus. In particular, [2] does not allow in general to construct a viscosity solution to the elliptic equation:

$$(DMA)_{\omega,v} \qquad (\omega + dd^c\varphi)^n = e^{\varphi} v$$

where ω is a smooth Kähler form and v a smooth volume on a general n-dimensional compact Kähler manifold X. A unique smooth solution has been however known to exist for more than thirty years thanks to the work of Aubin and Yau, [1] [16]. This is a strong indication that the viscosity method should work in this case to produce easily weak solutions

We confirm this guess, define and study viscosity solutions to degenerate complex Monge-Ampère equations. Our main technical result is:

Theorem A. *Let X be a compact complex manifold, ω a continuous closed real (1,1)-form with $\mathcal{C}^2$ local potentials and $v > 0$ be a volume form with continuous density. Then the viscosity comparison principle holds for $(DMA)_{\omega,v}$.*

The viscosity comparison principle differs substantially from the pluripotential comparison principle of [3] which is the main tool in [11], [10], [9]. This technical statement is based on the Alexandroff-Bakelmann-Pucci maximum principle. We need however to modify the argument in [6] by a localization technique.

Although we need to assume v is positive in Theorem A, it is then easy to let it degenerate to a non negative density in the process of constructing weak solutions to degenerate complex Monge-Ampère equations. We obtain this way the following:

Corollary B. *Assume X is as above, v is merely semi-positive with $\int_X v > 0$. If $\omega \geq 0$ and $\int_X \omega^n > 0$, then there is a unique viscosity solution $\varphi \in C^0(X)$ to $(DMA)_{\omega,v}$.*

If X is a compact complex manifold in the Fujiki class, it coïncides with the unique locally bounded ω-psh function φ on X such that $(\omega + dd^c\varphi)^n_{BT} = e^{\varphi}v$ in the pluripotential sense [9].

Recall that φ is ω-plurisubharmonic (ω-psh for short) if it is an u.s.c. integrable function such that $\omega + dd^c\varphi \geq 0$ in the weak sense of currents.

It was shown in this context by Bedford and Taylor [3] that when φ is bounded, there exists a unique positive Radon measure $(\omega + dd^c\varphi)^n_{BT}$ with the following property: if φ_j are smooth, locally ω-psh and decreasing to φ, then the smooth measures $(\omega + dd^c\varphi_j)^n$ weakly converge towards the measure $(\omega + dd^c\varphi)^n_{BT}$. If the measures $(\omega + dd^c\varphi_j)^n$ (locally) converge to $e^{\varphi}v$, we say that $(\omega + dd^c\varphi)^n_{BT} = e^{\varphi}v$ holds *in the pluripotential sense.*

Combining pluripotential and viscosity techniques, we can push our results further and obtain the following:

Theorem C. *Let X be a compact complex manifold in the Fujiki class. Let v is be a semi-positive probability measure with L^p-density, $p > 1$, and fix $\omega \geq 0$ a smooth closed real semipositive $(1,1)$-form such that $\int_X \omega^n = 1$. The unique locally bounded ω-psh function on X normalized by $\int_X \varphi = 0$ such that its Monge-Ampère measure satisfies $(\omega + dd^c\varphi)^n_{BT} = v$ is continuous.*

This continuity statement was obtained in [9] under a regularization statement for ω-psh functions that we were not able to obtain in full generality. It could have been obtained using [2] in the cases covered by this reference. However, for rational homogeneous spaces, the regularization statement is easily proved by convolution [7] and [2] does not give anything new. A proof of the continuity when X is projective under mild technical assumptions has been obtained in [8].

Let us stress some advantages of our method:

- it gives an alternative proof of Kolodziej's C^0-Yau theorem which does not depend on [16].

- it allows us to easily produce the unique negatively curved singular Kähler-Einstein metric in the canonical class of a projective manifold of general type, a result obtained first in [9] assuming [4], then in [5] by means of asymptotic Zariski decompositions.

We hope that the technique developed here will have further applications. In a forthcoming work it will be applied to the Kähler-Ricci flow.

References

[1] Aubin, T: *Equation de type Monge-Ampère sur les variétés kählériennes compactes.* Bull. Sci. Math. **102** (1978), 63–95.

[2] Azagra, D., Ferrara, D., Sanz, B. *Viscosity Solutions to second order partial differential equations on Riemannian manifolds.* J. Diff. Equations **245** (2008), 307–336.

[3] Bedford,E. Taylor,A. *A new capacity for psh functions.* Acta Math. **149** (1982), 1-40.

[4] Birkar, C., Cascini, P., Hacon, C. D., McKernan, J. *Existence of minimal models for varieties of log general type, I,II.*, J. Amer. Math. Soc. 23 (2010) 405–468, 469–490.

[5] Boucksom, S., Eyssidieux, P., Guedj, V., Zeriahi, A., *Monge Ampère equations in big cohomology classes.*, Acta Mathematica, to appear.

[6] Crandall, M., Ishii, H. , Lions, P.L. *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. **27** (1992), 1-67.

[7] Demailly, J.-P. *Regularization of closed positive currents and intersection theory.* J. Algebraic Geom. **1** (1992), no. 3, 361–409.

[8] Dinew, S., Zhang, Z. *Stability of Bounded Solutions for Degenerate Complex Monge-Ampère Equations.* Adv. Math. (2010), doi:10.1016/j.aim.2010.03.001.

[9] Eyssidieux, P., Guedj, V., Zeriahi, A. *Singular Kähler-Einstein metrics.* J. Amer. Math. Soc. **22** (2009), 607-639.

[10] Guedj, V., Zeriahi, A. *Intrinsic capacities on compact Kähler manifolds.* J. Geom. Anal. **15** (2005), no. 4, 607-639.

[11] Kolodziej, S. *The complex Monge-Ampère equation.* Acta Math. **180** (1998), 69-117.

[12] Harvey, F. R. Lawson, H. B. *Dirichlet Duality and the non linear Dirichlet problem on Riemannian manifolds.* Comm. Pure Appl. Math. **62** (2009), 396-443.

[13] Ishii, H. , Lions, P.L. *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations.* Journ. Diff. Equations **83** (1990), 26-78.

[14] Mok, Ngaiming *The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature.* J. Differential Geom. **27** (1988), 179–214.

[15] Tian, G., Zhang, Z. *On the Kähler-Ricci flow of projective manifolds of general type.* Chin. Ann. Math. **27** (2006), no. 2, 179-192.

[16] Yau, S.T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation.*, Comm. Pure Appl. Math. **31**, 339-441 (1978).

Cerf's theorem and other applications of the filling with holomorphic discs

Hansjörg Geiges
(joint work with K. Zehmisch)

The abelian group Γ_n of orientation preserving diffeomorphisms of the $(n-1)$-dimensional sphere S^{n-1} modulo those that extend to a diffeomorphism of the n-ball D^n plays an important role in differential topology. By the classical work of

Kervaire–Milnor on homotopy spheres and Smale's solution of the higher-dimensional Poincaré conjecture, Γ_n can be identified with the set of oriented smooth structures on the topological n-sphere for $n \geq 5$. The correspondence is given by associating with $[f] \in \Gamma_n$ the smooth structure on S^n obtained by using the diffeomorphism f of S^{n-1} to glue two copies of D^n along their boundary.

It is easy to see that Γ_1 and Γ_2 are trivial. The result $\Gamma_3 = 0$ is due independently to Munkres and Smale. For $n \geq 5$, the groups Γ_n are amenable to computation by the results of Kervaire–Milnor, for instance $\Gamma_5 = \Gamma_6 = 0$, and Γ_7 is the cyclic group of order 28.

The statement $\Gamma_4 = 0$ is known as *Cerf's theorem* [1]. One consequence of this result is that there are no exotic smooth structures on S^4 that can be obtained by gluing two 4-discs.

In [3] Eliashberg proposed an ingenious proof of Cerf's theorem based on his classification of contact structures on S^3 and his method of filling with holomorphic discs [2].

In this talk I present Eliashberg's idea for proving Cerf's theorem and our alternative approach [4] to the filling with holomorphic discs in a moduli-theoretic framework. Eliashberg's key observation is that it suffices to prove the extension result for contactomorphisms of the standard contact structure on S^3. In our setting, this extension is given 'explicitly' by an evaluation map on a suitable moduli space of holomorphic discs with totally real boundary conditions.

In current work [5] we apply the method of filling with holomorphic discs to a 4-dimensional symplectic cobordism with the standard contact 3-sphere as a convex boundary component. The corresponding moduli space of holomorphic discs is either compact, in which case the symplectic cobordism has to be the 4-ball, or there is non-compactness caused by bubbling-off of holomorphic discs or breaking, in which case there have to be periodic Reeb orbits in the concave boundary of the symplectic cobordism.

As corollaries we can derive a number of classical results in 4-dimensional symplectic resp. 3-dimensional contact topology. These include various instances of the Weinstein conjecture and, via the definition of a new symplectic capacity, Gromov's symplectic non-squeezing.

References

[1] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Math. **53**, Springer-Verlag, Berlin (1968).

[2] Ya. Eliashberg, *Filling by holomorphic discs and its applications*, in: Geometry of Low-Dimensional Manifolds, Vol. 2 (Durham, 1989), London Math. Soc. Lecture Note Ser. **151**, Cambridge University Press (1990), 45–67.

[3] Ya. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), 165–192.

[4] H. Geiges and K. Zehmisch, *Eliashberg's proof of Cerf's theorem*, J. Topol. Anal. **2** (2010), 543–579.

[5] H. Geiges and K. Zehmisch, *How to recognise a 4-ball when you see one*, preprint (2011), `arXiv: 1104.1543`.

Entropy for hyperbolic Riemann surface laminations

Nessim Sibony
(joint work with T.-C. Dinh, V.-A. Nguyên)

Consider the polynomial differential equation in $\mathbb{C}^2$

$$\frac{dz}{dt} = P(z,w), \qquad \frac{dw}{dt} = Q(z,w).$$

The polynomials P and Q are holomorphic, the time is complex. We want to study the global behavior of the solutions. It is convenient to consider the extension as a foliation in the projective plane P^2. Our main goal is to introduce a notion of entropy for possibly singular compact hyperbolic foliations by Riemann surfaces in complex manifolds. We also study the transverse regularity of the Poincaré metric and the finiteness of the entropy.

The multidimensional example to have in mind is the case of a polynomial vector field in $\mathbb{C}^k$. It induces, as above, a foliation by Riemann surfaces in the complex projective space $\mathbb{P}^k$. We can consider that this foliation is the image of the foliation in $\mathbb{C}^{k+1}$ given by the vector field

$$F(z) := \sum_{j=0}^{k} F_j(z)\frac{\partial}{\partial z_j}$$

with F_j homogeneous polynomials of degree $d \geq 2$. The singular set corresponds to the union of the indeterminacy points of $F = [F_0 : \cdots : F_k]$ and the fixed points of F in $\mathbb{P}^k$. The nature of the leaves as abstract Riemann surfaces has received much attention. Glutsyuk [12] and Lins Neto [18] have shown that on a generic foliation $\mathscr{F}$ of degree d the leaves are covered by the unit disc. We then say that the foliation is hyperbolic. More precisely, Lins Neto has shown that this is the case when all singular points $\mathrm{sing}(\mathscr{F})$ have non degenerate linear part. He constructs on $\mathbb{P}^k \setminus \mathrm{sing}(\mathscr{F})$ a metric which has strictly negative curvature along leaves. In [6] Candel-Gomez-Mont have shown that if all the singularities are hyperbolic, the Poincaré metric on leaves is transversally continuous.

Some dynamical results on such foliations are obtained in [10, 11, 8]. In particular, in [8] we obtain a geometric Birkhoff type theorem for harmonic measures.

Concerning the Poincaré metric we give two results about the transverse regularity. In the first one we show that if $(X, \mathscr{L})$ is a $\mathscr{C}^{2+\delta}$-Riemann surface lamination without singularities in a complex manifold M, then the Poincaré metric is transversally Hölder continuous. The result holds also for abstract compact $\mathscr{C}^{2+\delta}$-hyperbolic Riemann surface laminations. We give an estimate of the exponent of Hölder continuity in geometric terms. When we consider a foliation with linearizable singularities we give a precise estimate of the modulus of continuity of the Poincaré metric in the transverse directions. The main tool is to use Beltrami equation in order to construct the universal covering of any leaf L_b near a given leaf L_a. We first construct a non-holomorphic parametrization Φ from L_a to L_b with geometric estimates and then modify Φ, using Laplace-Beltrami, to obtain a holomorphic map that we can explicitly compare with a universal covering map $\tau_b : \mathbb{D} \to L_b$.

Our second concern is to define the entropy of hyperbolic foliation possibly with singularities. A notion of geometric entropy for regular Riemannian foliations was introduced by Ghys-Langevin-Walczak [13], see also Candel-Conlon [4, 5] and Walczak [19]. It is related to the entropy of the holonomy pseudogroup, which depends on the chosen generators. The basic idea here is to quantify how much leaves get far apart transversally. The transversal regularity of the metric on leaves and the lack of singularities play a role in the finiteness of the entropy. Ghys-Langevin-Walczak show in particular that when the geometric entropy vanishes, the foliation admits a transverse measure. The survey by Hurder [15] gives an account on many important results in foliation theory and contains a large bibliography.

Our notion of entropy joins a universal concept which contains a large number of classical situations. An interesting fact is that this entropy is related to an increasing family of distances as in Bowen's point of view. This allows us for example to introduce other dynamical notions like local entropies or Lyapounov exponents.

We first introduce a general notion of entropy on a metric space (X, d). To a given family of distances $(\mathrm{dist}_t)_{t\geq 0}$ is associated an entropy which measures the growth rate of the number of balls of small radius ϵ in the metric dist_t needed in order to cover the space X when t tends to infinity. For hyperbolic Riemann surface foliations we define

$$\mathrm{dist}_t(a, b) := \inf_{\theta\in\mathbb{R}} \sup_{\zeta\in\mathbb{D}_t} \mathrm{dist}_X(\tau_a(e^{i\theta}\zeta), \tau_b(\zeta)).$$

Here, τ_a, τ_b are universal covering maps for the leaves through a and b respectively (i.e. $\tau_a(0) = a$, $\tau_b(0) = b$), and $\mathbb{D}_t$ is, as usual, the disc of center 0 and of radius t with respect to the Poincaré metric on $\mathbb{D}$. The metric dist_t measures how far two leaves get apart before the hyperbolic time t. It takes into account the time parametrization. So, we are not just concerned with geometric proximity. We introduce a general notion of geometric and metric entropy, which permits to describe some natural situations in dynamics and in foliation theory.

Let X be a metric space and dist_X a distance on X. Consider a family $\mathscr{D} = \{\mathrm{dist}_t\}$ of distances on X indexed by $t \in \mathbb{R}^+$ such that $\mathrm{dist}_0 = \mathrm{dist}_X$ and dist_t is increasing with respect to $t > 0$. Let Y be a non-empty subset of X. Denote by $N(Y, t, \epsilon)$ the minimal number of balls of radius ϵ with respect to the distance dist_t needed to cover Y. Define the *entropy* of Y with respect to $\mathscr{D}$ by

$$h_{\mathscr{D}}(Y) := \sup_{\epsilon>0} \limsup_{t\to\infty} \frac{1}{t} \log N(Y, t, \epsilon).$$

Observe that $N(Y, t, \epsilon)$ is increasing with respect to $t > 0$, and that

$$\limsup_{t\to\infty} \frac{1}{t} \log N(Y, t, \epsilon)$$

is increasing when ϵ decreases.

We will show that our entropy is finite for compact hyperbolic lamination which are transversally of class $\mathscr{C}^{2+\delta}$ for some $\delta > 0$ and also for compact foliations with hyperbolic singularities in complex surfaces. The notion of entropy can be extended to Riemannian foliation and a priori it is bigger than or equal to the geometric entropy introduced by Ghys, Langevin and Walczak.

As for the tranverse regularity of the Poincaré metric, the main tool is to estimate the distance between leaves using the Beltrami equation in order to go from geometric estimates to the analytic ones needed in our definition. The advantage here is that the hyperbolic time we choose is canonical. So, the value of the entropy is unchanged under homeomorphisms between laminations which are holomorphic along leaves.

The proof that the entropy is finite for singular foliations is quite delicate and requires a careful analysis of the dynamics around the singularities. If we consider a neighborhood of a singular point, there are infinitely many leaves which are geometrically ϵ-apart. But if we use hyperbolic time, they do not get ϵ-apart at a bounded hyperbolic time. The number of Bowen balls (t-balls) of radius ϵ, close to the separatrices is finite and it has however to be estimated carefully. Indeed, we need that the orbits of two points in the same ball stay ϵ-close until time R. Near a singularity this requires the ball to be of diameter, with respect to the ambient metric, smaller than ϵ^{e^R}. The shape of balls changes close to separatrices and away from separatrices; it also changes as R increases.

To overcome this difficulty, at each time R we construct a partition of the space in a controllable number of cells such that given two points in the same cell we can define a parametrization from one point to another. The technical point is the concept of closedness to a family of points. The interesting point here is that the shape of such a cell reflects how often the orbits of its points travel near the singularity set of the foliation. Given $0 < \epsilon < 1$ and $R \in \mathbb{N}$, we construct a special system of flow boxes of diameter smaller than ϵ. They have the property that two points in the same flow box are (R, ϵ)-close. To such a flow box we associate a special transversal. Then, we introduce a system of translations on leaves and by induction on R we construct the subdivision valid until time $R + 1$ The problem is the counting of the number of flow boxes needed. We have to take into account the geometry near a singularity, its influence when the time R increases and also the fact that the flow boxes have different shape according to their situation with respect to the separatrices. We get the final estimate using Laplace-Beltrami equation. So far, we have been able to do this for singular foliations only in dimension 2.

References

[1] Bowen, Rufus. Topological entropy for noncompact sets. *Trans. Amer. Math. Soc.* **184** (1973), 125–136.

[2] Brin, M.; Katok, A., On local entropy. Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[3] Candel Alberto. Uniformization of surface laminations, *Ann. Sci. École Norm. Sup.* (4) **26** (1993), no. 4, 489–516.

[4] Candel, Alberto; Conlon, Lawrence. *Foliations. I.* Graduate Studies in Mathematics, **23**. American Mathematical Society, Providence, RI, 2000. xiv+402 pp.
[5] Candel, Alberto; Conlon, Lawrence. *Foliations. II.* Graduate Studies in Mathematics, **60**. American Mathematical Society, Providence, RI, 2003. xiv+545 pp.
[6] Candel A. and Gómez-Mont X., Uniformization of the leaves of a rational vector field, *Ann. Inst. Fourier (Grenoble)*, **45** (1995), no. 4, 1123-1133.
[7] Dinh T.-C., Nguyen V.-A. and Sibony N., Dynamics of horizontal-like maps in higher dimension. *Adv. Math.* **219** (2008), no. 5, 1689–1721.
[8] Dinh T.-C., Nguyen V.-A. and Sibony N., Heat equation and ergodic theorems for Riemann surface laminations, math.CV (math.DS) arXiv:1004.3931, 47 pages.
[9] Dinh, T.-C. and Sibony N., Upper bound for the topological entropy of a meromorphic correspondence, *Israel J. Math.* **163** (2008), 29–44.
[10] Fornæss, John Erik; Sibony, Nessim, Harmonic currents of finite energy and laminations, *Geom. Funct. Anal.* **15** (2005), no. 5, 962–1003.
[11] Fornæss, John Erik; Sibony, Nessim, Riemann surface laminations with singularities, *J. Geom. Anal.* **18** (2008), no. 2, 400–442.
[12] Glutsyuk A. A., Hyperbolicity of the leaves of a generic one-dimensional holomorphic foliation on a nonsingular projective algebraic variety. (Russian) *Tr. Mat. Inst. Steklova* **213** (1997), Differ. Uravn. s Veshchestv. i Kompleks. Vrem., 90–111; translation in *Proc. Steklov Inst. Math.* 1996, no. 2 **213**, 83–103.
[13] Ghys, Étienne; Langevin, Rémi; Walczak, Pawel. Entropie géométrique des feuilletages, *Acta Math.* **160** (1988), no. 1-2, 105–142.
[14] Gromov M., On the entropy of holomorphic maps, *Enseignement Math.*, **49** (2003), 217-235. *Manuscript* (1977).
[15] Hurder, Steven. Classifying foliations. *Foliations, geometry, and topology,* 1–65, Contemp. Math., **498**, *Amer. Math. Soc., Providence, RI,* 2009
[16] Kaimanovich V., Brownian motions on foliations: entropy, invariant measures, mixing, *Funct. Anal. Appl.*, **22** (1989) 326-328.
[17] Katok, Anatole; Hasselblatt, Boris, *Introduction to the modern theory of dynamical systems.* With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, **54**. Cambridge University Press, Cambridge, 1995. xviii+802 pp.
[18] Lins Neto A., Uniformization and the Poincaré metric on the leaves of a foliation by curves, *Bol. Soc. Brasil. Mat. (N.S.),* **31** (2000), no. 3, 351–366.
[19] Walczak, Pawel. *Dynamics of foliations, groups and pseudogroups.* Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), **64**. *Birkhäuser Verlag, Basel,* 2004. xii+225 pp.

On fixed points and convergence of meromorphic mappings

Sergey Ivashkovich

1. Fixed points of meromorphic self-maps

Let U and X be complex manifolds, X will be supposed to be compact. A meromorphic map $f : U \to X$ can be viewed as a holomorphic map $f : U \setminus A \to X$, where A is an analytic subset of U of codimension ≥ 2, such that the graph of f extends to an analytic subvariety of the product $U \times X$. This extension will be denoted as Γ_f and called the graph of the meromorphic mapping f. The smallest A such that f is holomorphic on $U \setminus A$ is called the indeterminacy set of f and is denoted as I_f.

Let $f\colon X \to X$ be a meromorphic self-map of a compact complex manifold. The topological degree of f is the number of preimages of a generic point. A meromorphic fixed point of f is a point $p \in X$ such that $p \in f[p]$. Here by $f[p]$ one means the full image of p by f: $f[p] := \Gamma_f \cap (\{p\} \times X)$. If $X = \mathbb{P}^N$ then by obvious homological reasons Γ_f intersects the diagonal D in $\mathbb{P}^N \times \mathbb{P}^N$. Therefore meromorphic fixed points for any $f\colon \mathbb{P}^N \to \mathbb{P}^N$ always exist.

A point $p \in X$ is said to be a holomorphic fixed point of f, if f is holomorphic in a neighborhood of p and $f(p) = p$. Our first goal in this talk is to explain the following:

Theorem 1.1. *For any given integer $d \geq 1$ there exist rational self-maps $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ of degree d without holomorphic fixed points.*

One of the reasons for the interest in fixed points of meromorphic maps lies in the attempt to understand what should be an analog of a Lefschetz Fixed Point Formula in meromorphic case, see [1]. Another one comes from higher dimensional meromorphic dynamics.

Note. I heard of the question whether any rational self-map of $\mathbb{P}^2$ has a holomorphic fixed point for the first time in the talk of J.-E. Fornaess on the “Colloque en l'honneur de P. Dolbeault” in Paris, June 1992.

2. Topologies on the space of meromorphic maps

One might ask what is the structure of the set $FFix(X)$ of holomorphic fixed point free meromorphic self-maps of X? An attempt to answer this question confronts with the observation that there are several natural topologies on the space $\mathcal{M}(U, X)$ of meromorphic maps between complex manifolds. The first is the strong one: f_n strongly converge to f (s-converge) if their graphs Γ_{f_n} converge to the graph Γ_f of f in Hausdorff metric (or, in the cycle topology, which is equivalent in this case). Denote by $DFix(X)$ the subset of the space $\mathcal{M}(X, X)$ of meromorphic self-maps of X which consists of the maps with a curve of holomorphic fixed points (i.e., in some sense they are degenerate maps). $DFix(X)$ is a closed subset of $\mathcal{M}(X, X)_s$. One can prove the following:

Theorem 2.1. *The set $FFix(X) \cup DFix(X)$ is closed in $\mathcal{M}(X, X)$ in strong topology. I.e., a sequence of holomorphic fixed point free meromorphic mapping converge either to a holomorphic fixed point free meromorphic map, or to a map with a curve of fixed points.*

Notice that by the Theorem 1.1 the set $FFix(\mathbb{P}^2) \cup DFix(\mathbb{P}^2)$ is a proper subset of $\mathcal{M}(\mathbb{P}^2, \mathbb{P}^2)$. Strong topology has several nice features. A part of Theorem 2.1 one can prove that for compact U and X the space $\mathcal{M}(U, X)_s$ is a finite dimensional analytic space in a neighborhood of each of its points. Also s-convergence is well related with the usual notion of convergence of holomorphic mappings:

Theorem 2.2 (Rouché principle)**.** *Let a sequence of meromorphic mappings $\{f_n\}$ between complex manifolds U and X strongly converge to a meromorphic map f. Then:*

(a) If f is holomorphic then for any relatively compact open subset $U_1 \subset U$ all restrictions $f_n|_{U_1}$ are holomorphic for n big enough, and $f_n \to f$ on compacts in U.
(b) If f_n are holomorphic then f is also holomorphic and $f_n \to f$ in the usual sense.

The bad thing about the strong topology is that the domains of strong convergence can be arbitrary, in particular non-pseudoconvex. This can be corrected in the following way. We say that f_n converge weakly to f (w-converge) if there exists an analytic subset A in X of codimension at least two such that f_n converge strongly to f on $U \setminus A$.

Remark 2.3. *From the Rouché Principle it follows that f_n converge weakly to f if and only if for every compact of $U \setminus I_f$ all f_n are holomorphic in a neighborhood of this compact for n big enough and converge there uniformly to f as holomorphic mappings.*

Domains of weak convergence of meromorphic mappings turn to be pseudoconvex for a large class of target manifolds. This follows from the following "propagation principle". Suppose that the compact complex manifold X possesses a dd^c-closed (or Gauduchon) metric form (ex. X is Kähler, or any compact complex surface).

Theorem 2.4. *Let U be a domain in a Stein manifold and let $f_n : U \to X$ be a weakly converging to $f : U \to X$ sequence of meromorphic mappings.*
(a) If all f_n meromorphically extend to the envelope of holomorphy $\wedge U$ of U then f extends to $\wedge U$ to and these extensions weakly converge on $\wedge U$ to the extension of f;
(b) And vice versa, if the weak limit f on f_n meromorphically extends to $\wedge U$ then all f_n extend to $\wedge U$ and weakly converge there.

Using certain area-volume estimates for meromorphic graphs one can prove that the graphs of a weakly converging sequence of meromorphic maps with values in Gauduchon manifold have uniformly bounded volume (over compacts in U). That means that the limit $\Gamma := \lim \Gamma_{f_n}$ naturally decomposes as

$$\Gamma = \Gamma_f \cup \bigcup_j \Gamma_j \tag{1}$$

where Γ_f is a graph of some (uniquely defined) meromorphic mapping and each Γ_j is an analytic set in $U \times X$ of pure dimension $\dim U$ which properly projects to an analytic subset γ_j of U of dimension $0 \le j \le \dim U - 2$. We call Γ_j the exceptional components of the limit.

This leads to one more natural notion of convergence: we say that f_n Γ-converge if the sequence of graphs Γ_{f_n} converge in the topology of cycles. Then we have the same decomposition as in (1), only the dimension of γ_j can reach $\dim U - 1$, i.e., the bubbling can occur over a divisor. Moreover one can determine the structure of the exceptional components of the limit:

Theorem 2.5. *Let $f_n\colon U \to X$ converge to $f\colon U \to X$ in Γ-topology, then for every $a \in \gamma := \bigcup_{0\le j\le \dim U-1}\gamma_j$ the fiber $pr|_{\Gamma}^{-1}(a)$ is rationally connected.*

3. The case of projective manifolds

It is instructive to understand the various notions of convergence of meromorphic mappings on the example $X = \mathbb{P}^N$. Every meromorphic mapping f with values in complex projective space can be locally represented as

$$f(z) = [f^0(z) : \dots : f^N(z)] \tag{2}$$

with holomorphic $f^0, \dots f^N$. This representation we call reduced if f^j have no common divisors. In that case

$$I_f = \{f^0(z) = \dots = f^N(z) = 0\}. \tag{3}$$

Proposition 3.1. *(a) A sequence $\{f_n\}$ of meromorphic mappings from a complex manifold D to $\mathbb{P}^N$ converge weakly to a meromorphic map f if and only if for any point $x_0 \in D$ there exists a neighborhood $U \ni x_0$ and reduced representations $f_n = [f_n^0 : \dots : f_n^N], f = [f^0 : \dots : f^N]$ such that for every $0 \le j \le N$ the sequence f_n^j converge to f^j uniformly on U.*
(b) $\{f_n\}$ converge in Γ-topology if the limit $f = [f^0 : \dots : f^N]$ is not necessarily reduced.

The convergence as in (b) for the case of $\mathbb{P}^N$ was studied by H. Fujimoto, see [2].

4. Fatou sets

The case of special interest is when the family $\mathcal{F}$ is the family of iterates $f_n := f \circ \dots \circ f$ of some fixed meromorphic self-map of a compact complex manifold X. The maximal open subset of X where $\{f_n\}$ is relatively compact is called the Fatou set of f. Depending on the sense of convergence that one wishes to consider one gets several different Fatou sets: strong, weak or gamma Fatou sets. We denote them as Φ_s, Φ_w and Φ_Γ respectively.

Theorem 4.1. *Let f be a meromorphic self map of a compact complex surface. Then the weak Fatou set Φ_w of f is pseudoconvex. If Φ_s is different from Φ_w then:*
a) X is bimeromorphic to $\mathbb{P}^2$,
b) $\Phi_w = \mathbb{P}^2 \setminus C$, where C is a chain of rational curves;
c) the weak limit of any weakly converging subsequence $\{f_{n_k}\}$ of iterates is degenerate.

It should be pointed out that our Fatou sets are different from the Fatou sets as they are usually understood in meromorphic dynamics, see ex. [2]. There the Fatou set of f is the maximal open subset Φ of $X \setminus \overline{\bigcup_{n=0^\infty} f^{-n}(I_f)}$ where the family f^n is equicontinuous. If, for example, $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ is the Cremona transformation $[z_0 : z_1 : z_2] \mapsto [z_1z_2 : z_0z_2 : z_0z_1]$ then $\Phi_s = \Phi_w = \Phi_\Gamma = \mathbb{P}^2$ but $\Phi = \mathbb{P}^2 \setminus \{\text{three lines}\}$.

References

[1] E. Bedford, *Dynamics of Rational Surface Automorphisms*, Lecture Notes.

[2] J.-E. Fornaess, N. Sibony, *Complex dynamics in higher dimension. II.*, Modern methods in complex analysis (Princeton, NJ, 1992), 135182, Ann. of Math. Stud., **137** (1995), Princeton Univ. Press, Princeton, NJ.

[3] H. Fujimoto., *On families of meromorfphic maps into the complex projective space* Nagoya Math. J. **54** (1974), 21–51.

[4] S. Ivashkovich, *On convergence properties of meromorphic functions and mappings*, B. Shabat Memorial Volume, Moscow, FASIS (2001), 145–163 (English version in arXiv: math/9804007).

[5] S. Ivashkovich, *On fixed points of rational self-maps of complex projective plane*, to appear in Indiana Univ. Math. J., math.CV/0911.5084v2.

Uniform algebras and approximation on manifolds

Hakan Samuelsson

(joint work with E. F. Wold)

Wermer's classical maximality theorem, [6], can be stated as follows: Let $f \in C^0(\partial\mathbb{D})$ be a continuous function on the boundary of the unit disc $\mathbb{D} \subset \mathbb{C}$. Then either f is the boundary value of a holomorphic function or the uniform algebra $[z, f]_{\partial\mathbb{D}}$ generated by z and f on $\partial\mathbb{D}$ equals $C^0(\partial\mathbb{D})$. Closely related is the following result by Čirka, [1]: Let $h \in C^0(\overline{\mathbb{D}})$ be harmonic but non-holomorphic on $\mathbb{D}$. Then the uniform algebra $[z, h]_{\overline{\mathbb{D}}}$ generated by z and h on $\overline{\mathbb{D}}$ equals $C^0(\overline{\mathbb{D}})$.

I will discuss generalizations of these results to several complex variables; this is a joint work, [5], with Erlend Fornæss Wold. Let $\Omega \subset \mathbb{C}^n$ be a domain such that $\overline{\Omega}$ is polynomially convex and let $h_1, \ldots, h_N \in C^0(\overline{\Omega})$ be pluriharmonic in Ω. If there is a holomorphic disc $\Delta \subset \Omega$ such that $h_j|_\Delta$ are holomorphic for all j, then clearly any function $\varphi \in C^0(\overline{\Omega})$ such that $\varphi|_\Delta$ is not holomorphic cannot be uniformly approximated on $\overline{\Omega}$ by polynomials in $z_1, \ldots, z_n$ and $h_1, \ldots, h_N$. On the other hand, our first result says that this is essentially the only obstruction. This generalizes results by Izzo, [2], [3], and is in the same spirit as the main result in [4].

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^1-smooth boundary and such that $\overline{\Omega}$ is polynomially convex. Let also $h_1, \ldots, h_N \in C^0(\overline{\Omega})$ be pluriharmonic in Ω and assume that there is no holomorphic disc in Ω where all h_j are holomorphic. Then*

$$[z_1, \ldots, z_n, h_1, \ldots, h_N]_{\overline{\Omega}} = C^0(\overline{\Omega}) \cap [z_1, \ldots, z_n, h_1, \ldots, h_N]_{\partial\Omega}.$$

In the case that the domain is the bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$ we have more complete results.

Theorem 2. *Let $h_1, \ldots, h_N \in C^1(\overline{\mathbb{D}}^2)$ be pluriharmonic in $\mathbb{D}^2$. Then either there is a holomorphic disc in $\overline{\mathbb{D}}^2$ where all h_j are holomorphic or $[z_1, z_2, h_1, \ldots, h_N]_{\overline{\mathbb{D}}^2} = C^0(\overline{\mathbb{D}}^2)$.*

Theorem 3. *Let $f_1, \ldots, f_N$ be continuous functions on the distinguished boundary $\mathbb{T}^2 \subset \partial\mathbb{D}^2$ and assume that the f_j have pluriharmonic extensions h_j to $\mathbb{D}^2$. Then* *either* $[z_1, z_2, f_1, \dots, f_N]_{\mathbb{T}^2} = C^0(\mathbb{T}^2)$ *or there is an algebraic subvariety* $Z \subset \mathbb{C}^2$ *such that* $Z \cap \partial\mathbb{D}^2 \subset \mathbb{T}^2$, $Z \cap \mathbb{D}^2$ *is non-trivial, and all the* h_j *are holomorphic along* $Z \cap \mathbb{D}^2$.

The last result generalizes Wermer's maximality theorem in the sense that analyticity is the only obstruction to the full algebra being generated.

A simple but useful observation when proving our results is the following: Let $\mathcal{G}_h = \{(z, h(z)); z \in \overline{\Omega})\} \subset \mathbb{C}^{n+N}$ be the graph of $h = (h_1, \dots, h_N)$ over $\overline{\Omega}$ and let $\pi \colon \mathcal{G}_h \to \overline{\Omega}$ be the projection. Then a function φ is in $[z_1, \dots, z_n, h_1, \dots, h_N]_{\overline{\Omega}}$ if and only if $\pi^*\varphi$ can be uniformly approximated on $\mathcal{G}_h$ by polynomials in $\mathbb{C}^{n+N}$. Our results are thus reduced to polynomial approximation on graphs. Since in our cases h is a pluriharmonic mapping it is possible to stratify $\mathcal{G}_h$ in a certain way so that each strata essentially is totally real. The technical part of our proofs is then contained in the following result. It relies on approximation results on totally real manifolds proved by, e.g., Henkin-Leiterer, Čirka, Berndtsson, and Manne-Øvrelid-Wold.

Theorem 4. *Let* $X \subset \mathbb{C}^n$ *be compact and polynomially convex. If there are closed sets* $X_0 \subset \cdots \subset X_m = X$ *such that* $X_j \setminus X_{j-1}$, $j = 1, \dots, m$, *are totally real then any* $\varphi \in C^0(X) \cap \mathcal{O}(X_0)$ *can be approximated uniformly on* X *by polynomials in* $\mathbb{C}^n$.

The proof of Theorem 3 is a bit more involved. One observes that since $C^0(\mathbb{T}^2) = \mathcal{O}(\mathbb{T}^2)$ one has $C^0(\mathcal{G}_f(\mathbb{T}^2)) = \mathcal{O}(\mathcal{G}_f(\mathbb{T}^2))$, where $\mathcal{G}_f(\mathbb{T}^2)$ is the graph of $f = (f_1, \dots, f_N)$ over $\mathbb{T}^2$. Hence, one has polynomial approximation on $\mathcal{G}_f(\mathbb{T}^2)$ if and only if $\mathcal{G}_f(\mathbb{T}^2)$ is polynomially convex. Now, if there is a non-trivial analytic variety $Z' \subset \mathbb{D}^2$ attached to $\mathbb{T}^2$ and all the pluriharmonic extensions h_j are holomorphic along Z' then clearly the polynomial hull of $\mathcal{G}_f(\mathbb{T}^2)$ must contain the graph of h over $\mathbb{T}^2 \cup Z'$. Conversely we show, by using Rossi's local maximum principle as well as Wermer's and Čirka's results, that if $\mathcal{G}_f(\mathbb{T}^2)$ is not polynomially convex then there is such a variety $Z' \subset \mathbb{D}^2$. Theorem 4 then follows from a result by Tornehave implying that such a variety in fact extends to an algebraic variety in $\mathbb{C}^2$.

References

[1] E. M. Čirka: *Approximation by holomorphic functions on smooth manifolds in* $\mathbb{C}^n$. Mat. Sb. (N.S.) **78** (1969), 101–123.

[2] A. Izzo: *Uniform approximation by holomorphic and harmonic functions.* J. London Math. Soc. **47** (1993), 129–141.

[3] A. Izzo: *Uniform algebras generated by holomorphic and pluriharmonic functions.* Trans. Amer. Math. Soc. **339** (1993), no. 2, 835–847.

[4] A. Izzo: *Uniform approximation on Manifolds.* Annals of Math., to appear.

[5] H. Samuelsson, E. F. Wold: *Uniform algebras and approximation on manifolds.* Preprint Oslo, 2011. Available at arXiv:1101.4840 [math.CV].

[6] J. Wermer: *On algebras of continuous functions.* Proc. Amer. Math. Soc. **4** (1953), 866–869.

Holomorphic factorization of mappings into $\mathbf{SL}_n(\mathbb{C})$

Frank Kutzschebauch
(joint work with B. Ivarsson)

1. Introduction

It is standard material in a Linear Algebra course that the group $\mathrm{SL}_m(\mathbb{C})$ is generated by elementary matrices $E + \alpha e_{ij},\ i \neq j$, i.e., matrices with 1's on the diagonal and all entries outside the diagonal are zero, except one entry. Equivalently, every matrix $A \in \mathrm{SL}_m(\mathbb{C})$ can be written as a finite product of upper and lower diagonal unipotent matrices (in interchanging order). The same question for matrices in $\mathrm{SL}_m(R)$ where R is a commutative ring instead of the field $\mathbb{C}$ is much more delicate. For example, if R is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a space X the problem amounts to finding for a given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ a factorization as a product of upper and lower diagonal unipotent matrices

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

where the G_i are maps $G_i : X \to \mathbb{C}^{m(m-1)/2}$.

Since any product of (upper and lower diagonal) unipotent matrices is homotopic to a constant map (multiplying each entry outside the diagonals by $t \in [0,1]$ we get a homotopy to the identity matrix), one has to assume that the given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ is homotopic to a constant map or as we will say *null-homotopic.* In particular this assumption holds if the space X is contractible.

This very general problem has been studied in the case of polynomials of n variables. For $n = 1$, i.e., $f : \mathbb{C} \to \mathrm{SL}_m(\mathbb{C})$ a polynomial map (the ring R equals $\mathbb{C}[z]$) it is an easy consequence of the fact that $\mathbb{C}[z]$ is an Euclidean ring that such f factors through a product of upper and lower diagonal unipotent matrices. For $m = n = 2$ the following counterexample was found by Cohn [1]: the matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices.

For $m \geq 3$ (and any n) it is a deep result of Suslin [22] that any matrix in $\mathrm{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices. More results in the algebraic setting can be found in [22] and [15]. For a connection to the Jacobian problem on $\mathbb{C}^2$ see [26].

In the case of continuous complex valued functions on a topological space X the problem was studied and partially solved by Thurston and Vaserstein [23] and then finally solved by Vaserstein [25].

It is natural to consider the problem for rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$. Explicitly this problem was posed by Gromov in his groundbreaking paper [14] where he extends the classical Oka-Grauert theorem from bundles with homogeneous fibers to fibrations with elliptic fibers, e.g., fibrations admitting a dominating spray. In spite of the above mentioned result of Vaserstein he calls it the

Vaserstein problem: (see [14, sec 3.5.G])

Does every holomorphic map $\mathbb{C}^n \to SL_m(\mathbb{C})$ decompose into a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $SL_m(\mathbb{C})$?

Gromov's interest in this question comes from the question about s-homotopies (s for spray). In this particular example the spray on $\mathrm{SL}_m(\mathbb{C})$ is that coming from the multiplication with unipotent matrices. Of course one cannot use the upper and lower diagonal unipotent matrices only to get a spray (there is no submersivity at the zero section!), there need to be at least one more unipotent subgroup to be used in the multiplication. Therefore the factorization in a product of upper and lower diagonal matrices seems to be a stronger condition than to find a map into the iterated spray, but since all maximal unipotent subgroups in $\mathrm{SL}_m(\mathbb{C})$ are conjugated and the upper and lower diagonal matrices generate $\mathrm{SL}_m(\mathbb{C})$ these two problems are in fact equivalent. We refer the reader for more information on the subject to Gromov's above mentioned paper.

The main result of this paper is a complete positive solution of Gromov's Vaserstein problem, namely we prove

Main Theorem. *Let X be a finite dimensional reduced Stein space and $f\colon X \to SL_m(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number K and holomorphic mappings $G_1, \ldots, G_K\colon X \to \mathbb{C}^{m(m-1)/2}$ such that f can be written as a product of upper and lower diagonal unipotent matrices*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

The method of proof is an application of the Oka-Grauert-Gromov-principle to certain stratified fibrations. The existence of a topological section for these fibrations we deduce from Vaserstein's result.

We need the principle in it's strongest form suggested by Gromov, completely proven by Forstnerič and Prezelj [9] and also Forstnerič [8, Theorem 8.3]. After the Gromov-Eliashberg embedding theorem for Stein manifolds (see [2], [21]) this is to our knowledge the second time this holomorphic h-principle has an application which goes beyond the classical results of Grauert, Forster and Rammspott [13], [12], [11], [3], [7], [6], [5], [4].

2. On the number of factors

A natural question to ask is how the number of factors needed in the factorization depends on the space X and the map f. In the algebraic setting there is no such uniform bound as proved by van der Kallen in [24]. However in the holomorphic setting (exactly as in the topological setting) it is easy to see that there is an upper bound depending only on the dimension of the space X $(= m)$ and the size of the matrix $(= n)$.

A way to prove the existence of such a uniform bound is the following. Suppose it would not exist, i.e., for all natural numbers i there are Stein spaces X_i of dimension m and holomorphic maps $f_i\colon X_i \to \mathrm{SL}_n(\mathbb{C})$ such that f_i does not factor over a product of less than i unipotent matrices. Set $X = \cup_{i=1}^{\infty} X_i$ the disjoint union of the spaces X_i and $F\colon X \to \mathrm{SL}_n(\mathbb{C})$ the map that is equal to f_i on X_i. By our main result F factors over a finite number of unipotent matrices. Consequently all f_i factor over the same number of unipotent matrices which contradicts the assumption on f_i.

Thus we proved

Theorem 2.1. *There is a natural number K such that for any reduced Stein space X of dimension m and any null-homotopic holomorphic mapping $f\colon X \to SL_n(\mathbb{C})$ there exist holomorphic mappings $G_1, \dots, G_K\colon X \to \mathbb{C}^{n(n-1)/2}$ such that*

$$f(x) = M_1(G_1(x)) \dots M_K(G_K(x))$$

for every $x \in X$.

Let us denote by $K_{\mathcal{C}}(m,n)$ the number of matrices needed to factorize any null-homotopic map from a Stein space of dimension m into $\mathrm{SL}_n(\mathbb{C})$ by continuous triangular matrices and the number needed in the holomorphic case by $K_{\mathcal{O}}(m,n)$. We know that the Cohn example can be factored as 4 matrices with continuous entries but if one wants to factor it using matrices with holomorphic entries one needs 5 matrices. For 2×2 matrices we have the following three preliminary results which can be in short written as

$$K(2,m,\mathcal{O}) \le 2 + K(2,m,\mathcal{C},\mathcal{O}), \quad K(2,1,\mathcal{O}) = 4, \quad K(2,2,\mathcal{O}) = 5$$

Theorem 2.2. *Let X be a one-dimensional Stein space and $f\colon X \to SL_2(\mathbb{C})$ be a holomorphic mapping. Then there exists holomorphic mappings $g_1, \dots, g_4\colon X \to \mathbb{C}$ such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ g_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & g_2(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_3(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & g_4(x) \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.3. *Let X be a two-dimensional Stein space and $f\colon X \to SL_2(\mathbb{C})$ be a holomorphic mapping. Then there exists holomorphic mappings $g_1, \dots, g_5\colon X \to \mathbb{C}$ such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ g_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & g_2(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_3(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & g_4(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g_5(x) & 1 \end{pmatrix}.$$

Theorem 2.4. *Let X be a finite dimensional Stein space and $f\colon X \to SL_2(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Assume that there exists continuous mappings $g_1, \dots, g_K\colon X \to \mathbb{C}$ such that*

$$f(x) = M_1(g_1(x)) M_2(g_2(x)) \dots M_K(g_K(x)).$$

Then there exists holomorphic mappings $h_1, \dots, h_{K+2}\colon X \to \mathbb{C}$ such that

$$f(x) = M_1(h_1(x)) M_2(h_2(x)) \dots M_{K+2}(h_{K+2}(x)).$$

References

[1] P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53.

[2] Y. Eliashberg and M. Gromov, Embeddings of Stein manifolds of dimension n into the affine space of dimension $3n/2+1$, *Ann. of Math. (2)* **136** (1992), no. 1, 123–135.

[3] O. Forster, Topologische Methoden in der Theorie Steinscher Räume, *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2,* Gauthier-Villars, Paris 1971, pp. 613–618.

[4] O. Forster and K. J. Ramspott, Analytische Modulgarben und Endromisbündel, *Invent. Math.* **2** (1966), 145–170.

[5] O. Forster and K. J. Ramspott, Okasche Paare von Garben nicht-abelscher Gruppen, *Invent. Math.* **1** (1966), 260–286.

[6] O. Forster and K. J. Ramspott, Homotopieklassen von Idealbasen in Steinschen Algebren, *Invent. Math.* **5** (1968), 255–276.

[7] O. Forster and K. J. Ramspott, Über die Anzahl der Erzeugenden von projektiven Steinschen Moduln, *Arch. Math. (Basel)* **19** (1968), 417–422.

[8] F. Forstnerič. The Oka principle for sections of stratified fiber bundles. *Pure Appl. Math. Q.*, 6(3, Special Issue: In honor of Joseph J. Kohn. Part 1):843–874, 2010.

[9] F. Forstnerič and J. Prezelj, Extending holomorphic sections from complex subvarieties, *Math. Z.* **236** (2001), no. 1, 43–68.

[10] F. Forstnerič and J. Prezelj, Oka's principle for holomorphic submersions with sprays, *Math. Ann.* **322** (2002), no. 4, 633–666.

[11] H. Grauert, Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen, *Math. Ann.* **133** (1957), 139–159.

[12] H. Grauert, Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen, *Math. Ann.* **133** (1957), 450–472.

[13] H. Grauert, Analytische Faserungen über holomorph-vollständigen Räumen, *Math. Ann.* **135** (1958), 263–273.

[14] M. Gromov, Oka's principle for holomorphic sections of elliptic bundles, *J. Amer. Math. Soc.* **2** (1989), no. 4, 851–897.

[15] F. Grunewald, J. Mennicke, and L. Vaserstein, On the groups $SL_2(\mathbb{Z}[x])$ and $SL_2(k[x,y])$, *Israel J. Math.* **86** (1994), no. 1-3, 157–193.

[16] W. Hurewicz, H. Wallman, Dimension Theory. Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941. vii+165 pp.

[17] B. Ivarsson, F. Kutzschebauch, A solution of Gromov's Vaserstein problem, *C.R. Acad. Sci., Paris* **346** (2008), no. 23-24, 1239" 1243

[18] B. Ivarsson, F. Kutzschebauch, Holomorphic factorization of mappings into $SL_n(\mathbb{C})$, arXiv:0812.0312

[19] B. Ivarsson, F. Kutzschebauch, On the number of factors in the unipotent factorization of holomorphic mappings into $SL_2(\mathbb{C})$, arXiv1012.1025, to appear in Proc. AMS

[20] J. Rosenberg, *Algebraic K-theory and its applications,* Graduate Texts in Mathematics, vol. 147, Springer-Verlag, New York, 1994.

[21] J. Schürmann, Embeddings of Stein spaces into affine spaces of minimal dimension, *Math. Ann.* **307** (1997), no. 3, 381–399.

[22] A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), no. 2, 235–252, 477, (English translation, *Math. USSR Izv.* **11** (1977), 221–238.)

[23] W. Thurston and L. Vaserstein, On K_1-theory of the Euclidean space, *Topology Appl.* **23** (1986), no. 2, 145–148.

[24] W. van der Kallen, $SL_3(\mathbf{C}[X])$ does not have bounded word length, *Algebraic K-theory*, Part I (Oberwolfach, 1980), Lecture Notes in Math., vol. 966, Springer, Berlin, 1982, pp. 357–361.

[25] L. Vaserstein, Reduction of a matrix depending on parameters to a diagonal form by addition operations, *Proc. Amer. Math. Soc.* **103** (1988), no. 3, 741–746.
[26] D. Wright, The amalgamated free product structure of $\mathrm{GL}_2(k[X_1, \dots, X_n])$ and the weak Jacobian theorem for two variables, *J. Pure Appl. Algebra* **12** (1978), no. 3, 235–251.

Strongly Gauduchon Manifolds and Deformation Limits of Moishezon Manifolds

Dan Popovici

We shall present the new concept of *strongly Gauduchon manifold* that we introduced recently in the resolution of two long-standing conjectures: if all the fibres, except one, of a holomorphic family of compact complex manifolds are supposed to be projective (or merely Moishezon, i.e. bimeromorphically equivalent to projective manifolds), then the remaining fibre is shown to be again Moishezon. Here are the precise statements.

Definition 1 ([1]). *Let $\omega > 0$ be a C^∞ $(1,1)$-form (i.e. a Hermitian metric) on a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. Then*

(a) the form ω is said to be a **strongly Gauduchon (sG) metric** *if $\partial\omega^{n-1}$ is $\bar\partial$-exact;*
(b) X is said to be a **strongly Gauduchon (sG) manifold** *if X carries a strongly Gauduchon metric ω.*

Theorem 2 ([1]). *Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family (i.e. π is a proper holomorphic submersion from a complex manifold $\mathcal{X}$ to an open disc $\Delta \subset \mathbb{C}$ containing the origin) of compact complex manifolds $X_t := \pi^{-1}(t)$, $t \in \Delta$.*

If X_t is projective for every $t \in \Delta \setminus \{0\}$, then X_0 is Moishezon.

Theorem 3 ([2]). *Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds $X_t := \pi^{-1}(t)$, $t \in \Delta$.*

If X_t is Moishezon for every $t \in \Delta \setminus \{0\}$, then X_0 is again Moishezon.

These results are optimal thanks to an example of Hironaka (1962). We shall briefly outline the two different strategies employed: by means of well-chosen Kähler metrics on the generic fibres (in the generically projective case), respectively by means of the Barlet space of relative divisors (in the generically Moishezon case). These two different approaches are finally brought down to the same major technical difficulty: proving that if the $\partial\bar\partial$-lemma (a topological assumption which implies the *strongly Gauduchon* metric property) holds on all fibres, except one, then the remaining fibre is a *strongly Gauduchon* manifold.

We shall emphasize the interplay between various topological notions (e.g. the $\partial\bar\partial$-lemma and the degeneration at E_1 of the Frölicher spectral sequence) and several metric notions (e.g. *balanced* and *strongly Gauduchon* compact complex manifolds) by providing examples (cf. [4]) of

(1) compact complex manifolds which are not *strongly Gauduchon* (e.g. all the Calabi-Eckmann, Hopf and Tsuji manifolds);

(2) compact complex manifolds which are *strongly Gauduchon* but do not enjoy either of the stronger $\partial\bar{\partial}$-lemma and *balanced* properties: they emerge from the stability of the *strongly Gauduchon* property under small deformations and from Nakamura's calculation of the Kuranishi family of the Iwasawa manifold via a result of Alessandrini and Bassanelli.

The notion of *sG manifold* enjoys remarkable stability properties under both deformations and modifications.

Theorem 4 ([3]). *If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds, then the following equivalence holds:*

$\widetilde{X}$ is an sG manifold if and only if X is an sG manifold.

The property is also *open* under deformations.

Theorem 5 ([2]). *Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds $X_t := \pi^{-1}(t)$, $t \in \Delta$.*

If X_0 is an sG manifold, then X_t is an sG manifold for all $t \in \Delta$ sufficiently close to 0.

We hope the notion is also *closed* under deformations.

Conjecture 6 (see [4]). *Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds $X_t := \pi^{-1}(t)$, $t \in \Delta$.*

If X_t is an sG manifold for all $t \in \Delta \setminus \{0\}$, then X_0 is again an sG manifold.

We shall also indicate how these methods are likely to lead to the resolution of other long-standing conjectures on holomorphic deformations.

References

[1] D. Popovici, *Limits of Projective Manifolds Under Holomorphic Deformations*, arXiv e-print AG 0910.2032v1

[2] D. Popovici, *Limits of Moishezon Manifolds Under Holomorphic Deformations*, arXiv e-print AG 1003.3605v1

[3] D. Popovici, *Stability of Strongly Gauduchon Manifolds under Modifications*, arXiv e-print CV 1009.5408v1

[4] D. Popovici, *Deformation Openness and Closedness of Various Classes of Compact Complex Manifolds; Examples*, arXiv e-print AG 1102.1687v1

Coverings of open Riemann surfaces and embeddings into disc bundles motivated by questions in knot theory

Burglind Jöricke

Call a knot in the unit sphere in $\mathbb{C}^2$ analytic (respectively, smoothly analytic) if it bounds a complex curve (respectively, a smooth complex curve) in the complex ball. By a deep theorem of Kronheomer and Mrowka its 4-ball genus is the genus of the complex curve.

Let K be an oriented knot, let $N(K)$ be a tubular neighbourhood of K and P a projection of $N(K)$ to K. For an oriented link L contained in $N(K)$ the winding number $n = w_K(L)$ with respect to the knot K is the degree of the restriction $P|L$. Analytic knots and links are oriented as boundaries of the complex curves in the ball which are bounded by them.

The following theorem is an analog of Schubert's classical theorem [3] concerning the genus of a knot and its satellite.

Theorem 1. *Let K be a smoothly analytic knot in $\partial\mathbb{B}^2$. There exists a tubular neighbourhood $N(K) \subset \partial\mathbb{B}^2$ of K such that the following statements hold. For any analytic link $L \subset N(K)$ the number $n = w_K(L)$ is non-negative. If n is positive then the following lower bound for the 4-ball genus holds*

$$g_4(L) \geq n g_4(K) - (n-1)\,.$$

If L is itself a knot then

$$g_4(L) \geq n g_4(K) - \left[\frac{n-1}{2}\right].$$

($[x]$ denotes the largest integer not exceeding the real number x).
Both estimates are sharp.

Moreover, the links L contained in this tubular neighbourhood can be described.

The theorem is related to *ramified* holomorphic coverings of open Riemann surfaces, embeddings into disc bundles and the closed braid formed by the related embedding of the boundary of the covering surface. We state some of the corresponding theorems for *unramified* coverings. (The effect of ramification occurs over a simply connected domain and has been known).

Consider a connected Riemann surface X of positive genus g with smooth or empty boundary. It is well known that (*unramified* smooth) coverings $p : \overline{Y} \to \overline{X}$ (that fix a base point z and the fiber over it) are in one to one correspondence to homomorphisms p_* of the fundamental group $\pi_1(\overline{X})$ (with base point z) into the symmetric group $\mathcal{S}_n$. We always assume that Y is equipped with the structure of a Riemann surface such that the covering map is holomorphic on Y.

A smooth embedding $i : \overline{Y} \to \overline{X} \times \mathbb{D}$ into the disc bundle lifts p if $P_X \circ i = p$ for the projection P_X of the disc bundle to the first factor. We call a smooth embedding of the closure of a Riemann surface into $\overline{X} \times \mathbb{D}$ horizontal if it lifts certain unramified covering. Smooth horizontal embeddings $i : \overline{Y} \to \overline{X} \times \mathbb{D}$ are in one to one correspondence to homomorphisms $i_* : \pi_1(\overline{X}) \to \mathcal{B}_n$ to the braid group $\mathcal{B}_n$ on n strands (again base point and fiber are fixed). Moreover, i lifts p iff i_*

lifts p_* (i.e. $\tau_n \circ i_* = p_*$ for the canonical projection $\tau_n : \mathcal{B}_n \to S_n$). (For details on the braid group see e.g. [1].)

Let the base X be of genus g with connected non-empty smooth boundary. Then the mapping p_* associated to a covering $p : \overline{Y} \to \overline{X}$ maps the class $[\partial X]$ of the boundary ∂X to the product of g commutators in $\mathcal{S}_n$. Respectively, $i_*([\partial X])$ is a product of g commutators in $\mathcal{B}_n$.

By a theorem of Ore each even permutation is a commutator. We are interested in the case when the covering surface Y is connected (this case is the building block for the general case). The following strengthening of Ore's theorem holds.

Theorem 2. *Each even permutation in S_n is the commutator $[s_1, s_2]$ of two permutations such that the generated subgroup $< s_1, s_2 >$ acts transitively on the set of n elements.*

Theorem 2 is due to A.Gleason. The first part of the following corollary is contained in D.Husemoller, Duke 29(1962),167-174.

Corollary 1. *For each natural n and k, with $n-k$ non-negative and even, there is an unramified n-covering $\overline{Y} \to \overline{X}$ by a connected surface Y with k boundary components. The covering can be lifted to a smooth embedding $i : \overline{Y} \to \overline{X} \times \mathbb{D}$ such that the closed braid $\partial Y \to \partial X \times \mathbb{D}$ is represented by a word of minimal length (i.e. of length $n - k$) in the generators of $\mathcal{B}_n$ and their inverses.*

Theorem 1 is related to *holomorphic* horizontal embeddings into disc bundles rather than to *smooth* embeddings. The following proposition holds.

Proposition 1. *Each smooth horizontal embedding $i : \overline{Y} \to \overline{X} \times \mathbb{D}$ can be approximated by a holomorphic embedding over a neighbourhood X' of a skeleton of X.*

(A skeleton is a bouquet of circles contained in X such that X retracts to this set.) We may think of $\overline{X}'$ as diffeomorphic to $\overline{X}$. The proof of theorem 1 uses corollary 1 and proposition 1.

The following problem arises. Over which subsets $X' \subset X$ a given embedding i is isotopic to holomorphic? The precise formulation of this problem will be given below in terms of the Teichmüller space $\mathcal{T}(X)$ modeled on X (see [2]). Note that in our case the Riemann surface X has non-empty ideal boundary.

Recall the definition of the configuration space: $C_n(\mathbb{C}) = \{(\zeta_1, ..., \zeta_n) \in \mathbb{C}^n : \zeta_i \neq \zeta_j \, for \, i \neq j\}$. The symmetrized configuration space (the unordered n-tuples of pairwise different complex numbers) is the factor of $C_n(\mathbb{C})$ by the symmetric group. It carries a complex structure in the following way. Unordered n-tuples of complex numbers are in one to one correspondence with the elementary symmetric functions in n variables, namely, the coefficients of monic polynomials of degree n. (A polynomial is monic if the highest order term has coefficient equal to one.) Hence, there is an identification of $C_n(\mathbb{C})/\mathcal{S}_n$ with $\mathbb{C}^n \setminus V_{\Delta_n}$, where V_{Δ_n} denotes the discriminant set, i.e. the set of coefficients of those monic polynomials which have multiple zeros. V_{Δ_n} is the zero set of the polynomial Δ_n in $\mathbb{C}^n$, hence $\mathbb{C}^n \setminus V_{\Delta_n}$ has the structure of a Stein manifold. Note that it is not Brody hyperbolic, in other words, there are non-constant holomorphic mappings of the complex plane $\mathbb{C}$ into $\mathbb{C}^n \setminus V_{\Delta_n}$.

A smooth (respectively, holomorphic) horizontal embedding $i : Y \to X \times \mathbb{D}$ can be considered as a smooth (respectively, holomorphic) mapping from X into symmetrized configuration space. A closed braid is a loop in $\mathbb{C}^n \setminus V_{\Delta_n}$ and the braid group $\mathcal{B}_n$ can be identified with the fundamental group of $\mathbb{C}^n \setminus V_{\Delta_n}$.

Problem. *Let $\varphi : \pi_1(X) \to \pi_1(\mathbb{C}^n \setminus V_{\Delta_n}) = \mathcal{B}_n$ be a homomorphism. Which elements $X' \in \mathcal{T}(X)$ have the following property: there is a holomorphic mapping from X' into n-dimensional symmetrized configuration space which induces φ on fundamental groups (for short, X' admits a holomorphic φ-mapping into n-dimensional symmetrized configuration space)?*

Theorem 3. *Let X be an open Riemann surface with smooth boundary.*
1. For every natural number n there exists a nontrivial homomorphism φ : $\pi_1(X) \to \pi_1(\mathbb{C}^n \setminus V_{\Delta_n}) = \mathcal{B}_n$ such that each $X' \in \mathcal{T}(X)$ admits a holomorphic φ-mapping into n-dimensional symmetrized configuration space. This is true for all homomorphisms into $\mathcal{B}_2$.
2. For $n \geq 3$ there exists a homomorphism φ into $\mathcal{B}_n$ such that for some $X' \in \mathcal{T}(X)$ there is no holomorphic φ-mapping into n-dimensional configuration space.

More concrete and stronger statements related to theorem 3 can be given. Many questions are still open.

References

[1] Birman, Joan, Braids, *Links and Mapping Class Groups*, Ann. Math. Studies **82**, Princeton Univ. Press, (1975).
[2] Nag, Subhashis, *The Complex Analytic Theory of Teichmüller Spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication. John Wiley Sons, Inc., New York, (1988).
[3] Schubert, Horst, *Knoten und Vollringe*, Acta Math. **90** (1953), 131-286.

Small families of complex lines for testing holomorphic extendibility from spheres

Josip Globevnik

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^2$. Let f be a continuous function on $b\mathbb{B}$. If L is a complex line that meets $\mathbb{B}$ then we say that the function f extends holomorphically into $\mathbb{B}$ along L if $f|(L \cap b\mathbb{B})$ extends holomorphically through $L \cap \mathbb{B}$. We consider the question about along how many complex lines should f extend holomorphically into $\mathbb{B}$ in order that f extends holomorphically through $\mathbb{B}$. Denote by $\mathcal{L}(a)$ the set of all complex lines passing through a.

We present two recent results.

Theorem 1 *Let a, b be two points in $\mathbb{C}^2$ such that the complex line through a and b meets $\mathbb{B}$ and such that $< a|b > \neq 1$ if one of the points is contained in $\mathbb{B}$ and the other in $\mathbb{C}^2 \setminus \overline{\mathbb{B}}$. If a function $f \in \mathcal{C}^\infty(b\mathbb{B})$ extends holomorphically into $\mathbb{B}$ along each $L \in \mathcal{L}(a) \cup \mathcal{L}(b)$ then f extends holomorphically through $\mathbb{B}$.*

When $a, b \in \overline{\mathbb{B}}$ and when f is real analytic such a theorem was proved by M. Agranovsky. Such a theorem fails to hold for functions in $\mathcal{C}^k(b\mathbb{B})$. The proof of Theorem 1 is contained in "Small families of complex lines for testing holomorphic extendibility", to appear in Amer. J. Math., http://arxiv.org/abs/0911.5088.

Our second result deals with continuous functions.

Theorem 2 *Let a, b, c be three points in $\mathbb{C}^2$ which do not lie in a complex line, such that the complex line through a, b meets $\mathbb{B}$ and such that if one of the points a, b is in $\mathbb{B}$ and the other in $\mathbb{C}^2 \setminus \overline{\mathbb{B}}$ then $< a|b > \neq 1$ and such that at least one of the numbers $< a|c >$, $< b|c >$ is different from 1. We prove that if a continuous unction f on $b\mathbb{B}$ extends holomorphically into $\mathbb{B}$ along each $L \in \mathcal{L}(a) \cup \mathcal{L}(b) \cup \mathcal{L}(c)$ then f extends holomorphically through $\mathbb{B}$.*

This generalizes the recent result of L. Baracco who proved such a theorem if a, b, c are contained in $\mathbb{B}$. Our proof is quite different from the one of Baracco and uses the following one variable result which we also prove and which in the real analytic case was proved by M. Agranovsky:

Theorem 3 *Let Δ be the open unit disc in $\mathbb{C}$. Given $\alpha \in \Delta$ let $\mathcal{C}_\alpha$ be the family of all circles in Δ obtained as the images of circles centered at the origin under an automorphism of Δ that maps 0 to α. Given $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, and $n \in \mathbb{N}$, a continuous function f on $\overline{\Delta}$ extends meromorphically from every circle $\Gamma \in \mathcal{C}_\alpha \cup \mathcal{C}_\beta$ through the disc bounded by Γ with the only pole at the center of Γ of degree not exceeding n if and only if f is of the form $f(z) = a_0(z) + a_1(z)\overline{z} + \cdots + a_n(z)\overline{z}^n$ $(z \in \Delta)$ where the functions a_j, $0 \leq j \leq n$, are holomorphic on Δ.*

The proofs of Theorems 2 and 3 are contained in "Meromorphic extensions from small families of circles and holomorphic extensions from spheres", http://arxiv.org/abs/1101.0136

The proof of Theorem 3, a one variable theorem, uses again analysis in several complex variables. Via semiquadrics, introduced into this context by M. Agranovsky and J. Globevnik in 2003, one transforms the problem to a problem of a function on a CR submanifold of $\mathbb{C}^2$ which consists of two families of semiquadrics, and then applying, following A.Tumanov, an idea of H.Lewy modified by H.Rossi to prove that the function extends as a holomorphic function of two variables which is a polynomial in the second variable.

Model pseudoconvex domains and bumping

Gautam Bharali

The Levi geometry at weakly pseudoconvex boundary points of domains in $\mathbb{C}^n$, $n \geq 3$, is sufficiently complicated that, in general, there are no universal model domains with which to compare a given domain near such points. On the other hand, a rather successful strategy, especially in $\mathbb{C}^2$, for understanding a pseudoconvex domain involves carefully deforming its boundary outwards about some boundary point without destroying pseudoconvexity so that the new domain "well approximates" the original but has a much simpler defining function. This procedure is formalised as follows:

(∗) Given a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, and $\zeta \in \partial\Omega$, find a neighbourhood U_ζ of ζ and a $\mathcal{C}^2$-smooth function $\rho_\zeta \in \mathsf{psh}(U_\zeta)$ such that
- $\rho_\zeta^{-1}\{0\}$ is a smooth hypersurface in U_ζ that is pseudoconvex from the side $U_\zeta^- := \{z \rho_\zeta(z) < 0\}$; and
- $\rho_\zeta(\zeta) = 0$, but $(\overline{\Omega} \setminus \{\zeta\}) \bigcap U_\zeta \subsetneq U_\zeta^-$.

We shall call the triple $(\partial\Omega, U_\zeta, \rho_\zeta)$ a *local bumping of* Ω *about* ζ. Diederich and Fornaess [2] have shown that if Ω is bounded and has real-analytic boundary, then local bumpings always exist about each $\zeta \in \partial\Omega$. However, much of the success in using bumpings relies on a second ingredient: that the bumpings constructed are, in some sense, well-adapted to the pair (Ω, ζ). To be more specific, it is highly desirable for a local bumping $(\partial\Omega, U_\zeta, \rho_\zeta)$ to have the following two properties:

(B1) The orders of contact of $\partial\Omega \cap U_\zeta$ with $\rho_\zeta^{-1}\{0\}$ at ζ along the various directions $V \in T_\zeta(\partial\Omega) \cap iT_\zeta(\partial\Omega)$ are the lowest possible.
(B2) The function ρ_ζ is as simple as possible and is explicitly known.

The difficulty with the Diederich–Fornaess construction is that when $\Omega \Subset \mathbb{C}^3$ and $n \geq 3$, the order of contact between $\partial\Omega$ and $\rho_\zeta^{-1}\{0\}$ at ζ along certain complex-tangential directions can be very high. Furthermore, the great generality of the scope of [2] makes it very hard for an explicit equation for ρ_ζ to deduced at the end of the Diederich–Fornaess construction.

Given a pair (Ω, ζ) as above, let P denote the sum of the lowest-weight (non-pluriharmonic) terms (depending on the complex-tangential variables) in the Catlin normal-form for the pair (Ω, ζ). If the point 0 lying in the boundary of the model domain $\Omega_P := \{(w, z) \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathfrak{Re}\, w + P(z) < 0\}$ is of finite type, then it is known that (B1) and (B2) are very precisely achievable: in fact, after a holomorphic change of coordinates from (z, w) to (Z, W), the bumping has the form $\{(W, Z) \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathfrak{Re}\, W + (P - H)(Z) < 0\}$, where H is a function that is positive away from $0 \in \mathbb{C}^{n-1}$ and has the same (weighted) homogeneity as P. Whenever this is possible, we say that (Ω, ζ) *is h–extendible*. While it is *demonstrably impossible* to construct such simple models when the pair (Ω, ζ) is not h–extendible, the evidence in [1] suggests that models that are only slightly more complicated can be constructed in $\mathbb{C}^3$ in the non-h-extendible case under some natural restrictions on P. For the pairs (Ω, ζ) that we shall describe/discuss, the phrase "only slightly more complicated" turns out to mean the defining function of the bumped model contains an extra, but precisely described, non-homogeneous term in addition to the term $(P - H)$ mentioned above.

Given a pair (Ω, ζ) in $\mathbb{C}^3$ that is non-h–extendible, and given the P associated to the Catlin normal-form for (Ω, ζ), set

$$\begin{aligned} \boldsymbol{\mathcal{E}}(P) &:= \text{the set of all irreducible complex curves } X \subset \mathbb{C}^2 \\ &\quad\ \text{such that } P \text{ is harmonic along the smooth part of } X, \\ \boldsymbol{\mathcal{E}}_0(P) &:= \text{the class of irreducible algebraic curves in } \boldsymbol{\mathcal{E}}(P) \\ &\quad\ \text{that pass througn } 0 \in \mathbb{C}^2. \end{aligned}$$

When (Ω, ζ) is non-h–extendible, $\boldsymbol{\mathcal{E}}_0(P) \neq \varnothing$. This is the consequence of work done independently by Diederich–Herbort [3] and Yu [4]. The structure of the set $\boldsymbol{\mathcal{E}}(P)$ appears to control the ability to construct a bumping at ζ that satisfies (B1) and (B2). In this talk, we shall consider pairs (Ω, ζ) (in $\mathbb{C}^3$) that are *almost h–extendible*. By this we mean that (Ω, ζ) is such that, with P as explained above:

- $\boldsymbol{\mathcal{E}}(P) = \boldsymbol{\mathcal{E}}_0(P)$; and
- $\cup_{X \in \boldsymbol{\mathcal{E}}_0(P)} X$ is well-separated (in a sense that will be made precise in the talk) from all other points at which the complex-Hessian of P is degenerate.

In this talk, we will discuss a theorem to the effect that if the pair (Ω, ζ) is almost h–extendible, then (Ω, ζ) admits a local bumping at ζ that has the properties (B1) and (B2) and which has a simple defining function that can be described explicitly.

References

[1] G. Bharali and B. Stensønes, *Plurisubharmonic polynomial polynomials and bumping*, Math. Z. **261** (2009), 39–63.

[2] K. Diederich and J.E. Fornaess, *Proper holomorphic maps onto pseudoconvex domains with real-analytic boundary*, Ann. of Math. **110** (1979), 575–592.

[3] K. Diederich and G. Herbort, *Pseudoconvex domains of semiregular type*, Contributions to Complex Analysis and Analytic Geometry, Aspects Math. (1994), Vieweg, Braunschweig, pp. 127–161.

[4] J.Y. Yu, *Peak functions on weakly pseudoconvex domains*, Indiana Univ. Math. J. **43** (1994), 1271–1295.

Geodesics in the space of Kähler metrics

László Lempert
(joint work with L. Vivas)

Let X be a connected, compact, complex manifold and ω_0 a smooth Kähler form on it. It was discovered by Mabuchi, and rediscovered by Semmes and Donaldson, that the set $\mathcal{H}_0$ of smooth Kähler forms cohomologous to ω_0, and the set $\mathcal{H}$ of smooth, strongly ω_0–plurisubharmonic functions on X carry natural Riemannian manifold structures, see [11, 12, 7]. A function $u \colon X \to \mathbb{R}$ is (strongly) ω_0–plurisubharmonic if $\omega_0 + i\partial\overline{\partial}u \geq 0$ (resp. > 0). Mabuchi shows that in fact $\mathcal{H}$ is isometric to the Riemannian product $\mathcal{H}_0 \times \mathbb{R}$, and both he and Donaldson point out that understanding geodesics in these spaces is important for the study of special Kähler metrics. Donaldson then raises the obvious question whether any pair of points in $\mathcal{H}$ (or $\mathcal{H}_0$) can be connected by a smooth geodesic. In the talk we gave a negative answer:

Theorem 1. *Suppose (X, ω_0) is a positive dimensional compact Kähler manifold and $h\colon X \to X$ is a holomorphic isometry with an isolated fixed point such that $h^2 = id_X$. Then there is a Kähler form $\omega_1 \in \mathcal{H}_0$ which cannot be connected to ω_0 by a smooth geodesic.*

Concretely, one can take X to be a torus $\mathbb{C}^m/\Gamma$, ω_0 a translation invariant Kähler form, and h induced by reflection $z \mapsto -z$ in $\mathbb{C}^m$.

According to Semmes, geodesics in $\mathcal{H}$ (and therefore in $\mathcal{H}_0$) are related to a Monge–Ampère equation as follows, [S]. Let $S = \{s \in \mathbb{C} : 0 < \operatorname{Im} s < 1\}$, and ω the pull back of ω_0 by the projection $\overline{S} \times X \to X$. With any smooth curve $[0,1] \ni t \mapsto v_t \in \mathcal{H}$ associate the smooth function $u(s,x) = v_{\operatorname{Im} s}(x)$, $(s,x) \in \overline{S} \times X$. Set $m = \dim X$. Then $t \mapsto v_t$ is a geodesic if and only if u satisfies

$$(\omega + i\partial\overline{\partial} u)^{m+1} = 0. \tag{1}$$

Since $\omega + i\partial\overline{\partial}u$, restricted to fibers $\{s\} \times X$, is positive, (1) is equivalent to $\operatorname{rk} \omega + i\partial\overline{\partial}u \equiv m$; and so a smooth geodesic connecting $0, v \in \mathcal{H}$ gives rise to an ω–plurisubharmonic $u \in C^\infty(\overline{S} \times X)$ solving

$$\begin{aligned} &\operatorname{rk}\ \omega + i\partial\overline{\partial}u \equiv m, \\ &u(s+\sigma, x) = u(s,x) \qquad \text{for } \sigma \in \mathbb{R},\ (s,x) \in \overline{S} \times X, \\ &u(s,x) = \begin{cases} 0, & \text{if } \operatorname{Im} s = 0 \\ v(x), & \text{if } \operatorname{Im} s = 1. \end{cases} \end{aligned} \tag{2}$$

Therefore Theorem 1 follows from the following more precise result:

Theorem 2. *If (X, ω_0) and h are as in Theorem 1, there is a $v \in \mathcal{H}$ for which* (2) *admits no real valued solution $u \in C^3(\overline{S} \times X)$. One can choose v so that $h^*v = v$.*

When $m = 1$, the v in Theorem 2 even form an open subset of the space of h–invariant functions in $\mathcal{H}$, but we do not know if this holds when $m > 1$.

The idea that symmetries help in the analysis of solutions of Monge–Ampère equations is not new. The first examples of irregularity of certain boundary value problems in $\mathbb{C}^m$ were constructed by Bedford and Fornaess using symmetries, see [3]. Our approach, based on the study of the so called Monge–Ampère foliation, is different from theirs. The symmetry is used to identify a leaf of the foliation associated to a C^3 solution u of (2). By analyzing the first order behavior of the foliation about this particular leaf we obtain a condition on the Hessian of u at $(1, x_0)$, where x_0 is an isolated fixed point of h. The proof is concluded by finding a boundary value v which is incompatible with this condition.

Studying solutions of the homogeneous Monge–Ampère equation through the associated foliation is not new, either. This approach first appeared in [1, 2, 9, 10], and still seems to be the only way to prove smoothness of the solution. More recently, in [8] Donaldson used the foliation method in a variant of the boundary value problem (2) to prove, resp. disprove, regularity, depending on the boundary data.

Generalized solutions to (2) and to rather more general boundary value problems for the homogeneous Monge–Ampère equation (1) are known to exist, see [6], with complements in [5].

References

[1] E. Bedford, *Extremal plurisubharmonic functions for certain domains in* $\mathbb{C}^2$*,* Indiana U. Math. J. **28** (1979) 613–626

[2] E. Bedford, *Stability of envelopes of holomorphy and the degenerate Monge–Ampère equation,* Math. Ann. **259** (1982) 1–28

[3] E. Bedford, J. Fornaess, *Counterexamples to regularity for the complex Monge–Ampère equation,* Invent. Math. **50** (1979) 129–134

[4] E. Bedford, M. Kalka, *Foliations and complex Monge–Ampère equations,* Comm. Pure Appl. Math. **30** (1977) 543–571

[5] Z. Błocki, *On geodesics in the space of Kähler metrics,* Proc. Conf. in Geometry dedicated to Shing–Tung Yau (to appear)

[6] X. X. Chen, *The space of Kähler metrics,* J. Diff. Geom. **56** (2000) 189–234

[7] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonian dynamics,* Amer. Math. Soc. Transl. Ser. 2 **196**, Amer. Math. Soc. Providence RI (1999) 13–33

[8] S. K. Donaldson, *Holomorphic discs and the complex Monge–Ampère equation,* J. Sympl. Geom. **1** (2002) 171–196

[9] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule,* Bull. Soc. Math. France **109** (1981) 427–474

[10] L. Lempert, *Solving the degenerate complex Monge–Ampère equation with one concentrated singularity,* Math. Ann. **263** (1983) 515–532

[11] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds I,* Osaka J. Math. **24** (1987) 227–252

[12] S. Semmes, *Complex Monge–Ampère and symplectic manifolds,* Amer. J. Math. **114** (1992) 495–550

Holomorphic foliations and hyperbolicity

Erwan Rousseau

(joint work with C. Gasbarri and G. Pacienza)

In the last decades, many efforts have been done to understand the geometry of subvarieties of varieties of general type. One of the main motivation is the fascinating conjectural relation between analytic aspects and arithmetic ones. On the geometric side, the philosophy (Green-Griffiths, Lang, Vojta, Campana) is that positivity properties of the canonical bundle of a projective manifold should impose strong restrictions on its subvarieties.

One of the first striking results is the following theorem of Bogomolov for surfaces.

Theorem 1 (Bogomolov)**.** *There are only finitely many rational and elliptic curves on a surface of general type with* $c_1^2 > c_2$.

In this theorem, the hypothesis $c_1^2 > c_2$ ensures that the cotangent bundle is big, so that rational and elliptic curves are shown to be leaves of a foliation and then, one can use results on algebraic leaves of foliations.

Two decades later, this result was extended to transcendental leaves of foliations by McQuillan.

Theorem 2 (McQuillan). *Let X be a surface of general type and $\mathcal{F}$ a holomorphic foliation on X. Then $\mathcal{F}$ has no entire leaf which is Zariski dense.*

As a consequence he obtains the following.

Corollary 3 (McQuillan). *On a surface X of general type with $c_1^2 > c_2$, there is no entire curve $f : \mathbb{C} \to X$ which is Zariski dense.*

It is of course of great interest to generalize these results, even partially, to higher dimension. On the algebraic side, this was investigated by Lu and Miyaoka.

Theorem 4 (Lu-Miyaoka). *Let X be a nonsingular projective variety. If X is of general type, then X has only a finite number of nonsingular codimension-one subvarieties having pseudoeffective anticanonical divisor. In particular, X has only a finite number of nonsingular codimension-one Fano, Abelian, and Calabi-Yau subvarieties.*

This can be seen as a generalization to higher dimension of the aforementioned theorem of Bogomolov.

In a joint work with C. Gasbarri and G. Pacienza, we study holomorphic mappings $f : \mathbb{C}^p \to X$ of generic maximal rank into a projective manifold of dimension n, such that the image of f is tangent to a holomorphic foliation $\mathcal{F}$ on X. We discuss a generalized Green-Griffiths-Lang conjecture and obtain several results of algebraic degeneracy in the strong sense (i.e. the existence of a proper closed subset of X containing all such maps).

Variational characterization of Kähler-Einstein metrics and application to the Kähler-Ricci flow

Sébastien Boucksom

(joint work with R. Berman, P. Eyssidieux, V. Guedj and A. Zeriahi)

We propose to present a series of joint works with Robert Berman, Philippe Eyssidieux, Vincent Guedj and Ahmed Zeriahi in which we develop in a systematic way a variational approach to complex Monge-Ampère equations.

Let X be a smooth projective variety and L be an ample line bundle on X. For each metric $e^{-\phi}$ on L with weight ϕ denote by $dd^c\phi$ its curvature current, whenever it is defined. The Monge-Ampère operator is defined on smooth weights ϕ by setting $\mathrm{MA}(\phi) := (dd^c\phi)^n$ with $n := \dim X$. A basic property of this operator is the existence of a primitive, i.e. a functional E on smooth weights such that

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi + tv) = \int v \,\mathrm{MA}(\phi)$$

for each smooth function v. As a consequence, ϕ satisfies the Monge-Ampère equation $\mathrm{MA}(\phi) = \mu$ for some measure μ of mass 1 iff ϕ is a critical point of $F_\mu(\phi) := E(\phi) - \int_X \phi\, d\mu$. Similarly, when $L = \pm K_X$ ϕ satisfies the Kähler-Einstein equation $\mathrm{MA}(\phi) = e^{\pm\phi + c}$ for some constant $c \in \mathbb{R}$ iff it is a critical point of $F_\pm(\phi) := E(\phi) - \pm \log \int e^{\pm\phi}$.

The functional E becomes non-decreasing and concave when restricted to smooth psh weights, and it may thus be extended by monotonicity to a non-decreasing, concave and usc function $E : \mathsf{psh}(X, L) \to [-\infty, +\infty[$ of psh weights on L. The domain $\mathcal{E}^1(X, L) := \{E > -\infty\}$ of E consists of those $\phi \in \mathsf{psh}(X, L)$ such that $\mathrm{MA}(\phi)$ is well-defined as a non-pluripolar probability measure and ϕ is integrable with respect to $\mathrm{MA}(\phi)$.

1. Variational characterization of Monge-Ampère equations

Our first main result is the following:

Theorem 1.1. *Let μ be a measure with finite energy, in the sense that the concave function F_μ is finite valued on $\mathcal{E}^1(X, L)$. Then F_μ achieves its maximum at $\phi \in \mathcal{E}^1(X, L)$ iff $\mathrm{MA}(\phi) = \mu$. Such a maximizer ϕ exists and is unique up to a constant.*

In case where $L = K_X$ is ample we similarly have

Theorem 1.2. *The concave functional F_+ is bounded above on $\mathcal{E}^1(X, L)$ and it achieves its maximum exactly at the Kähler-Einstein metric of X.*

The case where $L = -K_X$ is ample, i.e. X is Fano, is more involved since the functional F_-, first considered by Ding and Tian, is not concave anymore. It is however *geodesically concave* with respect to the L^2 metric on $\mathsf{psh}(X, -K_X)$ as a consequence on results on psh variations of Bergman kernels. Using this fact we also obtain a variational characterization in that case.

The strength of these variational characterizations is that in each case *any* maximimzing sequence ϕ_j of the functional must converge (in the weak topology and up to an additive constant) to the unique maximizer. Indeed, the functional F is in each case *proper* with respect to E in an appropriate sense.

2. Convergence of the Kähler-Ricci flow

Assume now that $L = \pm K_X$ is ample and let $\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) \pm \omega$ be the (normalized) Kähler-Ricci flow starting at a given Kähler metric ω_0 in $c_1(L)$. The solution ω is then C^∞ on $[0, +\infty[\times X$, and the functional $F_\pm$ is known to be non-decreasing along the flow.

Our second main result is the following:

Theorem 2.1. *Let $\pm K_X$ be ample, and assume that F_- is proper in the Fano case. Then $F_\pm$ converges to $\sup_{\mathcal{E}^1(X,L)} F_\pm$ along the flow as $t \to +\infty$. In particular, the flow converges weakly to the Kähler-Einstein metric in case the latter exists and is unique.*

We thus recover a weak form of a theorem of Perelman asserting the C^∞ convergence of the flow to the Kähler-Einstein metric on a Kähler-Einstein Fano manifold with holomorphic vector fields.

This result as well as the previous ones admit appropriate extensions to the case where X may admit singularities. In particular, the last result hold as well when X has log-terminal singularities.

Pseudoconvex domains over Kähler manifolds and Bergman kernels with parameters

Takeo Ohsawa

1. General settings

Let D be a locally pseudoconvex domain with twice continuously differentiable boundary in a complex manifold X. A general question asked by H. Grauert is whether or not D admits a plurisubharmonic exhaustion function if X admits a Kähler metric. Let Y be a complex analytic space such that there exists a proper surjective morphism π from X to Y. A general program is to extend the results for compact complex manifolds to X. Given a compact complex manifold with underlying differentiable manifold M and complex structure J, we consider a triple (M, J, ω), where ω is the fundamental form of a Kähler metric on M with respect to J. In the set M of all such triples for fixed M, one naturally defines an equivalence relation $\sim$ (resp. $\approx$) by $(M, J, \omega) \sim$ (resp. $\approx$) $(M, J', \omega') \Leftrightarrow$ "There exists a diffeomorphism φ from M onto itself whose differential commutes with J and J' (resp. pulls back ω' to ω')". Geometry of the triple $(M/\sim \leftarrow M \rightarrow M/\approx)$ is of general interest. These three questions are related to each other in a loose way as one can see below.

2. L^2 cohomology and the relative Bergman kernels

Let D and X be as above. Assume that D is relatively compact and X admits a Kähler form (:= the fundamental form of a Kähler metric) ω. Then there exists a twice continuously differentiable defining function ρ of D which is smooth on D. For any such ρ, there exists a positive number c such that $\omega(\rho) := \omega + L(c/\log(\rho))$ is a complete Kähler metric on D. Here $L(\cdot)$ denotes the Levi form.

Theorem 1. *∂D is connected unless it is Levi flat.*

Proof: If ∂D is not Levi flat, then the L^2 Dolbeault cohomology of D with respect to $\omega(\rho)$ vanishes at the bidegree (0,1), which implies that the Hartogs type continuation holds true.

Combining Theorem 1 with the main result in [5], one has

Theorem 2. *If $\dim X = 2$ and ∂D is real analytic in the above situation, then D is holomorphically convex unless ∂D is Levi flat.*

Let $\pi\colon X \to Y$ be as in 1, let $0 \in Y$ and let n be the dimension of the preimage of 0. As well as Theorem 1, a generalization of the following result is contained in [9], since its proof is based on a principle on the L^2 cohomology similarly as in Theorem 1.

Theorem 3. *(cf. [7]). Let E be an effective divisor on X whose support $|E|$ does not contain any branch of X_0, the preimage of 0. (X is not assumed to be Kähler.) If $\pi(|E|)$ contains 0, then the stalk at 0 of the n-th direct image by π of the tensor product of the canonical bundle of X and the line bundle $[E]$ vanishes.*

Assuming that π is a submersion, let $K[y]$ denote the diagonalized Bergman kernel of the preimage of y. The collection of the reciprocal of $K[y]$ (as y runs through Y) is naturally identified with a fiber metric of the relative canonical bundle of π. It was shown that the curvature form of this fiber metric is semipositive by Maitani-Yamaguchi [8] when $n = 1$ and by Berndtsson [2] when π is a projective morphism. In [3] it was shown that the curvature form of the direct image by π of the relative canonical bundle with respect to the L^2 metric is Nakano semipositive provided that X admits a Kähler metric.

3. Deformations of tori

In view of the above mentioned result of Berndtsson in [3], where π is supposed to be a submersion, we want to study the set of "Kähler directions" in the space of infinitesimal deformations $H^1(X_0, \Theta)$, where Θ denotes the holomorphic tangent bundle of X_0. For that, given any compact complex manifold M, we have introduced in [10] a subset KID(M) (KID is for Kähler infinitesimal deformations) of $H^1(M, \Theta)$ consisting of the images of the Kodaira-Spencer maps from the Zariski tangent spaces of the parameter spaces of Kähler deformations of M. Here Kähler deformation of M means an analytic family of M such that the total space admits a Kähler metric. A question on KID(M) arises when M is a complex torus of dimension ≥ 2 (cf. [6]). In fact, for any complex torus T with $\dim T \geq 2$, Berndtsson's theorem implies that KID(T) does not coincide with the total space $H^1(T, \Theta)$ because the relative canonical bundle of the Kuranishi family of T has the reciprocal of the diagonalized Bergman kernels of the fibers as the canonical fiber metric, and its curvature form is easily seen to be indefinite. This phenomenon may be regarded as a part of a somewhat vague principle that the Kähler condition implies pseudoconvexity.

The following provides a more precise information on KID(T) than what Berndtsson's theorem predicts in general.

Theorem 4. *Let T be a complex torus of dimension ≥ 2. Then, with respect to a standard identification of $H^1(T, \Theta)$ with the set of $n\times n$ matrices,* KID(T) *coincides with the set $\{\Xi | \Xi Y^{-1} H$ is symmetric for some positive definite Hermitian matrix $H\}$, where Y denotes a real matrix Y with $\det Y > 0$ such that T has a period matrix $(I, X + iY)$ for some real matrix X.*

Thus, loosely speaking, KID(T) is swept out by complex hyperplanes that are parametrized by a convex cone. The proof of Theorem 4 is done by computing the

(2,0)-components of the pull-back of Kähler forms by harmonic diffeomorphisms. Based on this result, it will be observed that KID(T) is properly contained in the subset of $H^1(T, \Theta)$ consisting of the elements for which the curvature of the L^2 metric is semipositive.

By analyzing harmonic maps that pulls back a Kähler form to a Kähler form, the following is obtained, too.

Theorem 5. *Let T be a complex torus of dimension ≥ 2 equipped with a Kähler form ω and let η be any nonzero (2,0) form on T. Then, for any complex structure on T such that ω is a positive (1,1) form, $\eta + \eta^-$ is not of type (1,1). Here η^- denotes the complex conjugate of η.*

Corollary 6. *For any torus $\check{T}$ of dimension ≥ 4 equipped with a symplectic form ω, the set of elements of $H^2(\check{T}, R)$ which are represented by (1,1) forms for some complex structure compatible with ω has a nonempty complement.*

As a work preceding to ours, it was known by Calabi [4] that, for any translation invariant Kähler metric g on a complex torus T and for any nondegenerate translation invariant real (1,1) form σ of index j on T, there exist exactly $n!/(n-j)!j!$ complex structures on T for which g is Hermitian and σ is of type (1,1) and of index j. A work of Bartolomeis [1] describes the deformation of Kählerian tori from a viewpoint different from ours.

References

[1] P. Bartolomeis, *Symplectic deformations of Kähler manifolds*, J. Symplectic Geom. 3 (2005), 341355.

[2] B. Berndtsson, *Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains*, Ann. Inst. Fourier **56** (2006), 16331662.

[3] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Ann. Math. 169 (2009), 531560.

[4] E. Calabi, *Isometric families of Kähler structures*, The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp. 2339, Springer, New York-Berlin, 1980.

[5] K. Diederich, T. Ohsawa, *A Levi problem on two-dimensional complex manifolds*, Math. Ann. **261** (1982), 255261.

[6] A. Fujiki, *Coarse moduli space for polarized compact Kähler manifolds*, Publ. RIMS. 20 (1984), 9771005.

[7] H. Fuse, T. Ohsawa, *On a curvature property of effective divisors and its application to sheaf cohomology*, Publ. Res. Inst. Math. Sci. 45 (2009), 10331039.

[8] F. Maitani, H. Yamaguchi, *Variation of Bergman metrics on Riemann surfaces*, Math. Ann. **330** (2004), 477489.

[9] T. Ohsawa, *L^2 estimates for the d-bar operator in the top degrees and application to pseudoconvex hypersurfaces*, preprint.

[10] T. Ohsawa, *On the deformations of tori with Kähler metrics*, preprint.

Almost complex Borel's and Bloch's theorems

Benoît Saleur

The hyperbolicity, in the different acceptations of the term, of the complementary set in the projective complex plane $\mathbb{P}^2(\mathbb{C})$ of a union of projective lines, has been the subject of many studies. The well known Green's theorem (see [8]), derived from a result due to E. Borel (see [3]), asserts that the complementary set of five projective lines in general position is hyperbolic, in both Brody's and Kobayashi's senses. The case of four lines is more complicated. It has been studied by A. Bloch and H. Cartan, who proved, grounding again in Borel's result, that an entire curve missing four lines in general position is linearly degenerate (see [2] and [5]). According to the heuristic Bloch's principle, a similar statement should hold for non normal sequences of holomorphic discs. It has not been proved until 1928, when H. Cartan perfected a result obtained one year earlier by A. Bloch, stating that a non normal sequence of holomorphic discs avoiding four lines in general position has a subsequence that converges in Hausdorff's sense to a certain divisor (see again [2], [5]).

Recently, the notion of hyperbolicity in an almost complex context has been the subject of several studies (see for examples [6], [7], [10]). Given a smooth almost complex structure J on an even-dimensional manifold M, ie. an automorphism $J : TM \to TM$ such that $J^2 = -Id$, a J-curve is a differentiable map $f : (\Sigma, i) \to (M, J)$, where (Σ, i) is a Riemann surface, such that $df \circ i = J \circ df$. It can be seen as a solution of a generalized Cauchy-Riemann equation. Locally, there are many J-curves: every vector X tangent to M at a point P is tangent to the image of a J-disc $f : \mathbb{D} \to M$. This allows us to define a *Kobayashi-Royden pseudometric*:

$$\forall P \in M,\ \forall X \in T_P M :$$

$$K_P(X) = \inf \left\{ \frac{1}{|\lambda|} \mid \exists f : \mathbb{D} \to M,\ f(0) = P,\ d_0 f(\partial/\partial z) = \lambda X \right\}.$$

The manifold M is called Kobayashi-hyperbolic if K is non-degenerate. When M is compact, Brody's Lemma holds (see [4], [10]) and proves that this notion of hyperbolicity is equivalent to the non-existence of non constant entire J-curves in M, ie. of non constant J-curves $f : \mathbb{C} \to M$.

A formulation of Green's theorem in an almost complex context has been ivestigated by R. Debalme and S. Ivashkovich, and then J. Duval. Given a smooth almost complex structure J on $\mathbb{P}^2(\mathbb{C})$ which is tame by the Fubini-Study 2-form ω, ie. such that $\omega_P(X, J_P X) > 0$ for all point $P \in \mathbb{P}^2(\mathbb{C})$ and all vector $X \in T_P\mathbb{P}^2(\mathbb{C}) \setminus \{0\}$, the triple $(\mathbb{P}^2(\mathbb{C}), \omega, J)$ is called an almost complex projective plane. In this context, a compact 2-dimensional almost complex submanifold diffeomorphic to $\mathbb{P}^1(\mathbb{C})$ which has degree one in homology is called a J-line. These J-lines are the analogue, in the almost complex projective plane, of projective lines, and M. Gromov prooed that there are many of them (see [9]). More precisely, the the set of all J-lines can be given a structure of smooth manifold

of real dimension 4, diffeomorphic to $\mathbb{P}^2(\mathbb{C})$, called the *dual space* of $(\mathbb{P}^2(\mathbb{C}), J)$. Moreover, two distinct points in $\mathbb{P}^2(\mathbb{C})$ are contained in a unique J-line, and given any point P in $\mathbb{P}^2(\mathbb{C})$, a non-zero vector of $T_P\mathbb{P}^2(\mathbb{C})$ is tangent to a unique J-line. The set of all J-lines containing the point P is a submaniflod of the dual space, diffeomorphic to $\mathbb{P}^1(\mathbb{C})$, and is called a *pencil* of J-lines.
The existence of such pencils allows us to define, for any point $P \in \mathbb{P}^2(\mathbb{C})$, the *almost complex blow-up* of $(\mathbb{P}^2(\mathbb{C}), J)$ and a central projection $\pi_P : \mathbb{P}^2(\mathbb{C}) \setminus \{P\} \to \mathbb{P}^1(\mathbb{C})$. If $f : \mathbb{D} \to \mathbb{P}^2(\mathbb{C}) \setminus \{P\}$ is a J-holomorphic disc missing the point P, the map: $\pi_P \circ f : \mathbb{D} \to \mathbb{P}^1(\mathbb{C})$ may not be holomorphic, but it is always *quasiconformal*, which is almost as good.

It seems then rather natural to study the hyperbolicity of the complementary set of a five or four J-lines in general position (ie. such that three of them never intersect in one point). A first step has been made by R. Debalme and S. Ivashkovich, who proved that the property "$(\mathbb{P}^2(\mathbb{C}) \setminus C, J)$ is hyperbolic" (in both Kobayashi's ans Brody's senses) is open in the spaces of tame almost complex structures J and configurations C of five J-lines (see [6]).

Theorem 1. *The set*

$$\mathcal{M} = \{(J, \{L_K, 1 \leq k \leq 5\}) \mid J \text{ is tamed by } \omega, \text{ the } J\text{-lines } L_k \text{ are in general position}\}$$

may be given a structure of Banach manifold. The subset

$$\mathcal{H} = \{(J, \{L_k, 1 \leq k \leq 5\}) \mid (\mathbb{P}^2(\mathbb{C}) \setminus \bigcup_{1\leq k\leq 5} L_k, J) \text{ is hyperbolically embbeded in } (\mathbb{P}^2(\mathbb{C}), J)\}$$

is a non-empty open subset of $\mathcal{M}$.

More recently, J. Duval proved that Green's theorem is actually always true in an almost complex projective plane (see [7]):

Theorem 2. *Let be* $C = \bigcup_{1\leq j\leq 5}$ *a configuration of five J-lines L_j in general position. A J-curve* $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ *missing these five J-lines must be constant. Therefore,* $\mathbb{P}^2(\mathbb{C}) \setminus C$ *is hyperbolically embedded in* $(\mathbb{P}^2(\mathbb{C}), J)$.

It seems natural to investigate the hyperbolicity of the complementary set of four J-lines. Of course, stricly speeking, the complementary set of four J-line is never hyperbolic, since it contains non constant entire J-curves. But theese J-curves are degenarate, like in the standard complex projective plane: this is an almost complex Borel's theorem.

Theorem 3. *Let* $C = L_1 \cup L_2 \cup L_3 \cup L_4$ *a configuration o four J-lines in general position. The diagonal divisor Δ is defined as the union of the three J-lines Δ_1, Δ_2 and Δ_3 which intersects C in only two points. Then if* $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}) \setminus C$ *is an entire J-line avoiding C, its image $f(\mathbb{C})$ is contained in Δ.*

Bloch's theorem is the counterpart of Borel's for sequences of J-discs, and is generally considered much deeper.

Theorem 4. *If for $n \in \mathbb{N}$, $f_n : \mathbb{D} \to \mathbb{P}^2(\mathbb{C}) \setminus C$ is a J-disc in $\mathbb{P}^2(\mathbb{C})$ avoiding the configuration C, then the following alternative is verified:*

(1) *The sequence (f_n) is normal (i.e. every subsequence of (f_n) has a subsequence that converges to a J-disc uniformly on compact sets).*

(2) *For all $r \in]0,1[$, the sequence (f_n) has a subsequence (f_{n_p}) such that $f_{n_p}(D(0,r))$ converges, in Hausdorff's sense, to the diagonal divisor Δ.*

As a consequence, the zero set of the Kobayashi-Royden's pseudo metric of the complementary set of the four J-lines L_j is the diagonal divisor Δ. More precisely, $\mathbb{P}^2(\mathbb{C}) \setminus \bigcup_{1 \le j \le 4} L_j$ is hyperbolically embedded in $\mathbb{P}^2(\mathbb{C})$ modulo Δ.

These two theorems can be seen as geometric versions of their holomorphic counterparts. To proove them, one must use only geometric tools, such as central projections and blow-ups. The general idea of the proof is to use central projections from the double points of the configuration $\bigcup_{1 \le j \le 4} L_j$, in order to work not with J-curves in $\mathbb{P}^2(\mathbb{C})$ but with quasiconformal maps from $\mathbb{C}$ to $\mathbb{P}^1(\mathbb{C})$. This class of functions is very convenient, since *Ahlfors' theory of covering surfaces* gives rise to a value distribution theory for quasiconformal maps. This theory is the second main tool of the proof. Therefore, Borel's and Bloch's theorems can be reduced to simple expressions of elementary geometric properties of the projective plane and to Ahlfors' theory.

Suppose given, for example, an entire curve $f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ missing four lines L_j in general position. The currents $[f(D(0,r))]$ of integration over the image by f of the disc $D(0,r)$ can be used to build a positive closed current T, obtained as a limit of currents of the form $\left(\frac{T_{f,R}}{T_{f,R}(\omega)}\right)$, where $T_{f,R}$ is Nevanlinna's characteristic current:

$$T_{f,R} = \int_0^R [f(D(0,r))] \, \frac{dr}{r}.$$

Using central projections and Ahlfors' theory, it can be proved that the current T is singular, supported by the diagonal divisor. This can be seen as a weak formulation of Borel's theorem. Then, blowing up the almost complex plane $(\mathbb{P}^2(\mathbb{C}), J)$ at the double points of the configuration $C = \bigcup_{1 \le j \le 4} L_j$, it can be proved that the image of the entire J-curve f is contained in the diagonal divisor.

References

[1] M. Audin, J. Lafontaine ed., *Holomorphic curves in symplectic geometry*, Progress in Math., vol 117, Birkhä user, Basel, 1994.

[2] A. Bloch, *Sur les systèmes de fonctions holomorphes à variétés lacunaires*, Ann. Sci. Ecole. Norm. Sup. **43** (1926), 309-362.

[3] E. Borel, *Sur les zéros des fonctions entières*, Acta. Math. **20** (1897), 357-396

[4] R. Brody, *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213-219.

[5] H. Cartan, *Sur les systèmes de fonctions holomorphes à variétés lacunaires*, Ann. Sci. Ecole. Norm. Sup. **45** (1928), 255-346.
[6] R. Debalme and S. Ivashkovich, *Complete hyperbolic neighborhoods in almost complex surfaces*, Int. J. of Math. **12** (2001), 211-221.
[7] J. Duval, *Un théorème de Green presque complexe*, Annales de l'institut Fourier **54** no 7 (2004), 2357-2367.
[8] M. Green, *Some Picard theorems for holomorphic maps to algebraic varieties*, Amer. J. Math. **97** (1975), 43-75.
[9] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Ivent.Math. **83** (1985), 3007-347.
[10] B. Kruglikov and M. Overholt, *Pseudoholomorphic mappings and Kobayashi hyperbolicity*, Diff. Geom. Appl. **11** (1999), 265-277.
[11] J.-C. Sikorav, *Dual elliptic Planes*, preprint 2000 arXiv math.SG/008234.

Classification of systems of orthogonal polynomials in two variables

Stepan Orevkov
(joint work with D. Bakry and M. Zani)

For $n = 2$ we give a complete solution to the following problem: find all triples (D, ρ, g) where $D = \operatorname{Int} \overline{D}$ is a domain in $\mathbb{R}^n$, ρ is a smooth positive function in D, and $g = (g_{ij})$ a smooth metric in D such that the second order differential operator

$$L(f) = \frac{1}{\rho} \sum_{i,j} \partial_i \rho g^{ij} \partial_j f$$

satisfies the following conditions (here $x = (x^1, \dots, x^n)$ and $dx = dx^1 \dots dx^n$):

(1) $\mathbb{R}[x] \subset L_1(D, \rho\, dx)$ and $\mathbb{R}[x]$ is dense in $L_2(D, \rho\, dx)$;
(2) for any k, the space $\mathbb{R}[x]_k := \{P \in \mathbb{R}[x] \mid \deg P \leq k\}$ is invariant under L;
(3) L is symmetric, i.e., $\int_D P\, LQ\, \rho\, dx = \int_D Q\, LP\, \rho\, dx$ for any $P, Q \in \mathbb{R}[x]$.

In this case, the eigenfunctions of L are the orthogonal polynomials.

Given a domain D, one can easily find ρ and g by solving simultaneous linear equations. The problem is to determine all domains for which a solution exists.

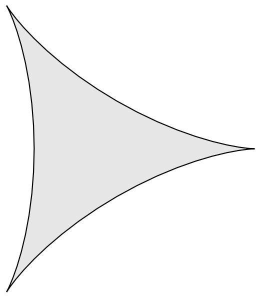
For $n = 1$, a solution always exists. It gives Jacobi polynomials for $D =]-1, 1[$, Laguerre polynomials for $D = \mathbb{R}_+$, and Hermite polynomials for $D = \mathbb{R}$.

A complete list of domains admitting a system of orthogonal polynomials for $n = 2$ is as follows.

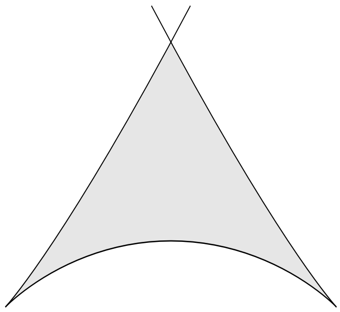
Noncompact domains:

$$\{x^3 > y^2\}, \qquad \{y > x^2\}, \qquad I_1 \times I_2, \quad I_1 =]0,1[,\ \mathbb{R}_+,\ \mathbb{R}, \quad I_2 = \mathbb{R}_+,\ \mathbb{R}$$

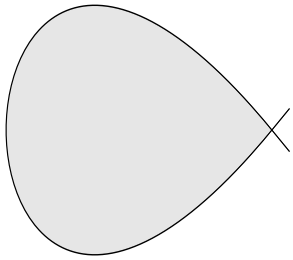
Compact domains:



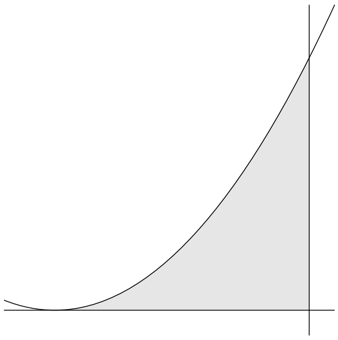
Deltoid
(3-hypocicliod)



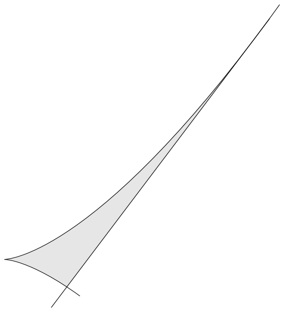
Swallow tail
(dual of $\{y = x^4 - x^2\}$)



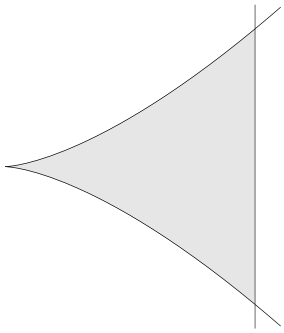
Nodal cubic
$\{y^2 = x^3 - x^2\}$



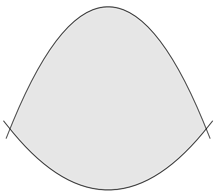
parabola with
tangent and axis



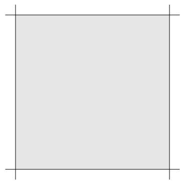
cuspidal cubic
with tangent line



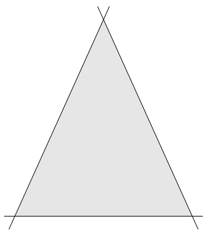
cuspidal cubic
ith bisecant



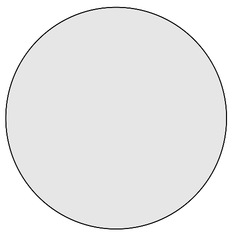
parabolic lens
(2 coaxial parabolas)



rectangle



triangle



circle

All solutions are rigid up to affine linear transformations except the parabolic lens which depends on one real parameter.

Idea of the proof. The condition of the invariance of $\mathbb{R}[x]_k$ under L implies $g^{ij} \in \mathbb{R}[x]_2$. The symmetricity condition combined with Stokes' formula implies

$$\sum_i g^{ij} \partial_j F = 0, \quad i = 1, \dots, n, \tag{$*$}$$

on the bounfary bD of D where $F = 0$ is a local equation of bD. In particular, $\Delta := \det g^{ij}$ vanishes on bD. Hence, D is bounded by a fourth degree curve $C = \{\Delta = 0\}$ where Δ.

Let $n = 2$. Given a power series expansion of Δ at any point p of C, the condition $(*)$ provides a system of simultaneous linear equations on the coefficients of g^{ij}. It can be easily checked that this system has no solution under certain local conditions on C at p, in particular, if C has a flex point in $\mathbb{C}^2$, a flex point at the infinity with the tangent line different from the line at infinity and in some other cases.

Combining the local restrictions on C with Plücker formulas for the projectively dual curve, we obtain a short list of a priori possible domains. Analysing these domains one by one, we select those ones for which g^{ij} can be found.

It remains to find ρ. It is always of the form

$$\rho = \exp(h) \prod \Delta_i^{a_i}, \qquad h \in \mathbb{R}[x], \quad \deg h = 2n - \sum \deg \Delta_i$$

where $\prod \Delta_i$ is the facorization of Δ and a_i any constants.

Runge approximation for pseudo-holomorphic maps

Antoine Gournay

The Runge approximation theorem for holomorphic maps ($U \to \mathbb{C}$) is a fundamental result in complex analysis. We present here a similar result for (pseudo-) holomorphic maps from a compact Riemann surface to a compact (almost-) complex manifold M under certain (strong) assumptions. Though the setting is definitively that of pseudo-holomorphic maps, it also covers some complex varieties.

Basic concepts. A manifold M of even real dimension is said to be almost complex when it is endowed with a section $J \in \mathrm{End}\mathrm{T}M$ such that $\forall x \in M$, $J_x^2 = -\mathrm{Id}_{\mathrm{T}_x M}$. Complex multiplication gives rise to such a structure, and when M is of real dimension 2 an almost complex structure is a complex structure (as can be seen from the vanishing of the Nijenhuis tensor). M will be assumed compact and Σ will denote a compact Riemann surface whose complex structure will be written j.

A map $u : \Sigma \to (M, J)$ will be said pseudo-holomorphic or J-holomorphic if $du \circ j = J \circ du$, or, equivalently, if

$$\forall v \in \mathrm{T}_z\Sigma, \qquad \bar{\partial}_J u(v) := \frac{1}{2}(du_z(v) + J_{u(z)} \circ du_z \circ j_z(v)) = 0.$$

Problem. The Runge approximation problem can, in this setting, be formulated as follows: given a J-holomorphic map $f : U \to (M, J)$ for U an open subset of Σ, a compact $K \subset U$, some small $\delta \in \mathbb{R}_{>0}$, under which conditions is it possible to find a J-holomorphic map $h : \Sigma \to (M, J)$ such that $\|h - f\|_{C^0} < \delta$?

Let us say that Runge approximation holds for $f : U \to (M, J)$ (where $U \subset \Sigma$) if the above question has a positive answer.

Though the interest of the problem lies in the fact that h is defined on the whole of Σ, this in not actually so much an extension result (which is in general impossible even for holomorphic maps $\mathbb{C} \to \mathbb{C}$) as an approximation result (whence the name). But even then, there are choices of (M, J) and Σ where it is impossible (see below).

The subject matter of this article is to show that under certain assumptions on (M, J), the aforementioned question has a positive answer for any Σ.

Assumption. Let a map u *realize the tangent* $v \in \mathrm{T}_m M$ if v is in the image of the differential, $u(z) = m$ and $\bar{\partial}_J u_z = 0$, or, as expressed in local charts, if it can be written as $vz + O(|z|^2)$ (see Sikorav's characterization of local behavior in [9, Proposition 3]). Obviously, if there is a pseudo-holomorphic map $\mathbb{C}\mathrm{P}^1 \to (M, J)$ realizing v then, $\forall \lambda \in \mathbb{C}$, there is a map realizing λv. Denote by $\mathrm{S}M$ the unit tangent bundle of M.

Furthermore, the almost complex structure has to be assumed regular (as described in McDuff and Salamon's book [7, Theorem 3.1.5]). Regularity is important to ensure that the linearization of the $\bar{\partial}$ operator at a pseudo-holomorphic curve ($\mathbb{C}\mathrm{P}^1 \to (M, J)$) is surjective, hence invertible. If this is not assumed, then each grafting might generate additional problem. From an algebraic viewpoint, this implies that fusion of rational curves (the construction which to two curves $x = 0$ and $y = 0$ associates the curve $xy = \epsilon$) is possible.

Theorem 1: *Let (M, J) be an almost complex manifold. Assume $J \in \mathsf{C}^{1,1}$ and is regular. Assume there is a dense set $R \subset \mathrm{S}M$ such that all $v \in R$ is realized by a pseudo-holomorphic map $u : \mathbb{C}\mathrm{P}^1 \to (M, J)$. Then the Runge approximation holds for $f : U \to (M, J)$ (where $U \subset \Sigma$) if f can be extended C^0 to Σ.*

It is worth noting that the hypothesis of theorem 1 are as minimal as can be reasonably expected.

Proposition 2: *Let Σ be a Riemann surface. Suppose that J is Lipschitz. Assume that Runge approximation holds for map $f : D_r \to (M, J)$ (for discs D_r which can be seen as open subsets of Σ). Then there exists a dense subset $R \subset \mathrm{S}M$, so that $\forall v \in R$ there is a J-holomorphic map $g_v : \mathbb{C}\mathrm{P}^1 \to M$ realizing the tangent v.*

Specializing this proposition at $\Sigma = \mathbb{C}\mathrm{P}^1$ shows that the hypothesis of theorem 1 only differ by the assumptions on J with the minimal possible hypothesis.

Examples. A simple example in which the hypothesis in theorem 1 are easily verified is $M = \mathbb{C}\mathrm{P}^n$ with its usual complex structure (note that the classical Runge theorem may, of course, directly be applied in this case). The same can be said of product of projective spaces. If the complex structure is not standard but is still tamed by the standard symplectic form on $\mathbb{C}\mathrm{P}^n$ then theorem 1 holds (by Gromov's results [5]).

On the other hand, $M = \mathbb{T}^n$ with their usual complex structures are clearly cases where it fails, as there can be no holomorphic maps from $\mathbb{C}\mathrm{P}^1 \to \mathbb{T}^n$. In this particular example, this is not only that the hypothesis of Theorem 1 cannot be fulfilled. The Runge approximation in $\mathbb{T}^n$ cannot exist for $\Sigma = \mathbb{C}\mathrm{P}^1$; it could however still be true for other Riemann surfaces Σ, *e.g.* $\Sigma = \mathbb{T}^1$.

The conditions of Theorem 1) also hold in a Grassmanian $\mathscr{G}(k, E)$. Indeed $\mathrm{T}_A \mathscr{G} \simeq \mathrm{Hom}(A, B)$ for B a supplement of $A = [a_1 \wedge \ldots \wedge a_k] \in \mathscr{G}$. For $p \in \mathrm{Hom}(A, B)$, the map $u : z \mapsto \big[\big(a_1 + zp(a_1)\big) \wedge \ldots \wedge \big(a_k + zp(a_k)\big)\big]$ extends to $\mathbb{C}\mathrm{P}^1$ and realize the tangent p.

For a more general approach to complex varieties that will satisfy the assumptions, see Debarre's book [3, Chapter 4], more precisely the part about rationally connected varieties. As such, if there is a "very free" curve (see [3, Definition 4.5]; this is equivalent to being rationally connected, see [3, Definition 4.3 and Corollary 4.17]) in an algebraic compact variety, the conditions will also hold. Over $\mathbb{C}$, rationally connected varieties are exactly those for which a general pair of points (outside a subvariety of codimension at least 2) can be joined by a rational curve.

Applications. Compactified moduli spaces of curves of genus g (we speak of the Deligne-Mumford compactification), $\overline{\mathcal{M}}_g$, are unirational when $g \leq 14$, and rationally connected for $g \leq 15$. As a consequence theorem 1 will apply for these spaces. However, if $g \geq 24$, the moduli space is then of general type (see the survey of Farkas [4] on the topic).

A case of interest for the application of theorem 1 are Lefschetz fibrations; this idea is due to S. Donaldson. The aim is to partially recover the results of Auroux (see [1] and [2]) and Siebert-Tian [8]. A fibration $p : V \to \mathbb{C}\mathrm{P}^1$ can be seen in terms of its classifying map $\mathbb{C}\mathrm{P}^1 \to \mathcal{M}_2^c$ where $\overline{\mathcal{M}}_2$, the moduli space of genus 2 curves, is (almost-)smooth and complex (actually Kählerian). In this context, the Runge theorem 1 applies: as mentioned above $\overline{\mathcal{M}}_2$ satisfies the hypothesis. Taking $U = \emptyset$, one gets that any Lefschetz fibrations becomes, after sufficiently many fibred sum (stabilization) and a small deformation, holomorphic. Thence

Corollary 3: *Let $p : M \to \mathbb{C}\mathrm{P}^1$ be a genus $g \leq 15$ differentiable Lefschetz fibration. Then, after fiber sum with sufficiently many copies of some holomorphic Lefschetz fibrations (*a.k.a. *stabilization), it becomes isomorphic to a holomorphic Lefschetz fibration.*

This is perhaps even more striking in view of [2]. Indeed, Auroux's method do not require any hypothesis on the genus of the surface; the methods are in fact much more direct (the "universal" fibration f_g^0 is quite explicit). This could, perhaps, either hint at the fact that there might be a dense set of tangents realized by rational curves in $\overline{\mathcal{M}}_g$, while this space remains of generic type or that it could be possible to restrict the problem on a part of $\overline{\mathcal{M}}_g$ having this property.

In the classical Runge theorem, the number of poles of the approximating map is related to the topology of the set U. Unfortunately, the notion of a pole does not have a meaning in the compact setting. What will obviously happen however is that one expects that the energy (the L^2 norm of the differential) of the approximating map may be very big. A consequence of Taubes result [10, Theorem 1.1] is that the minimal number of necessary connect sums of $\overline{\mathbb{C}\mathrm{P}^2}$ required to make a metric structure anti-self-dual is defined. It is an invariant of the conformal metric, but not a simple one to compute (LeBrun and Singer [6, §1] gave a bound of 14 in the case of $\mathbb{C}\mathrm{P}^2$ with its usual metric). Though again probably not an easy question to answer, it would, in the context of the present article, be interesting to look for the minimal energy of a J-holomorphic map realizing a given approximation.

References

[1] D. Auroux. Fiber sums of genus 2 Lefschetz fibrations. *Turkish J. Math.*, 27(1):1–10, 2003.

[2] D. Auroux. A stable classification of Lefschetz fibrations. *Geom. Topol.*, 9:203–217 (electronic), 2005.
[3] O. Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.
[4] G. Farkas. The global geometry of the moduli space of curves. (English summary) *Algebraic geometry—Seattle 2005*. Part 1, 125–147. *Proc. Sympos. Pure Math.*, 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.
[5] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2), 307–347, 1985.
[6] C. LeBrun and M. Singer. A Kummer-type construction of self-dual 4-manifolds. Math. Ann., 300(1):165–180, 1994.
[7] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
[8] B. Siebert and G. Tian. On the holomorphicity of genus two Lefschetz fibrations. *Ann. of Math.*, 161(2):959–1020, 2005.
[9] J.-C. Sikorav. Singularities of J-holomorphic curves. *Math. Z.*, 226(3):359–373, 1997.
[10] C. H. Taubes. The existence of anti-self-dual conformal structures. *J. Differential Geom.*, 36(1):163–253, 1992.

Radon transform on complex projective varieties and applications

Gennadi Henkin

Radon type transforms on complex projective varieties were introduced in different forms and with different purposes in the works of Fantappiè (1943), Martineau (1962), Andreotti, Norguet (1967, 1971), Penrose (1969, 1977), Gindikin, Henkin (1978),...

This report gives survey of recent works in the theory of complex Radon transforms.

1. In the work (Henkin, Polyakov, arXiv, dec. 2010) we have shown that complex Radon transform realizes isomorphism between the space of residual $\bar{\partial}$-cohomologies of algebraic (not necessary reduced) locally complete intersections in a linearly concave domain of $\mathbb{C}P^n$ and the space of holomorphic solutions of the associated homogenous system of linear differential equations with constant coefficients in a dual domain of $(\mathbb{C}P^n)^*$.

2. In addition, for algebraic complete intersection V in a linearly concave polyhedral domain $D \subset \mathbb{C}P^n$ we have constructed an explicit inversion formula for Radon transform on V, which implies an explicit formulas for solutions of natural boundary value problems for the associated system of differential equations in a dual domain $D^* \subset (\mathbb{C}P^n)^*$. This formula develops an "explicit fundamental principle" of Berndtsson, Passare (1989).

3. These results can be applied, in particularly,
for characterization in terms of Radon transform of those elements of cohomology space $H^{n,n-1}(\Omega)$ in a linearly concave domain $\Omega \subset \mathbb{C}P^n$, which can be represented by residual currents with support in algebraic subvarieties of Ω,
for extension of inverse Abel theorems, Saint-Donat (1975), Griffiths (1976),

Henkin (1992), Fabre (2007), to the case of datas on arbitrary (not necessary reduced) complex curves in linearly concave domains of $\mathbb{C}P^n$.

1. Introduction and main result.

We consider two related problems: the one is to describe infinite-dimensional spaces of $\bar{\partial}$-cohomologies of subvarieties in linearly concave domains of $\mathbb{C}P^n$ in terms of the spaces of holomorphic solutions of associated systems of differential equations in dual linearly convex domains, the inverse problem is to realize the spaces of holomorphic solutions of systems of linear homogeneous differential equations with constant coefficients on linearly convex domains of $\mathbb{C}P^n$ as spaces of $\bar{\partial}$-cohomologies of associated subvarieties of dual linearly concave domains.

The investigation of these problems was started by Martineau (1962).

The main result of Martineau was interpretated in Gindikin, Henkin (1978) as the statement about isomorphism by complex Radon transform between the space of $(n, n-1)$ $\bar{\partial}$-cohomology of a linearly concave domain $D \subset \mathbb{C}P^n$ and the space of holomorphic functions on the dual linearly convex domain $D^* \subset (\mathbb{C}P^n)^*$.

The study of these questions in Henkin, Polyakov (1986) and Henkin (1995) for the case of complex submanifolds in a linearly concave domains of $\mathbb{C}P^n$ brings to the following result.

Let $(z_0, \ldots, z_n)$and $(\xi_0, \ldots, \xi_n)$ be the homogeneous coordinates of points $z \in \mathbb{C}P^n$ and $\xi \in (\mathbb{C}P^n)^*$ such that $\xi \cdot z \overset{\text{def}}{=} \sum\limits_{k=0}^{n} \xi_k \cdot z_k = 0$.

Put $\mathbb{C}P^{n-1}_\xi = \{z \in \mathbb{C}P^n : \ \xi \cdot z = 0\}$.

By linearly concave domain in $\mathbb{C}P^n$ we will call a domain D in $\mathbb{C}P^n$, such that there exists continuous mapping $w \to \xi(w)$, $w \in D$, $\xi(w) \in D^*$ with the property $w \in \mathbb{C}P^{n-1}_{\xi(w)}$ $\forall\ w \in D$.

Theorem 1.1.

Let D be a linearly concave domain in $\mathbb{C}P^n$, $n \geq 2$. Let V be a (n-m)-dimensional algebraic manifold of the form

$$V = \{z \in \mathbb{C}P^n : \ P_1(z) = \ldots = P_r(z) = 0\},$$

where homogeneous polynomials $P_1, \ldots, P_r$ are such that everywhere in V $rank\, [grad\, P_1, \ldots, grad\, P_r] = m$. Let $H^{n-m,n-m-1}(D \cap V)$ denote the cohomology space $H^{n-m-1}(V \cap D, \omega_V)$, where ω_V be canonical bundle on V.

Then:

i) the transform

$$F \mapsto f = \sum_{j=0}^{n} \int\limits_{z \in \mathbb{C}P^{n-1}_\xi \cap V} < \xi dz > \rfloor z_j F,$$

$F \in H^{n-m,n-m-1}(D \cap V)$ determines the linear continuous transform $R:\ H^{n-m,n-m-1}(D \cap V) \to H^{1,0}(P^*)$ with finitedimensional kernel, consisting of restrictions on $D \cap V$ of $\bar{\partial}$-cohomologies from $H^{n-m,n-m-1}(V)$

ii) the transform R has the following image

$$\left\{f \in H^{1,0}(D^*):\ f = d\varphi, \text{where } \varphi \in H^{0,0}(D^*) \text{ and } P_j\left(\frac{\partial}{\partial \xi}\right)\varphi = 0, j = 1, \ldots, r\right\}$$

where $\{P_j\}$ are the homogeneous polynomes defining the manifold V.

Remark. Under condition $n - m = 1$ statement i) of Theorem 1.1 is a consequence of inverse Abel theorem (see Saint-Donat (1975), Griffiths (1976)). Under conditions: $n - m > 1$ and V is complete intersection, statement i) of Theorem 1.1 is a consequence of Theorem 3.3 from Henkin, Polyakov (1986). In the complete form Theorem 1.1 was obtain in Henkin (1995).

The goal of this research is to obtain natural generalization of Theorem 1.1 to the case of arbitrary locally complete intersections in linearly concave domains.

Definition 1.1 (locally complete intersections).

An analytic subvariety $V \subset \mathbb{C}P^n$ is called locally complete intersection subvariety in $\mathbb{C}P^n$ of pure dimension $n - m$, if there exists a finite open cover $\{U_\alpha\}_{\alpha=1}^N$ of $\mathbb{C}P^n$ and collection of holomorphic functions $\{F_k^{(\alpha)}\}$ in U_α such that

$$V \cap U_\alpha = \{z \in U_\alpha :\ F_1^{(\alpha)}(z) = \ldots = F_m^{(\alpha)}(z) = 0\}. \quad (1)$$

In our construction of $\bar\partial$-closed residual currents on a locally complete intersection variety $V \cap D$ we will use conormal bundle $N(V)$ and dualizing bundle ω_V^0 on V.

Definition 1.2 (conormal bundle and dualizing bundle).

Conormal bundle $N(V)$ on V is defined by the nondegenerate holomorphic transition matrices $\Delta_{\alpha\beta}(z) \in H(U_\alpha \cap U_\beta)$ such that

$$F^{(\alpha)}(z) = A_{\alpha\beta}F^{(\beta)}(z) \text{ on } U_\alpha \cap U_\beta, \text{ where } F^{(\alpha)} \text{ is column } {}^t(F_1^{(\alpha)}, \ldots, F_m^{(\alpha)}). \quad (2)$$

By dualizing bundle following Grothendieck (1958) and Hartshorne (1977) we will call line bundle
$\omega_V^0 = \omega_{\mathbb{C}P^n} \times \det\ N(V)^{-1}$, where $\det\ N(V) = \wedge^m N(V)$.

We define further the spaces of residual currents and of residual $\bar\partial$-cohomologies on $V \cap D$, where D is linearly cncave domain in $\mathbb{C}P^n$.

Definition 1.3 (residual currents).

For a subvariety $V \cap D$ of pure dimension $n - m$ locally satisfying (1) develloping Coleff, Herrera (1978) and Passare (1988) we will write that a $\bar\partial$-closed current φ with support in $V \cap D$ is a $\bar\partial$-closed residual current, belonging to $Z^{n-m-1}(V \cap D, \omega_V^0)$, if there exists a finite collection of open neighborhood $\{U_\alpha \subset \mathbb{C}P^n\}_{\alpha=1}^N$ and of differential forms $\Phi_\alpha \in C_{n,n-m-1}^{(\infty)}(U_\alpha \cap D)$ such that $\cup_{\alpha=1}^N U_\alpha \supset V$ and for any (0,1)-form $\psi \in C_{0,1}^{(\infty)}$ with compact support in $U_\alpha \cap D$ we have

$$< \varphi, \psi >= \int\limits_{U_\alpha} \psi \wedge \Phi_\alpha \wedge \bar\partial(\frac{1}{F_1^{(\alpha)}}) \wedge \ldots \wedge \bar\partial(\frac{1}{F_m^{(\alpha)}}) \stackrel{\text{def}}{=} \lim_{t\to 0} \int\limits_{T^\varepsilon_{\{F^{(\alpha)}\}}(t)} \frac{\psi \wedge \Phi_\alpha}{F_1^{(\alpha)} \ldots F_m^{(\alpha)}}, \quad (3)$$

where

$$\bar{\partial}\Phi_\alpha = \sum_{k=1}^{m} F_k^{(\alpha)}\Omega_k^{(\alpha)} \quad \text{on} \;\; U_\alpha \cap D,$$

$$\Phi_\alpha = (\det\, A_{\alpha,\beta})^{-1}\Phi_\beta + \sum_{k=1}^{m} F_k^{(\alpha)}\Omega_k^{(\alpha\beta)} \;\text{on}\quad U_\alpha \cap U_\beta \cap D,$$

$A_{\alpha,\beta}$ are holomorphic matrices from (2),

$$T^{\varepsilon}_{\{F^{(\alpha)}\}}(t) = \{z:\; |F_1^{(\alpha)}(z)| = \varepsilon_1(t), \dots, |F_m^{(\alpha)}(z)| = \varepsilon_m(t)\}$$

is a family of tubular varieties depending on the real parameter t, and the limit in the right-hand side of (3) is taken along an admissible path $\varepsilon = \{\varepsilon_k(t)\}_1^m$ in the sense of Coleff-Herrera (1978), i.e. ε is analytic map $\varepsilon:\; [0,1] \to \mathbb{R}^m$ satisfying the conditions

$$\lim_{t\to 0} \varepsilon_m(t) = 0, \;\; \lim_{t\to 0} \varepsilon_j(t)/\varepsilon^q_{j+1}(t) = 0, \;\; j = 1, \dots, m-1, \; \forall\, q \in \mathbb{Z}. \qquad (3')$$

The collection $\{\Phi_\alpha\}_{\alpha=1}^N$ define a $\bar{\partial}$-closed (n,n-m-1) differential form on $V \cap D$ with coefficients in det $N(V)^{-1}$. Looking very technical condition $(3')$ can not be replaced by simpler condition $\varepsilon_j(t) \to 0$, $t \to 0$, $j = 1, \dots, m$, (see Passare, Tsikh (1996)).

Definition 1.4 (residual $\bar{\partial}$-cohomologies).
For $n - m \geq 2$, a $\bar{\partial}$-closed residual current $\varphi \in Z^{n-m-1}(V \cap D, \omega^0_V)$ is called $\bar{\partial}$-exact with notation $\varphi \in B^{n-m-1}(V \cap D, \omega^0_V)$, if there exists a residual current $g \in Z^{n-m-2}(V \cap D, \omega^0_V)$ such that $\bar{\partial} g = \varphi$ on V. Therefore the spaces of residual $\bar{\partial}$-cohomologies

$$H^{n-m-1}(V \cap D, \omega^0_V) = Z^{n-m-1}(V \cap D, \omega^0_V)/B^{n-m-1}(V \cap D, \omega^0_V) \qquad (4)$$

are well defined if $n - m \geq 2$. For the case $n - m = 1$ we put
$H^0(V \cap D, \omega^0_V) = Z^0(V \cap D, \omega^0_V)$.

Before defining the complex Radon transform we introduce additional notations. We denote by $H^{0,0}(D^*)$ and $H^{1,0}(D^*)$ the spaces of holomorphic functions and respectively holomorphic 1-forms on D^*.

Let V^0 denote the reduced version of the variety V. Let $S^*_{V^0}$ denote the subset of D^*, corresponding to the hyperplanes $\mathbb{C}P^{n-1}_\xi$, having tangency points with $Reg\, V^0$ or having intersection with $Sing\, V^0$. This notation implies that $S^*_{V^0}$ is analytic subset in D^* of dimension less than n and $\forall \xi \in D^* \backslash S^*_{V^0}$ $dim_{\mathbb{C}}(V^0 \cap \mathbb{C}P^{n-1}_\xi) = n - m - 1$.

Definition 1.5 (complex Radon transform).
Let $V \subset \mathbb{C}P^n$ be a locally complete intersection subvariety of pure dimension $n - m$. Then we define Radon transform

$$R_V:\; Z^{n-m-1}(V \cap D, \omega^0_V) \to C^{1,0}(D^* \backslash S^*_{V^0})$$

for $\bar{\partial}$-closed residual currents $\varphi \in Z^{n-m-1}(V \cap D, \omega_V^0)$ by the formula

$$R_V[\varphi](\xi) = (\frac{1}{2\pi i})^{n+1} \sum_{j=0}^{n} (\sum_{\alpha=1}^{N} \int_{V\cap D} \theta_\alpha(z) z_j \Phi_\alpha \bar{\partial} \frac{1}{<\xi, z>} \wedge \bar{\partial} \frac{1}{F_1^{(\alpha)}} \wedge \ldots \wedge \bar{\partial} \frac{1}{F_m^{(\alpha)}}) d\xi_j, \tag{5}$$

where $\{\theta_\alpha\}_1^N$ is a partition of unity subordinate to a cover $\{U_\alpha\}_1^N$ of D by open subdomains in $\mathbb{C}P^n$, $\xi \in D^* \backslash S^*_{V^0}$, the forms $\Phi_\alpha \wedge \bar{\partial} \frac{1}{F_1^{(\alpha)}} \wedge \ldots \wedge \bar{\partial} \frac{1}{F_m^{(\alpha)}}$ are the local representatives of the current φ.

The main result. Theorem 1.2. (Henkin, Polyakov, arXiv, dec.2010)
Let $V = \{z \in \mathbb{C}P^n : \ P_1(z) = \ldots = P_r(z) = 0\}$ be a locally complete intersection subvariety of pure dimension (n-m), defined by the homogeneous polynomials $\{P_k\}_{k=1}^r$, $r \geq m$. Let $D \subset \mathbb{C}P^n$ be a linearly concave domain and D^* be a dual domain. Then the Radon transform R_V defined by (5) induces a continuous linear operator on the space of cohomologies:

$$R_V : \ H^{0,n-m-1}(V \cap D, \omega_V^0) \to H^{1,0}(D^*).$$

Moreover, the following properties are satisfied:
1) the $Ker\, R_V$ is the finite-dimensional subspace in $H^{0,n-m-1}(V \cap D, \omega_V^0)$, consisting of restrictions to $V \cap D$ of classes of cohomologies from $H^{0,n-m-1}(V, \omega_V^0)$
ii) the image R_V is the following subspace in $H^{1,0}(D^*)$:

$$\{f \in H^{1,0}(D^*) : f = dg \text{ with } g \in H^{0,0}(D^*) \text{ such that } P_j(\frac{\partial}{\partial \xi})g = 0, j = 1, ..., m\}.$$

Remark. Under condition $m = n-1$ the statement i) of Theorem 1.2 is a consequence of result of B.Fabre (2007).

On the effective membership problem on algebraic varieties

MATS ANDERSSON
(joint work with E. Wulcan)

Let X be a, not necessarily smooth, n-dimensional subvariety of $\mathbb{P}^N$ and let $X_{aff} = X \cap \mathbb{C}^N$ be the affine part. If $F_1, \ldots, F_m$ are polynomials on X_{aff} of degree (at most) d with no common zeros on X_{aff}, then one can find polynomials Q_j such that $F_1Q_1 + \cdots + F_mQ_m = 1$ on X_{aff} and

$$\deg(F_jQ_j) \leq c_m d^\mu \deg X,$$

where $c_m = 1$ if $m \leq n$, $c_m = 2$ if $m > n$, and

$$\mu := \min(m, n).$$

This estimate was proved by Kollár, [13], in case $X = \mathbb{P}^n$ (even without the factor 2), and by Jelonek, [12], in general, and it is (almost) optimal.

For any given polynomials F_j of degree (at most) d and a polynomial Φ in the ideal (F_j) generated by F_j the best degree estimate of a solution

$$F_1Q_1 + \cdots + F_mQ_m = \Phi \tag{1}$$

is doubly exponential in d, i.e., like d^{2^n}. However, in more special situations one can obtain sharper results.

In an ongoing joint work with E. Wulcan, [7], we study global division problems on algebraic varieties, and obtain generalizations to singular varieties of various results previously known for smooth varieties, due to Hickel, Ein-Lazarsfeld, and others. For instance we have:

Theorem 1. *Assume that X is an n-dimensional projective subvariety of $\mathbb{P}^N$ and let $X_{aff} = \mathbb{P}^N \cap \mathbb{C}^N$.*
There exists a number μ_0 such that if $F_1, \ldots, F_m$ are polynomials of degree $\leq d$ on X_{aff} and Φ is a polynomial such that

$$|\Phi| \leq C|F|^{\mu+\mu_0} \tag{2}$$

locally on X_{aff}, then one can solve (1) *on X_{aff} with*

$$\deg(F_jQ_j) \leq \max\big(\deg\Phi + (\mu+\mu_0)d^{c_\infty}\deg X, \beta). \tag{3}$$

Here $\mu = \min(m,n)$, c_∞ is a number that depends on the common zero set Z_∞ of F_j at infinity, it always holds that $c_\infty \leq \mu$. Moreover, β is a number that is usually small compared to the first entry. If F_j have no common zeros at all at infinity, then $c_\infty = -\infty$, and in that case we get an extension of the classical results of Max Noether and Macaulay. If X is smooth one can take $\mu_0 = 0$, and if $X = \mathbb{P}^n$, one then gets back the theorem of Hickel in [10].

This result can be seen as a global Briançon-Skoda-Huneke theorem, see below.

We also have a more abstract variant, generalizing the effective Nullstellensatz of Ein-Lazarsfeld in [9] to a non-smooth X. Recall that if $L \to X$ is an ample line bundle then there is a (smallest) number ν_L such that $H^i(X, L^{\otimes s}) = 0$ for $i \geq 1$ and $s \geq \nu_L$. (If X is smooth then, in view of Kodaira's theorem, ν_L is less than or equal to the least number ν such that $L^\nu \otimes K_X^{-1}$ is positive.)

Theorem 2. *Let X be a projective variety. There is a number μ_0, only depending on X, such that the following holds:*

Let $f_1, \ldots, f_m$ be global holomorphic sections of an ample line bundle $L \to X$, and let ϕ be a section of $L^{\otimes s}$, where $s \geq \nu_L + \min(m, n+1)$. If

$$|\phi| \leq C|f|^{\mu+\mu_0} \tag{4}$$

on X, then there are holomorphic sections q_j of $L^{\otimes(s-1)}$ such that

$$f_1q_1 + \cdots + f_mq_m = \phi. \tag{5}$$

If X is smooth one can take μ_0 and then one gets back the corresponding result in [9].

To begin with, via homogenization one can reformulate Theorem 1 on a similar form as Theorem 2, with L as the pull-back to X of $\mathcal{O}(d)_{\mathbb{P}^N} \to \mathbb{P}^N$. The starting point for the proof of Theorem 2 is the geometric set-up introduced in [2] to solve global membership problems by means of residue theory. As long as X is smooth, the proof is reduced to two things: The first one is to show that the right hand side ϕ annihilates a certain residue current obtained from the generators f_j. One then obtains the desired global holomorphic solution q_j after solving a sequence of $\bar\partial$-equations on X.

However, this approach breaks down when X is not smooth because the residue current must be defined in an embedding of X in a smooth manifold Y. Moreover, we cannot assume in general that f_j and ϕ admit holomorphic extensions to Y so we must introduce a surrogate for that. Come so far we then have to use some recent ideas to show that the residue current that appears in Y is annihilated by ϕ. By a regularization principle the problem then boils down to a $\bar\partial$-problem on X itself that can be solved. To perform these steps we make use of some quite recent results on residue theory:

Together with Wulcan we introduced in [5], to any ideal sheaf $\mathcal{J}$, a (vector-valued) residue current $R^{\mathcal{J}}$, such that the annihilator ideal of $R^{\mathcal{J}}$, i.e., the ideal of holomorphic functions ϕ such that the current $\phi R^{\mathcal{J}}$ vanishes, is precisely the ideal $\mathcal{J}$ itself. This is a generalization of the classical duality principle for a complete intersection ideal, due to Passare and Dickenstein-Sessa.

With Wulcan, [6], we also introduced the class (sheaf) of pseudomeromorphic currents; this class includes all principal value currents like $1/f$ where f is holomorphic as well as $\bar\partial(1/f)$, it is closed under multiplication with smooth forms and under push-forwards by proper mappings. In particular all Coleff-Herrera currents and the current $R^{\mathcal{J}}$ above are pseudomeromorphic. There is a very useful analogue for pseudomeromorphic currents of the dimension principle for normal (p,p)-currents:

If μ is a pseudomeromorphic current of bidegree $(*,p)$ that has support on a variety of codimension greater than p, then μ must vanish.

By means of this residue theory we (together with Samuelsson and Sznajdman) found, [4], an analytic proof of the Briançon-Skoda-Huneke theorem, [11]:

Let X be a germ of an analytic space. Then there is a number $\mu_0 = \mu_0(X)$ such that the following holds: If $a \subset \mathcal{O}^X$ is an ideal and $\phi \in \mathcal{O}^X$, then

$$|\phi| \leq C|a|^{\ell+1-\mu_0}$$

implies that $\phi \in a^\ell$.

Together with Samuelsson we have combined the residue theory in [5], [6], with integral formulas, [1], and obtained semiglobal Koppelman formulas for $\bar\partial$ on an analytic space X. By means of such formulas we have found sheaves $\mathcal{A}_q$ of $(0,q)$-currents on X with the following properties: $\mathcal{A}_q$ coincide with the sheaves of smooth $(0,q)$-forms on the regular part X_{reg} of X, $\mathcal{A}_q$ is closed under multiplication by smooth $(0,*)$-forms, and the sequence

$$0 \to \mathcal{O}^X \to \mathcal{A}_0 \xrightarrow{\bar\partial} \mathcal{A}_1 \xrightarrow{\bar\partial} \mathcal{A}_2 \xrightarrow{\bar\partial}$$

is exact, i.e., it is a fine resolution of $\mathcal{O}^X$. As a consequence we get a generalization of the classical Dolbeault isomorphism to a singular space, [3].

References

[1] M. Andersson, *Integral representation with weights I*, Math. Ann. **326** (2003), 1-18.

[2] M. Andersson, *The membership problem for polynomial ideals in terms of residue currents*, Ann. Inst. Fourier **56** (2006), 101–119.

[3] M. Andersson & H. Samuelsson, *A Dolbealt-Grothendieck lemma on a complex space via Koppelman formulas*, arXiv:1010.6142.

[4] M. Andersson & H. Samuelsson & J. Sznajdman, *On the Briançon-Skoda theorem on a singular variety*, Ann. Inst. Fourier **60** (2010), 417–432.

[5] M. Andersson, E. Wulcan, *Residue currents with prescribed annihilator ideals*, Ann. Sci. École Norm. Sup. **40** (2007), 985–1007.

[6] M. Andersson & E. Wulcan, *Decomposition of residue currents*, J. Reine Angew. Math. **638** (2010), 103–118.

[7] M. Andersson & E. Wulcan, *On the membership problem on algebraic varieties*, In preparation.

[8] J. Briançon & H. Skoda, *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de* $\mathbb{C}^n$, C. R. Acad. Sci. Paris Sér. A 278 (1974), 949–951.

[9] L. Ein & R. Lazarsfeld, *A geometric effective Nullstellensatz*, Invent. Math. **135** (1999), 427–448.

[10] M. Hickel, *Solution d'une conjecture de C. Berenstein-A. Yger et invariants de contact à l'infini*, Ann. Inst. Fourier **51** (2001), 707–744.

[11] C. Huneke, *Uniform bounds in Noetherian rings*, Invent. Math. **107** (1992) 203–223.

[12] Z. Jelonek, *On the effective Nullstellensatz*, Invent. Math. **162** 1–17 (2005).

[13] J. Kollár, *Sharp effective Nullstellensatz*, J. American Math. Soc. **1** (1988), 963–975.

[14] F.S. Macaulay, *The algebraic theory of modular systems*, Cambridge Univ. Press, Cambridge 1916.

[15] M. Nöther, *Über einen Satz aus der Theorie der algebraischen Functionen*, Math. Ann. (1873), 351–359.

Extension of plurisubharmonic functions with growth control

Dan Coman

(joint work with V. Guedj and A. Zeriahi)

Let $X \subset \mathbb{C}^n$ be a (closed) analytic subvariety. In the case when X is smooth it is well known that a plurisubharmonic (psh) function on X extends to a psh function on $\mathbb{C}^n$ [5]. Using different methods, Colţoiu generalized this result to the case when X is singular [2]. We recall that a function $\varphi : X \to [-\infty, +\infty)$ is called psh if $\varphi \not\equiv -\infty$ on X and if every point $z \in X$ has a neighborhood U in $\mathbb{C}^n$ so that φ extends to a psh function on U.

Following Colţoiu's approach we show here that it is possible to obtain extensions with global growth control:

Theorem A. *Let X be an analytic subvariety of a Stein manifold M and let φ be a psh function on X. Assume that u is a continuous psh exhaustion function on M so that $\varphi(z) < u(z)$ for all $z \in X$. Then for every $c > 1$ there exists a psh function $\psi = \psi_c$ on M so that $\psi|_X = \varphi$ and $\psi(z) < c \max\{u(z), 0\}$ for all $z \in M$.*

We then look at a similar problem on a compact Kähler manifold V. Given a Kähler form ω, let

$$PSH(V,\omega) = \{\varphi \in L^1(V,[-\infty,+\infty)) : \varphi \text{ upper semicontinuous, } dd^c\varphi \geq -\omega\}$$

denote the set of ω-plurisubharmonic (ω-psh) functions. If $X \subset V$ is an analytic subvariety, we define similarly the class $PSH(X, \omega|_X)$ of $\omega|_X$-psh functions on X: a function $\varphi : X \to [-\infty,+\infty)$ is called $\omega|_X$-psh if $\varphi \not\equiv -\infty$ on X and if there exist an open cover $\{U_i\}_{i\in I}$ of X and psh functions φ_i, ρ_i defined on U_i, where ρ_i is smooth and $dd^c\rho_i = \omega$, so that $\rho_i + \varphi = \varphi_i$ holds on $X \cap U_i$, for every $i \in I$.

By restriction, ω-psh functions on V yield $\omega|_X$-psh functions on X. If ω is a *Hodge form*, i.e. a Kähler form with integer cohomology class, our second result is:

Theorem B. *Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. Then any $\omega|_X$-psh function on X is the restriction of an ω-psh function on V.*

Note that in the assumptions of Theorem B there exists a positive holomorphic line bundle L on V whose first Chern class $c_1(L)$ is represented by ω. In this case the ω-psh functions are in one-to-one correspondence with the set of (singular) positive metrics of L (see [4]). Thus an alternate formulation of Theorem B is the following:

Theorem B'. *Let X be a subvariety of a projective manifold V and L be an ample line bundle on V. Then any (singular) positive metric of $L|_X$ is the restriction of a (singular) positive metric of L on V.*

Recall that it is possible to regularize quasipsh functions on $\mathbb{P}^n$, since it is a homogeneous manifold. Hence Theorem B has the following immediate corollary:

Corollary C. *Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. If $\varphi \in PSH(X, \omega|_X)$ then there exists a sequence of smooth functions $\varphi_j \in PSH(V,\omega)$ which decrease pointwise on V so that* $\lim \varphi_j = \varphi$ *on X.*

When X is smooth this regularization result is well known to hold even when the cohomology class of ω is not integral (see [3], [1]).

As an application of Theorem B, we conclude by discussing the extension problem for psh functions in the Lelong classes. If X is an analytic subvariety of $\mathbb{C}^n$ and $\gamma > 0$, we denote by $\mathcal{L}_\gamma(X)$ the *Lelong class* of psh functions φ on X which verify $\varphi(z) \leq \gamma \log^+ \|z\| + O(1)$ on X. By Theorem A, functions $\varphi \in \mathcal{L}(X)$ admit a psh extension in each class $\mathcal{L}_\gamma(\mathbb{C}^n)$, for every $\gamma > 1$.

We assume next that X is an *algebraic* subvariety of $\mathbb{C}^n$ and address the question whether it is necessary to allow the arbitrarily small additional growth. Consider the standard embedding $z \in \mathbb{C}^n \hookrightarrow [1 : z] \in \mathbb{P}^n$, where $[t : z]$ denote the homogeneous coordinates on $\mathbb{P}^n$. Let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. It is well known that the class $PSH(\mathbb{P}^n, \omega)$, where ω is the Fubini-Study form, is in one-to-one correspondence with $\mathcal{L}(\mathbb{C}^n)$. Similarly, a function $\eta \in \mathcal{L}(X)$ induces a function $\widetilde{\eta}$ on $\overline{X}$ defined by

$$\widetilde{\eta}([t:z]) = \begin{cases} \eta(z) - \log\sqrt{1+\|z\|^2}, & \text{if } t=1,\ z \in X, \\ \limsup\limits_{[1:\zeta]\to[0:z],\zeta\in X} (\eta(\zeta) - \log\sqrt{1+\|\zeta\|^2}), & \text{if } t=0,\ [0:z] \in \overline{X}\setminus X. \end{cases}$$

The function $\widetilde{\eta}$ is in general only *weakly ω-psh on $\overline{X}$*, i.e. it is bounded above on $\overline{X}$ and ω-psh at the regular points of $\overline{X}$. Theorem B has the following consequence:

Proposition. *Let $\eta \in \mathcal{L}(X)$. The following are equivalent:*
(i) There exists $\psi \in \mathcal{L}(\mathbb{C}^n)$ so that $\psi = \eta$ on X.
(ii) $\widetilde{\eta} \in PSH(\overline{X}, \omega\,|_{\overline{X}})$.
(iii) For every point $a \in \overline{X}\setminus X$ the values

$$\limsup_{X_j \ni [1:\zeta]\to a} (\eta(\zeta) - \log\sqrt{1+\|\zeta\|^2})$$

are independent of j, where X_j are the irreducible components of the germ $(\overline{X}, a)$.

In view of this proposition, it is easy to construct examples of algebraic curves $X \subset \mathbb{C}^2$ and functions in $\mathcal{L}(X)$ which do not admit an extension in $\mathcal{L}(\mathbb{C}^2)$.

References

[1] Z. Błocki and S. Kołodziej, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc **135** (2007), 2089–2093.
[2] M. Colţoiu, *Traces of Runge domains on analytic subsets*, Math. Ann. **290** (1991), 545–548.
[3] J. P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), 361–409.
[4] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. **15** (2005), 607–639.
[5] A. Sadullaev, *Extension of plurisubharmonic functions from a submanifold*, (Russian), Dokl. Akad. Nauk USSR **5** (1982), 3–4.

L^2-cohomology of singular spaces

Jean Ruppenthal

In the 1960s, the L^2-theory for the $\overline{\partial}$-operator has become an important, indispensable part of complex analysis through the fundamental work of Hörmander on L^2-estimates and existence theorems for the $\overline{\partial}$-operator and the related work of Andreotti and Vesentini. One should also mention Kohn's solution of the $\overline{\partial}$-Neumann problem, which implies existence and regularity results for the $\overline{\partial}$-complex, as well. But whereas the theory is very well developed on complex manifolds, it has been an open problem ever since to create an appropriate L^2-theory for the $\overline{\partial}$-operator on singular complex spaces. We will report on the latest developments in this direction.

When we consider the $\overline{\partial}$-operator on singular complex spaces, the first problem is to define an appropriate Dolbeault complex in the presence of singularities. It turns out that it is very fruitful to investigate the $\overline{\partial}$-operator in the L^2-category (simply) on the complex manifold consisting of the regular points of a complex space. One reason lies in Goresky and MacPherson's notion of intersection (co-) homology and the conjecture of Cheeger, Goresky and MacPherson, which states that the L^2-deRham cohomology on the regular part of a projective variety Y (with respect to the restriction of the Fubini-Study metric and the exterior derivate in the sense of distributions) is naturally isomorphic to the intersection cohomology of middle perversity $IH^*(Y)$ of Y:

Conjecture 1. (Cheeger-Goresky-MacPherson)
Let $Y \subset \mathbb{CP}^N$ be a projective variety. Then there is a natural isomorphism

$$H^k_{(2)}(Y - \operatorname{Sing} Y) \cong IH^k(Y).$$

The early interest in this conjecture was motivated in large parts by the hope that one could then use the natural isomorphism and a classical Hodge decomposition for $H^k_{(2)}(Y - \operatorname{Sing} Y)$ to put a pure Hodge structure on the intersection cohomology of Y. The conjecture was proved by Ohsawa in the case of isolated singularities (see [2]), while it is still open in general.

It is also interesting to have a look at the arithmetic genus of complex varieties. When M is a compact complex manifold of dimension n, the arithmetic genus

$$\chi(M) := \sum_{q=0}^{n} (-1)^q \dim H^{n,q}(M)$$

is a birational invariant of M. The conjectured extension of the classical Hodge decomposition to projective varieties led MacPherson also to ask whether the arithmetic genus $\chi(M)$ extends to a birational invariant of all projective varieties.

This turns out to be actually true if we consider the L^2-arithmetic genus

$$\chi_{(2)}(X) := \sum_{q=0}^{n} (-1)^q \dim H^{n,q}_{(2)}(X - \operatorname{Sing} X),$$

because the L^2-Dolbeault cohomology groups with respect to the $\overline{\partial}$-operator in the sense of distributions $H^{n,q}_{(2)}(\operatorname{Sing} X)$ themselves are invariant under modifications for all $0 \leq q \leq n$:

Theorem 2. (Pardon-Stern [4]**, R.** [8]**)** *Let X be a Hermitian compact complex space of pure dimension, and $0 \leq q \leq n = \dim X$. Then*

$$H^{n,q}_{(2)}(X - \operatorname{Sing} X) \cong H^{n,q}(M) \tag{1}$$

for any resolution of singularities $\pi : M \to X$.

This settles MacPherson's conjecture in the more general setting of Hermitian compact complex spaces. The proof is due to Pardon-Stern in the case of projective varieties and can be deduced from Takegoshi's vanishing theorem (and arguments due to Donelly-Fefferman, Ohsawa, Pardon-Stern) in the general situation (see [8]).

On the contrary, the L^2-Dolbeault cohomology groups $H^{0,q}_{(2)}(X - \operatorname{Sing} X)$ of a singular Hermitian complex space are not birational invariant. Nevertheless, it seems possible to determine also the behavior of these groups under modifications, and to describe them by use of a resolution of singularities. This seems very interesting in context of the Cheeger-Goresky-MacPherson Conjecture 1 or maybe the minimal model program.

For spaces with only isolated singularities, the task of characterizing the L^2 Dolbeault cohomology groups $H^{0,q}_{(2)}(X - \operatorname{Sing} X)$ by use of a resolution of singularities has been completed very recently by Øvrelid-Vassiliadou ([3]). Their final result is based on intermediate steps due to Pardon-Stern [4, 5], Fornæss-Øvrelid-Vassiliadou [1] and Ruppenthal [6, 7, 8].

To explain the results, let X be a Hermitian compact complex space of pure dimension n with isolated singularities only. For reasons of simplicity of the exposition, we restrict the presentation here to compact complex spaces, though the most techniques are of local nature and allow to treat much more general situations. Let $\pi : M \to X$ be a resolution of singularities with only normal crossings (of the exceptional divisor). We denote by $Z := \pi^*(\operatorname{Sing} X)$ the unreduced exceptional divisor and by $|Z|$ the underlying reduced divisor (i.e. the reduced divisor associated to the exceptional set E). Then:

Theorem 3. (Øvrelid-Vassiliadou [3])
Under the assumptions as above:

$$\begin{aligned} H^{0,q}_{(2)}(X - \operatorname{Sing} X) &\cong H^q(M, \mathcal{O}), \quad 0 \le q \le n-2, \\ H^{0,n}_{(2)}(X - \operatorname{Sing} X) &\cong H^n(M, \mathcal{O}(Z - |Z|)), \end{aligned}$$

and there exists a surjective homomorphism

$$\Psi : H^{n-1}(M, \mathcal{O}(Z - |Z|)) \to H^{0,n-1}_{(2)}(X - \operatorname{Sing} X)$$

such that

$$\ker \Psi \cong H^{n-1}_E(M, \mathcal{O}(Z - |Z|)),$$

where H^q_E denotes the cohomology with support on the exceptional set E.

If the sheaf $\mathcal{O}(Z - |Z|)$ is locally semi-negative with respect to the base-space X, i.e. if each point $x \in X$ has a neighborhood U_x such that $\mathcal{O}(Z - |Z|)$ is semi-negative on $\pi^{-1}(U_x)$, then $\ker \Psi = 0$. This situation has been treated already before in [8]. It occurs especially (trivially) if the divisor Z has multiplicity 1, yielding that $Z = |Z|$. This is the case for conical singularities which can be resolved by a single blow-up (see also [6, 7]). Another interesting special situation is the case of complex surfaces, $\dim X = 2$. Then, also $\ker \Psi = 0$ in Theorem 3 (see [3]).

One of the key ideas in the proof of Theorem 3 – and for an L^2-theory for the $\overline{\partial}$-operator on singular spaces in general – is to consider different closed L^2-extensions of the $\overline{\partial}$-operator. Besides the $\overline{\partial}$-operator in the sense of distributions, one also considers the minimal closed extension $\overline{\partial}_s$ given by the L^2-closure of the graph of the $\overline{\partial}$-operator acting on smooth forms with compact support

$$\overline{\partial}_{cpt} : C^\infty_{*,cpt}(X - \operatorname{Sing} X) \to C^\infty_{*,cpt}(X - \operatorname{Sing} X).$$

One can then use L^2-Serre duality

$$H^{0,q}_{(2)}(X - \operatorname{Sing} X) \cong H^{n,n-q}_{(2),s}(X - \operatorname{Sing} X) \tag{2}$$

to relate the L^2-cohomology groups with respect to the $\overline{\partial}$-operator in the sense of distributions $H^{0,q}_{(2)}$ to the L^2-cohomology with respect to the $\overline{\partial}_s$-operator $H^{n,n-q}_{(2),s}$.

A crucial idea introduced in [8] is then that the $\overline{\partial}_s$-operator can be localized, and that this operator $\overline{\partial}_{s,loc}$ is locally exact in the L^2-sense for (n, q)-forms on a Hermitian space with isolated singularities. One can then deduce

$$H^{n,n-q}_{(2),s}(X - \operatorname{Sing} X) \cong H^{n-q}(X, \mathcal{K}^s_X) \tag{3}$$

where $\mathcal{K}^s_X := \ker \overline{\partial}_{s,loc} \subset L^2_{n,0}(X - \operatorname{Sing} X)$ is a new kind of canonical sheaves introduced in [8]. By use of (3), it is then possible to express (2) in terms of a resolution of singularities.

References

[1] J. E. Fornæss, N. Øvrelid, S. Vassiliadou,*Local L^2 results for $\overline{\partial}$: the isolated singularities case*,Internat. J. Math. **16** (2005), *no. 4*, 387–418.

[2] T. Ohsawa, *Cheeger-Goresky-MacPherson's conjecture for the varieties with isolated singularities*, Math. Z. **206** (1991), 219–224.

[3] N. Øvrelid, S. Vassiliadou, *L^2 $\overline{\partial}$- cohomology groups of some singular complex spaces*, Preprint 2011, arXiv:1101.1860, submitted.

[4] W. Pardon, M. Stern, *L^2-$\overline{\partial}$-cohomology of complex projective varieties*, J. Amer. Math. Soc. **4** (1991), no. 3, 603–621.

[5] W. Pardon, M. Stern, *Pure Hodge structure on the L^2-cohomology of varieties with isolated singularities*, J. reine angew. Math. **533** (2001), 55–80.

[6] J. Ruppenthal, *About the $\overline{\partial}$-equation at isolated singularities with regular exceptional set*, Internat. J. Math. **20** (2009), no. 4, 459–489.

[7] J. Ruppenthal, *The $\overline{\partial}$-equation on homogeneous varieties with an isolated singularity*, Math. Z. **263** (2009), 447–472.

[8] J. Ruppenthal,*L^2-theory for the $\overline{\partial}$-operator on compact complex spaces*, Preprint 2009/2010, arXiv:1004:0396, ESI-Preprint **2202**, submitted.

Invariant Fatou components in 2 complex variables

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(joint work with M. Lyubich)

The Fatou set $\mathcal{F}$ of a holomorphic map $f : X \to X$ is the set of all points $z \in X$ for which the family of iterates $\{f^{\circ n}\}$ is a normal family in a neighborhood of z. A connected component of the Fatou set is called a Fatou component.

For rational self-maps of the Riemann sphere, the behavior of the orbits on Fatou components is very well understood. It was proved by Sullivan [4] that every Fatou component is preperiodic, and periodic Fatou components have been completely classified.

Our understanding of Fatou components in 2 and more complex dimensions is much smaller. In this presentation we will focus on the case of (generalized) Hénon maps, a family of polynomial automorphisms of $\mathbb{C}^2$ whose dynamics have been studied extensively in the last two decades. In general it is not known whether a Fatou component is necessarily preperiodic, we will consider the classification of invariant Fatou components.

Following Bedford-Smillie [1] we say that an invariant Fatou component $\Omega = f(\Omega)$ is recurrent if there exists an orbit in Ω that accumulates at a point in Ω. The following result combines the work of Bedford-Smillie [1], Fornæss-Sibony [2] and Ueda [5]:

Theorem 1. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a Hénon map and let Ω be a recurrent Fatou component. Then Ω is one of the following types:*

(1) *There is an attracting fixed point $p \in \Omega$ and Ω is biholomorphic to $\mathbb{C}^2$,*
(2) *there exists a one-dimensional closed complex submanifold Σ of Ω and $f^n(K) \mapsto \Sigma$ for any compact set K in Ω. The Riemann surface Σ is biholomorphic to a disc or an annulus and $f|_\Sigma$ is conjugate to an irrational rotation, or*
(3) *the domain Ω is a Siegel domain and all convergent subsequences of $\{f^n\}$ converge to an automorphism of Ω.*

Although there are still a few important open questions regarding recurrent Fatou components, for example whether the annulus in case 2 can really occur, Theorem 1 gives a relatively clear picture of the possible recurrent Fatou components that can occur for Hénon maps.

Much less understood are the non-recurrent Fatou components. Such components have been studied by Weickert [6] and Jupiter-Lilov [3], but many fundamental questions remain unanswered. The main problem with these components is that it is not known whether the limit set is unique.

If Ω is a non-recurrent Fatou component then all orbits converge to the boundary of Ω. By normality there exists a sequence of iterates $\{f^{n_j}\}$ that converges on Ω to a limit map $h : \Omega \to \partial\Omega$. It is clear that the map h does not need to be unique and may depend on the sequence $\{n_j\}$. It is unknown whether the limit set $h(\Omega)$ is unique. If we do assume that the limit set $h(\Omega)$ is unique, then we can give the following accurate description of the Fatou component Ω.

Theorem 2. *Let Ω be a non-recurrent Fatou component, let $h = \lim f^{n_j}|_\Omega$ and suppose that the limit set $h(\Omega)$ does not depend on the sequence $\{n_j\}$. Then $h(\Omega)$ is a fixed point p. Moreover, the eigen values λ_1, λ_2 of $Df(p)$ satisfy $|\lambda_1| < 1$ and $\lambda_2 = 1$.*

The proof of Theorem 2 relies upon several subresults, most of which hold in greater generality. First we show that if the generic rank of h is 1, then $h(\Omega)$ is a smoothly embedded Riemann surface. Then, we introduce a (not necessarily anti-symmetric) partial ordering on the set of all possible limit maps $h : \Omega \to \partial\Omega$

and show that there must exist a minimal limit map h_{min}. Then we show that h_{min} must have rank 0, so its image is a single point p. Here we use the fact that a Hénon map with a non-recurrent Fatou component is volume-decreasing. The first part of Theorem 2 is now proved.

It follows immediately that p is a semi-attracting fixed point, that is, the eigenvalues λ_1, λ_2 of $Df(p)$ satisfy $|\lambda_1| < 1$ and $|\lambda_2| = 1$. To show that $\lambda_2 = 1$ we prove a two-dimensional version of the snail lemma, which is a local statement and only uses the existence of an invariant connected open set on which the orbits converge uniformly to the fixed point.

It follows from the characterization of the eigenvalues means that we obtain a precise picture of the behavior of orbits on the Fatou component.

References

[1] Bedford, E. Smillie, J. *Polynomial diffeomorphisms of* $\mathbf{C}^2$*: II. Stable manifolds and recurrence*, J. Amer. Math. Soc., **4** (1991), no. 4, 657–679.

[2] Fornæss, J. E., Sibony, N., *Classification of recurrent domains for some holomorphic maps*, Math. Ann., **301** (1995), no. 4, 813-820.

[3] Jupiter, D., Lilov, K., *Invariant Nonrecurrent Fatou components of automorphisms of* $\mathbb{C}^2$, Far East J. Dyn. Syst., **6** (2003), no. 1, 49–65.

[4] Sullivan, D., *Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math., **122**, (1985) 401–418.

[5] Ueda, T., *Holomorphic maps on projective spaces and continuations of Fatou maps*, Michigan Math J., **56**, (2008), no. 1, 145-153.

[6] Weickert, B. J., *Nonwandering, nonrecurrent Fatou components in* $\mathbb{P}^2$, Pac. J. Math., **211**, (2003), no. 2, 391–397.

The diffeotopy group of rational or ruled 4-manifolds

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A 4-manifold X is *rational* or *ruled* if it is diffeomorphic to a rational or resp. ruled complex surface, possibly blown-up several times. In particular, $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 = S^2 \times S^2$ are rational, and the product $Y \times S^2$ of a Riemann surface Y with the sphere is ruled. Such manifolds can be characterized from the point of view of the symplectic geometry [5]: A compact symplectic 4-manifold (X, ω) is rational or ruled if and only if it contains a symplectic surface $\Sigma \subset X$ such that $c_1(X) \cdot [\Sigma] > 0$ and Σ is not an exceptional sphere. Further "symplectic" properties of rational or ruled manifolds are [1, 2, 4, 3]: For every symplectic form ω on such X there exists an integrable complex structure J such that ω is a Kähler form for J. For every pair of symplectic forms ω_1, ω_2 on such X with equal cohomology class $[\omega_1] = [\omega_2]$ there exists a diffeomorphism $F : X \to X$ with $F_* \omega_1 = \omega_2$.

The main result of my talk is [6]:

Theorem 1. *Let* (X, ω) *be a rational symplectic* 4*-manifold and* $F : X \to X$ *a symplectomorphism which is* homotopically trivial*, ie., acts trivially on the homology group* $\mathsf{H}_2(X, \mathbb{Z})$*. Then* F *is isotopic to identity.*

The meaning of the result is that the smooth isotopy class of a symplectomorphism of some rational complex surface is determined by its action in homology. It allows to give an almost complete description of the *diffeotopy group* $\Gamma = \Gamma(X)$ of rational 4-manifolds X, ie., the quotient group $\Gamma(X) := \mathcal{Diff}(X)/\mathcal{Diff}_0(X)$ of all diffeomorphisms of X by the group of isotopies.

Corollary 2. *Let* (X, ω) *be a rational symplectic* 4*-manifold and* $\Gamma_{\blacksquare}$ *the group of isotopy classes of homotopically trivial diffeomorphisms. Then* $\Gamma_{\blacksquare}$ *acts* simply transitively *on the set of connected components of symplectic forms having given cohomology class* $[\omega_0]$.

The latter result can be formulated as follows: On a rational complex surface there are as many mutually non-isotopic homotopically trivial diffeomorphisms as many mutually deformationally non-equivalent Kähler structures.

Theorem 3. *The group* $\Gamma_{\blacksquare}$ *remains unchanged under blow-ups. In particular,* $\Gamma_{\blacksquare}(\mathbb{CP}^2) = \Gamma_{\blacksquare}(S^2 \times S^2) = \Gamma_{\blacksquare}(X)$ *for every rational* 4*-manifold* X.

The proof of the results is given in the preprint [6]. It contain also the description of the action of the diffeomorphism group $\mathcal{Diff}(X)$ on the homology. Here we recall that the intersection form on $\mathsf{H}_2(X, \mathbb{R})$ has Lorentz signature and therefore the set $\mathcal{K} := \{[A] \in \mathsf{H}_2(X, \mathbb{R}) : [A]^2 > 0\}$ is a quadratic cone, called the *positive cone* of X.

Theorem 4. *Let* X *be a rational* 4*-manifold and* Γ_{H_2} *the image of the diffeomorphism group* $\mathcal{Diff}(X)$ *in the group* $\mathsf{Aut}(\mathsf{H}_2(X, \mathbb{Z}))$. *Further, let* $\boldsymbol{L}; \boldsymbol{E}_1, \ldots, \boldsymbol{E}_\ell \in \mathsf{H}_2(X, \mathbb{Z})$ *be the homology classes of the line and resp. the exceptional spheres with respect to some contraction map* $\pi : X \to \mathbb{CP}^2$. *Then* Γ_{H_2} *is generated by reflections with respect to hyperplanes in* $\mathsf{H}_2(X, \mathbb{R})$ *orthogonal to the classes* $\boldsymbol{L} - (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3)$, $\boldsymbol{E}_i - \boldsymbol{E}_{i+1}$ *with* $i = 1, \ldots, \ell - 1$, *and* $\boldsymbol{E}_\ell$.

Moreover, the action of Γ_{H_2} *on the positive cone* $\mathcal{K}$ *admit a fundamental domain consisting of those classes* $[A] \in \mathcal{K}$ *which have non-negative intersection with the classes* $\boldsymbol{L} - (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3)$, $\boldsymbol{E}_i - \boldsymbol{E}_{i+1}$ *with* $i = 1, \ldots, \ell - 1$, *and* $\boldsymbol{E}_\ell$.

The meaning of the latter result is as follows. Let X be a rational complex surface and D an ample divisor on X. Then there exists a (holomorphic) contraction map $\pi : X \to \mathbb{CP}^2$ whose exceptional divisor E is the sum $E_1 + \cdots + E_l$ of rational curves with the homology classes $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_\ell$ such that the divisors $\boldsymbol{L} - (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3)$, $\boldsymbol{E}_i - \boldsymbol{E}_{i+1}$ have positive intersection with D.

The preprint [6] contains also a description of the diffeotopy group $\Gamma(X) = \mathcal{Diff}(X)/\mathcal{Diff}_0(X)$ of irrational ruled 4-manifolds X. The main difference from the rational case is appearance of a new differential invariant, *secondary Stiefel-Whitney class* $\tilde{w}_2(F)$ of homotopically trivial diffeomorphisms. The main result is generalized in the following form:

Theorem 5. *Let* (X, ω) *be a ruled symplectic* 4*-manifold and* $F : X \to X$ *a symplectomorphism. Then* F *is isotopic to the identity if and only if* F *is homotopically trivial, ie., acts trivially on the groups* $\mathsf{H}_2(X, \mathbb{Z})$, $\pi_1(X)$, $\pi_2(X)$, *and has trivial secondary Stiefel-Whitney class* $\tilde{w}_2(F)$.

One also obtains the counterparts of **Corollary 2** and **Theorems 3, 4.**

References

[1] F. Lalonde, *Isotopy of symplectic balls, Gromov's radius and the structure of ruled symplectic* 4*-manifolds.* Math. Ann., **300**(1994), 273–296.

[2] F. Lalonde, D. McDuff, *The classification of ruled symplectic manifolds.* Math. Res. Lett., **3**(1996), 769–778.

[3] D. McDuff, *Blow ups and symplectic embeddings in dimension 4.* Topology, **30** (1991), 409-421.

[4] D. McDuff, *The structure of rational and ruled symplectic manifolds.* J. AMS, **3**(1990), 679–712; Erratum: J. AMS, **5**(1992), 987–988.

[5] D. McDuff, D. Salamon, *A survey of symplectic* 4*-manifolds with* $b_+ = 1$. Turk. J. Math. **20**, 47–60, (1996).

[6] V. Shevchishin, *Secondary Stiefel-Whitney class and diffeomorphisms of rational and ruled symplectic* 4*-manifolds.*, Preprint, 52 p., arXiv:0904.0283v2, (2010).

Periodicities and Positive Entropy for Linear Fractional Recurrences in 3-space

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(joint work with K. Kim)

We consider the family of birational maps of 3-space which may be written in affine coordinates as

$$f_{\alpha,\beta} : (x_1, x_2, x_3) \mapsto \left(x_2, x_3, \frac{\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3} \right). \tag{1}$$

The algebraic iterates $f^n_{\alpha,\beta} := f_{\alpha,\beta} \circ \cdots \circ f_{\alpha,\beta}$ are rational maps for all $n \in \mathbf{Z}$. Here we study the dynamics of $f = f_{\alpha,\beta}$, by which we mean the behavior of f^n as $n \to \pm\infty$. We have invertible dynamics since f has a rational inverse, but it does not behave like a diffeomorphism (or even a homeomorphism). There are two difficulties if we want to regard f as a mapping of points. First, there is the set of indeterminacy $\mathcal{I}(f)$; f blows up each point of $\mathcal{I}(f)$ to a variety of positive dimension. Second, there can be hypersurfaces E which are exceptional, in the sense that the codimension of $f(E - \mathcal{I}(f))$ is at least 2. We will say that f is a pseudo-automorphism if neither f nor f^{-1} has an exceptional hypersurface. In dimension 2, every pseudo-automorphism is in fact an automorphism. However, for pseudo-automorphisms, indeterminate behaviors are possible in higher dimension which have no analogue in dimension 2.

Given a rational map $f : X \dashrightarrow X$ there is a well-defined pullback map on cohomology, $f^* : H^*(X) \to H^*(X)$. Passage to cohomology, however, may not be compatible with iteration because the identity $(f^*)^n = (f^n)^*$ may not be valid. Given a birational map f in dimension 2, Diller and Favre [7] showed that there is a new manifold $\pi : Y \to X$ such that the iterates of the induced map f_Y behave naturally on cohomology, in the sense that $(f^*_Y)^n = (f^n_Y)^*$. In dimension greater than 2, however, no such theorem is known.

Given a rational map of $\mathbf{P}^n$ we may consider modifications $\pi : X \to \mathbf{P}^n$, where π is a morphism which is birational. This induces a rational map $f_X := \pi^{-1} \circ f \circ \pi$ of X, which might have pointwise properties which are different from those of the original f. If f_X is a pseudo-automorphism, then f_X acts naturally on $H^{1,1}(X)$. The exponential rate of growth of f^n on $H^{p,p}$: $\delta_p(f) := \lim_{n\to\infty} ||f^{n*}|_{H^{p,p}(X)}||^{1/n}$ is known as the pth dynamical degree and is a birational invariant (see [8]).

Within the family (1) we find the first known examples of pseudo-automorphisms of positive entropy on blowups of $\mathbf{P}^3$:

Theorem 1. *Suppose that $\alpha = (a, 0, \omega, 1)$ and $\beta = (0, 1, 0, 0)$ where $a \in \mathbf{C} \setminus \{0\}$ and ω is a non-real cube root of the unity. Then there is a modification $\pi : Z \to \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. The dynamical degrees $\delta_1(f) = \delta_2(f) \cong 1.28064 > 1$ are equal and are given by the largest root of $t^8 - t^5 - t^4 - t^3 + 1$. The entropy of f_Z is the logarithm of the dynamical degree and is thus positive.*

Theorem 2. *For the mappings in Theorem 1, there is a 1-parameter family of surfaces $S_c \subset Z$, $c \in \mathbf{C}$ which have the invariance $fS_c = S_{\omega c}$. For generic c, S_c is $K3$, and the restriction $f^3|_{S_c}$ is an automorphism. For generic c and c', the surfaces S_c and $S_{c'}$ are biholomorphically inequivalent, and the automorphisms $f^3|_{S_c}$ and $f^3|_{S_{c'}}$ are not smoothly conjugate.*

The surface S_0 is invariant, and the restriction f_{S_0} is an automorphism which has the same entropy as f. This is smaller than the entropy of the automorphism constructed in [15, Theorem 1.2] and is thus the smallest known entropy for a *projective* $K3$ surface automorphism.

The following mappings have quadratic degree growth and complete integrability:

Theorem 3. *Suppose that $\beta = (0, 1, 0, 0)$ and either $\alpha = (0, 0, \omega, 1)$ or $\alpha = (a, 0, 1, 1)$ where $a \in \mathbf{C} \setminus \{1\}$, $\omega \neq 1$, and $\omega^3 = 1$. Then the degree of f^n grows quadratically in n. Further, there is a modification $\pi : Z \to \mathbf{P}^3$ such that f_Z is a pseudo-automorphism. There is a two-parameter family of surfaces S_c, $c = (c_1, c_2) \in \mathbf{C}^2$ which are invariant under f^3. For generic c and c', S_c is a smooth $K3$ surface, and $S_c \cap S_{c'}$ is a smooth elliptic curve.*

For the mappings in Theorems 1 and 3, f is reversible on the level of cohomology: f_Z^* is conjugate to $(f_Z^{-1})^* = (f_Z^*)^{-1}$. The identity $\delta_1(f) = \delta_2(f)$ for such maps is a consequence of the duality between $H^{1,1}$ and $H^{2,2}$, so they are not cohomologically hyperbolic, in the terminology of [10]. For each of these maps, the family of invariant $K3$ surfaces becomes singular at an invariant 8-cycle $\mathcal{R}$ of rational surfaces (see (7.2)).

Theorem 4. *Let f be a map from Theorems 1 and 3. If $a \neq 1$, then the restriction $f^8|_{\mathcal{R}}$ is not birationally equivalent to a surface automorphism. Thus there is no proper modification $\pi : W \to \mathbf{P}^3$ such that the induced map f_W is an automorphism.*

We will also determine which mappings $f_{\alpha,\beta}$ are periodic, or finite order, in the sense that $f^p = id$ for some $p > 0$. In contrast to Theorem 4, it was shown by de Fernex and Ein [6] that if f is a rational map of finite order, then there is a modification f_X as above, which is an automorphism of X. If f_X is periodic, then f_X^* will also be periodic.

In (4.1) and (4.2) we identify conditions which are necessary for f to be periodic and are sufficient for the existence of a space $Z = Z_{\alpha,\beta}$ such that f_Z is a pseudo-automorphism. We show that for a map in (1), if f_Z^* is periodic, then f also turns out to be periodic. The birational map (1) may also be considered as a 3-step linear fractional recurrence: given z_0, z_1, z_2, we define a sequence $\{z_n\}$ by

$$z_{n+3} = \frac{\alpha_0 + \alpha_1 z_n + \alpha_2 z_{n+1} + \alpha_3 z_{n+2}}{\beta_0 + \beta_1 z_n + \beta_2 z_{n+1} + \beta_3 z_{n+2}}. \tag{2}$$

The recurrence (2) is said to be periodic if the sequence $\{z_n\}$ is periodic for all choices of initial terms z_0, z_1 and z_2. Equivalently, $f^p_{\alpha,\beta} = id$ for some p. For all $r > 0$ there are r-step recurrences of the form (2). In [1] we determined the possible periods for 2-step linear fractional recurrences. McMullen [14] has explained the periods that arise by showing that the corresponding (2-dimensional) $f_{\alpha,\beta}$ represent certain Coxeter elements.

Here we determine all possible periods for 3-step recurrences (2). To rule out trivial cases, we assume that the coefficients satisfy (2.3), and we have:

Theorem 5. *The only nontrivial periods for (2) are 8 and 12. Each periodic recurrence is equivalent to one of the following:*

$$z_{n+3} = \frac{1 + z_{n+1} + z_{n+2}}{z_n} \qquad z_{n+3} = \frac{-1 - z_{n+1} + z_{n+2}}{z_n} \qquad \text{(period 8)}$$

$$z_{n+3} = \frac{\eta/(1-\eta) + \eta z_{n+1} + z_{n+2}}{\eta^2 + z_n} \qquad \eta^3 = -1 \qquad \text{(period 12)}$$

In the notation of (1), the first case corresponds to $\beta = (0, 1, 0, 0)$, $\alpha = (\pm 1,\ 0,\ \pm 1,\ 1)$, *and the second case to* $\beta = (\eta^2, 1, 0, 0)$, $\alpha = (\eta/(1-\eta), 0, \eta, 1)$.

Each of these mappings has a different structure; these structures are described in Theorems 6.10 and 6.11. The first period 8 recurrence above was found by Lyness [13], and the second one was found by Csörnyei and Laczkovic [5] (see also [4]). We note that the period 12 recurrences are the case $k = 3$ of a general phenomenon exhibited in [2]: *For each k, there are k-step linear fractional recurrences with period $4k$.* There is a literature dealing with r step recurrences of the form (2). We refer to the books [11], [12], [9], [3] and the extensive bibliographies they contain. That direction of research is largely concerned with the case where the structural parameters α, β, as well as the dynamical points, are real and positive. This avoids the difficulty that the denominator in (2) might vanish, causing the expression to be undefined; but the restriction to positive numbers leads to a subdivision into a large number of distinct cases to be treated separately.

In working with the family $f_{\alpha,\beta}$, we work with the pointwise iterates as much as possible, but this runs into difficulties if the orbit enters the indeterminacy locus. We can often deal with this by blowing up certain subsets. In this way we convert these subsets into hypersurfaces, and we then deal with the hypersurfaces by passing to f^* on Pic. This allows us to convert many difficulties with indeterminate orbits into more tractable problems of Linear Algebra.

References

[1] E. Bedford and KH Kim, *Periodicities in linear fractional recurrences: Degree growth of birational surface maps*, Michigan Math. J. **54** (2006), 647-670.

[2] E. Bedford and KH Kim, *Linear fractional recurrences: periodicities and integrability*, arXiv:0910.4409

[3] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC Press, 2008.

[4] A. Cima, A. Gasull and F. Mañosas, *On periodic rational difference equations of order k*, J. Difference Equ. Appl. **10** (2004), no. 6, 549–559.

[5] M. Csörnyei and M. Laczkovich, *Some periodic and non-periodic recursions*, Monatshefte für Mathematik **132** (2001), 215-236.

[6] T. de Fernex and L. Ein, *Resolution of indeterminacy of pairs*, Algebraic geometry, 165–177, de Gruyter, Berlin, 2002.

[7] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. of Math., **123** (2001), 1135–1169.

[8] T.-C. Dinh and N. Sibony, *Une borne supérieure pour l'entropie topologique d'une application rationnelle*, Ann. of Math. (2) **161** (2005), no. 3, 1637–1644.

[9] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Kluwer Academic Publishers, 2005.

[10] V. Guedj, *Théorie ergodique des transformations rationnelles*, arXiv:math/0611302

[11] V.I. Kocic and G. Ladas, *Global Behaviour of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers 1993.

[12] M. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations*, CRC Press, 2002.

[13] R.C. Lyness, Notes 1581,1847, and 2952, Math. Gazette **26** (1942), 62, **29** (1945), 231, and **45** (1961), 201.

[14] C.T. McMullen, *Dynamics on blowups of the projective plane*, Publ. Math. I.H.E.S. No. **105** (2007), 49–89.

[15] C.T. McMullen, *K3 surfaces, entropy and glue.*

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