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New Directions in Algebraic K-Theory

Organised by
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May 15th – May 21st, 2011

Abstract. This meeting brought together algebraic geometers, algebraic topologists and geometric topologists, all of whom use algebraic K-theory. The talks and discussions involved all the participants.

Mathematics Subject Classification (2000): 19xx.

Introduction by the Organisers

There have been dramatic advances in algebraic K-theory recently, especially in the computation and understanding of negative K-groups and of nilpotent phenomena in algebraic K-theory. Parallel advances have used remarkably different methods. Quite complete computations for the algebraic K-theory of commutative algebras over fields have been obtained using algebraic geometric techniques. On the other hand, the Farrell-Jones conjecture implies results on the K-theory for arbitrary rings. Proofs here use controlled topology and differential geometry.

Given the diversity of interests and backgrounds of the 28 participants in our mini-workshop, we encouraged everyone to make their talks accessible to a wide audience and scheduled five expository talks. The opening talk of the conference was an inspiring talk by Charles Weibel, on the work of Daniel Quillen, the creator of higher algebraic K-theory, who died at the end of April. Wolfgang Lück spoke on the Farrell-Jones conjecture. Jim Davis applied the Farrell-Jones conjecture to give a foundational result on algebraic K-theory, showing that geometric techniques have algebraic consequences for the iterated N^pK-groups. Bjorn Dundas gave a survey of trace methods on algebraic K-theory, focusing on topological cyclic homology and his new integral homotopy cartesian square. Christian Haesemeyer gave a survey of algebraic K-theory of singularities and new techniques for computing negative K-theory and NK-theory for commutative $\mathbb{Q}$-algebras. The idea of the expository talks worked quite well; it was remarkable how many of the speakers relied on them.

The mini-workshop had a full schedule; in addition to the five expository talks there were seventeen research talks. There were computational talks (Teena Gerhardt, Charles Weibel, Daniel-Juan Pineda), foundational talks (Bruce Williams, Lars Hesselholt, Max Karoubi, Guilermo Cortiñas, Jens Hornbostel, Andrew Blumberg, Thomas Geisser), applications of ideas from K-theory to geometric topology (Ib Madsen, Frank Connolly, Qayum Khan, Ian Hambleton, Michael Weiss, Wolfgang Steimle), as well as the proof of the Farrell-Jones Conjecture for the group $SL_n(\mathbb{Z})$ (Holger Reich). The talk of Charles Weibel was notable since the topic was research done at the workshop. Weibel's talk connected and compared two different computations of the N^pK_qR groups, one done by algebraic geometry and one done by geometric topology. This was emblematic of a successful implementation of the original goal of the workshop to compare and contrast two powerful but quite distinct approaches to algebraic K-theory.

Timetable

Monday 16^{th} **May, 2011**

9:00-10:00	Chuck Weibel	*On the work of Daniel Quillen (1940-2011)*
10:15-11:05	Ib Madsen	*On the homological structure of BDiff(M)*
11:25-12:15	Bruce Williams	*K-Theory and Endomorphisms*
16:00-16:50	Teena Gerhardt	*On the algebraic K-theory of truncated polynomials in multiple variables*
17:15-18:05	Wolfgang Lück	*The Farrell-Jones conjecture and its applications*

Tuesday 17^{th} **May, 2011**

9:00-9:50	Jim Davis	*Some remarks on Nil groups in algebraic K-theory*
10:15-11:05	Bjørn Dundas	*A survey of trace methods in algebraic K-theory*
11:25-12:15	Lars Hesselholt	*Algebraic K-theory and reality*
16:00-16:50	Frank Connolly	*An equivariant rigidity theorem for certain discrete groups (Part I)*
17:15-18:05	Qayum Khan	*An equivariant rigidity theorem for certain discrete groups (Part II)*

Wednesday 18^{th} **May, 2011**

9:30-10:20	Christian Haesemeyer	*Algebraic K-theory of singularities, a survey*
11:00-11:50	Michael Weiss	*Smooth maps to the plane and Pontryagin classes*

Thursday 19^{th} **May, 2011**

9:00-10:00	Max Karoubi	*Twisted bundles and twisted K-theory*
10:15-11:05	Holger Reich	*The Farrell-Jones conjecture for $SL(n, \mathbb{Z})$*
11:25-12:15	Andrew Blumberg	*Localisation in THH of Waldhausen categories*
16:00-16:50	Chuck Weibel	*NK and N^pK of commutative algebras*
17:15-18:05	Wolfgang Steimle	*Higher Whitehead torsion and the geometric assembly map*

Friday 20^{th} **May, 2011**

9:00-9:50	Guillermo Cortiñas	*Isomorphism conjectures with proper coefficients*
10:15-11:05	Thomas Geisser	*Rational K-theory in characteristic p*
11:25-12:15	Daniel Juan-Pineda	*Algebraic K-theory of $\mathbb{Z}[\Gamma]$ for Γ the braid group of a surface*
16:00-16:50	Jens Hornbostel	*Preorientations of the derived motivic multiplicative group*
17:15-18:05	Ian Hambleton	*Cocompact discrete group actions and the assembly map*

Workshop: New Directions in Algebraic K-Theory

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Abstracts

The work of Daniel Quillen (1940-2011)

Charles Weibel

This was a survey talk about the mathematical works of Daniel Gray Quillen, who died on April 30, 2011. I focussed on the period 1967-1972, with emphasis on his paper cited below. The notes for this lecture may be found at:

`http://www.maths.ed.ac.uk/~aar/confer/quillen.pdf`

References

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On the homological structure of $BDiff(\mathcal{M})$

Ib Madsen

(joint work with Alexander Berglund)

1. Introduction

The traditional method to obtain homotopical information about diffeomorphism groups of high dimensional manifolds is a two step procedure: the surgery exact sequence gives information on the group of block diffeomorphisms and Waldhausen's K-theory of spaces connects block diffeomorphisms and diffeomorphisms.

With the solution of the generalized Mumford conjecture [MW] a new method was introduced for diffeomorphisms of surfaces, based on embedded surfaces and the Pontryagin-Thom collapse map. The method was generalized first in [GMTW], which determined the homotopy type of the embedded cobordism category. Most recently, Galatius and Randal-Williams used geometric methods (surgery) to calculate the homological structure of the "stable" diffeomorphism groups of $(d-1)$-connected $2d$-dimensional manifolds. As a special case, consider the manifolds

$$\mathcal{M}_g^{2d} = (S^d \times S^d)\sharp \dots \sharp(S^d \times S^d), \quad g \text{ summands.}$$

Let $D^{2d} \subset \mathcal{M}_g^{2d}$ be an embedded disk. Boundary connected sum induces a map

$$BDiff(\mathcal{M}_g^{2d}, D^{2d}) \times BDiff(\mathcal{M}_h^{2d}, D^{2d}) \to BDiff(\mathcal{M}_{g+h}^{2d}, D^{2d})$$

so that

$$\mathscr{M} = \coprod_g BDiff(\mathcal{M}_g, D)$$

becomes a topological monoid. This monoid can be viewed as a subcategory of the embedded cobordism category $\mathscr{C}_{2d}^{\theta}$ with suitable tangential structure θ. Galatius and Randal-Williams show that the inclusion of $\mathscr{M}$ in $\mathscr{C}_{2d}^{\theta}$ induces a homotopy equivalence of classifying spaces by performing surgery ("in families") below the middle dimension. They then use the group completion theorem and results from [K] to show

Theorem 1.1 ([GR-W]). *There is a homology equivalence*

$$\operatorname*{colim}_g H_*(BDiff(\mathcal{M}_g^{2d}, D^{2d})) \to H_*(\Omega_0 B\mathscr{C}_{2d}^{\theta})$$

for $d \neq 2$.

The right-hand side was determined in [GMTW]. For surfaces ($d = 1$), the result was proved in [MW] but with an argument that used Harer type stability theorems [H]. In contrast, the proof of the above theorem for $d > 2$ is not based on homological stability. However, it does raise the question if there is a range of dimensions where

$$\sigma_k : H_k(BDiff(\mathcal{M}_g^{2d}, D^{2d})) \to H_k(BDiff(\mathcal{M}_{g+1}^{2d}, D^{2d}))$$

is an isomorphism. The present report gives a partial answer to this question, but only for homology with rational coefficients and for large d. The result is

Theorem 1.2. *For odd $d > 2$, the stabilization homomorphism*

$$\sigma_k : H_k(BDiff(\mathcal{M}_g^{2d}, D^{2d}); \mathbb{Q}) \to H_k(BDiff(\mathcal{M}_{g+1}^{2d}, D^{2d}); \mathbb{Q})$$

is an isomorphism if $k < \min(\frac{1}{2}(g-4), d-2)$.

Remark 1.3. *In the case of surfaces ($d = 1$) there is a stability range for the forgetful map*

$$H_k(BDiff(\mathcal{M}_g^2, D^2)) \to H_k(BDiff(\mathcal{M}_g^2)).$$

For our high-dimensional analogue $\mathcal{M}_g^{2d}$ we cannot have stability of this kind. This follows from the fibration

$$Emb(D, \mathcal{M}) \to BDiff(\mathcal{M}, D) \to BDiff(\mathcal{M})$$

and the homotopy equivalence

$$Emb(D, \mathcal{M}) \simeq Fr(T\mathcal{M}),$$

the frame bundle. For $\mathcal{M} = \mathcal{M}_g^{2d}$, $SO(2d) \to Fr(T\mathcal{M})$ is $(d-1)$-connected and the fibration prevents a stability range for

$$H_k(BDiff(\mathcal{M}_g^{2d}, D^{2d})) \to H_k(BDiff(\mathcal{M}_{g+1}^{2d}, D^{2d}))$$

2. Block Diffeomorphisms

The simplicial group (or Δ-group) $\widetilde{Diff}(\mathcal{M})$ of block diffeomorphisms has k-simplices equal to the set of face preserving diffeomorphisms

$$\varphi : \Delta^k \times \mathcal{M} \to \Delta^k \times \mathcal{M}. \tag{1}$$

In comparison, the simgular complex of the topological group $Diff(\mathcal{M})$ has k-simplices given by commutative diagrams

$$\begin{array}{ccc} \Delta^k \times \mathcal{M} & \xrightarrow{\varphi} & \Delta^k \times \mathcal{M} \\ & \searrow \quad \swarrow & \\ & \Delta^k & \end{array} \tag{2}$$

with φ a diffeomorphism. There is an analogue of (1) where we require φ to be a face preserving homotopy equivalence. This defines the monoid $\widetilde{Aut}(\mathcal{M})$. The geometric realizations give topological groups

$$Diff(\mathcal{M}) \subset \widetilde{Diff}(\mathcal{M}) \subset \widetilde{Aut}(\mathcal{M}). \tag{3}$$

There are homotopy equivalences

$$Diff(\mathcal{M}) \simeq Diff^W(\mathcal{M}), \quad \widetilde{Aut}(\mathcal{M}) \simeq aut(\mathcal{M}),$$

where $Diff^W(\mathcal{M})$ denotes the diffeomorphism group of $\mathcal{M}$ equipped with the Whitney topology, and $aut(\mathcal{M})$ the monoid of self homotopy equivalences of $\mathcal{M}$ in the compact-open topology.

The Δ-groups (monoids) above are fibrant, so the homotopy theory of their geometric realizations and the simplicial homotopy theory agree. The homogeneous spaces of (3) are defined to be the fibers in the fibrations

$$\begin{gathered} \widetilde{Diff}(\mathcal{M})/Diff(\mathcal{M}) \longrightarrow BDiff(\mathcal{M}) \longrightarrow B\widetilde{Diff}(\mathcal{M}), \\ \widetilde{Aut}(\mathcal{M})/\widetilde{Diff}(\mathcal{M}) \longrightarrow B\widetilde{Diff}(\mathcal{M}) \longrightarrow B\widetilde{Aut}(\mathcal{M}). \end{gathered} \tag{4}$$

If $X \subset \mathcal{M}$ is a closed subset, we write $Diff(\mathcal{M}, X)$ for the subgroup that fixes a neighbourhood germ of X. The first fiber in (4) can be examined via Morlet's lemma of disjunction, see e.g. [BLR], p.31. Let V be a compact n-manifold, and let $D_0 \subset \operatorname{int}(V)$ be an n-disk. There is a diagram of inclusions

$$\begin{array}{ccc} Diff(D_0, \partial D_0) & \rightarrowtail & Diff(V, \partial V) \\ \downarrow & & \downarrow \\ \widetilde{Diff}(D_0, \partial D_0) & \rightarrowtail & \widetilde{Diff}(V, \partial V) \end{array}$$

where the horizontal inclusions extend a diffeomorphism by the identity in the complement $V \backslash D_0$. Let

$$\begin{array}{ccc} BDiff(D_0, \partial D_0) & \rightarrowtail & BDiff(V, \partial V) \\ \downarrow & & \downarrow \\ B\widetilde{Diff}(D_0, \partial D_0) & \rightarrowtail & B\widetilde{Diff}(V, \partial V) \end{array} \tag{5}$$

be the associated diagram

Theorem 2.1 (Morlet). *Suppose V is k-connected with $k+1 < \frac{1}{2}\dim V$. Then*

$$\pi_j(Diff(V,\partial V), Diff(D_0,\partial D_0)) \to \pi_j(\widetilde{Diff}(V,\partial V), \widetilde{Diff}(D_0,\partial D_0))$$

is $(2k-2)$-connected.

It follows that the vertical fibers in (5) are compared by a $(2k-2)$-connected map. This implies

Corollary 2.2. *The stabilization map*

$$\sigma : \widetilde{Diff}(\mathcal{M}_g^{2d}, D^{2d})/Diff(\mathcal{M}_g^{2d}, D^{2d}) \to \widetilde{Diff}(\mathcal{M}_{g+1}^{2d}, D^{2d})/Diff(\mathcal{M}_{g+1}^{2d}, D^{2d})$$

is $(2d-4)$-connected.

3. The proof of theorem 1.2

In this paragraph, $\mathcal{M}_g = \mathcal{M}_g^{2d}$ and I will assume $d > 2$ is an odd number. There are two homotopy fibrations

$$\text{(I)}\quad [\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)} \xrightarrow{\eta} Map_*(\mathcal{M}_g, G/O)_{(1)} \xrightarrow{\lambda} \mathbb{L}(\mathcal{M}_g)_{(1)},$$

$$\text{(II)}\quad \widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D) \longrightarrow B\widetilde{Diff}(\mathcal{M}_g, D) \longrightarrow B\widetilde{Aut}(\mathcal{M}_g, D).$$

The subscript (1) in (I) indicates the connected component of the identity and Map_* the space of pointed maps. The homotopy exact sequence of (I) is the surgery exact sequence (above degree zero) by a theorem of F. Quinn.
In particular,

$$L_{k+2d} = \pi_k \mathbb{L}(\mathcal{M}_g)_{(1)} = \begin{cases} \mathbb{Z}, & if\ k+2d \equiv 0\ (4) \\ \mathbb{Z}/2, & if\ k+2d \equiv 2\ (4) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.1. *For $k > 0$,*

$$\pi_k[\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}] \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}^{2g}, & if\ k+d \equiv 0\ (4) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We remember the rational homotopy equivalences

$$(G/O)_{\mathbb{Q}} \simeq (G/TOP)_{\mathbb{Q}} \simeq \prod K(\mathbb{Q}, 4l)$$

and that by the plumbing construction

$$\lambda_* : [\mathcal{M}_g \times S^k, G/TOP] \to L_{k+2d}$$

is surjective. The theorem follows from the π_*-sequence of (I). □

The proof above shows that

$$\pi_k[\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}] \otimes \mathbb{Q} \to \pi_k Map_*(\mathcal{M}, G/O) \otimes \mathbb{Q}$$

is injective. This in turn implies that the rational homotopy type of $[\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}$ has vanishing k-invariants so that

$$H_*([\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}; \mathbb{Q}) = \Lambda(\pi_* \otimes \mathbb{Q}) \tag{6}$$

with $\pi_* \otimes \mathbb{Q}$ given in (3.1) and $\Lambda(\pi_* \otimes \mathbb{Q})$ being the free commutative graded algebra generated by $\pi_* \otimes \mathbb{Q}$.

Lemma 3.2. *The image of*

$$\widetilde{J}_0 : \pi_0 \widetilde{Diff}(\mathcal{M}, D) \to \pi_0 \widetilde{Aut}(\mathcal{M}, D)$$

has finite index.

Proof. The standard surgery exact sequence,

$$* \longrightarrow \mathscr{S}(\mathcal{M}_g, D) \xrightarrow{\eta_*} [\mathcal{M}_g, G/O] \xrightarrow{\lambda_*} L_{2d}$$

(c.f. [B]) shows that the structure set $\mathscr{S}(\mathcal{M}_g, D)$ is finite. On the other hand, the cokernel of $\widetilde{J}_0$ injects into the structure set. □

The rest of the argument is basically an application of the Serre spectral sequence of (II), but we need to replace the fiber in (II) by its connected component in order to apply theorem 3.1. Let

$$\overline{B}\widetilde{Aut}(\mathcal{M}, D) \to B\widetilde{Aut}(\mathcal{M}, D)$$

be the covering associated to the subgroup $\mathrm{im}\widetilde{J}_0$ of $\pi_1 B\widetilde{Aut}(\mathcal{M}, D)$. Then

$$\widetilde{J} : B\widetilde{Diff}(\mathcal{M}, D) \to B\widetilde{Aut}(\mathcal{M}, D)$$

lifts to $\overline{B}\widetilde{Aut}(\mathcal{M}, D)$ and gives rise to the homotopy fibration

$$[\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)} \to B\widetilde{Diff}(\mathcal{M}_g, D) \to \overline{B}\widetilde{Aut}(\mathcal{M}_g, D).$$

Its Serre spectral sequence has E^2-term

$$E^2_{r,s} = H_r(\overline{B}\widetilde{Aut}(\mathcal{M}_g, D); H_s([\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}; \mathbb{Q})),$$

and we want to prove that stabilization

$$\sigma : E^2_{r,s}(\mathcal{M}_g) \to E^2_{r,s}(\mathcal{M}_{g+1})$$

is an isomorphism in a range of dimensions.

The E^2-term has local coefficients in the sense that $\pi_1 \overline{B}\widetilde{Aut}(\mathcal{M}_g, D)$ acts on the fiber through the map

$$\pi_1 \overline{B}\widetilde{Aut}(\mathcal{M}_g, D) \to \pi_0 \widetilde{Aut}(\mathcal{M}_g, D).$$

We need to understand this action. First a general lemma from surgery theory.

A homotopy equivalence $f : \mathcal{M}_1 \to \mathcal{M}_2$ (with $\partial f : \partial\mathcal{M}_1 \to \partial\mathcal{M}_2$ a diffeomorphism if $\partial\mathcal{M}_1 \neq \emptyset$) determines an element of the structure set $\mathscr{S}(\mathcal{M}_2)$ and defines an element

$$\eta(f) \in [\mathcal{M}_2/\partial\mathcal{M}_2, G/O]_*.$$

Lemma 3.3. *Consider degree one homotopy equivalences*

$$f : \mathcal{M}_1 \to \mathcal{M}_2,\ g : \mathcal{M}_2 \to \mathcal{M}_3$$

which restrict to diffeomorphisms on the boundaries. Then

$$\eta(g \circ f) = (g^{-1})^*(\eta(f)) + \eta(g).$$

Since $\pi_0\widetilde{Diff}(\mathcal{M}_g, D) \to \pi_1\overline{B}\widetilde{Aut}(\mathcal{M}_g, D)$ is surjective, the lemma implies that the normal invariant

$$\eta_* : \pi_k(\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D), \mathbf{1}) \to [\mathcal{M}_g, \Omega^k(G/O)]_*$$

is $\pi_1\overline{B}\widetilde{Aut}(\mathcal{M}_g, D)$-equivariant. This in turn yields that the action of $\pi_1\overline{B}\widetilde{Aut}(\mathcal{M}_g, D)$ on

$$\pi_s([\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}; \mathbb{Q}) = \begin{cases} \mathbb{Q}^{2g} & s + 2d \equiv 0\,(4) \\ 0 & \text{otherwise} \end{cases}$$

is the standard action, induced from the action of $\pi_0\widetilde{Aut}(\mathcal{M}_g, D)$ on $H_d(\mathcal{M}_g; \mathbb{Q}) = \mathbb{Q}^{2g}$.

We need information on the rational type of $\widetilde{Aut}(\mathcal{M}_g, D) \simeq aut^{(1)}(\mathcal{M}_g, *)$. To this end, notice that

$$\pi_*(\Omega\mathcal{M}_g) \otimes \mathbb{Q} = \mathbb{L}(\alpha_1, \ldots, \alpha_{2g})/(R), \quad |\alpha_i| = d - 1.$$

Here $\mathbb{L}$ denotes the free Lie algebra on the listed generators and (R) is the ideal generated by

$$R = \sum_{i=1}^{g} [\alpha_i, \alpha_{i+g}].$$

The rational type of $\mathcal{M}_g$ is both formal and coformal in the sense of Sullivan and Quillen. In particular

$$\begin{aligned} \pi_0 aut((\mathcal{M}_g)_\mathbb{Q}) &= Aut_{Lie}(\pi_*(\Omega\mathcal{M}_g) \otimes \mathbb{Q}), \\ \pi_i(aut((\mathcal{M}_g)_\mathbb{Q}, \mathbf{1})) &= 0 \text{ for } 1 \le i < d - 2. \end{aligned} \tag{7}$$

Let φ be a Lie algebra automorphism of $\pi_*(\Omega\mathcal{M}_g)\otimes\mathbb{Q}$, determined by the equations

$$\varphi(\alpha_i) = \sum \omega_{ij}\alpha_j, \quad \Omega = (\omega_{ij}).$$

Since φ preserves the ideal (R), $\varphi(R) = \lambda R$ for some $\lambda \in \mathbb{Q}^*$. This is equivalent to the matrix relation

$$\Omega J \Omega^t = \lambda J,$$

where (since d is odd)

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The scalar λ is the degree, so

$$\pi_0(aut^{(1)}((\mathcal{M}_g)_\mathbb{Q})) = Sp_g(\mathbb{Q}).$$

Moreover, $\pi_0(aut^{(1)}(\mathcal{M}_g)) \to \pi_0(aut^{(1)}((\mathcal{M}_g)_\mathbb{Q}))$ is commensurable with $Sp_g(\mathbb{Z})$. In conclusion,

$$E^2_{r,s}(\mathcal{M}_g) \cong H_r(BSp_g(\mathbb{Z}); H_s([\widetilde{Aut}(\mathcal{M}_g, D)/\widetilde{Diff}(\mathcal{M}_g, D)]_{(1)}; \mathbb{Q}))$$

for $r < d - 2$.

Let $H_g = \mathbb{Q}^{2g}$ be the standard representation of $Sp_g(\mathbb{Z})$. We need the following result from [C].

Theorem 3.4 (Charney). *The stabilization map*

$$\sigma_* : H_*(Sp_g(\mathbb{Z}), H_g^{\otimes r}) \to H_*(Sp_{g+1}(\mathbb{Z}), H_{g+1}^{\otimes r})$$

is $\frac{1}{2}(g - r - 4)$*-connected.*

Altogether the above results imply that the stabilization

$$\sigma_* : E^2_{r,s}(\mathcal{M}_g) \to E^2_{r,s}(\mathcal{M}_{g+1})$$

is an isomorphism in total degrees less than $\min(\frac{1}{2}(g-4), d-2)$. The same is then the case for the E^∞-term, so that

$$\sigma_* : H_*(\widetilde{BDiff}(\mathcal{M}_g, D); \mathbb{Q}) \to H_*(\widetilde{BDiff}(\mathcal{M}_{g+1}, D); \mathbb{Q})$$

is an isomorphism in the same range of dimensions. Together with corollary 2.2 this proves theorem 1.2.

Remark 3.5. *There are similar theorems for* $\mathcal{M}_g^{2d}$ *where* d *is even, where* $Sp_g(\mathbb{Z})$ *is replaced by* $SO_g(\mathbb{Z})$*. The only problem is to give a different proof of lemma 3.2 when* $d \equiv 0(4)$.

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K-theory of Endomorphisms

Bruce Williams

(joint work with Andrew Blumberg, Lars Hesselholt, John Klein, Mike Mandell)

Suppose A is a ring and M is an A-bimodule. Let $End(A; M)$ be the category with objects, pairs $(P, \alpha : P \to P \otimes M)$ where P is a finitely generated projective A-module and α is A-linear. For example if $\phi : A \to A$ is a ring endomorphism, then A_ϕ is the A-bimodule where $a_1 a_2 a_3 = a_1 \cdot a_2 \cdot \phi(a_3)$. We denote $End(A; A_\phi)$ by $End(A, \phi)$, and $End(A, id)$ by $End(A)$. Let $Nil(A; M)$ denote the subcategory of $End(A; M)$ given by nilpotent endomorphisms.

Our goal is to analyze $\tilde{K}(End(A;M)$, the homotopy fiber of the forgetful map $KEnd(A;M) \to KA$.

Let $T(A;M)$ denote the tensor algebra with augmentation $\epsilon : T(A;M) \to A$. Let $\tilde{K}T(A;M)$ be the homotopy fiber of ϵ_*; $KT(A;M) \to KA$. Recall that if M is a left free A-module, then Waldhausen showed that $\tilde{K}Nil(A;M)$ is homotopy equivalent to $\Omega\tilde{K}T(A;M)$. (This generalized earlier work of Bass-Heller-Swan, Quillen, Farrell-Hsiang, and Grayson.) Let I be the kernel of ϵ and let $\hat{T}(A;M)$ be the completion of $T(A;M)$ with respect to I. This ring is called the *formal power series ring* for $T(A;M)$.

Following Ranicki we let Σ denote the set of square $T(A;M)$-matrices which are $\hat{T}(A;M)$-invertible. (Ranicki introduced the key idea of using localization with respect to Σ in order to study endomorphisms.) Let $\hat{\mathbb{T}}(A;M)$ denote the derived noncommutative Cohen localization of $T(A;M)$ with respect to Σ. The notion of derived localization is due to Dwyer and his associated localization fibration also plays a key role in the proof of the following theorem.

Theorem: When $M = A_\phi$, then $\tilde{K}(End(A;M))$ is homotopy equivalent to $\Omega\tilde{K}\hat{\mathbb{T}}(A;M)$.

This theorem extends results of Grayson(A commutative and $\phi = id$) and Ranicki (induced isomorphism on π_0 when $\phi = id$). Also this theorem is a response to the challenge from the introduction to "On the algebraic K-theory of formal power series" by Lindenstrauss and McCarthy. They study $End(A;M)$ when M is a simplicial bimodule using Goodwillie calculus and trace methods. It would be very interesting to understand how this theorem and their results fit together. See also the related paper by Betley.

This work is part of an ongoing project with Andrew Blumberg, Lars Hesselholt, John Klein and Mike Mandell to use trace methods to study invariants for families of endomorphisms of spaces.

On the algebraic K-theory of truncated polynomials in multiple variables

Teena Gerhardt

(joint work with Vigleik Angeltveit, Michael Hill, Ayelet Lindenstrauss)

About 15 years ago, Hesselholt and Madsen [3] computed the relative algebraic K-theory groups $K_q(\mathbb{F}_p[x]/x^a, (x))$. We consider the algebraic K-theory of truncated polynomials in multiple commuting variables. We study

$$K_q(k[x_1, \ldots, x_n]/(x_1^{a_n}, \ldots, x_n^{a_n}), (x_1), (x_2), \ldots, (x_n)),$$

the appropriate multi-relative version of $K_q(k[x_1, \ldots, x_n]/(x_1^{a_n}, \ldots, x_n^{a_n}))$. In the case where $k = \mathbb{F}_p$ and p does not divide any of the truncations $a_1, a_2, \ldots, a_n$ we compute these multi-relative algebraic K-groups explicitly.

Theorem 1. *If $p \nmid a_i$ for all $1 \le i \le n$,*

$$K_{2q-1}(\mathbb{F}_p[x_1, \dots x_n]/(x_1^{a_n}, \dots x_n^{a_n}), (x_1), \dots (x_n)) \cong \bigoplus_{\substack{s_1,\dots s_n \ge 0 \\ a_i \nmid s_i \\ p \nmid \gcd(s_i)}} \bigoplus_{0 \le l < q} (\mathbb{Z}/p^m\mathbb{Z})^{\oplus \binom{n-1}{2l}}$$

$$K_{2q}(\mathbb{F}_p[x_1, \dots x_n]/(x_1^{a_n}, \dots x_n^{a_n}), (x_1), \dots (x_n)) \cong \bigoplus_{\substack{s_1,\dots s_n \ge 0 \\ a_i \nmid s_i \\ p \nmid \gcd(s_i)}} \bigoplus_{0 \le l < q} (\mathbb{Z}/p^m\mathbb{Z})^{\oplus \binom{n-1}{2l+1}}$$

where m is the unique integer such that

$$\sum_{i=1}^{n} \lfloor \frac{p^{m-1}s_i - 1}{a_i} \rfloor < q - l \le \sum_{i=1}^{n} \lfloor \frac{p^m s_i - 1}{a_i} \rfloor.$$

To prove this theorem we relate the multi-relative algebraic K-theory groups in question to multi-relative topological cyclic homology groups using the cyclotomic trace map of Bökstedt, Hsiang, and Madsen [2]. By a theorem of McCarthy [5], this map is an isomorphism. Thus, we aim to compute the relative topological cyclic homology groups $\mathrm{TC}_q(\mathbb{F}_p[x_1, \dots, x_n]/(x_1^{a_n}, \dots x_n^{a_n}), (x_1), \dots, (x_n))$.

Topological cyclic homology can be defined for any cyclotomic spectrum [4]. A cyclotomic spectrum Y is an S^1-spectrum, and the topological cyclic homology of Y, $\mathrm{TC}(Y)$, is defined as a limit of fixed point spectra Y^{C_n} where $C_n \subset S^1$ is the cyclic subgroup of order n. For a ring A, the topological Hochschild homology of A, $T(A)$, is cyclotomic, and the topological cyclic homology of A is defined to be $\mathrm{TC}(T(A))$. Thus the first step toward understanding the topological cyclic homology of A is understanding the topological Hochschild homology of A. In the case where the ring in question is a pointed monoid algebra, $A(\Pi)$, in order to understand $T(A(\Pi))$ we take advantage of the following equivalence

$$T(A(\Pi)) \simeq T(A) \wedge B^{cy}(\Pi).$$

Here $B^{cy}(\Pi)$ denotes the cyclic bar construction on the pointed monoid Π. In particular, for our computation this equivalence allows us to write

$$T(k[x_1, x_2, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n})) \simeq T(k) \wedge B^{cy}(\Pi_{a_1} \wedge \dots \wedge \Pi_{a_n})$$

where Π_a is the pointed multiplicative monoid $\{0, 1, x, \dots, x^{a-1}\}$, $x^a = 0$, and k is any ring. To compute the fixed points of $T(A(\Pi))$ one must first understand the S^1-equivariant homotopy type of $B^{cy}(\Pi)$. In the case of the pointed monoid Π_a, Hesselholt and Madsen [3] described $B^{cy}(\Pi_a)$ as a homotopy cofiber of a map of S^1-spaces. They then smashed the cofiber with $T(\mathbb{F}_p)$ and applied topological cyclic homology to compute $\mathrm{TC}_q(\mathbb{F}_p[x]/x^a, (x))$. Hesselholt and Madsen's description of $B^{cy}(\Pi_a)$ can be used to express $B^{cy}(\Pi_{a_1} \wedge \dots \wedge \Pi_{a_n})$ as an iterated homotopy cofiber of an n-cube of S^1-spaces. We will denote this n-cube by $X = \{X_I\}_{I \subset \{1,2,\dots,n\}}$. To compute the multirelative topological cyclic homology

$$\mathrm{TC}_q(k[x_1, \dots, x_n]/(x_1^{a_n}, \dots, x_n^{a_n}), (x_1), \dots, (x_n)),$$

one needs to compute TC(hocofib($T(k) \wedge X$)). We show using properties of cyclotomic spectra that

$$\mathrm{TC}(\mathrm{hocofib}(T(k) \wedge X)) \simeq \mathrm{hocofib}(\mathrm{TC}(T(k) \wedge X))$$

In the case $k = \mathbb{F}_p$, we then compute $\mathrm{TC}(T(k) \wedge X_I)$ for each $I \subset \{1, 2, \ldots, n\}$. These computations use the $RO(S^1)$-graded equivariant homotopy groups of $T(\mathbb{F}_p)$ [1]. If p does not divide any of the truncations $a_1, a_2, \ldots, a_n$, we also compute the maps in the n-cube $\mathrm{TC}(T(\mathbb{F}_p) \wedge X)$. These computations yield the main theorem above.

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The Farrell-Jones Conjecture and its applications

Wolfgang Lück

Let G be a discrete group and let R be an associative ring with unit. We explain and state the following conjectures and discuss their relevance.

Kaplanski Conjecture. If G is torsionfree and R is an integral domain, then 0 and 1 are the only idempotents in RG.

Conjecture. Suppose that G is torsionfree. Then $K_n(\mathbb{Z}G)$ for $n \leq -1$, $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish.

Novikov Conjecture. Higher signatures are homotopy invariants.

Borel Conjecture. An aspherical closed manifold is topologically rigid.

Conjecture. If G is a finitely presented Poincaré duality group of dimension $n \geq 5$, then it is the fundamental group of an aspherical homology ANR-manifold.

Conjecture If G is a hyperbolic group with S^n as boundary, then there is a closed apsherical manifold M whose fundamental group is G.

Farrell-Jones Conjecture. Let G be torsionfree and let R be regular. Then the assembly maps for algebraic K- and L-theory

$$\begin{array}{rcl} H_n\big(BG; \mathbb{K}_R\big) & \to & K_n(RG); \\ H_n\big(BG; \mathbb{L}_R^{\langle -\infty \rangle}\big) & \to & L_n^{\langle -\infty \rangle}(RG), \end{array}$$

are bijective for all $n \in \mathbb{Z}$.

There is a more complicate version of the Farrell-Jones Conjectures which makes sense for all groups and rings and allows twistings of the group ring. We explain that it implies all the other conjectures mentioned above provided that in the Kaplanski Conjecture R is a field of characteristic zero, in the Novikov Conjecture and the Borel Conjecture the dimension is greater or equal to five and in the conjecture about boundaries of hyperbolic groups the dimension of the sphere is greater or equal to five.. We present the following result which summarizes joint work with Bartels, Echterhoff, Farrell, Reich, Rüping and Weinberger.

Theorem. Let $\mathcal{FJ}$ be the class of groups for which the Farrell-Jones Conjecture is true in its general form. Then:

(1) Hyperbolic groups belong to $\mathcal{FJ}$;
(2) CAT(0) groups belong to $\mathcal{FJ}$;
(3) Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;
(4) $SL_n(\mathbb{Z})$ belongs to $\mathcal{FJ}$;
(5) Fundamental groups of (not necessarily compact) 3-manifolds possibly with boundary) belong to $\mathcal{FJ}$;
(6) If G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 * G_1$ and $G_0 \times G_1$;
(7) If G belongs to $\mathcal{FJ}$, then any subgroup of G belongs to $\mathcal{FJ}$;
(8) Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps). If each G_i belongs to $\mathcal{FJ}$, then also the direct limit of $\{G_i \mid i \in I\}$.
(9) Let $1\colon H \to G \xrightarrow{p} Q \to 1$ be an extension of groups. If Q and for all virtually cyclic subgroups $V \subseteq Q$ the preimage $p^{-1}(V)$ belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;

Since certain prominent constructions of groups yield colimits of hyperbolic groups, the class $\mathcal{FJ}$ contains many interesting groups, e.g. limit groups, Tarski monsters, groups with expanders and so on. Some of these groups were regarded as possible counterexamples to the conjectures above but are now ruled out by the theorem above.

There are also prominent constructions of closed aspherical manifolds with exotic properties, e.g. whose universal covering is not homeomorphic to Euclidean space, whose fundamental group is not residually finite or which admit no triangulation. All these constructions yield fundamental groups which are CAT(0) and hence yield topologically rigid manifolds.

However, the Farrell-Jones Conjecture is open for instance for solvable groups, $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups or automorphism groups of finitely generated free groups.

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Some remarks on Nil groups in algebraic K-theory

James F. Davis

Theorem 1: For any ring R and integer q, if $K_qR[x] = K_qR$ and $K_{q-1}R[x] = K_{q-1}R$, then $K_qR[x,y] = K_qR$.

Under the hypothesis of the theorem, this implies that $(N^2K_q)R = 0$. More generally, one has a formula for the iterated Nil term:

Theorem 2: For any ring R and integer q,

$$(N^nK_q)R = \bigoplus_{\mathcal{M}_+} \bigoplus_{i=0}^{n-1} \binom{n-1}{i} NK_{q-i}R$$

where $\mathcal{M}_+ = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_i > 0, \gcd = 1\}$.

The lecture gave a complete proof of these theorems by comparing the Fundamental Theorem of Algebraic K-theory due to Bass [1] and Quillen [4, 3] with a computation based on the Farrell-Jones conjecture in K-theory for the group $\mathbb{Z}^n$.

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A survey of trace methods in algebraic K-theory

Bjørn Ian Dundas

The cyclotomic trace was introduced by Bökstedt, Hsiang and Madsen in their proof of the algebraic K-theory Novikov conjecture, and has since been one of the most prominent invariants for calculating algebraic K-theory.

I was asked to answer the questions: what is the cyclotomic trace, and can you make it seem like a plausible invariant? We review some of the major calculations, starting off with the structural theorems that the fiber of the cyclotomic trace from algebraic K-theory of connective ring spectra to Goodwillie's integral version of topological cyclic homology is "locally constant" and satisfies "closed excision".

Trying to make the construction of the cyclotomic trace credible, we review the prehistory briefly, and finally the cyclotomic trace is presented as an approximation to categorical S^1-fixed points by homotopy-theoretical means.

Algebraic K-theory and reality

Lars Hesselholt
(joint work with Ib Madsen)

By analogy with Atiyah's K-theory with reality [1], we associate to an exact category with strict duality $(\mathscr{C}, T)$ a real symmetric spectrum $KR(\mathscr{C}, T)$, the real algebraic K-theory spectrum. The construction uses a new modified version of Waldhausen's S-construction that we call the real Waldhausen construction.

Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We define a real set to be a left G-set and a real map of real sets to be a G-equivariant map. The category Real Set of real sets and real maps has a cartesian closed structure with the internal **Hom**-object from X to Y defined to be the set of all maps from X to Y with the conjugation left G-action. We define a real category to be a category enriched in Real Set and a real functor to be an enriched functor. The real simplicial index category is the real category ΔR whose objects are the categories $[n] = 0 \to 1 \to \cdots \to n$, where $n \geq 0$, and whose real set of morphisms from $[m]$ to $[n]$ consists of all functors $\theta\colon [m] \to [n]$ with the generator $T \in G$ acting by $(T\theta)(i) = n - \theta(m-i)$. We define a real simplicial set to be a real functor $X[-]\colon \Delta R^{\mathrm{op}} \to \mathrm{Real\,Set}$. The geometric realization $|X[-]|_R$

of a real simplicial set is a real space. For instance, the geometric realization of the real standard n-simplex $\Delta R[n][-] = \Delta R([-],[n])$ is given by the topological standard n-simplex with $T \in G$ acting through the affine map that takes the ith vertex to the $(n-i)$th vertex. The real nerve of a category with strict duality is the real simplicial set $N(\mathscr{C}, T)[-]$ that in degree n is given by the set of all functors $c\colon [n] \to \mathscr{C}$ with $T \in G$ acting by $(Tc)(i) = T(c(n-i))$.

A real symmetric spectrum is a symmetric spectrum in the category of pointed real spaces with respect to the object

$$S^{2,1} = |S^{2,1}[-]|_R = |\Delta R[2][-]/\partial \Delta R[2][-]|_R.$$

We define the real algebraic K-theory spectrum of the exact category with strict duality $(\mathscr{C}, T)$ to be the real symmetric spectrum $KR(\mathscr{C}, T)$ defined as follows. The real Waldhausen construction associates to $(\mathscr{C}, T)$ the real simplicial exact category with strict duality $(S^{2,1}\mathscr{C}[-], T)$, where

$$S^{2,1}\mathscr{C}[n] \subset \mathrm{Cat}(\mathrm{Cat}([2],[n]), \mathscr{C})$$

is the full subcategory of functors $A\colon \mathrm{Cat}([2],[n]) \to \mathscr{C}$ that satisfy that

(i) for all $\theta\colon [1] \to [n]$,

$$A(s_0\theta) = A(s_1\theta) = 0,$$

a fixed null-object; and

(ii) for all $\theta\colon [3] \to [n]$, the sequence

$$A(d_3\theta) \rightarrowtail^{f} A(d_2\theta) \xrightarrow{g} A(d_1\theta) \overset{h}{\twoheadrightarrow} A(d_0\theta)$$

is 4-term exact;

and where $T\colon S^{2,1}\mathscr{C}[n]^{\mathrm{op}} \to S^{2,1}\mathscr{C}[n]$ is defined by $(TA)(\theta) = T(A^{\mathrm{op}}(T^{-1}\theta))$. That the sequence in (ii) is 4-term exact means that the maps f and g are an admissible monomorphism and an admissible epimorphism, respectively, and that g induces an isomorphism $\mathrm{coker}(f) \to \ker(f)$. Now, if $w\mathscr{C} \subset \mathscr{C}$ is a subcategory of weak equivalences with $T(w\mathscr{C}^{\mathrm{op}}) = w\mathscr{C}$, we define

$$KR(\mathscr{C}, T)_n = |NwS^{2n,n}\mathscr{C}[-]|_R,$$

where $S^{2n,n}(-)$ indicates the real Waldhausen construction iterated n times, and define the real spectrum structure map

$$\sigma_{n,1}\colon |NwS^{2n,n}\mathscr{C}[-]|_R \wedge S^{2,1} \to |NwS^{2n+2,n+1}\mathscr{C}[-]|_R$$

to be the inclusion of the 2-skeleton in the last $S^{2,1}$-direction.

To understand the equivariant homotopy type of $KR(\mathscr{C}, T)$, we compare it to the real direct sum K-theory spectrum $KR^{\oplus}(\mathscr{C}, T)$ defined as follows. If (X, x) is a finite pointed real set, we define $Q(X, x)$ to be the category with strict duality, where the objects are all pointed (not necessarily real) subsets $x \in U \subset X$, where the morphisms from U to V are all pointed subsets $x \in F \subset V \cap U$, and where the duality functor $T\colon Q(X,x)^{\mathrm{op}} \to Q(X,x)$ takes $F\colon U \to V$ to $TF\colon TV \to TU$. We define $\{F_\alpha\colon U_\alpha \to U\}$ to be a covering if $\cup_\alpha F_\alpha = U$, define

$$\mathscr{C}(X,x)^{\sim} \subset \mathrm{Cat}_*(Q(X,x), \mathscr{C})$$

to be the full subcategory of sheaves, and define $T\colon \mathscr{C}(X,x)^{\sim\mathrm{op}} \to \mathscr{C}(X,x)^{\sim}$ to be the strict duality functor given by $T \circ A^{\mathrm{op}} \circ T^{-1}$. If $f\colon (X,x) \to (Y,y)$ is a pointed map, we let $f_*\colon \mathscr{C}(X,x)^{\sim} \to \mathscr{C}(Y,y)^{\sim}$ be the direct image functor, define

$$KR^{\oplus}(\mathscr{C},T)_n = |Nw\mathscr{C}(S^{2n,n}[-])^{\sim}[-]|_R,$$

where $S^{2n,n}[-]$ is the n-fold smash power of $S^{2,1}[-]$, and define

$$\sigma^{\oplus}_{n,1}\colon |Nw\mathscr{C}(S^{2n,n}[-])^{\sim}[-]|_R \wedge S^{2,1} \to |Nw\mathscr{C}(S^{2n+2,n+1}[-])^{\sim}[-]|_R$$

to be the inclusion of the 2-skeleton in the last $S^{2,1}[-]$-direction. It follows from a theorem of Shimakawa [3, Theorem B] that the adjoint structure map

$$\tilde{\sigma}^{\oplus}_{n,1}\colon KR^{\oplus}(\mathscr{C},T)_n \to \Omega^{2,1}KR^{\oplus}(\mathscr{C},T)_{n+1}$$

is an equivariant weak equivalence, for $n \geqslant 1$, and an equivariant group completion, for $n = 0$. There is a map of real symmetric spectra

$$\phi^*\colon KR^{\oplus}(\mathscr{C},T) \to KR(\mathscr{C},T)$$

induced by the functors $\phi^*\colon \mathscr{C}(S^{2,1}[n])^{\sim} \to S^{2,1}\mathscr{C}[n]$ that, in turn, are induced by the functors $\phi\colon \mathrm{Cat}([2],[n]) \to Q(S^{2,1}[n])$ that take the morphism $\theta \to \theta'$ to the morphism $\phi(\theta) \cap \phi(\theta')\colon \phi(\theta) \to \phi(\theta')$, where

$$\phi([2] \xrightarrow{\theta} [n]) = \{[n] \xrightarrow{\rho} [2] \mid \rho \circ \theta = \mathrm{id}_{[2]}\} \cup \{*\}.$$

Following Quillen's proof of [2, Theorem 2], we prove:

Theorem. *Let $(\mathscr{C},T)$ is a split-exact category with strict duality, and let $w\mathscr{C} = i\mathscr{C}$ be the subcategory of isomorphisms. Then the map of real symmetric spectra*

$$\phi^*\colon KR^{\oplus}(\mathscr{C},T) \to KR(\mathscr{C},T)$$

is a level weak equivalence.

For all integers p and q, we define the real algebraic K-group

$$KR_{p,q}(\mathscr{C},T) = [S^{p,q}, KR(\mathscr{C},T)]_R$$

to be the abelian group of maps in the homotopy category of real symmetric spectra from a (choice of virtual) sphere $S^{p,q}$ of dimension p and weight q. (The weight counts the number of sign representations.) The theorem identifies the group $KR_{p,0}(\mathscr{C},T)$ with the pth hermitian K-group of $(\mathscr{C},T)$ defined to be the pth homotopy group of the group completion of the classifying space of the groupoid $\mathrm{Sym}(i\mathscr{C},T)$ of non-degenerate symmetric spaces in $(\mathscr{C},T)$. Moreover, for every integer q, the cofibration sequence of real symmetric spectra

$$S^{q,q} \wedge G_+ \xrightarrow{f_q} S^{q,q} \xrightarrow{i_q} S^{q+1,q+1} \xrightarrow{h_q} \Sigma S^{q,q} \wedge G_+$$

induces a long-exact sequence

$$\cdots \to K_{p+1}(\mathscr{C}) \xrightarrow{H_q} KR_{p+1,q+1}(\mathscr{C},T) \xrightarrow{I_q} KR_{p,q}(\mathscr{C},T) \xrightarrow{F_q} K_p(\mathscr{C}) \to \cdots$$

relating the real algebraic K-groups of $(\mathscr{C},T)$ and the algebraic K-groups of $\mathscr{C}$.

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An Equivariant Rigidity Theorem For Certain Discrete Groups, Parts I and II

Frank Connolly, Qayum Khan
(joint work with Jim Davis)

Let Γ be a discrete cocompact group of isometries of an n-dimensional CAT(0) manifold X^n, with $n \geq 5$. We make the following assumption about Γ: For each element of finite order $\gamma \in \Gamma$, with $\gamma \neq 1$, the centralizer of γ is a finite subgroup of Γ.

Let $\mathcal{S}(\Gamma)$ denote the set of equivariant homeomorphism classes of contractible n-manifolds equipped with a proper Γ action, (M, Γ). We prove that $\mathcal{S}(\Gamma)$ has the structure an abelian group and that there is an isomorphism,

$$\mathcal{S}(\Gamma) \cong \sum_{(mid)\Gamma} UNil_{n+1}(\mathbf{Z}, \mathbf{Z}^{\epsilon}, \mathbf{Z}^{\epsilon})$$

Here $\epsilon = (-1)^n$, and $(mid)\Gamma$ denotes the set of conjugacy classes of maximal infinite dihedral subgroups of Γ.

In particular then, $\mathcal{S}(\Gamma)$ consists of a single element (X, Γ) if Γ has no element of order 2, or if $n = 0, 1 \pmod 4$.

K-theory of singularities - a survey

Christian Haesemeyer

The algebraic K-theory of a singular scheme X of finite type over a field of characteristic zero can be computed from two pieces of information: the algebraic K-theory of smooth schemes involved in resolving the singularities of X (this is, of course, very hard in general); and an "error term" that can be attacked by comparing with cyclic homology via the Jones-Goodwillie Chern character. To be more precise, there is a fibration sequence

$$\mathcal{F}_K(X) \to K(X) \to KH(X)$$

where KH denotes Weibel's homotopy K-theory and an equivalence

$$\mathcal{F}_K(X) \simeq \mathcal{F}_{HC}(X)[1]$$

where $\mathcal{F}_{HC}(X)$ is the homotopy fiber of the natural map from the cyclic homology of X to its *cdh*-hypercohomology.

Using this comparison, the Hochschild-Kostant-Rosenberg theorem and the λ operation on cyclic homology, one can compute $\mathcal{F}_K$ and hence the Bass Nil groups $NK_i(X)$. For example, one can conclude the following theorem (see [1]; for more results see for example [2], [3]; for related results in characteristic p see [4]):
Theorem: Let X be a scheme of finite type over a field F of characteristic 0, and suppose that the dimension of X is d. Then $NK_i(X) = 0$ for all $d \leq -i$, and in fact X is K_{-d}-regular. Moreover, $K_i(X) = 0$ for $i < -d$ and $K_{-d}(X)$ can be computed explicitly in terms of a resolution of the singularities of X.

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Smooth maps to the plane and Pontryagin classes

Michael Weiss

(joint work with Rui Reis)

Novikov and Thom proved long ago that the Pontryagin classes of vector bundles extend rationally to characteristic classes defined for fiber bundles with fiber $\mathbb{R}^n$, that is, bundles with structure group $\mathrm{TOP}(n)$, the group of homeomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$. It emerged some years later that the inclusion $B\mathrm{O} \to B\mathrm{TOP}$ induces an isomorphism in rational cohomology. By contrast, the rational cohomology of $B\mathrm{TOP}(n)$ is well understood only in dimensions $\leq 4n/3$ approximately, and in those dimensions it often deviates from the rational cohomology of $B\mathrm{O}(n)$, as shown in [1]. The following is not obvious.

Hypothesis A. We have $p_m = e^2 \in H^{4m}(B\mathrm{STOP}(2m); \mathbb{Q})$, where e is the Euler class and p_m is the Pontryagin class.
(Corollary: $p_m = 0 \in H^{4m}(B\mathrm{TOP}(n); \mathbb{Q})$ if $n < 2m$.)

One reason for being interested in $B\mathrm{TOP}(n)$ for finite n is the homotopy equivalence $\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}(\mathrm{TOP}(n)/\mathrm{O}(n))$, where $\mathrm{Diff}_\partial(D^n)$ denotes the group of diffeomorphisms $D^n \to D^n$ which extend the identity on $\partial D^n = S^{n-1}$. This comes from smoothing theory [3],[2].

Smoothing theory also leads to the following reformulation of Hypothesis A. Let $\mathcal{R}$ be the space of smooth regular (= nonsingular) smooth maps from $D^n \times D^2$ to D^2 which, on the entire boundary of $D^n \times D^2$, agree with the standard projection. By viewing the derivative of $f \in \mathcal{R}$ as a map from $D^n \times D^2$ to $\mathrm{GL}(n+2)/\mathrm{GL}(n)$

taking the entire boundary to the base point, and by writing $\mathrm{O}(n{+}2)/\mathrm{O}(n)$ instead of $\mathrm{GL}(n+2)/\mathrm{GL}(n)$ for convenience, we obtain a map

$$\mathcal{R} \xrightarrow{\;\nabla\;} \Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n))\ .$$

This can be viewed as an $\mathrm{O}(2)$-map for certain conjugation type actions of $\mathrm{O}(2)$ on source and target. The target is rationally an Eilenberg-MacLane space $K(\mathbb{Z}, n-3)$ when n is even, $n \geq 4$.

Hypothesis B. For all even $n \geq 4$, the map ∇ is rationally nullhomotopic with derived $\mathrm{O}(2)$-invariance.

It was explained in the talk how this is equivalent to Hypothesis A. Meanwhile Hypothesis B belongs to differential topology and resembles a well-known, integrally correct and easily proved statement from concordance theory (replace D^2 by D^1 and $\mathrm{O}(2)$ by $\mathrm{O}(1)$ where applicable). It is natural to use ideas from concordance theory to approach Hypothesis B. So we look for $\mathcal{W}$, a subspace of the space of all smooth maps $D^n \times D^2 \to D^2$ which agree with the projection on the boundary, such that $\mathcal{R} \subset \mathcal{W}$ and

- $\mathcal{W}$ is invariant under $\mathrm{O}(2)$;
- $\mathcal{W}$ is large enough to be homologically computable and such that the boundary map $\delta : H^{n-3}_{S^1}(\mathcal{R};\mathbb{Q}) \longrightarrow H^{n-2}_{S^1}(\mathcal{W},\mathcal{R};\mathbb{Q})$ is injective;
- $\mathcal{W}$ is small in the sense that elements $f \in \mathcal{W}$ are not far from being regular (only a few singularity types permitted, etc.).

We define our $\mathcal{W}$ by disallowing all but the most common singularity types of smooth maps to the plane (fold, cusp, swallowtail, lips, beak-to-beak), and also (new idea) by disallowing certain singular features *in the target*, such as two cusps in the source $D^n \times D^2$ with the same value in D^2. For more details, see our recent articles (arXiv). Why this is the best choice ... we hope to defend that in forthcoming articles.

Why is $\mathcal{W}$ so defined homologically computable? To explain that I gave an overview on h-principles and manifold calculus. Let $\mathcal{E}_d$ be the category of smooth d-manifolds, with codimension zero embeddings as morphisms. (Here we assume that all objects of $\mathcal{E}$ have empty boundary; in applications it is often more realistic to assume that all objects of $\mathcal{E}$ have "the same" fixed boundary of dimension $d-1$.) For each $k = 0, 1, 2, 3, \ldots$ let $\mathcal{O}_d(k)$ be the full subcategory of $\mathcal{E}_d$ with objects $\mathbb{R}^d \times \{1, 2, \ldots, j\}$ where $0 \leq j \leq k$. Also, let $\mathcal{O}_d(\infty)$ be the union of the $\mathcal{O}_d(k)$. *Definition*: A contravariant continuous functor F from $\mathcal{E}_d$ to spaces is said to be *polynomial of degree* $\leq k$ if it is determined by its restriction to $\mathcal{O}_d(k)$. More precisely it is required that the obvious natural transformation from F to T_kF, the homotopy right Kan extension (to $\mathcal{E}_d$) of $F|\mathcal{O}_d(k)$, is an equivalence.

Here are some extreme cases. If F is polynomial of degree ≤ 1, and $F(\emptyset)$ is contractible, we also say that F *satisfies the h-principle.* Such F can also be characterized by homotopy sheaf properties: they take unions to homotopy pullbacks. If F is polynomial of degree $\leq \infty$, we also say that *the functor is analytic* (manifold calculus jargon). Examples: Let $F(M)$ be the space of smooth immersions from M to a fixed N, where $\dim(N) > d$; then F satisfies the h-principle. Or let $F(M)$ be the space of smooth embeddings from M to a fixed N, where $\dim(N) > d+2$; then F is analytic.

The functors F which appear in Vassiliev's *first main theorem* [4],[5] constitute a curious in-between case. Here $F(M)$ is defined to be the space of smooth maps $f: M \to \mathbb{R}^j$, for fixed j, which only have certain allowed singularity types. (For example $F(M)$ could be the space of smooth maps from M to $\mathbb{R}$ which have only Morse and birth-death singularities.) Vassiliev's statement is that, for many such F, the natural map from F to T_1F induces an isomorphism in homology. (It may or may not be a homotopy equivalence, as in the standard h-principle.) Manifold calculus explains what is going on. Namely, Vassiliev essentially works with the functor (*singular chain complex*) $\circ F$ or $SP^\infty \circ F$ or something equivalent. Here SP^∞ is the Dold-Thom infinite symmetric product, so that the homotopy groups of $SP^\infty(X)$ are the homology groups of X. Composition with SP^∞ fails badly to preserve the property of being polynomial of degree ≤ 1, because SP^∞ fails badly to preserve homotopy pullback squares. Vassiliev's arguments show nevertheless that $SP^\infty \circ F$ and $SP^\infty \circ T_1F$ are analytic. It is clear that the natural map from $SP^\infty \circ F$ to $SP^\infty \circ T_1F$ is an equivalence on the subcategory $\mathcal{O}_d(\infty)$, and since both functors are determined by their restrictions to $\mathcal{O}_d(\infty)$, it is an equivalence on all of $\mathcal{E}_d$. This completes the proof.

So it emerges that any sheaf properties which F may have are not essential to Vassiliev's argument, no matter how much the history of the h-principle may suggest that they are crucial. Keeping that in mind, we were able to generalize Vassiliev's theorem and so to understand some of the homological properties of spaces like $\mathcal{W}$ above.

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Twisted bundles and twisted K-theory

Max Karoubi

Many papers have been devoted recently to twisted K-theory. We offer here a more direct approach based on the notion of "twisted vector bundles". In the same vein, twisted Hilbert bundles may be used to define extended twisted K-groups.

More generally, we also analyse the notion of "twisted principal bundles" with structural group G. Under favourable circumstances, we show that the associated category is equivalent to the category of locally trivial fibrations, with an action of a bundle of groups with fibre G; which is simply transitive on each fibre. When the bundle of groups is trivial, we recover the usual notion of principal G-bundle.

As is well known, twisted K-theory is a graded group, indexed essentially by the third cohomology of the base space. The twisted vector bundles are also indexed by elements of the same group up to isomorphism. Roughly speaking, twisted K-theory appears as the Grothendieck group of the category of twisted vector bundles (or suitable Hilbert bundles). This provides a geometric description of this theory, very close in spirit to Steenrod's definition of coordinate bundles.

The usual operations on vector bundles (exterior powers, Adams operations...) are easily extended to twisted vector bundles.

Finally we define connections on twisted vector bundles in a quite elementary way. From this analog of Chern-Weil theory, we deduce a "Chern character" from twisted K-theory to twisted cohomology (indexed by a 3-dimensional de Rham class) which is an isomorphism up to torsion, as in the classical framework. The contents of this lecture are detailed on the Web site
`http://www.math.jussieu.fr/~karoubi/Publications.html`

The Farrell-Jones conjecture for $SL_n(\mathbb{Z})$

Holger Reich

(joint work with Arthur Bartels, Wolfgang Lück, Henrik Rüping)

The Farrell-Jones conjecture predicts that the algebraic K-theory of a group ring RG can be assembled from the algebraic K-theory of the rings RH, where H runs over all virtually cyclic subgroups of G. There is an analogous conjecture for L-theory. Among the most important applications of this conjecture is the Borel conjecture, which asserts that aspherical manifolds with fundamental group G are topologically rigid. Many other applications and the state of the art concerning these conjectures have been surveyed by Wolfgang Lück in his talk at the same conference.

We prove the conjecture for the groups $SL_n(\mathbb{Z})$. In fact we prove both the K- and L-theory conjecture in the more general version with coefficients in an additive category (with involutions). Since these generalized versions have good inheritance properties one can easily deduce the following statement.

The K- and L-theoretic Farrell-Jones conjecture with coefficients in an additive category holds for $GL_n(S)$, where S is any ring (not necessarily commutative) whose underlying abelian group is finitely generated. Moreover it also holds for subgroups of such groups and for finite index overgroups, i.e. groups which contain $GL_n(S)$ as a subgroup of finite index.

We would like to remark that the Borel conjecture for torsionfree discrete subgroups of $GL_n(\mathbb{R})$ is covered by work of Farrell and Jones and that the Baum-Connes conjecture is not known for $SL_n(\mathbb{Z})$ if $n \geq 4$.

Inspired by results of Farrell and Jones, earlier work of Bartels and Lück had dealt with the case of groups acting properly, cocompactly by isometries on a $CAT(0)$-space. The group $SL_n(\mathbb{Z})$ acts by isometries on a symmetric space X, which can be identified with the space of all inner products on $\mathbb{R}^n$. This is a Riemannian manifold with nonpositive sectional curvature, but the main new technical difficulty arises from the fact that the action is not cocompact. Grayson describes how to cut out certain neighbourhoods at infinity from X in order to obtain a cocompact space. A detailed understanding of these neighbourhoods is necessary in order to prove the theorem.

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Localization in THH of Waldhausen categories

Andrew J. Blumberg

(joint work with Michael A. Mandell)

In the last two decades, trace methods have proved remarkably successful at making algebraic K-theory computations tractable via the methods of equivariant stable homotopy theory. These methods proceed by studying the cyclotomic trace map from K-theory to topological Hochschild homology (THH) and topological cyclic homology (TC). In favorable cases, the fiber of the map $K \to TC$ is well-understood after p-completion, and TC is a comparatively tractable theory.

This success focuses attention on the structural properties of THH and TC. However, K-theory and THH appear to take different inputs and have very different formal properties. For algebraic K-theory, the input is typically a Waldhausen category: A category with subcategories of cofibrations and weak equivalences. For THH, the basic input is a spectral category: A category enriched in spectra. While THH shares K-theory's additivity properties, THH seems to lack K-theory's approximation and localization properties [2]. From the perspective of the algebraic K-theory of rings and connective ring spectra, where THH is the stabilization of K-theory, this lack is surprising, as one might expect THH to inherit the fundamental properties of K-theory.

A specific example of this failure was studied at great length in the paper [4]. In order to study the localization sequence

$$K(k) \to K(A) \to K(K)$$

via trace methods, Hesselholt-Madsen established a localization cofiber sequence

$$THH(k) \to THH(A) \to THH(A|K),$$

where A is a discrete valuation ring, K its quotient field (with characteristic 0), and k the residue field (with characteristic p). These sequences sit in a commutative diagram

$$\begin{array}{ccccc} K(k) & \longrightarrow & K(A) & \longrightarrow & K(K) \\ \downarrow & & \downarrow & & \downarrow \\ THH(k) & \longrightarrow & THH(A) & \longrightarrow & THH(A|K). \end{array}$$

Both localization sequences arise from the sequence of categories

$$(C_z^b(A))^q \to C_z^b(A) \to C_q^b(A),$$

where $C_z^b(A)$ is the Waldhausen category of bounded complexes of finitely-generated projective A-modules and quasi-isomorphisms, $C_q^b(A)$ denotes the same category with rational quasi-isomorphisms as weak equivalences, and $(C_z^b(A))^q$ is the rationally acyclic objects in $C_z^b(A)$.

The lefthand terms are identified as $K(k)$ and $THH(k)$ via devissage. However, there is a discrepancy on the right: $K(C_q^b(A)) \simeq K(K)$ via Waldhausen's approximation theorem, but "$THH(C_q^b(A))$" is not equivalent to $THH(K)$. This result is hard to reconcile with the general theory of localization in THH [1]. Specifically, when

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is a sequence of pre-triangulated spectral categories such that

$$\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{B}) \to \mathrm{Ho}(\mathcal{C})$$

is a quotient sequence (i.e., the map from the Verdier quotient $\mathrm{Ho}(\mathcal{B})/\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{C})$ is cofinal), then there exists a localization cofiber sequence

$$THH(\mathcal{A}) \to THH(\mathcal{B}) \to THH(\mathcal{C}).$$

Since the data for Waldhausen's localization theorem (i.e., two categories of weak equivalences $v\mathcal{C} \subset w\mathcal{C}$ and acyclics $\mathcal{C}^v$) encodes the same Bousfield localization [5], the discrepancy is puzzling.

In this paper, we construct THH for a general class of Waldhausen categories, and show that much of the apparent mismatch of formal properties is a consequence of the former mismatch of input data. We provide a theory of THH of Waldhausen categories, with connective ($WTHH^\Gamma$) and non-connective ($WTHH$) variants. Both $WTHH^\Gamma$ and $WTHH$ agrees with usual THH for rings and connective ring spectra, and there is a version of the cyclotomic trace $K(\mathcal{C}) \to WTHH^\Gamma(\mathcal{C})$.

Our construction allows us to show that the "Theorems of K-theory" hold for $WTHH^\Gamma$, $WTHH$, including Waldhausen's approximation theorem. Most interestingly, we find two different analogues of the localization sequence in Waldhausen K-theory (the "Fibration Theorem" [7]). The localization sequences for $WTHH$ agrees with the one developed in our companion paper on localization in THH of spectral categories [1]; when applied to the K-theory of schemes, this sequence produces an analogue of the localization sequence of Thomason-Trobaugh [6]. The other localization sequence for $WTHH^\Gamma$ generalizes the localization sequence of

Hesselholt-Madsen [4] above. One of the principal contributions of this paper is to provide a complete conceptual explanation of the two localization sequences of THH in relation to the localization sequence of K-theory.

As a primary application, we establish the THH localization sequences

$$THH(\mathbb{Z}) \to THH(\ell) \to WTHH^{\Gamma}(\ell|L)$$

and

$$THH(\mathbb{Z}) \to THH(ku) \to WTHH^{\Gamma}(ku|KU)$$

conjectured by Ausoni-Rognes and Hesselholt. Identifying the terms in this sequences requires in particular a devissage theorem for the left-hand term.

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NK and N^pK of commutative algebras

Charles Weibel

In Jim Davis' talk, he gave the following formula for any ring R:

$$N^{p+1}K_q(R) \cong \oplus_{j=0}^{p} \binom{p}{j} \oplus_C NK_{q-j}(R),$$

where C ranges over all positive rays in $\mathbb{Q}^{p+1}$. On the other hand, if R is a commutitive $\mathbb{Q}$-algebra then the following formula occurs in [1]:

$$N^{p+1}K_q(R) \cong \oplus_{j=0}^{p} \wedge^j \mathbb{Q}^p \otimes \left[V^{\otimes p-j} \otimes dV^{\otimes j}\right] \otimes NK_{q-j}(R),$$

Here V is $x\mathbb{Q}[x]$ and dV is $\Omega^1_{\mathbb{Q}[x]}$ (and $d : V \to dV$ is an isomorphism). The point of this talk was to explain how the summands correspond in these formulas

To compare these, we fix a commutative ring R and an R-algebra A. Recall that for $r \in R$ the substitution $t \mapsto rt$ induces a map $NK_q(A) \to NK_q(A)$, and the inclusion $i : A[t] \cong A[t^n] \subset A[t]$ induces the Frobenius operator $F_n = i^*$ and the Verschiebung operator $V_n = i_*$. By [2], the row-and-column finite sums $\sum V_m[r_{mn}]F_n$ act on $NK_q(A)$ and make it a module over the ring Carf(R) of finite

Cartier operators. This ring contains the ring $W(R)$ of big Witt vectors as the sums $\sum V_n[r_n]F_n$. When $\mathbb{Q} \subset R$, $W(R) \cong \prod_1^\infty R$ as a ring.

There is a notion of *continuous* Carf(R)-module — one in which every element is annihilated by F_n for all large enough n — and $NK_q(R)$ is such a continuous module. These modules also arise in the context of group rings; see [3]

When $\mathbb{Q} \subset R$, there is an equivalence between this category and the category of R-modules; to an R-module M we associate $M \otimes x\mathbb{Q}[x]$ with $[r](x^n) = r^n x^n$, $V_d(x^n) = x^{nd}$ and $F_d(x^n) = dx^{n/d}$ when $d \mid n$ ($F_d(x^n) = 0$ otherwise).

Thus there are R-modules T_q such that $NK_q(R) = T_q \otimes x\mathbb{Q}[x]$. The modules T_q are described in [1] in terms of *cdh*-cohomology of the structure sheaf $\mathcal{O}_X$ and the Kähler differentials Ω^i; for example if $\dim(R) = 2$ we have $T_{-1} = H^1_{cdh}(R, \mathcal{O})$ and $NK_{-1}(R) \cong H^1_{cdh}(R, \mathcal{O}) \otimes x\mathbb{Q}[x]$.

For a positive ray C generated by $(a_0, ..., a_p) \in \mathbb{N}^{p+1}$ with $gcd\{a_i\} = 1$, the summand $T_q \otimes x^n$ of $NK_q(R)$ in Davis' formula corresponds to the summand

$$(T_q \otimes x_0^{na_0}) \otimes (x_1^{na_1} x_2^{na_2} \cdots x_p^{na_p})$$

of the second formula. Thus the ring map $R[t] \to R[x_0, ..., x_p]$ sending t to $x_0^{a_0} x_1^{a-1} \cdots x_p^{a_p}$ induces the injection $NK_q(R) \to N^{p+1}K_q(R)$ sending $T_q \otimes t\mathbb{Q}[t]$ to the sum of the terms indexed by the integer lattice points on the ray in $\mathbb{Q}^{p+1}$ of slope $(a_0, ..., a_p)$. A similar formula holds for the terms $NK_{q-j}(R)$.

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Higher Whitehead torsion and the geometric assembly map

Wolfgang Steimle

Given a space X, the *structure space* of compact n-manifold structures is, roughly speaking, the space $\mathcal{S}_n(X)$ of all pairs (M, h) where M is a compact (topological) n-manifold and $h\colon M \to X$ is a homotopy equivalence. Crossing M with the unit interval I defines a stabilization map $\mathcal{S}_n(X) \to \mathcal{S}_{n+1}(X)$; define the *stable structure space* as

$$\mathcal{S}_\infty(X) := \operatorname{hocolim}\big(\mathcal{S}_n(X) \to \mathcal{S}_{n+1}(X) \to \dots\big).$$

In the case where X is itself a compact manifold (so $\mathcal{S}_\infty(X) \neq \emptyset$), Hoehn [2] used the stable parametrized h-cobordism theorem [5] to describe the homotopy type of $\mathcal{S}_\infty(X)$ by a homotopy equivalence

$$\mathcal{S}_\infty(X) \xrightarrow{\tau \times T} \Omega\mathrm{Wh}(X) \times \mathrm{map}(X, B\mathrm{TOP})$$

where $\mathrm{Wh}(X)$ is the topological Whitehead space as defined by Waldhausen [6]. The map τ is a version of a higher Whitehead torsion, defined using the parametrized A-theory characteristic [1], while the map T describes the tangent bundle.

More generally, if $p\colon E \to B$ is a fiber bundle whose fibers are compact manifolds, there is a structure space $\mathcal{S}_n(p)$ which consists of commutative diagrams

$$\begin{array}{ccc} E' & \xrightarrow[\simeq]{\varphi} & E \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

where q is a bundle of compact n-manifolds and φ is a fiber homotopy equivalence. This space can also be stabilized and Hoehn's result says more generally that there is a homotopy equivalence

$$\mathcal{S}_\infty(p) \xrightarrow{\tau\times T} \Gamma\begin{pmatrix} \Omega\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \times \mathrm{map}(E, B\mathrm{TOP}).$$

Here $\Gamma(\dots)$ denotes the section space of a fibration $\Omega\mathrm{Wh}_B(E) \to B$ which is obtained from p, loosely speaking, by applying the functor $\Omega\mathrm{Wh}$ "fiber-wise".

Suppose now that B happens to be a compact topological manifold.

Definition. The geometric assembly map

$$\alpha\colon \mathcal{S}_\infty(p) \to \mathcal{S}_\infty(E)$$

sends a fiber homotopy equivalence of bundles over B to the underlying homotopy equivalence of compact manifolds.

Suppose that B is connected and choose $b \in B$. Let

$$\beta\colon \Gamma\begin{pmatrix} \Omega\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \xrightarrow{\text{Restr.}} \Omega\mathrm{Wh}(p^{-1}(b)) \xrightarrow{\chi(B)\cdot i_*} \Omega\mathrm{Wh}(E),$$

$$\gamma\colon \mathrm{map}(E, B\mathrm{TOP}) \to \mathrm{map}(E, B\mathrm{TOP}), \quad \xi \mapsto \xi \oplus p^*TB$$

where $\chi(B) \in \mathbb{Z}$ denotes the Euler characteristic and i_* is the inclusion-induced map.

Theorem 2 ([3]). *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \mathcal{S}_\infty(p) & \xrightarrow{\tau\times T} & \Gamma\begin{pmatrix} \Omega\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \times \mathrm{map}(E, B\mathrm{TOP}) \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta\times\gamma} \\ \mathcal{S}_\infty(E) & \xrightarrow{\tau\times T} & \Omega\mathrm{Wh}(E) \times \mathrm{map}(E, B\mathrm{TOP}) \end{array}$$

Theorem 2, in combination with the "Riemann-Roch theorem with converse" by Dwyer-Weiss-Williams [1], may be applied to fibering questions: Given a map $f\colon M \to B$ between compact manifolds, say that f *fibers stably* if there exists an $n \in \mathbb{N}$ such that the composite

$$M \times D^n \xrightarrow{\text{proj}} M \xrightarrow{f} B$$

is homotopic to the projection map of a fiber bundle with manifold fibers.

Theorem 3 ([4])**.** *Let $f\colon M \to B$ be a map between compact manifolds where B is connected. Then f fibers stably if and only if*

(1) the homotopy fibers of f are finitely dominated,
(2) a "parametrized Wall obstruction"

$$\mathrm{Wall}(f) \in \Gamma\begin{pmatrix} \mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix}$$

is nullhomotopic, and
(3) a secondary obstruction

$$o(f) \in \operatorname{coker} \pi_0(\beta)$$

is zero.

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Isomorphism conjectures for proper actions

Guillermo Cortiñas
(joint work with Eugenia Ellis)

This is a report on joint work with Eugenia Ellis. Let G be a group, $\mathcal{F}$ a family of subgroups, A a ring on which G acts by automorphisms (i.e. a G-ring), and E a functor from $\mathbf{Z}$-linear categories to spectra. Under very general conditions on E and A, there is defined an assembly map

$$H_*^G(\mathcal{E}(G,\mathcal{F}), E(A)) \to E_*(A \rtimes G) \qquad (1)$$

from G-equivariant E-homology with coefficients in A of the classifying space of G relative to $\mathcal{F}$, to the E-groups of the crossed product of A by G. We introduce a notion of properness (modelled on the analogous notion for C^*-algebras) and give sufficient conditions on E so that the assembly map be an isomorphism when A is proper. For example if X is a locally finite simplicial set with an action of G such that all stabilizers are in $\mathcal{F}$ (i.e. X is a $(G;\mathcal{F})$-*complex*), then the ring $\mathbf{Z}^{(X)}$ of finitely supported integral polynomial functions on X is $(G;\mathcal{F})$-proper. In general a G-ring A is $(G;\mathcal{F})$-proper if there is a $(G;\mathcal{F})$-complex X such that A is proper over X; this means that A is an algebra over $\mathbf{Z}^{(X)}$ in such a way that the actions of G on A and $\mathbf{Z}^{(X)}$ are compatible, and that $\mathbf{Z}^{(X)} \cdot A = A$. We show that if E is a sufficiently good theory, such as Weibel's homotopy K- theory KH, then (1) is an isomorphism for every proper $(G;\mathcal{F})$-ring A. We view this as an algebraic analogue of the fact that the Baum-Connes conjecture with proper coefficients holds ([1]). A key property that we use is that KH satisfies excision; if $I \triangleleft R$ is an ideal then the fiber $KH(R:I)$ of $KH(R) \to KH(R/I)$ depends only on I, and not on R. We remark that Quillen's K-theory does not satisfy excision; the not necessarily unital rings I for which $K(R:I)$ depends only on I are called K-excisive. For example $\mathbf{Z}^{(X)}$ is K-excisive if X is locally finite. We show that the K-theory assembly map (1) is a rational isomorphism for every $(G;\mathcal{F})$-proper K-excisive ring A, and an integral isomorphism if in addition A is a $\mathbb{Q}$-algebra. We also show that for a functor E satisfying rather mild assumptions (which hold when $E=K$), the assembly map (1) is an isomorphism when A is E-excisive and proper over a 0-dimensional $(G;\mathcal{F})$-space. This is already enough to give an algebraic analogue of the Dirac-dual Dirac method from Baum-Connes' theory which applies to the $(G,\mathcal{F},E)$-isomorphism conjecture with E-excisive coefficients.

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The Beilinson-Soule vanishing conjecture and rational K-theory in characteristic p

Thomas Geisser

This is a survey on open problems in algebraic K-theory. No new results were presented.

Recently, many of the fundamental properties and conjectures on algebraic K-theory of schemes have been established, most notably the Beilinson-Lichtenbaum conjectures and the spectral sequence from motivic cohomology to algebraic K-theory. Among the remaining properties, the following two conjectures seem to be the most important and least accessible:

Conjecture 0.1. *(Bass, 1973) For any finitely generated (commutative) algebra over the integers, the groups $K_i'(R)$ are finitely generated abelian groups.*

Here $K_i'(R)$ is the K-theory of finitely generated R-modules. Except in special cases, this is only known for R of dimension 1, by work of Quillen.

Conjecture 0.2. *(Beilinson-Soule vanishing conjecture, 1985) For any regular ring, R we have* $K_m(R) = F_\gamma^n K_m(R)$ *for* $m \geq \max\{2n-1, 1\}$.

In other words, the Adams eigenspace $K_m(R)^{(n)}$ vanishes for $n < \frac{2m+1}{2}$ except for $K_0(R)^{(0)}$. Beilinson gave an ad hoc definition for motivic cohomology

$$H^i_M(X, \mathbb{Q}(n)) := K_{2n-i}(X)^{(n)},$$

and in this language the conjecture says that $H^i_M(X, \mathbb{Q}(n))$ vanishes for $i < 1$, except in case $i, n = 0$. As candidates for motivic cohomology, Bloch constructed higher Chow groups $CH_n(X, i) = H^{BM}_{i+2n}(X, \mathbf{Z}(n))$ (as indicated, they actually form a Borel-Moore homology theory), and later Voevodsky gave a definition of motivic cohomology groups $H^i_M(X, \mathbf{Z}(n))$. The analog of Poincaré duality holds: For X smooth of dimension d one has $H^{BM}_i(X, \mathbf{Z}(n)) \cong H^{2d-i}_M(X, \mathbf{Z}(d-n))$. By definition, $H^i_M(X, \mathbf{Z}(n))$ vanishes for $i > 2n$ or $i > \dim X + n$ if X is smooth, and $H^{2n}_M(X, \mathbf{Z}(n)) = CH^n(X)$ is the usual Chow group in codimension n. There is a spectral sequence from Borel-Moore homology to algebraic K'-theory for X of finite type over a field or Dedekind ring,

$$E_2^{p,q} = H^{BM}_{p+q}(X, \mathbf{Z}(-q)) \Rightarrow K'_{-p-q}(X).$$

This spectral sequence degenerates modulo small torsion because the Adams operator ψ^r acts like r^n on the term $H^{BM}_{p+q}(X, \mathbf{Z}(d-n))$, so that Beilinson's ad hoc definition becomes a theorem. In characteristic p, one has the following strengthening of the vanishing conjecture:

Conjecture 0.3. *(Parshin) For* X *smooth and proper over a finite field, the group* $K_i(X)$ *is torsion for* $i > 0$.

This conjecture is motivated by the idea that higher algebraic K-groups are related to extensions in a conjectural category of mixed motives, whereas over a finite field, such a category would be semi-simple. Using the niveau spectral sequence

$$E_1^{pq} = \bigoplus_{x \in X^{(p)}} H^{q-p}_M(k(x), \mathbf{Z}(n-p)) \Rightarrow H^q_M(X, \mathbf{Z}(n)),$$

where $X^{(p)}$ denotes the points x of X whose closure has codimension p, one can show that Parshin's conjecture implies

a) $H^i_M(U, \mathbb{Q}(n)) = 0$ for all smooth U over $\mathbb{F}_q$ and all $i < n$.
b) For all fields over $\mathbb{F}_q$, $H^i_M(F, \mathbb{Q}(n)) = 0$ unless $i = n$ and $n \leq trdeg_{\mathbb{F}_q} F$.

The last statement means that the niveau spectral sequence rationally collapses to one line $q = n$.

Since $K_i(X)_\mathbb{Q} = \oplus_n H^{2n-i}_M(X, \mathbb{Q}(n))$, Parshin's conjecture is equivalent to the vanishing of $H^i_M(X, \mathbb{Q}(n))$ for all smooth and projective schemes X over a finite field, and all $i \neq 2n$. Except in special cases, the conjecture is only known in the case $\dim X = 1$ by work of Harder and Soule. Soule's argument is as follows: A curve C decomposes in the category of (pure, Chow) motives into a direct sum $\mathbb{P}^1 \oplus C^+$, and the K-theory of the projective line is known. By work of Weil, we know that the characteristic polynomial P_{C^+} of the Frobenius endomorphism F_{C^+} has roots of absolute value $\sqrt{q}$ on the one hand, and acts like q^n on $H^i_M(C^+, \mathbb{Q}(n))$ on the other hand. Hence $0 \neq P_{C^+}(q^n) = P_{C^+}(F_{C^+}) = 0$ on motivic cohomology, so that the group is torsion.

Parshin's conjecture on higher K-theory is complemented by a conjecture of Beilinson for $K_0(X)_{\mathbb{Q}} \cong \oplus_n CH^n(X)$:

Conjecture 0.4. *For all smooth and proper schemes X over a finite field, the intersection pairing*

$$CH^n(X)_{\mathbb{Q}} \times CH_n(X)_{\mathbb{Q}} \to CH_0(X) \to \mathbb{Q}$$

is non-degenerate.

Non-degeneration implies finite dimensionality of the rational Chow groups, hence perfectness of the pairing. A uniform formulation of Parshin's and Beilinson's conjecture is the statement that for all smooth and projective schemes X over a finite field $\mathbb{F}_q$, the cup product pairing

$$H^i_M(X, \mathbb{Q}(n)) \times H^{2d-i}_M(X, \mathbb{Q}(d-n)) \to \mathbb{Q}$$

is perfect. The role of the Hodge conjecture in characteristic p is played by

Conjecture 0.5. *(Tate, 1965) For X smooth and proper over a finitely generated field k, the cycle map to the fixed set under the Galois group of etale cohomology,*

$$CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}_{et}(X \times_k \bar{k}, \mathbb{Q}_l(n))^{Gal\ k}$$

is surjective.

Note that the conjecture of Beilinson stated above implies the injectivity of the map. In [3], we showed that if Tate's conjecture and Beilinson's conjecture hold, then Parshin's conjecture holds, giving some evidence for Parshin's conjecture. The method is a modification of Soule's original argument.

For more general fields of characteristic p, one can ask the following question (generalization of Parshin's conjecture):

If X is smooth and proper over a field F of characteristic p, is the rational K-theory multiplicatively generated by $K_0(X)_{\mathbb{Q}}$ and the Milnor K-theory $K^M_*(F)_{\mathbb{Q}}$?

In terms of motivic cohomology, this would mean that there is a surjection

$$CH^n(X)_{\mathbb{Q}} \otimes K^M_{i-n}(F)_{\mathbb{Q}} \to H^{i+n}_M(X, \mathbb{Q}(n)).$$

What can one say about the kernel?

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Algebraic K-theory of $\mathbb{Z}[\Gamma]$ for Γ the braid group of a surface.

Daniel Juan-Pineda
(joint work with J. Guaschi., S. Millán)

Let Γ be the braid group of a closed connected surface. When the surface is not the sphere nor the projective plane, it was proven by S. Aravinda, T. Farrell and K. Roushon, [1], that the Whitehead group vanishes for both the pure and the full braid groups in any number of strands. The technique is to prove that the Farrell-Jones isomorphism conjecture holds for these groups. In recent work we study the case when the surface is the sphere or the projective plane. In case of the pure braid groups, Juan-Pineda and S. Millán proved that most of the Whitehead groups vanish. In fact we proved that these groups vanish when we have at least four strands.

The case of the full braid groups of the sphere is much more complex, this is mainly due to the following facts:

(1) The family of finite subgroups is bigger: it has cyclic groups of order that grows with the number of strands, it has also generalized quaternion groups whose orders grow with the number of strands.
(2) The family of infinite virtually cyclic groups has elements of arbitrary large order, this implies that there will be non-trivial Nil groups in the K-theory of $\mathbb{Z}$ for almost all groups.

We will present general results for the K groups and outline in detail the case of four strands, we prove that the Whitehead group is not finitely generated in this case.

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Preorientations of the derived motivic multiplicative group

Jens Hornbostel

Recently, Jacob Lurie [Lu1] gave a description of the spectrum tmf (= "topological modular forms") as the solution of a moduli problem in derived algebraic geometry. The latter here is constructed with commutative ring spectra as the affine derived schemes, and the moduli problem is to classify derived oriented elliptic curves with all terms defined appropriately. Lurie sketches a proof of this theorem in the language of infinity categories in [Lu1].

The above description of tmf corresponding to height 2 and the second chromatic layer has an analog in height 1 which is much easier to state and to prove, and is also due to Lurie [Lu1, §3]. Namely, real topological K-theory KO classifies oriented derived multiplicative groups. The key step for proving this is to show that the suspension spectrum of $\mathbf{CP}^\infty$ classifies preorientations of the derived multiplicative group. Here the derived multiplicative group is by definition $\mathbf{G}_m := \Sigma^\infty \mathbf{Z}_+$, in analogy with the multiplicative group $Spec(k[\mathbf{Z}])$ in classical algebraic geometry over a base field k. As usual, the object $Rmap_{AbMon(Sp^\Sigma)}(\Sigma^\infty \mathbf{Z}_+, -))$ it represents via the derived version of the Yoneda embedding will still be called the multiplicative group.

We are able to provide a proof of this result in the language of model categories and symmetric spectra Sp^Σ, and present some of its ingredients in our talk. The result reads as follows in general, the special case $N = \mathbf{CP}^\infty$ being the one discussed above:

Theorem 1. *(Lurie) For any abelian monoid A in symmetric spectra Sp^Σ (based on simplicial sets) and any simplicial abelian group N, we have a natural isomorphism of abelian groups*

$$\begin{aligned} &Hom_{Ho(AbMon(Sp^\Sigma))}(\Sigma^\infty N_+, A) \\ \simeq\ &Hom_{Ho(AbMon(\Delta^{op}Sets))}(N, Rmap_{AbMon(Sp^\Sigma)}(\Sigma^\infty \mathbf{Z}_+, A)) \\ =\ &Hom_{Ho(AbMon(\Delta^{op}Sets))}(N, \mathbf{G}_m(A)). \end{aligned}$$

Here $Ho(-)$ denotes the homotopy category, $Rmap$ means the derived mapping space and the weak equivalences between abelian monoids are always the underlying ones, forgetting the abelian monoid structure. We explain the model structures involved in this theorem, which are due to Hovey-Shipley-Smith, Harper and others (see in particular [HSS], [Sh], [Ha]). Among the ingredients of the proof we then discuss are a model category refinement of the recognition principle and a new non-positive model structure for E-modules in Sp^Σ where E is the Barratt-Eccles operad, thus avoiding the "Lewis paradoxon". Using a theorem of Snaith [Sn], Lurie's definition of an orientation and the above theorem then imply his above theorem about KO.

We then discuss the motivic generalization of this theorem, that is to motivic symmetric spectra $Sp^{\Sigma,T}(\mathcal{M})$ on the site $\mathcal{M} = (Sm/S)_{Nis}$ with S an arbitrary noetherian base scheme. For this, we must establish various motivic model structures on categories built from motivic symmetric spectra with respect to both circles S^1 and $\mathbf{P}^1$ and suitable model structures, the first results here being due to Hovey and Jardine. Once we have established all necessary model structures and some of their properties, the main theorem then is as follows.

Theorem 2. *Let $\mathcal{M} = (Sm/S)_{Nis}$ and $T = S^1$ or $T = \mathbf{P}^1$. Then for any abelian monoid A in motivic symmetric T-spectra $Sp^{\Sigma,T}(\mathcal{M})$ and any abelian group N in the category $\Delta^{op}PrShv(\mathcal{M})$ of simplicial presheaves on $\mathcal{M}$, we have a natural isomorphism of abelian groups*

$$Hom_{Ho(AbMon(Sp^{\Sigma,T}(\mathcal{M})))}(\Sigma_T^\infty N_+, A)$$

$$\simeq Hom_{Ho(AbMon(\Delta^{op}PrShv(\mathcal{M})))}(N, Rmap_{AbMon(Sp^{\Sigma,T}(\mathcal{M}))}(\Sigma_T^\infty \mathbf{Z}_+, A))$$

Appliying this theorem to $T = \mathbf{P}^1$ pointed at ∞ and to $N = \mathbf{P}^\infty$ which is not a variety but still a simplicial presheaf, and using the recently established motivic version of Snaith's theorem [GS], [SO], this implies that *algebraic K-theory represents motivic orientations of the derived motivic multiplicative group*, provided one works with the correct motivic generalizations of the concept of derived algebraic groups and of orientations. As a corollary of the motivic model structures we establish, we see that these satisfy both the conditions for an HA-context in the sense of Toën-Vezzosi [TV] and the axioms of Goerss-Hopkins [GH] for doing E_∞-obstruction theory.

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Discrete cocompact group actions and the assembly map

Ian Hambleton
(joint work with Erik Pedersen)

We are interested in a question of Farrell and Wall: which discrete groups can act freely, properly and cocompactly on some product $S^n \times \mathbb{R}^k$? In [1], Connolly and Prassidis proved that any countable discrete group Γ with $\operatorname{vcd}\Gamma < \infty$ and periodic Farrell cohomology can act freely and properly on some $S^n \times \mathbb{R}^k$, but did not produce cocompact actions. Connolly and Prassidis also proved:

Theorem ([1])**.** *Suppose that Γ is a virtual Poincaré duality group with periodic Farrell cohomology. Then there exists a finite Poincaré complex X of dimension $n+k$, where $k = \operatorname{cd}\Gamma$, with fundemental group Γ and universal covering $\widetilde{X} \simeq S^n$.*

This result leads to questions in surgery theory: (i) can X be constructed so that its Spivak normal fibre space is reducible (is fibre-homotopy equivalent to a topological sphere bundle), and (ii) does there exist such an X which is homotopy equivalent to a closed manifold?

A positive answer to (i) would lead to a degree 1 normal map $f\colon M \to X$, with a surgery obstruction in the Wall group $L_n(\mathbb{Z}\Gamma)$. If this surgery problem could be solved, then (M, f) would be normally cobordant to a manifold, whose universal covering would then answer the existence question for the given group Γ. The main result presented in the talk was:

Theorem A. *Suppose that Γ has periodic Farrell cohomology and a normal finite index subgroup Γ_0 which is the fundamental group of a closed aspherical manifold. Then there is a finite Poincaré complex X as above, with reducible Spivak normal fibre space.*

The method of proof for Theorem A gives an equivariant surgery problem, with target X, blocked over the finite dimensional classifying space $\mathcal{E}\Gamma$ for proper Γ-actions. We obtain an element in the controlled L-group over $\mathcal{E}\Gamma$ defined by Hambleton and Pedersen [3, §7]. The controlled assembly map applied to this element gives the surgery obstruction to existence of a free cocompact Γ action on $S^n \times \mathbb{R}^k$. This is similar to the blocked surgery construction in our proof of Theorem 8.1 in [2].

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