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Theory and Applications of Discontinuous Galerkin Methods

Organised by
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Abstract. Theory and Application of Discontinuous Galerkin Methods.

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Introduction by the Organisers

The workshop was organized by Susanne C. Brenner (Louisiana State University), Ronald Hoppe (University of Houston, Augsburg University) and Béatrice Rivière (Rice University). There were twenty-one lectures. The meeting was well attended with participants from Europe, England, North America and India. Both theoretical and computational talks on state-of-the-art discontinuous Galerkin methods were given by leading experts as well as by promising young mathematicians. The workshop addressed important issues in the development of discontinuous Galerkin methods. These issues include (but are not limited to) the formulation of more efficient methods with fewer number of degrees of freedom, the formulation of methods for coupled problems, the analysis of guaranteed a posteriori error estimation, and faster and accurate solvers. Scientific discussions continued in the evening among smaller groups of participants.

Workshop: Theory and Applications of Discontinuous Galerkin Methods

Table of Contents

Abstracts

Schwarz methods for a preconditioned WOPSIP discretization of elliptic problems

Paola F. Antonietti

(joint work with Blanca Ayuso de Dios, Susanne C. Brenner, Li-yeng Sung)

The aim of this talk is to propose and analyze a class of non-overlapping Schwarz methods for a preconditioned weakly over-penalized symmetric interior penalty (WOPSIP) discretization of a second order boundary value problem.

The WOPSIP method has been introduced in [6] for the discretization of the weak form of the Poisson equation with homogeneous Dirichlet boundary conditions:

$$\text{Find } u_h \in V_h : \quad \mathcal{A}_h(u_h, v) = \int_\Omega f v \, \mathrm{dx} \quad \forall v \in V_h, \tag{1}$$

where V_h is the space of piecewise linear discontinuous functions defined on a triangulation $\mathcal{T}_h$ of granularity h. Denoting by $\mathcal{E}_h$ the set of all edges of the partition $\mathcal{T}_h$, the WOPSIP bilinear form $\mathcal{A}_h(\cdot,\cdot) : V_h \times V_h \longrightarrow \mathbb{R}$ reads

$$\mathcal{A}_h(w,v) := \sum_{T\in\mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, \mathrm{dx} + \sum_{e\in\mathcal{E}_h} \frac{\alpha}{h_e^3} \int_e \Pi_e^0([\![w]\!]) \cdot \Pi_e^0([\![v]\!]) \, \mathrm{ds} \quad w, v \in V_h.$$

Here $\alpha \geq 1$ is a parameter at our disposal, h_e is the length of an edge $e \in \mathcal{E}_h$, $[\![\cdot]\!]$ is the jump operator defined according to [3], and $\Pi_e^0(\cdot)$ is the L^2-orthogonal projection onto the space of constant functions over $e \in \mathcal{E}_h$, i.e.,

$$\Pi_e^0(v) := \frac{1}{h_e} \int_e v \, \mathrm{ds} \quad \forall\, e \in \mathcal{E}_h \quad \forall v \in V_h.$$

As shown in [6], the WOPSIP method (1) is stable for any choice of the penalty parameter α, and exhibits optimal error estimates both in the L^2 norm and in a suitable energy norm. However, due to the over-penalization, the condition number of the WOPSIP stiffness matrix grows as $\mathcal{O}(h^{-4})$. For such a reason, a block-diagonal, symmetric and positive definite operator can be introduced with the aim of effectively reduce the condition number of the WOPSIP method [6]. More precisely, let the bilinear form $\mathcal{B}_h(\cdot,\cdot) : V_h \times V_h \longrightarrow \mathbb{R}$ be defined as

$$\mathcal{B}_h(w,v) := \sum_{T\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_h} w|_T(m_e) v|_T(m_e) + \sum_{e\in\mathcal{E}_h} \frac{\alpha}{h_e} \int_e \Pi_e^0([\![w]\!]) \cdot \Pi_e^0([\![v]\!]) \, \mathrm{ds}$$

for all $w, v \in V_h$, where m_e denotes the midpoint of an edge $e \in \mathcal{E}_h$. Denoting by $\mathsf{B}_h : V_h \longrightarrow V_h'$ the discrete operator associated with the bilinear form $B_h(\cdot,\cdot)$, i.e.,

$$< \mathsf{B}_h w, v > := \mathcal{B}_h(w,v) \quad w, v \in V_h,$$

we consider the following P-WOPSIP method:

$$\text{(2)} \qquad \text{Find } u_h \in V_h : \quad \mathcal{A}_h(\mathsf{B}_h^{-1/2} u_h, \mathsf{B}_h^{-1/2} v) = \int_\Omega f\, \mathsf{B}_h^{-1/2} v \, \mathrm{dx} \quad \forall v \in V_h.$$

Notice that the operator $\mathsf{B}_h^{-1/2} : V_h' \longrightarrow V_h$ is symmetric and positive definite, thanks to the fact that $\mathcal{B}_h(\cdot,\cdot)$ is symmetric and positive definite. Moreover, in [6] it was shown that

$$\mathcal{B}_h(v,v) \lesssim \mathcal{A}_h(v,v) \lesssim h^{-2}\mathcal{B}_h(v,v) \quad \forall v \in V_h.$$

The above estimate guarantees that the condition number of the stiffness matrix associated with the P-WOPSIP method is of order $\mathcal{O}(h^{-2})$ as typical finite element discretizations of second order elliptic problems. Moreover, in [5] it is shown that, by a suitable ordering of the degrees of freedom, the P-WOPSIP method has an intrinsic high-level of parallelism, making the P-WOPSIP discretization technique an ideal method for parallel computations.

In this talk we will show that, by employing a suitable ordering of the degrees of freedom and exploiting the orthogonal decomposition of the DG space proposed in [4], the action of the operator $\mathsf{B}_h^{-1/2}$ can be fully characterized, and that the P-WOPSIP bilinear form is continuous and coercive in the following (standard) DG norm

$$\|v\|_{\mathrm{DG}}^2 := \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_h} \frac{1}{h_e} \|\Pi_e^0([\![v]\!])\|_{0,e}^2 \quad \forall v, w \in V_h.$$

Lemma 1 (Continuity and coercivity of the P-WOPSIP method [2])**.** *It holds*

$$\mathcal{A}_h(\mathsf{B}_h^{-1/2} w, \mathsf{B}_h^{-1/2} v) \lesssim \|w\|_{\mathrm{DG}} \|v\|_{\mathrm{DG}} \quad \forall w, v \in V_h.$$

Moreover, there exists $h_0 > 0$ such that

$$\mathcal{A}_h(\mathsf{B}_h^{-1/2} v, \mathsf{B}_h^{-1/2} v) \gtrsim \|v\|_{\mathrm{DG}}^2 \quad \forall v \in V_h.$$

provided the mesh size $h < h_0$.

The above result guarantees, in particular, that problem (2) is well posed. Moreover, Lemma 1 also suggests a spectral equivalence of the P-WOPSIP method with all the stable and strongly consistent DG discretizations proposed so far in the literature.

In order to reduce further the condition number of the matrix arising from the P-WOPSIP discretization, in this talk we will also propose and analyze several two-level non-overlapping Schwarz methods for the P-WOPSIP method. We will consider both *exact* and *inexact* local solvers as the ones proposed in [7] and [1], respectively. Following the abstract theory of Schwarz methods [8], we will show that the preconditioners are scalable (i.e., the performance are independent on the number of suddomains), and that the condition number of the resulting preconditioned linear systems of equations is of order $O(Hh^{-1})$, being H and h the granularity of the coarse and fine partitions, respectively. Moreover, our

condition number estimates are independent of the penalty parameter. This is indeed a novelty in the framework of solution techniques for DG approximations. Finally, numerical experiments that illustrate the performance of the proposed two-level Schwarz methods will be also presented.

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DG$-H$(div) Conforming Approximations for Stokes Problem: A Simple Preconditioner

Blanca Ayuso de Dios

(joint work with Franco Brezzi, L. Donatella Marini, Jinchao Xu and L.T. Zikatanov)

Block preconditioners are generally employed to solve the linear algebraic systems of saddle point type that arise from Finite Element (FE) discretizations of fluid flow problems, such as the Stokes problem. Although in many cases they can be shown to be optimal (guaranteeing uniform convergence with respect to the mesh size of the discretization, in the asymptotic limit), a very slow convergence of the resulting iterative method is often observed. A possible reason for this could be related to the treatment of the divergence-free condition on the velocity field at the discretization level, which is typically enforced only weakly. In this talk, following an approach similar to that of [5], we construct Discontinuous Galerkin (DG) approximations to the Stokes problem where the velocity field is $H(\mathrm{div}, \Omega)$-conforming. This implies that the velocity solution is divergence-free in the whole domain. We show how this property can be exploited to design a simple and effective preconditioner (actually one of the blocks), assuming one has an optimal and efficient solver for elliptic second order problems (AMG, GMG or any of your

favorite methods).

• **Model Problem:** Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a polygonal or polyhedral domain with boundary $\Gamma = \partial\Omega$ and $\mathbf{f} \in [L^2(\Omega)]^d$. The Stokes equations we consider read

$$\begin{cases} -\mathrm{div}(2\nu\boldsymbol{\varepsilon}(\mathbf{u})) + \boldsymbol{\nabla}\mathrm{p} = \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{1}$$

$\mathbf{u}$ is the velocity field, p the pressure, ν the viscosity of the fluid, and $\boldsymbol{\varepsilon}(\mathbf{u}) \in [L^2(\Omega)]^{d\times d}_{\mathrm{sym}}$ is the symmetric (linearized) strain rate tensor defined by $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T)$. Equations (1) are completed with no-slip boundary conditions on $\mathbf{u}$ and a natural condition on the tangential component of the normal stresses:

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \text{and} \quad ((2\,\nu\,\boldsymbol{\varepsilon}(\mathbf{u}) - p\mathbf{I})\cdot\mathbf{n})\cdot\mathbf{t} = 0 \quad \text{on } \Gamma. \tag{2}$$

From now, we present the ideas in $d = 2$, pointing out the differences that arise for $d = 3$ in the end.

• **Abstract Setting:** The finite element partition $\mathcal{T}_h$ is assumed to be shape-regular and with no hanging nodes. The main ingredient is the choice of a triplet of finite element spaces denoted by $(\mathbf{V}_h, \mathcal{Q}_h, \mathcal{N}_h)$ and defined as:

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) \ : \ \mathbf{v}|_K \in \mathcal{R}(K)\ \forall\, K \in \mathcal{T}_h,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\},$$

$$\mathcal{Q}_h := \left\{q \in L^2(\Omega)/\mathbb{R} \ : \ q|_K \in \mathcal{Q}(K)\ \forall\, K \in \mathcal{T}_h\right\},$$

$$\mathcal{N}_h := \left\{\varphi \in H^1_0(\Omega) \ : \ \varphi|_K \in \mathcal{M}(K)\ \forall\, K \in \mathcal{T}_h\right\},$$

where $\mathcal{R}(K)$ is a vector FE space and $\mathcal{Q}(K)$ and $\mathcal{M}(K)$ are scalar FE spaces. Our basic assumptions on the triplet of spaces $(\mathbf{V_h}, \mathcal{Q}_h, \mathcal{N}_h)$ is that they have to be related by the following exact sequence:

$$0 \longrightarrow \mathcal{N}_h \ \xrightarrow{\mathbf{curl}}\ \mathbf{V}_h \ \xrightarrow{\mathrm{div}}\ \mathcal{Q}_h \longrightarrow 0, \tag{3}$$

and each operator in (3) must have a uniformly bounded (in h) continuous right inverse. The spaces are also assumed to have some approximation property.
From the triplet $(\mathbf{V}_h, \mathcal{Q}_h, \mathcal{N}_h)$, only the first two spaces are used in the construction of the DG $-H(\mathrm{div};\Omega)$ methods; the third one will be only used for the design of the preconditioner. We note also that $\mathcal{N}_h$ arises naturally in the construction of the discrete Helmholtz decomposition of $\mathbf{V}_h$:

$$\mathbf{V}_h = \mathcal{G}_h(\mathcal{Q}_h) \oplus \mathbf{curl}\,\mathcal{N}_h, \tag{4}$$

where the discrete gradient operator $\mathcal{G}_h : \mathcal{Q}_h \longrightarrow \mathbf{V}_h$ is defined by [3]

$$(\mathcal{G}_h q_h, \mathbf{v}_h)_{0,\Omega} := -(q_h, \mathrm{div}\,\mathbf{v}_h)_{0,\Omega} \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h.$$

Examples of triplets are easily given by choosing $(\mathbf{V_h}, \mathcal{Q}_h)$ as one of the $H(\mathrm{div};\Omega)$ conforming mixed FE methods for second order problems: say Raviart-Thomas or Brezzi-Douglas-Marini elements [2]. In both cases, $\mathcal{N}_h$ reduces to the standard conforming FE space of Lagrange polynomials (of one degree higher than that of $\mathbf{V}_h$).

• **The DG-$H(\mathrm{div};\Omega)$ methods read:** *Find* $(\mathbf{u}_h, p_h)$ *in* $\mathbf{V}_h \times \mathcal{Q}_h$ *such that*

$$\text{(5)} \qquad \begin{cases} a_h(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}) & \forall\, \mathbf{v} \in \mathbf{V}_h \\ b(\mathbf{u}_h, q) = 0 & \forall\, q \in \mathcal{Q}_h, \end{cases}$$

where since $\mathbf{V}_h \subset H(\mathrm{div};\Omega)$ the bilinear forms reduce to

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) &:= 2\nu \left[(\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{T}_h} - \langle \{\{\boldsymbol{\varepsilon}(\mathbf{u})\}\} : [\![\mathbf{v}_t]\!] \rangle_{\mathcal{E}_h^o} - \langle [\![\mathbf{u}_t]\!] : \{\{\boldsymbol{\varepsilon}(\mathbf{v})\}\} \rangle_{\mathcal{E}_h^o} \right] \\ &\quad + 2\nu\alpha \sum_{e \in \mathcal{E}_h^o} h_e^{-1} \int_e [\![\mathbf{u}_t]\!] : [\![\mathbf{v}_t]\!]\, ds \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}_h, \end{aligned} \tag{6}$$

$$b(\mathbf{v}, q) := -(q, \mathrm{div}\, \mathbf{v})_\Omega \qquad \forall\, \mathbf{v} \in \mathbf{V}_h,\ \forall q \in \mathcal{Q}_h.$$

We prove stability following the classical framework in [3]; i.e, showing coercivity of $a_h(\cdot,\cdot)$ and an inf-sup condition for $b(\cdot,\cdot)$, in appropriate norms. However, unlike in [5], coercivity of $a_h(\cdot,\cdot)$ does not follow straightforwardly and we have to prove a discrete Korn's inequality. This is so, since the results in [4] do not apply directly here because the jumps of $\mathbf{u}_h$ are not penalized on boundary edges/faces (see definition of $a_h(\cdot,\cdot)$ in (6)). Then, the error analysis can be completed following [3], showing optimal error estimates for both velocity and pressure.

• **Preconditioning strategy:** The preconditioning strategy is based on a key property of the methods: that the solution of (5) $\mathbf{u}_h \in \mathbf{V}_h$ is divergence free on the whole Ω, i.e, $\mathbf{u}_h \in \mathbf{V}_h \cap \mathbf{H}(\mathrm{div}^0;\Omega)$. Therefore, taking into account the discrete Helmholtz decomposition (4) it follows that for such $\mathbf{u}_h$

$$\text{there exists a unique } \psi_h \in \mathcal{N}_h \quad \text{such that} \quad \mathbf{u}_h = \mathbf{curl}\, \psi_h. \tag{7}$$

Then, restricting the bilinear form $a_h(\cdot,\cdot)$ to $\mathbf{V}_h \cap \mathbf{H}(\mathrm{div}^0;\Omega)$ corresponds to restrict trial and test spaces to $\mathbf{curl}(\mathcal{N}_h)$; that is *Find* $\psi_h \in \mathcal{N}_h$ *such that*

$$\widetilde{a}_h(\psi_h, \varphi_h) := a_h(\mathbf{curl}\, \psi_h, \mathbf{curl}\, \varphi_h) = (\mathbf{f}, \mathbf{curl}\, \varphi_h) \qquad \forall\, \varphi_h \in \mathcal{N}_h. \tag{8}$$

Observe now that, in order to compute the approximate velocity $\mathbf{u}_h \in \mathbf{V}_h$, one only needs to solve (8). As a consequence, it turns out that the question of preconditioning the linear system arising from (5) can be reduced to devise a good preconditioner for the linear system associated to the above problem.
Denoting by $\widetilde{\mathcal{A}}$ the operator associated to $\widetilde{a}_h(\cdot,\cdot)$ as defined in (8), we construct a preconditioner for $\widetilde{\mathcal{A}}$ using the auxiliary space method [6]. As auxiliary space, we choose the larger space $\mathbf{V}_h$, and we consider as auxiliary operator the one associated to $a_h(\cdot,\cdot)$ defined in (6) (without the incompressibility constraint), say $\mathcal{A}$. To link both problems and spaces, we define the operator $\Pi_h : \mathbf{V}_h \longrightarrow \mathbf{V}_h \cap \mathbf{H}(\mathrm{div}^0;\Omega) \equiv \mathbf{curl}(\mathcal{N}_h)$ which associates to each $\mathbf{v}_h \in \mathbf{V}_h$ its divergence free part. We show then that if problem (1) is $\mathbf{H}^2$-regular (say Ω convex), the operator Π_h is stable in the DG norm and as a consequence we have the following result:

Theorem: *In the above assumptions, let* $\mathcal{B}$ *be an optimal preconditioner for* $\mathcal{A}$. *Then, the preconditioner* $\widetilde{\mathcal{B}} := \Pi_h \mathcal{B} \Pi_h^*$ *is an optimal preconditioner for* $\widetilde{\mathcal{A}}$.

• **Case** $d = 3$**:** Only the design and analysis of the preconditioner needs to be modified. The third space in the triple of FE spaces, is now defined by

$$\boldsymbol{\mathcal{N}}_h := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) : \mathbf{v}|_K \in \mathcal{M}(K)\ \forall\, K \in \mathcal{T}_h,\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma\}$$

guaranteeing that the resulting triplet $(\mathbf{V}_h, \mathcal{Q}_h, \mathcal{N}_\mathbf{h})$ still satisfies the same basic assumptions and that the corresponding discrete Helmholtz decomposition (4) can be established. The main difficulty now is that uniqueness cannot be guaranteed in the corresponding equation to (7). As a consequence, the operator $\widetilde{\mathcal{A}}$ associated to the (8), now defined in $\mathbf{curl}(\boldsymbol{\mathcal{N}}_h)$ is singular. We show how the same ideas can still be applied by extending the Auxiliary space techniques to singular problems.

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Recent Developments in the Numerical Methods for the Chemotaxis Models

Yekaterina Epshteyn

In this talk we will discuss recent progress in the development of the numerical methods for the chemotaxis models and closely related problems in physics and biology. We consider the most common formulation of the classical Patlak-Keller-Segel system [1] with the 'parabolic-parabolic' coupling, which can be written in the dimensionless form as

$$\begin{cases} \rho_t + \nabla \cdot (\chi \rho \nabla c) = \Delta \rho, \\ c_t = \Delta c - c + \rho, \end{cases} \qquad (x, y) \in \Omega, t > 0, \tag{1}$$

subject to the Neumann boundary conditions:

$$\nabla \rho \cdot \mathbf{n} = \nabla c \cdot \mathbf{n} = 0, \quad (x, y) \in \partial\Omega. \tag{2}$$

Here, $\rho(x, y, t)$ is the cell density, $c(x, y, t)$ is the chemoattractant concentration, χ is a chemotactic sensitivity constant, Ω is a bounded domain in $\mathbb{R}^2$, $\partial\Omega$ is its boundary, and $\mathbf{n}$ is a unit normal vector.

Chemotaxis refers to mechanisms by which cellular motion occurs in response to an external stimulus, usually a chemical one. Chemotaxis is an important process in many medical and biological applications, including bacteria/cell aggregation and

pattern formation mechanisms, as well as tumor growth. There exists an extensive literature about chemotaxis models and their mathematical analysis a first place to start is [2], as well as [4, 3], and for a deeper background [11, 15, 9, 10, 1, 13, 14, 12]. The first descriptions of the mechanism owe to Keller and Segel, [7, 6, 5] and Patlak [8]. In this description, the organism or migrating enzyme chooses a direction upwards of a chemical signal which leads to aggregation.

Although there is an extensive literature on this subject, only a few numerical methods have been proposed for these models. Chemotaxis models are usually highly nonlinear due to the density dependent cross diffusion term (attracting force) that models chemotactic behavior, and hence, any realistic chemotaxis model is too difficult to solve analytically. Therefore, development of accurate and efficient numerical methods is crucial for the modeling and analysis of chemotaxis systems. Furthermore, a common property of all existing chemotaxis systems is their ability to model a concentration phenomenon that mathematically results in rapid growth of solutions in small neighborhoods of concentration points/curves. The solutions may blow up or may exhibit a very singular, spiky behavior. This blow-up represents a mathematical description of a cell concentration phenomenon that occurs in real biological systems, see, e.g., [10, 11, 13, 14, 15, 12]. In either case, capturing such solutions numerically is a very challenging problem.

Let us briefly review the numerical methods that have been proposed in the literature. A finite-volume, [16], and finite-element, [22, 17], methods have been proposed for a simplified version of the Patlak-Keller-Segel model with 'parabolic-elliptic' coupling: the equation for concentration of chemical signals c has been replaced by an elliptic equation using an assumption that c changes over much smaller time scales than the density of the cells ρ. A fractional step numerical method for a fully time-dependent chemotaxis system from [23] has been proposed in [24]. However, the operator splitting approach may not be applicable when a convective part of the chemotaxis system is not hyperbolic, which is a generic situation for the original Patlak-Keller-Segel model with 'parabolic-parabolic' coupling. In [25], a finite-volume central-upwind scheme was derived for the original Patlak-Keller-Segel model and extended to some other chemotaxis models. Recently, in [18], an implicit flux-corrected finite element method has been developed for the original Patlak-Keller-Segel model as well. In our recent work [28, 26, 27] we developed a family of high-order Finite Element Methods (Discontinuous Galerkin Methods) for the original Patlak-Keller-Segel chemotaxis model. However, among the methods that have been proposed, only [22, 17, 18] were designed to treat complex geometry by the use of unstructured meshes. Finally, a different approach has been proposed in [19]. The authors considered the measure-valued global in time solutions of the simplified Patlak-Keller-Segel system in $\mathbb{R}^2$ and proposed a stochastic particle approximation. The advantage of their method is that it captures the solution even after the (possible) blow-up. However, the method was only designed for the simplified Patlak-Keller-Segel model with 'parabolic-elliptic' coupling. Moreover, at least in the 2D case, methods based on particle simulation

are usually less efficient than 'conventional' finite element or finite volume methods for solving convection-diffusion equations.

Often, modeling of real biomedical problems has to deal with the complex structure of the computational domains. Therefore there is a need for accurate, fast, and computationally efficient numerical methods for different chemotaxis models that can handle arbitrary geometries. In our recent paper [20] we develop novel and efficient upwind-difference potentials method which can handle complex geometry without the use of unstructured meshes and it can be combined with fast Poisson solvers. Our method combines the simplicity of the positivity-preserving upwind scheme on Cartesian meshes [21] with the flexibility of the Difference Potentials method [29].

Therefore, in this talk we will discuss and compare recently developed high-order Discontinuous Galerkin Methods [28, 26, 27] and Upwind-Difference Potentials method [20] for the original Patlak-Keller-Segel chemotaxis model.

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Discontinuous Galerkin methods for fully nonlinear second order partial differential equations

Xiaobing Feng
(joint work with Thomas Lewis)

Fully nonlinear partial differential equations (PDEs) refer to the class of PDEs which are nonlinear in the highest order of derivatives of unknown functions appearing in the equations. For example, the general first and second order fully nonlinear PDEs, respectively, have the form $H(\nabla u, u, x) = 0$ and $F(D^2u, \nabla u, u, x) = 0$, where ∇u and D^2u denote the gradient vector and Hessian matrix of the unknown function u. Fully nonlinear PDEs, which have experienced extensive analytical developments in the past thirty years (cf. [2, 6, 14]), arise from many scientific and engineering applications such as differential geometry, astrophysics, antenna design, image processing, optimal control, optimal mass transport, and geostrophical fluid dynamics. Fully nonlinear PDEs play a critical role for the solutions of these application problems because they appear one way or another in the governing equations of these problems. As expected, this class of nonlinear PDEs are the most difficult to study analytically and to solve numerically. On the one hand, great successes have been achieved in the past 30 years on the analysis of these PDEs. On the other hand, numerical approximations of fully nonlinear

second order PDEs was an untouched area just 4–5 years ago. There are two main reasons for the lack of progress. First, the notion of viscosity (weak) solutions is *nonvariational*, which prevents any direct construction of Galerkin-type numerical methods and forces one to use indirect approaches as done in [4, 5, 7, 9] for approximating viscosity solutions. Second, fully nonlinear PDEs often have multiple solutions, the uniqueness of viscosity (weak) solutions may only hold in a restrictive function class. This conditional uniqueness is difficult to handle at discrete level and prevents any straightforward construction of finite difference methods because such a method does not have a mechanism to enforce the conditional uniqueness and often fails to capture the sought-after viscosity solution.

The goal of this talk was to present a newly developed general framework for directly constructing discontinuous Galerkin (DG) methods for fully nonlinear second order elliptic and parabolic PDEs. The bulk of the materials of this talk are taken from references [12, 13] and are inspired by the finite difference work [10]. The proposed DG framework consists of the following five key ingredients/ideas. First, the given fully nonlinear PDE must be rewritten into a nonstandard mixed formulation in which all second order derivatives (i.e., the Hessian matrix) are treated as independent variables. Second, due to possible discontinuity of second order partial derivatives, each of these second order derivatives must be approximated by three functions representing some well-defined left, right and central derivatives. Third, the original differential operator F must be approximated by a "numerical operator" $\widehat{F}$ which satisfies some consistency and generalized monotonicity properties. Fourth, in order to construct the desired "numerical operator" $\widehat{F}$, the main idea is to introduce a key concept called "numerical moment" to achieve the goal. Fifth, when formulating DG methods, different numerical fluxes must be used to discretize the three (linear) equations satisfied by three variables which represent the left, right and central (second order) derivatives. Several one-dimensional numerical experiments were presented to guage the performance of the proposed DG framework and methods. Generalizations to high dimensional case and the formulation of local discontinuous Galerkin (LDG) methods were also discussed. Connections between the proposed DG framework with the finite difference work [10] and with the finite difference and DG results [15, 16] for the first order fully nonlinear Hamilton-Jacobi equations were briefly discussed as well.

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Convergence Analysis of an Adaptive Interior Penalty Discontinuous Galerkin Method for the Biharmonic Problem

Thomas Fraunholz
(joint work with R. H. W. Hoppe, M. A. Peter)

For the biharmonic problem, we study the convergence of adaptive C^0-Interior Penalty Discontinuous Galerkin (C^0-IPDG) methods of any polynomial order. We note that C^0-IPDG methods for fourth order elliptic boundary value problems have been suggested in [1], whereas a residual-type a posteriori error estimator for a quadratic C^0-IPDG method applied to the biharmonic equation has been developed and analyzed in [2].

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$. For a given function $f \in L^2(\Omega)$ we consider the biharmonic problem

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

A weak formulation of (1) requires the computation of $u \in V := H_0^2(\Omega)$ such that

$$a(u,v) = (f,v)_{0,\Omega}, \quad v \in V, \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(v,w) = (D^2 v, D^2 w)_{0,\Omega} := \sum_{|\beta|=2} (D^\beta v, D^\beta w)_{0,\Omega}, \quad v,w \in V.$$

Let $\mathcal{T}_h$ be a geometrically conforming simplicial triangulation of Ω. We denote by $\mathcal{E}_h^\Omega$ and $\mathcal{E}_h^\Gamma$ the set of edges of $\mathcal{T}_h$ in the interior of Ω and on the boundary Γ, respectively, and set $\mathcal{E}_h := \mathcal{E}_h^\Omega \cup \mathcal{E}_h^\Gamma$. For $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we denote by h_T and h_E the diameter of T and the length of E, and we set $h := \max\limits_{T\in\mathcal{T}_h} h_T$.
Denoting by $P_k(T), k \in \mathbb{N}$, the linear space of polynomials of degree $\le k$ on T, for $k \ge 2$ we refer to

$$V_h := \{v_h \in H_0^1(\Omega) \mid v_h|_T \in P_k(T),\ T \in \mathcal{T}_h\}$$

as the finite element space of Lagrangian finite elements of type k. Given a penalty parameter $\alpha > 1$, the C^0-IPDG method for the approximation of (2) requires the computation of $u_h \in V_h$ such that

$$a_h^{IP}(u_h, v_h) = (f, v_h)_{0,\Omega}, \quad v_h \in V_h. \tag{3}$$

Here, the mesh-dependent bilinear form $a_h^{IP}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is given according to

$$\begin{aligned} a_h^{IP}(v_h, w_h) := & \sum_{T\in\mathcal{T}_h} (D^2 v_h, D^2 w_h)_{0,T} + \sum_{E\in\mathcal{E}_h} (\{\frac{\partial^2 v_h}{\partial n^2}\}_E, [\frac{\partial w_h}{\partial n}]_E)_{0,E} \\ & + \sum_{E\in\mathcal{E}_h} ([\frac{\partial v_h}{\partial n}]_E, \{\frac{\partial^2 w_h}{\partial n^2}\}_E)_{0,E} + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} ([\frac{\partial v_h}{\partial n}]_E, [\frac{\partial w_h}{\partial n}]_E)_{0,E}. \end{aligned}$$

For adaptive mesh refinement we consider the residual-type a posteriori error estimator

$$\eta_h := \Big(\sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{E\in\mathcal{E}_h^\Omega} \eta_E^2 \Big)^{1/2},$$

where the element residuals $\eta_T, T \in \mathcal{T}_h$, and the edge residuals $\eta_E, E \in \mathcal{E}_h$, are given by

$$\begin{aligned} \eta_T^2 &:= h_T^4 \ \|f - \Delta^2 u_h\|_{0,T}^2, \quad T \in \mathcal{T}_h, \\ \eta_E^2 &:= h_E \ \|[\frac{\partial^2 u_h}{\partial n^2}]_E\|_{0,E}^2 + h_E^3 \ \|[\frac{\partial}{\partial n}\Delta u_h]_E\|_{0,E}^2, \quad E \in \mathcal{E}_h^\Omega. \end{aligned}$$

Following the convergence analysis of adaptive IPDG methods for second order elliptic problems [3], we prove a contraction property for a weighted sum of the C^0-IPDG energy norm of the global discretization error and the estimator. The proof of the contraction property is based on the reliability of the estimator, a quasi-orthogonality result, and an estimator reduction property.

Theorem. *Let $u \in H_0^2(\Omega)$ be the unique solution of (2). Further, let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the C^0-IPDG solutions of (3) and error estimators, respectively. Then, there exist constants $0 < \delta < 1$ and $\rho > 0$, depending only on the local geometry of the triangulations and the parameter Θ from the Dörfler marking, such that for sufficiently large penalty parameter α the fine mesh and coarse mesh discretization errors $e_h := u - u_h$ and $e_H = u - u_H$ satisfy*

$$a_h^{IP}(e_h, e_h) + \rho\ \eta_h^2 \leq \delta\ \Big(a_H^{IP}(e_H, e_H) + \rho\ \eta_H^2 \Big).$$

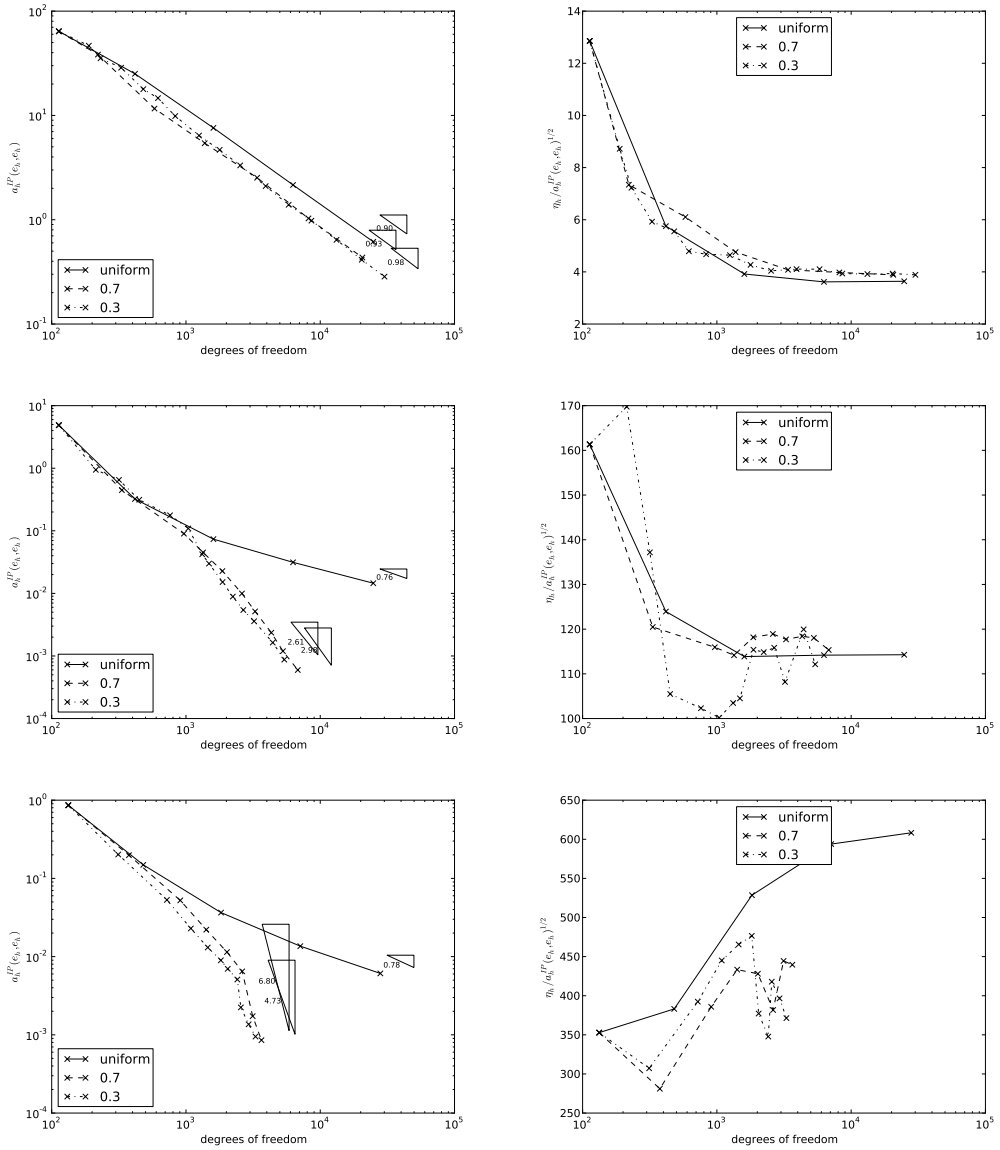

FIGURE 1. Error reduction (left) and effectivity index (right) for $k = 2$ (top), $k = 4$ (middle) and $k = 6$ (bottom).

In order to provide a detailed documentation of the performance of the adaptive C^0-IPDG method we take an illustrative example from [2]. We have run simulations for polynomial degrees $2 \leq k \leq 6$ with penalty parameter $\alpha = 2.5(k+1)^2$. For $k = 2, k = 4$, and $k = 6$, Figure 1 shows the convergence histories in terms of the broken C^0-IPDG energy norm of the error $a_h^{IP}(u - u_h, u - u_h)$ (left), as well as the computed effectivity indices $\eta_h / a_h^{IP}(u - u_h, u - u_h)^{1/2}$ (right) as a function of the total number of degrees of freedom (DOF) on a logarithmic scale for uniform refinement and adaptive refinement with $\Theta = 0.7$ and $\Theta = 0.3$ in the Dörfler marking. As far as the convergence rates and the estimator reduction are concerned, the benefits of adaptive versus uniform refinement can be clearly recognized, in particular for increasing polynomial degree. Moreover, as in case of IPDG methods for second order elliptic boundary value problems and H-IPDG methods for Maxwell's equations we observe a different convergence behavior depending on the choice of Θ in the Dörfler marking. The effectivity indices show a clear dependence on the polynomial degree k.

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DG Methods and Local Time-Stepping For Wave Propagation

Marcus J. Grote

(joint work with Michaela Mehlin, Teodora Mitkova)

The efficient simulation of time-dependent wave phenomena is of fundamental importance in a wide variety of applications from acoustics, electromagnetics and elasticity, for which the scalar damped wave equation

$$u_{tt} + \sigma u_t - \nabla \cdot (c^2 \nabla u) = f \qquad \text{in } \Omega \times (0, T)\,, \tag{1}$$

often serves as a model problem. Here, Ω is a bounded domain, $f(x,t)$ is a (known) source term, whereas the damping coefficient $\sigma(x) \geq 0$ and the speed of propagation $c(x) > 0$ are piecewise smooth.

Three popular spatial discretizations for (1) are standard continuous (H^1-conforming) finite elements with mass lumping [1], the symmetric IP-DG formulation [5], or nodal DG finite elements [7]. All three lead to a system of ordinary differential equations with an essentially diagonal mass matrix. Thus, when combined with explicit time integration, the resulting fully discrete scheme for the solution of (1) will be truly explicit.

Locally refined meshes impose severe stability constraints on explicit time-stepping methods for the numerical solution of (1). Local time-stepping (LTS)

methods overcome that bottleneck by using smaller time-steps precisely where the smallest elements in the mesh are located. In [2, 3], explicit second-order LTS integrators for transient wave motion were developed, which are based on the standard leap-frog scheme. In the absence of damping, i.e. $\sigma = 0$, these time-stepping schemes yield methods of arbitrarily high (even) order when combined with the modified equation approach. By blending the leap-frog and the Crank-Nicolson methods, a second-order LTS scheme was also derived there for (damped) electromagnetic waves in conducting media, i.e. $\sigma > 0$, yet this approach cannot be readily extended beyond order two. To achieve arbitrarily high accuracy even in the presence of damping, while remaining fully explicit, explicit LTS methods for (1) based on Adams-Bashforth multi-step schemes can be used [4, 7].

Here we propose explicit LTS methods based either on standard or low-storage Runge-Kutta (RK) schemes of arbitrarily high order. In contrast to Adams-Bashforth methods, RK methods are one-step methods; hence, they do not require a starting procedure and easily accommodate adaptivity in time. Although Runge-Kutta methods require more computational work per time-step, that additional work is typically compensated by a less stringent CFL stability restriction.

Clearly, the idea of using different time steps for different components in the context of ordinary different equations is not new [8]. However, RK methods achieve higher accuracy not by extrapolating farther from the (known) past but instead by including further intermediate stages from the current time step. Thus for the numerical solution of partial differential equations, the derivation of high-order local time-stepping methods that are based on RK schemes is generally more difficult.

To illustrate the versatility of our approach, we now consider a computational rectangular domain of size $[0, 2] \times [0, 1]$ with two rectangular barriers inside forming a narrow gap. We use continuous P^2 elements on a triangular mesh, which is highly refined in the vicinity of the gap, as shown in Fig. 1. For the time discretization, we choose an LTS method based on an explicit third-order low-storage Runge-Kutta scheme. Thus, the numerical method is third-order accurate both in space and time with respect to the L^2-norm. Since the typical mesh size inside the refined region is about $p = 7$ times smaller than that in the surrounding coarser region, we take p local time steps of size $\Delta\tau = \Delta t/p$ for every time step Δt. In Fig. 2, a Gaussian pulse initiates two plane waves which propagate horizontally in opposite directions. As the right-moving wave impinges upon the obstacle, a small fraction of the incoming wave penetrates the gap and generates multiple circular waves on both sides of the obstacle, which further interact with the wave field.

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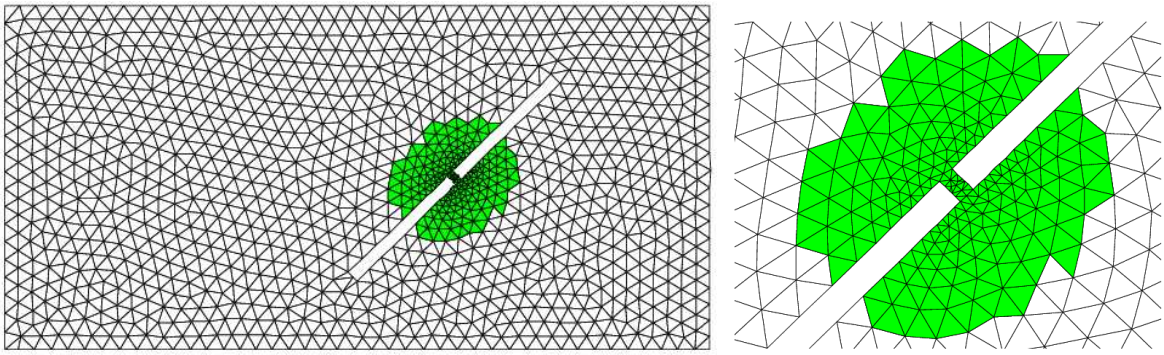

FIGURE 1. The initial triangular mesh (left); zoom on the "fine" mesh indicated by the darker (green) triangles (right).

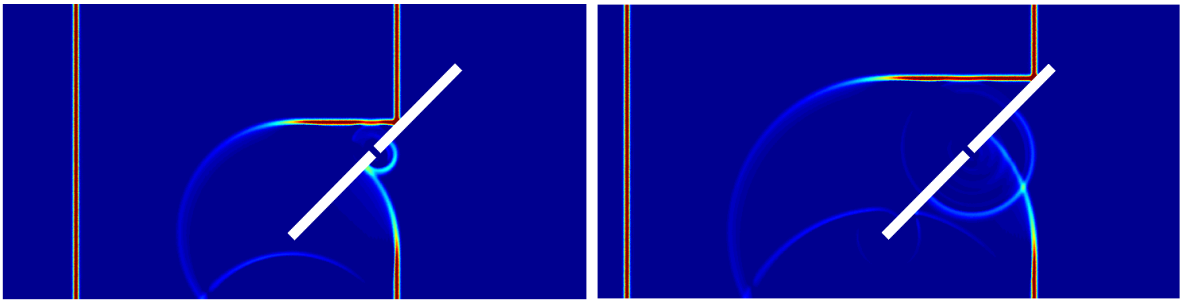

FIGURE 2. The solution at times t =0.55 and 0.7.

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A Posteriori Error Control of Discontinuous Galerkin Methods for Elliptic Obstacle Problem

THIRUPATHI GUDI
(joint work with Kamana Porwal)

Variational inequalities form an important class of nonlinear problems for which obstacle problem may be considered as a prototype model. For given $f \in L_2(\Omega)$ and $\psi \in H^1(\Omega) \cap C^0(\overline{\Omega})$ with $\psi \leq 0$ on $\partial\Omega$, we consider the model problem of finding $u \in \mathcal{K} = \{u \in H_0^1(\Omega) : u \geq \psi \text{ in } \Omega\}$ such that

$$a(u, v-u) \geq (f, v-u) \qquad \forall v \in \mathcal{K},$$

where $a(u,v) = (\nabla u, \nabla v)$ and $(\cdot,\cdot)$ denotes the $L_2(\Omega)$ inner-product. The convergence of finite element approximations for obstacle problems is known for many years [7, 5, 13, 8]. *A priori* error estimates for conforming linear and quadratic finite element methods are derived in [7] and [5, 13], respectively. For general convergence analysis and error estimates, we refer to [8]. However, the *a posteriori* error analysis of conforming finite element methods for obstacle problems has been received attention since a decade [6, 12, 2, 3, 11]. In [10], Hierarchical error estimator is derived. In [6], the first residual based *a posteriori* error estimator is derived using a positivity preserving interpolation operator. The *a posteriori* error analysis in [12] is derived without using positivity preserving interpolation operator. The estimates in [6] and [12] are slightly different from each other but it is shown therein that both the estimators are reliable and efficient. In [2], error estimates based on averaging techniques are derived for conforming finite element method. Error estimates for conforming finite element methods in an abstract framework can be found in [3, 11] when the obstacle is a $P_1(\Omega)$ function. Recently, convergence of adaptive conforming finite element method for obstacle problem is studied in [4].

During the past decade and half, discontinuous Galerkin methods have been attractive for the numerical approximation of variety of applications [1]. In this article, we are concerned on the application of DG methods for variational inequalities. Recently, *a priori* error analysis of various discontinuous Galerkin(DG) methods for elliptic variational inequalities of the first and the second kind is derived in [14]. The discontinuous Galerkin methods in [14] are based on the discontinuous Galerkin formulations in [1] for the Laplace equation.

In this article, we derive *a posteriori* error estimates for various DG finite element methods for second order elliptic obstacle problems. Using a key property of discontinuous Galerkin methods, we perform the analysis in a unified framework. The error estimator for discontinuous Galerkin methods is comparable with the estimator for conforming finite element method. This is achieved by using a nonlinear smoothing function mapping discontinuous finite element space to conforming finite element space. The error estimator consists of a discrete Lagrange multiplier associated with the obstacle constraint. It is shown for non over-penalized discontinuous Galerkin methods that the discrete Lagrange multiplier is uniformly stable on non-uniform meshes. The results are verified by numerical experiments.

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Two-Grid *hp*–Adaptive Discontinuous Galerkin Finite Element Methods for Second–Order Quasilinear Elliptic PDEs

Paul Houston
(joint work with Scott Congreve and Thomas Wihler)

In this talk we present an overview of some recent developments concerning the a posteriori error analysis and adaptive mesh design of h– and hp–version discontinuous Galerkin finite element methods for the numerical approximation of second–order quasilinear elliptic boundary value problems. In particular, we consider the derivation of computable bounds on the error measured in terms of an appropriate (mesh–dependent) energy norm in the case when a two-grid approximation is employed. In this setting, the fully nonlinear problem is first computed on a coarse finite element space $V_{H,P}$. The resulting 'coarse' numerical solution is then exploited to provide the necessary data needed to linearise the underlying discretization on the finer space $V_{h,p}$; thereby, only a linear system of equations is solved on the richer space $V_{h,p}$. Here, an adaptive hp–refinement algorithm is proposed which automatically selects the local mesh size and local polynomial degrees on both the coarse and fine spaces $V_{H,P}$ and $V_{h,p}$, respectively. Numerical experiments confirming the reliability and efficiency of the proposed mesh refinement algorithm are presented.

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Error Analysis for a Monolithic Discretization of Coupled Darcy and Stokes Problem

Guido Kanschat
(joint work with Vivette Girault and Béatrice Rivière)

Recently, we proposed a monolithic discretization of coupled Darcy/Stokes flow problems in [4], based on a weak formulation of the coupled problem with velocities in a subspace of $H^{\mathrm{div}}(\Omega)$. Let us briefly summarize the setting: The computational domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is divided into two subdomains Ω_S and Ω_D with Stokes and Darcy flow, respectively. Their outer boundaries are denoted by Γ_S and Γ_D, and the interface between the two as Γ_{SD}. On Γ_{SD}, we impose the Beavers-Joseph-Saffman interface conditions (see e. g. [5, 3]), consisting of the conservation of mass and normal momentum, and a tangential friction term. Using the function spaces

$$\begin{aligned} H_0^{\mathrm{div}}(\Omega) &= \big\{ v \in L^2(\Omega;\mathbb{R}^d) \big| \nabla\cdot v \in L^2(\Omega), \quad (v\cdot n)_{|\partial\Omega} = 0 \big\}, \\ V &= \big\{ v \in H_0^{\mathrm{div}}(\Omega) \big| v_{|\Omega_S} \in H^1(\Omega_S), \quad v_{|\Gamma_S} = 0 \big\}, \\ Q &= \big\{ q \in L^2(\Omega) \big| (q,1)_\Omega = 0 \big\}, \end{aligned}$$

we strive to approximate a solution $(u, p) \in V \times Q$ of the weak formulation

$$\begin{aligned} (1) \quad & 2\mu(Du, Dv)_{\Omega_S} + (\rho^2 u, v)_{\Omega_D} + (\gamma\rho u_{S;\tau}, v_{S;\tau})_{\Gamma_{SD}} - (p, \nabla\cdot v)_\Omega \\ & \qquad - (q, \nabla\cdot u)_\Omega = (f, v)_\Omega + (g, q)_\Omega \qquad \forall v \in V, q \in Q. \end{aligned}$$

Here μ is the dimensionless viscosity, Du is the strain tensor of u, $\rho^2 = \mu/K$, K the permeability of the porous media, and γ the friction coefficient introduced by Saffman. For our discretization, we propose a finite element pressure space $Q_h \subset Q$. The velocity space $V_h \subset H_0^{\mathrm{div}}(\Omega)$ is chosen such that $\nabla\cdot V_h = Q_h$; without confining ourselves to this case, consider for instance the space of discontinuous polynomials P_k and the matching Raviart-Thomas RT_k on simplicial elements. Since $V_h \not\subset V$, we choose a discontinuous Galerkin method in order to fix the inconsistency on Ω_S. The method is required to be consistent and adjoint consistent in order such that we can apply duality arguments.

While we observed optimal and balanced convergence rates of order $k + 1$ for the velocity error in $L^2(\Omega_D)$ and $L^2(\Omega_S)$, respectively, the energy error analysis

only yields rates suboptimal by one order in Ω_D, namely

$$\|u-u_h\|_{1,h;\Omega_S} + \|\rho(u-u_h)\|_{\Omega_D} + \|\nabla\cdot(u-u_h)\|_{\Omega} + \|\sqrt{\gamma\rho}(u_{S;\tau}-u_{h,S;\tau})\|_{\Gamma_{SD}} \leq ch^s|u|_{H^{s+1}(\Omega)}. \tag{2}$$

This is due to the fact that the derivatives on the Stokes side pollute the error on the Darcy side. Since we have no derivatives of the Darcy velocity, a straight forward duality argument does not help.

More recently, we presented the optimal L^2 analysis in [2]; the outcome was, that the errors in Ω_D and Ω_S are indeed balanced. We review the main steps of the proof and omit the details. One important feature needs to be pointed out though: as soon as the interface is just Lipschitz, the solution u cannot have full H^2-regularity anymore, since one of the two subdomains has a reentrant corner/edge. Therefore, all estimates are presented assuming that all solutions to (1) with right hand side in L^2 are bounded in $H^{s+1}(\Omega;\mathbb{R}^d)$ with $1/2 < s \leq 1$. We assume further, that $\nabla\cdot u \in H^{s+1}(\Omega)$.

(I) In a standard duality argument, we solve (1) with right hand side $f = \chi(\Omega_S)(u-u_h)$ and $g = 0$. Without much ado, we obtain the estimate

$$\|u-u_h\|_{L^2(\Omega_S)} \leq Ch^{2s}|u|_{H^{s+1}(\Omega_S)}. \tag{3}$$

(II) In the second step, we prove improved estimates for the error on the interface Γ_{SD}. To this end, we observe that $\nabla\cdot u_h$ is the L^2-projection of $\nabla\cdot u$, and thus

$$\|\nabla\cdot u - \nabla\cdot u_h\|_{L^2(\Omega_S)} \leq Ch^{s+1}|\nabla\cdot u|_{H^{s+1}(\Omega_S)}. \tag{4}$$

Combining estimates (3) and (4), a standard trace estimate yields the same convergence order for the error in $H^{-1/2}(\partial\Omega)$, which can be localized by an argument in [1]. By a more technical argument in order to avoid requiring mesh uniformity, we also get an estimate in $L^2(\Gamma_{SD})$:

$$\begin{aligned} \|u\cdot n - u_h\cdot n\|_{H^{-1/2}(\Gamma_{SD})} &\leq ch^{2s}|u|_{H^{s,\mathrm{div}}(\Omega_S)}, \\ \|u\cdot n - u_h\cdot n\|_{L^2(\Gamma_{SD})} &\leq ch^{\frac{3}{2}s}|u|_{H^{s+1}(\Omega_S)}, \end{aligned} \tag{5}$$

where we define $|u|^2_{H^{s+1,\mathrm{div}}(\Omega_S)} = |u|^2_{H^{s+1}(\Omega_S)} + |\nabla\cdot u|^2_{H^{s+1}(\Omega_S)}$.

(III) We introduce the divergence free lifting w of the error on the interface into the Darcy subdomain, and we have

$$\begin{aligned} \|w\|_{L^2(\Omega_D)} &\leq C\|u\cdot n - u_h\cdot n\|_{H^{-1/2}(\Gamma_{SD})} \\ \|w\|_{H^{\frac{1}{2}}(\Omega_D)} &\leq C\|u\cdot n - u_h\cdot n\|_{L^2(\Gamma_{SD})}. \end{aligned} \tag{6}$$

This lifting allows us to effectively decouple the Darcy subproblem from the Stokes subproblem, since $u-u_h-w \in H_0^{\mathrm{div}}(\Omega_D)$. On the other hand, the estimates (6) (where the second is needed to apply the Raviart-Thomas interpolation operator to w) guarantee that the pollution of the right hand side due to this decoupling is sufficiently small. Finally, we get the estimate

$$\|u-u_h\|_{L^2(\Omega_D)} \leq Ch^{2s}\Big(|u|_{H^{s+1,\mathrm{div}}(\Omega_S)} + |u|_{H^{s+1}(\Omega_D)}\Big). \tag{7}$$

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Tools and Methods for Discontinuous Galerkin Solvers on Modern Computer Architectures

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(joint work with Timothy Warburton, Jan S. Hesthaven)

The solution of partial differential equations (PDEs) by numerical methods is an endeavor that is severely constrained by computational cost. Very few simulation codes are able to achieve the fidelity that would be desirable from the point of view of their target application. To determine the cost of finding a solution to a PDE, one needs to know the application problem, along with the desired accuracy. If one then chooses a numerical method that will be employed along with its parameters, one can begin to estimate cost. Still, considerable uncertainty remains in how mathematical expressions are to be evaluated, and algorithmic choices (such as the use of stored vs. recomputed quantities) may make a large amount of difference. The final, *actual* cost is only known once the solver software *and* the target machine is in place.

Curiously, the literature on PDE solvers abounds with claims of optimality, but largely ignores algorithmic and machine concerns. Previously, computational time was roughly comparable between machines, but this is no longer true. Computer design has encountered restrictive economical bounds on power as well as bandwidth and latency in memory access and communication. Many different machine types have emerged, each optimized for a different workload. For example, a few hundred US dollars will buy a parallel computer that is capable of performing, in each second, $\sim 4 \cdot 10^{12}$ floating point operations ("flops"), but only of reading $\sim 5 \cdot 10^{10}$ values from memory. From the point of view of numerical analysis, this means that the traditional cost measure of "flops" has become mostly irrelevant. Claims of optimality derived from this measure therefore need to be reevaluated.

Necessarily, this situation increases the burden on numerical analysts and software implementers, and if practical cost is adopted as the measure of success, then the creation of PDE solvers becomes an interdisciplinary effort in a *joint design space* where complex interdependencies exist. For example, the type of workload that is well-supported by actual machines may determine what numerical schemes

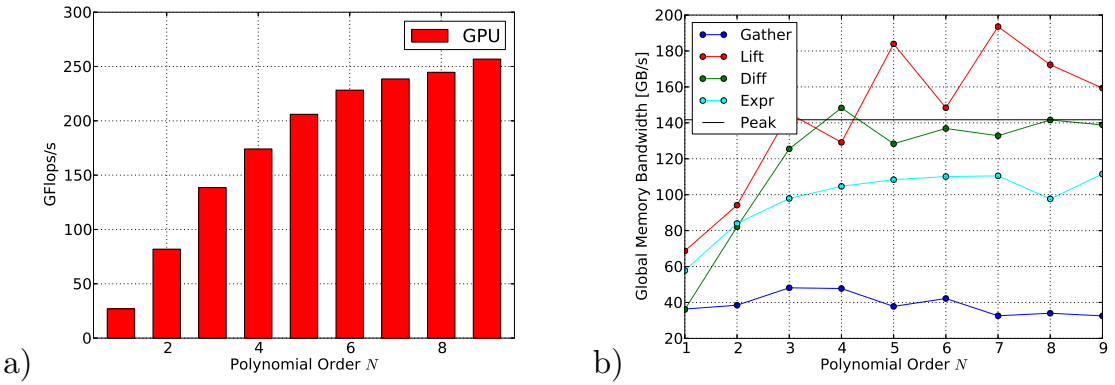

FIGURE 1. Performance of GPU-DG on Nvidia GTX295 as shown in [1]. a) Aggregate, wall-clock performance measured timestep-to-timestep. b) Achieved memory bandwidth, broken down by computational component. (Above-peak values due to texture cache effects.)

for the solution of PDEs are sensible and which ones are not. As a numerical analyst, one may ignore such facts at one's own peril. In the context of a discontinuous Galerkin (DG) scheme, this contribution presents infrastructure and tools we propose to alleviate some of this burden.

In 2009, we were able to demonstrate [1] that nodal DG methods for hyperbolic systems of conservation laws $\partial_t u + \nabla \cdot F(u) = S$, when implemented with consideration for modern processors, can achieve excellent utilization of Nvidia GPUs. More specifically, the evolution of the solution on a finite element D_k is governed by

$$\partial_t u^k = (M^k)^{-1} \left[\sum_\nu S^{\partial_\nu, k}[F(u^k)] - \sum_{A \subset \partial \mathsf{D}_k} M_A^k [(\hat{n} \cdot F)^*]|_A \right], \tag{1}$$

where $S^{\partial_\nu, k}$ is a one-sided stiffness matrix, M^k is the mass matrix, M_A^k is the mass matrix on a face A, and $(n \cdot F)^*$ is the numerical flux. Exploiting, among other things, the block-diagonal structure of all these matrices, one may rewrite this expression such that there are only two matrix-vector products per element to be computed. Using either an affine mapping between D_k and the reference element, or Warburton's low-storage curvilinear scheme [3], these matrices can all be related back to template matrices (with scaling factors) that are usable for the entire computational problem.

It is thus reasonable to expect that the main computational expense of the expression (1) will be the evaluation of element-wise matrix-vector products, even at low-to-moderate polynomial order N. Unfortunately, even the seemingly simple implementation of matrix-vector products becomes slightly non-trivial, as the machine presents its user with a large array of granularities that need to be taken into account. For instance, as the polynomial degree N varies, the amount of storage for each element-local matrix grows as $O(N^6)$. Thus, any finite amount of

on-chip temporary storage will likely be exhausted as N increases. The matrix is clearly the most-used piece of information in the expression, so bringing as much of it as possible close to the computation is clearly a priority. Further granularity restrictions emerge at the core-parallel level (or rather the programming model's abstraction thereof, variously called "thread block" or "work group"), where one is forced to decide how large a task is undertaken by each such "core", be it in the form of vector parallelism, instruction-level parallelism, or sequential processing.

It is expedient to automatically tune for optimal values of the choices described above within some predetermined parameter space. This is a step in the right direction, as it emphasizes tuning *ideas* over ephemeral tuning results. Yet, even this approach falls short, although it naturally leads to a key to addressing the issues outlined–namely, the *generation of code at run time* ("RTCG"). This amounts to inserting another layer of reasoning between the user and the code expressing the computation (which is often very redundant and not genuinely "human-readable"). It further enables the creation of tools operating on this code.

This approach is embodied in our packages PyOpenCL and PyCUDA [2], which were first written to support our implementation of the DG method. In addition to facilitating RTCG by features such as object code caches, they provide elementary linear algebra and basic primitives for parallel programming. The idea of RTCG and its use through these packages have struck a chord with a global community, as we have so far counted over 80,000 source-code downloads of these tools, in addition to many available third-party binary distributions.

In the context of DG methods, the approach enables auto-tuning for memory layout, work partition and other granularities, but it also yields a number of further important benefits. It allows high-performance code to be written generically, so that an input language close to mathematical notation may be used. Since kernel code is now single-use, it may be emitted with constants instead of variables. This often helps compilers generate more efficient code.

Performance obtained using this approach is shown in Figure 1. A hand-written reference code that does not do automatic tuning achieves similar performance from $N \geq 6$, however in the intermediate regime of $N = 3, \ldots, 5$, the automated code is faster by up to 60% on identical hardware. This regime is practically important because of the DG CFL restriction $\Delta t \leq h/N^2$, and it is challenging because the small number of degrees of freedom (and the associated matrix sizes) frequently clash with machine-imposed granularities. It is all the more encouraging that for $N \geq 3$, our method achieves near-peak memory bandwidth utilization.

Despite the undoubted success of the effort described above, we acknowledge that the transformations required on even a simple concept such as matrix-vector multiplication are not appealing to carry out by hand. Seeking to enable more researchers to face this challenge, we have therefore created a next-generation tool called "loo.py", which takes in a semi-mathematical description of an expression to be evaluated and accepts commands that gradually transform the expression into memory-efficient, granularity-respecting parallel C code. In a sense, loo.py is the

smallest, simplest possible increment in tool abstraction over our existing infrastructure. It is capable of non-redundantly expressing a large range of workloads ranging from dense linear algebra, to DG, to n-body problems.

As we have tried to outline above, numerical analysis for PDEs finds itself at a crossroads, with great challenge, but also great opportunity. We believe that innovation in tools as we have described in this contribution may help enable additional progress towards high-fidelity PDE solvers.

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Trefftz-Discontinuous Galerkin Methods for Acoustic Scattering

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(joint work with Ralf Hiptmair, Andrea Moiola)

Trefftz discontinuous Galerkin (TDG) methods are finite element methods based on approximation spaces locally made by finite linear combinations of solutions to the homogeneous PDE to be discretized. They have recently been applied to problems for which standard polynomial-based finite elements present computational challenges, like time-harmonic wave propagation problems. For these problems, TDG methods using plane wave approximation spaces generalize the ultra-weak variational formulation (UWVF) of [2], as observed in [1, 4].

Here, following [7], we focus on TDG methods applied to the time-harmonic acoustic scattering. In previous papers, we developed h- and p-version error analysis of TDG methods (see [4] and [6]), restricting to convex domains and quasi-uniform meshes. On the other hand, practical experience (see [8]) suggests that TDG methods should be used on *locally refined meshes* together with *spatially varying* local dimension of trial spaces. Moreover, in order to consider acoustic scattering problems, we need to extend our analysis to non-convex domains.

We consider the following acoustic scattering problem:

$$
\begin{aligned}
-\Delta u - k^2 u &= 0 \qquad \text{in } \Omega := \Omega_R \setminus \overline{\Omega}_D \,, \\
u &= 0 \qquad \text{on } \Gamma_D := \partial\Omega_D \,, \\
\frac{\partial u}{\partial \mathbf{n}} + iku &= g_R \qquad \text{on } \Gamma_R := \partial\Omega_R \,,
\end{aligned}
\tag{1}
$$

where k is the wave number, Ω_D is a bounded, *star-shaped*, Lipschitz polyhedron occupied by a sound-soft scatterer, and Ω_R is an artificial bounded, either smooth or polyhedral Lipschitz domain such that $\operatorname{dist}(\Gamma_R, \Gamma_D) > 0$.

Well-posedness of (1) can be proved by Fredholm alternative, while stability estimates in weighted H^1-norm has been proved in [9, 3, 5]. On the other hand, the duality argument in our error analysis requires refined regularity and stability results: provided that $g_R \in H^r(\Gamma_R)$, $0 < r < 1/2$, we prove that the solution to (1), with a general L^2 right-hand side in the first equation, belongs to $H^{\frac{3}{2}+s}(\Omega)$; we also prove stability in $H^{\frac{3}{2}+s}$-norm with explicit dependence of the constant on k.

Given a mesh $\mathcal{T}_h$ and local resolutions $(p_K)_{K\in\mathcal{T}_h}$, define

$$V_p(\mathcal{T}_h) = \{v_{hp} \in L^2(\Omega) : \ v_{hp}|_K \in V_{p_K}(K) \ \forall K \in \mathcal{T}_h\} \ ,$$

where $V_{p_K}(K)$ is a p_K-dimensional space of functions such that $\Delta v + k^2 v = 0$ for all $v \in V_{p_K}(K)$. With standard DG notation, we write our TDG methods as follows: find $u_{hp} \in V_p(\mathcal{T}_h)$ such that, for all $v_{hp} \in V_p(\mathcal{T}_h)$ and for all $K \in \mathcal{T}_h$,

$$\int_{\partial K} \widehat{u}_{hp} \, \nabla \overline{v}_{hp} \cdot \mathbf{n}_K \, \mathrm{d}S - \int_{\partial K} ik \widehat{\boldsymbol{\sigma}}_{hp} \cdot \mathbf{n}_K \, \overline{v}_{hp} \, \mathrm{d}S = 0 \ ,$$

with numerical fluxes defined by

$$ik\widehat{\boldsymbol{\sigma}}_{hp} = \begin{cases} \{\!\{\nabla_h u_{hp}\}\!\} - \alpha \, ik \, [\![u_{hp}]\!]_N & \text{on interior faces,} \\ \nabla_h u_{hp} - (1-\delta)\,(\nabla_h u_{hp} + ik\vartheta u_{hp}\mathbf{n} - g_R\mathbf{n}) & \text{on faces on } \Gamma_R, \\ \nabla_h u_{hp} - \alpha \, ik \, u_{hp}\mathbf{n} & \text{on faces on } \Gamma_D, \end{cases}$$

$$\widehat{u}_{hp} = \begin{cases} \{\!\{u_{hp}\}\!\} - \beta \, (ik)^{-1} [\![\nabla_h u_{hp}]\!]_N & \text{on interior faces,} \\ u_{hp} - \delta \left((ik\vartheta)^{-1} \nabla_h u_{hp} \cdot \mathbf{n} + u_{hp} - (ik\vartheta)^{-1} g_R \right) & \text{on faces on } \Gamma_R, \\ 0 & \text{on faces on } \Gamma_D. \end{cases}$$

The flux parameters α, β, δ are piecewise constant positive functions on suitable unions of edge/faces; we require $0 < \delta \leq 1/2$. Well-posedness and quasi-optimal *hp*-error estimates in a mesh-dependent norm are obtained as in [6], while in order to avoid the quasi-uniformity mesh assumption, instead of using constant numerical flux parameters, we define them on each edge/face f as

$$\alpha_{|f} = \mathtt{a} \, \frac{h}{h_f} \ , \qquad \beta_{|f} = \mathtt{b} \, \frac{h}{h_f} \ , \qquad \delta_{|f} = \mathtt{d} \, \frac{h}{h_f} \ ,$$

with $h_f = \min\{h_K, h_{K'}\}$ if $f = \partial K \cap \partial K'$, $h_f = h_K$ if $f = \partial K \cap \partial\Omega$, and $\mathtt{a}, \mathtt{b}, \mathtt{d} > 0$ independent of the mesh width, the local Trefftz spaces, and k. With this choice of flux parameters, we derive *hp*-error estimates in the L^2-norm assuming quasi-uniformity of the meshes only close to the outer artificial boundary, while allowing local refinement near the scatterer. Then, for plane wave or circular/spherical wave approximation spaces, the actual convergence rates are derived from the best approximation estimates proved in [10].

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Comparison of Finite Element Methods for the Poisson Model Problem

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(joint work with C. Carstensen and M. Schedensack)

In the recent preprint [7], the authors establish the equivalence of conforming Courant finite element method (CFEM) and nonconforming Crouzeix-Raviart finite element method (CRFEM) in the sense that the respective energy error norms are equivalent up to generic constants and higher-order data oscillations in a Poisson model problem. The Raviart-Thomas mixed finite element method is better than the previous methods whereas the conjecture of the converse relation is proved to be false. Those results complete the analysis of comparison initiated by Braess [2]. This note extends the comparison to several Discontinuous Galerkin FEM (DGFEM), e.g., symmetric interior penalty method (SIPG) [10, 12, 1], nonsymmetric interior penalty method (NIPG) [14], and local DG (LDG) [9, 8].

Given a bounded polygonal domain Ω in the plane and data $f \in L^2(\Omega)$, the Poisson model problem seeks $u \in V := H^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \text{for all } v \in V.$$

Let $\mathcal{T}$ be some shape regular triangulation of Ω with associated mesh size function $h_\mathcal{T}$. The Courant finite element space of H^1-conforming $\mathcal{T}$-piecewise affine functions is denoted $V_\mathrm{C}(\mathcal{T}) := P_1(\mathcal{T}) \cap V$. The corresponding (unique) Galerkin approximation $u_\mathrm{C} \in V_\mathrm{C}(\mathcal{T})$ satisfies

$$a(u_\mathrm{C}, v_\mathrm{C}) = \textstyle\int_\Omega f v_\mathrm{C} \,\mathrm{d}x \quad \text{for all } v_\mathrm{C} \in V_\mathrm{C}(\mathcal{T}).$$

Abstract DGFEM. Consider the space $V_{\mathrm{DG}}(\mathcal{T}) := P_1(\mathcal{T})$ of $\mathcal{T}$-piecewise affine functions with associated norm $\| \bullet \|_{\mathrm{DG}} := (\|\nabla \bullet \|^2_{L^2(\Omega)} + | \bullet |^2_{\mathrm{J}})^{1/2}$ and jump seminorm

$$| \bullet |^2_{\mathrm{J}} := \sum_{E \in \mathcal{E}} |E|^{-1} \|[\bullet]_E\|^2_{L^2(E)};$$

$[v_{\mathrm{DG}}]_E$ denotes the jump of $v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T})$ across the edge $E \in \mathcal{E}$ as usual.

The bounded and coercive (with respect to $\| \bullet \|_{\mathrm{DG}}$) DG bilinear form $a_{\mathrm{DG}} : V_{\mathrm{DG}}(\mathcal{T}) \times V_{\mathrm{DG}}(\mathcal{T}) \to \mathbb{R}$ extends $a|_{V_{\mathrm{C}}(\mathcal{T}) \times V_{\mathrm{C}}(\mathcal{T})}$ to $V_{\mathrm{DG}}(\mathcal{T}) \times V_{\mathrm{DG}}(\mathcal{T})$ and satisfies

$$|a(v, v_{\mathrm{C}}) - a_{\mathrm{DG}}(v_{\mathrm{DG}}, v_{\mathrm{C}})| \le C_1 \|v - v_{\mathrm{DG}}\|_{\mathrm{DG}} \|\nabla v_{\mathrm{C}}\|_{L^2(\Omega)} \tag{1}$$

for all $v_{\mathrm{C}} \in V_{\mathrm{C}}(\mathcal{T})$, $v \in V$, and $v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T})$ with some universal positive constant C_1 independent of $h_{\mathcal{T}}$. The (unique) DG approximation $u_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T})$ satisfies

$$a_{\mathrm{DG}}(u_{\mathrm{DG}}, v_{\mathrm{DG}}) = \textstyle\int_\Omega f v_{\mathrm{DG}} \,\mathrm{d}x \quad \text{for all } v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T}).$$

Assume further that there exists some bounded linear operator $\mathrm{I}_{\mathrm{C}} : V_{\mathrm{DG}}(\mathcal{T}) \to V_{\mathrm{C}}(\mathcal{T})$ and some positive constant C_2 that does not depend on $h_{\mathcal{T}}$ such that

$$\|v_{\mathrm{DG}} - \mathrm{I}_{\mathrm{C}} v_{\mathrm{DG}}\|_{\mathrm{DG}} \le C_2 |v_{\mathrm{DG}}|_{\mathrm{J}} \quad \text{holds for all } v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T}). \tag{2}$$

It is shown in [11, Section 3.2] that SIPG, NIPG, and LDG fit into this abstract framework with some operator I_{C} based on nodal averaging [3, 4, 5, 13].

Main result. The comparison is stated in terms of $A \lesssim B$ which abbreviates the existence of some constant C which only depends on the minimal angle in $\mathcal{T}$, but *not* on the domain Ω and *not* on the mesh-size $h_{\mathcal{T}}$, such that $A \le CB$. The comparison includes data oscillations $\mathrm{osc}(f, \mathcal{T}) := \|h_{\mathcal{T}}(f - \Pi_0 f)\|_{L^2(\Omega)}$, where Π_0 denotes the L^2 orthogonal projection onto the piecewise constants.

The comparison result for CFEM and DGFEM reads

$$\|\nabla u - \nabla u_{\mathrm{C}}\|_{L^2(\Omega)} \lesssim \|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \lesssim \|\nabla u - \nabla u_{\mathrm{C}}\|_{L^2(\Omega)} + \mathrm{osc}(f, \mathcal{T}). \tag{3}$$

Needless to say that (3), by transitivity, establishes the equivalence of SIPG, NIPG, and LDG as well as CRFEM. It is remarkable that those results do *not* rely on any regularity assumption and hold for arbitrary coarse triangulations and not just in an asymptotic regime.

Sketch of proof. The inclusion $V_{\mathrm{C}}(\mathcal{T}) \subset V_{\mathrm{DG}}(\mathcal{T})$ and the triangle inequality yield

$$\|\nabla(u - u_{\mathrm{C}})\|_{L^2(\Omega)} = \|u - u_{\mathrm{C}}\|_{\mathrm{DG}} \le \|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} + \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|_{\mathrm{DG}}.$$

Coercivity of a_{DG} (with respect to $\| \bullet \|_{\mathrm{DG}}$), Galerkin orthogonality, boundedness of a_{DG}, and the property (2) of the averaging operator I_{C} lead to

$$\begin{aligned} \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|^2_{\mathrm{DG}} &\lesssim a_{\mathrm{DG}}(u_{\mathrm{DG}} - u_{\mathrm{C}}, u_{\mathrm{DG}} - u_{\mathrm{C}}) = a_{\mathrm{DG}}(u_{\mathrm{DG}} - u_{\mathrm{C}}, u_{\mathrm{DG}} - \mathrm{I}_{\mathrm{C}} u_{\mathrm{DG}}) \\ &\lesssim \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|_{\mathrm{DG}} |u_{\mathrm{DG}}|_{\mathrm{J}} \lesssim \|u_{\mathrm{DG}} - u_{\mathrm{C}}\|_{\mathrm{DG}} |u - u_{\mathrm{DG}}|_{\mathrm{J}}. \end{aligned}$$

The combination of the previous estimates proves the first inequality in (3). The proof of the second inequality follows directly from [11, Section 3.2], which requires the condition (1). □

Various generalizations. The equivalence of CFEM and DGFEM immediately generalizes to its higher-order variants. Let $V_{\mathrm{C}}^k(\mathcal{T}) := P_k(\mathcal{T}) \cap V$ be the conforming subspace of $\mathcal{T}$-piecewise polynomials of degree at most $k \in \mathbb{N}$; $V_{\mathrm{DG}}^k(\mathcal{T}) := P_k(\mathcal{T})$ denotes the corresponding DG space of the same order. Then

$$\|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} \lesssim \|u - u_{\mathrm{DG}}^k\|_{\mathrm{DG}} \lesssim \|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} + \mathrm{osc}_k(f, \mathcal{T})$$

holds with $\mathrm{osc}_k(f, \mathcal{T}) := \|h_{\mathcal{T}}^k (f - \Pi_{k-1} f)\|$ where Π_{k-1} denotes the L^2 orthogonal projection onto $P_{k-1}(\mathcal{T})$. The hidden generic constants may depend on the polynomial degree k but *not* on the mesh size $h_{\mathcal{T}}$.

Often, the large number of degrees of freedom in DGFEM compared to CFEM is justified by the possibility of using non-conforming meshes that may contain some finite number of hanging nodes per edge. Define $V_{\mathrm{DG}}^k(\mathcal{T}) := P_k(\mathcal{T})$ for some non-conforming triangular mesh $\mathcal{T}$. It is shown in [13] that also for such meshes there exists an averaging operator $\mathrm{I}_{\mathrm{C}} : V_{\mathrm{DG}}^k(\mathcal{T}) \to V$ that satisfies (2) with suitably redefined jump seminorm. The image $\mathrm{I}_{\mathrm{C}}(V_{\mathrm{DG}}^k(\mathcal{T})) = V_{\mathrm{DG}}^k(\mathcal{T}) \cap V$ defines some conforming space $V_{\mathrm{C}}^k(\mathcal{T})$. One might not want to use $V_{\mathrm{C}}^k(\mathcal{T})$ for actual computations but the corresponding Galerkin solution u_{C}^k serves for a comparison. The proof of (3) remains valid in this setting and establishes the comparison

$$\|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} \lesssim \|u - u_{\mathrm{DG}}^k\|_{\mathrm{DG}} \lesssim \|\nabla u - \nabla u_{\mathrm{C}}^k\|_{L^2(\Omega)} + \mathrm{osc}_k(f, \mathcal{T})$$

for non-conforming meshes. Hence, SIPG, NIPG, and LDG are equivalent also on non-conforming meshes and their accuracy is limited by the accuracy that is provided by its largest conforming subspace.

These new results on DGFEM will be included in an upcoming revised version of [7]. Similar comparison results can be achieved for 3-dimensional domains, non-simplicial meshes, or other DG schemes (e.g., WOPSIP [6]). Applications of comparison results include least-squares finite element methods and equality of approximation classes for concepts of optimality for adaptive finite element methods.

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Exponential Convergence of *hp*-Version Discontinuous Galerkin Methods for Elliptic Problems in Polyhedral Domains

Dominik Schötzau
(joint work with Christoph Schwab, Thomas Wihler)

In a series of land mark papers in the mid eighties, Babuška and Guo proved that using *hp*-version finite element methods for the numerical approximation of elliptic problems with piecewise analytic data in polygonal domains leads to exponential rates of convergence in the number of degrees of freedom. The convergence bounds are typically of the form

$$\|u - u_N\|_E \leq C \exp(-b\sqrt[3]{N}),$$

where u is the solution of the boundary-value problem, u_N its *hp*-version finite element approximation, $\|\cdot\|_E$ a suitable (energy) norm to measure the error, N the dimension of the *hp*-version finite element space, and C and b are constants independent of N; see [2, 3, 4] and the references therein.

Starting in the nineties, steps were undertaken to extend these results to polyhedral domains in $\mathbb{R}^3$; see [1] and the references therein. It was asserted and confirmed numerically that the errors decay exponentially as $C \exp(-b\sqrt[5]{N})$, i.e., with an exponent that contains the fifth root of N.

In this talk, we prove this convergence rate for *hp*-version discontinuous Galerkin (DG) discretizations of the model problem

$$\begin{aligned} -\nabla \cdot (\boldsymbol{A} \nabla u) &= f \qquad \text{in } \Omega \subset \mathbb{R}^3, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where Ω is an axiparallel Lipschitz polyhedron, $\boldsymbol{A}$ a constant symmetric positive definite coefficient matrix, and f is a right-hand side which we assume to be analytic in $\overline{\Omega}$.

The solution u now exhibits not only isotropic corner singularities as in polygons, but possesses also anisotropic edge and corner-edge singularities near edges and corners of the domain Ω. In the very recent work [5], the precise behaviour of these singularities has been characterized in terms of analytic regularity results in anisotropically weighted Sobolev spaces; this will be a key ingredient in our analysis.

To numerically resolve these singularities, we shall present a specific construction of families $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 1}$ of geometrically and anisotropically refined hexahedral meshes in Ω; see [6]. The parameter $\sigma \in (0,1)$ is the geometric refinement factor, and the index ℓ represents the number of geometric layers. In Figure 1, we show an example of an axiparallel geometric mesh $\mathcal{M}_\sigma^{(\ell)}$ on the Fichera domain with $\sigma = 0.5$. Notice the high aspect ratio elements along edges, which are mandated by the regularity of the solution u. Following [6], we shall further introduce accompanying polynomial degree distributions on each geometric mesh $\mathcal{M}_\sigma^{(\ell)}$. We allow for elemental polynomial degree distributions which are nonuniform between elements and anisotropic in elements along edges, but whose ratio across interfaces of hexahedral elements is uniformly bounded. As in the two-dimensional case, the polynomials degrees are linearly increased away from corners and edges.

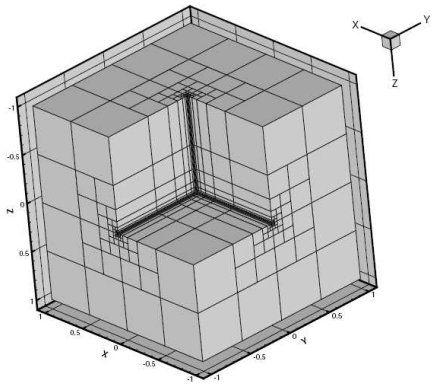

FIGURE 1. Geometric mesh on Fichera domain with $\sigma = 0.5$.

The *hp*-meshes and polynomial degree distributions lead to *hp*-version DG finite element spaces of the form

$$V(\mathcal{M}_\sigma^{(\ell)}) = \left\{ u \in L^2(\Omega) \,:\, u|_K \in \mathbb{Q}^{\boldsymbol{p}_K}(K),\, K \in \mathcal{M}_\sigma^{(\ell)} \right\},$$

where $\mathbb{Q}^{\boldsymbol{p}_K}(K)$ denotes the usual mapped tensor-product polynomial space on element K, and $\boldsymbol{p}_K = (p_{K,1}, p_{K,2}, p_{K,3})$ is the polynomial degree in each coordinate direction. The discontinuous Galerkin approximation of our model problem now

reads as follows:

$$\text{Find } u_{DG} \in V(\mathcal{M}_\sigma^{(\ell)}) \text{ s.t. } A_{DG}(u_{DG}, v) = \int_\Omega f v \, dx \text{ for all } v \in V(\mathcal{M}_\sigma^{(\ell)})$$

Here, we take $A_{DG}(u, v)$ as an *hp*-version interior penalty form; cf. [6]. Due to the presence of anisotropic elements and polynomial degrees, the penalization terms will have to be defined suitably in terms of discretization parameters perpendicular to faces.

For our error analysis, we introduce the broken energy norm

$$\|u\|_{DG}^2 = \sum_{K \in \mathcal{M}_\sigma^{(\ell)}} \|\nabla u\|_{L^2(K)}^2 + \sum_{F \in \mathcal{F}} \alpha|_F \|[\![u]\!]\|_{L^2(F)}^2,$$

where we denote by $\mathcal{F}$ the set of all mesh faces, $[\![\cdot]\!]$ signifies the jump across elements, and $\alpha|_F$ is the interior penalty parameter restricted to face F.

We shall show that the DG method is well-defined on geometric mesh families, that the associated bilinear form A_{DG} is bounded and coercive over the discrete spaces, and that we have the property of Galerkin orthogonality. With these results, it follows now straightforwardly that the error in the DG norm can be bounded by quantities involving only the interpolation error $u - \Pi u$, for a properly chosen elementwise interpolation operator Πu.

First, we shall derive an abstract error estimate of the form

$$\|u - u_{DG}\|_{DG} \leq C \left(E_{\mathcal{I}}[u - \Pi u] + E_{\mathcal{B}}[u - \Pi u] \right).$$

The first error term $E_{\mathcal{I}}$ only involves elements away from the boundary of Ω, while the second contribution $E_{\mathcal{B}}$ is related to elements at the boundary of Ω; cf. [6]. To bound $E_{\mathcal{I}}[u - \Pi u]$, we shall take Π as an elementwise tensorized *hp*-interpolation operator. Hence, combining the regularity results of [5] with classical *hp*-approximation techniques yields

$$E_{\mathcal{I}}[u - \Pi u] \leq C e^{-b\ell}.$$

On the boundary, we take the zero interpolation operator $\Pi u = 0$. Since the elements there are exponentially small and by exploiting the structure of the weighted spaces in [5], we obtain

$$E_{\mathcal{B}}[u] \leq C e^{-b\ell}.$$

Since $N = \dim(V(\mathcal{M}_\sigma^{(\ell)})) \simeq \ell^5$, these two bounds imply the main result of this talk; see [7]:

Theorem 1. *Let $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 1}$ be a geometric mesh family on Ω. Then for each $\ell \geq 1$, the DG approximation $u_{DG} \in V(\mathcal{M}_\sigma^{(\ell)})$ is well-defined, and as $\ell \to \infty$, it satisfies the error bound*

$$\|u - u_{DG}\|_{DG} \leq C \exp\left(-b \sqrt[5]{N}\right),$$

with constants C and b independent of $N = \dim\left(V(\mathcal{M}_\sigma^{(\ell)})\right)$.

Let us conclude by mentioning possible extensions that are the subject of ongoing research: mixed boundary conditions which are considerably more involved than Dirichlet boundary conditions, more general element mappings and geometries, non-constant coefficients, as well as more complex elliptic boundary-value problems (such as elasticity).

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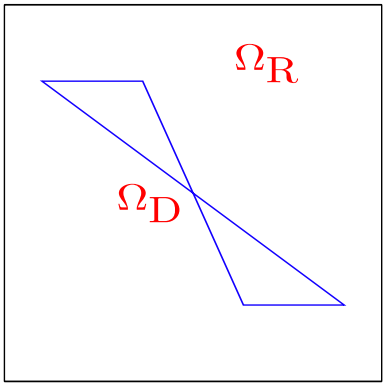
Convergence Analysis of an Adaptive Interior Penalty Discontinuous Galerkin Method for the Helmholtz Equation

Natasha S. Sharma
(joint work with R. H. W. Hoppe)

We are concerned with a convergence analysis of an adaptive Interior Penalty Discontinuous Galerkin (IPDG) method for the numerical solution of acoustic wave propagation problems as described by the Helmholtz equation. The mesh adaptivity relies on a residual-type a posteriori error estimator that does not only control the approximation error but also the consistency error caused by the nonconformity of the approach. As in the case of IPDG for standard second order elliptic boundary value problems [1, 2, 4], the convergence analysis is based on the reliability of the estimator, an estimator reduction property, and a quasi-orthogonality result. However, in contrast to the standard case, special attention has to be paid to a proper treatment of the lower order term in the equation containing the wavenumber which is taken care of by an Aubin-Nitsche type argument for the associated conforming finite element approximation. This idea, which can be traced back to [6], has been used in the convergence analysis of adaptive conforming finite element approximations of general second order elliptic PDE [5] and of adaptive conforming edge element approximations of the time-harmonic Maxwell equations [8]. Numerical results are given for a screen problem illustrating the performance



of the adaptive IPDG method.
The screen problem for the Helmholtz equation (1a)-(1c) describes the propagation of an acoustic wave with wave number k in the computational domain $\Omega := \Omega_R \backslash \Omega_D$ and its scattering at a soft-sound screen Ω_D.

$$-\Delta u - k^2 u = f \quad \text{in } \Omega, \tag{1a}$$

$$\frac{\partial u}{\partial \nu_R} + iku = g \quad \text{on } \Gamma_R, \tag{1b}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{1c}$$

where f, g, u are complex-valued functions.

Under the following assumption on the data of the problem

$$f \in L^2(\Omega), \quad g \in L^2(\Gamma_R),$$

the weak formulation of (1a)-(1c) amounts to the computation of $u \in V, V := H^1_{0,\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}$ such that for all $v \in V$ it holds

$$a(u,v) - k^2 c(u,v) + ik\, r(u,v) = \ell(v). \tag{2}$$

Here, the sesquilinear forms a, c, r and the linear functional ℓ are given by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla \bar{v}\, dx, \quad c(u,v) := \int_\Omega u\bar{v}\, dx,$$

$$r(u,v) := \int_{\Gamma_R} u\bar{v}\, ds, \quad \ell(v) := \int_\Omega f\bar{v} dx + \int_{\Gamma_R} g\bar{v}\, ds.$$

The convergence of the proposed adaptive IPDG method is analyzed by proving the following contraction property for a weighted sum of the global discretization error in the IPDG energy norm and the residual-type a posteriori error estimator.

Theorem. *Let $u \in H^1_{0,\Gamma_D}(\Omega)$ be the unique solution of (2). Further, let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the associated IPDG approximations solutions and error estimators, respectively. Then, there exist constants $0 < \delta < 1$ and $\rho > 0$, depending only on the shape regularity of the triangulations and the parameter θ from the Dörfler marking, such that for sufficiently large penalty parameter α and*

sufficiently small mesh widths h, H the fine mesh and coarse mesh discretization errors $e_h := u - u_h$ and $e_H = u - u_H$ satisfy

$$a_h^{IP}(e_h, e_h) + \rho\ \eta_h^2 \leq \delta\ \Big(a_H^{IP}(e_H, e_H) + \rho\ \eta_H^2\Big).$$

Numerical Experiments. The benefits of the adaptive IPDG approach have been illustrated by various experiments in [3, 7]. In particular, the screen problem (1a)-(1c) has been investigated for $f \equiv 0$ and an incoming wave $g(x, y) = \exp^{iky}$ on the exterior boundary Γ_R of the domain. Figure 1 below depicts the real part of the computed solution and the refined mesh after 11 refinement steps.

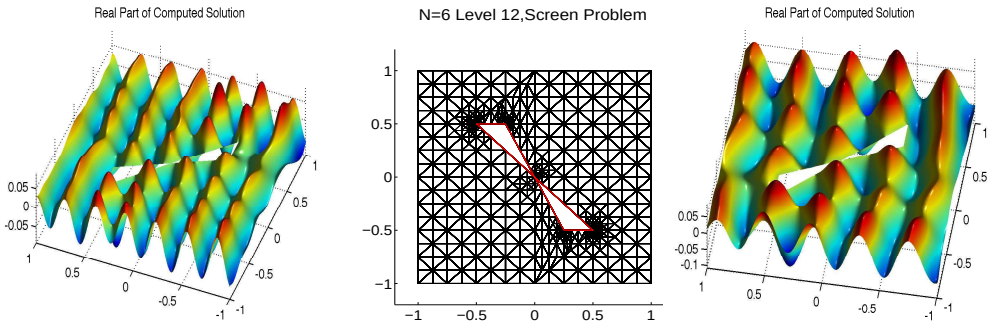


FIGURE 1. Real part of the computed IPDG solution for $k = 15$ (left); refined mesh for $\theta = 0.1$ in Dörfler marking for $k = 15$ (center); Real part of the computed IPDG solution for $k = 20$ (right) .

As expected, refinement is concentrated around the corner singularities present at the boundary Γ_D of the acoustic screen.
For a fixed wavenumber $k = 10$ and polynomial degrees $N = 2$ (left), $N = 4$ (center), and $N = 6$ (right), Figure 2 displays the convergence history for different values of the universal constant θ in the Dörfler marking. Since an analytic solution is not available, the decrease in the error estimator is shown as a function of the total number of degrees of freedom on a logarithmic scale. As expected, the benefits of the adaptive approach can be observed for increasing polynomial degree N. Moreover, higher polynomial degrees can handle higher wave numbers better at the expense of increased computational work.

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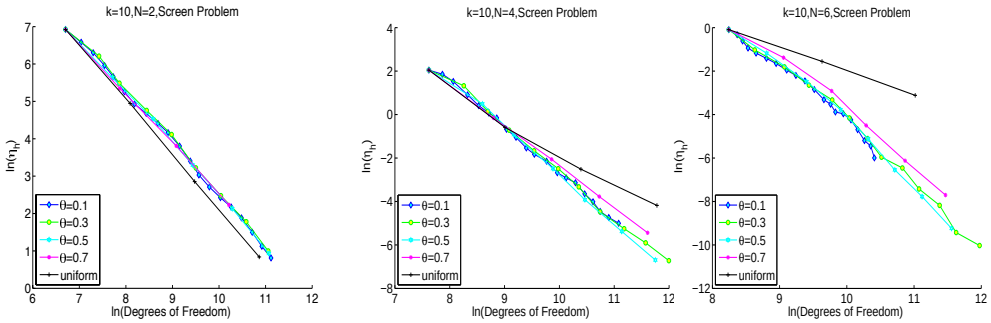

FIGURE 2. Convergence history of the adaptive IPDG method. Error estimator as a function of the DOF (degrees of freedom) on a logarithmic scale: $k = 10, N = 2$ (left), $k = 10, N = 4$ (center), and $k = 10, N = 6$ (left).

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BDDC for SIPG

LI-YENG SUNG
(joint work with Susanne C. Brenner and Eun-Hee Park)

The balancing domain decomposition by constraints (BDDC) method was introduced in [5] for classical finite element methods for second order elliptic boundary value problems. It is a nonoverlapping domain decomposition algorithm closely related to the balancing domain decomposition (BDD) algorithm [8] and the finite element tearing and interconnecting dual-primal (FETI-DP) algorithm [7]. In this talk we discuss the extension of the BDDC algorithm to the symmetric interior penalty Galerkin (SIPG) method [6, 11, 1].

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$, $\Omega_1, \ldots, \Omega_J$ be a shape regular nonoverlapping decomposition of Ω and $\rho_1, \ldots, \rho_J$ be positive constants. We consider the following model problem: Find $u \in H_0^1(\Omega)$ such that

$$\sum_{j=1}^{J} \rho \int_{\Omega_j} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \qquad \forall\, v \in H_0^1(\Omega), \tag{1}$$

where $\rho = \rho_j$ on Ω_j for $1 \le j \le J$ and $f \in L_2(\Omega)$.

Let $\mathcal{T}_h$ be a conforming triangulation of Ω aligned with the boundaries of the subdomains, $\mathcal{E}_h$ be the set of the edges of the triangles in $\mathcal{T}_h$, and V_h be the discontinuous P_1 finite element space associated with $\mathcal{T}_h$. The SIPG method for (1) is to find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{2}$$

where

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T \rho \nabla w \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}_h} \int_e \big(\{\!\{\rho \nabla w\}\!\} \cdot [\![v]\!] + \{\!\{\rho \nabla v\}\!\} \cdot [\![w]\!] \big) ds + \eta \sum_{e\in\mathcal{E}_h} \frac{\rho_e}{|e|} \int_e [\![v]\!] \cdot [\![w]\!] \, dx,$$

$|e|$ is the length of e and η is a (sufficiently large) positive penalty parameter. On an interior edge e shared by two triangles $T_\pm$, we define

$$\rho_e = \frac{2\rho_-\rho_+}{\rho_- + \rho_+}, \quad \{\!\{\rho\nabla v\}\!\} = \frac{\rho_-\rho_+}{\rho_- + \rho_+}(\nabla v_- + \nabla v_+) \quad \text{and} \quad [\![v]\!] = v_- n_- + v_+ n_+,$$

where $\rho_\pm = \rho\big|_{T_\pm}$ and $n_\pm$ is the outward unit normal of $T_\pm$. On a boundary edge they reduce to ρ, $\rho\nabla v$, and v.

There are two main difficulties in extending the analysis of BDDC in [9, 10, 4] to SIPG: (i) the coupling along the interface $\Gamma = \cup_{j=1}^J \Gamma_j = \cup_{j=1}^J (\partial\Omega_j \setminus \partial\Omega)$ prevents the representation of the bilinear form $a_h(\cdot,\cdot)$ as the sum of bilinear forms defined on the subdomains; (ii) the discontinuous nature of the finite element functions prevents the analysis of discrete harmonic functions through the $H^{\frac{1}{2}}(\partial\Omega_j)$ norm of their traces on the boundaries of the subdomains.

To avoid the first difficulty, we introduce a space decomposition

$$V_h = V_{h,C} \oplus V_{h,D} \tag{3}$$

where

$$\begin{aligned} V_{h,C} &= \{v \in V_h : [\![v]\!] = 0 \quad \text{on the edges along } \Gamma\}, \\ V_{h,D} &= \{v \in V_h : \{\!\{v\}\!\} = 0 \quad \text{on the edges along } \Gamma \text{ and } v \text{ vanishes} \\ &\qquad \text{at all the other nodes of } V_h\}. \end{aligned}$$

Here the average $\{\!\{v\}\!\}$ is defined by

$$\{\!\{v\}\!\} = \Big(\frac{\rho_-}{\rho_- + \rho_+}\Big) v_- + \Big(\frac{\rho_+}{\rho_- + \rho_+}\Big) v_+$$

for an interior edge. The original problem (2) is reduced to a subproblem on $V_{h,D}$ and a subproblem on $V_{h,C}$. Since the dimension of $V_{h,D}$ is small and the system matrix for the subproblem on $V_{h,D}$ is block diagonal with small blocks of fixed sizes, the subproblem on $V_{h,D}$ can be easily solved. Hence we can focus on the subproblem on $V_{h,C}$.

Let $a_{h,j}(\cdot,\cdot)$ be defined by

$$a_{h,j}(w,v) = \sum_{T\in\mathcal{T}_{h,j}} \int_T \rho\nabla w\cdot\nabla v\,dx - \sum_{e\in\mathcal{E}_{h,j}} \int_e \big(\{\!\{\rho\nabla w\}\!\}\cdot[\![v]\!] + \{\!\{\rho\nabla v\}\!\}\cdot[\![w]\!]\big)ds + \eta\sum_{e\in\mathcal{E}_{h,j}} \frac{\rho_e}{|e|}\int_e [\![v]\!]\cdot[\![w]\!]\,dx,$$

where $\mathcal{T}_{h,j}$ is the set of the triangles in $\mathcal{T}_h$ that are subsets of Ω_j and $\mathcal{E}_{h,j}$ is the set of the edges in $\mathcal{E}_h$ that are subsets of $\bar{\Omega}_j\setminus\Gamma_j$. Then we have

$$a_h(v_C,v_C) = \sum_{j=1}^{J} a_{h,j}(v_{C,j},v_{C,j}) \qquad \forall\, v_C\in V_{h,C}, \tag{4}$$

where $v_{C,j}$ is the restriction of v_C to Ω_j.

We now apply the BDDC methodology to construct a nonoverlapping domain decomposition preconditioner for the problem on $V_{h,C}$. Because of the relation (4), the first difficulty mentioned above disappears and we can show that the minimum eigenvalue of the preconditioned system is greater than or equal to 1 as in the case of classical finite element methods. The estimate for the maximum eigenvalue of the preconditioned system requires the equivalence of the energy norm and a trace norm on the space of discrete harmonic functions so that the effect of truncating the nodal values along the interface, which appears in many nonoverlapping domain decomposition algorithms, can be controlled. Here we encounter the second difficulty mentioned above, namely, the equivalence of the energy norm and a trace norm is not obvious for discrete harmonic functions that are discontinuous. Using the techniques developed in [3], we can overcome this difficulty by showing that for discrete harmonic functions the energy norm is equivalent to a norm involving continuous piecewise cubic polynomials on the boundaries of the subdomains constructed from the traces of the discrete harmonic functions.

With the two difficulties circumvented, we can recover the condition number estimate that holds for classical finite element methods. Let $A_h : V_h \longrightarrow V_h'$ be the system operator for (2) and $B_h : V_h' \longrightarrow V_h$ be the preconditioner constructed from the space decomposition (3) and the BDDC preconditioner for the subproblem associated with $V_{h,C}$. Then we have

$$\kappa(B_hA_h) = \frac{\lambda_{\max}(B_hA_h)}{\lambda_{\min}(B_hA_h)} \le C\Big(1+\ln\frac{H}{h}\Big)^2,$$

where H represents the typical diameter of a subdomain and the positive constant C is independent of ρ, h, H, J and η. Details can be found in [2].

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HP-Multigrid as Smoother Algorithm for Higher Order Discontinuous Galerkin Discretizations of Advection Dominated Flows

J. J. W. van der Vegt
(joint work with S. Rhebergen)

Higher order accurate space-time discontinuous Galerkin methods are well suited for the solution of the compressible and incompressible Navier-Stokes equations on time dependent domains, such as occur in fluid structure interaction and free surface problems [1, 3]. The space-time DG algorithm achieves higher order accuracy on general unstructured deforming meshes and is well suited to be combined with local mesh refinement, resulting in a versatile solution adaptive finite element method.

The space-time DG method is, however, implicit in time and requires the efficient solution of a generally large system of algebraic equations. This also applies to DG discretizations in space in combination with an implicit time integration method. For second order accurate space-time DG discretizations of advection dominated flows a reasonably efficient algorithm for the solution of the algebraic system resulting from a space-time DG discretization can be obtained using h-multigrid in combination with an explicit Runge-Kutta method as smoother [2]. Unfortunately, this is not the case for higher order accurate DG discretizations. In order to achieve also good multigrid convergence rates for higher order accurate discretizations we developed the hp-Multigrid as Smoother algorithm (hp-*MGS*) [4, 5]. This algorithm combines p-multigrid with h-multigrid at all p-levels, where the h-multigrid acts as smoother in the p-multigrid. The performance of the hp-*MGS* algorithm is further improved using semi-coarsening in combination with a

new semi-implicit Runge-Kutta method as smoother. A detailed multilevel analysis of the *hp-MGS* algorithm is used to obtain more insight into the theoretical performance of the algorithm. For the multilevel analysis a fourth order accurate space-time DG discretization of the advection-diffusion equation is used as model problem. The multilevel analysis shows that the *hp-MGS* algorithm has excellent convergence rates, both for low and high cell Reynolds numbers and also on highly stretched meshes.

The multilevel analysis also gives the opportunity to optimize the Runge-Kutta smoother in the *hp*-Multigrid as Smoother algorithm since it provides accurate predictions of the spectral radius and operator norms of the multigrid error transformation operator. These results then can be used to search for Runge-Kutta coefficients which minimize the spectral radius of the multigrid error transformation operator under properly chosen constraints. In particular, it is required that the Runge-Kutta smoothers are stable at each polynomial level in order to ensure a sufficiently robust multigrid algorithm. The Runge-Kutta coefficients have been computed for a wide range of cell Reynolds numbers and during numerical simulations the optimal coefficients are used based on the local cell Reynolds number. This results in a very efficient multigrid algorithm which is suitable for higher order accurate discretizations of advection dominated flows, including thin boundary layers and stretched meshes.

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Aspects of the a priori convergence analysis for the low storage curvilinear discontinuous Galerkin method

Timothy Warburton

Modern many-core processing units, including graphics processing units (GPU), presage a new era in on-chip massively parallel computing. The advent of processors with $\mathcal{O}(1000)$ floating point units (FPU) raises new issues challenging conventional measures of "optimality" of numerical methods. The ramp up in FPU counts for each new generation of GPU over the past four years has been accompanied by a slower increase in the the memory capacity of the GPU. Even more importantly the bandwidth for data transfer between the GPU chip and on board memory has also not increased in line with the number of FPU.

In [2] Warburton introduced the low storage curvilinear discontinuous Galerkin method (LSC-DG) as an extension to the GPU accelerated DG methods introduced by Klöckner et al [1]. The idea behind LSC-DG is to build customized approximation spaces for each curvilinear element that removes the dependence of the elemental mass matrix on the non-constant determinant of the Jacobian of the coordinate transform between the physical and reference element. This reduces the memory requirement for meshes with a large number of curvilinear elements.

In this abstract we consider a reference element, with local coordinates (r, s), given by $\hat{D} = \{-1 \leq r, s; r + s \leq 0\}$, in the case of a triangle or $\hat{D} = \{-1 \leq r, s \leq 1\}$, in the case of a quadrilateral element. The physical element $D^k \subset \mathbb{R}^2$ is the image of $(x^k, y^k) \in \left(\mathbb{S}^N(\hat{D})\right)^2$ where $\mathbb{S}^N(\hat{D})$ is the space of polynomials of total degree at most N in the case of triangles and the space of polynomials of maximum degree N in either variable in the case of quadrilateral elements. The superscript k emphasizes the specialization of the transformation to element k. The determinant of the transform between $\hat{D}$ and D^k is given by $J^k = \frac{\partial(x^k, y^k)}{\partial(r,s)}$.

The LSC-DG ansatz involves the numerical approximation of the solution on element k by choosing functions from element specific function spaces

$$V^k = \left\{ \phi : \phi = \frac{\tilde{\phi}}{\sqrt{J^k}} \text{ for some } \tilde{\phi} \in \mathbb{P}^N(\hat{T}) \right\},$$

i.e. each function is a linear combination of polynomials on the reference element divided by the determinant of the Jacobian. The following holds for all $\phi, \psi \in V^k$

$$(\phi, \psi)_{D^k} = \left(\phi, \psi J^k\right)_{\hat{D}} = \left(\frac{\tilde{\phi}}{\sqrt{J^k}}, \frac{\tilde{\psi}}{\sqrt{J^k}} J^k \right)_{\hat{D}} = \left(\tilde{\phi}, \tilde{\psi}\right)_{\hat{D}},$$

where $(.,.)$ is the usual L^2 inner-product on $\hat{D}$. Thus when we choose a basis for the polynomial numerator space $\mathbb{S}^N(\hat{D})$ on the reference element the associated mass matrix does not depend on the geometry of the curvilinear element.

Numerical experiments presented by Warburton [2] indicate that the novel choice of approximation spaces V^k does not greatly impact accuracy when compared with the standard DG method. As a first step in a full a priori analysis

of the LSC-DG method we sketch an L^2 approximation result when using these spaces. Given $u \in H^s(D^k)$ we define a weighted L^2 projection operator Π^k by

$$\Pi^k u = \frac{1}{\sqrt{J^k}} \hat{\Pi} \left(u\sqrt{J^k} \right),$$

where $\hat{\Pi}$ is the reference element L^2 projection operator defined implicitly by

$$(\phi, \hat{\Pi} u)_{\hat{D}} = (\phi, u)_{\hat{D}} \text{ for all } \phi \in \mathbb{S}^N(\hat{D}).$$

The weighted projection operator error can be estimated by observing

$$\left\| u - \Pi^k u \right\|_{D^k} = \left\| u\sqrt{J^k} - \hat{\Pi} \left(u\sqrt{J^k} \right) \right\|_{\hat{D}} \leq C \left| u\sqrt{J^k} \right|_{H_2^{N+1}(\hat{D})},$$

which follows from the Bramble-Hilbert lemma [4] applied on the reference element $\hat{D}$ with generic constant C independent of $u\sqrt{J^k}$. We next assume that the elements are quasi-regular and that the following scaling argument holds

$$\left| u\sqrt{J^k} \right|_{H_2^{N+1}(\hat{D})} \leq C h^N \left\| u\sqrt{J^k} \right\|_{H_2^{N+1}(D^k)},$$

for representative element size h. Applying standard arguments we separate the Sobolev norm on the right hand side into the product of two norms to obtain

$$\left\| u - \Pi^k u \right\|_{D^k} \leq C h^N \left\| \sqrt{J^k} \right\|_{H_\infty^{N+1}(D^k)} \|u\|_{H_2^{N+1}(D^k)}.$$

This suggests an extra condition for asymptotically optimal order approximation: that the first norm on the right hand side scales as

$$\left\| \sqrt{J^k} \right\|_{H_\infty^{N+1}(D^k)} \approx Ch. \tag{1}$$

To examine the relevance of this constraint we include a numerical convergence study for transverse mode, time-domain Maxwell's equation in a perfectly electrically conducting cavity. We used two sequences of quadrilateral meshes inspired by Arnold, Boffi, & Falk [3] as shown in Figure 1.

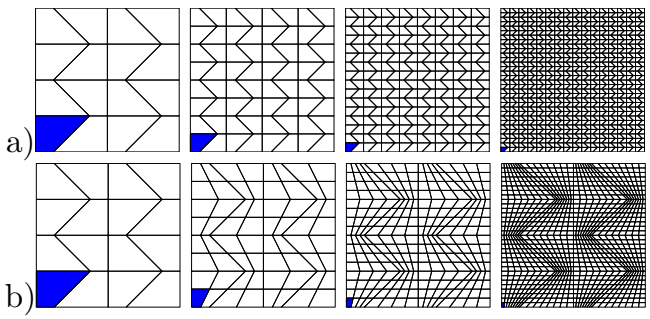

FIGURE 1. Sequence of meshes used to test convergence rates of standard DG and LSC-DG. Top: meshes obtained through self-similar refinement. Bottom: meshes obtained through bisection.

The first sequence of meshes is generated through self-similar refinement (Figure 1a). In Figure 2a we show that the standard DG method converges as expected

but LSC-DG convergence stalls on this mesh sequence. The quadrilateral elements used have non-constant Jacobians and it is straightforward to show that for the self-similar sequence of elements shaded blue in Figure 1a satisfy

$$\left\| \sqrt{J^k} \right\|_{H_\infty^{N+1}(D^k)} \approx Ch^{-N},$$

strongly violating the sufficient geometric scaling constraint (Equation 1) and the initial condition is approximated in a sub-optimal way.

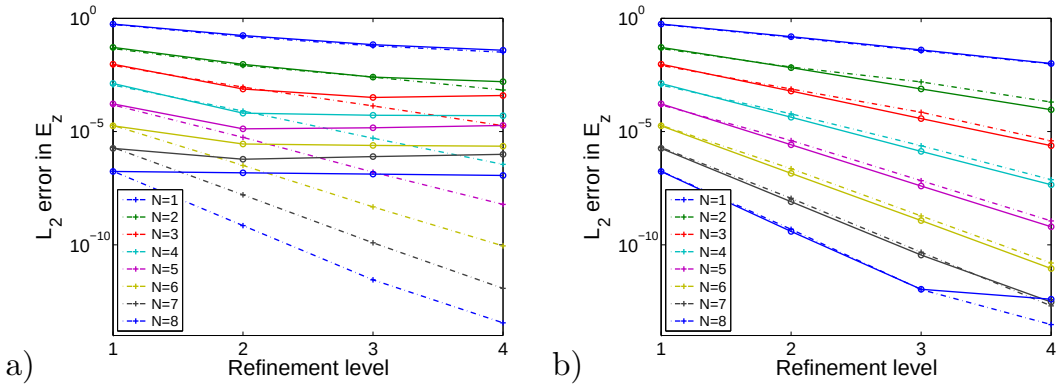

FIGURE 2. Numerical convergence of standard DG (dashed lines) and LSC-DG (solid lines) of the error in the electric field for a two-dimensional transverse mode time-dependent Maxwell's simulation. a) convergence on sequence of self-similar meshes. b) convergence on meshes obtained through bisection.

In the second sequence of meshes obtained through bisection, it is straightforward to show that the element geometry constraint of Equation 1 is satisfied for the representative blue elements highlighted in Figure 1b. The results for this experiment shown in Figure 2b reveal that the standard DG and LSC-DG converge at similar rates with h-refinement using bisection.

In summary, these and previous results reported in [2] prompt a conjecture: assuming the exact solution satisfies usual regularity assumptions of standard DG analysis and that the sequence of meshes satisfy the geometric scaling constraint of Equation 1 then LSC-DG will converge at the same order as standard DG.

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Discontinuous Galerkin for Geophysical Applications

Mary Fanett Wheeler

We discuss discontinuous Galerkin (DG) algorithms for modeling three important and complex geophysical applications. These include: (1) domain decomposition algorithms for elasticity and poroelasticity; (2) a linear slip model for elastic wave propagation; (3) two phase flow with DG-DG IMPES using H(div)-projection and enforcing continuous capillary pressure.

The coupling of geomechanics and multiphase flow is an important research area for both energy and environmental applications. In stress-sensitive reservoirs, variation of the effective stress resulting from fluid production may induce deformation of the rocks and cause permeability reduction. This effect may significantly reduce expected productivity. In other applications, when CO_2 is injected in saline aquifers, the impact on uplifting is still not completely understood. Here we discuss a parallel domain decomposition method for solving both linear elasticity and poroelasticity systems. In linear elasticity [1], data are transmitted by jumps, as in the discontinuous Galerkin method, and mortars are introduced at the interfaces to dissociate the computation between neighboring subdomains. A decoupling algorithm condenses the unknowns on the interface. The matrix of the system is constructed by parallel computation. Numerical experiments are presented to confirm the theoretical convergence rates. In poroelasticity [2], we build upon this work using domain decomposition, by coupling a time-dependent poroelastic model in a localized region with an elastic model in adjacent regions. Each model is discretized independently on non-matching grids and the systems are coupled using DG jumpso n the interface. The unknowns are condensed on the interface, so that at each time step, the computation in each subdomain can be performed in parallel. We also propose an algorithm where the computation of the displacement is time-lagged. We show that in each case, the matrix of the interface problem is positive definite. Error estimates are established for this scheme. This work was done in collaboration with V. Girault, G. Pencheva, and T. Wildey.

In the second application [3], we formulate and implement a discontinuous Galerkin method for elastic wave propagation that allows for discontinuities in the displacement field to simulate fractures or faults using the linear-slip model. We show numerical results using a two dimensional model with one linear-slip discontinuity and different frequencies. The results show a good agreement with analytic solutions. This work was done in collaboration with J. De Basabe and M. Sen.

In the third application [4], we discuss a slightly compressible two-phase flow in porous media based on an iteratively coupled DG-DG scheme which includes capillary pressure that varies over fault blocks. The problem is solved using 'implicit pressure, explicit saturation'-method (IMPES), and the convergence is accelerated with iterative coupling of the equations. We use discontinuous Galerkin to discretize both the pressure and the saturation equations. We propose two improvements, namely projecting the flux to a mass conservative H(div)-space and penalizing the jump in capillary pressure in the saturation equation. We show

that these modifications stabilize the method. We also discuss the need and use of slope limiters. The method is validated with numerical examples. This work was done in collaboration with T. Arbogast, M. Juntunen, and J. Pool.

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A Posteriori Error Analysis for Linear Parabolic PDE based on *hp*-Discontinuous Galerkin Time-Stepping

Thomas P. Wihler

(joint work with M. Georgoulis, O. Lakkis, D. Schötzau)

In this work we are interested in the numerical solution of linear parabolic evolution problems of the form

$$u'(t) + \mathsf{A}u(t) = g(t), \quad t \in (0,T), \qquad u(0) = u_0,$$

with given data

$$u_0 \in H, \qquad g \in L^2((0,T);X^\star).$$

Here, X and H are Hilbert spaces with dense embedding $X \hookrightarrow H$, and $\mathsf{A} : X \to X^\star$ is a linear elliptic operator that is bounded and coercive:

$$|a(u,v)| \le \alpha \|u\|_X \|v\|_X \qquad \forall u, v \in X,$$
$$a(u,u) \ge \beta \|u\|_X^2 \qquad \forall u \in X.$$

This problem has a unique solution satisfying

$$u \in L^2((0,T);X) \cap C^0([0,T];H), \qquad u' \in L^2((0,T);X^\star),$$

with continuous dependence on the data.

In order to discretize the above initial-value problem in time, the use of Galerkin time-stepping methods is proposed. They are based on variational formulations and provide piecewise polynomial approximations in time. The approximation can be chosen to be either continuous or discontinuous at the time discretization points, thereby giving rise to the so-called continuous Galerkin (cG) and discontinuous Galerkin (dG) time-stepping methods, respectively. For both approaches, the discrete Galerkin formulations decouple into local problems on each time-step, and the discretizations can therefore be understood as implicit one-step schemes.

Due to its dissipative nature, the dG time-stepping scheme has proved to be particularly attractive in the context of parabolic PDEs. The variational character of Galerkin time-stepping methods allows for arbitrary variation in the size of the time-steps and the local approximation orders. Therefore, they can be extended naturally to *hp*-version Galerkin schemes. The main feature of these *hp*-methods is their ability to approximate smooth solutions—with possible local singularities—at high algebraic or even exponential rates of converge. In particular, exponential convergence can be achieved in the numerical approximation of problems with start-up singularities.

Choosing, on each time step $I_m = (t_{m-1}, t_m)$, $0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m < \cdots < t_M = T$, a conforming spatial subspace $X_m \subset X$, $\dim(X_m) < \infty$, and a polynomial degree $r_m > 0$ for the discretization of the time variable, the dG time-stepping method is to find a solution $U_{\mathrm{dG}}|_{I_m} \in \mathbb{P}_{r_m}(I_m; X_m)$ such that

$$\int_{I_m} \{(U'_{\mathrm{dG}}, v)_H + a(U_{\mathrm{dG}}, v)\}\,\mathsf{d}t + (U_{\mathrm{dG}}(t^+_{m-1}), v(t^+_{m-1}))_H = \int_{I_m} \langle g, v\rangle_{X^\star \times X}\,\mathsf{d}t + (U_{\mathrm{dG}}(t^-_{m-1}), v(t^+_{m-1}))_H$$

for any $v \in \mathbb{P}_{r_m}(I_m; X_m)$. In order to obtain a posteriori error estimates for this scheme several techniques have been proposed in the literature. A recent approach [1] is based on an appropriate time reconstruction of the dG solution which allows the dG variational formulation to be rewritten in strong form, and subsequently, enables the application of natural energy arguments to obtain suitable error estimates. This idea has been worked out in the *hp*-context in the paper [2], where a posteriori error estimates for semi-discrete *hp*-version Galerkin schemes (discontinuous dG and continuous cG time stepping) of linear parabolic PDE have been derived within a fully abstract Hilbert space setting. They are completely explicit with respect to the size of the time-steps and polynomial degrees.

In our current work, a posteriori error estimates for the full-discretization in time and space of linear parabolic problems by combined *hp*-dG time stepping and conforming spatial FEM, respectively, are investigated in [3]. Again, the time discretization is dealt with by means of the above-mentioned time reconstruction; moreover, the spatial variables are analyzed with the aid of an elliptic reconstruction. This yields the following strong form for the full-discrete problem on each time step $I_m = (t_{m-1}, t_m)$:

$$\widehat{U}'_{\mathrm{dG}}(t) + \mathsf{A}\widetilde{U}_{\mathrm{dG}}(t) = \Pi_m g(t), \qquad t \in I_m.$$

Here, $\widehat{U}_{\mathrm{dG}}$ and $\widetilde{U}_{\mathrm{dG}}$ are parabolic and elliptic reconstructions, respectively, of the numerical solution $U_{\mathrm{dG}}|_{I_m}$. Moreover, $\Pi_m : X^\star \to X_m$ is a suitable L^2-projection. Now, subtracting the exact problem and applying energy techniques, this approach allows to derive computable bounds for the errors $\|u - \widehat{U}_{\mathrm{dG}}\|_{L^2(I_m;X)}$

and $\|u - \widetilde{U}_{\mathrm{dG}}\|_{L^2(I_m;X)}$. Then, we obtain

$$\|u - U_{\mathrm{dG}}\|_{L^2(I_m;X)} \leq \epsilon\|u - \widehat{U}_{\mathrm{dG}}\|_{L^2(I_m;X)} + (1-\epsilon)\|u - \widetilde{U}_{\mathrm{dG}}\|_{L^2(I_m;X)} + \epsilon\|U_{\mathrm{dG}} - \widehat{U}_{\mathrm{dG}}\|_{L^2(I_m;X)} + (1-\epsilon)\|\widetilde{U}_{\mathrm{dG}} - U_{\mathrm{dG}}\|_{L^2(I_m;X)}$$

for any $\epsilon \in (0,1)$. The last two terms on the right-hand side of the above estimate are time and space reconstruction errors of the numerical solution, respectively. For the former term a simple explicit representation in terms of the discontinuity jumps over time steps has been derived in [2]; the latter term can be bounded by means of a posteriori error techniques for elliptic PDE. In summary, we obtain a computable $L^2(X)$-type a posteriori error bound for the fully discrete dG-in-time-FEM-in-space method on each time step. In our future work, suitable adaptive procedures, which will be based upon the a posteriori error estimates derived here, will be investigated.

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Finite Element Methods for the Displacement Obstacle Problem of Clamped Plates

Yi Zhang

(joint work with Susanne C. Brenner, Li-yeng Sung, Hongchao Zhang)

The displacement obstacle problem of clamped plates is a classical example of a fourth order variational inequality. In this talk we present a unified convergence analysis for C^1 conforming finite element methods, classical nonconforming finite elements methods and discontinuous Galerkin methods for this problem.

Suppose $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, $f(x) \in L_2(\Omega)$ and $\psi(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $\psi(x) < 0$ on $\partial\Omega$. We formulate the displacement obstacle problem as the following variational inequality: Find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u) \qquad \forall v \in K, \tag{1}$$

where

$$K = \{v \in H_0^2(\Omega) : \psi \leq v \text{ in } \Omega\}, \tag{2}$$

$$a(w,v) = \int_\Omega D^2 w : D^2 v \, dx = \int_\Omega \sum_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j} \, dx, \tag{3}$$

and $(\cdot,\cdot)$ denotes the L_2 inner product.

It is well known that the problem (1) possesses a unique solution (cf. [12, 11]). However, the solution only belongs to $H^3(\Omega)\cap C^2(\Omega)$ on a convex polygonal domain (cf. [10, 7, 8]). Since the solution does not have H^4 regularity, we cannot obtain the optimal error estimate if we use the complementarity form of the variational inequality in the convergence analysis. In the following, we will introduce a new approach to prove the optimal error estimate.

First, we consider the finite element methods for (1). Let V_h be a finite element space and $a_h(\cdot,\cdot)$ be a symmetric bilinear form such that it is bounded and coercive on V_h. The discrete obstacle problem is: Find $u_h \in K_h$ such that

$$a_h(u_h, v-u_h) \geq (f, v-u_h) \qquad \forall v \in K_h, \tag{4}$$

where

$$K_h = \{v \in V_h : \psi(p) \leq v(p) \quad \forall p \in \mathcal{V}_h\}, \tag{5}$$

and $\mathcal{V}_h$ is the set of the vertices of the triangulation.

In general, the sets K_h and K are not subsets of each other. To connect K_h and K, we make use of the enriching operator E_h (cf. [2, 3]). In fact, E_h maps functions from the finite element space V_h to the Sobolev space $H_0^2(\Omega)$. Moreover, it preserves function values at the vertices and also has the correct approximation properties. By introducing

$$\tilde{K}_h = \{v \in H_0^2(\Omega) : \psi(p) \leq v(p) \quad \forall p \in \mathcal{V}_h\} \tag{6}$$

we can connect K_h and K through the relation $E_hK_h \subset \tilde{K}_h$ and $K \subset \tilde{K}_h$.

It leads us to consider an auxiliary obstacle problem: Find $\tilde{u}_h \in \tilde{K}_h$ such that

$$a(\tilde{u}_h, v-\tilde{u}_h) \geq (f, v-\tilde{u}_h) \quad \forall v \in \tilde{K}_h. \tag{7}$$

We show that the auxiliary solution $\tilde{u}_h$ converges uniformly to u. Combining this with the fact that $K \subset \tilde{K}_h$, $\|u-\tilde{u}_h\|_{H^2(\Omega)}$ can be bounded by the square root of the distance between $\tilde{u}_h$ and K (cf. [1]). In [4], we show the distance between $\tilde{u}_h$ and K is of order $O(h^2)$. Hence $\|u-\tilde{u}_h\|_{H^2(\Omega)}$ is bounded by $O(h)$.

By using the approximation properties of the enriching operator and the properties of the auxiliary obstacle problem, we show in [4] that the convergence rate is $O(h)$ for C^1 conforming finite element methods, classical nonconforming finite elements methods and C^0 interior penalty methods. The results can be extended to two-sided obstacle problems on general polygonal domains with general Dirichlet boundary conditions (cf. [5, 6]).

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