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Representations of Finite Groups

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Abstract. The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Markus Linckelmann (Aberdeen), Gunter Malle (Kaiserslautern) and Jeremy Rickard (Bristol). It covered a wide variety of aspects of the representation theory of finite groups and related objects like Hecke algebras.

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Introduction by the Organisers

The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Markus Linckelmann (Aberdeen), Gunter Malle (Kaiserslautern) and Jeremy Rickard (Bristol). It was attended by 55 participants with broad geographic representation. It covered a wide variety of aspects of the representation theory of finite groups and related objects like Hecke algebras.

In 23 lectures of 50 minutes each, recent progress in representation theory was presented and interesting new research directions were proposed. Besides the lectures, there was plenty of time for informal discussion between the participants, either continuing ongoing research cooperation or starting new projects.

The talks presented several results which were considered as breakthroughs on important, long time open problems. In the first talk, Kessar reported on her solution (joint with Malle) of one direction of the long-standing Brauer height zero conjecture and the completion of the classification of all blocks of all finite simple groups. In related development, Navarro among other results announced a reduction (joint with Späth) of the other half of Brauer's height zero conjecture to a question on finite simple groups, using a generalization (joint with Tiep) of the difficult Gluck–Wolf result to non-solvable groups. Späth explained her reductions of two other famous conjectures of representation theory of finite groups to statements on finite simple groups: the Alperin-McKay conjecture on characters of height 0, and the Alperin weight conjecture. Puig announced a reduction of Alperin's weight conjecture to quasi-simple $k^\times$-groups. These very recent results gave rise to the feeling that the proofs of several of the long standing conjectures in this field might finally have come into reach.

Michel, Dudas and Craven reported on various results about properties of ℓ-adic cohomology of Deligne-Lusztig varieties which, among others, lead to the determination of some of the few remaining unknown Brauer trees of finite simple groups and which, according to the talk by Rouquier, might also open the way to the computation of decomposition matrices even in cases of non-cyclic defect groups.

Williamson gave an introduction to rational representations of reductive algebraic groups in positive characteristic and explained how Lusztig's conjecture on the characters of simple representations is related to modular reduction in an integral version of Bernstein-Gelfand-Gelfand category O (as defined by generators and relations in joint work with Elias).

Several speakers presented substantially simplified new approaches to key results in cohomological representation theory. Benson gave a new algebraic approach to the famous theorem of Mislin (stating that the restriction map in mod p cohomology to a subgroup H of a finite group G is an isomorphism if and only if H controls fusion of p-subgroups). Symonds talked about a new, and surprisingly simple, proof of his theorem implying that the symmetric algebra of a finite-dimensional representation of a finite group contains only finitely many isomorphism classes of indecomposable direct summands. Carlson presented a new approach to the classification of thick subcategories of the stable module category theory of a finite p-group (due to Benson, Rickard and Carlson) that avoids the use of infinite-dimensional modules.

Mathas presented a proof that the decomposition numbers of cyclotomic Hecke algebras and Schur algebras at non-roots of unity are independent of the characteristic, as part of an overview of work with Hu on Khovanov-Lauda-Rouquier algebras and their quasi-hereditary covers.

Geck presented a generalisation of the Frobenius-Schur indicator to arbitrary symmetric algebras. As an application, the well-known fact that the number of involutions in a finite Coxeter group W is equal to the sum of the character degrees is refined to a statement involving Kazhdan-Lusztig cells of W.

Sambale gave an overview on classical block invariants using and refining methods going back to Brauer. He extended in the process the calculations of these invariants to a number of new families of defect groups whose possible fusion systems are calculated.

Tiep described work with Larsen and Shalev on a non-commutative analogue of the Waring problem, culminating in the statement that every element in a quasi-simple finite group is a product of two squares.

Workshop: Representations of Finite Groups

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Abstracts

Around a theorem of Mislin in the cohomology of finite groups

David Benson

This talk is about joint work I carried out in February and early March of this year in Copenhagen with Jesper Grodal and Ellen Henke. Let G be a finite group, p be a prime dividing $|G|$, k a field of characteristic p and S a Sylow p-subgroup of G. An element $x \in H^*(S,k)$ is said to be *stable* with respect to G if for all P, $Q \leq S$ and $g \in G$ with $Q = gPg^{-1}$ we have $\mathrm{res}_{S,P}(x) = c_g^*\mathrm{res}_{S,Q}(x)$, where $c_g^*\colon H^*(Q,k) \to H^*(P,k)$ is induced by conjugation. A theorem of Cartan and Eilenberg from their 1956 book on homological algebra shows that $H^*(G,k)$ is identified by restriction with the subset of $H^*(S,k)$ consisting of elements which are stable with respect to G.

A subgroup $H \leq G$ containing S is said to *control fusion* of subgroups of S in G if for all P, $Q \leq S$ and $g \in G$ such that $gPg^{-1} = Q$ there exists $h \in H$ such that $h^{-1}g \in C_G(P)$. In other words, every conjugation in G between subgroups of S already happens in H. It follows from the theorem of Cartan and Eilenberg that if H controls fusion of subgroups of S in G then the restriction map $H^*(G,k) \to H^*(H,k)$ is an isomorphism.

Mislin's theorem is the converse of this. It says that if $S \leq H \leq G$ as above, and the restriction map $H^*(G,k) \to H^*(H,k)$ is an isomorphism then H controls fusion of subgroups of S in G. Mislin's proof uses Sullivan's conjecture on maps between classifying spaces, which had been proved by Haynes Miller in 1984, and a theorem of Dwyer and Zabrodsky from 1987. Peter Symonds partially algebraised the proof in 2006. He reduced it to a problem about cohomology of trivial source modules. Hida and Okuyama independently completed this algebraisation process in 2007.

Our first theorem provides an independent route to algebraising Mislin's theorem when p is odd. Namely, we prove that if p is an odd prime and H controls fusion of elementary abelian p-subgroups of S in G then it controls fusion of subgroups of S in G. The proof is purely group theoretic, and makes use of two results of John Thompson. One is the $A \times B$ lemma, and the other is a theorem guaranteeing the existence of a characteristic subgroup of a p-group that has exponent p and detects p'-automorphisms, provided p is odd. This lemma is false for $p = 2$, as is easy to see by looking at the quaternion group Q_8.

A theorem of Quillen expresses the cohomology variety in terms of the elementary abelian p-subgroups of G. It is easy to see from Quillen's theorem that if $H^*(G,k) \to H^*(H,k)$ is an F-isomorphism then H controls fusion of elementary abelian subgroups of G. Combining this with our theorem not only reproves Mislin's theorem for p odd, but also shows that $H^*(G,k) \to H^*(H,k)$ is an isomorphism if and only if it is an F-isomorphism. It is easy to see that this last statement is false for $p = 2$, by looking at the inclusion of Q_8 into $SL(2,3) = Q_8 \rtimes \mathbb{Z}/3$.

Let $\mathcal{U}$ be the category of unstable modules over the Steenrod algebra $\mathcal{A}$, and let $\mathcal{N}il$ be the full subcategory consisting of locally nilpotent modules in $\mathcal{U}$. This is the same as the localising subcategory generated by suspensions of modules. More generally, we write $\mathcal{N}il_n$ for the localising subcategory generated by the n-fold suspensions of modules. Then an F-isomorphism is the same thing as an isomorphism in the quotient $\mathcal{U}/\mathcal{N}il$. So what we have proved is that for p odd, $H^*(G,k) \to H^*(H,k)$ is an isomorphism in $\mathcal{U}$ if and only if it is an isomorphism in $\mathcal{U}/\mathcal{N}il$. We have examples to show that for $p = 2$, given any $n > 0$ there are groups $H \leq G$ with $|G : H|$ odd, such that $H^*(G,k) \to H^*(H,k)$ is an isomorphism in $\mathcal{U}/\mathcal{N}il_n$ but not in $\mathcal{U}/\mathcal{N}il_{n+1}$. So the situation for $p = 2$ is as different as it can be from p odd.

On the other hand, we prove that if $p = 2$ and H controls fusion of abelian 2-subgroups of S of exponent at most 4 then it controls fusion of subgroups of S in G. In particular, for any prime, if H controls fusion of abelian subgroups of S in G then it controls fusion of subgroups of S in G.

We can combine this statement with the work of Hopkins, Kuhn and Ravenel (1992/2000) on "character theory" for Morava $K(n)$ theory of BG for the prime p. Their work implies the following. Suppose that $S \leq H \leq G$ as above and $K(n)^*(BG) \to K(n)^*(BH)$ is an isomorphism for $n \geq r_p(G)$. Then H controls fusion of abelian subgroups of S in G.

This gives us a new topological proof of Mislin's theorem for any prime, and indeed it gives the further information that $H^*(G,k) \to H^*(H,k)$ is an isomorphism if and only if $K(n)^*(BG) \to K(n)^*(BH)$ is an isomorphism, provided that p does not divide $|G : H|$.

Finally, we prove the following non-obvious group theoretic statement. Let $H \to G$ be a homomorphism of finite groups. Suppose that $\mathrm{Hom}(A,H)/H \to \mathrm{Hom}(A,G)/G$ is a bijection whenever A is a finite abelian p-group. Then the kernel has order prime to p, the image has index prime to p (hence contains a Sylow p-subgroup S of G), and the image controls fusion of subgroups of S in G. The proof is again entirely group theoretic.

Using this, we can remove the coprime index hypothesis and deduce that for any subgroup H of G, $H^*(G,k) \to H^*(H,k)$ is an isomorphism if and only if $K(n)^*(BG) \to K(n)^*(BH)$ is an isomorphism for $n \geq r_p(G)$. In particular, since the former implies that p does not divide $|G : H|$, so does the latter.

The Roquette category of finite p-groups

Serge Bouc

Let p be a prime number. *The Roquette category* $\mathcal{R}_p$ of finite p-groups is an additive tensor category with the following properties :

- Every finite p-group can be viewed as an object of $\mathcal{R}_p$. The tensor product of two finite p-groups P and Q in $\mathcal{R}_p$ is the direct product $P \times Q$.

- In $\mathcal{R}_p$, any finite p-group has a direct summand ∂P, called *the edge* of P, such that
$$P \cong \bigoplus_{N \trianglelefteq P} \partial(P/N)\ .$$
Moreover, if the center of P is not cyclic, then $\partial P = 0$.
- In $\mathcal{R}_p$, every finite p-group P decomposes as a direct sum
$$P \cong \bigoplus_{R \in \mathcal{S}} \partial R\ ,$$
where $\mathcal{S}$ is a finite sequence of *Roquette p-groups*, i.e. of p-groups of normal p-rank 1, and such a decomposition is essentially unique. Given the group P, such a decomposition can be obtained explicitly from the knowledge of a *genetic basis* of P.
- The tensor product $\partial P \times \partial Q$ of the edges of two Roquette p-groups P and Q is isomorphic to a direct sum of a certain number $\nu_{P,Q}$ of copies of the edge $\partial(P \diamond Q)$ of another Roquette group (where both $\nu_{P,Q}$ and $P \diamond Q$ are known explicitly).
- The additive functors from $\mathcal{R}_p$ to the category of abelian groups are exactly the *rational p-biset functors* introduced in [1].

This yields a construction of a genetic basis of $P \times Q$, for finite p-groups P and Q, knowing a genetic basis of P and a genetic basis of Q. It allows for a quick computation of the evaluations $F(P)$ and their faithful part $\partial F(P)$, when F is a rational p-biset functor : this applies in particular to the functor R_K, where K is a field of characteristic 0, or to the functor $B^\times$ of units of Burnside rings ([2]), but also to the torsion part D_t of the Dade group ([3]).

Another possibly interesting fact is that some non-isomorphic p-groups may become isomorphic in the category $\mathcal{R}_p$. When $p = 2$, there are even examples where this occurs for groups of different orders.

References

[1] S. Bouc, *Rational p-biset functors*, J. of Algebra **319** (2008), 1776–1800.
[2] S. Bouc, *The functor of units of Burnside rings for p-groups*, Comm. Math. Helv. **82** (2007), 583–615.
[3] S. Bouc, *The Dade group of a p-group*, Inv. Math. **164** (2006), 189–231.

Some questions of equivariance and extendibility in finite groups of Lie type

Marc Cabanes
(joint work with Britta Späth)

Several important problems relating global and local information on representations of finite groups have been reduced recently to questions about quasi-simple groups. Examples are the McKay conjecture about characters of p'-degrees, or Alperin's weight conjecture (see [6], [7] respectively, and the talks of this volume by Navarro and Späth). The verification to be made is often called the *inductive* condition corresponding to that conjecture.

When trying to solve the problems for finite quasi-simple groups of Lie type, at least three classes of problems arise. Equivariance of correspondences of characters, extendibility of characters to natural overgroups, and a 3rd class of problems, less well acknowledged, deriving from the fact that the representation theory of quasi-simple groups (special linear or unitary groups, for instance) is in poorer shape than the one of finite groups of rational points of groups with connected center (general linear groups, for instance).

In this talk we report on recent research by the author, mostly in joint work with B. Späth (Aachen), on those questions in relation with the McKay conjecture.

In what follows, let $\mathbf{G}$ be a reductive group over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$, assume it is defined over a finite subfield $\mathbb{F}_q$, thus giving rise to a Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$. An instance is $\mathbf{G} = \mathrm{SL}_n(\mathbb{F})$ and F being be the raising of matrix entries to the q-th power or the same composed with transposition-inversion. Let $\mathbf{G} \subseteq \widetilde{\mathbf{G}} = \mathbf{G}.\mathrm{Z}(\widetilde{\mathbf{G}})$ an inclusion into a group with connected center and same derived group, also defined over $\mathbb{F}_q$. Let $\widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ the corresponding dual groups. In the case of $\mathbf{G} = \mathrm{SL}_n$, one would take $\widetilde{\mathbf{G}} = GL_n$, $\widetilde{\mathbf{G}}^* = \mathrm{GL}_n \to \mathbf{G}^* = \mathrm{PGL}_n$.

Define $\mathrm{Aut}(\mathbf{G}, F)$ as the set of bijective algebraic group morphisms $\mathbf{G} \to \mathbf{G}$ commuting with F.

Any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{P}_u \rtimes \mathbf{L}$ satisfying $F(\mathbf{L}) = \mathbf{L}$ gives rise to a Deligne-Lusztig variety $\{x \in \mathbf{G}/\mathbf{P}_u \mid g^{-1}F(g) \in \mathbf{P}_u.F(\mathbf{P}_u)\}$ on which $\mathbf{G}^F$ acts on the left and $\mathbf{L}^F$ acts on the right, thus allowing to define Deligne-Lusztig induction

$$\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}\colon \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F) \to \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F).$$

As a consequence of equivariance of étale cohomology, one has $\sigma.\mathrm{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = \mathrm{R}^{\mathbf{G}}_{\sigma\mathbf{L}\subseteq\sigma\mathbf{P}}.\sigma$ for any $\sigma \in \mathrm{Aut}(\mathbf{G}, F)$.

Recall

Theorem. (Deligne-Lusztig, 1976) $\mathrm{Irr}(\mathbf{G}^F) = \coprod_{[s]} \mathcal{E}(\mathbf{G}^F, [s])$ *where the sum is over* ${\mathbf{G}^*}^F$*-conjugacy classes of semi-simple elements* $s \in {\mathbf{G}^*_{\mathrm{ss}}}^F$ *and where the subset* $\mathcal{E}(\mathbf{G}^F, [s])$ *gathers all components of characters* $\mathrm{R}^{\mathbf{G}}_{\mathbf{T}}(\theta)$ *where* $\mathbf{T}$ *is an* F*-stable maximal torus of* $\mathbf{G}$ *with the pair* $(\mathbf{T}, \theta)$ *corresponding to a pair* $(\mathbf{T}^* \ni s)$ *by duality.*

When $s = 1$ the elements of $\mathcal{E}(\mathbf{G}^F, 1)$ are called unipotent characters. One has a Jordan decomposition of characters of $\mathbf{G}^F$.

Theorem. (Lusztig, 1984) *If* $\mathrm{Z}(\widetilde{\mathbf{G}}) = \mathrm{Z}(\widetilde{\mathbf{G}})^\circ$, $s \in \widetilde{\mathbf{G}}^F_{\mathrm{ss}}$, *then* $\mathrm{C}_{\widetilde{\mathbf{G}}^*}(s)$ *is connected and there is a bijection*

$$\mathcal{E}(\widetilde{\mathbf{G}}^F, [s]) \leftrightarrow \mathcal{E}(\mathrm{C}_{\widetilde{\mathbf{G}}^*}(s)^F, 1)$$

which commutes with Deligne-Lusztig inductions.

Remark. (C.-Späth, 2011, see [4]) The above can be taken $\mathrm{Aut}(\widetilde{\mathbf{G}}, F)$-equivariant.

Theorem. (Lusztig, 1988) *The above Jordan decomposition holds for any reductive* $\mathbf{G}$ *with compatibility with* $\mathrm{R}_{\mathbf{T}}^{\mathbf{G}}$ *maps and defining* $\mathcal{E}(\mathrm{C}_{\mathbf{G}^*}(s)^F, 1)$ *as the set of characters of* $\mathrm{C}_{\mathbf{G}^*}(s)^F$ *lying over an element of* $\mathcal{E}(\mathrm{C}_{\mathbf{G}^*}(s)^{\circ F}, 1)$.

In the case of type A, Bonnafé has proved more precise statements relating the above with Weyl groups. Assume $\mathbf{G} = \mathrm{SL}_n$, $s \in {\mathbf{G}^*_{\mathrm{ss}}}^F$.

Theorem. (Bonnafé, 2006 [1], [2]) *Let* $\mathbf{T}$ *be an* F*-stable maximal torus of* $\mathrm{C}_{\mathbf{G}^*}(s)^\circ$, $W(s) := (\mathrm{N}_{\mathbf{G}^*}(\mathbf{T})/\mathbf{T})^{w_s F}$ *where* $w_s \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{T})$ *is chosen so that* $W(s)$ *has maximal order. Then*

(i) There is an explicit bijection $\mathrm{Irr}(W(s)) \leftrightarrow \mathcal{E}(\mathbf{G}^F, [s])$ *compatible with* $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ *functors on the RHS and ordinary induction on the LHS.*

(ii) When $\mathbf{G}^F = \mathrm{GL}_n(\mathbb{F}_q)$, *there is a bijection* $\mathrm{Irr}(W(s)) \leftrightarrow \mathcal{E}(\mathrm{C}_{\mathbf{G}^*}(s)^F, 1)$ *with the same kind of compatibilities.*

Proposition. (C. 2011 [3]) *Point (ii) above is also true for* $\mathbf{G}^F = \mathrm{SU}_n(\mathbb{F}_q)$.

The formulas given by Bonnafé are linear combinations of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ generalized characters, so the equivariance mentioned above allows to prove

Theorem. (C.-Späth 2012 [5]) *The above Jordan decomposition of characters when* $\mathbf{G} = \mathrm{SL}_n$ *is* $\mathrm{Aut}(\mathbf{G}, F)$*-equivariant.*

Application to the McKay conjecture. If (S, ℓ) is a pair consisting of a finite non-abelian simple group S and a prime ℓ, one says that it satisfies the inductive McKay condition (iMK) if and only if there is a universal covering $1 \to M(S) \to H \to S \to 1$ (where $M(S)$ is the Schur multiplier of S), a Sylow ℓ-subgroup $P \subseteq H$, a subgroup $N \subsetneq H$ which is $\mathrm{Aut}(H)_P$-stable and contains $\mathrm{N}_H(P)$, and a bijection

$$\Omega\colon \{\chi \in \mathrm{Irr}(H) \mid \ell \nmid \chi(1)\} \to \{\psi \in \mathrm{Irr}(N) \mid \ell \nmid \psi(1)\}$$

which is $\mathrm{Aut}(H)_P$-equivariant, preserves characters of $M(S)$ and such that each pair $(\chi, \Omega(\chi))$ satisfies a condition (Cohom) as in [8] (see also the talk by B. Späth, this volume). Recall

Theorem. (Isaacs-Malle-Navarro 2007 [6]) *If every finite simple group* S *is such that* (S, ℓ) *satisfies iMK, then the McKay conjecture on character degrees prime to* ℓ *holds.*

Many simple groups have been shown to satisfy iMK for any prime ℓ. Specifically types PSL_2, ${}^2B_2$, ${}^2G_2$ ([6]), alternating groups (Malle). We also show

Theorem. *(i) For* $S = \mathrm{PSL}_n(q)$ *there is a bijection* Ω *as in iMK satisfying equivariance (*[5]*).*

(ii) $S = \mathrm{PSU}_n(q)$ *satisfies iMK (*[5]*).*

(iii) $S = {}^3D_4(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(2^{2a+1})$, $G_2(q)$ *satisfy iMK (*[4]*)*

References

[1] C. Bonnafé, *Produits en couronne de groupes linéaires,* J. Algebra **211** (1999), 57–98.
[2] C. Bonnafé, *Sur les caractères des groupes réductifs finis à centre non connexe : applications aux groupes spéciaux linéaires et unitaires,* Astérisque **306**, 2006.
[3] M. Cabanes, *On Jordan decomposition of characters for $SU(n,q)$,* preprint (2011) arXiv:1112.6035.
[4] M. Cabanes and B. Späth, *Equivariance and extendibility in finite reductive groups with connected center,* preprint (2012).
[5] M. Cabanes and B. Späth, *Equivariance and extendibility in finite reductive groups of type A,* in preparation (2012).
[6] M. Isaacs, G. Malle and G. Navarro, *A reduction theorem for the McKay conjecture,* Invent. Math. **170** (2007), 33–101
[7] G. Navarro, P.H. Tiep, *A reduction theorem for the Alperin weight conjecture, Invent. Math.,* **184** (2011), 529–565.
[8] B. Späth, The inductive McKay condition in the maximally split case, *Bull. London Math. Soc.,* (2011).

Thick subcategories of the bounded derived category

Jon F. Carlson

(joint work with Srikanth B. Iyengar)

Throughout, k denotes a field of characteristic p. Let $S = k[x_1, \dots, k_n]$ be a polynomial ring in n variables, and let G be a finite group. For any ring R, the derived category $\mathbf{D}(R)$ is the category consisting of objects which are complexes of R-modules, with morphisms being chain maps of complexes – except that we invert any quasi-isomorphism (a chain map that induces an isomorphism on homology). We let $\mathbf{D}^{\mathrm{b}}(R)$ be the full subcategory of bounded, finitely generated complexes.

The stable category, $\mathbf{stmod}(kG)$, has objects consisting of all finitely generated kG-modules. If M and N are kG-modules then the set of morphisms from M to N in the stable category is the group

$$\mathrm{Hom}_{\mathbf{stmod}(kG)}(M,N) = \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N),$$

where $\mathrm{PHom}_{kG}(M,N)$ is the set of all homomorphisms from M to N that factor through a projective module. There is a functor $\mathbf{D}^{\mathrm{b}}(kG) \longrightarrow \mathbf{stmod}(kG)$, which collapses a complex down to a single module concentrated in degree zero.

The derived category and the stable category are triangulated categories, meaning that each has a translation functor τ on the category and the category has triangles. In the case of $\mathbf{D}^{\mathrm{b}}(R)$, the translation functor τ is the shift $\tau(F_*) = F[1]$, where $F[1]_r = F_{r-1}$ and the boundary map on F is multiplied by -1. In $\mathbf{stmod}(kG)$, the translation functor is Ω^{-1}. If M is a finitely generated kG-module, $\Omega^{-1}(M)$ is the cokernel of the inclusion of M in its injective hull.

If $\mathcal{S}$ is a subcategory of a triangulated category $\mathcal{C}$, then $\mathcal{S}$ is thick if it is triangulated, closed under finite direct sums (coproducts) and direct summands.

The purpose of this work is to tie together two theorems, which sound the same, but originally had very different proofs. The first was actually proved more generally for commutative noetherian rings.

Theorem 1. *(Hopkins [5]) The thick subcategories of* $\mathbf{D}^{\mathrm{b}}(S)$ *(or* $\mathbf{D}^{\mathrm{b}}(R)$ *for any commutative noetherian ring* R*) are determined by the support varieties of the objects.*

The second was proved several years later.

Theorem 2. *(Benson-Carlson-Rickard [2]) The (tensor ideal) thick subcategories of* $\mathbf{stmod}(kG)$ *are determined by the support varieties of the objects. ("Tensor ideal" means that if* M *is in the subcategory, then so is* $M \otimes_k N$ *for any module* N*.)*

While the statements of the two theorems are similar, the proofs were completely different because the notions of support varieties in the two categories, seemingly could not be reconciled. The proof in [2] relied on the theory of idempotent modules. These are infinitely generated modules associated to thick subcategories and thus, the theorem in [2] which spoke only of finitely generated modules relied on infinite constructions. One aspect of the current work is that it does not resort to infinite constructions.

The support varieties for $\mathbf{D}^{\mathrm{b}}(S)$ are described in the following fashion. Let $\mathrm{Spec}(S)$ denote the spectrum of S, the space of all prime ideals in S with the Zariski topology. The support of an object $F_* \in \mathbf{D}^{\mathrm{b}}(S)$ is the variety of the annihilator of the homology of F_*. Viewed this way, Hopkins' Theorem says: If $\mathcal{S}$ is a thick subcategory of $\mathbf{D}^{\mathrm{b}}(S)$, then there exists a subset $\mathcal{W} \subseteq \mathrm{Spec}(S)$, which is closed under specialization (if $U \subset V$ and $V \in \mathcal{W}$, then $U \in \mathcal{W}$) such that $\mathcal{S} = \mathbf{D}^{\mathrm{b}}(S)_{\mathcal{W}}$, the full subcategory of objects whose supports are in $\mathcal{W}$.

For the group algebra the support varieties are defined by way of the cohomology ring. Suppose that M is a finitely generated kG-module or bounded complex of finitely generated modules. Then the ring $\mathrm{Ext}^*_{kG}(M,M)$ is a finitely generated module over the cohomology ring $\mathrm{H}^*(G,k) \cong \mathrm{Ext}^*_{kG}(k,k)$. Let $J(M)$ be the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$ in $\mathrm{H}^*(G,k)$, and let $V_G(M)$ be the variety of $J(M)$, the set of all ideals that contain $J(M)$.

Thus, the theorem of BCR says: If $\mathcal{S}$ is a tensor ideal thick subcategory of $\mathbf{stmod}(kG)$, then there exists a subset $\mathcal{W} \subseteq \mathrm{Spec}^*(\mathrm{H}^*(G,k))$, which is closed under specialization, such that $\mathcal{S} = \mathbf{stmod}(kG)_{\mathcal{W}}$, the full subcategory of objects whose supports are in $\mathcal{W}$.

The thick subcategory generated by an object M, $\mathrm{Thick}(M)$, is the smallest thick subcategory that contains M. In $\mathbf{D}(S)$, $\mathrm{Thick}(S) = \mathbf{D}^{\mathrm{b}}(S)$. If G is a p-group, then in $\mathbf{D}(kG)$, $\mathrm{Thick}(k) = \mathbf{D}^{\mathrm{b}}(kG)$.

For the moment, we assume that $G = \langle g_1, \dots, g_n\rangle \cong (\mathbb{Z}/2\mathbb{Z})^n$ is an elementary abelian group of order 2^n. ($g_ig_j = g_jg_i$, $g_i^2 = 1$.) Note that if $z_i = g_i - 1$, then $z_i^2 = g_i^2 - 1^2 = 0$. So $kG = k[z_1, \dots, z_n]/(z_1^2, \dots, z_n^2)$. Let $\Lambda = \Lambda(z_1, \dots, z_n)$ denote the exterior algebra generated by $z_1, \dots, z_n$. If $p = 2$, then $\Lambda = kG$.

We define a complex $J = \Lambda \otimes \mathrm{Hom}_k(S,k)$ with differential $\delta = \sum_{i=1}^n z_i \otimes x_i$. Let S^n denote the space of homogeneous polynomials of degree n in S. Let $T_n =$

$\mathrm{Hom}_k(S^n, k)$. Then J has the form.

$$\dots \longrightarrow \Lambda \otimes T_2 \longrightarrow \Lambda \otimes T_1 \longrightarrow \Lambda \otimes T_0 \longrightarrow 0$$

An easy check shows that this is a Λ-projective resolution of the trivial module k.

Following work of Avramov, Buchweitz, Iyengar and Miller [1] we connect this with the support varieties for the polynomial ring S. This is accomplished by regarding S and Λ as differential Graded (DG) algebras with zero differential. Then J is a DG $(\Lambda \otimes S)$-module. Consider $\mathbf{D}(\Lambda)$ and $\mathbf{D}(S)$ as the categories of DG modules over Λ and S. Define a functor

$$h: \ \mathbf{D}(\Lambda) \ \longrightarrow \ \mathbf{D}(S)$$

$$M \ \mapsto \ \mathrm{Hom}_\Lambda(J, M) = \mathrm{Hom}_\Lambda(\Lambda \otimes \mathrm{Hom}_k(S, k), M)$$

Note that $h(k) = \mathrm{Hom}_\Lambda(\Lambda \otimes \mathrm{Hom}_k(S, k), k) \cong S$. So $h(\mathbf{D}^{\mathrm{b}}(\Lambda)) = h(\mathrm{Thick}(k)) = \mathrm{Thick}(S) = \mathbf{D}^f(S)$.

Theorem 3. [1] *The functor h induces an equivalence of categories $\mathbf{D}^{\mathrm{b}}(\Lambda) \to \mathbf{D}^{\mathrm{f}}(S)$.*

So define the support of a Λ-module M to be the support of the S-module $h(M) = \mathrm{Hom}_\Lambda(J, M)$, which is the annihilator in S of the homology of $h(M)$. Now because J is a Λ-projective resolution of k, $S = h(k) = \mathrm{Ext}^*_\Lambda(k, k)$ which (when $p = 2$) is $\mathrm{H}^*(G, k)$. Moreover, $h(M) = \mathrm{Ext}^*_{kG}(k, M)$. So ($p = 2$), the support of $h(M)$ is the subvariety of $\mathrm{Spec}^*(\mathrm{H}^*(G, k))$ of the annihilator of $\mathrm{Ext}_{kG}(k, M)$. This is equivalent to the usual definition of $V_G(M)$. Hence, if $p = 2$ and G is an elementary abelian 2-group, we have Hopkins' Theorem implies the theorem of BCR.

If $p > 2$, then we can prove the same, but we need an intermediate step through a Koszul algebra whose homology is the exterior algebra Λ as in the work of Benson, Iyengar and Krause [3]. The functor we get is not an equivalence – but it is good enough to establish the desired connection of support varieties.

If G is not an elementary abelian p-group, then the theorem that we need is the following. Its original purpose was to prove that the stable category is generated by modules induced from elementary abelian p-subgroups. But it relates the stable and derived categories of kG-modules to those of the elementary abelian subgroups of G is many ways.

Theorem 4. [4] *For M a finitely generated kG-module or complex of kG-modules, there exists an integer τ, depending only on G, and a finitely generated kG-module V such that the direct sum $M \oplus V$ has a filtration*

$$\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_\tau = M \oplus V$$

with the property that for each $i = 1, \dots, \tau$, there is an elementary abelian subgroup $E_i \subseteq G$ and a finitely generated kE_i-module W_i such that

$$L_i/L_{i-1} \cong W_i^{\uparrow G}.$$

References

[1] L. L. Avramov, R.-O. Buchweitz, S. B. Iyengar and C. Miller, *Homology of perfect complexes*, Advances in Math. **223** (2010), 1731–1781. `Corrigendum`: Advances in Math. **225** (2010), 3576–3578.
[2] D. Benson, J. Carlson and J. Rickard, *Thick subcategories of the stable category*, Fund. Math. **153** (1997), 59–80.
[3] D. J. Benson, S. B. Iyengar, and H. Krause, *Stratifying modular representations of finite groups*, Annals of Math **174** (2011), 1643–1684.
[4] J. F. Carlson, *Cohomology and induction from elementary abelian subgroups*, Quart. J. Math. **51** (2000), 169–181.
[5] M. J. Hopkins, *Global methods in homotopy theory*, Homotopy Theory, Durham 1985, Lecture Notes in Mathematics, vol. 117, Cambridge University Press, 1987.

Broué's Conjecture: The story so far

David A. Craven

Broué's abelian defect group conjecture, posited in 1988, states that a block of a finite group is derived equivalent to its Brauer correspondent, whenever the defect group of the block is abelian. It is the most structural of a variety of local-global conjectures that relate a block and its Brauer correspondent, and remains one of the most difficult problems in group representation theory.

In this talk I described the new approach started by the author and Raphaël Rouquier, which started in [3], and which has been expanded in [1] and [2]. This has as its starting point the geometric version of Broué's conjecture, which suggests that a particular derived equivalence between the two blocks is given by the cohomology of a Deligne–Lusztig variety; this version of the conjecture is only stated for unipotent blocks of finite groups of Lie type, but since these form most of the finite simple groups, this is a fundamental case that needs to be established.

The approach suggested in [3] is to focus on the derived equivalence given by the geometric form, without proving that it is actually given by the cohomology of a Deligne–Lusztig variety. This derived equivalence should be *perverse*, a special type of derived equivalence defined combinatorially. Furthermore, the combinatorial information needed to define a perverse equivalence can (conjecturally) be extracted from the (specialization of the) cyclotomic Hecke algebra associated to a given unipotent block; this version of Broué's conjecture is called the *combinatorial form*.

These perverse equivalences have been constructed for various blocks of groups, in some cases proving new cases of Broué's conjecture, and so these methods can be used to prove new instances of Broué's conjecture.

In [2] I constructed all perverse equivalences between a block with cyclic defect group and the Brauer tree algebra of the star, which is the Brauer correspondent; I then prove that the combinatorial form of Broué's conjecture holds whenever the Brauer tree of a block with cyclic defect group is known. The collection of Brauer trees of unipotent blocks that are still not known is becoming ever smaller, and it is a reasonable goal that soon all Brauer trees for all principal blocks of all finite groups will be known, at least except for a small collection of primes corresponding to large sporadic groups.

I then ended the talk by applying the methods of [2] to construct a sample Brauer tree corresponding to the non-existent group $H_3(q)$ and $\ell \mid \Phi_6(q)$; the combinatorial Broué conjecture can be described for H_3, H_4 and $I_2(n)$, and so we can construct, up to Morita equivalence, the unipotent blocks of these 'groups'. The mathematical significance behind these objects is still shrouded in mystery.

References

[1] David A. Craven, *Deligne–Lusztig varieties, Broué's conjecture, and Brauer trees*, submitted.

[2] David A. Craven, *Perverse equivalences and Broué's conjecture II: The cyclic case*, preprint.

[3] David A. Craven and Raphaël Rouquier, *Perverse equivalences and Broué's conjecture*, submitted.

Ghost algebras for double Burnside algebras

Susanne Danz

(joint work with Robert Boltje)

Given a finite group G, the *double Burnside ring* of G is defined as the Grothendieck ring of the category of finite (G, G)-bisets with respect to disjoint unions and the tensor product of bisets. In recent years, double Burnside rings and the underlying concept of bisets have developed into objects of great interest, having connections with modular representation theory of finite groups, with algebraic topology, and with the theory of fusions systems.

In this talk I focussed on the the rational double Burnside algebra $\mathbb{Q}B(G, G) := \mathbb{Q} \otimes_{\mathbb{Z}} B(G, G)$. I introduced *ghost algebras* for $\mathbb{Q}B(G, G)$, as analogues of the classical ghost ring of the usual (commutative) Burnside ring, in order to translate the complicated multiplication in $B(G, G)$ into a more transparent one.

The ghost algebras.

As an abelian group, the double Burnside ring $B(G, G)$ can be canonically identified with the usual Burnside group $B(G \times G)$, by viewing (G, G)-bisets as left $G \times G$-sets and vice versa. Using this identification and denoting by $\mathscr{S}_{G\times G}$ the set of all subgroups of $G \times G$ we obtain an embedding of $\mathbb{Q}$-vector spaces

$$\alpha : \mathbb{Q}B(G, G) \to \mathbb{Q}\mathscr{S}_{G\times G},\ [X] \mapsto \sum_{x\in X} \mathrm{Stab}_{G\times G}(x)\,,$$

where X is any (G, G)-biset and $[X]$ is its image in $B(G, G)$. Every subgroup of $G\times G$ can be interpreted as a relation on G, and the composition of relations turns $\mathscr{S}_{G\times G}$ into a monoid. In [1] we defined a 2-cocycle of this monoid, and, denoting the resulting twisted monoid algebra by A, we showed the following

Theorem 1. *There is an idempotent e in the twisted monoid algebra A such that the map α yields a $\mathbb{Q}$-algebra isomorphism*

$$\mathbb{Q}B(G, G) \xrightarrow{\sim} eAe \subseteq A = \mathbb{Q}\mathscr{S}_{G\times G}\,.$$

As a set, eAe simply equals the set $(\mathbb{Q}\mathscr{S}_{G\times G})^{G\times G}$ of fixed points under the $G \times G$-conjugation action.

In [1] we also defined a second ghost algebra of $\mathbb{Q}B(G, G)$, which is obtained from the one in Theorem 1 by composing α with the $\mathbb{Q}$-vector space isomorphism $\zeta : \mathbb{Q}\mathscr{S}_{G\times G} \to \mathbb{Q}\mathscr{S}_{G\times G}$, $L \mapsto \sum_{L' \leqslant L} L'$. In fact, the composition $\zeta \circ \alpha$, viewed as a $\mathbb{Q}$-linear map $\mathbb{Q}B(G \times G) \to (\mathbb{Q}\mathscr{S}_{G\times G})^{G\times G}$, is precisely the classical mark homomorphism. An important feature of this alternative ghost algebra is the fact that many products of basis elements are 0.

Application 1 – Simple $\mathbb{Q}B(G, G)$-modules.

By Theorem 1 we, can now view the double Burnside algebra as a Schur algebra of the twisted monoid algebra A. As a first application of this result, we have studied the simple $\mathbb{Q}B(G, G)$-modules. From recent work of Linckelmann–Stolorz [7] on twisted category algebras we first obtained a parametrization of the isomorphism classes of simple A-modules. By Green's theory of Schur functors (cf. [5, Section 6.2]), it is known that there is an injective map from the set of isomorphism classes of simple eAe-modules into the set of isomorphism classes of simple A-modules; the isomorphism class of a simple A-module S is contained in the image of this map if and only if $eS \neq \{0\}$. In [1] we have given an explicit criterion, formulated in terms of character theory of finite groups, for the last condition to be satisfied. This also enabled us to improve an earlier result of Bouc, given in [3], on the parametrization of simple $\mathbb{Q}B(G, G)$-modules.

Application 2 – Double Burnside algebras of cyclic groups.

Another application of our ghost algebra approach concerns the double Burnside algebra $\mathbb{Q}B(G, G)$ in the case where G is a finite cyclic group. We have shown that $\mathbb{Q}B(G, G)$ is then isomorphic to the (untwisted) category algebra of a finite *inverse* category. Thus, invoking work of Linckelmann [6] and Steinberg [8], $\mathbb{Q}B(G, G)$ is isomorphic to a direct product of matrix algebras over finite group algebras over $\mathbb{Q}$. We have determined these matrix algebras as well as an explicit isomorphism between their product and the algebra $\mathbb{Q}B(G, G)$.

In this way we, in particular, recover one half of Bouc's result [3, Prop. 6.1.7], which states that the double Burnside algebra $\mathbb{Q}B(G, G)$ is semisimple if and only if G is cyclic.

Application 3 – Simple biset functors.

Although we have only mentioned our results concerning the double Burnside ring of a fixed group G here, our theory developed in [1] uses a more general setup. For any finite groups G, H and K, one defines the double Burnside groups $B(G, H)$, $B(H, K)$ and $B(G, K)$, and a $\mathbb{Z}$-bilinear tensor product $\times_H : B(G, H) \times B(H, K) \to B(G, K)$. Following work of Bouc [2], the double Burnside groups $B(G, H)$ are the morphism sets in a category whose objects are the finite groups, and the composition of morphisms is the tensor product above. Representation groups of finite groups can be seen as additive functors (*biset functors*) on this category, and these functors in turn form themselves a category, which is equivalent to the module category of the algebra $\bigoplus_{G,H} B(G,H)$. Generalizing Theorem 1, in [1] we obtained a description of the category of biset functors (over suitable commutative rings) as the module category of a Schur algebra of a twisted category algebra. Moreover, we gained new information on the simple biset functors over fields of characteristic 0.

References

[1] R. Boltje, S. Danz, *Ghost algebras of double Burnside algebras via Schur functors*, preprint 2012, arXiv:1203.1346v1 [math.RT].

[2] S. Bouc, *Foncteurs d'ensembles munis d'une double action*, J. Algebra **183** (1996), 664–736.

[3] S. Bouc, *Biset functors for finite groups*, Lecture Notes in Mathematics, 1990. Springer-Verlag, Berlin, 2010.

[4] O. Ganyushkin, V. Mazorchuk, B. Steinberg, *On the irreducible representations of a finite semigroup*, Proc. Amer. Math. Soc. **137** (2009), 3585–3592.

[5] J. A. Green, *Polynomial representations of* GL_n*. Second corrected and augmented edition. With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, Green and M. Schocker*, Lecture Notes in Mathematics, 830. Springer-Verlag, Berlin, 2007.

[6] M. Linckelmann, *On inverse categories and transfer in cohomology*, to appear in Skye conference proceedings.

[7] M. Linckelmann, M. Stolorz, *On simple modules over twisted finite category algebras*, preprint, 2011.

[8] B. Steinberg, *Möbius functions and semigroup representation theory*, II. Character formulas and multiplicities. Adv. Math. **217** (2008), 1521–1557.

Quotients of Deligne-Lusztig varieties

Olivier Dudas

Let $\mathbf{G}$ be a connected reductive algebraic group and F be a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. The group of fixed points $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ is a *finite reductive group*. Examples of such groups are the classical groups $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q)$ and the exceptional groups of Lie type, such as $E_8(q)$.

The first approach for studying the representation theory of finite reductive groups is a variant of the classical induction/restriction for representations of abstract groups. Given an F-stable Levi subgroup $\mathbf{L}$ which is the Levi complement of an F-stable parabolic subgroup $\mathbf{P} = \mathbf{L}\mathbf{U}$ of $\mathbf{G}$ one can define a pair of adjoint functors, called *Harish-Chandra induction/restriction functors* as follows.

$$\begin{array}{rcccl} \mathrm{R}_{\mathbf{L}}^{\mathbf{G}} & : & \mathcal{O}\mathbf{L}^F\text{-mod} & \longrightarrow & \mathcal{O}\mathbf{G}^F\text{-mod} \\ & & N & \longmapsto & \mathcal{O}[\mathbf{G}^F/\mathbf{U}^F] \otimes_{\mathcal{O}\mathbf{L}^F} N \end{array}$$

$$\begin{array}{rcccl} {}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} & : & \mathcal{O}\mathbf{G}^F\text{-mod} & \longrightarrow & \mathcal{O}\mathbf{L}^F\text{-mod} \\ & & M & \longmapsto & M^{\mathbf{U}^F}. \end{array}$$

A key tool for working with these functors is the so-called *Mackey formula* which gives the following isomorphism of functors

$${}^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}} \circ \mathrm{R}_{\mathbf{L}}^{\mathbf{G}} \simeq \sum \mathrm{R}_{\mathbf{L}\cap {}^x\mathbf{M}}^{\mathbf{L}} \circ {}^*\mathrm{R}_{\mathbf{L}\cap {}^x\mathbf{M}}^{{}^x\mathbf{M}} \circ \mathrm{ad}\, x$$

where x runs over a set of representatives in $\mathbf{L}^F\backslash\mathbf{G}^F/\mathbf{M}^F$ of elements such that $\mathbf{L}\cap {}^x\mathbf{M}$ contains a maximal torus of $\mathbf{G}$. This can be used for example to prove that $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ do not depend on $\mathbf{P}$ (but only on $\mathbf{L}$).

Deligne and Lusztig have generalised this construction to the case where $\mathbf{P}$ is no longer assumed to be F-stable [6]. The permutation module $\mathcal{O}[\mathbf{G}^F/\mathbf{U}^F]$ is replaced by the cohomology of a quasi-projective variety $\mathrm{Y}_{\mathbf{G}}(\mathbf{U})$ (with coefficients in a finite extension of $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$). The price to pay is that the new functors $\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ are no longer defined on the module category but on its bounded derived category. Furthermore, the naive Mackey formula does not hold for these derived functors, even though it holds for the morphisms induced on the Grothendieck groups [6, 1, 2].

The purpose of this note is to explain how to solve this problem in the specific case where $\mathbf{L}$ is any F-stable Levi subgroup and $\mathbf{M}$ is a Levi complement of an F-stable parabolic subgroup $\mathbf{Q} = \mathbf{M}\mathbf{V}$. In that situation, the composition of induction/restriction is given by the cohomology of a quotient of the Deligne-Lusztig variety

$$ {}^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}}\circ\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \;\simeq\; \mathrm{R}\Gamma_c(\mathbf{V}^F\backslash\mathrm{Y}_{\mathbf{G}}(\mathbf{U}))\otimes_{\mathbf{L}^F} - $$

and having a Mackey formula amounts to expressing the cohomology of this quotient in terms of "smaller" Deligne-Lusztig varieties. This provides an inductive method for computing the cohomology of Deligne-Lusztig varieties. We will detail the example of $\mathrm{GL}_n(q)$ for which the representation theory is well-known.

1. Unipotent ℓ-blocks of $\mathrm{GL}_n(q)$

Recall that the unipotent characters of $\mathrm{GL}_n(q)$ are parametrised by partitions of n. The trivial character $1 = \chi_{(n)}$ corresponds to the partition (n) whereas the Steinberg character $\mathrm{St} = \chi_{(1,\ldots,1)}$ corresponds to the conjugate partition $(1,1,\ldots,1)$.

Most of the properties of the unipotent characters (dimensions, restriction/induction...) can be read off from the associated partition. The Nakayama conjectures give the partition of the unipotent characters into ℓ-blocks.

Theorem 1.1 (Brauer-Robinson). *Let ℓ be a prime number. Assume ℓ and q are coprime and let d be the order of q modulo ℓ. Then χ_λ and $\chi_{\lambda'}$ are in the same ℓ-block if and only if λ and λ' have the same d-core.*

Examples.

(i) If λ is a d-core, then χ_λ is the character of a projective module over $\mathbb{Z}_\ell$.

(ii) Assume $n = 3$. Then $\{1, \mathrm{St}, \chi_{(2,1)}\}$ is a single 3-block whereas the partition of unipotent characters into 2-blocks is $\{1, \mathrm{St}\}, \{\chi_{(2,1)}\}$.

(iii) Assume $n = d$. Then the unipotent character in the principal block correspond to n-hooks $(n-i, 1^i)$.

2. Deligne-Lusztig varieties associated with blocks

Let $d \in \{1, \ldots, n\}$. There exists a Deligne-Lusztig variety $\mathrm{X}_{n,d}$ of dimension $2n-d-1$ whose cohomology affords a "minimal" d-induction [3]. More precisely, if $\mu \vdash n-d$, we can form the local system $\mathcal{F}_\mu$ associated to the representation χ_μ of $\mathrm{GL}_{n-d}(q)$ and the constituents of the virtual character $\sum(-1)^i \mathrm{H}_c^i(\mathrm{X}_{n,d}, \mathcal{F}_\mu)$ are exactly the unipotent characters χ_λ where λ is obtained from μ by adding a d-hook. In particular, if μ is a d-core, the cohomology of $\mathrm{X}_{n,d}$ with coefficients in $\mathcal{F}_\mu$ gives the unipotent characters in the ℓ-block associated to μ.

Although there are general methods for computing the alternating sum of the cohomology (such as Lefschetz trace formula), it is a difficult problem to determine each individual cohomology group. When μ is trivial, the cohomology of $\mathrm{X}_{n,d}$ has only been determined when $d = n$ [10], $d = n-1$ [7] and $n = 2$ [8]. Craven has formulated in [5] a conjecture giving the degree in the cohomology where a given unipotent character should appear. Using a good quotient of $\mathrm{X}_{n,d}$ one can prove the following.

Theorem 2.1 (2011 [9]). *Craven's formula holds for $\mathrm{X}_{n,d}$ when μ is trivial. Furthermore, Craven's formula holds for any unipotent local system if it holds for $d = 1$.*

The case $d = 1$ corresponds to the Deligne-Lusztig variety $\mathrm{X}(\boldsymbol{\pi})$ associated with the central element $\boldsymbol{\pi} = \mathbf{w_0}^2$ of the Braid group. A precise conjecture for the cohomology of this variety was already formulated in [4].

3. Quotients of $\mathrm{X}_{n,d}$

In this section we assume that $\mathbf{M} = \mathrm{GL}_{n-1}(\overline{\mathbb{F}}_q)$ is the standard Levi subgroup of $\mathbf{G}$. The Harish-Chandra restriction ${}^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}}$ of a unipotent character χ_λ is given by the usual branching rule for representations of the symmetric group. In particular, if λ is obtained from μ by adding a d-hook, then the restricting χ_λ amounts to

- restricting μ to obtain a character $\chi_{\lambda'}$ which occurs in the cohomology of $\mathrm{X}_{n-1,d}$ with coefficients in the restriction of $\mathcal{F}_\mu$;
- restricting the d-hook (in general in two different ways) to obtain a character $\chi_{\lambda''}$ which occurs in the cohomology of $\mathrm{X}_{n-1,d-1}$ with coefficients in $\mathcal{F}_\mu$.

To understand geometrically why two copies of $\chi_{\lambda''}$ should occur we use Lusztig's result on the case $d = n$. He showed in [10] that the quotient $\mathbf{V}^F \backslash \mathrm{X}_{n,n}$ is isomorphic to $\overline{\mathbb{F}}_q^\times \times \mathrm{X}_{n-1,n-1}$. The cohomology of $\overline{\mathbb{F}}_q^\times$ is given by two copies of the coefficient ring in two consecutive degrees. Note that this part does not contribute to the alternating sum as the two terms cancel out.

Theorem 3.1 (2011 [9]). *Assume $d \geq 2$. There is a decomposition of the quotient $\mathbf{V}^F \backslash \mathrm{X}_{n,d} = U \cup Z$ into a disjoint union of $\mathbf{L}^F$-subvarieties such that*

- *Z is a closed subvariety whose cohomology is given by*

$$\mathrm{H}_c^i(Z, \mathcal{F}_{\mu|Z}) \simeq \mathrm{H}_c^{i-2}(\mathrm{X}_{n-1,d}, \mathcal{F}_{\mathrm{Res}\,\mu})(1)$$

- *U is an open subvariety whose cohomology is given by*

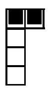
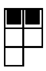
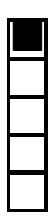
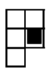
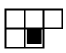
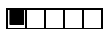
$$\mathrm{H}_c^i(U, \mathcal{F}_{\mu|U}) \simeq \mathrm{H}_c^{i-2}(\mathrm{X}_{n-1,d-1}, \mathcal{F}_\mu)(1) \oplus \mathrm{H}_c^{i-1}(\mathrm{X}_{n-1,d-1}, \mathcal{F}_\mu)$$

From this decomposition we obtain a long exact sequence relating the cohomology of $\mathbf{V}^F \backslash \mathrm{X}_{n,d}$ to the cohomology of $\mathrm{X}_{n-1,d}$ and $\mathrm{X}_{n-1,d-1}$. Together with the action of the Frobenius this determines completely the cohomology of $\mathrm{X}_{n,d}$.

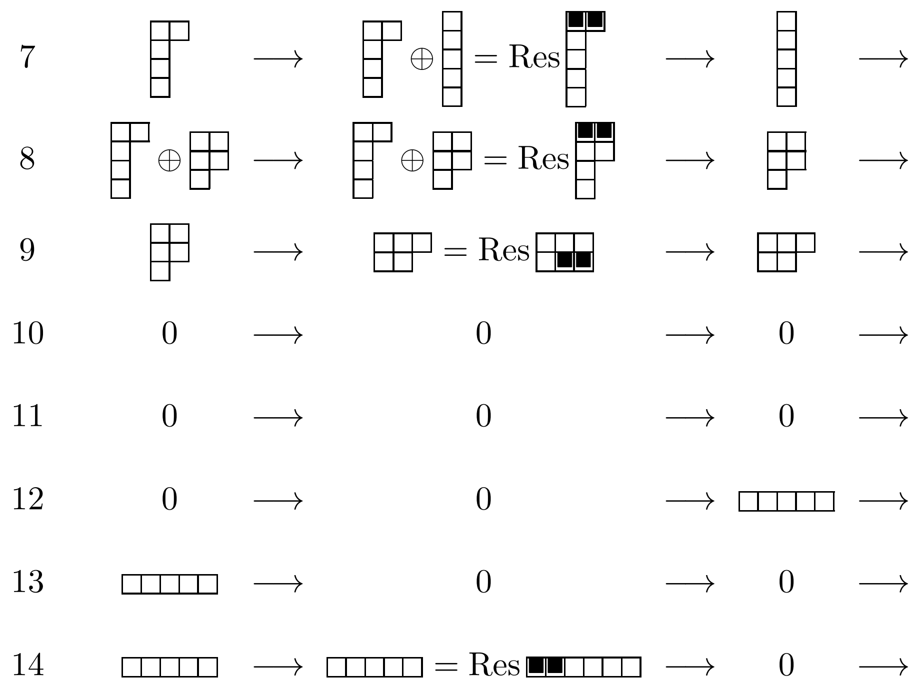
The minimal cases correspond to $d = n$ and $d = 1$. Lusztig solved the first one in [10]. For the second one, we can only prove that Craven's formula hold when μ is the trivial partition. The other cases are work in progress.

4. An example

Assume that Craven's formula hold for the cohomology of $\mathrm{X}_{5,3}$ and $\mathrm{X}_{5,4}$ with coefficients in the trivial local system. We give here the cohomology with compact support; the black boxes in Young diagrams correspond to the partition μ we started with, that is (2) for $\mathrm{X}_{5,3}$ and (1) for $\mathrm{X}_{5,4}$. The white boxes represent the d-hook that we have added.

	5	6	7	8	9	10	11	12
$\mathrm{H}_c^\bullet(\mathrm{X}_{5,3})$								
$\mathrm{H}_c^\bullet(\mathrm{X}_{5,4})$								

Using Theorem 3.1 we write the long exact sequence in cohomology and deduce the cohomology of $\mathrm{X}_{6,5}$. The groups of degree 10 and 11 are obviously zero, the group of degree 14 is also easily obtained, and the group of degree 7 follows from equivariance of the boundary map $\mathrm{H}_c^7(Z) \longmapsto \mathrm{H}_c^8(U)$. For the remaining ones, we need to use the action of F together with the fact that the cohomology of $\mathbf{V}^F \backslash \mathrm{X}_{6,5}$ should be the restriction of unipotent characters of $\mathrm{GL}_6(q)$.



References

[1] C. Bonnafé. *Formule de Mackey pour q grand.* J. Algebra **201**(1) (1998); 207–232.
[2] C. Bonnafé and J. Michel. *Computational proof of the Mackey formula for* $q > 2$. J. Algebra **327** (2011), 506–526.
[3] M. Broué, G. Malle and J. Michel. *Generic blocks of finite reductive groups.* Astérisque **212** (1993), 7–92.
[4] M. Broué and J. Michel. *Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées* in Finite reductive groups, Progr. Math. **141** (1994), 73–139.
[5] D. Craven. *On the cohomology of Deligne-Lusztig varieties.* preprint (2011).
[6] P. Deligne and G. Lusztig. *Representations of reductive groups over finite fields.* Ann. of Math. (2) **103**(1) (1976), 103–161.
[7] F. Digne and J. Michel. *Endomorphisms of Deligne-Lusztig varieties.* Nagoya Math. J. **183** (2006), 35–103.
[8] F. Digne, J. Michel and R. Rouquier. *Cohomologie des variétés de Deligne-Lusztig.* Adv. Math. **209**(2) (2007), 749–822.
[9] O. Dudas. *Cohomology of Deligne-Lusztig varieties for groups of type A* preprint (2011).
[10] G. Lusztig. *Coxeter orbits and eigenspaces of Frobenius.* Invent. Math. **38**(2) (1976), 101–159.

Some recent developments in the study of homomorphisms between Specht modules

Matt Fayers

The Specht modules S^λ are very natural combinatorially defined modules for the group algebra of the symmetric group over any field. The problem of computing the homomorphism space $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}\big(S^\lambda, S^\mu\big)$ is an interesting one, and is closely connected to the fundamental problem of computing the decomposition numbers for the symmetric groups.

In the talk I give some history of the problem, beginning with the set-up developed by Gordon James, and outlining some key results from the last thirty years. I then highlight three important developments from 2011.

An algorithm for computing $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu)$**:**
Techniques developed over the last ten years have been used to construct nonzero homomorphisms between Specht modules in a variety of situations. However, actually computing the space of all homomorphisms between two Specht modules has been difficult to do in general. However, a 'semistandardising' result of the speaker [F] yields a fast algorithm for computing $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu)$, provided either λ is 2-regular or the characteristic is not 2. This has been implemented in GAP programs available at my web site [F-GAP].

Multi-dimensional homomorphism spaces:
Although examples abound of decomposition numbers greater than 1, it has been very difficult to find examples of homomorphism spaces $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, S^\mu)$ of dimension greater than 1. This has now been achieved independently and almost simultaneously by Dodge [D] and Lyle [L]. In Dodge's case, the Specht modules lie in *Rouquier blocks* of symmetric groups. These are unusually well understood blocks; in particular, the radical filtrations of the Specht modules in these blocks are known (provided the defect is abelian) [CT], and this enables Dodge to construct his examples (in characteristic at least 5). Lyle just finds explicit (much smaller) examples, using the speaker's GAP calculator. Her first example for a given characteristic p is

$$\lambda = (6p-5, 4p-3, 3p-3),$$
$$\mu = (4p-3, 4p-3, 3p-3, p-1, p-1).$$

Decomposable Specht modules:
Specht modules can be decomposable in characteristic 2 only, and examples are hard to find. Remarkably, after a thirty-year hiatus, progress was made by Dodge and the speaker, who (independently and simultaneously) discovered that the Specht module $S^{(4,3,1,1)}$ is decomposable in characteristic 2. In joint work [DF], we have exhibited a new family of decomposable Specht modules of the form $S^{(a,3,1,1,\dots,1)}$, using established techniques to construct homomorphisms between Specht modules to show that certain smaller Specht modules arise as summands of these decomposables.

References

[CT] J. Chuang & K. M. Tan, *Filtrations in Rouquier blocks of symmetric groups and Schur algebras*, Proc. London Math. Soc. (3) **86** (2003), 685–706.

[D] C. Dodge, *Large dimension homomorphism spaces between Specht modules for symmetric groups*, J. Pure Appl. Algebra **215** (2011), 2949–2956.

[DF] C. Dodge & M. Fayers, *Some new decomposable Specht modules*, J. Algebra **357** (2012) 235–262.

[F] M. Fayers, *An algorithm for semistandardising homomorphisms*, submitted.
[F-GAP] www.maths.qmul.ac.uk/∼mf/homcalculator.html
[L] S. Lyle, *Large-dimensional homomorphism spaces between Weyl modules and Specht modules*, J. Pure Appl. Algebra, to appear.

Cells, symmetric algebras, and the Frobenius–Schur indicator

Meinolf Geck

Let W be a finite Coxeter group. It is well-known that all irreducible characters of W can be realized over $\mathbb{R}$ and, hence, the number of involutions in W (that is, elements $w \in W$ such that $w^2 = 1$) equals the sum of the degrees of the irreducible characters. Following a suggestion of Lusztig, we show [2] that the equality "number of involutions = sum of character degrees" admits a refinement with respect to left, right and two-sided Kazhdan–Lusztig cells of W (both in the equal and in the general unequal parameter case). The proof uses a generalisation of the Frobenius–Schur indicator to symmetric algebras, which may be of independent interest.

Let H be a finite-dimensional associative algebra over a sufficiently large field K of characteristic 0. Assume that H is split semisimple and write $\mathrm{Irr}(H)$ for the set of irreducible characters of H. Assume further that H is symmetric with trace form $\tau \colon H \to K$ and write $\tau = \sum_\chi c_\chi^{-1}\chi$ where $0 \neq c_\chi \in K$ for all $\chi \in \mathrm{Irr}(H)$. Finally, assume that there is an anti-algebra automorphism $* \colon H \to H$ of order 2 such that $\tau(h^*) = \tau(h)$ for all $h \in H$. Under these assumptions, there exists a basis B_0 of H such that $\tau(b'b^*) = \delta_{b,b'}$ for all $b, b' \in B_0$. Then it turns out that, for any $\chi \in \mathrm{Irr}(H)$, the number

$$\nu_\chi := \frac{1}{c_\chi \chi(1)} \sum_{b \in B_0} \chi(b)^2$$

has properties analogous to the familiar Frobenius–Schur indicator for the characters of a finite group; for example, we have

$$\nu_\chi \in \{0, 1, -1\} \quad \text{and} \quad \sum_{\chi \in \mathrm{Irr}(H)} \nu_\chi \chi(1) = |\{b \in B_0 \mid b^* = b\}|.$$

There is also a characterisation in terms of the existence of H-invariant bilinear forms on a module affording χ. In particular, this shows that ν_χ does not depend on the choice of B_0; furthermore, if $K \subseteq \mathbb{R}$, then $\nu_\chi = 1$ for all $\chi \in \mathrm{Irr}(H)$. The proofs are easy adaptations of the original proofs of Frobenius and Schur, as presented by Curtis [1].

Note that the standard symmetric algebra structure of the group algebra of a finite group G satisfies the above properties where $K = \mathbb{C}$ and $g^* = g^{-1}$ for all $g \in G$. Our application to left cells in a finite Coxeter group W uses the symmetric algebra structure on the corresponding generic Iwahori–Hecke algebra and Lusztig's asymptotic algebra J (see [4]). Marberg [6] noticed that these results imply a weak version of a conjecture due to Kottwitz [3] which predicts that the number of involutions in a left cell Γ of W is given by the scalar product of the character afforded by Γ with the character afforded by a certain "involution module" which re-appeared in recent work of Lusztig–Vogan [5].

References

[1] C. W. Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*, History of Mathematics, 15. Amer. Math. Soc., Providence, RI; London Math. Soc., 1999.

[2] M. Geck, *Kazhdan-Lusztig cells and the Frobenius-Schur indicator*, preprint (2011); see arXiv:1110.5672.

[3] R. Kottwitz, *Involutions in Weyl groups*, Represent. Theory **4** (2000), 1–15.

[4] G. Lusztig, *Leading coefficients of character values of Hecke algebras*, Proc. Symp. Pure Math. **47**, Amer. Math. Soc., Providence, RI, 1987, pp. 235–262.

[5] G. Lusztig and D. A. Vogan, Jr., *Hecke algebras and involutions in Weyl groups*, Bull. Inst. Math. Acad. Sinica (N.S.), to appear; see also arXiv:1109.4606.

[6] E. Marberg, *How to compute the Frobenius–Schur indicator of a unipotent character of a finite Coxeter system*, preprint (2012); see arXiv:1202.1311.

Quasi-isolated blocks and Brauer's height zero conjecture

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(joint work with Gunter Malle)

We complete the parametrisation of all p-blocks of finite quasi-simple groups by describing quasi-isolated blocks of exceptional groups of Lie type for bad primes in terms of generalised e-Harish Chandra theory [2]. Our description is analogous to that obtained by Cabanes-Enguehard [3],[4] for good primes and by Enguehard [5] for unipotent blocks and bad primes. Our second main result is the following:

If a p-block of a finite group has abelian defect groups, then the p-parts of the degree of any two ordinary irreducible characters in the block are equal.

The above statement and its converse were conjectured by Brauer in 1955 and are known as the height zero conjecture. Our proof relies upon a reduction of (the relevant direction of) the conjecture by Berger and Knörr [1] to the quasi-simple groups. We also draw on previous results by several authors proving the conjecture for special cases.

References

[1] T. R. Berger, R. Knörr, *On Brauer's height* 0 *conjecture*, Nagoya Math. J. **109** (1988), 109–116.

[2] M. Broué, G. Malle, J. Michel, *Generic blocks of finite reductive groups*, Astérisque **212** (1993), 7–92.

[3] M. Cabanes, M. Enguehard, *On unipotent blocks and their ordinary characters*, Invent. Math. **117** (1994), 149–164.

[4] M. Cabanes, M. Enguehard, *On blocks of finite reductive groups and twisted induction*, Adv. Math. **145** (1999), 189–229.

[5] M. Enguehard, *Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais*, J. Algebra **230** (2000), 334–377.

[6] R. Kessar, G.Malle *Quasi-isolated blocks and Brauer's height zero conjecture*, preprint.

Quiver Schur algebras and decomposition numbers

Andrew Mathas
(joint work with Jun Hu)

Building on work of Khovanov and Lauda [14, 13] and Rouquier [16], Brundan and Kleshchev [6] showed that the degenerate and non-degenerate cyclotomic Hecke algebras of type A are $\mathbb{Z}$-graded algebras. For the Hecke algebras of type B at a non-root of unity, these algebras were first introduced by Brundan and Stroppel [8] in their study of the Khovanov diagram algebras.

Fix an integer $e \in \{0, 2, 3, 4, \dots\}$. Let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \longrightarrow i+1$, for all $i \in I$. To the quiver Γ_e we attach the standard Lie theoretic data of a Cartan matrix $(a_{ij})_{i,j\in I}$, fundamental weights $\{\, \Lambda_i \mid i \in I \,\}$ and the positive weight lattice $P^+ = \sum_{i\in I} \mathbb{N}\Lambda_i$,

For each dominant weight $\Lambda \in P^+$ and each integer $n \geq 0$, Brundan and Kleshchev [6] defined the **cyclotomic quiver Hecke algebra** $\mathscr{R}_n^\Lambda$ by generators and relations. In the rapidly growing literature about these algebras they are also often called **Khovanov-Lauda-Rouquier algebras**.

Remarkably, Brundan and Kleshchev showed that the cyclotomic quiver Hecke algebra $\mathscr{R}_n^\Lambda$ is isomorphic to the corresponding degenerate and non-degenerate cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda$ of type $G(\ell, 1, n)$. The cyclotomic Schur algebras $\mathscr{S}_n^{\mathrm{DJM}}$, introduced in [9, 5], are quasi-hereditary covers of these Hecke algebras. It is natural to ask whether the KLR grading on the cyclotomic Hecke algebras extends to the cyclotomic Schur algebras.

Theorem 1 (Ariki [1], Hu-M. [10, 12], Stroppel-Webster [17])**.** *Suppose that K is a field. Then there exists a $\mathbb{Z}$-graded algebra $\dot{\mathscr{S}}_n^\Lambda$ which is a quasi-hereditary graded cellular algebra, which has graded standard modules $\{\, \Delta^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{P}_n^\Lambda \,\}$ and irreducible modules $\{\, L^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{P}_n^\Lambda \,\}$. Moreover, there is an equivalence of (ungraded) highest weight categories $\mathsf{E}_n^{DJM} : \dot{\mathscr{S}}_n^\Lambda-\mathbf{Mod} \longrightarrow \mathscr{S}_n^{DJM}-\mathbf{Mod}$ which sends standard modules to standard modules and simple modules to simple modules in the obvious way.*

Ariki [1] first constructed these algebras in the special case when $e > 3$ and $\Lambda = \Lambda_0$, so that $\mathscr{S}_n^{\mathrm{DJM}}$ is the Dipper-James q-Schur algebra. Stroppel and Webster [17] constructed their quiver Schur algebras in characteristic zero using the geometry of quiver varieties. In [12] we use an algebraic construction to show, over an arbitrary ring, that the KLR grading on $\mathscr{R}_n^\Lambda \cong \mathscr{H}_n^\Lambda$ induces a grading on the 'permutation modules' which are used to define $\mathscr{S}_n^{\mathrm{DJM}}$. This naturally induces a grading on $\dot{\mathscr{S}}_n^\Lambda \cong \mathscr{S}_n^{\mathrm{DJM}}$. Moreover, we show that the algebras constructed by Ariki and Stroppel-Webster are canonically isomorphic, as graded algebras, to the cyclotomic quiver Schur algebras that we obtain.

When $e = 0$ and $e \geq n$, in [10] we gave a very different construction of a cyclotomic quiver Schur algebra $\mathscr{S}_n^\Lambda$ which, it turns out, is both a graded subalgebra of $\dot{\mathscr{S}}_n^\Lambda$ and graded Morita equivalent to $\dot{\mathscr{S}}_n^\Lambda$. These 'smaller' cyclotomic quiver Schur algebras have many remarkable properties. Quite surprisingly, given that their definitions are so different, in the special case when Λ is a weight of level 2 at a non-root of unity, $\mathscr{S}_n^\Lambda$ is graded isomorphic to the graded quasi-hereditary algebras introduced by Brundan and Stroppel [8] in their study of the Khovanov diagram algebras.

In the degenerate case, Brundan and Kleshchev [5] have shown that the blocks of $\mathscr{S}_n^{\mathrm{DJM}}$ are Morita equivalent to a sum of blocks $\mathcal{O}_n^\Lambda$ of parabolic category $\mathcal{O}$ for the general linear groups. We obtain a non-trivial extension of the main result of [5] to the graded setting.

Theorem 2 (Hu-M. [10])**.** *Suppose that $\Lambda \in P^+$, $e = 0$ and $K = \mathbb{C}$. Then there are graded exact Schur functors $\mathsf{F}_n^{\mathcal{O}} : \mathcal{O}_n^\Lambda \longrightarrow \mathscr{R}_n^\Lambda\text{–}\mathbf{Mod}$ and $\mathsf{F}_n^\Lambda : \mathscr{S}_n^\Lambda\text{–}\mathbf{Mod} \longrightarrow \mathscr{R}_n^\Lambda\text{–}\mathbf{Mod}$ and an equivalence of graded categories $\mathsf{E}_n^\Lambda : \mathcal{O}_n^\Lambda \longrightarrow \mathscr{S}_n^\Lambda\text{–}\mathbf{Mod}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_n^\Lambda & \xrightarrow{\ \mathsf{E}_n^\Lambda\ } & \mathscr{S}_n^\Lambda\text{–}\mathbf{Mod} \\ & \searrow^{\mathsf{F}_n^{\mathcal{O}}} & \downarrow{\scriptstyle \mathsf{F}_n^\Lambda} \\ & & \mathscr{R}_n^\Lambda\text{–}\mathbf{Mod} \end{array}$$

In particular, $\mathscr{S}_n^\Lambda\text{–}\mathbf{Mod}$ is Koszul. Moreover, $[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_q = [\Delta^{\boldsymbol{\lambda}} : L^{\boldsymbol{\mu}}]_q$ whenever $D^{\boldsymbol{\mu}} \neq 0$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\Lambda$.

Applying the results of [5], the grading on $\mathcal{O}_n^\Lambda$ also induces a grading on the cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda$. The key to proving Theorem 2 is to show that the KLR and category $\mathcal{O}$ gradings on $\mathscr{H}_n^\Lambda$ coincide. We prove this in [10] by artificially manufacturing a positive grading on the indecomposable prinjective $\mathscr{S}_n^\Lambda$-modules and then using a delicate counting argument which exploits the rigidity of the projective modules (since category $\mathcal{O}_n^\Lambda$ is Koszul), and the fact that the graded decomposition numbers of $\mathscr{R}_n^\Lambda$ are known through the work of Brundan and Kleshchev [7]. Ultimately, however, our argument relies on Ariki's categorification theorem [2] and the Koszulity of parabolic category $\mathcal{O}_n^\Lambda$ [3, 4], both of which are proved using geometric machinery.

The direct sum of the Grothendieck groups of finitely generated $\mathscr{S}_n^\Lambda$-modules, for $n \geq 0$, can be identified with the full higher level combinatorial Fock space $\mathfrak{F}^\Lambda$. There are two natural representation theoretically defined 'bar involutions' on $\mathfrak{F}^\Lambda$. By Theorem 2 the graded decomposition numbers of $\mathscr{S}_n^\Lambda$ are polynomials with non-negative coefficients, so it follows that the irreducible, projective indecomposable and tilting modules of $\mathscr{S}_n^\Lambda$ correspond to the canonical bases (and dual canonical bases) determined by these involutions. This allows us to give a very simple LLT-like algorithm for computing the graded decomposition matrices of the quiver Schur algebras when $e = 0$. What is most remarkable about this 'LLT algorithm' is that it quickly computes the entire graded decomposition matrix for $\mathscr{S}_n^\Lambda$, unlike the LLT algorithm which only computes the (graded) decomposition matrix for $\mathscr{H}_n^\Lambda$ when $\Lambda = \Lambda_0$.

Example The quiver Schur algebra $\mathscr{S}_n^\Lambda = \bigoplus_\beta \mathscr{S}_\beta^\Lambda$ naturally decomposes into a direct sum of blocks which are indexed by the positive root lattice. Each block $\mathscr{S}_\beta^\Lambda$ is an indecomposable quasi-hereditary graded cellular algebra with an explicit grading.

Suppose that $e = 0$, $\Lambda = 3\Lambda_0$ and that $\beta = \alpha_{-1} + 3\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Then $\mathscr{S}_\beta^\Lambda$ is a block of defect 4. The maximal multipartition in $\mathcal{P}_\beta^\Lambda$ is $(4,2|1|0)$ so $P^{(4,2|1|0)} = \Delta^{(4,2|1|0)}$. Taking $\boldsymbol{\mu} = (4,1|1|1)$ our LLT algorithm says that we should look at the following four standard tableaux because these tableaux index a Δ-filtration of a particular graded projective $\mathscr{S}_n^\Lambda$-module $Z^{\boldsymbol{\mu}}$, which has the projective indecomposable graded module $P^{\boldsymbol{\mu}}$ as a summand:

$$\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5\\\cline{1-1}\end{array}\;\Bigg|\;\begin{array}{|c|}\hline 6\\\hline\end{array}\;\Bigg|\;\begin{array}{|c|}\hline 7\\\hline\end{array}\right) \qquad \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\;\Bigg|\;\begin{array}{|c|}\hline -\\\hline\end{array}\;\Bigg|\;\begin{array}{|c|}\hline 7\\\hline\end{array}\right)$$

$$\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&7\\\cline{1-2}\end{array}\;\Bigg|\;\begin{array}{|c|}\hline 6\\\hline\end{array}\;\Bigg|\;\begin{array}{|c|}\hline -\\\hline\end{array}\right) \qquad \left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\;\Bigg|\;\begin{array}{|c|}\hline 7\\\hline\end{array}\;\Bigg|\;\begin{array}{|c|}\hline -\\\hline\end{array}\right)$$

This implies that $[Z^{\boldsymbol{\mu}}] = [\Delta^{(4,1|1|1)}] + q[\Delta^{(4,2|0|1)}] + (q^2+1)[\Delta^{(4,2|1|0)}]$ in the Grothendieck group of $\mathscr{S}_\beta^\Lambda$. Applying our LLT algorithm, $[Z^{\boldsymbol{\mu}}] = [P^{\boldsymbol{\mu}}] + [P^{(4,2|1|0)}]$. In general, our results say that there exist bar invariant Laurent polynomials $p_{\boldsymbol{\lambda\mu}}(q) = p_{\boldsymbol{\lambda\mu}}(q^{-1}) \in \mathbb{N}[q,q^{-1}]$ such that $[Z^{\boldsymbol{\mu}}] = [P^{\boldsymbol{\mu}}] + \sum_{\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}} p_{\boldsymbol{\lambda\mu}}(q)[P^{\boldsymbol{\lambda}}]$. By induction on the dominance order, and Gaussian elimination, this determines $[P^{\boldsymbol{\mu}}]$ uniquely because

$$[Z^{\boldsymbol{\mu}}] = \sum_{\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}} \sum_{\mathfrak{t} \in \operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})} q^{\deg \mathfrak{t} - \deg \mathfrak{t}^{\boldsymbol{\mu}}} [\Delta^{\boldsymbol{\lambda}}],$$

where $\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda}) = \{\, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid \mathfrak{t} \trianglerighteq \mathfrak{t}^{\boldsymbol{\mu}} \text{ and } \operatorname{res}(\mathfrak{t}) = \operatorname{res}(\mathfrak{t}^{\boldsymbol{\mu}}) \,\}$ and $\mathfrak{t}^{\boldsymbol{\mu}}$ is the minimal $\boldsymbol{\mu}$-tableau under dominance. The full graded decomposition matrix of $\mathscr{S}_\beta^\Lambda$ in characteristic zero (and, in fact, any field) is:

$(0\|\ 1\ \|4,2)$	1														
$(0\|4,2\|1)$	q	1													
$(1\|\ 0\ \|4,2)$	q	.	1												
$(1\|\ 1\ \|4,1)$	q^2	.	q	1											
$(1\|\ 1^2\ \|4)$	.	.	.	q	1										
$(1\|\ 4\ \|1^2)$	.	.	.	q	.	1									
$(1\|4,1\|1)$	q^2	q	q	q^2	q	q	1								
$(1\|4,2\|0)$	q^3	q^2	q^2	.	.	.	q	1							
$(1^2\|\ 1\ \|4)$	.	.	q	q^2	q	.	.	.	1						
$(1^2\|\ 4\ \|1)$	.	.	q^2	q^3	q^2	q^2	q	.	q	1					
$(4\|\ 1\ \|1^2)$	.	.	q	q^2	.	q	.	.	.	.	1				
$(4\|\ 1^2\ \|1)$	.	.	q^2	q^3	q^2	q^2	q	.	.	.	q	1			
$(4,1\|\ 1\ \|1)$	q^2	q	q^3+q	q^4	q^3	q^3	q^2	.	q^2	q	q^2	q	1		
$(4,2\|\ 0\ \|1)$	q^3	q^2	q^2	.	.	.	.	.	.	.	.	.	q	1	
$(4,2\|\ 1\ \|0)$	q^4	q^3	q^3	.	.	.	q^2	q	.	q	.	q	q^2	q	1

The algebra $\mathscr{S}_\beta^\Lambda$ is one of the smallest examples of a block which has a graded decomposition number which is not a monomial. ◊

The graded decomposition numbers of $\mathscr{S}_n^\Lambda$ are the polynomials $[\Delta^{\boldsymbol{\lambda}}:L^{\boldsymbol{\mu}}]_q = \sum_{z\in\mathbb{Z}}[\Delta^{\boldsymbol{\lambda}}:L^{\boldsymbol{\mu}}\langle z\rangle]q^z$. Part of the content of Theorem 2 is that the graded decomposition numbers are polynomials rather than Laurent polynomials. The coefficients of these polynomials describe the grading and Jantzen filtrations of the Weyl modules and the projective indecomposable modules. The Kleshchev multipartitions in this block, which index the simple $\mathscr{H}_\beta^\Lambda$-modules, are $(0|1|4,2)$ and $(1|1|4,1)$. If $\boldsymbol{\mu}$ is Kleshchev then $[\Delta^{\boldsymbol{\lambda}}:L^{\boldsymbol{\mu}}]_q = [S^{\boldsymbol{\lambda}}:D^{\boldsymbol{\mu}}]_q$.

If Λ is any weight of level $\ell = 2$ then it is not too hard to show that $|\operatorname{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})| = 1$ and hence that the graded decomposition numbers of $\mathscr{S}_n^\Lambda$ are monomials in q. It follows that, in this case, $\mathscr{S}_n^\Lambda$ is a positively graded quasi-hereditary and Koszul basic algebra. This allows us to give explicit bases for the graded Young modules and the indecomposable projective $\mathscr{S}_n^\Lambda$-modules. If $\ell > 2$ then, in general, $\mathscr{S}_\beta^\Lambda$ is neither basic nor positively graded.

The next result was announced in [10].

Theorem 3 (Hu-M. [11]). *Suppose that $e = 0$ and that K is a field. Then the graded decomposition numbers of $\mathscr{S}_n^\Lambda$ (and hence of $\mathscr{S}_n^{DJM}$ and $\mathscr{H}_n^\Lambda$), are independent of the characteristic of K.*

Equivalently, this result says that the graded dimensions and the formal characters of the simple $\mathscr{S}_n^\Lambda$-modules are independent of the characteristic of the base field when $e = 0$. Applying Theorem 2, this implies that $\mathscr{S}_n^\Lambda-\mathbf{Mod}$ is Koszul whenever K is a field.

Using the graded Schur functor from Theorem 2, it follows that the graded decomposition numbers and the graded dimensions of the simple $\mathscr{H}_n^\Lambda$-modules are independent of the characteristic when $e = 0$. This result establishes a special case of a conjecture of Kleshchev and Ram [15, Conjecture 7.3] for the formal characters of the KLR algebras of finite type.

Let $\mathscr{H}_n^{\mathrm{aff}}(K,\xi)$ be the extended affine Hecke algebra of type A defined over the field K with Hecke parameter ξ. Then Theorem 3, via the last paragraph, implies that the decomposition numbers and the dimensions of the simple module of $\mathscr{H}_n^{\mathrm{aff}}(K,\xi)$ are independent of the characteristic of K, and independent of ξ, whenever ξ is a non-root of unity in K.

References

[1] S. Ariki, *Graded q-Schur algebras*, preprint 2009. arXiv:0903.3453.

[2] ———, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ., **36** (1996), 789–808.

[3] E. Backelin, *Koszul duality for parabolic and singular category $\mathscr{O}$*, Represent. Theory, **3** (1999), 139–152 (electronic).

[4] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., **9** (1996), 473–527.

[5] J. Brundan and A. Kleshchev, *Schur-Weyl duality for higher levels*, Selecta Math., **14** (2008), 1–57.

[6] ———, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.

[7] ———, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math., **222** (2009), 1883–1942.

[8] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra I: cellularity*, Mosc. Math. J., to appear. arXiv.RT/0806.1532.

[9] R. Dipper, G. James, and A. Mathas, *Cyclotomic q–Schur algebras*, Math. Z., **229** (1999), 385–416.

[10] J. Hu and A. Mathas, *Cyclotomic quiver Schur algebras I: linear quivers*, preprint 2011. arXiv:1110.1699.

[11] ———, *Cyclotomic quiver Schur algebras II: linear quivers in positive characteristic*, in preparation.

[12] ———, *Cyclotomic quiver Schur algebras III: cyclic quivers*, in preparation.

[13] M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc., **363** (2008), 2685–2700. arXiv:0804.2080.

[14] ———, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory, **13** (2009), 309–347.

[15] A. Kleshchev and A. Ram, *Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words*, Math. Ann., **349** (2011), 943–975.

[16] R. Rouquier, *2-Kac-Moody algebras*, preprint 2008. arXiv:0812.5023.

[17] C. Stroppel and B. Webster, *Quiver Schur algebras and q-Fock space*, preprint 2011. arXiv:1110.1115.

Parabolic Deligne-Lusztig varieties

Jean Michel

(joint work with François Digne)

The geometric version of Broué's conjecture for finite reductive groups goes through Deligne-Lusztig varieties.

Let $\mathbf{G}$ be a connected quasi-simple reductive group over the algebraic closure of a finite field of characteristic p, and let F be an isogeny such that the fixed points $\mathbf{G}^F$ are finite. Then $\mathbf{G}^F$ is called a finite group of Lie type.

Let $\ell \neq p$ be a prime such that the Sylow ℓ-subgroup S of $\mathbf{G}^F$ is abelian. Then $\mathbf{L} = C_{\mathbf{G}}(S)$ is a Levi subgroup; Broué's conjecture for the principal block predicts a derived equivalence between the principal ℓ-block of $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}}(\mathbf{L})^F$ and that of $\mathbf{G}^F$. Specifically, there should exist a parabolic subgroup with Levi decomposition $\mathbf{P} = \mathbf{L}\mathbf{V}$ such that the cohomology complex $H^i_c(\mathbf{X}_{\mathbf{V}}, \overline{\mathbb{Z}}_\ell)$ of $\mathbf{G}^F \times \mathbf{L}^F$-bimodules, cut by the principal block of $\mathbf{G}^F$ (or equivalently by that of $\mathbf{L}^F$) is a tilting complex for the sought derived equivalence, where $\mathbf{X}_{\mathbf{V}}$ is the variety $\{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g\mathbf{V} \cap F(g\mathbf{V}) \neq \emptyset\}$.

The variety $\mathbf{X}_{\mathbf{V}}$ is a $\mathbf{L}^F$-torsor over $\mathbf{X}_{\mathbf{P}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\}$, and the conjecture reflects in the following question:

Conjecture 1.

(1) The modules $H^i_c(\mathbf{X}_{\mathbf{P}}, \overline{\mathbb{Q}}_\ell)$ are pairwise disjoint.

(2) $End_{\mathbf{G}^F}(\oplus_i H^i_c(\mathbf{X}_{\mathbf{P}}, \overline{\mathbb{Q}}_\ell)) \simeq \overline{\mathbb{Q}}_\ell W_S$, where $W_S = (N_{\mathbf{G}}(\mathbf{L})/\mathbf{L})^F$ is a complex reflection group.

The problem (1) above is quite difficult and was not dealt with in the talk. We explained how to construct a monoid of $\mathbf{G}^F$-endomorphisms of the variety $\mathbf{X}_{\mathbf{P}}$, which should be a monoid for the braid group of W_S, and induce a cyclotomic Hecke algebra for W_S on the cohomology complex.

References

[1] F. Digne and J. Michel, *Parabolic Deligne-Lusztig varieties*, arXiv:1110.4863v2 [math.GR]

On the heights of 2-Brauer characters of symmetric groups

John Murray

Quadratic Forms and Involutions. Let G be a finite group and let F be a perfect field of characteristic 2. C. Martínez-Perez and W. Willems [MPW] have noted that if $\Lambda^2(FG^*)^G$ is the space of G-invariant symplectic forms on FG, then the cohomology group $H^1(G, \Lambda^2(FG^*)^G)$ can be given the structure of a FG-module, and this module is naturally isomorphic to the *involution module* of G (the F-permutation module induced by the conjugation action of G on its elements of order ≤ 2). Moreover, $H^1(G, \Lambda^2(FG^*)^G)$ is a quotient of the space of G-invariant quadratic forms on FG. Earlier [W], Willems used a similar approach to give a transparent proof of Fong's Lemma:

Lemma. *Suppose that M is a non-trivial self-dual absolutely irreducible FG-module. Then M has a symplectic geometry. In particular, M has even dimension.*

R. Gow and W. Willems [GW93] used closely related ideas to show that a principal indecomposable FG-module M affords a quadratic geometry if and only if there is an idempotent $e \in FG$ and an involution $t \in G$ such that $M \cong eFG$ and $e^t = e^o$.

Proof of a result of Kiyota, Okuyama and Wada. Let B be a 2-block of a symmetric group $\mathfrak{S}_n$, with weight w. Then $n - 2w = k(k+1)/2$ is a triangular number and the partitions in B are the partitions of n that have 2-core $\delta = [k, k-1, \ldots, 2, 1]$. Now $\mathfrak{S}_{2w} \times \mathfrak{S}_{n-2w}$ contains the normalizer in $\mathfrak{S}_n$ of a defect group of B. The Brauer corresponding block is $B_0 \otimes B_\delta$, where B_0 is the principal 2-block of $\mathfrak{S}_{2w}$ and B_δ is the 2-block of $\mathfrak{S}_{n-2w}$ whose 2-core is δ. In particular B_δ has defect 0. Set e_δ as the block idempotent of B_δ.

It is known that the unique irreducible B_δ-module D^δ has a quadratic geometry (this comes from lifting to characteristic 0). As D^δ is projective, it follows from the result of Gow and Willems that there exists a primitive idempotent $e_1 \in F\mathfrak{S}_{n-2w}$ and an involution $t \in \mathfrak{S}_{n-2w}$ such that $e_1 = e_1 e_\delta$ and $e_1^t = e_1^o$.

Let λ be a 2-regular partition in B. Then $D^\lambda = S^\lambda / J(S^\lambda)$ is an irreducible B-module, where S^λ is the Specht module and $J(S^\lambda)$ is the radical of the standard $\mathfrak{S}_n$-invariant form on S^λ. Now $S^\lambda e_\delta \cong (S^\lambda e_1) \otimes D^\delta$, as $F\mathfrak{S}_{2w} \times \mathfrak{S}_{n-2w}$-modules. We can use the fact that $e_1^t = e_1^o$ to define an $\mathfrak{S}_{2w}$-invariant symmetric bilinear form on $S^\lambda e_1$. Moreover, this induces a nondegenerate $\mathfrak{S}_{2w}$-invariant symmetric bilinear form on $D^\lambda e_1$.

Suppose that λ is not the most dominant partition in B. Then the $F\mathfrak{S}_{n-2w}$-module $\mathrm{Hom}_{F\mathfrak{S}_{n-2w}}(F_{\mathfrak{S}_{2w}}, S^{\lambda *}{\downarrow_{\mathfrak{S}_{2w}}})$ has no submodule isomorphic to D^λ. Here

$S^{\lambda *}$ is the linear dual of S^{λ}. This follows from D. Hemmer's computation [H] of 'fixed-point functors' and well-known results about semi-standard homomorphisms and skew-Specht modules in characteristic 0. It follows that the symmetric bilinear form on $D^{\lambda}e_1$ is symplectic.

We can use this approach to give a cleaner and more conceptual proof of a new result of M. Kiyota, T. Okuyama and T. Wada [KOW]:

Theorem. *Let $n \geq 1$ and let B be a 2-block of the symmetric group $\mathfrak{S}_n$. Then B has a unique irreducible Brauer character of height zero.*

References

[MPW] C. Martínez-Pérez, W. Willems, Involutions, cohomology and metabolic spaces, *J. Algebra* **327** (2011) 366–385.

[GW93] R. Gow, W. Willems, Quadratic geometries, projective modules and idempotents, *J. Algebra* **160** (1993) 257–272.

[H] D. J. Hemmer, Fixed-point functors for symmetric groups and Schur algebras, *J. Algebra* **280** (2004) 295–312.

[KOW] M. Kiyota, T. Okuyama, T. Wada, The heights of irreducible Brauer characters in 2-blocks of the symmetric groups, preprint, April 2011.

[W] W. Willems, Duality and forms in representation theory, *Progress in Math.* **95** (1991) 509–520.

Cohomology for Finite Groups of Lie Type

Daniel K. Nakano

Let G be a reductive algebraic group scheme defined over $\mathbb{F}_p$, $G(\mathbb{F}_q)$ be the corresponding finite Chevalley group ($q = p^r$), and G_r be the rth Frobenius kernel. Moreover, let k be an algebraically closed field of characteristic p.

My talk was motivated by two questions that arose during the Cohomology of Finite Groups meeting in Oberwolfach in 2005. The first question was posed by Eric Friedlander:

1) In which degree does the first non-trivial cohomology class occur in the cohomology, $\mathrm{H}^{\bullet}(G(\mathbb{F}_q), k)$, of the finite Chevalley group $G(\mathbb{F}_q)$?

The second question was motivated by the seminal cohomological calculations by Cline, Parshall and Scott [CPS] on $\mathrm{H}^1(G(\mathbb{F}_q), L(\lambda))$ where $L(\lambda)$ is a simple module of minimal dominant highest weight. These computations were employed by Wiles in his proof of Fermat's Last Theorem. One is tempted to ask:

2) Can one compute the first cohomology groups and second cohomology groups for simple modules with fundamental dominant weight?

In this talk I will present new techniques invented by Bendel, Nakano and Pillen (cf. [BNP1, BNP2]) for computing cohomology for the finite Chevalley group $G(\mathbb{F}_q)$ directly in terms of cohomology for the ambient algebraic group G and its associated Frobenius kernels. These methods can be used to answer Question (1) above when p is larger than the Coxeter number. For instance, when the root system is of Type C_n one has a relatively nice statement:

Theorem: Let G be a simple simply connected algebraic group over an algebraically closed field k of characteristic $p > 0$. Suppose that the underlying root system is of type C_n and $p > 2n$. Then

a) $\mathrm{H}^j(G(\mathbb{F}_q), k) = 0$ for $0 < j < r(p-2)$

b) $\mathrm{H}^{r(p-2)}(G(\mathbb{F}_q), k) \cong k$.

These aforementioned techniques were also utilized to compute the first and second cohomology group when M is a simple $G(\mathbb{F}_q)$-module. This project was undertaken by the University of Georgia VIGRE Algebra Group during the academic years 2009-10 and 2010-11 (cf. [UGA1, UGA2]). A salient feature of these results is that no twisting of the coefficient module by the Frobenius morphism is necessary, which enabled calculations to be made for relatively small values of p and q.

My lecture and abstract are dedicated in honor of Edward T. Cline (1940-2012) whose career had a profound influence on this work and on the field of algebraic and finite group representation theory.

References

[BNP1] C.P. Bendel, D.K. Nakano, C. Pillen, *On the vanishing ranges for the cohomology of finite groups of Lie type*, International Math. Research Notices, doi:10.1093/imrn/rnr30, (2011).

[BNP2] C.P. Bendel, D.K. Nakano, C. Pillen, *On the vanishing ranges for the cohomology of finite groups of Lie type II*, to appear in the Proc. Symp. Pure Math., arXiv:1112.2367

[CPS] E. Cline, B. Parshall, L. Scott, *Cohomology of finite groups of Lie type. I*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 169–191.

[UGA1] University of Georgia VIGRE Algebra Group, *First cohomology for finite groups of Lie type: simple modules with small dominant weights*, to appear in the Transactions of the AMS, arXiv:1010.1203.

[UGA2] University of Georgia VIGRE Algebra Group, *Second cohomology for finite groups of Lie type*, Journal of Algebra, 360 (2012), 21–52.

Character Degrees

Gabriel Navarro

Many relevant problems in the Representation Theory of Finite Groups involve character degrees. If p is a prime and G is a finite group, then the McKay conjecture, for instance, counts locally the number of irreducible characters of G of degree not divisible by p. The Alperin Weight Conjecture counts the number of irreducible characters of G whose p-part is $|G|_p$, while Dade's conjecture counts the number of irreducible characters of G with a fixed p-part p^d. But there are more problems, of course. For instance, we do not know if the knowledge of the set of the degrees of G determines its solvability, nor if the complex group algebra of

G determines if G has a normal Sylow p-subgroup. Nor we know if the nilpotent blocks are the blocks in which all height zero characters have the same degree ([MN]), to mention a more recent problem.

In the first part of our talk we review the different refinements of the McKay conjecture, some of them leading to a reduction of this conjecture to a problem of simple groups ([IMN]). These refinements include a relative version with respect to a normal subgroup; the equivariant version with respect to automorphisms; the corresponding version where certain Galois automorphisms are acting (Navarro); blocks (Alperin-McKay); the inclusion of Brauer characters (T. Wolf); of sets of primes (T. Wolf); or congruences of degrees (Isaacs-Navarro). We also point out that in groups with a nilpotent Hall π-subgroup H, all the counting conjectures seem to hold with respect to π. For instance, the number of π'-degree irreducible characters of G and of $\mathbf{N}_G(H)$ seem to coincide. (This is a joint observation with M. Isaacs.)

In the second part of my talk, I consider all these variations (relative to a normal subgroup, fields, blocks, actions, Brauer characters) in the Itô-Michler theorem, a fundamental result in the theory of character degrees. The Itô-Michler theorem asserts that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal and abelian Sylow p-subgroup. When introducing fields, for instance, we obtained that if 2 does not divide the real irreducible character degrees, then the Sylow 2-subgroup of G is normal in G ([DNT]). This result is now generalized to every prime: if p does not divide $\chi(1)$ for all real irreducible characters of G, then $\mathbf{O}^{2'}(G)$ has a normal Sylow p-subgroup ([IN], [T]). Also, if p does not divide the degrees of the p-rational characters of G, then a Sylow p-subgroup of G is normal ([NT1]). The Itô-Michler theorem holds for p-Brauer characters (same p), and we mention that the case $p = 2$ is due to T. Okuyama and does not involve the classification of finite simple groups. The Itô-Michler theorem for the principal block is specially appealing: p does not divide $\chi(1)$ for all χ in the principal block of G if and only if P is abelian. (This is, of course, Brauer's Height Zero Conjecture for the principal block.) And we leave for the end the relative version of the Itô-Michler theorem with respect to a normal subgroup, that states the following: if $N \triangleleft G$, $\theta \in \mathrm{Irr}(N)$ and p does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G|\theta)$, then a Sylow p-subgroup of G/N is abelian.

If G/N is p-solvable, this latter result was proved by D. Gluck and T. Wolf and gave a proof of the Height Zero Conjecture for p-solvable groups ([GW]). To prove this result for every finite group was one of the major obstacles to obtain Brauer Height Zero conjecture. Now P. H. Tiep and the author have recently accomplished this in [NT2], using the recent classification of the groups having a faithful p-module such that all the orbits have p'-length ([GLPST]).

Finally, using the work of Murai in [M] (inspired by results in [R]), our generalized Gluck-Wolf ([NT2]), and the proof of the Brauer Height Zero Conjecture for quasisimple groups in [KM], then it can be seen that Dade's Projective Conjecture implies Brauer Height Zero Conjecture, and also, that if all finite simple involved in G satisfy the inductive Alperin-McKay condition, then Brauer's Height Zero

Conjecture is true ([NS]). This reduces Brauer Height Zero Conjecture to checking the Alperin-McKay conjecture via the method that is proposed in [S].

References

[DNT] S. Dolfi, G. Navarro, P. H. Tiep, *Primes dividing the degrees of the real characters*, Math. Z. **259** (2008), no. 4, 755–774.

[GLPST] M. Giudici, M. Liebeck, C. Praeger, J. Saxl, P. H. Tiep, *Arithmetic results on orbits of linear groups*, (submitted).

[GW] D. Gluck and T. R. Wolf, *Brauer's height conjecture for p-solvable groups* Trans. Am. Math. Soc., **282**, (1984) 137–152.

[IMN] I. M. Isaacs, G. Malle, and G. Navarro, *A reduction theorem for the McKay conjecture*, Invent. Math. **170**(1), (2007) 33–101.

[IN] I. M. Isaacs, G. Navarro, *Groups whose real irreducible characters have degrees coprime to p*, to appear in J. Algebra.

[KM] R. Kessar, G. Malle. *Brauer's height zero conjecture for universal covering groups of simple groups*, private correspondence, 2011.

[M] M. Murai, *On Brauer's height zero conjecture*, Proc. Japan. Acad., **88**, Series A, (2012) 38–40.

[MN] G. Malle, G. Navarro, *Blocks with equal height zero degrees*, Trans. Amer. Math. Soc. **363** (2011), 6647–6669.

[NS] G. Navarro, B. Späth, *A bijection above height zero characters and Brauer's height zero conjecture*, in preparation.

[NT1] G. Navarro, P. H. Tiep, *Degrees of p-rational characters*, to appear in Bull. London Math. Soc.

[NT2] G. Navarro, P. H. Tiep, *Characters of relative p'-degree over a normal subgroup*, submitted, 2012.

[R] G. Robinson, *Cancellation theorems related to conjectures of Alperin and Dade*, J. of Algebra, **249**(1) (2002) 196 – 219.

[S] B. Späth, *A reduction theorem for the Alperin-McKay conjecture*, to appear, Crelle's, 2012.

[T] P. H. Tiep, *Real ordinary characters and real ordinary Brauer characters*, in preparation.

Equivariant Alperin-Robinson's Conjecture reduces to almost-simple k^*-groups

Lluís Puig

Our reduction result[†] concerns *Alperin's Conjecture for blocks* in an *equivariant* formulation which goes back to Geoffrey Robinson in the eighties (it appears in his joint work [4] with Reiner Staszewski). In the introduction of [2] — from I29 to I37 — we consider a *refinement* of Alperin-Robinson's Conjecture for blocks; but, only in [3] we really show that its verification can be reduced to check that the *same* refinement holds on the so-called *almost-simple k^* groups*. To carry out this checking obviously depends on admitting the *Classification of the Finite Simple Groups*, and our proof of the reduction itself uses the *solvability* of the *outer automorphism group* of a finite simple group (SOFSG), a known fact whose actual proof depends on this classification.

[†]We thank Britta Späth for pointing us a mistake in an earlier version of this paper.

Our purpose here is, from our results in [2] and [3], to show that the Alperin-Robinson's Conjecture for blocks can be reduced to checking the *same* statement on any central k^*extension of any finite group H containing a finite non-abelian simple group S such that H/S is a cyclic p'group and we have $C_H(S) = \{1\}$, and moreover, that it may be still reduced to any central k^*extension of any finite simple group provided we check an "almost necessary" condition in any finite group H as above.

Explicitly, let p be a prime number, k an algebraically closed field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring of characteristic zero admitting k as the *residue* field, and $\mathcal{K}$ the field of fractions of $\mathcal{O}$. Moreover, let $\hat{G}$ be a k^*central extension of a finite group G — simply called *finite k^*group* of k^*quotient G [2, 1.23] — b a block of $\hat{G}$ [2, 1.25] and $\mathcal{G}_k(\hat{G}, b)$ the *scalar extension* from $\mathbb{Z}$ to $\mathcal{O}$ of the *Grothendieck group* of the category of finitely generated $k_*\hat{G}b$modules [2, 14.3]. Choose a maximal Brauer $(b,\hat{G})$pair (P,e); denote by $\mathcal{F}_{(b,\hat{G})}$ the category — called the *Frobenius Pcategory* of $(b,\hat{G})$ [2, 3.2] — formed by all the subgroups of P and, if Q and R are subgroups of P, by the group homomorphisms $\mathcal{F}_{(b,\hat{G})}(Q,R)$ from R to Q determined by all the elements $x \in G$ fulfilling $(R,g)\imath(Q,f)^x$ where (Q,f) and (R,g) are the corresponding Brauer $(b,\hat{G})$pairs contained in (P,e); in particular, we set

$$\mathcal{F}_{(b,\hat{G})}(Q) = \mathcal{F}_{(b,\hat{G})}(Q,Q) \cong N_G(Q,f)/C_G(Q) \qquad .$$

Recall that the Brauer $(b,\hat{G})$pair (Q,f) is called *selfcentralizing* if the image $\bar{f}$ of f in $\bar{C}_{\hat{G}}(Q) = C_{\hat{G}}(Q)/Z(Q)$ is a block of defect *zero* and then we denote by $(\mathcal{F}_{(b,\hat{G})})^{\text{sc}}$ the full subcategory of $\mathcal{F}_{(b,\hat{G})}$ over the *selfcentralizing* Brauer $(b,\hat{G})$pairs contained in (P,e).

Recall that an $(\mathcal{F}_{(b,\hat{G})})^{\text{sc}}$*chain* is just a functor $\mathfrak{q} : \Delta_n \to (\mathcal{F}_{(b,\hat{G})})^{\text{sc}}$ from the n-simplex Δ considered as a category with the morphisms given by the order relation; then, the *proper category of* $(\mathcal{F}_{(b,\hat{G})})^{\text{sc}}$*chains* — denoted by $\mathfrak{ch}^*\big((\mathcal{F}_{(b,\hat{G})})^{\text{sc}}\big)$ — is formed by the $(\mathcal{F}_{(b,\hat{G})})^{\text{sc}}$*chains* as objects and by the pairs of ordering-preserving maps and natural isomorphisms of functors as morphisms [2, A2.8]. Denoting by $\mathfrak{Gr}$ the category of finite groups, we actually have a functor [2, Proposition A2.10]

$$\mathfrak{aut}_{(\mathcal{F}_{(b,\hat{G})})^{\text{sc}}} : \mathfrak{ch}^*\big((\mathcal{F}_{(b,\hat{G})})^{\text{sc}}\big) \longrightarrow \mathfrak{Gr}$$

mapping the $(\mathcal{F}_{(b,\hat{G})})^{\text{sc}}$chain $\mathfrak{q}$ on the stabilizer $\mathcal{F}_{(b,\hat{G})}(\mathfrak{q})$ of $\mathfrak{q}$ in $\mathcal{F}_{(b,\hat{G})}\big(\mathfrak{q}(n)\big)$. Moreover, setting $Q = \mathfrak{q}(n)$ and denoting by f the block of $C_{\hat{G}}(Q)$ such that (P,e) contains (Q,f), it is clear that $N_{\hat{G}}(Q,f)$ acts on the simple $k$algebra $k_*\bar{C}_{\hat{G}}(Q)\bar{f}$ and it is well-known that this action determines a central k^*extension $\hat{\mathcal{F}}_{(b,\hat{G})}(Q)$ of $\mathcal{F}_{(b,\hat{G})}(Q)$ [2, 7.4]; in particular, by restriction we get a central k^*extension $\hat{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})$ of $\mathcal{F}_{(b,\hat{G})}(\mathfrak{q})$.

Denoting by $k^*\mathfrak{Gr}$ the category of k^*groups with finite k^*quotient, in [2, Theorem 11.32] we prove the existence of a suitable functor

$$\widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}} : \mathfrak{ch}^*\big((\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}\big) \longrightarrow k^*\mathfrak{Gr}$$

lifting $\mathfrak{aut}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}$ and mapping $\mathfrak{q}$ on $\hat{\mathcal{F}}_{(b,\hat{G})}(\mathfrak{q})$; then, still denoting by $\mathcal{G}_k$ the functor mapping any k^*group with finite k^*quotient $\hat{G}$ on the *scalar extension* from $\mathbb{Z}$ to $\mathcal{O}$ of the *Grothendieck group* of the category of finitely generated $k_*\hat{G}$-modules, and any k^*group homomorphism on the corresponding restriction, we consider the inverse limit

$$\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) = \lim_{\longleftarrow} (\mathcal{G}_k \circ \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) \qquad ,$$

called the *Grothendieck group of* $\mathcal{F}_{(b,\hat{G})}$ [2, 14.3.3 and Corollary 14.7]; it follows from [2, I32 and Corollary 14.32] that Alperin's Conjecture for blocks is actually equivalent to the existence of an $\mathcal{O}$module isomorphism

$$\mathcal{G}_k(\hat{G}, b) \cong \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}})$$

which actually amounts to saying that both members have the same $\mathcal{O}$rank.

Denote by $\mathrm{Out}_{k^*}(\hat{G})$ the group of *outer* k^*automorphisms of $\hat{G}$ and by $\mathrm{Out}_{k^*}(\hat{G})_b$ the stabilizer of b in $\mathrm{Out}_{k^*}(\hat{G})$; on the one hand, it is clear that $\mathrm{Out}_{k^*}(\hat{G})_b$ acts on $\mathcal{G}_k(\hat{G}, b)$; on the other hand, an easy *Frattini argument* shows that the stabilizer $\mathrm{Aut}_{k^*}(\hat{G})_{(P,e)}$ of (P,e) in $\mathrm{Aut}_{k^*}(\hat{G})_b$ *covers* $\mathrm{Out}_{k^*}(\hat{G})_b$ and it is clear that it acts on $(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}$, so that finally $\mathrm{Out}_{k^*}(\hat{G})_b$ still acts on the inverse limit $\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{nc}}})$ [2, 16.3 and 16.4]. A stronger question is whether or not there exists above a $\mathrm{Out}_{k^*}(\hat{G})_b$*stable* isomorphism and in [3, Theorem 1.6] we prove that it suffices to check this statement in the *almost-simple k^*groups* considered above.

Here, we are interested in a weaker form of this question, namely in whether or not there exists a $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$module isomorphism

$$\mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\hat{G}, b) \cong \mathcal{K} \otimes_{\mathcal{O}} \mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) \qquad ;$$

as a matter of fact, it is a *numerical* question since it amounts to saying the $\mathrm{Out}_{k^*}(\hat{G})_b$characters of both members coincide and note that they are actually *rational* characters. Thus, it makes sense to relate this statement with the old Robinson's *equivariant condition* recalled below. We still need some notation; for any Brauer $(b,\hat{G})$pair (Q,f) contained in (P,e) , the group $\mathcal{F}_Q(Q)$ of inner automorphisms of Q is a normal subgroup of $\mathcal{F}_{(b,\hat{G})}(Q)$ and we set

$$\tilde{\mathcal{F}}_{(b,\hat{G})}(Q) = \mathcal{F}_{(b,\hat{G})}(Q)/\mathcal{F}_Q(Q) \cong N_G(Q,f)/Q \cdot C_G(Q) \qquad ;$$

moreover, if (Q,f) is selfcentralizing then $\mathcal{F}_Q(Q)$ can be identified to a normal subgroup of $\hat{\mathcal{F}}_{(b,\hat{G})}(Q)$; then, we also set

$$\hat{\bar{\mathcal{F}}}_{(b,\hat{G})}(Q) = \hat{\mathcal{F}}_{(b,\hat{G})}(Q)/\mathcal{F}_Q(Q)$$

and denote by $o_{(Q,f)}$ the sum of blocks of defect zero of $\hat{\bar{\mathcal{F}}}_{(b,\hat{G})}(Q)$; note that, since the stabilizer $\mathrm{Aut}_{k^*}(\hat{G})_{(P,e)}$ of (P,e) in $\mathrm{Aut}_{k^*}(\hat{G})_b$ *covers* $\mathrm{Out}_{k^*}(\hat{G})_b$ and acts on $(\mathcal{F}_{(b,\hat{G})})^{^{\mathrm{sc}}}$, the stabilizer $C_{(Q,f)}$ in a (cyclic) subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$ of the G-conjugacy class of (Q,f) acts naturally on $\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(b,\hat{G})}(Q), o_{(Q,f)}\big)$.

Following Robinson, let us consider the following *equivariant condition*: (E) *For any cyclic subgroup C of $\mathrm{Out}_{k^*}(\hat{G})_b$ we have*

$$\mathrm{rank}_{\mathcal{O}}\big(\mathcal{G}_k(\hat{G},b)^C\big) = \sum_{(Q,f)} \mathrm{rank}_{\mathcal{O}}\Big(\mathcal{G}_k\big(\hat{\bar{\mathcal{F}}}_{(b,\hat{G})}(Q), o_{(Q,f)}\big)^{C_{(Q,f)}}\Big)$$

where (Q,f) runs over a set of representatives contained in (P,e) for the set of C-orbits of G-conjugacy classes of selfcentralizing Brauer $(b,\hat{G})$-pairs and, for such a (Q,f) , we denote by $C_{(Q,f)}$ the stabilizer in C of the G-conjugacy class of (Q,f) . We are ready to state our first main result.

Theorem. *Assume* (SOSFG) *and that any block c of any central k^*-extension $\hat{H}$ of any finite group H , containing a finite non-abelian simple group S such that H/S is a cyclic p'-group and we have $C_H(S) = \{1\}$, fulfills the equivariant condition* (E)*. Then, any block b of any central k^*-extension $\hat{G}$ of any finite group G fulfills the equivariant condition* (E) *and, in particular, we have a $\mathcal{K}\mathrm{Out}_{k^*}(\hat{G})_b$-module isomorphism*

$$\mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k(\hat{G},b) \cong \mathcal{K}\otimes_{\mathcal{O}}\mathcal{G}_k(\mathcal{F}_{(b,\hat{G})}, \widehat{\mathfrak{aut}}_{(\mathcal{F}_{(b,\hat{G})})^{\mathrm{sc}}}) \quad .$$

References

[1] Gabriel Navarro and Pham Huu Tiep, *A reduction theorem for the Alperin weight conjecture*, Inventiones math., 184(2011), 529–565.

[2] Lluís Puig, *Frobenius categories versus Brauer blocks*, Progress in Math. 274(2009), Birkhäuser, Basel.

[3] Lluís Puig, *On the reduction of Alperin's Conjecture to the quasi-simple groups*, Journal of Algebra, 328(2011), 372–398.

[4] Geoffrey Robinson and Reiner Staszewski, *More on Alperin's Conjecture*, Astérisque, 181–182(1990), 237–255

Pseudoweights for finite groups

Geoffrey R. Robinson

Alperin's weight conjecture (in its non-blockwise version) predicts an equality between the number of simple and the number of conjugacy classes of weights of a finite group. Recall that a weight for a finite group is a pair (Q, S), where Q is a p-subgroup of G and S is a Q-projective simple module for $N_G(Q)$. We start with a naive question: where are all the weights? Clearly, a Sylow p-subgroup of G gives rise to weights, but are there others we can easily see? A suggestive result is one of Brauer and Fowler: if p is odd and t is an involution of a finite group which neither inverts nor centralizes an element of order p, then there is a p-block of defect 0 of G.

It follows easily from this that if p is odd and we have a pair (Q, t) where t is an involution of G and Q is a maximal t-invariant p-subgroup of G, then $N_G(Q)$ has a Q-projective simple module so the pair (Q, t) gives rise to a weight. It's also possible to show that (Q, x) is a pair such that Q is a maximal x-invariant p-subgroup of G where x is a p-regular element of $O_{p,p'}(N_G(Q))$, then $N_G(Q)$ has a Q-projective simple module. When $p = 2$, we are able to prove, using a recent theorem of John Murray that if (Q, x) is a pair where x is an element of order 3 and Q is a maximal x-invariant 2-subgroup of G, then there is a Q-projective simple module for $N_G(Q)$. The crucial case for the last result is when $Q = 1$, for then note that the product of a conjugate of x and a conjugate of x^{-1} can never be an involution, otherwise x lies in a subgroup isomorphic to A_4 and normalizes a Klein 4-group.

This motivates the following definition: A p-pseudoweight is a pair (Q, x) where x is a p-regular element of G, Q is a maximal x-invariant p-subgroup of G and one or more of the following holds:
i) $x \in O_{p,p'}(N_G(Q))$.
ii) p is odd and x is an involution, or:
iii) $p = 2$ and x has order 3.

Then we are able to prove:

Theorem. *The number of conjugacy classes of p-weights for G is greater than or equal to the number of conjugacy classes of p-pseudoweights.*

It might be tempting to assume that the number of conjugacy classes of pairs (Q, x) such that x is p-regular and Q is a maximal x-invariant p-subgroup of G would be equal to the number of conjugacy classes of p-weights of G. This is not true in general. In groups for which Alperin's conjecture holds, it would force all maximal x-invariant p-subgroups of G to be $C_G(x)$-conjugate. However when $p = 3$ and $G = \mathrm{PSL}(2, 11)$ and x is an involution, then all maximal x-invariant 3-subgroups of G have order 3. However, there is one which is inverted by x and another which is centralized by x, so they are certainly not conjugate via $C_G(x)$. Hence x gives rise to more than one conjugacy class of 3-weights.

For example, let G be a $\{2, p\}$-group with elementary Abelian Sylow 2-subgroups, Then we see that the number of p-weights of G is at least as great as the number of p-regular conjugacy classes. The fact that Alperin's weight conjecture holds for p implies that for each involution $t \in G$, all maximal t-invariant p-subgroups of G are conjugate via $C_G(t)$. This is relatively easy to see group-theoretically. We can assume that $O_p(G) = 1$, and then G has a normal Sylow 2-subgroup. Let $p = 2$, G be a $\{2, 3\}$-group with Sylow 3-subgroups of exponent 3. Then the number of 2-weights of G is at least the number of conjugacy classes of 3-elements. Again this tells us that whenever x has order 3, we should expect all maximal x-invariant 2-subgroups to be conjugate via $C_G(x)$ (and this is equivalent to Alperin's weight conjecture for such a group). This is not immediately obvious, but we are grateful to C.W. Parker for providing a direct group-theoretic proof (his proof generalizes easily to $\{2, p\}$-groups whose Sylow p-subgroups do not involve $C_p \wr C_p$).

Perverse equivalences and applications

Raphaël Rouquier
(joint work with Joseph Chuang, David Craven and Olivier Dudas)

Perverse equivalences have been introduced in joint work with Joe Chuang in 2003 [1, 8]. They correspond to a situation where there are two abelian categories filtered by Serre subcategories and a derived equivalence respecting the filtrations such that the induced equivalences on each slice of the filtration of the derived categories comes from a shifted equivalence of the corresponding abelian category slices. The function recording the shifts is the perversity function. In the case of blocks of finite groups, the data is encoded in a filtration of the set of simple modules and a $\mathbf{Z}$-valued function on that set. Given a block together with the filtration and the perversity function, there is a unique symmetric algebra, up to Morita equivalence, perversely equivalent to the given block.

We showed that two blocks of symmetric groups with same local structures have equivalent derived categories through a composition of perverse equivalences (corresponding to the reduced decomposition of an element of an affine Weyl group). Two blocks with cyclic defect and same local structure are related by a perverse equivalence. For finite groups of Lie type in non-describing characteristic, Broué conjectures the complex of cohomology of an appropriate Deligne-Lusztig variety will provide a derived equivalence between a block with abelian defect and its Brauer correspondent. We conjecture that this equivalence will be perverse.

More recently, in collaboration with David Craven, we started looking systematically at perverse equivalences in the setting of Broué's abelian defect group conjecture [4]. Recent work of Craven [2] has provided a conjectural description of the perversity function in the setting of Broué's conjecture, for finite groups of Lie type — equivalently, a conjectural description of the degrees of cohomology in which a given unipotent character of a finite group of Lie occurs in an appropriate Deligne-Lusztig variety. This opens the way to the study of genericity phenomena for categories of modular representations of finite groups of Lie type and has led David Craven and myself to conjectures on independence of q of Green correspondents and images of simple modules under the conjectural derived equivalences.

This will hopefully lead to the construction of an actual category, depending only on the multiplicative order e of q modulo ℓ, that gives rise to representations of a corresponding finite group of Lie type in a given unipotent block.

A first instance of that type of genericity was provided by a conjecture of Hiß, Lübeck and Malle on Brauer trees for principal blocks, when e is the Coxeter number. Work of Olivier Dudas [5, 6] and joint work [7] has led to a solution of that conjecture, as well as a solution of Broué's geometric conjecture, for such blocks. This is the first instance where the study of the mod-ℓ cohomology of Deligne-Lusztig varieties has led to the determination of new decomposition matrices for principal blocks of finite groups. The techniques developed by Dudas should lead to further progress on the determination of decomposition matrices and the classification of simple modules for finite groups of Lie type.

The work of Craven [3] on cyclic defect for unipotent blocks of finite groups of Lie type provides a conjectural description of the perversity function and the compatibilities of parametrizations of unipotent characters. In particular, he describes the expected unknown Brauer trees. Combined with the techniques of Dudas, this might lead to a determination of all Brauer trees of finite groups of Lie type.

References

[1] J. Chuang and R. Rouquier, *Calabi-Yau algebras and perverse equivalences*, in preparation.

[2] D. Craven, *On the cohomology of Deligne-Lusztig varieties*, arXiv:1107.1871v1.

[3] D. Craven, *Perverse equivalences and Broué's conjecture II: the cyclic case*, in preparation, 2012.

[4] D. Craven and R. Rouquier, *Perverse Equivalences and Broué's Conjecture*, preprint arXiv:1010.1378v1.

[5] O. Dudas, *Coxeter Orbits and Brauer trees*, Adv. Math. **229** (2012), 3398–3435.

[6] O. Dudas, *Coxeter Orbits and Brauer trees II*, arXiv:math/1011.5478.

[7] O. Dudas and R. Rouquier, *Coxeter orbits and Brauer trees III*, arXiv:1204.1606v1.

[8] R. Rouquier, *Derived equivalences and finite dimensional algebras*, International Congress of Mathematicians, Vol. II, 191–221, Eur. Math. Soc., 2006.

Survey on invariants of blocks of finite groups

Benjamin Sambale

Many open conjectures in modular representation theory relate the invariants of a block of a finite group to its defect group. Here we consider a p-block B of a finite group G with respect to a splitting p-modular system for G. The important invariants of B are the number $k(B)$ of ordinary irreducible characters of B and the number $l(B)$ of irreducible Brauer characters. Moreover, the ordinary characters split in $k_i(B)$ characters of height $i \geq 0$. Here the height describes the p-part of the degree of the character. We denote the inertial group of B by $I(B)$ and its order by $e(B) := |I(B)|$.

A long standing task in representation theory is the determination of the block invariants if the defect group is given. In the talk I present a general method in several steps to complete this task. After that I give a table which lists many cases in which the block invariants are known (see end of this abstract).

In cases where the invariants are known one can ask if the conjectures are satisfied. Sometimes this is even possible if only inequalities are known. In the last part of my talk I present some general inequalities. For example in [5] we proved the following.

Theorem 1. *Let (u, b_u) be a B-subsection such that b_u has Cartan matrix $C_u = (c_{ij})$ up to basic sets. Then for every positive definite, integral quadratic form $q(x_1, \ldots, x_{l(b_u)}) = \sum_{1 \le i \le j \le l(b_u)} q_{ij} x_i x_j$ we have*

$$k_0(B) \le \sum_{1 \le i \le j \le l(b_u)} q_{ij} c_{ij}.$$

In particular

$$k_0(B) \le \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$

If (u, b_u) is major, we can replace $k_0(B)$ by $k(B)$ in these formulas.

As a consequence one gets Brauer's $k(B)$-Conjecture in some small cases.

Theorem 2. *Brauer's $k(B)$-Conjecture holds for defect groups which contain a central, cyclic subgroup of index at most* 9.

Theorem 3. *Let B be a block with a defect group which is a central extension of a group Q of order* 16 *by a cyclic group. If Q is not elementary abelian or if* 9 *does not divide the inertial index of B, then Brauer's $k(B)$-conjecture holds for B.*

I also show some results concerning Olsson's Conjecture and Brauer's Height Zero Conjecture for odd primes. Here the following inequality is useful (see [5]).

Theorem 4. *Let $p > 2$, and let (u, b_u) be a B-subsection such that $l(b_u) = 1$ and b_u has defect d. Moreover, let $|\mathrm{Aut}_{\mathcal{F}}(\langle u \rangle)| = p^s r$ where $p \nmid r$ and $s \ge 0$. Then we have*

$$k_0(B) \le \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle| r} p^d \le p^d.$$

If (in addition) (u, b_u) is major, we can replace $k_0(B)$ by $\sum_{i=0}^{\infty} p^{2i} k_i(B)$.

p	D	$I(B)$	classification used?	references
any	cyclic	arbitrary	no	[2]
any	abelian	$e(B) \leq 4$	no	[20, 13, 12]
any	abelian	S_3	no	[21]
≥ 7	abelian	$C_4 \times C_2$	no	[23]
$\notin \{2, 7\}$	abelian	C_3^2	no	[22]
2	metacyclic	arbitrary	no	[1, 11, 13, 16]
2	maximal class $*$ cyclic, incl. $* = \times$	arbitrary	only for $D \cong C_2^3$	[6, 19, 14, 15]
2	minimal nonabelian	arbitrary	only for one family where $\lvert D\rvert = 2^{2r+1}$	[17, 3]
2	minimal nonmetacyclic	arbitrary	only for $D \cong C_2^3$	[18]
2	$\lvert D\rvert \leq 16$	$\ncong C_{15}$	yes	[7, 10]
2	$C_4 \wr C_2$	arbitrary	no	[9]
2	$D_8 * Q_8$	C_5	yes	manuscript
2	$C_{2^n} \times C_2^3$, $n \geq 2$	arbitrary	yes	based on [24, 6, 7]
3	C_3^2	$\notin \{C_8, Q_8\}$	no	[8, 25]
3	p_-^{1+2}	arbitrary	no	based on [4]
5	p_-^{1+2}	C_2	no	based on [4]

References

[1] R. Brauer, *On 2-blocks with dihedral defect groups*, in Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), 367–393, Academic Press, London, 1974.

[2] E. C. Dade, *Blocks with cyclic defect groups*, Ann. of Math. (2) **84** (1966), 20–48.

[3] C. W. Eaton, B. Külshammer and B. Sambale, *2-Blocks with minimal nonabelian defect groups II*, J. Group Theory (to appear), DOI: 10.1515/JGT.2011.110.

[4] S. Hendren, *Extra special defect groups of order p^3 and exponent p^2*, J. Algebra **291** (2005), 457–491.

[5] L. Héthelyi, B. Külshammer and B. Sambale, *A note on Olsson's Conjecture*, submitted.

[6] R. Kessar, S. Koshitani and M. Linckelmann, *Conjectures of Alperin and Broué for 2-blocks with elementary abelian defect groups of order 8*, J. Reine Angew. Math. (to appear), DOI: 10.1515/CRELLE.2011.

[7] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, arXiv:1112.2642v1.

[8] M. Kiyota, *On 3-blocks with an elementary abelian defect group of order 9*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 33–58.

[9] B. Külshammer, *On 2-blocks with wreathed defect groups*, J. Algebra **64** (1980), 529–555.

[10] B. Külshammer and B. Sambale, *The 2-blocks of defect 4*, submitted.

[11] J. B. Olsson, *On 2-blocks with quaternion and quasidihedral defect groups*, J. Algebra **36** (1975), 212–241.

[12] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and Klein four inertial quotients*, J. Algebra **160** (1993), 192–225.

[13] L. Puig and Y. Usami, *Perfect isometries for blocks with abelian defect groups and cyclic inertial quotients of order 4*, J. Algebra **172** (1995), 205–213.

[14] B. Sambale, *Blocks with central product defect group $D_{2^n} * C_{2^m}$*, Proc. Amer. Math. Soc. (to appear), arXiv:1105.4977v1.

[15] B. Sambale, *Blocks with defect group $Q_{2^n} \times C_{2^m}$ and $SD_{2^m} \times C_{2^m}$*, submitted.

[16] B. Sambale, *Fusion systems on metacyclic 2-groups*, Osaka J. Math. (to appear), arXiv:0908.0783v3.
[17] B. Sambale, *2-Blocks with minimal nonabelian defect groups*, J. Algebra **337** (2011), 261–284.
[18] B. Sambale, *Cartan matrices and Brauer's $k(B)$-conjecture II*, J. Algebra **337** (2011), 345–362.
[19] B. Sambale, *Blocks with defect group $D_{2^n} \times C_{2^m}$*, J. Pure Appl. Algebra **216** (2012), 119–125.
[20] Y. Usami, *On p-blocks with abelian defect groups and inertial index 2 or 3. I*, J. Algebra **119** (1988), 123–146.
[21] Y. Usami, *Perfect isometries for blocks with abelian defect groups and dihedral inertial quotients of order 6*, J. Algebra **172** (1995), 113–125.
[22] Y. Usami, *Perfect isometries and isotypies for blocks with abelian defect groups and the inertial quotients isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$*, J. Algebra **182** (1996), 140–164.
[23] Y. Usami, *Perfect isometries and isotypies for blocks with abelian defect groups and the inertial quotients isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_2$*, J. Algebra **181** (1996), 727–759.
[24] A. Watanabe, *Notes on p-blocks of characters of finite groups*, J. Algebra **136** (1991), 109–116.
[25] A. Watanabe, *Appendix on blocks with elementary abelian defect group of order 9*, in Representation Theory of Finite Groups and Algebras, and Related Topics (Kyoto, 2008), 9–17, Kyoto University Research Institute for Mathematical Sciences, Kyoto, 2010.

Reduction theorems for blockwise conjectures of Alperin and McKay

Britta Späth

Two old conjectures in the modular representation theory of finite groups are the Alperin weight and the (Alperin-)McKay conjecture. Both conjectures have blockwise and non-blockwise versions. By work of Isaacs, Malle, Navarro and Tiep it has been shown that the non-blockwise versions of the conjectures hold if certain strong forms of these conjectures are satisfied for all quasi-simple groups. For a verification of these conjectures it is sufficient to check the inductive McKay or Alperin weight condition for simple groups.

We discussed how by a generalization of their methods the results can be adapted to the blockwise version of the conjectures. Starting point is the reformulation of the inductive McKay condition in [Spä10] using projective representations which leads to a simpler proof of the reduction theorem in [IMN07].

Let us denote by $\mathrm{Irr}_0(G \mid D)$ the set of characters that lie in a p-block of the finite group G with defect group D and have height zero. For a subgroup $D \leq G$ we denote by $\mathrm{Aut(G)_D}$ the group of automorphisms of G that stabilize D. The *inductive Alperin-McKay condition* for a simple group S and a prime p is concerned with the representation theory of the universal covering group G of S. For every radical p-subgroup D of G there should exists an $\mathrm{Aut(G)_D}$-stable group N with $\mathrm{N_G(D) \leq N \lneq G}$ and a bijection

$$\Omega_D : \mathrm{Irr}_0(G \mid D) \to \mathrm{Irr}_0(N \mid D),$$

that is $\mathrm{Aut(G)_D}$-equivariant. Furthermore for every $\chi \in \mathrm{Irr}_0(G \mid D)$ the characters χ and $\Omega_D(\chi)$ satisfy a technical condition on associated projective representations that implies that these characters have the "same" Clifford theory in all situations and the characters χ and $\Omega_D(\chi)$ lie in blocks with a common Brauer correspondent.

These conditions enable explicit constructions of associated projective representations and imply the following result. For a finite group X with normal subgroup K and an irreducible character χ of K the set $\mathrm{Irr}(X \mid \chi)$ is considered, where $\mathrm{Irr}(X \mid \chi)$ is the set of all irreducible constituents of the induced character χ^X.

Theorem 1. *Let S be a simple non-abelian group, which satisfies the inductive Alperin-McKay condition. Let X be a finite group, $K \lhd X$, such that $K/\mathrm{Z}(K)$ is isomorphic to a direct product of simple groups isomorphic to S. Let $D \leq K$ be a non-central radical p-subgroup. Then there exist*

(a) an $\mathrm{N}_X(D)$-stable group $N \leq K$ with $\mathrm{N_K(D) \leq N \lneq K}$, and
(b) an $\mathrm{N}_X(D)$-equivariant bijection

$$\Omega_D : \mathrm{Irr}_0(K \mid D) \to \mathrm{Irr}_0(N \mid D),$$

such that for every $\chi \in \mathrm{Irr}_0(K \mid D)$ and every p-block B of X there exists a bijection

$$\Psi_\chi : \mathrm{Irr}_0(B) \cap \mathrm{Irr}(X \mid \chi) \to \mathrm{Irr}_0(b) \cap \mathrm{Irr}(N\mathrm{N}_X(D) \mid \Omega_D(\chi)),$$

where b is the block of $N\mathrm{N}_X(D)$ such that B and b have a common Brauer correspondent.

These bijections are used to prove the following reduction theorem. Recall that a group S is *involved* in X if there exist groups $H_2 \lhd H_1 \leq X$ such that H_1/H_2 is isomorphic to S.

Theorem 2 ([Spä12])**.** *Let X be a finite group and p a prime. Assume that every simple group involved in X satisfies the inductive Alperin-McKay condition. Then the Alperin-McKay conjecture holds for X and the prime p.*

The starting point of the proof is a first reduction due to Murai. Besides the bijections from Theorem 1, results about nilpotent blocks are applied to deal with specific cases.

One can also extend the methods from [NT11] in order to obtain a reduction theorem for the blockwise Alperin weight conjecture. The inductive blockwise Alperin weight condition is a refinement of the inductive Alperin weight condition given by Navarro and Tiep, and enables to establish correspondences between various sets of Brauer characters as in Theorem 1.

Theorem 3 ([Spä11])**.** *Let X be a finite group and p a prime. Assume that every simple group involved in X satisfies the inductive blockwise Alperin weight condition. Then the blockwise Alperin weight conjecture holds for X and the prime p.*

The proof is based on an induction. A central role is played by a relative version of the blockwise Alperin weight conjecture that is closely related to relative projective modules. Besides bijections obtained from the inductive blockwise

Alperin weight condition for all involved simple groups the proof uses also the Dade-Glauberman-Nagao correspondence.

During the proof the defect group of the considered blocks can be controlled. Accordingly, we obtain as a offshoot a reduction theorem for blocks with a given defect group D: the blockwise Alperin weight conjecture holds for all blocks with defect group D if the inductive blockwise Alperin weight condition holds with respect to all p-blocks of involved simple groups, whose defect group are involved in D.

The above results give rise to the question whether the inductive conditions hold for all simple groups and all primes. As a first step to a general verification of the conditions we considered cases where the inductive McKay condition or the inductive Alperin weight condition hold.

Theorem 4 (Breuer, S.). *The inductive Alperin-McKay condition holds for S and p in the following cases*

(a) S is one of the sporadic groups,
(b) S is an alternating group, and
(c) S is a simple group of Lie type and $p \geq 5$ the characteristic of the underlying field.

In analogy to this result one can also check the blockwise Alperin weight condition in several cases using the earlier results on simple groups satisfying the inductive Alperin weight condition.

Theorem 5 (Breuer, Malle, S.). *The inductive blockwise Alperin weight condition holds for S and p in the following cases:*

(a) S is one of the 13 smallest sporadic groups,
(b) S is an alternating group,
(c) S has an abelian Sylow 2-subgroup and p is any prime, and
(d) S is a simple group of Lie type and $p \geq 5$ the characteristic of the underlying field.

References

[IMN07] I. M. Isaacs, G. Malle, and G. Navarro. A reduction theorem for the McKay conjecture. *Invent. Math.*, 170(1):33–101, 2007.

[NT11] G. Navarro and P. Tiep. A reduction theorem for the Alperin weight conjecture. *Invent. Math.*, 184:529–565, 2011.

[Spä10] B. Späth. Inductive McKay condition in defining characteristic. to appear in *Bull. London Math. Soc.*

[Spä11] B. Späth. A reduction theorem for the blockwise Alperin weight conjecture. submitted, 2011.

[Spä12] B. Späth. A reduction theorem for the Alperin-McKay conjecture, 2011. to appear in *J. reine angew. Math.*

Local cohomology of rings of invariants and the Cech complex

Peter Symonds

In a paper with Dikran Karagueuzian [1] we gave a structure theorem for a group action on a polynomial ring. This had various corollaries, among them that only finitely many isomorphism classes of indecomposable summands occur. Later we showed how this structure theorem could be used to give a bound on the degrees of the generators of the invariant subring of $n(|G|-1)$, where n is the number of variables and G is the group (provided $n, |G| \geq 2$) [2].

However, the proof of the structure theorem is long and complicated. Here we sketch a more conceptual approach that proves both of these results at the same time. It also works for a slightly more general class of rings than just polynomial rings. It works by considering the Cech complex of the polynomial ring with respect to an invariant system of parameters as a complex of modules for the group.

Theorem *The Cech complex is split as a module for the group in degrees greater than $-n$.*

By taking invariants, we see that the Cech complex of the ring of invariants is exact in degrees greater than $-n$, hence its regularity is at most 0. It is shown in [2] that this proves the result about degrees of generators. The result about the indecomposable summands also follows easily.

References

[1] D. Karagueuzian, P. Symonds, *The module structure of a group action on a polynomial ring: a finiteness theorem.* Jour. Amer. Math. Soc. **20** (2007), 931–967.

[2] P. Symonds, *On the Castelnuovo-Mumford Regularity of rings of polynomial invariants*, Ann. Math. **174** (2011) 499-517.

Representations of finite quasisimple groups and the Waring problem

Pham Huu Tiep

(joint work with Michael Larsen and Aner Shalev)

A classical result of Lagrange, respectively Wieferich, shows that every positive integer is a sum of (at most) four squares, respectively nine cubes. The *Waring problem* in number theory generalizes this, asking whether there is a function $g(k)$ such that every positive integer is a sum of (at most) $g(k)$ k^{th} powers. This problem was solved in the affirmative by Hilbert in 1909.

In the past 15 years non-commutative analogues of the Waring problem have been considered, and various interesting results have been obtained, with particular emphasis on finite (non-abelian) simple groups. Martinez and Zelmanov [9], and independently Saxl and Wilson [10], showed that any element of a finite simple group G is a product of $f(k)$ k^{th} powers, provided there are non-trivial k^{th} powers in G.

Recall that a word $w = w(x_1, \ldots, x_d)$ is an element of the free group F_d on $x_1, \ldots, x_d$. Given a word w and a group G we consider the word map $w_G : G^d \to G$ obtained by substituting group elements $g_1, \ldots, g_d$ in $x_1, \ldots, x_d$ respectively. Let $w(G) \subseteq G$ denote the image of this map. For subsets $S, T \subseteq G$ we set $ST = \{st : s \in S, t \in T\}$; in particular $S^k = \{s_1 \ldots s_k : s \in S\}$.

More generally, the *non-commutative Waring problem* asks whether, given a word w, there is $c = c(w)$ such that $w(G)^c = G$ for all finite simple groups G with $w(G) \neq \{1\}$. Extending the aforementioned results on powers and commutators, Liebeck and Shalev [6] showed in 2001 that such $c(w)$ indeed exists, but no explicit bounds on $c(w)$ were given.

Later it turned out that, if G is large enough (given $w \neq 1$), then $c(w)$ does not depend on w, and is in fact surprisingly small. Indeed Shalev [11] showed that for any non-trivial word w there exists a number $N(w)$ such that if G is a finite non-abelian simple group of order at least $N(w)$ then $w(G)^3 = G$.

Building on the results of [11], [2], and [3], we have been able to prove the following theorem in [4], which gives a best possible solution to the non-commutative Waring problem:

Theorem 1. [4] (i) *Let $w \in F_d$ be a non-trivial word in the free group on d generators. Then there exists a constant $N = N(w)$ depending on w such that for all finite non-abelian simple groups G of order greater than N we have $w(G)^2 = G$.*

(ii) *Let $w_1, w_2 \in F_d$ be non-trivial words in the free group on d generators. Then there exists a constant $N = N(w_1, w_2)$ depending on w_1, w_2 such that for all finite non-abelian simple groups G of order greater than N we have $w_1(G)w_2(G) = G$.*

Our proof of Theorem 1 involves both algebro-geometric and representation-theoretic tools. In particular, we prove a *Chebotarev Density Theorem* for word maps (see [4, Theorem 5.3.2]). We also define the notion of the *support* of any element in a finite classical group which measures how far g is from being scalar, cf. [4, Definition 4.1.1], and prove the following result which is also of independent interest:

Theorem 2. [4] *If G is a finite quasisimple classical group over $\mathbb{F}_q$ and $g \in G$ is an element of support at least N, then $|\chi(g)|/\chi(1) < q^{-\sqrt{N}/481}$ for all non-principal irreducible characters χ of G.*

Extending the aforementioned results to quasisimple groups, we prove in [5]:

Theorem 3. [5] (i) *Let $w \in F_d$ be a non-trivial word in the free group on d generators. Then there exists a constant $N = N(w)$ depending on w such that for all finite quasisimple groups G of order greater than N we have $w(G)^3 = G$.*

(ii) *Let $w_1, w_2, w_3 \in F_d$ be non-trivial words in the free group on d generators. Then there exists a constant $N = N(w_1, w_2, w_3)$ depending on w_1, w_2, w_3 such that for all finite quasisimple groups G of order greater than N we have $w_1(G)w_2(G)w_3(G) = G$.*

(iii) *Let $k > 2$ be any integer and let $w(x) = x^k$. Then there are finite quasisimple groups G of arbitrarily large order such that $\{1\} \neq w(G)^2 \neq G$.*

For the non-commutative Waring problem in the case of *powers*, i.e. when $w(x) = x^k$, or *commutators*, i.e. $w(x,y) = xyx^{-1}y^{-1}$, one can in fact establish stronger statements. Indeed, the main result of [7] shows that *the Ore conjecture holds for every finite non-abelian simple group G, that is, every element in G is a commutator.* Furthermore, it is proved in [1] that *if $k > 1$ is a power either of a prime or of* 6*, then every element in any finite non-abelian simple group G is a product of two k^{th} powers.* (The latter result in the case $k \neq 3, 5, 7$ is a prime has been proved independently in [8].)

It turns out that, among all words, the word x^2y^2 behaves the best in quasisimple groups:

Theorem 4. [5] *Every element in any finite quasisimple group G is a product of two squares.*

References

[1] R. M. Guralnick and G. Malle, *Product of conjugacy classes and fixed point spaces*, J. Amer. Math. Soc. **25** (2012), 77–121.

[2] M. Larsen and A. Shalev, *Word maps and Waring type problems*, J. Amer. Math. Soc. **22** (2009), 437–466.

[3] M. Larsen and A. Shalev, *Characters of symmetric groups: sharp bounds and applications*, Invent. Math. **174** (2008), 645–687.

[4] M. Larsen, A. Shalev, and Pham Huu Tiep, *The Waring problem for finite simple groups*, Annals of Math. **174** (2011), 1885–1950.

[5] M. Larsen, A. Shalev, and Pham Huu Tiep, *Waring problem for finite quasisimple groups*, Int. Math. Res. Notices (to appear).

[6] M. W. Liebeck and A. Shalev, *Diameters of finite simple groups: sharp bounds and applications*, Annals of Math. **154** (2001), 383–406.

[7] M. W. Liebeck, E. A. O'Brien, A. Shalev, and Pham Huu Tiep, *The Ore conjecture*, J. Europ. Math. Soc. **12** (2010), 939–1008.

[8] M. W. Liebeck, E. O'Brien, A. Shalev, and Pham Huu Tiep, *Products of squares in finite simple groups*, Proc. Amer. Math. Soc. **140** (2012), 21–33.

[9] C. Martinez and E. I. Zelmanov, *Products of powers in finite simple groups*, Israel J. Math. **96** (1996), 469–479.

[10] J. Saxl and J.S. Wilson, *A note on powers in simple groups*, Math. Proc. Camb. Phil. Soc. **122** (1997), 91–94.

[11] A. Shalev, *Word maps, conjugacy classes, and a non-commutative Waring-type theorem*, Annals of Math. **170** (2009), 1383–1416.

Checking Lusztig's conjecture around the Steinberg weight

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Let $(R \subset X, R^\vee \subset X^\vee)$ be a root datum and $G \supset T$ the corresponding split Chevalley group scheme over $\mathbb{Z}$. Fix an algebraically closed field k. A fundamental question in representation theory is to determine the simple rational modules of G_k. Here "determine" means: Can they be parametrised? What are their dimensions? What are their characters? Can one give a uniform construction/description? etc.

Let p denote the characteristic of k. In characteristic $p = 0$ there exists a uniform construction and description. To describe it we choose a system of positive roots $R^+ \subset R$, a basis $\Delta \subset R^+$ and let $B \subset G$ be the Borel subgroup corresponding to the negative roots $R^- = -R^+$. To each $\lambda \in X$ one can associate a line bundle $\mathcal{L}(\lambda) := G \times_B k_\lambda$ over the flag variety $(G/B)_k$. Here k_λ denotes the B-module which is obtained by inflation from the one-dimensional T-module given by the character $\lambda \in X$ of T. If we set

$$\nabla(\lambda) = H^0(G/B, \mathcal{L}(\lambda))$$

then it is known that $\nabla(\lambda)$ is non-zero if and only if λ belongs to the cone of dominant characters:

$$X^+ = \{\lambda \in X \mid \langle \alpha^\vee, \lambda \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Moreover, in the later case $\nabla(\lambda)$ is a simple G_k-module. One obtains in this way a bijection

$$X^+ \xrightarrow{\sim} \mathrm{Irr} G_k$$

where $\mathrm{Irr}G_k$ denotes the set of isomorphism classes of simple rational G_k-modules. Moreover, the characters are given by Weyl's character formula.

If $p > 0$ the situation is more complicated. Let us assume for simplicity that k is an algebraic closure of $\mathbb{F}_p$, the finite field with p elements. It is a priori obvious that things will be more complicated than (or at the very least different to) the characteristic zero case. The reason is that we have a Frobenius map

$$\mathrm{Fr} : G_k \to G_k$$

obtained by elevating coordinates to the p^{th} power. Hence given any G_k-module V, we can produce another G_k-module $V^{(1)}$ (or in fact infinitely many new modules $V^{(m)}$) by precomposing (m times) with the Frobenius morphism. This operation is called "Frobenius twist". It is easy to see, for example, that it preserves simple modules. It leads to a recursive structure on the category of rational representations of G_k which is only partly understood, and is a big part of the fascination of the subject.

In positive characteristic one may still define the modules $\nabla(\lambda)$ as above, but they are not in general simple. However they contain a unique simple module $L(\lambda)$. It turns out that $L(\lambda)$ still has highest weight λ and hence the above bijection between simple modules and X^+ and $\mathrm{Irr}G_k$ remains true. However the dimensions and characters of $L(\lambda)$ are not known. By Kempf's vanishing theorem one still knows the characters of the module $\nabla(\lambda)$ (as in the case $p = 0$ they are given by Weyl's character formula). Much of the recent work on determining the simple G_k modules focuses on understanding the composition series for the modules $\nabla(\lambda)$.

A cornerstone of the subject are two theorems of Steinberg. Consider the set

$$X_r^+ = \{\lambda \in X^+ \mid \langle \alpha^\vee, \lambda + \rho \rangle < p^r \text{ for all } \alpha \in \Delta\}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$. Steinberg's tensor product theorem asserts that if we have weights λ_0, λ_1, ..., λ_m all belonging to X_1^+ then the module

$$L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_m)^{(m)}$$

is simple (and is hence isomorphic to $L(\lambda_0 + p\lambda_1 + \cdots + p^m\lambda_m)$). Hence, in order to understand the simple representations of G_k it is enough to understand $L(\lambda)$ with $\lambda \in X_1^+$.

The second important theorem is Steinberg's restriction theorem. It states that for any $\lambda \in X_r^+$, the restriction of $L(\lambda)$ to the finite group $G(\mathbb{F}_{p^r})$ is simple. Moreover on obtains all simple $kG(\mathbb{F}_{p^r})$-modules in this way. This theorem is probably why we are discussing rational G_k-modules at a conference on finite groups!

To get to Lusztig's conjecture we need to recall two more pieces of structure theory. Consider W_p, the subgroup of all affine transformations of X generated by reflections in the hyperplanes $H_{\alpha,pm} = \langle\alpha^\vee, \lambda+\rho\rangle \in pm$ for all $\alpha \in R^+$ and $m \in \mathbb{Z}$. The linkage principle is the statement:

$$\mathrm{Ext}^1(L(\lambda), L(\mu)) \neq 0 \Rightarrow \lambda \in W_p \cdot \mu.$$

Denote by $\mathrm{Rep}G_k$ the category of all rational representations of G_k. Given $\pi \subset X^+$ let $\mathrm{Rep}_\pi G_k$ denote the full subcategory of all objects whose composition factors belong to $\{L(\lambda) \mid \lambda \in \pi\}$. The linkage principle implies that we have a decomposition:

$$\mathrm{Rep}G_k = \bigoplus_{\pi \in X/W_p} \mathrm{Rep}_{\pi\cap X^+}G_k.$$

Finally, the translation principle states that, as long as $p > h$ (so that 0, the weight of the trivial module, lies on no hyperplane $H_{\alpha,pm}$) we understand all multiplicities $[\nabla(\lambda) : L(\mu)]$ as long as we understand the multiplicities $[\nabla(x\cdot 0) : L(y \cdot 0)]$ for $x, y \in W_p$ with $x \cdot 0, y \cdot 0 \in X^+$. Steinberg's tensor product theorem even allows us to assume that $x \cdot 0, y \cdot 0 \in X_1^+$.

Lusztig's conjecture then expresses

$$[\nabla(x \cdot 0) : L(y \cdot 0)] \text{ for } x \cdot 0, y \cdot 0 \in X_1^+$$

in terms of an affine Kazhdan-Lusztig polynomial. It is important to note that the assumption that $x \cdot 0, y \cdot 0 \in X_1^+$ is essential: even though the statement makes sense for any $x \cdot 0, y \cdot 0 \in X^+$ it is certainly false in full generality (even though it will remain true within the "Jantzen region").

Let us pause to note that even if Lusztig's conjecture is true this is a slightly unsatisfactory state of affairs. For weights outside of the X_1^+ there is no conceptual understanding of the situation: in order to calculate a character many iterations of Steinberg's tensor product theorem and Lusztig's conjecture may be necessary.

It is a result due to Andersen, Jantzen and Soergel that Lusztig's conjecture is true for large p, and an effective (but enormous) bound has recently been provided by Fiebig. On the other hand, there is very little experimental evidence for the validity of Lusztig's conjecture. It is known in rank 2 (using Jantzen's sum formula), for a few rank 3 cases, for a few primes in higher rank ...

In the late 1990's Soergel suggested that a useful toy-model for Lusztig's conjecture would be provided by looking "around the Steinberg weight". To be precise, assume from now on that $p > h$, let W denote the Weyl group of our root system acting on X in the standard way and let $\mathrm{st} = (p-1)\rho$ denote the Steinberg weight (the extremal vertex of the fundamental box X_1). Consider the sets

$$\Omega = \{\mathrm{st} + x\rho \mid x \in W\},\ \leq \Omega = \{\lambda \mid \lambda \leq p\rho\} \text{ and } < \Omega = \leq \Omega \setminus \Omega.$$

Now consider the quotient category

$$\mathcal{O}_p = \mathrm{Rep}_{\leq \Omega} G_k / \mathrm{Rep}_{<\Omega} G_k.$$

We let

$$L(x) := \overline{L(\mathrm{st} + x\rho)}, \quad \nabla(x) := \overline{\nabla(\mathrm{st} + x\rho)}$$

denote the images in $\mathcal{O}_p$. Then $\mathcal{O}_p$ is a finite length, abelian, highest weight category and Lusztig's conjecture predicts

$$[\nabla(x) : L(y)] = h_{w_0 y, w_0 x}(1)$$

where w_0 denotes the longest element of W and $h_{w_0 y, w_0 x} \in \mathbb{Z}[v]$ is a Kazhdan-Lusztig polynomial (this time for the finite Weyl group). Let us emphasise that (as far as we know) the above statement is weaker than the original statement. As we mentioned above, the above should be thought of as a toy model or "sanity check" for Lusztig's original conjecture.

Recently (building on the work of Soergel and Elias-Khovanov) Elias and the author proved the existence of a $\mathbb{Z}$-algebra A which is a free and finitely generated over $\mathbb{Z}$ such that

i) $A_{\mathbb{C}} - \mathrm{mod} \cong \mathcal{O}_0$ "principal block of category $\mathcal{O}$" for $\mathfrak{g} = \mathrm{Lie}\, G_{\mathbb{C}}$.
ii) $A_k - \mathrm{mod} \cong \mathcal{O}_k$ "modular category $\mathcal{O}$".

Moreover, A may be described by generators and relations. Let us make the following remakes:

a) One can think of A as interpolating between characteristic zero representation theory ($\mathcal{O}_0$ is where Kazhdan-Lusztig polynomials made their first appearance "in nature") and modular representation theory. Hence one can think of the above result as "freeing p".
b) A admits a grading and hence $\mathcal{O}_p$ admits a grading $\widetilde{\mathcal{O}_p}$. With Riche and Soergel we have recently proved a "modular Koszul duality":

$$D^b(\widetilde{\mathcal{O}_p}) \cong D^b_{(B^\vee_{\mathbb{C}})}\widetilde{(G^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}, k)}$$

c) A_k is Morita equivalent to an ext algebra of parity sheaves on $G^\vee_{\mathbb{C}}/B^\vee_{\mathbb{C}}$.
d) By results of Fiebig there is also a version of A which controls the full Lusztig conjecture (related to the affine Weyl group, rather than the finite Weyl group). It is unclear to the author what this category has to do with the whole principal block of Rep G_k.

Following the "freeing p" reasoning, one can also study the representation theory of A in characteristics below the Coxeter number (where $\mathcal{O}_p$ stops behaving well). Consider the statement

$$(*)_p : \qquad \text{the decomposition matrix of } A \text{ is trivial.}$$

Because of ii) above, Lusztig's conjecture would imply that $(*)_p$ is true for $p > h$.

Using the explicit description of A, we can do computer calculations to check $(*)_p$ in low rank. Here is a summary of the cases where $(*)_p$ holds:

A_n	B_n	D_n	F_4	G_2	E_6 (partial)
all p for $n < 6$ $p \neq 2$ for $n = 7$	$p \neq 2$ for $n < 6$	$p \neq 2$ for $n < 6$	$p \neq 2, 3$	$p \neq 2, 3$	$p \neq 2, 3$

The entry $p \neq 2$ in A_7 is due to Braden (2002). The exclusions $p \neq 2, 3$ in E_6 are due to Polo and Riche. The entries $p \neq 3$ for F_4 and E_6 give a counterexample to Fiebig's "GKM-conjecture". Thanks are also due to Jean Michel for help speeding up my programs significantly.

Recently Polo has found an example to show that $(*)_p$ fails in A_{4p-1}. So the situation is more complicated than one might have thought ...

References

[1] B. Elias, M. Khovanov, *Diagrammatics for Soergel bimodules*, Int. J. Math. Math. Sci., 2011.

[2] B. Elias, G. Williamson, *Generators and relations for Soergel bimodules*, in preparation.

[3] N. Libedinsky, *Presentation of right-angled Soergel categories by generators and relations*, J. Pure Appl. Algebra 214 (2010), no. 12, 2265–2278.

[4] S. Riche, W. Soergel, G. Williamson, *Modular Koszul duality*, in preparation.

[5] W. Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*. J. Pure Appl. Algebra, 2000.

[6] G. Williamson, (with an appendix by T. Braden) *Modular intersection cohomology complexes on flag varieties*, Math. Z. (2011).

[7] G. Williamson, *Some examples of p-canonical bases*, in preparation.

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