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Mini-Workshop: Endomorphisms, Semigroups and C*-Algebras of Rings

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Abstract. The main aim of the workshop was to explore recent progress in the study of endomorphisms of C^*-algebras, semigroup crossed products, graph algebras, ring C^*-algebras, purely infinite C^*-algebras and related algebraic constructions, such as dilations or Leavitt path algebras, by bringing together experts from several different fields.

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Introduction by the Organisers

The study of actions of semigroups and groups on C^*-algebras as well as the study of individual endomorphisms and automorphisms of C^*-algebras is of central importance in operator algebra theory and has a long tradition in the subject. An important basis for this is the fact that one can associate a C^*-algebra, the crossed product, to such an action which reflects the dynamical behaviour of the action and of the algebra. This construction is an inexhaustible source of interesting examples of C^*-algebras. Whilst for automorphic actions the concept of a crossed product is well understood and established, the theory of crossed products by endomorphisms is much more subtle and has been subject of research in the past 30 years. There are many related constructions in the literature, such as Pimsner algebras, which can be regarded as crossed products by correspondences, and the intensively studied class of graph algebras.

Recently there is a renewed interest in the study of endomorphisms of C^*-algebras and in related constructions such as crossed products by semigroups or the C^*-algebras associated with the left or right regular representation of a semigroup. This new interest is mainly due to many intriguing explicit examples provided by structures from other fields such as number theory or ergodic theory. In a recent development the dilation theory for such semigroup crossed products made it possible to use the Baum-Connes conjecture to determine the K-theory of such semigroup crossed products in many cases.

Many of these constructions produce examples of interesting C^*-algebras which tend to be purely infinite simple as well as nonsimple and thus provide important potential examples for the classification of simple and non-simple purely infinite C^*-algebras.

There have also been remarkable advances in the study of individual endomorphisms and automorphisms as well as groups and subgroups of the automorphism group of special C^*-algebras in particular Cuntz algebras and other related C^*-algebras such as graph algebras.

As a mostly new development during the past decade, algebraists study algebraic counterparts of the constructions of C^*-algebras listed above. They have extended K-theory computations from the realm of C^*-algebras to this algebraic setting. An interesting line of investigations is to find out to what extent the classification of the C^*-algebraic constructions via K-theory prevails in this purely algebraic setting. Many further developments in this and the previously mentioned areas are to be expected in the near future.

The workshop gave the unique opportunity to bring together experts from different areas to discuss and lecture about all these different developments. It is a pleasure to thank the Mathematisches Forschungsinstitut Oberwolfach for providing this opportunity and a fantastic environment for this meeting with an ever smooth organisation and support. Special thanks go to the very competent and helpful staff of the institute and to the chef de cuisine.

It is also a pleasure for the organisers to thank all participants of the workshop for their contributions in lectures held at the workshop and the stimulating discussions following or in between the lectures, which made this workshop an ample success.

Mini-Workshop: Endomorphisms, Semigroups and C*-Algebras of Rings

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Abstracts

Dilations and full corners

Marcelo Laca

This is a summary of the contents of my two talks at the workshop. The main results discussed here have appeared in [5] and in further recent work [6, 9, 7, 2, 3]. The first section corresponds roughly to the first talk and consists of general facts about crossed products by semigroups of endomorphisms and their relation to crossed products by groups of automorphisms. Via this relation, much of the powerful machinery available for group crossed products can be used on semigroup crossed products. Several examples of such applications are given in the second section, which corresponds, roughly again, to the second talk.

By a monoid or a semigroup (with identity) we mean a multiplicatively closed subset of a group. Such a subset defines a left and a right partial orders on the group. When the group G acts on a C*-algebra B by automorphisms, the role of the semigroup can be assimilated by considering a distinguished subalgebra $A \subset B$ that is invariant under the action restricted to S; alternatively, one may simply consider the endomorphic action of S on the subalgebra A (by injective endomorphisms). We shall see that under some assumptions, these two points of view are equivalent. When one cuts down to the invariant subalgebra and restricts the attention to the action of the positive cone in the left order, automorphisms become endomorphisms. The process is reversed by a localization-type construction that enlarges the C*-algebra A and extends the endomorphisms to automorphisms.

These results are also valid for projective isometric representations and twisted crossed products with circle-valued multipliers. Indeed, one of the original motivations was the problem of extending multipliers from a semigroup to a group, following work of Arveson and Dinh on twisted units of product systems. For simplicity, twists are not discussed here; those details are to be found in [5].

1. Semigroup crossed products

1.1. **Covariant representations.** Suppose A is a unital C*-algebra and let α be an action of the semigroup S by endomorphisms of A (not necessarily unit-preserving). A *covariant representation* of the semigroup dynamical system (A, S, α) on a C*-algebra C is a pair (π, V) in which

(1) π is a unital homomorphism of A to C,
(2) $V : S \to C$ is a representation of S by isometries in C, i.e., $V_s V_t = V_{st}$, $V_s^* V_s = I$ and
(3) the covariance condition $\pi(\alpha_t(a)) = V_t \pi(a) V_t^*$ holds for every $a \in A$ and $t \in S$.

The *(semigroup) crossed product* associated to (A, S, α) is a C*-algebra $A \rtimes_\alpha S$ together with a unital homomorphism $i_A : A \to A \rtimes_\alpha S$ and a representation of S by isometries $i_S : S \to A \rtimes_\alpha S$ such that

(1) (i_A, i_S) is a covariant representation for (A, S, α),
(2) for any other covariant representation (π, V) there is a representation $\pi \times V$ of $A \rtimes_\alpha S$ such that $\pi = (\pi \times V) \circ i_A$ and $V = (\pi \times V) \circ i_S$, and
(3) $A \rtimes_\alpha S$ is generated by $i_A(A)$ and $i_S(S)$ as a C*-algebra.

This definition was originally motivated by (and tailored for) the study of Toeplitz algebras [4]. Specifically, if S is a cancellative semigroup with identity, the C*-algebra generated by the left regular representation can often be realized as a semigroup crossed product $B_S \rtimes S$ where B_S is an abelian (diagonal) algebra generated by certain projections in $\ell^\infty(S)$.

1.2. **Ore semigroups.** A classical result of Ore asserts that a monoid S embeds in a group G with $S^{-1}S = G$ iff S is cancellative and $Ss \cap St \neq \emptyset$ for every pair $s, t \in S$, i.e. S is an *Ore monoid*. In this case, the group G is determined by S, and every monoid morphism φ of S to a group $\mathcal{G}$ extends uniquely to a group homomorphism $\tilde{\varphi} : G \to \mathcal{G}$ defined by $\tilde{\varphi}(x^{-1}y) = \varphi(x)^{-1}\varphi(y)$. An equivalent form of the Ore condition is that the *right (partial) order* defined by

$$s \preceq_r t \quad \text{if} \quad t \in Ss$$

is *cofinal* in the sense that for every pair $x, y \in S$ there exists $z \in S$ such that $x \preceq_r z$ and $y \preceq_r z$.

1.3. **Examples.** Various Ore monoids that appear in the context of semigroup actions include: subsemigroups of abelian groups; pullbacks of positive cones from totally ordered quotients; normal semigroups (i.e those for which $xS = Sx$ for all $x \in S$); semidirect products such as the "$ax + b$" (or affine) semigroup $R \rtimes R^\times$ with R the ring of algebraic integers in a number field; integer matrices with positive determinant; and Artin groups of finite type (i.e. those with finite Coxeter quotients) as introduced by Brieskorn and Saito and by Deligne. In contrast, nonabelian free monoids, and more generally, Artin monoids of rectangular type do not satisfy the Ore condition.

It is immediate that the monomials $v_x^* a v_y$ with $a \in A$, and $x, y \in S$, generate the crossed product $A \rtimes S$ as a C*-algebra, but when S is an Ore semigroup these monomials are closed under multiplication. Specifically, given y and r there exist t and z in S such that $ty = zr$, hence

$$\begin{aligned} v_x^* a v_y v_r^* b v_s &= v_x^* a v_y v_{ty}^* v_{zr} v_r^* b v_s = v_x^* a v_y v_y^* v_t^* v_z v_r v_r^* b v_s \\ &= v_x^* v_t^* \alpha_t(a\alpha_y(1))\alpha_z(\alpha_r(1)b) v_z v_s = v_{tx}^* \alpha_t(a\alpha_y(1))\alpha_z(\alpha_r(1)b) v_{zs}. \end{aligned}$$

The linear span of such monomials is thus a dense *-subalgebra of $A \rtimes S$.

1.4. **Dilations and extensions.** The first main result is a dilation theorem for semigroups of isometries and the second one is an extension of endomorphisms to automorphisms. Then a dilation of the covariant representations leads to the realization of the semigroup crossed product as a corner. The proofs rely on (twisted) direct limit constructions of the relevant structures.

Theorem 1. *Suppose S is an Ore semigroup and let $G = S^{-1}S$. Let $\{V_s : s \in S\}$ be an isometric representation of S on a Hilbert space H. Then there exists a unitary representation of G on a Hilbert space $\mathcal{H}$ containing a copy of H such that*

(i) U_s leaves H invariant and $U_s|_H = V_s$; and
(ii) $\bigcup_{s\in S} U_s^* H$ is dense in $\mathcal{H}$.

Moreover, U and $\mathcal{H}$ are unique up to canonical isomorphism.

Let us recall that if p is a projection in a C*-algebra A then the algebra pAp is a *corner* in A; the projection p is said to be *full* if the linear span of ApA is dense in A. The most relevant feature of full corners is that if pAp is a full corner in A, then pA is a full Hilbert bimodule implementing the Morita equivalence, in the sense of Rieffel, of pAp to A. This Morita equivalence allows one to transfer the technology of crossed products by group actions to the semigroup crossed products.

There are two steps in realizing a semigroup crossed product as a corner in a crossed product by a group action. The first one is the extension of a semigroup of (injective) endomorphisms to a group of automorphisms, and the second one is the corresponding dilation-extension of covariant representations from the semigroup dynamical system to the dilated system. This generalizes results from [10].

Theorem 2. *Assume S is an Ore monoid and let $G = S^{-1}S$. Let α be an action of S by injective endomorphisms of the unital C*-algebra A. Then*

(1) there exists a C-dynamical system (B, G, β), unique up to canonical isomorphism, consisting of an action β of G by automorphisms of a C*-algebra B and an embedding $i : A \to B$ such that*
  (a) β dilates α, that is, $\beta_s \circ i = i \circ \alpha_s$ for $s \in S$, and
  (b) (B, G, β) is minimal, that is, $\bigcup_{s\in S} \beta_s^{-1}(i(A))$ is dense in B;
(2) the projection $i(\mathbf{1}_A)$ is full in $B \rtimes_\beta G$;
(3) the semigroup crossed product $A \rtimes_\alpha S$ is canonically isomorphic to the full corner $i(\mathbf{1}_A)(B \rtimes_\beta G)i(\mathbf{1}_A)$.

The system (B, G, β) characterized by conditions (1)(a) and (1)(b) in the theorem is called the *minimal automorphic dilation* of (A, S, α). The crossed product $B \rtimes_{\beta,\mu} G$ of this minimal automorphic dilation is thus Morita equivalent to the semigroup crossed product $A \rtimes_{\alpha,\lambda} S$.

Without the injectivity assumption it is still possible to carry out the above constructions; however, the resulting homomorphism $i : A \to B$ may not be an embedding any more, e.g. see Example 2.1(a) of [11] where the limit algebra B is trivial. As in [11, Proposition 2.2] which deals with the case $S = \mathbb{N}$, we conclude that the crossed product $A \rtimes_\alpha S$ is nontrivial if and only if $B \neq \{0\}$. Clearly, this is the case when, for instance, the endomorphisms are injective.

2. Examples and Applications

2.1. **Integral adeles under multiplication by $\mathbb{N}^\times$.** It follows from [8, Corollary 2.10] that the Bost-Connes Hecke C*-algebra $\mathcal{C}_\mathbb{Q}$ is canonically isomorphic to the semigroup crossed product $C(\mathcal{Z}) \rtimes \mathbb{N}^\times$ where $\mathcal{Z} := \prod_p \mathbb{Z}_p$ is the ring of integral adeles, and the endomorphisms α_n consist of 'division by n' in $\prod_p \mathbb{Z}_p$:

$$\alpha_n(f)(x) = \begin{cases} f(x/n) & \text{if } x \in n\mathcal{Z} \\ 0 & \text{otherwise.} \end{cases}$$

The diagonally embedded natural numbers $\mathbb{N}^\times \hookrightarrow \mathcal{Z}$ form a multiplicative set with no zero-divisors, so we may localize $\mathcal{Z}$ at $\mathbb{N}^\times$, obtaining a locally compact ring $(\mathbb{N}^\times)^{-1}\mathcal{Z}$ in which division by an element of $\mathbb{N}^\times$ is always possible. This allows us to extend the endomorphism α_n to an automorphism β_n. Clearly $(\mathbb{N}^\times)^{-1}\mathcal{Z}$ is the locally compact ring $\mathbb{A}_f$ of finite adeles, which has $\mathcal{Z}$ as its maximal compact open subring, and there is an action β of $\mathbb{Q}_+^* = (\mathbb{N}^\times)^{-1}\mathbb{N}^\times$ by automorphisms of $C_0(\mathbb{A}_f)$ arising from the diagonal embedding $\mathbb{Q}_+^* \hookrightarrow \mathbb{A}_f$:

$$\beta_r(f)(a) = f(r^{-1}a), \quad a \in \mathbb{A}_f, r \in \mathbb{Q}_+^*.$$

Since $\mathcal{Z}$ is compact and open, its characteristic function $\mathbf{1}_\mathcal{Z}$ is a projection in $C_0(\mathbb{A}_f)$ and there is an obvious embedding i of $C(\mathcal{Z})$ as the corresponding ideal of $C_0(\mathbb{A}_f)$, namely, for $f \in C(\mathcal{Z})$, let

$$i(f)(a) = \begin{cases} f(a) & \text{if } a \in \mathcal{Z} \\ 0 & \text{if } a \notin \mathcal{Z}. \end{cases}$$

A direct application of Theorem 2, using uniqueness, gives the following.

Theorem 3. *The system $(C_0(\mathbb{A}_f), \mathbb{Q}_+^*, \beta)$ is the minimal automorphic dilation of $(C(\mathcal{Z}), \mathbb{N}^\times, \alpha)$. Hence the Bost-Connes algebra $\mathcal{C}_\mathbb{Q}$ is (isomorphic to) the full corner of $C_0(\mathbb{A}_f) \rtimes_\beta \mathbb{Q}_+^*$ determined by the projection $\mathbf{1}_\mathcal{Z}$.*

Since the discrete multiplicative group $\mathbb{Q}_+^*$ acts by homotheties on the locally compact (additive) group $\mathbb{A}_f$, and since the self-duality of $\mathbb{A}_f$ satisfies $\langle rx, y\rangle = \langle x, ry\rangle$ for $r \in \mathbb{Q}_+^*$, then $C^*(\mathbb{A}_f)$ is covariantly isomorphic to $C_0(\mathbb{A}_f)$ and hence $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^* \cong C^*(\mathbb{A}_f) \rtimes \mathbb{Q}_+^* \cong C^*(\mathbb{A}_f \rtimes \mathbb{Q}_+^*)$. Hence $\mathcal{C}_\mathbb{Q}$ is a full corner also in the group C*-algebra of $\mathbb{A}_f \rtimes \mathbb{Q}_+^*$, corresponding to the Fourier transform $e_\mathcal{Z} \in C^*(\mathbb{A}_f)$ of $\mathbf{1}_\mathcal{Z} \in C_0(\mathbb{A}_f)$.

2.2. **Neshveyev's approach to uniqueness and type of KMS states.** Each KMS_β state of the BC algebra $\mathcal{C}_\mathbb{Q}$ gives a probability measure on $\mathcal{Z}$ through the unital embedding $C(\mathcal{Z}) \hookrightarrow C(\mathcal{Z}) \rtimes \mathbb{N}^\times \cong \mathcal{C}_\mathbb{Q}$. Using the scaling property of KMS states this probability can be uniquely extended to an (infinite) positive measure μ_β on $\mathbb{A}_f$ quasi-invariant under $\mathbb{Q}_+^*$. For each of these measures μ_β, the (measure theoretic) dynamical system $(\mathbb{A}_f, \mathbb{Q}_+^*, \mathrm{Mult.}, \mu_\beta)$ determines a von Neumann algebra crossed product $L^\infty(\mathbb{A}_f, \mu_\beta) \rtimes \mathbb{Q}_+^*$. These are examples of the Murray-von Neumann group-measure space construction, for which there are known criteria of factoriality and type. In [9] Neshveyev showed that for $\beta \in (0, 1]$ the dilated action of $\mathbb{Q}_+^*$ is ergodic with respect to μ_β since μ_β is quasi invariant but has no atoms and no equivalent invariant measure, the associated KMS_β state of the full corner is unique and a factor of type III. Further analysis of the action of $\mathbb{Q}_+^*$ on the full adeles show that the type is III_1, providing a simplification of the original proof of Bost and Connes.

2.3. **Affine monoids of rings.** Let R be the ring of integers in an algebraic number field, and let $R^\times := R \setminus \{0\}$ be the nonzero elements. In [2] we studied the C*-algebra $\mathcal{T}(R \rtimes R^\times)$ generated by the left regular representation of $R \rtimes R^\times$. Note that $S^{-1}S$ is a group but SS^{-1} is not, so the C*-algebras of the left and the right regular representations of S are quite different. We show that $\mathcal{T}(R \rtimes R^\times)$ is semigroup crossed product of the form $C(\Omega_R) \rtimes (R \rtimes R^\times)$. To construct the compact space Ω_R, we first let $\hat{R}$ be the profinite completion of R under the projective system $R/bR \to R/aR$ when $a^{-1}b \in R$ (this projective limit is the space of integral adeles over K; e.g. it yields $\hat{R} = \prod_p \mathbb{Z}_p$ when $K = \mathcal{Q}$). Next take $\hat{R} \times \hat{R}$ modulo the equivalence relation

$$(r,a) \sim (s,b) \quad \Longleftrightarrow \quad a\hat{R}^* = b\hat{R}^* \text{ and } s - r \in a\hat{R}, \tag{3.1}$$

where $\hat{R}^*$ denotes the group of units (i.e. invertible elements) in $\hat{R}$. The resulting compact space Ω_R has an obvious action of $R \rtimes R^\times$ given by

$$(x,k) \cdot \omega_{r,a} = \omega_{x+kr,ka} \qquad (x,k) \in R \rtimes R^\times.$$

It is now easy to find a good candidate for the minimal dilation. Let $\mathbb{A}_f$ be the locally compact, totally disconnected ring of finite adeles over K, which can be regarded as the localization $(R^\times)^{-1}\hat{R}$ of the ring $\hat{R}$ at the multiplicative set $R^\times$. The space Ω_A is the quotient of $\mathbb{A}_f \times \mathbb{A}_f$ by (3.1), now with r, a, s, b in $\mathbb{A}_f$. If we denote the class of $(r,a) \in \mathbb{A}_f \times \mathbb{A}_f$ by $\omega_{r,a}$, then the group $K \rtimes K^\times$ acts on Ω_A in the obvious way by $(x,k) \cdot \omega_{r,a} = \omega_{x+kr,ka}$, and Theorem 2 yields

Theorem 4 (cf. [2])**.** $\mathcal{T}(R \rtimes R^\times) \cong \mathbf{1}_{\Omega_R}(C(\Omega_A) \rtimes K \rtimes K^\times)\mathbf{1}_{\Omega_R}$

The primitive ideal space of $(R \rtimes R^\times)$ can be described using this full corner realization.

Theorem 5 (cf. [3])**.** *Let $2^{\mathcal{P}}$ denote the power set of the set of prime ideals of R with the power-cofinite topology, in which the basic open sets are indexed by finite sets $G \subseteq \mathcal{P}$ and are given by $U_G = \{T \in 2^{\mathcal{P}} : T \cap G = \emptyset\}$. For each subset A of $\mathcal{P}$ let I_A be the kernel of the compression to $\mathcal{T}(R \rtimes R^\times)$ of the induced representation corresponding to a point $\omega_{r,a}$ with $A = \{p \in \mathcal{P} : a_p = 0\}$ and trivial stabilizer. Then the map $A \mapsto I_A$ is a homeomorphism of $2^{\mathcal{P}}$ to* $\operatorname{Prim}\mathcal{T}(R \rtimes R^\times)$.

2.4. **Further research.** Similar dilation and full corner results have been obtained in [1] at the purely algebraic level and it is very likely that more general versions and further applications lie ahead. Current work of X. Li on the realization of the C*-algebras associated to general cancellative semigroups also points to generalizations that go beyond Ore semigroups.

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Endomorphisms, transfer operators, interactions and interaction groups

Ruy Exel

The purpose of this two-lecture presentation is to describe a new concept of crossed-products introduced in [2] (see also [1], [3], [4], [5] and [6]). Given a unital C*-algebra A and a discrete group G, we say that the triple (A, G, V) is an *interaction group* [2, Definition 3.1] if V is a positive linear map

$$V : G \to B(A),$$

where $B(A)$ refers to the set of all bounded linear maps on A, such that, for every g and h in G, one has

i) $V_1 = id_A$,
ii) $V_{g^{-1}} V_g V_h = V_{g^{-1}} V_{gh}$,
iii) $V_g V_h V_{h^{-1}} = V_{gh} V_{h^{-1}}$,
iv) $V_g(1) = 1$,
v) $V_g(ab) = V_g(a)V_g(b)$, for every a in A and every b in the range of $V_{g^{-1}}$.

If $V : G \to \mathrm{Aut}(A)$ is an action of G by automorphisms of A, then evidently (A, V, G) is an interaction group. Besides this intensively studied class of examples, interaction groups are meant to generalize group actions to situations in which the dynamical system involves irreversible processes. Perhaps the most basic example to include the idea of irreversibility originates from transfer operators.

*Partially supported by CNPq.

Recall from [1] that, given a unital endomorphism α of a C*-algebra A, we say that a *transfer operator* for α is a positive, unital, linear map

$$L : A \to A$$

such that $L(a\alpha(b)) = L(a)b$, for all a and b in A.

Once such a transfer operator is given, define, for every n in $\mathbf{Z}$,

$$V_n : A \to A,$$

by

$$V_n = \begin{cases} \alpha^n, & \text{when } n \geq 0, \text{ and} \\ L^{-n}, & \text{when } n < 0. \end{cases}$$

It is an easy exercise to check that V provides an interaction group.

Other interesting examples are obtained as follows: Let X be a compact space and let $T, S : X \to X$ be surjective local homeomorphisms which commute. As in [6, Definition 10.1] we say that T and S **-commute* if, whenever $T(x) = S(y)$, there exists a unique z in X such that $x = S(z)$ and $y = T(z)$. Let us therefore assume that T and S satisfy this property.

To simplify we will also assume that the number of inverse images of any point in X under T is constant, and that the same holds for S (see [6, section 13] for the general case). For each $n \in \mathbf{Z}$, let V_n be the linear transformation of $C(X)$ given by

$$V_n(f)|_y = \begin{cases} \dfrac{1}{\#(T^n)^{-1}(y)} \displaystyle\sum_{T^n(x)=y} f(S^n(x)) \ , & \text{if } n \geq 0, \\ \frac{1}{\#(S^m)^{-1}(y)} \displaystyle\sum_{S^m(x)=y} f(T^m(x)), & \text{if } m := -n > 0, \end{cases}$$

for every $f \in C(X)$, and $y \in X$. Assuming that neither T nor S are injective, this example of interaction group is interesting because V_n is not an endomorphism for every n. Please see [6, section 13] for more details.

Given an interaction group (A, G, V), one may prove that the range of each V_g is a closed *-subalgebra of A and that $V_g V_{g^{-1}}$ is a conditional expectation onto $V_g(A)$. When all such conditional expectations are faithful, we say that the interaction group is *faithful*.

Let us assume from now on that (A, G, V) is a given faithful interaction group.

1. Definition. [2, Definition 4.1] A *covariant representation* of (A, G, V) in a unital C*-algebra B is a pair (π, v), where $\pi : A \to B$ is a unital *-homomorphism and $v : G \to B$ is a **-partial representation*, meaning that for every g and h in G,

i) $v_1 = 1$,
ii) $(v_g)^* = v_{g^{-1}}$,
iii) $v_{g^{-1}} v_g v_h = v_{g^{-1}} v_{gh}$, and moreover such that

$$v_g \pi(a) v_{g^{-1}} = \pi\big(V_g(a)\big) v_g v_{g^{-1}},$$

for every $a \in A$.

2. Definition. [2, Definition 5.1] The Toeplitz algebra of (A, G, V), denoted $T(A, G, V)$, is the universal unital C*-algebra generated by the set

$$A \cup \{v_g : g \in G\}$$

subject to the relations making the standard maps $a \mapsto a$, and $g \mapsto v_g$, a covariant representation.

By the universal property of $T(A, G, V)$, its representations correspond bijectively to the covariant representations of (A, V, G).

One rather uninteresting example of covariant representation is obtained by taking any *-homomorphism $\pi : A \to B$, and setting $v_g \equiv 0$. As already mentioned, this uninteresting representation will give rise to a representation of $T(A, G, V)$ into B. One should interpret this as saying that $T(A, G, V)$ is too big, allowing for too many representations. Therefore, should we want to construct a manageable algebra, the excess fat of $T(A, G, V)$ should be modded out. This is precisely the motivation for introducing the concept of *redundancies*, which we shall now describe.

Let $\alpha = (g_1, g_2, \ldots, g_n)$ be a finite sequence of elements of G. Working within $T(A, G, V)$, let

$$Z_\alpha = \overline{\text{span}}\{a_0 v_{g_1} a_1 v_{g_2} a_2 \ldots v_{g_n} a_n : a_0, a_1, \ldots, a_n \in A\},$$

and

$$M_\alpha = \overline{\text{span}}\{a v_{g_1} v_{g_2} \ldots v_{g_n} b : a, b \in A\}.$$

If $\beta = (h_1, h_2, \ldots, h_m)$ is another finite sequence of elements of G, one may prove [2, proposition 4.9] that

$$Z_\beta M_\alpha \subseteq M_\alpha, \tag{†}$$

whenever

(i) $h_1 h_2 \ldots h_m = 1$, and

(ii) $\mu(\beta) \subseteq \mu(\alpha)$,

where $\mu(\alpha)$ is the set of all products of initial segments of α, namely

$$\mu(\alpha) = \{1,\ g_1,\ g_1 g_2,\ g_1 g_2 g_3,\ \ldots,\ g_1 g_2 g_3 \cdots g_n\},$$

and similarly for $\mu(\beta)$.

For a fixed α, define K_α to be the closed sum of all Z_β, for β satisfying conditions (i) and (ii) above. By [2, proposition 4.7], one has that K_α is a closed *-subalgebra of $T(A, G, V)$. In addition, by (†) one clearly has that

$$K_\alpha M_\alpha \subseteq M_\alpha.$$

3. Definition. [2, Section 6]

a) By an α-redundancy we shall mean any element $k \in K_\alpha$, such that

$$k M_\alpha = \{0\}.$$

b) The *redundancy ideal* is the ideal of $T(A, G, V)$ generated by all α-redundancies, for all finite sequences α of elements of G.
c) The *crossed product* of A by G under V, denoted $A \rtimes_V G$, is the C*-algebra obtained by taking the quotient of $T(A, G, V)$ by the redundancy ideal.

Should the reader be unimpressed by the notion of redundancies, there is a special case in which an alternate and much more elementary construction of $A \rtimes_V G$ may be obtained. The special case we have in mind is characterized by the existence of a faithful state ϕ on A which is invariant under V, in the sense that

$$\phi\big(V_g(a)\big) = \phi(a), \quad \forall g \in G, \ \forall a \in A.$$

Once ϕ is given, let π be the GNS representation of A associated to ϕ. Denoting by H the representation space and by ξ the standard cyclic vector, one may prove that, for each g in G, the correspondence

$$\pi(a)\xi \mapsto \pi\big(V_g(a)\big)\xi$$

extends to a bounded linear map on H, which we shall denote by v_g.

It may then be proved [2, proposition 11.4] that (π, v) is a covariant representation of (A, G, V), which therefore extends to $T(A, G, V)$. In fact, this representation is *strongly covariant* in the sense that it vanishes on the redundancy ideal and hence factors through the crossed product $A \rtimes_V G$. In many cases this representation is faithful, but not always. The reason it should not always be expected to be faithful is quite elementary: if our interaction group consists of a true group action, and should this action happen to be the trivial action, the above representation will clearly not be faithful, not least because one would then have $v_g = I$, for all g.

Fortunately there is a simple way to fix this. Let λ be the left regular representation of G on $\ell_2(G)$.

4. Theorem. [2, Theorem 11.7] If ϕ is a V-invariant state as above, then $(\pi \otimes 1, v \otimes \lambda)$ is a covariant representation of (A, G, V) on $H \otimes \ell_2(G)$. Assuming that G is amenable, the kernel of the associated representation of $T(A, G, V)$ is precisely the redundancy ideal. Therefore $A \rtimes_V G$ is isomorphic to the closed *-subalgebra of $B\big(H \otimes \ell_2(G)\big)$ generated by

$$\{\pi(a) \otimes 1 : a \in A\} \cup \{v_g \otimes \lambda_g : g \in G\}.$$

In other words, this means that one may use the above to *define* the crossed product algebra without worrying about redundancies at all!

To conclude, let us mention another interesting class of examples for which we may describe the crossed product algebra in a very concrete way.

Given a compact space X, consider the semigroup $\mathrm{End}(X)$ formed by all surjective, local homeomorphisms from X to itself. Suppose we are moreover given a subsemigroup P of a group G, and a right action of P on X, meaning an anti-homomorphism of semigroups

$$\theta : P \to \mathrm{End}(X).$$

By the standard process of dualization we get a left semigroup action

$$\alpha : P \to \mathrm{End}\big(C(X)\big),$$

where, this time, $\mathrm{End}\big(C(X)\big)$ refers to the semigroup of *-endomorphisms of $C(X)$.

If P is such that $P^{-1}P \subseteq PP^{-1}$, one may prove [6, section 3] that the set

$$\mathcal{G} = \{(x, g, y) \in X \times G \times X : \exists n, m \in P,\ g = nm^{-1},\ \theta_n(x) = \theta_m(y)\}$$

is a groupoid under the operations

$$(x, g, y)(y, h, z) = (x, gh, z), \quad \text{and} \quad (x, g, y)^{-1} = (y, g^{-1}, x).$$

One may make $\mathcal{G}$ an étale groupoid by introducing the topology generated by the sets

$$\Sigma(n, m, A, B) = \{(x, nm^{-1}, y) \in A \times G \times B : \theta_n(x) = \theta_m(y)\},$$

where A and B are open subsets of X, and $n, m \in P$. This groupoid is called the *transformation groupoid* for θ.

Returning to our interaction groups, it is a delicate problem to determine under which conditions the semigroup action α above extends to an interaction group defined on the whole of G, but under suitable conditions this may be achieved [6, Corollary 8.8]. Assuming these conditions are met we have:

5. Theorem. [6, Theorem 6.6] Under the conditions set forth in [6, 4.1], and moreover assuming G to be amenable, the crossed product $A \rtimes_V G$ is isomorphic to the groupoid C*-algebra of $\mathcal{G}$.

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Semigroup C*-algebras

Xin Li

This is a very brief summary of the talks. The reader may find the details, concrete examples and more references in [3], [4] and [1] (and also [2]).

1. Constructions

The first talk was about the construction of reduced and full semigroup C*-algebras.

1.1. **Reduced semigroup C*-algebras.** Let P be a left cancellative semigroup (with identity element). On $\ell^2(P)$ with the canonical orthonormal basis $\{\varepsilon_x\colon x \in P\}$, we define for every $p \in P$ an isometry V_p by setting $V_p\varepsilon_x = \varepsilon_{px}$. As in the group case, we simply take the C*-algebra generated by the left regular representation of our semigroup:

Definition 1.1. $C_r^*(P) := C^*\left(\{V_p\colon p \in P\}\right) \subseteq \mathcal{L}(\ell^2(P))$.

1.2. **Towards full semigroup C*-algebras.** Unfortunately, the most obvious construction of the universal C*-algebra

$$C^*_{\mathrm{Mur}}(P) := C^*\left(\{v_p\colon p \in P\}\middle|\ \begin{array}{c} v_p^* v_p = 1 \\ v_p v_q = v_{pq} \end{array}\right)$$

leads to intractable objects. Indeed, Murphy observed that for the semigroup $\mathbb{N}_0 \times \mathbb{N}_0$, the C*-algebra $C^*_{\mathrm{Mur}}(\mathbb{N}_0 \times \mathbb{N}_0)$ is not nuclear, and hence, the canonical homomorphism $C^*_{\mathrm{Mur}}(\mathbb{N}_0 \times \mathbb{N}_0) \to C^*_r(\mathbb{N}_0 \times \mathbb{N}_0)$ cannot be an isomorphism. This is a problem because this semigroup is abelian, so that we would expect its semigroup C*-algebra to have nice properties. So we need more relations besides the canonical ones which define $C^*_{\mathrm{Mur}}(P)$.

A first step in this direction is due to Nica. He considers positive cones in so-called quasi-lattice ordered groups. A pair (G, P) consisting of a subsemigroup P of a group G is a quasi-lattice ordered group if $P \cap P^{-1} = \{e\}$ where e is the unit element in G, and for every $g \in G$,

(QL) $P \cap (g \cdot P)$ is either empty or of the form pP for some $p \in P$.

Nica then defines the full semigroup C*-algebra of P as the universal C*-algebra

$$C^*_{\mathrm{Nica}}(P) := C^*\left(\{v_p\colon p \in P\}\middle|\ \begin{array}{c} v_p^* v_p = 1 \\ v_p v_q = v_{pq} \end{array} \ \&\ (\mathrm{NICA})\right)$$

where Nica's relation is

$$(\mathrm{NICA})\quad v_p v_p^* v_q v_q^* = \begin{cases} v_r v_r^* \text{ if } pP \cap qP = rP \text{ for some } r \in P, \\ 0 \text{ if } pP \cap qP = \emptyset. \end{cases}$$

Note that it follows from (QL) that there are only these two possibilities $pP \cap qP = rP$ for some $r \in P$ or $pP \cap qP = \emptyset$.

It turns out that Nica's construction leads to tractable C*-algebras. So Nica's idea was to impose extra relations which make the range projections of our isometries reflect the ideal structure of our semigroup. Nica only looks at quasi-lattice ordered pairs because in that case, it is enough to consider principal ideals. The question remains what to do for general semigroups.

1.3. **Full semigroup C*-algebras.** The idea is to impose extra relations in the definition of full semigroup C*-algebras which should reflect the ideal structure of our semigroup. First, let us introduce the ideals we will take into account.

Definition 1.2. *Let $\mathcal{J}$ be the smallest family of right ideals of P such that*

- $\emptyset, P \in \mathcal{J}$;
- *$\mathcal{J}$ is closed under left multiplication and pre-images under left multiplication ($X \in \mathcal{J}, p \in P \Rightarrow pX, p^{-1}X \in \mathcal{J}$).*

The ideals in $\mathcal{J}$ are called constructible (right) ideals of P.

This family $\mathcal{J}$ is automatically closed under finite intersections.

Definition 1.3.

$$C^*(P) := C^*\left(\{v_p\colon p \in P\} \cup \{e_X\colon X \in \mathcal{J}\} \;\middle|\; \begin{array}{c} v_p \text{ are isometries} \\ \text{and } e_X \text{ are projections} \\ \text{satisfying I, II and III.} \end{array}\right)$$

with the following relations:

I. $v_{pq} = v_p v_q$;
II. $e_\emptyset = 0$, $e_P = 1$, $e_{X_1 \cap X_2} = e_{X_1} \cdot e_{X_2}$;
III. $v_p e_X v_p^* = e_{pX}$.

These relations are satisfied by the concrete isometries V_p and projections E_X, where E_X is the orthogonal projection onto $\ell^2(X) \subseteq \ell^2(P)$.

However, there is another extra relation which we will need later on and which might not be automatic from these three relations defining $C^*(P)$. So we have to modify our construction.

Definition 1.4. *Let P be a subsemigroup of a group G. We let $C^*_s(P)$ be the universal C^*-algebra*

$$C^*_s(P) := C^*\left(\{v_p\colon p \in P\} \cup \{e_X\colon X \in \mathcal{J}\} \;\middle|\; \begin{array}{c} v_p \text{ are isometries} \\ \text{and } e_X \text{ are projections} \\ \text{satisfying } I_s,\ II_s \text{ and } III_s. \end{array}\right)$$

with the following relations:

I_s. $v_{pq} = v_p v_q$,
II_s. $e_\emptyset = 0$,
III_s. *whenever $p_1, q_1, \ldots, p_m, q_m \in P$ satisfy $p_1^{-1}q_1 \cdots p_m^{-1}q_m = e$ in G, then*

$$v_{p_1}^* v_{q_1} \cdots v_{p_m}^* v_{q_m} = e_{\left[q_m^{-1}p_m \cdots q_1^{-1}p_1 P\right]}.$$

These relations are satisfied by the concrete operators V_p and E_X so that we obtain a canonical homomorphism $\lambda : C^*_s(P) \to C^*_r(P)$. We call λ the left regular representation. Moreover, the relations I_s, II_s and III_s are stronger then the relations I, II and III we had before. This means that there is a canonical homomorphism $C^*(P) \to C_s^*(P)$ which sends generators to generators. The situation can be summarized in the following commutative diagram:

$$\begin{tikzcd} C^*(P) \arrow[d] \arrow[dr] & \\ C_s^*(P) \arrow[r, "\lambda"] & C_r^*(P). \end{tikzcd}$$

It is not known for which semigroups the canonical homomorphism $C^*(P) \to C_s^*(P)$ is an isomorphism. But for quasi-lattice ordered $P \subseteq G$, both of these constructions coincide with the one by Nica.

1.4. **Independence.**

Definition 1.5. *We call $\mathcal{J}$ independent (or we also say that the constructible right ideals of P are independent) if whenever $X = \bigcup_{i=1}^n X_i$ holds for $X, X_1, \ldots, X_n \in \mathcal{J}$, then we must have $X = X_i$ for some $1 \leq i \leq n$.*

Here is why this condition is useful. Let $D(P) = C^*(\{e_X \colon X \in \mathcal{J}\}) \subseteq C_s^*(P)$ and $D_r(P) = C^*(\{E_X \colon X \in \mathcal{J}\}) \subseteq C_r^*(P)$ denote the canonical commutative subalgebras. The left regular representation restricts to a homomorphism from $D(P)$ to $D_r(P)$.

Lemma 1.6. *$\lambda : D(P) \to D_r(P)$ is an isomorphism if and only if $\mathcal{J}$ is independent.*

1.5. **The Toeplitz condition.** The full semigroup C*-algebra $C_s^*(P)$ consists of two ingredients, the isometries v_p and the projections e_X. Another way of saying this is that $C_s^*(P)$ is built out of the semigroup dynamical system given by P acting on $D(P)$ by conjugation with the v_ps. Now our semigroup sits inside a group G. Our goal is to find a group dynamical system for G so that $C_s^*(P)$ embeds as a full corner into a the corresponding crossed product by G. This would then allow us to apply known results about group crossed products to our semigroup C*-algebras.

It turns out that in our specific situation, there is a canonical candidate for the dilated system. The idea is to look at the reduced case first. As above, let P be a subsemigroup of a group G.

Definition 1.7. *We let $D_{P \subseteq G}$ be the smallest sub-C^*-algebra of $\ell^\infty(G) \subseteq \mathcal{L}(\ell^2(G))$ which is G-invariant and contains $D_r(P)$.*

Here we let G act on $\ell^\infty(G)$ by left translations. As for $D_r(P)$, we know that $D_{P \subseteq G}$ is a C*-algebra generated by projections: $D_{P \subseteq G} = C^*(\{E_Y \colon Y \in \mathcal{J}_{P \subseteq G}\})$ where $\mathcal{J}_{P \subseteq G}$ is the smallest family of subsets of G which contains $\mathcal{J}$ and which is closed under left translations by group elements as well as finite intersections. The reduced crossed product $D_{P \subseteq G} \rtimes_r G$ can be realized on $\ell^2(G)$ with $D_{P \subseteq G}$ acting by multiplication operators and G acting through its left regular representation. Moreover, the projection E_P is by construction a full projection in $D_{P \subseteq G} \rtimes_r G$.

And because we have for all $p \in P$ that $V_p = E_P \lambda_p^G E_P$ (λ^G denotes here the left regular representation of G), we always have that $C_r^*(P) \subseteq E_P(D_{P\subseteq G} \rtimes_r G)E_P$.

Definition 1.8. *We say that $P \subseteq G$ satisfies the Toeplitz condition (or simply that $P \subseteq G$ is Toeplitz) if for every $g \in G$ with $E_P\lambda_g^G E_P \neq 0$, there exist $p_1, q_1, \dots, p_n, q_n$ in P such that $E_P\lambda_g^G E_P = V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}$.*

The Toeplitz condition implies that $C_r^*(P) = E_P(D_{P\subseteq G} \rtimes_r G)E_P$.

1.6. **Alternative descriptions of semigroup C*-algebras.** Now we can come to alternative descriptions of semigroup C*-algebras. First, the *-semigroup $S := \{v_{p_1}^* v_{q_1} \cdots v_{p_n}^* v_{q_n} \colon p_i, q_i \in P\} \cup \{0\} \subseteq C_s^*(P)$ is an inverse semigroup of partial isometries. The reason why we need the extra relation III_s in the definition of $C_s^*(P)$ is that it ensures that the left regular representation restricts to an isomorphism (of inverse semigroups) $\lambda : S \to \lambda(S)$. We obtain the following

Theorem 1.9. *Let P be a subsemigroup of a group G. Assume that $\mathcal{J}$ is independent and that $P \subseteq G$ satisfies the Toeplitz condition. With $\Omega := \mathrm{Spec}\, D_r(P)$ and $\Omega_{P\subseteq G} := \mathrm{Spec}\,(D_{P\subseteq G})$, we obtain the following commutative diagram:*

$$\begin{array}{ccc}
C_s^*(P) & \xrightarrow{\lambda} & C_r^*(P) \\
\downarrow\cong & & \cong\downarrow \\
C^*(S) & \longrightarrow & C_r^*(S) \\
\downarrow\cong & & \cong\downarrow \\
C^*((\Omega_{P\subseteq G} \rtimes G)_\Omega^\Omega) & \longrightarrow & C_r^*((\Omega_{P\subseteq G} \rtimes G)_\Omega^\Omega) \\
\downarrow\cong & & \cong\downarrow \\
E_P(D_{P\subseteq G} \rtimes G)E_P & \longrightarrow & E_P(D_{P\subseteq G} \rtimes_r G)E_P.
\end{array}$$

All the horizontal arrows are given by the regular representations.

Here $C^*(S)$ is the universal C*-algebra for *-representations of S which send $0 \in S$ to 0. Moreover, $C_r^*(S)$ is the reduced C*-algebra of S, i.e. the C*-algebra generated by the left regular representation of S on $\ell^2(S^\times)$ where $S^\times = S \setminus \{0\}$. The transformation groupoid $\Omega_{P\subseteq G} \rtimes G$ is built out of the action of G on $D_{P\subseteq G}$ by left translations. The Toeplitz condition allows us to identify Ω with a subspace of $\Omega_{P\subseteq G}$ in a canonical way, and the restriction of $\Omega_{P\subseteq G} \rtimes G$ to this subspace is $(\Omega_{P\subseteq G} \rtimes G)_\Omega^\Omega$.

2. Structure

The second talk was about the structure of semigroup C*-algebras. The first result expresses amenability of semigroups in terms of semigroup C*-algebras.

Theorem 2.1. *Let P be a subsemigroup of a group G and assume that P has independent constructible ideals. Then the following are equivalent:*

- *P is left amenable,*
- *the left regular representation $\lambda : C_s^*(P) \to C_r^*(P)$ is an isomorphism and there exists a non-zero character on $C_s^*(P)$,*
- *$C_s^*(P)$ is nuclear and there exists a non-zero character on $C_s^*(P)$.*

It turns out that amenability is a very strong assumption on the semigroup which interesting examples do not satisfy. So we really would like to characterize nuclearity of semigroup C*-algebras and faithfulness of left regular representations independently from the existence of non-zero characters.

Theorem 2.2. *Let P be a subsemigroup of a group G and assume that P has independent constructible ideals as well as that $P \subseteq G$ is Toeplitz. Then the following are equivalent:*

- *$C_s^*(P)$ is nuclear,*
- *$C_r^*(P)$ is nuclear,*
- *whenever given a G-action α on a C*-algebra A, the canonical homomorphism $\lambda_{(A,P,\alpha)} : A \rtimes_{\alpha,s}^a P \to A \rtimes_{\alpha,r}^a P$ is an isomorphism,*
- *G acts amenably on $\Omega_{P\subseteq G}$.*

Here $A \rtimes_{\alpha,s}^a P$ stands for the universal C*-algebra generated by a copy of A and a copy of $C_s^*(P)$ such that the commutation relation $v_p a = \alpha_p(a) v_p$ is satisfied for all $a \in A$ and $p \in P$. And $A \rtimes_{\alpha,r}^a P$ is the reduced crossed product which is constructed in the same way as in the group case.

Furthermore, every semigroup C*-algebra has a distinguished quotient, the so-called boundary quotient. For example, in the case of the free semigroup $\mathbb{N}_0^{*n}$, the C*-algebra $C_s^*(\mathbb{N}_0^{*n})$ is canonically isomorphic to $\mathcal{E}_n$, the universal C*-algebra generated by n isometries with pairwise orthogonal range projections. The boundary quotient of $C_s^*(\mathbb{N}_0^{*n})$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$. We now construct the boundary quotient in the general case.

Lemma 2.3. *Let P be a left cancellative semigroup with independent constructible ideals. Then we can identify Ω with the set of non-empty $\mathcal{J}$-valued filters by sending a character χ to $\{X \in \mathcal{J}\colon \chi(E_X) = 1\}$.*

Definition 2.4. *Let P be a left cancellative semigroup with independent constructible ideals. We set $\Omega_{\max}$ as the subset of Ω corresponding to the set of maximal $\mathcal{J}$-valued filters under the identification in the lemma above, and $\partial\Omega := \overline{\Omega_{\max}} \subseteq \Omega$. The vanishing ideal is given by $V(\partial\Omega) = \{d \in D(P)\colon \chi(\lambda(d)) = 0 \text{ for all } \chi \in \partial\Omega\}$. And finally, the boundary quotient is $C_s^*(P)/\langle V(\partial\Omega)\rangle$.*

Theorem 2.5. *Assume that P is a subsemigroup of a countable group G such that P has independent constructible ideals and that $P \subseteq G$ is Toeplitz. Let $G_0 := \big\{g \in G\colon (g \cdot P) \cap (pP) \neq \emptyset \text{ and } (g^{-1} \cdot P) \cap (pP) \neq \emptyset \text{ for all } p \in P\big\}$. If $P \neq \{e\}$, if G acts amenably on $(\partial\Omega) \cdot G \subseteq \Omega_{P\subseteq G}$ and if G_0 acts topologically freely on $(\partial\Omega) \cdot G$, then the boundary quotient $C_s^*(P)/\langle V(\partial\Omega)\rangle$ is a unital UCT Kirchberg algebra.*

Note that by the Toeplitz condition, we can embed Ω canonically in $\Omega_{P\subseteq G}$. Therefore, we can form $(\partial\Omega)\cdot G$ in $\Omega_{P\subseteq G}$.

In the case of the free semigroup $\mathbb{N}_0^{*n}$, the proof of this theorem gives a new description of $\mathcal{O}_n$ as a full corner in a group crossed product of a commutative C*-algebra by the free group $\mathbb{F}_n$. This description then gives a new explanation why $\mathcal{O}_n$ is a UCT Kirchberg algebra.

3. K-theory

The third talk was about joint work with J. Cuntz and S. Echterhoff on the K-theory for semigroup C*-algebras. We start with a general result:

Theorem 3.1. *Suppose that τ is an action of a (discrete, countable) group G on a commutative C*-algebra D. Further assume that D is generated by a family of non-zero commuting projections $\{e_i\colon i\in I\}$ such that $\{e_i\colon i\in I\}\cup\{0\}$ is multiplicatively closed. Here I is a countable index set. Moreover, we assume that there exists a G-action on I such that $\tau_g(e_i)=e_{g\cdot i}$. If we have*

- *whenever $e=\bigvee_{i=j}^n e_j$ holds for projections in $\{e_i\colon i\in I\}$, then we must have $e=e_j$ for some $1\leq j\leq n$,*
- *G satisfies the Baum-Connes conjecture for $c_0(I)$ and D,*

then

$$\bigoplus_{[i]\in G\backslash I} K_*(C_r^*(G_i))\cong K_*(D\rtimes_{\tau,r}G).$$

Here $G_i=\{g\in G\colon g\cdot i=i\}$.

Applying this general statement to our situation of semigroup C*-algebras (i.e. $D=D_{P\subseteq G}$ and $I=\mathcal{J}_{P\subseteq G}^{\times}:=\mathcal{J}_{P\subseteq G}\setminus\{\emptyset\}$), we obtain

Theorem 3.2. *Let P be a subsemigroup of a countable group G such that P has independent constructible ideals and that $P\subseteq G$ is Toeplitz. If G satisfies the Baum-Connes conjecture for $c_0(\mathcal{J}_{P\subseteq G}^{\times})$ and $D_{P\subseteq G}$, then*

$$\bigoplus_{[X]\in G\backslash\mathcal{J}_{P\subseteq G}^{\times}} K_*(C_r^*(G_X))\cong K_*(C_r^*(P)).$$

Here $G_X=\{g\in G\colon g\cdot X=X\}$. Moreover, there is also a version of this result involving coefficients.

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Endomorphisms of the Cuntz algebras

Roberto Conti
(joint work with J. H. Hong, J. Kimberley, M. Rørdam, W. Szymański)

This is a partial report on recent work [10, 9, 7, 8, 15, 2, 3, 5, 6] (see also the announcements/overviews [18, 4, 11]) aimed at studying in detail the properties of certain endomorphisms and automorphisms of the Cuntz algebras $\mathcal{O}_n$ (with n finite), [12], as explained below. The talk was mostly focused on combinatorial and enumerative aspects as well as the construction of notable examples, other more conceptual issues like actions on shift spaces (cf. [14]) being addressed in the companion report by W. Szymański (and generalized to graph algebras, *mutatis mutandis*).

In [13] it was pointed out that the study of the automorphism group of $\mathcal{O}_n$ displays intriguing analogies with the theory of semisimple Lie groups. However, at that time very few examples of automorphisms of $\mathcal{O}_n$ were known, especially for $n = 2$, cf. [1, 16], and many natural questions could not be answered. Following the ideas pioneered by Cuntz we defined certain Weyl groups of the Cuntz algebras, namely the restricted Weyl group, the Weyl group and their outer companions, constructed new nontrivial examples of automorphisms to identify concretely elements in these groups and even started a classification program based on combinatorial datas. Indeed calculations in the reduced Weyl group boil down to solving certain polynomial equations on symmetric groups of exponentially increasing size and can be further analyzed by an ingenious technique based on careful analysis of certain n-tuples of labeled rooted trees. In the case of $\mathcal{O}_2$, twelve new outer classes of automorphisms were discovered (associated to permutations over a set with 16 elements), and their relations were partly computed, showing e.g. the existence of infinite dihedral subgroups in the reduced Weyl group of $\mathrm{Aut}(\mathcal{O}_2)$. We also investigated some more conceptual aspects, especially in connection to symbolic dynamics, thus obtaining deep structural results for these Weyl groups.

All in all, our results provide a new insight into the structure of $\mathrm{Aut}(\mathcal{O}_n)$ (as well as of $\mathrm{Out}(\mathcal{O}_n)$) and its interplay with the automorphisms of the canonical MASA $\mathcal{D}_n$ and of the core UHF subalgebra $\mathcal{F}_n$.

In the talk we have also discussed notable explicit examples of endomorphisms and automorphisms associated to elements in the Higman-Thompson subgroup of $\mathcal{U}(\mathcal{O}_n)$ [17], that is those unitaries that can be written as finite sums of words in the canonical generating isometries of $\mathcal{O}_n$ and their adjoints. Finally, we remarked the existence of a recently discovered new powerful algorithmic criterion for deducing when such endomorphisms are surjective on the diagonal, i.e. they provide homeomorphisms of the full n-shift space, thereby establishing a bridge with dynamics on Cantor sets. This is an important necessary condition for a complete characterization of elements in the Weyl group. It has been also shown that the outer Weyl group strictly contains the outer reduced Weyl group. As a consequence, $\mathrm{Aut}(\mathcal{O}_n)$ does not admit a decomposition as $\mathrm{Ad}(\mathcal{U}(\mathcal{O}_n))\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$, leaving however open whether such decomposition may be achieved by replacing $\mathcal{F}_n$ with $\mathcal{D}_n$.

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Comparing Leavitt path algebras with graph C*-algebras

Gene Abrams

Introduction. Motivated in part by the construction of graph C*-algebras, in 2004 the definition of a *Leavitt path algebra of a graph* was presented in [2] and [9]. Striking, not-fully-understood similarities between the structure of Leavitt path algebras and graph C*-algebras became legion during the initial years of the investigation. Recently, however, some significant differences in the two theories have emerged. We describe some of these here.

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Throughout this note K always denotes a field. Let $E = (E^0, E^1, r, s)$ be a (directed) graph with vertex set E^0, edge set E^1, and range (resp., source) functions r and s. The *path algebra* KE is the K-algebra with basis $\{p_i\}$ consisting of the directed paths in E (vertices are paths of length 0). Multiplication is defined by setting, for paths p, q, $p \cdot q = pq$ if $r(p) = s(q)$, 0 otherwise. In particular, $s(e) \cdot e = e = e \cdot r(e)$ for $e \in E^1$. We construct the *double graph* $\widehat{E}$, gotten from E by inserting, for each $e \in E^1$, a new edge e^* having $s(e^*) = r(e)$ and $r(e^*) = s(e)$. We then construct the path algebra $K\widehat{E}$, and consider these relations in $K\widehat{E}$:

(CK1) $e^*e = r(e)$ for all $e \in E^1$; $f^*e = 0$ for all $f \neq e \in E^1$.

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ (for $v \in E^0$ having $0 < |\{e \in E^1 \mid s(e) = v\}| < \infty$)

Definition. *The Leavitt path algebra of E with coefficients in K* is

$$L_K(E) = K\widehat{E}/\langle (CK1), (CK2) \rangle.$$

Standard algebras arising as Leavitt path algebras include $M_n(K)$, $K[x, x^{-1}]$, and $L_K(1, n)$, the "Leavitt K-algebra of order n" [15]. We let $C^*(E)$ denote the well-studied *graph C*-algebra of E*, which is the universal C*-algebra generated by mutually orthogonal projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$, subject to the well-known Cuntz-Krieger E-family relations.

Proposition: The $*$-subalgebra $A = \text{span}_{\mathbb{C}}\{P_v, S_\mu S_\nu^* \mid v \in E^0,\ \mu, \nu \text{ paths in } E\}$ of $C^*(E)$ has $L_{\mathbb{C}}(E) \cong A$ as $*$-algebras.

Consequently, $C^*(E)$ may be viewed as the completion (in operator norm) of $L_{\mathbb{C}}(E)$. In particular, the Cuntz algebra $\mathcal{O}_n$ [13] is the completion of $L_{\mathbb{C}}(1, n)$. So it's perhaps not surprising that there are some close relationships between $L_{\mathbb{C}}(E)$ and $C^*(E)$.

Section 1: Similarities. Since any graph C*-algebra can be considered both as a ring with and without a topology, one can in general consider ring-theoretic properties of a graph C*-algebra in two different ways. Let $\mathcal{P}$ be some property of rings, and let $\mathcal{G}$ be some graph-theoretic property. A number of theorems of the following form have been established.

Theorem. The following five statements are equivalent for any finite graph E.

1. $L_{\mathbb{C}}(E)$ has property $\mathcal{P}$.
2. $L_K(E)$ has property $\mathcal{P}$ for any field K.
3. $C^*(E)$ has (topological) property $\mathcal{P}$.
4. $C^*(E)$ has (algebraic) property $\mathcal{P}$.
5. E has property $\mathcal{G}$

For instance, the properties $\mathcal{P} =$ "simple" ([2] and [18]), "purely infinite simple" ([3] and [12]), "exchange" ([10] and [16] with [8]), "primitive" ([4] and [11]), and "finite dimensional" each fit into a theorem of the above form. Additional similarities between the two types of algebras when E is finite include, for instance, that $K_0(L_K(E)) = K_0(C^*(E))$ [9], and that each of the algebras satisfies a version of the Cuntz-Krieger uniqueness theorem ([20] and [18]). It is of interest that in each case in which a theorem of this form has been established, the analysts show that statements (3) and (5) are equivalent, while, using completely different tools, the algebraists show that statements (2) and (5) are equivalent. Specifically, in none of these cases has there been a direct relationship established between the two algebraic statements (1) / (2), and the two C*-algebraic statements (3) / (4).

Section 2: Differences. Given the similarities observed above, one might suspect that there is some sort of Rosetta Stone which connects the two types of structures. However, there are significant differences; we point out two such here.

The graph E is called *downward directed* in case for each pair $v, w \in E^0$, there exists $u \in E^0$ and paths p, q in E for which $s(p) = v$, $s(q) = w$, and $r(p) = r(q) = u$. For any field K and graph E, $L_K(E)$ is prime if and only if E is downward directed [4]. On the other hand, any separable C*-algebra is (topologically, equivalently, algebraically) prime if and only if it is (topologically) primitive [14]. So in particular for finite E, $C^*(E)$ is prime precisely when $C^*(E)$ is primitive, so that $C^*(E)$ is prime if and only if E is downward directed **and** satisfies Condition (L) [11]. So for example $L_{\mathbb{C}}(\bullet \circlearrowright) \cong \mathbb{C}[x, x^{-1}]$ is prime, but $C^*(\bullet \circlearrowright) \cong C(\mathbb{T})$ is not prime.

Here is a second distinction between the two theories. It is well known that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. On the other hand, in early 2011 three different proofs (by Ara / Cortiñas [7]; Dicks; Bell / Bergman) were given which show that the two $\mathbb{C}$-algebras $L_{\mathbb{C}}(1,2) \otimes L_{\mathbb{C}}(1,2)$ and $L_{\mathbb{C}}(1,2)$ are *not* isomorphic.

Section 3: Similar or Different? We conclude this short note by presenting some properties for which we do not currently know whether these yield similarities or differences between the structure of $L_{\mathbb{C}}(E)$ and the structure of $C^*(E)$.

Most basically, does a ring isomorphism between two Leavitt path algebras $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ imply an isomorphism between $C^*(E)$ and $C^*(F)$ as C*-algebras, and conversely? Using tools from the Kirchberg-Phillips Classification Theorem [17], we can show that if E and F are finite graphs for which $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are simple and for which $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$, then $C^*(E) \cong C^*(F)$ [6]. The question remains open in the general case.

Perhaps the most interesting outstanding question in this context is the following. Suppose E and F are finite graphs for which $C^*(E)$ and $C^*(F)$ (equivalently, $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$) are purely infinite and simple. It is well-known (and deep) that if there is an isomorphism $\varphi : K_0(C^*(E)) \to K_0(C^*(F))$ for which $\varphi([1_{C^*(E)}]) = [1_{C^*(F)}]$, then $C^*(E) \cong C^*(F)$ ([19] or [17]). On the algebra side, if there is an isomorphism $\varphi : K_0(L_{\mathbb{C}}(E)) \to K_0(L_{\mathbb{C}}(F))$ for which $\varphi([1_{L_{\mathbb{C}}(E)}]) = [1_{L_{\mathbb{C}}(F)}]$, and if in addition the signs of $\det(I - A_E)$ and $\det(I - A_F)$ are equal (where A_E denotes the usual incidence matrix of E), then $L_{\mathbb{C}}(E) \cong L_C(F)$ [1]. It is currently not known whether the hypothesis on the sign of the determinants is necessary.

Two additional outstanding questions on which the author and others are working are: (1) Do the necessary and sufficient conditions for the primitivity of the Leavitt path algebra of an arbitrary graph given in [4] also apply to graph C*-algebras? (2) Do the criteria for the simplicity of the Lie algebra $[L_K(E), L_K(E)]$ given in [5] apply to the simplicity of the Lie algebra $[C^*(E), C^*(E)]$?

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Classification of Graph C^*-algebras and Leavitt path algebras.

Mark Tomforde

In the past few years there have been a number of efforts to classify Leavitt path algebras associated with directed graphs. Many of these efforts have been modeled on the classification of C^*-algebras — particularly graph C^*-algebras and Cuntz-Krieger algebras — and there have been varying levels of success importing these techniques to the algebraic setting.

Cuntz and Krieger introduced a class of C^*-algebras [5, 4] constructed from square matrices with entries in $\{0, 1\}$, and showed that the structure and many of the invariants of $\mathcal{O}_A$ may be read off from A. If A is an $n \times n$ matrix with

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entries in $\{0,1\}$, then $\mathcal{O}_{\mathcal{A}}$ is simple if and only if A is irreducible. In addition, $K_0(\mathcal{O}_A) \cong \mathbb{Z}^n/(I-A^t)\mathbb{Z}^n$ and $K_1(\mathcal{O}_{\mathcal{A}}) \cong \ker(I-A^t)$. In particular, $K_0(\mathcal{O}_{\mathcal{A}}) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \mathbb{Z}^{n-k}$ and $\ker \mathcal{O}_A \cong \mathbb{Z}^{n-k}$, so that $K_1(\mathcal{O}_A)$ is equal to the free part of $K_1(\mathcal{O}_A)$, and all of the K-theory information is contained in K_0-group.

In classification of C^*-algebras one typically wishes to obtain two classifications: A classification up to Morita equivalence; and a classification up to isomorphism. One of the first classifications up to isomorphism was obtained in 1981 by Enomoto, Fujii, and Watatani [6]. They showed that all simple Cuntz-Krieger algebras of 3×3 matrices are classified by $(K_0(\mathcal{O}_A), [1])$; i.e., the K_0-group together with the position of the unit. They accomplished this classification by describing moves on matrices that preserve the isomorphism class of the associated Cuntz-Krieger algebra.

In their original paper, Cuntz and Krieger proved that if A and B are irreducible $\{0,1\}$ matrices and X_E is flow equivalent to X_F, then $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_B$. They gave two ways to prove this: (1) By realizing $\mathcal{O}_A \otimes \mathcal{K}$ as a crossed product; and (2) By using the fact flow equivalence is generated by the Strong Shift Equivalence moves plus the Parry-Sullivan move, and showing these matrix moves preserve the Morita equivalence class of the associated Cuntz-Krieger algebra.

In 1984 Franks proved his famous theorem in symbolic dynamics, which states that if A and B are irreducible square $\{0,1\}$ matrices, with $\operatorname{coker}(I-A) \cong \operatorname{coker}(I-A)$ and $\det(I-A) = \det(I-B)$, then X_A is flow equivalent to X_B. Since $K_0(\mathcal{O}_A) \cong \operatorname{coker}(I-A^t) \cong \operatorname{coker}(I-A)$ and $\det(I-A^t) = \det(I-A)$, we may combine this with Cuntz and Krieger's result to obtain the following fact: If A and B are irreducible with $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$ and $\det(I-A^t) = \det(I-B^t)$, then $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_B$. For a number of years it was wondered whether the determinant condition was necessary, and in 1995 Rørdam proved it is superfluous. In particular, if A is a matrix, we define the "Cuntz splice" of A to be the matrix

$$A_- := \begin{pmatrix} & & & 0 & 0 \\ & A & & \vdots & \vdots \\ & & & 1 & 0 \\ 0 & \cdots & 1 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}.$$

If $\mathcal{O}_2$ is the Cuntz algebra, and $\mathcal{O}_{2_-}$ is the corresponding Cuntz-Krieger algebra obtained by performing the Cuntz splice, then Cuntz proved that $\mathcal{O}_2 \cong \mathcal{O}_{2_-}$ implies that $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_{A_-}$ for all irreducible A. In 1995 Rørdam proved, using KK-theory, that indeed $\mathcal{O}_2 \cong \mathcal{O}_{2_-}$, and hence it is true that $\mathcal{O}_A \cong \mathcal{O}_{A_-}$ for all irreducible A [8]. Since the Cuntz splice changes the sign of the determinant (i.e., $\det(I-A^t) = -\det(I-A^t_-)$, it follows from Rørdam's result that if A and B are irreducible with $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$, then $\mathcal{O}_A$ is Morita equivalent to $\mathcal{O}_B$ and one can turn A into B via a sequence of flow equivalence moves plus the Cuntz splice move. In addition, using a result of Huang, which states that an automorphism on the Bowen-Franks group is induced by a flow equivalence, Rørdam showed in [8] that if A and B are irreducible and $(K_0(\mathcal{O}_A), [1]) \cong (K_0(\mathcal{O}_B), [1])$, then $\mathcal{O}_A \cong \mathcal{O}_B$.

In 1997, generalizations of Cuntz-Krieger algebras, known as graph C^*-algebras were introduced. The Cuntz-Krieger algebras coincide with the C^*-algebras of finite directed graphs with no sinks or sources. If A_E is the vertex matrix of E, then $K_0(C^*(E)) \cong \operatorname{coker}(I - A_E^t)$ and $K_1(C^*(E)) \cong \ker(I - A_E^t)$. Unlike with the Cuntz-Krieger algebras, when the graph E is not finite the K_1-group of $C^*(E)$ is not determined by the K_0-group. There have been attempts to extend the Cuntz-Krieger classification results to the graph C^*-algebras and their algebraic counterparts, the Leavitt path algebras [3].

Purely algebraic classifications for the Leavitt path algebras were first initiated in the late 2000's. Interestingly, the development has been very similar to the classification of Cuntz-Krieger algebras. In 2008, Abrams, Ánh, Louly, and Pardo proved that $(K_0(L_K(E)), [1])$ is a complete isomorphism invariant for simple Leavitt path algebras of graphs with 3 vertices and no parallel edges [1]. (This may be though of as an analogue of the result of Enomoto, Fujii, and Watatani for Cuntz-Krieger algebras of 3×3 matrices.) They proved this result by exhibiting moves on the graphs that preserve the isomorphism class of the Leavitt path algebras.

In 2011, Abrams, Louly, Pardo, and Smith undertook a classification of Leavitt path algebras of finite graphs, in analogy with Rørdam's results for Cuntz-Krieger algebras. They were able to show that when the flow equivalence moves are performed on finite graphs, they preserve the Morita equivalence class of the associated Leavitt path algebra. Thus, using Franks' result, they prove that if E and F are finite graphs with no sinks and with simple Leavitt path algebras, then $K_0(L_K(E)) \cong K_0(L_K(F))$ and $\det(I - A_E^t) = \det(I - B_E^t)$ implies $L_K(E)$ is Morita equivalent to $L_K(F)$. Also, by applying Huang's result and using an argument similar to Rørdam's, they have shown that $(K_0(L_K(E)), [1]) \cong (K_0(L_K(F)), [1])$, then $L_K(E) \cong L_K(F)$. However, in all of the Leavitt path algebra results it is unknown if the determinant condition is necessary. As with Cuntz-Krieger algebras, one can perform a Cuntz splice move to a graph. Abrams, Louly, Pardo, and Smith have shown that if $L_2 \cong L_{2_-}$ and this isomorphism lifts to an isomorphism of certain subalgebras of the endomorphism rings, then it follows that the Cuntz splice preserves Morita equivalence for all finite graphs with no sinks that have simple Leavitt path algebras (see [2, Hypothesis on p.224] for a precise statement). Unfortunately, no one has been able to determine whether L_2 and L_{2_-} are isomorphic, much less whether there exists an isomorphism lifting to the subalgebras of the endomorphism rings. It remains an open and important question as to whether the determinant is a Morita equivalence invariant for Leavitt path algebras, and in particular whether the Cuntz splice preserves Morita equivalence of Leavitt path algebras.

Very recently, Sørensen has shown that if one uses the flow equivalence moves on graphs with finitely many vertices and infinitely many edges, then one does not need the Cuntz splice [9]. In particular, Sørensen has proven a Franks-type theorem that says the following: If E and F are each graphs with a finite number of vertices, an infinite number of edges, no sinks, and have graph C^*-algebras that are simple, and if $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F)$, then there is a sequence of "flow equivalence moves" from E to F, each of which preserves Morita equivalence of the associated graph C^*-algebra. In particular, this implies that (unlike in the finite graph case) when there are an infinite number of edges, the Cuntz splice may be obtained by a sequence of "flow equivalence moves" on the graph. Using these results, Ruiz and the author have shown that the "flow equivalence moves" described by Sørensen also preserve Morita equivalence of the Leavitt path algebras, and if E and F are each graphs with a finite number of vertices, an infinite number of edges, no sinks, and their associated Leavitt path algebras are simple, and if $K_0(L_K(E)) \cong K_0(L_K(F))$ and $K_1(L_K(E)) \cong K_1(L_K(F)$, then there is a sequence of "flow equivalence moves" from E to F, and hence $L_K(E)$ and $L_K(F)$ are Morita equivalent. This shows that in this case the determinant is not a Morita equivalence invariant and the Cuntz splice is unnecessary in the classifying moves. It also implies that L_∞ is Morita equivalent to L_{∞_-}.

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Purely infinite C^*-algebras arising from actions of groups on Cantor sets

Mikael Rørdam

The Elliott classification program has made it an important task to find natural ways of constructing simple nuclear C^*-algebras. In particular, with the classification of the Kirchberg algebras in the UCT class by Kirchberg and Phillips, [4, 6], one is interested in finding (many) models of Kirchberg algebras. (Recall that a Kirchberg algebra is a separable, nuclear, simple, purely infinite C^*-algebra.) One natural source of (simple, nuclear) C^*-algebras comes from dynamical systems. There is an extensive literature on the structure and classification of C^*-algebras arising from a dynamical systems $\Gamma \curvearrowright X$ with a group Γ acting on a space X, often with the group Γ being $\mathbb{Z}$ or $\mathbb{Z}^n$. There are also several papers describing ways of obtaining Kirchberg algebras as crossed products $C(X) \rtimes_{\mathrm{red}} \Gamma$ with Γ being a (necessarily) non-amenable group acting amenably on X, see for example [5].

Let us first recall some well-known facts about the structure of crossed product C^*-algebras. Archbold and Spielberg proved in [2] that if Γ is a discrete group acting on a locally compact Hausdorff space X, then $C_0(X) \rtimes_{\mathrm{full}} \Gamma$ is simple if and only if the action $\Gamma \curvearrowright X$ is minimal, topologically free and regular. (The latter means that the canonical epimorphism $C_0(X) \rtimes_{\mathrm{full}} \Gamma \to C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is injective.) Anathraraman-Delaroche introduced in [1] the notion of amenable actions and proved that any amenable action is regular. She also proved that $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is nuclear if and only if the action of Γ on X is amenable. This shows that:

Proposition 1 (Archbold–Spielberg, Anatharaman-Delaroche)**.** *Let Γ be a discrete group acting on a locally compact Hausdorff space X. Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is simple and nuclear if and only if the action of Γ on X is minimal, topologically free and amenable.*

One can combine the results (and method of proof) by Laca and Spielberg in [5] with the proposition above to obtain:

Proposition 2. *Let Γ be a countable discrete group acting on a metrizable, locally compact, totally disconnected space X. Then $C_0(X) \rtimes_{\mathrm{red}} \Gamma$ is a Kirchberg algebra in the UCT class if and only if the action of Γ on X is minimal, topologically free and amenable, and each non-zero projection in $C_0(X)$ is properly infinite in $C_0(X) \rtimes_{\mathrm{red}} \Gamma$.*

The latter condition is somewhat unpleasant because it is not directly expressed in terms of the dynamical system.

Definition 3. An action of a group Γ on a (totally disconnected) locally compact Hausdorff space X is said to be *purely infinite* if the following holds: For every compact-open subset E of X there are pairwise disjoint compact-open subsets $E_1, \ldots, E_{n+m}$ of E and $t_1, \ldots, t_{n+m} \in \Gamma$ such that

$$E = \bigcup_{j=1}^{n} t_j.E_j = \bigcup_{j=n+1}^{n+m} t_j.E_j.$$

It is easy to see that the last condition in Proposition 2 is satisfied if the action is purely infinite. Moreover, if the last condition in Proposition 2 holds, then there is no non-zero invariant Radon measure on X. It is not known if the action is purely infinite if and only if there is no non-zero invariant Radon measure in the case where X totally disconnected, which would imply that all three conditions are equivalent.

Let us also recall the following well-known facts about the Roe algebra $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ associated with a discrete group Γ:

Proposition 4. *(1) Γ always acts freely on $\ell^\infty(\Gamma)$.*
(2) (Ozawa) *$\Gamma \curvearrowright \ell^\infty(\Gamma)$ is amenable if and only if Γ is exact.*
(3) The unit of $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ is properly infinite if and only if Γ is non-amenable.

The latter statement was strengthened in [7] where it was shown that if $E \subseteq \Gamma$, then 1_E is properly infinite in $\ell^\infty(\Gamma) \rtimes_{\mathrm{red}} \Gamma$ if and only if E is paradoxical in Γ. This observation was one of the key ingredients in the proof of the following:

Theorem 5 (R.-Sierakowski, [7]). *Let Γ be a countable discrete exact non-amenable group. Then Γ admits a minimal free amenable action on the Cantor set X such that $C(X) \rtimes \Gamma$ is a (unital) Kirchberg algebra in the UCT class.*

It is well-known that a group Γ is amenable if and only if whenever it acts on a compact Hausdorff space, then there is an invariant probability measure. The situation is quite different if we instead consider actions of groups on locally compact non-compact Hausdorff spaces. Here we need to consider the so-called supramenable groups first considered by Rosenblatt in [8]. A discrete group Γ is said to be supramenable if it has no paradoxical subsets.

Monod observed that a group is supramenable if and only if whenever it acts co-compactly on a locally compact Hausdorff space, then there is a non-zero invariant Radon measure. This result can be sharpened as follows:

Theorem 6 (Kellerhals, Monod, R., [3]). *Let Γ be an exact non-supramenable countable discrete group. Then Γ admits a minimal free amenable purely infinite action on the locally compact, non-compact Cantor set Y. In particular, $C_0(Y) \rtimes \Gamma$ is a (non-unital) Kirchberg algebra in the UCT class.*

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Entropy and invariant abelian subalgebras for endomorphisms of Cuntz algebras

Adam Skalski

Topological entropy for a continuous transformation of a compact space (see [Wal]) is a numerical invariant which in a sense measures the degree of 'mixing' or 'chaotic' behaviour of the dynamical system in question. In [Voi] it was extended by Voiculescu to automorphisms of (nuclear) C^*-algebras, with the definition based on the growth of sizes of suitable completely positive approximations.

Let A be a nuclear (or exact) C^*-algebra, $\alpha \in \mathrm{End}(\mathsf{A})$ and let $\mathrm{ht}\,\alpha$ denote the Voiculescu's topological entropy of α. The usual method of computing $\mathrm{ht}\,\alpha$ is based on two steps. First one produces an explicit or semi-explicit approximating net for A through matrix algebras whose rank can be controlled and thus provides an estimate of the $\mathrm{ht}\,\alpha$ from above. Then, to obtain a lower bound, one looks for α-invariant commutative C^*-subalgebras $\mathsf{C} \subset \mathsf{A}$ in order to exploit the monotonicity of entropy with respect to passing to subalgebras and the fact that $\alpha|_{\mathsf{C}}$ is induced by a homeomorphism T of the spectrum of C and it was shown in [Voi] that $\mathrm{ht}\,\alpha|_{\mathsf{C}} = h_{\mathrm{top}}(T)$. The general difficulty in understanding how the positive Voiculescu entropy is produced is reflected in the fact that there is no direct proof of the inequality $\mathrm{ht}\,\alpha|_{\mathsf{C}} \geq h_{\mathrm{top}}(T)$, (the corresponding argument in [Voi] exploits the properties of the dynamical state entropy and classical variational principle).

On the other hand we have the following result.

Theorem ([Sk$_1$])**.** *There exist pairs* (A, α) *(certain bitstream shifts) such that*

$$\mathrm{ht}\,\alpha > \mathrm{ht}_c\,\alpha := \sup\{\mathrm{ht}\,\alpha|_{\mathsf{C}} : \mathsf{C} \text{ is an } \alpha - \text{invariant commutative subalgebra of } \mathsf{A}\}.$$

The above discussion leads to two natural questions related to the computations of the Voiculescu entropy:

- given an endomorphism of a C^*-algebra what are the (maximal) abelian subalgebras it leaves globally invariant?
- what other techniques, not based on the existence of invariant abelian subalgebras, yield lower bounds for the Voiculescu entropy?

Below we present some results related to these questions in the context of the endomorphisms of Cuntz algebras.

Let $\mathcal{O}_N$ denote the Cuntz algebra generated by N-isometries $S_1, \ldots, S_N$ whose range projections are orthogonal and sum to 1 ([Cu$_1$]). We use the symbol μ to denote a $\{1, \ldots, N\}$-valued multiindex and let $S_\mu := S_{\mu_1} S_{\mu_2} \ldots S_{\mu_k}$, if the length of μ, denoted by $|\mu|$, is k. The Cuntz algebra contains a so-called *diagonal masa* (maximal abelian subalgebra) $\mathcal{C}_N := \overline{\mathrm{Lin}}\{S_\mu S_\mu^*\}$, isomorphic to the algebra of continuous functions on a Cantor set (equivalently, a full Markov shift on N letters). Moreover, if we write $\mathcal{F}_N^k = \mathrm{Lin}\{S_\mu S_\nu^* : |\mu| = |\nu| \leq k\} \approx M_N^{\otimes k}$, $\mathcal{F}_N = \lim_{k\to\infty} \mathcal{F}_N^k$, we obtain natural inclusions

$$\mathcal{C}_N = \bigotimes_{n=1}^{\infty} D_N \subset \bigotimes_{n=1}^{\infty} M_N = \mathcal{F}_N \subset \mathcal{O}_N.$$

By 'changing coordinates' in M_N and replacing diagonals D_N by $\mathcal{U}^* D_N \mathcal{U}$ ($\mathcal{U} \in M_N$ - a unitary) we can construct other, so-called *standard masas* in $\mathcal{O}_N$.

In [Cu$_2$] it was shown that there is a bijective correspondence between unitaries in $\mathcal{O}_N$ and unital endomorphisms of $\mathcal{O}_N$, given by the formulas

$$\rho_U(S_i) = U S_i, \quad i = 1, \ldots, N$$

and

$$U_\rho = \sum_{i=1}^{n} \rho(S_i) S_i^*.$$

This correspondence makes the endomorphisms of $\mathcal{O}_N$ particularly amenable to study (see [CoS] and references therein). From the point of view of the entropy we have the following result.

Theorem ([SkZ]). *Let $k \in \mathbb{N}$ and $U \in \mathcal{U}(\mathcal{F}_N^k)$. Then*

$$\mathrm{ht}(\rho_U) \leq (k-1)\log N.$$

The above estimate is, perhaps surprisingly, analogous to the bounds on index appearing in the work of Doplicher, Longo, Roberts, Conti, Pinzari and others.

The canonical shift, implemented by the *flip unitary* $F = \sum_{i,j=1}^{N} S_i S_j S_i^* S_j^* \in \mathcal{F}_N^2$ gives an easy example of the bound being achieved (its Voiculescu entropy, equal to $\log N$, was computed in [Cho], using the fact that the canonical shift leaves $\mathcal{C}_N$ invariant and the corresponding restriction is dual to the usual shift transformation on the full shift of N-letters). Note that for *Bogolyubov automorphisms*, i.e. automorphisms associated with unitaries $U \in \mathcal{F}_N^1 \approx M_N$ we have $\mathrm{ht}\,\rho_U = 0$. The standard masas can be alternatively described as images of $\mathcal{C}_N$ with respect to Bogolyubov automorphisms.

In [SkZ] we present an example of an endomorphism ρ of $\mathcal{O}_2$ induced by a unitary in $\mathcal{F}_2^2$, which leaves the diagonal masa invariant, but $\mathrm{ht}\rho = \log 2$, and $\mathrm{ht}\rho|_{\mathcal{C}_2} = 0$. In fact ρ leaves all standard masas invariant, and in some of them reduces again to the (dual of the) classical full shift.

In [HSS] we analyse in detail the endomorphisms which 'look the same' in all standard masas, i.e. commute with all Bogolyubov automorphisms. Moreover we develop there several sufficient (and necessary) conditions for an endomorphism to preserve a given standard masa. Here we just sample some of the interesting examples:

- if $U \in \mathcal{N}_{\mathcal{C}_N} := \{U \in U(\mathcal{O}_N) : U\mathcal{C}_N U^* = \mathcal{C}_N\}$, then ρ_U leaves $\mathcal{C}_N$ invariant;
- there exists a unitary $U \notin \mathcal{N}_{\mathcal{C}_2}$ such that ρ_U leaves $\mathcal{C}_2$ invariant;
- there exists $\rho \in \mathrm{End}(\mathcal{O}_2)$ which leaves $\mathcal{C}_2$ invariant, but no other standard masa;
- there exists $\rho \in \mathrm{End}(\mathcal{O}_2)$ which leaves invariant each standard masa, but does not commute with all Bogolyubov automorphisms.

Results of [HSS] and [Sk$_2$] show also that there exists $\rho \in \mathrm{End}(\mathcal{O}_2)$ (originally studied in [Izu] in relation to Watatani indices of the subalgebras of Cuntz algebras) which leaves no standard masa invariant, but whose Voiculescu entropy is non-zero. The entropy computation is related to the following general facts.

Let H be a (finite-dimensional) Hilbert space. A *multiplicative unitary* is a unitary V on $\mathsf{H} \otimes \mathsf{H}$ satisfying the following relation (on $\mathsf{H} \otimes \mathsf{H} \otimes \mathsf{H}$) (in the leg notation, so for example $V_{12} := V \otimes I_{\mathsf{H}}$):

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

It is called *irreducible* if it cannot be non-trivially written as '$V_1 \otimes I_{\mathsf{H}_1}$' for some other multiplicative unitary V_1.

Theorem ([Sk$_2$]). *Let V be an irreducible multiplicative unitary on $\mathsf{H} \otimes \mathsf{H}$, where $\mathsf{H} \approx \mathbb{C}^N$; view V as a matrix in $M_N \otimes M_N$ and further via the usual isomorphism $M_N \otimes M_N \approx \mathcal{F}_N^2$ as a unitary in $\mathcal{O}_N$. Let F be the flip unitary in $M_N \otimes M_N$. The topological entropy of $\rho_{VF} \in \mathrm{End}(\mathcal{O}_N)$ is equal to $\log N$.*

The proof of the above theorem is based on the von Neumann algebraic techniques: one first identifies a certain extension of ρ_{VF} with a canonical endomorphism of the Longo type, then passes to certain finite von Neumann subalgebras, views the respective restricted endomorphism as the Ocneanu canonical shift for the tower of subfactors and finally uses some computations of the CNT entropy ([NSt]) in terms of the index due to Hiai ([Hia]).

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C^*-algebras geneated by two operations

Yasuo Watatani

We study some relations among C^*-algebras of rings by Cuntz and Li, C^*-algebras generated by Toeplitz operators and composition operators on a Hardy space, and C^*-algebras associated with complex dynamical systems. They are all generated by two operations.

1. C^*-algebras of rings

Cuntz [2] introduced C^*-algebras associated with the $ax+b$ semigroup over $\mathbb{N}$ .The algebra is gnerated by addition operators and multiplication operators The algebra contains Bost-Connes algebra construced in [1]. Cuntz and Li [3] extend the construction to an arbitary commutative ring R without zero divisors (an integral domain). Consider a Hilbert space $\ell^2(R)$ with a natural basis $\{\delta_k \mid k \in R\}$. For $r \in R$, define an addition operator U_r on $\ell^2(R)$ by

$$U_r \delta_k = \delta_{r+k}.$$

For $n \in R^\times := R \setminus \{0\}$, define a multiplication operator S_n on $\ell^2(R)$ by

$$S_n \delta_k = \delta_{nk}.$$

Then the reduced C^*-algebra $\mathcal{A}_r[R]$ of an ring (integral domain) R is the C^*-algebra generated by the addition operators $\{U_r \mid r \in R\}$ and the multiplication operators $\{S_n \mid n \in R^\times\}$ on $\ell^2(R)$.

A close relationship of the C^*-algebra $\mathcal{A}_r[R]$ of an ring R to classfield theory over the rational numbers is given in terms of KMS states. The aim of this note is to show that a similar situation occurs for C^*-algebras associated with complex dynamical sistems.

2. Toeplitz-composition C^*-algebras

Let $L^2(\mathbb{T})$ denote the square integrable measurable functions on $\mathbb{T}$ with respect to the nomalized Lebesgue measure. The Hardy space $H^2(\mathbb{T})$ is the closed subspace of $L^2(\mathbb{T})$ consisting of the functions whose negative Fourier coefficients vanish. We put $H^\infty(\mathbb{T}) := H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$.

The Hardy space $H^2(\mathbb{D})$ is the Hilbert space consisting of all analytic functions $g(z) = \sum_{k=0}^\infty c_k z^k$ on the open unit disk $\mathbb{D}$ such that $\sum_{k=0}^\infty |c_k|^2 < \infty$. The inner product is given by

$$(g|h) = \sum_{k=0}^{\infty} c_k \overline{d_k}$$

for $g(z) = \sum_{k=0}^\infty c_k z^k$ and $h(z) = \sum_{k=0}^\infty d_k z^k$.

We identify $H^2(\mathbb{D})$ with $H^2(\mathbb{T})$ by a unitary $U : H^2(\mathbb{D}) \to H^2(\mathbb{T})$. We note that $\tilde{g} = Ug$ is given as

$$\tilde{g}(e^{i\theta}) := \lim_{r \to 1-} g(re^{i\theta}) \ \ a.e.\theta$$

for $g \in H^2(\mathbb{D})$ by Fatou's theorem. Moreover the inverse $\check{f} = U^*f$ is given as a Poisson integral.

Let $P_{H^2} : L^2(\mathbb{T}) \to H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ be the projection. For $a \in L^\infty(\mathbb{T})$, the Toeplitz operator T_a on $H^2(\mathbb{T})$ is defined by $T_a f = P_{H^2} af$ for $f \in H^2(\mathbb{T})$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. Then the composition operator C_φ on $H^2(\mathbb{D})$ is defined by $C_\varphi g = g \circ \varphi$ for $g \in H^2(\mathbb{D})$. By the Littlewood subordination theorem, C_φ is always bounded.

For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, we denote by $\mathcal{TC}_\varphi$ the C^*-algebra generated by the Toeplitz operator T_z and the composition operator C_φ on $H^2(\mathbb{D})$. The C^*-algebra $\mathcal{TC}_\varphi$ is called the Toeplitz-composition C^*-algebra with symbol φ. Since $\mathcal{TC}_\varphi$ contains the ideal $K(H^2(\mathbb{D}))$ of compact operators, we define a C^*-algebra $\mathcal{OC}_\varphi$ to be the quotient C^*-algebra $\mathcal{TC}_\varphi/K(H^2(\mathbb{D}))$ in [5] and [4]

3. C^*-algebras associated with complex dynamical systems

We recall the construction of Cuntz-Pimsner algebras. Let A be a C^*-algebra and X be a Hilbert right A-module. We denote by $L(X)$ the algebra of the adjointable bounded operators on X. For ξ, $\eta \in X$, the operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)_A$ for $\zeta \in X$. The closure of the linear span of these operators is denoted by $K(X)$. We say that X is a Hilbert C^*-bimodule (or C^*-correspondence) over A if X is a Hilbert right A-module with a *-homomorphism $\phi : A \to L(X)$. We always assume that X is full and ϕ is injective. Let $F(X) = \bigoplus_{n=0}^\infty X^{\otimes n}$ be the full Fock module of X with a convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \quad \text{and} \quad T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz-Toeplitz algebra $\mathcal{T}_X$ is the C*-algebra acting on $F(X)$ generated by $i_{F(X)}(a)$ with $a \in A$ and T_ξ with $\xi \in X$.

Let $j_K|K(X) \to \mathcal{T}_X$ be the homomorphism defined by $j_K(\theta_{\xi,\eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of A. Let $\mathcal{J}_X$ be the ideal of $\mathcal{T}_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a)\,;\, a \in I_X\}$. Then the Cuntz-Pimsner algebra $\mathcal{O}_X$ is defined as the quotient $\mathcal{T}_X/\mathcal{J}_X$. Let $\pi : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. We set $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$.

Next we introduce the C^*-algebras associated with complex dynamical systems as in [7] and [8]. Let R be a rational function of degree at least two. The sequence $(R^n)_n$ of iterations of composition by R gives a complex dynamical system on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Fatou set F_R of R is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R^n)_n$ is equicontinuous (or a normal family), and the Julia set J_R of R is the complement of the Fatou set in $\hat{\mathbb{C}}$. We denote by $e(z_0)$ the branch index of R at z_0. Let $A = C(\hat{\mathbb{C}})$ and $X = C(\mathit{graph}\ R)$ be the set of continuous functions on $\hat{\mathbb{C}}$ and *graph* R respectively. Then X is an A-A bimodule by

$$(a \cdot \xi \cdot b)(x, y) = a(x)\xi(x, y)b(y), \quad a, b \in A, \ \xi \in X.$$

We define an A-valued inner product $(\ |\ )_A$ on X by

$$(\xi|\eta)_A(y) = \sum_{x \in R^{-1}(y)} e(x)\overline{\xi(x, y)}\eta(x, y), \quad \xi, \eta \in X, \ y \in \hat{\mathbb{C}}.$$

Since the Julia set J_R is completely invariant under R, i.e., $R(J_R) = J_R = R^{-1}(J_R)$, we can consider the restriction $R|_{J_R} : J_R \to J_R$. The C^*-algebra $\mathcal{O}_R(\hat{\mathbb{C}})$ on $\hat{\mathbb{C}}$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $X = C(\textit{graph } R)$ over $A = C(\hat{\mathbb{C}})$. We also define the C^*-algebra $\mathcal{O}_R(J_R)$ on the Julia set J_R similarly.

4. Relations among these C^*-algebras

These algebras are simple and purely infinite C^*-algebras in many cases. These algebras are also generated by two operations. In fact, the reduced C^*-algebra $\mathcal{A}_r[R]$ of an ring (integral domain) R is the C^*-algebra generated by the addition operators $\{U_r \mid r \in R\}$ and the multiplication operators $\{S_n \mid n \in R^\times\}$ on $\ell^2(R)$. The Toeplitz-composition C^*-algebra $\mathcal{TC}_\varphi$ with symbol φ is generated by the Toeplitz operator T_z and the composition operator C_φ on $H^2(\mathbb{D})$. The Cuntz-Toeplitz algebra $\mathcal{T}_X$ is the C^*-algebra generated by coeffcient operators $i_{F(X)}(a)$ with $a \in A$ and the creation operators T_ξ with $\xi \in X$ on the Fock space $F(X)$. If we consider the case $R(z) = z^n$, then these operators have natural correspondences. The structure of KMS states on these C^*-algebras like phase transition describes some properties in number theory or complex dynamical systems [6].

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C^*-algebras associated with endomorphisms of compact abelian groups

Joachim Cuntz

This talk is based on the first part of a recent joint preprint with A.Vershik [5]. Let H be a compact abelian group. Let α be a surjective endomorphism of H. For Haar measure on H, the transformation α will define an isometric operator s_α on the Hilbert space L^2H. We can form the C*-algebra $C^*(s_\alpha, C(H))$ generated in $\mathcal{L}(L^2H)$ by s_α together with $C(H)$ acting as multiplication operators on L^2H. In this talk we analyze the structure of such C*-algebras and study their K-theory. We will always assume that α is surjective with finite kernel and exact (i.e. the union of the kernels of the α^n is dense).

In fact, to study $C^*(s_\alpha, C(H))$, most of the time it is more useful to work, rather than with α, with the dual endomorphism $\varphi = \hat{\alpha}$ of the dual group $G = \hat{H}$. Fourier transform transforms $C^*(s_\alpha, C(H))$ isomorphically into the C*-algebra $\mathfrak{A}[\varphi]$ acting on ℓ^2G. As in [9] and [2], but still somewhat surprisingly, this C*-algebra $\mathfrak{A}[\varphi]$ which is originally defined by a concrete representation, can also be characterized as a universal algebra given by generators and relations. The structure of $\mathfrak{A}[\varphi]$ (and thus of $C^*(s_\alpha, C(H))$) is governed by two "complementary" maximal abelian subalgebras, one being the algebra of continuous functions on H, the other one the algebra of continuous functions on a compactification (with respect to φ) of G. The algebras $\mathfrak{A}[\varphi]$ all have a similar structure, in particular they are simple, nuclear and purely infinite. They therefore belong to a very well understood class of C*-algebras. In particular by the Kirchberg-Phillips classification they are completely determined by their K-theory.

We derive a Pimsner-Voiculescu type formula that can be used to determine the K-theory of $\mathfrak{A}[\varphi]$. We prove that there is an exact sequence of the form

$$\begin{tikzcd} K_*C(H) \arrow[r, "1-b(\varphi)"] & K_*C(H) \arrow[r] & K_*\mathfrak{A}[\varphi] \arrow[ll, bend left] \end{tikzcd} \tag{5.1}$$

We should point out that $b(\varphi)$ is not simply the map induced by α, even though it is related to this map by a simple equation. In many cases $b(\varphi)$ is determined by this equation, but in some examples its determination requires extra work. Since $K_*C(H)$ is always torsion-free, the exact sequence (5.1) is particularly useful for computations. It can be used to explicitly determine the K-theory of $\mathfrak{A}[\varphi]$ for many examples, including the case of endomorphisms of $\mathbb{T}^n$, of $\prod_k \mathbb{Z}/n$ and of a solenoid group.

We mention that in [1] the K-theory of the left regular semigroup C*-algebra $C^*_\lambda(G \rtimes \mathbb{N})$ is shown to be isomorphic to $K_*(C^*(G))$. The algebra $\mathfrak{A}[\varphi]$ is a natural quotient of $C^*_\lambda(G \rtimes \mathbb{N})$ (it is generated by a representation of the semigroup $G \rtimes \mathbb{N}$ on ℓ^2G rather than on $\ell^2(G \rtimes \mathbb{N})$). It can be shown that the exact sequence (5.1) is exactly the long exact sequence associated with this extension of $\mathfrak{A}[\varphi]$.

Algebras such as $\mathfrak{A}[\varphi]$ have been studied by quite a few authors. The simplicity of the algebra $\mathfrak{A}[\varphi]$ and its description as a universal algebra has been established already in [9] even in a more general setting. Constructions along the same lines are considered in the thesis of F.Vieira, [10]. As pointed out to us by R.Exel, the simplicity of $\mathfrak{A}[\varphi]$ could also be established using an approach as in [7] and [6]. One virtue of our approach here is its simplicity together with the fact that it reveals interesting structural properties of $\mathfrak{A}[\varphi]$ and its canonical subalgebras.

Special cases of the algebra $\mathfrak{A}[\varphi]$ for $H = \mathbb{T}^n$ or for $H = \prod_k \mathbb{Z}/p$ had also occurred before in [2, 3, 4], where again it was shown that in these examples $\mathfrak{A}[\varphi]$ is purely infinite simple and its K-theory was partially computed. Our proof here that $\mathfrak{A}[\varphi]$ is purely infinite simple is very similar to that in [2]. For the case of an endomorphism of $\mathbb{T}^n$, an algebra which is easily seen to be isomorphic to $\mathfrak{A}[\varphi]$ has been described as a Cuntz-Pimsner algebra in [8], using Exel's concept of a transfer operator [7]. In this paper, it was also proved that, for an expansive endomorphism of $\mathbb{T}^n$, the algebra is simple purely infinite and its K-theory was determined (using Pimsner's extension which leads to a sequence similar to (5.1)). Our computation of the K-theory is somewhat simpler and more general.

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On the positive Rokhlin property for dynamical systems

Joachim Zacharias

The Rokhlin property is a dynamical property of a group action α on a C^*-algebra A saying roughly that the action can be approximated by shifts. The motivation to study this property comes from the famous Rohklin Lemma for aperiodic measure preserving transformations of Lebesgue space, which shows in particular that such transformations can be approximated w.r.t. a generic topology on the automorphism group by cyclic shifts ([1]). There are different versions of the C^*-Rokhlin property in literature, mostly for actions of finite groups and $\mathbb{Z}$ (i.e. single automorphisms) but also for amenable groups. Rokhlin actions play an important role in the classification programme, not least because they often produce simple crossed products. In fact, the Rokhlin property is a strong form of outerness.

Our original motivation was to study the Rokhlin property in connection with noncommutative topological dimension, that is, decomposition rank and nuclear dimension. There are still no good general results about decomposition rank and nuclear dimension of crossed products. We show that nice estimates can be obtained for Rokhlin actions of finite groups and single automorphisms.

However, the current C^*-Rokhlin properties can only hold in fairly restrictive settings ([3]). For instance A must have many nontrivial projections. We thus introduce a new Rokhlin property involving several Rokhlin towers of positive elements rather than projections. The number of towers can be thought of as a Rokhlin dimension. Here is the definition for actions of finite groups and single automorphisms.

Definition 1. *Let G be a finite group and A a unital C^*-algebra. An action $\alpha : G \to \mathrm{Aut}(A)$ has the positive Rokhlin property (with Rokhlin dimension k) if for any $F \subset A$ finite, $\varepsilon > 0$ there are positive elements $f_g^{(l)} \in A$, $l = 0, \ldots, k$, $g \in G$ with*

- $\|f_g^{(l)} f_h^{(l)}\| < \varepsilon \qquad (g \neq h$ *in* G, *all* $l)$
- $\|\sum_{l,g} f_g^{(l)} - 1\| < \varepsilon$
- $\|[f_g^{(l)}, a]\| < \varepsilon \qquad (a \in F$, *all* $g, l)$
- $\|\alpha_g(f_h^{(l)}) - f_{gh}^{(l)}\| < \varepsilon$, *(all* l, $g, h \in G)$.

Definition 2. *$\alpha \in \mathrm{Aut}(A)$ has the positive Rokhlin property (with Rokhlin dimension k) if for any $n > 0$, $F \subset A$ finite $\varepsilon > 0$ there are positive elements $f_{r,i}^{(l)} \in A$, $l = 0, \ldots, k$, $r = 0, 1$, $i = 0, \ldots, n$ with*

- $\|f_{r,i}^{(l)} f_{s,j}^{(l)}\| < \varepsilon$ $((r, i) \neq (s, j)$, *all* $l)$
- $\|\sum_{l,r,i} f_{r,i}^{(l)} - 1\| < \varepsilon$
- $\|[f_{r,i}^{(l)}, a]\| < \varepsilon$ $(a \in F$, *all* $r, i, l)$
- $\|\alpha(f_{r,i}^{(l)}) - f_{r,i+1}^{(l)}\| < \varepsilon$, *(all* l, $j = 0, \ldots, n - 2 + r)$
- $\|\alpha(f_{0,n-1}^{(l)} + f_{1,n}^{(l)}) - f_{0,0}^{(l)} - f_{1,0}^{(l)}\| < \varepsilon$.

The cases $k = 0$ (Rokhlin dimension 0) are equivalent to the usual Rokhlin properties involving projections. For $k > 0$ our positive Rokhlin property is much more flexible. For instance irrational rotations on the circle verify the positive Rokhlin property with $k = 1$. In fact we have a topological density result which shows show that the positive Rokhlin property is fairly prevalent. Equip $\mathrm{Aut}(A)$ with the topology of pointwise convergence, more precisely, define basic neighborhoods by

$$V_{F,\varepsilon}(\alpha) = \{\beta \in \mathrm{Aut}(A) \mid \|\alpha(a) - \beta(a)\|, \|\alpha^{-1}(a) - \beta^{-1}(a)\| < \varepsilon \forall a \in F\}.$$

If A is separable then $\mathrm{Aut}(A)$ with this topology is a complete metric space.

Theorem 3. *If A is separable and $\mathcal{Z}$-stable then the set of automorphisms of A satisfying the positive Rokhlin property is G_δ in the above topology.*

We have good estimates for the nuclear dimension of crossed products.

Theorem 4. *Let be a finite group. If $\alpha : G \to \mathrm{Aut}(A)$ has the positive Rohklin property with Rokhlin dimension k then*

$$\mathrm{nd}(A \rtimes_\alpha G) \leq (\mathrm{nd}(A) + 1)(k + 1) - 1.$$

Theorem 5. *If $\alpha \in \mathrm{Aut}(A)$ has the positive Rohklin property with Rokhlin dimension k then*

$$\mathrm{nd}(A \rtimes_\alpha \mathbb{Z}) \leq 8(\mathrm{nd}(A) + 1)(k + 1) - 1.$$

Here $\mathrm{nd}(A)$ denotes the nuclear dimension of A. Moreover we can show that $\mathcal{Z}$-stability is preserved under taking crossed products with the positive Rokhlin property Our main result shows that every minimal homeomorphism of a compact metric space of finite covering dimension defines an automorphism of finite Rokhlin dimension.

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Fractional skew monoid rings

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Paschke [7] gave a construction of a C^*-algebraic crossed product $A \rtimes_\alpha \mathbb{N}$ associated to a not necessarily unital C^*-algebra endomorphism α on a C^*-algebra A, which has been generalized to other semigroups, see e.g. [5] and [6]. Here, we develop a purely algebraic analog with respect to monoid actions on rings: for a monoid T acting on a unital ring A by endomorphisms and a submonoid S of T satisfying the left denominator conditions, a fractional skew monoid ring $S^{\mathrm{op}} *_\alpha A *_\alpha T$ is constructed, which satisfy a universal property analogous to the one for the skew group ring. A general interesting result is provided in Section 2. Namely, assume that G is a group acting on a ring A by automorphisms, and that there are a submonoid S of G such that $G = S^{-1}S$ and a non-trivial idempotent e in A such that $\alpha_s(e) \in eAe$ for all $s \in S$. Then the corner ring $e(A *_\alpha G)e$ of the skew group ring $A *_\alpha G$ is isomorphic as a G-graded ring to a fractional skew monoid ring $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$ (Proposition 2.2). Under the standing assumption that S acts by corner isomorphisms, we prove that all $S^{\mathrm{op}} *_\alpha A *_\alpha S$ can be exhibited in the form $e(A *_\alpha G)e$ (Theorem 2.5).

This is a joint work [1] with P. Ara, K.R. Goodearl and M.A. González-Barroso.

1. The general construction

1.1. Let A be a (unital) ring, and $\mathrm{Endr}(A)$ the monoid of not necessarily unital ring endomorphisms of A. Let T be a monoid and $\alpha : T \to \mathrm{Endr}(A)$ a monoid homomorphism, written $t \mapsto \alpha_t$. For $t \in T$, set $p_t = \alpha_t(1)$, an idempotent in A. Let $S \subseteq T$ be a submonoid satisfying the left denominator conditions, i.e., the left Ore condition and the monoid version of left reversibility: whenever $t, u \in T$ with $ts = us$ for some $s \in S$, there exists $s' \in S$ such that $s't = s'u$. Then there exists a monoid of fractions, $S^{-1}T$, with the usual properties (e.g., see [2, §1.10] or [3, §0.8]). If S is cancellative and $S = T$, then left Ore condition trivially implies the left reversibility property. But S can satisfy left denominator conditions and not be cancellative (see e.g. [8, Example 1.5(2)]).

Definition 1.2. The label $S^{\mathrm{op}} *_\alpha A *_\alpha T$ stands for a (unital) ring R equipped with a (unital) ring homomorphism $\phi : A \to R$ and monoid homomorphisms $s \mapsto s_-$ from $S^{\mathrm{op}} \to R$ and $t \mapsto t_+$ from $T \to R$, universal with respect to the following relations:

(1) $t_+\phi(a) = \phi\alpha_t(a)t_+$ for all $a \in A$ and $t \in T$;
(2) $\phi(a)s_- = s_-\phi\alpha_s(a)$ for all $a \in A$ and $s \in S$;
(3) $s_-s_+ = 1$ for all $s \in S$;
(4) $s_+s_- = \phi(p_s)$ for all $s \in S$.

1.3. The existence of such a ring follows from a classical arguments. The construction above also applies when A is an algebra over a field k or a $*$-algebra and the ring endomorphisms α_t for $t \in T$ are k-linear or $*$-homomorphisms. Also, if s_- is an isometry for all $s \in S$, then (2) can be replaced by "$s_+\phi(a)s_- = \phi\alpha_s(a)$ for all $a \in A$ and $s \in S$", (3-4) become redundant, and hence we extends the covariant representation property for semigroup crossed products (see e.g. [4]).

Then we have the following:

Proposition 1.4.

(1) $R = \sum_{s\in S,\, t\in T} s_-\phi(A)t_+ = \sum_{s\in S,\, t\in T} s_-\phi(p_sAp_t)t_+$.
(2) The ring R has an $S^{-1}T$-grading $R = \bigoplus_{x\in S^{-1}T} R_x$ where each $R_x = \bigcup_{s^{-1}t=x} s_-\phi(A)t_+$.

Now assume that S is *left saturated* in T: whenever $s \in S$ and $t \in T$ such that $ts \in S$, we must have $t \in S$. Under this additional hypothesis, we can show the following result:

Proposition 1.5.

(1) Let $s \in S$, $t \in T$, and $a \in A$. Then $s_-\phi(a)t_+ = 0$ if and only if $p_sap_t \in \ker(\alpha_{s'})$ for some $s' \in S$. In particular, $\ker(\phi) = \bigcup_{s'\in S} \ker(\alpha_{s'})$.

(2) *The ideal $I = \ker(\phi)$ satisfies $\alpha_s^{-1}(I) = I$ for all $s \in S$ and $\alpha_t(I) \subseteq I$ for all $t \in T$.*
(3) *α induces a monoid homomorphism $\alpha' : T \to \mathrm{End}_{\mathbb{Z}}(A/I)$, and α'_s is injective for all $s \in S$.*
(4) *$S^{\mathrm{op}} *_\alpha A *_\alpha T = S^{\mathrm{op}} *_{\alpha'} (A/I) *_{\alpha'} T$.*

1.6. As Proposition 1.5 shows, we can reduce the problem to the case where α_s is injective for all $s \in S$. In that case, ϕ is injective by Proposition 1.5(a), and so we can identify A with the unital subring $\phi(A) \subseteq R$.

2. Fractional skew monoid rings versus corners of skew group rings

Paschke [7] has shown that a C*-algebra crossed product by an endomorphism corresponds naturally to a corner in a crossed product by an automorphism. In other words, the C*-algebra versions of fractional skew monoid rings ${\mathbb{Z}^+}^{\mathrm{op}} *_\alpha A *_\alpha \mathbb{Z}^+$ are isomorphic to corners $e(B *_{\alpha'} \mathbb{Z})e$ in certain skew group rings. We prove that $S^{\mathrm{op}} *_\alpha A *_\alpha S$ appear as corner rings $e(B * G)e$, where $B * G$ is some skew group ring over the group $G = S^{-1}S$. We will assume that all the morphisms are injective, and that the action is given by corner isomorphisms.

2.1. Let A be a unital ring, G a group, and $\alpha : G \to \mathrm{Aut}(A)$ an action. Assume that S is a submonoid of G with $G = S^{-1}S$ (thus, S is cancellative and satisfies the left Ore condition), and let $R = A *_\alpha G$. Suppose that there exists a nontrivial idempotent $e \in A$ such that $\alpha_s(e) \leq e$ for all $s \in S$.

Proposition 2.2. *Under the above assumptions, the following hold:*
(1) *The action α restricts to an action $\alpha' : S \to \mathrm{Endr}(eAe)$ by corner isomorphisms.*
(2) *There are natural monoid morphisms $S^{\mathrm{op}} \to eRe$, given by $s \mapsto es^{-1}$, and $S \to eRe$, given by $t \mapsto te$, satisfying the conditions* (1)–(4) *in Definition* 1.2 *with respect to α' and the inclusion map $\phi : eAe \to eRe$.*
(3) *Under the assumptions of* (2.1)*, the rings $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$ and $e(A *_\alpha G)e$ are isomorphic as G-graded rings.*

Now we go in the reverse direction, looking for a representation of a fractional skew monoid ring $S^{\mathrm{op}} *_\alpha A *_\alpha S$ as a corner ring of a skew group ring. Our approach used ideas in the work of Picavet [8].

2.3. Suppose that $\alpha : S \to \mathrm{Endr}(A)$ is an action of S on A by corner isomorphisms. We construct a ring $S^{-1}A$ as in [8], but with some changes of notation to fit our situation. First, define a relation $\sim$ on $S \times A$ as follows: $(s_1, a_1) \sim (s_2, a_2)$ if and only if there exist $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$ and $\alpha_{t_1}(a_1) = \alpha_{t_2}(a_2)$. This is an equivalence relation [8, Lemma 2.1], and we write $[s, a]$ for the equivalence class of a pair (s, a). Let $S^{-1}A = (S \times A)/{\sim}$ be the set of these equivalence classes. The left Ore condition guarantees “common denominators” in $S^{-1}A$. By [8, Lemma 2.2 ff.], there are well-defined associative multiplication and addition on $S^{-1}A$. For, given any $[s_1, a_1], [s_2, a_2] \in S^{-1}A$, choose $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$, and set: (i) $[s_1, a_1] \cdot [s_2, a_2] = [t_1 s_1, \alpha_{t_1}(a_1)\alpha_{t_2}(a_2)]$; (ii) $[s_1, a_1] + [s_2, a_2] = [t_1 s_1, \alpha_{t_1}(a_1) +$

$\alpha_{t_2}(a_2)]$. The distributive law is also routine, and so $S^{-1}A$ becomes a non-unital ring. This procedure can be seen as a different way for obtaining Laca's dilation construction [4]. Next, we extend α to an action of S on $S^{-1}A$.

Proposition 2.4.

(1) The action of α on A extends to an action $\alpha : S \to \mathrm{Aut}(S^{-1}A)$. Concretely, given any $s \in S$ and $[t, a] \in S^{-1}A$, set $\widehat{\alpha}_s([t, a]) = [s', \alpha_{t'}(a)]$ for $s', t' \in S$ such that $s's = t't$.

(2) The rule $a \mapsto [1, a]$ defines an S-equivariant ring embedding $\phi : A \to S^{-1}A$ with image $[1, 1] \cdot S^{-1}A \cdot [1, 1]$.

Theorem 2.5. *Let G be a group and S a submonoid of G such that $G = S^{-1}S$. Let $\alpha : S \to \mathrm{Endr}(A)$ be an action of S on A by corner isomorphisms. Then*

$$S^{\mathrm{op}} *_\alpha A *_\alpha S \cong [1, 1]((S^{-1}A) *_{\hat{\alpha}} G)[1, 1]$$

(as G-graded rings).

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Endomorphisms of graph C*-algebras

Wojciech Szymański

(joint work with Roberto Conti, Jeong Hee Hong)

A graph C^*-algebra $C^*(E)$ is a universal C^*-algebra generated by partial isometries $\{S_e\}$ with mutually commuting domain and range projections satisfying natural relations encoded in the underlying directed graph E, [13]. In the case of finite graphs the class of graph C^*-algebras coincides with Cuntz-Krieger algebras based on finite 0-1 matrices, [9], and thus contains the Cuntz algebras $\mathcal{O}_n$ ([7]), $2 \le n < \infty$, as particularly important examples.

In this talk we discuss recent advances in the study of endomorphisms of graph C^*-algebras (i.e. unital $*$-homomorphisms from $C^*(E)$ into itself). Our motivation comes from numerous applications of such endomorphisms, including semigroup crossed products, index theory ([11]), and symbolic dynamics. Until now, it was the theory of endomorphisms of the Cuntz algebras $\mathcal{O}_n$ that attracted most of attention, starting with Cuntz's seminal paper [8] and up to recent works on combinatorial approach to localized endomorphisms, [6], and exotic endomorphisms preserving the core UHF-subalgebra, [5]. In [4], we lay foundation for a systematic development of an analogous theory of endomorphisms of graph C^*-algebras and Cuntz-Krieger algebras.

In the case of $\mathcal{O}_n$, it is well-known that every unital $*$-endomorphism arises via $\lambda_u(S_i) = uS_i$, $i = 1, \dots, n$, for some unitary u, [8]. For graph algebras only those endomorphisms which fix the vertex projections arise this way, and the corresponding untaries commute with the vertex projections, [4]. In addition, each graph automorphism extends to an automorphism of $C^*(E)$ (the subgroup of these automorphisms of $C^*(E)$ we denote $\mathrm{Aut}(E)$) and of course there are inner automorphisms around. We denote by $\lambda(\mathcal{P}_E)^{-1}$ the group of those automorphisms λ_u of $C^*(E)$ which correspond to unitaries u which permute paths of certain fixed length (and fix the vertex projections).

Denote by $\mathcal{F}_E$ the core AF-subalgebra of $C^*(E)$ and by $\mathcal{D}_E$ the canonical diagonal MASA. Let $\mathrm{Aut}(C^*(E), \mathcal{D}_E) = \{\alpha \in \mathrm{Aut}(C^*(E)) : \alpha(\mathcal{D}_E) = \mathcal{D}_E\}$ and $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E)) = \{\alpha \in \mathrm{Aut}(C^*(E)) : \alpha|_{\mathcal{D}_E} = \mathrm{id}\}$. Let E be a finite graph without sinks, in which every loop has an exit. We also assume that the center of the corresponding graph C^*-algebra $C^*(E)$ is trivial. Then, analogously to the case of $\mathcal{O}_n$ studied by Cuntz in [8], $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E)) \cong \mathcal{U}(\mathcal{D}_E)$ is a maximal abelian subgroup of $\mathrm{Aut}(C^*(E))$, and the quotient $\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))/\,\mathrm{Aut}_{\mathcal{D}_E}(C^*(E))$ is a countable discrete group, called the *Weyl group* of $C^*(E)$, [4].

The subgroup of the Weyl group corresponding to those automorphisms which in addition globally preserve $\mathcal{F}_E$ is called the *restricted Weyl group* of $C^*(E)$, and its image in $\mathrm{Out}(C^*(E))$ the restricted outer Weyl group of $C^*(E)$. The subgroup $\langle \lambda(\mathcal{P}_E)^{-1}, \mathrm{Aut}(E)\rangle$ of $\mathrm{Aut}(C^*(E))$ generated by permutative automorphisms and graph automorphisms embeds into the restricted Weyl group. Furthermore, if an element of this subgroup has infinite order in $\mathrm{Aut}(C^*(E))$ then it automatically has infinite order in $\mathrm{Out}(C^*(E))$, [4].

Say a unital $*$-endomorphism $\alpha : \mathcal{D}_E \to \mathcal{D}_E$ *eventually commutes with the shift* if there exists a positive integer m such that $\alpha\varphi^m$ and φ commute. (Here $\varphi : \mathcal{D}_E \to \mathcal{D}_E$ is the usual shift: $\varphi(d) = \sum_{e \in E^1} S_e d S_e^*$.) We denote by $\mathfrak{A}_E$ the group of those $\alpha \in \mathrm{Aut}(\mathcal{D}_E)$ that both α and α^{-1} eventually commute with the shift. Let E be a finite graph without sinks and sources in which all loops have exits. In addition, suppose that the center of $\mathcal{F}_E$ is trivial. Then the restriction map gives rise to an embedding of the restricted Weyl group of $C^*(E)$ into $\mathfrak{A}_E$, [4]. This result provides a far reaching generalization of the known fact (cf. [9, 2, 12]) that automorphisms of a one-sided subshift of finite type extend to the corresponding Cuntz-Krieger algebra. In the special case of $\mathcal{O}_n$ this embedding is surjective, [3]. Further, there exists an embedding of the restricted outer Weyl group of $\mathcal{O}_n$ into the quotient by its center of the automorphism group of the full two-sided

n-shift, and this embedding is an isomorphism for n prime, [3]. This provides a new perspective on the much studied group of shift automorphisms, [10].

If a unitary $u \in C^*(E)$ belongs to the algebraic part of the core AF-subalgebra $\mathcal{F}_E$ then the endomorphism λ_u of $C^*(E)$ is called *localized.* Such endomorphisms may be viewed as maps on the Leavitt path algebra corresponding to graph E, [1]. Thus many of our results about localized endomorphisms of $C^*(E)$ may be applied in this purely algebraic context as well.

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