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Mini-Workshop: Generalizations of Symmetric Spaces

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Abstract. This workshop brought together experts from the areas of algebraic Lie theory, invariant theory, Kac–Moody theory and the theories of Tits buildings and of symmetric spaces. The main focus was on topics related to symmetric spaces in order to stimulate progress in current research projects or trigger new collaboration via comparison, analogy, transfer, generalization, and unification of methods. Specific topics that were covered include Kac–Moody symmetric spaces, double coset decompositions of (groups of rational points of) algebraic groups and Kac–Moody groups, and symmetric/Gelfand pairs in Lie algebras.

Mathematics Subject Classification (2000): 17Bxx, 20Gxx, 51E24, 53C35.

Introduction by the Organisers

The topics of this workshop all in some way evolved from the classical theory of real and complex Lie groups. One of the important mathematical goals during the 1950's was to find analogs of the semisimple Lie groups of exceptional type over arbitrary fields. Chevalley completed the first crucial step by producing his famous basis theorem for simple complex Lie algebras, and later Steinberg succeeded in describing these analogs group-theoretically. An important theory developed by Tits was the theory of groups with a BN-pair and the invention of buildings; these buildings belong to arbitrary Chevalley groups as naturally as the projective spaces belong to the special linear groups. Certain S-arithmetic groups in positive characteristic and the Kac–Moody groups also belong to the class of groups admitting BN-pairs.

Since then the various disciplines developed into different directions. However, due to their common origin the different theories often lead naturally to similar questions. The discussions during the workshop concerning Kac–Moody symmetric spaces may serve as a suitable example how different approaches and backgrounds can interact: the absence of a KAK decomposition for real Kac–Moody groups implies that real Kac–Moody symmetric spaces suffer from the shortcoming that there exist pairs of points that do not lie in a common flat; an identification of the set of points of a real Kac–Moody symmetric space with the set of anisotropic involutions of the corresponding real Kac–Moody group immediately shows that nevertheless each point of the Kac–Moody symmetric space can be joined with each point at infinity by a geodesic ray; investigations as to whether KNK (Kostant) decompositions hold in real Kac–Moody groups will provide insight on whether a reasonable concept of horospheres exists in Kac–Moody symmetric spaces; connections on real Kac–Moody symmetric spaces arise by abstract means from the underlying Kac–Moody Lie triple systems.

In total there have been 16 talks by 12 of the participants of the workshop that gave insight into different aspects of the theory of symmetric spaces, its generalizations, and neighbouring fields. These 16 talks are represented by the 15 attached reports, the two talks on hyperbolic Kac–Moody geometry having been subsumed into one report.

We are particularly pleased by the lively interaction between the participants during the long afternoon breaks (each morning's lectures finished at 11.30 a.m. while the afternoon sessions only started at 4 p.m.) and during the evening.

Mini-Workshop: Generalizations of Symmetric Spaces

Table of Contents

Abstracts

Real geometric invariant theory and double coset spaces

Gerald W. Schwarz
(joint work with Aloysius G. Helminck)

Let U be a compact connected Lie group and let $U_{\mathbb{C}}$ denote its complexification. Assume that we have a real form G of $U_{\mathbb{C}}$ with corresponding real involution φ which preserves U. Assume that we have holomorphic involutions σ and θ of $U_{\mathbb{C}}$, commuting with φ, such that they generate a finite group of automorphisms of the connected center of $U_{\mathbb{C}}$. Let H denote G^{σ}, the fixed points of σ, and let K denote G^{θ}. We are interested in the space of double cosets $H\backslash G/K$.

As in our previous work [HelS01], we identify G/K with a submanifold X of G via the mapping $g \mapsto \beta(g) := g\theta(g^{-1})$. Via this identification, the H action on G/K becomes the $*$-action on X where $h * x := hx\theta(h)^{-1}$, $h \in H$, $x \in X$. We show that there is a quotient $X/\!\!/H$ parameterizing the closed orbits (then one can, in principal, determine all orbits). We can assume that the Cartan involution δ of $U_{\mathbb{C}}$ commutes with σ and θ. Let G_0 denote $G \cap U$. Using the results of [HeiS07] we define a kind of moment mapping on X whose zero set $\mathcal{M}$ is $H_0 := (H \cap U)$-invariant and has the following properties:

- An orbit $H * x$ is closed if and only if it intersects $\mathcal{M}$.
- For every $x \in X$, the orbit closure $\overline{H * x}$ contains a unique H_0-orbit in $\mathcal{M}$.
- The inclusion $\mathcal{M} \to X$ induces a homeomorphism $\mathcal{M}/H_0 \simeq X/\!\!/H$.

Now $X_0 := \beta(G_0) \subset G_0$ has the $*$-action of $G_0 \supset H_0$. Let A be a (connected) torus in X. We say that A is (σ,θ)-split if $\sigma(a) = \theta(a) = a^{-1}$ for all $a \in A$. Now let A_0 be a maximal (σ,θ)-split torus in G_0 (so $A_0 \subset X_0$). Then it follows from [Mat97] that there is a finite Weyl group W_0^* acting on A_0 such that the inclusion $A_0 \to X_0$ induces a homeomorphism $A_0/W_0^* \simeq X_0/H_0$. The idea is to try to find a similar result for the H_0-action on $\mathcal{M}$.

Let $x \in X$. Then there is a natural submanifold P_x of X which is stable under conjugation by H_x such that $P_x x$ is transversal to the orbit $H * x$ at x. If $H * x$ is closed, then an H_x-stable open subset of $P_x x$ is a slice for the action of H. Moreover, P_x is a symmetric space for the action of H_x. We say that x is a *principal point* if the action of H_x on $\mathcal{S}_x := T_e P_x$ is trivial. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{r}_0$ be the Cartan decomposition of $\mathfrak{g}$. If $u \in G_0$, then $\mathcal{S}_u$ is δ-stable and decomposes into a compact part $\mathcal{S}_u \cap \mathfrak{g}_0$ and a noncompact part $\mathcal{S}_u \cap \mathfrak{r}_0$. There is a natural H_0-equivariant surjective map $\pi\colon \mathcal{M} \to X_0$ where the fiber of π above $u \in X_0$ is $\exp(\mathcal{S}_u \cap \mathfrak{r}_0)u$. Thus $\mathcal{M}$ fibers over X_0 with fiber over u the noncompact part of the transversal at u. We show that there is a natural and finite stratification of A_0 which is W_0^*-stable such that the mapping π is a fiber bundle over each stratum. This in turn implies that $X/\!\!/H \simeq \mathcal{M}/H_0$ is a fiber bundle over the images of the strata in A_0/W_0^*. If u lies in a stratum S of A_0, then the fiber over the image of S in A_0/W_0^* is $\exp(\mathcal{S}_u \cap \mathfrak{r}_0)/(H_0)_u$. Moreover, for any maximal

σ-split commutative subspace $\mathfrak{t}$ of $\mathcal{S}_u \cap \mathfrak{r}_0$ there is a finite Weyl group $W(S, \mathfrak{t})$ such that $\exp(\mathcal{S} \cap \mathfrak{r}_0)/(H_0)_u \simeq \exp(\mathfrak{t})/W(S, \mathfrak{t})$.

There is another way to parameterize the quotient $X/\!\!/H$. Let $u \in A_0$ and let A be a δ-stable maximal (σ, θ_u)-split torus. Here θ_u denotes θ followed by conjugation by u. Then $Au \subset \mathcal{M}$. We say that A (or Au) is *standard* if $A \cap U = A \cap A_0$. We say that maximal (σ, θ_{u_i})-split tori $A_i u_i$, $i = 1, 2$, are *equivalent* if there is an $h \in H$ such that $h * A_1 u_1 = A_2 u_2$. Then we show the following:

- For each stratum of A_0/W_0^* there is at most one associated standard maximal Au. Let $\{A_i u_i\}$ be a maximal collection of pairwise non-equivalent tori coming from the strata. Then $\cup_i A_i u_i \to X/\!\!/H$ is surjective. If x is a principal point, then $H * x$ intersects precisely one of the $A_i u_i$ and P_x is a maximal (σ, θ_x)-split torus.
- If $A_1 u_1$ and $A_2 u_2$ are standard maximal, then they are equivalent if and only if there is a $w \in W_0^*$ such that $w * (A_1 u_1 \cap A_0) = A_2 u_2 \cap A_0$.
- If A is a maximal (σ, θ_u)-split torus, then the group of self-equivalences of Au is a finite group. This group acts freely on the set of principal points of Au.

Our results are a natural follow up to [HelS01] where we considered the problem of determining the quotient $G^\sigma \backslash G / G^\theta$ (the complex case, or more generally the case of an algebraically closed field of characteristic not 2). Our techniques were those of invariant theory, i.e., slice theorems, isotropy type stratifications, etc. We also used these techniques here. The new ingredient is the use of moment-map techniques from [HeiS07]. The moment map techniques have also been used in a recent paper of Miebach [Mie07] who works in the setting of Matsuki's characterization of the double coset spaces. The main novelty of our results is the use of stratifications of A_0/W_0^* which help in determining the topological structure of $X/\!\!/H$.

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Cartan decomposition for p-adic symmetric spaces

Vincent Sécherre

1. p-adic reductive symmetric spaces

Let $\mathbf{G}$ be a connected reductive group defined over a non-Archimedean locally compact field k of odd residue characteristic, and let $\mathbf{H}$ be an open k-subgroup of the fixed point subgroup of an involutive k-automorphism σ of $\mathbf{G}$. We write $\mathbf{G}_k$ and $\mathbf{H}_k$ for the groups of k-points of $\mathbf{G}$ and $\mathbf{H}$, and consider the p-adic reductive symmetric space $X = \mathbf{H}_k \backslash \mathbf{G}_k$. Harmonic analysis on X is the study of the action of $\mathbf{G}_k$ on the space of complex smooth functions on X, that is, complex functions on X that are invariant under an open subgroup of $\mathbf{G}_k$.

This is related to the study of those representations of $\mathbf{G}_k$ that are smooth (that is, such that any vector has an open stabilizer) and $\mathbf{H}_k$-distinguished (that is, having a non-zero space of $\mathbf{H}_k$-invariant linear forms).

Connected reductive groups can be considered as reductive symmetric spaces. Indeed, if $\mathbf{G}'$ is such a group, the map:

$$\sigma : (x, y) \mapsto (y, x)$$

defines a k-involution of $\mathbf{G} = \mathbf{G}' \times \mathbf{G}'$ whose fixed point subgroup $\mathbf{H}$ is the diagonal image of $\mathbf{G}'$ in $\mathbf{G}$, and the symmetric space $X = \mathbf{H}_k \backslash \mathbf{G}_k$ naturally identifies with $\mathbf{G}'_k$ via the map $(x, y) \mapsto x^{-1}y$. Moreover, if K' is a subgroup of $\mathbf{G}'_k$, and if we set $K = K' \times K'$, then this map induces a bijective correspondence:

$$\{(\mathbf{H}_k, K)\text{-double cosets of } \mathbf{G}_k\} \quad \leftrightarrow \quad \{K'\text{-double cosets of } \mathbf{G}'_k\}.$$

If we choose for K' a special maximal compact open subgroup of $\mathbf{G}'_k$, then the decomposition of X into K-orbits corresponds to the Cartan decomposition of $\mathbf{G}'_k$ with respect to the subgroup K'.

Now, given X any p-adic reductive symmetric space and K a special maximal compact open subgroup of $\mathbf{G}_k$, we want to describe the decomposition of X into K-orbits.

2. Results

In this talk, I will give such a description in the case where the group $\mathbf{G}$ is k-split (this is joint work with Patrick Delorme [2]). This uses the maximal (σ, k)-split tori introduced by Helminck [3], that is, the maximal σ-anti-invariant k-split tori of $\mathbf{G}$. By [4], such tori are all conjugate under $\mathbf{G}_k$, and they have only finitely many $\mathbf{H}_k$-conjugacy classes.

Let $\{A_1, \ldots, A_N\}$ be a set of representatives of the $\mathbf{H}_k$-conjugacy classes of maximal (σ, k)-split tori of $\mathbf{G}$ (note that $N > 1$ in general) and S be a σ-stable maximal k-split torus of $\mathbf{G}$ containing a maximal (σ, k)-split torus A. For each $1 \leq i \leq N$, we choose $y_i \in \mathbf{G}_k$ such that $y_i A y_i^{-1} = A_i$.

Theorem 1. *Assume that* $\mathbf{G}$ *is k-split. Let* K *be the stabilizer in* $\mathbf{G}_k$ *of a special point in the apartment attached to* S *in the Bruhat-Tits building of* $\mathbf{G}$ *over* k. *Then:*

$$\mathbf{G}_k = \bigcup_{1 \le i \le N} \mathbf{H}_k y_i S_k K.$$

In a more general context (k any non-Archimedean locally compact field of odd characteristic and $\mathbf{G}$ any connected reductive group over k), Benoist and Oh [1] have obtained a polar decomposition for X. In the case where k has odd residue characteristic and $\mathbf{G}$ is k-split, our decomposition is a refinement of Benoist-Oh's polar decomposition.

3. Methods

To prove Theorem 1, we make a large use of the Bruhat-Tits theory. First, let $\mathbf{G}$ be any connected reductive group over k and write $\mathcal{B}$ for its Bruhat-Tits building. The latter is endowed with an action of σ. Our first task is to prove that $\mathcal{B}$ is the union of its σ-stable apartments.

Note that in the case where $\mathbf{G} = \mathbf{G}' \times \mathbf{G}'$ and $\sigma(x,y) = (y,x)$, the building $\mathcal{B}$ identifies with the product of two copies of the building of $\mathbf{G}'$ over k and this simply says that two arbitrary points in the building of $\mathbf{G}'$ are always contained in a common apartment.

When $\mathbf{G}$ is k-split, we obtain the following refinement (which is not true in general for non-split groups).

Proposition 2. *Assume* $\mathbf{G}$ *is k-split, and let* x *be a special point of* $\mathcal{B}$. *There is a* σ*-stable maximal k-split torus* S *of* $\mathbf{G}$ *such that the apartment corresponding to* S *contains* x *and the maximal* σ*-anti-invariant subtorus of* S *is a maximal* (σ, k)*-split torus of* $\mathbf{G}$.

Note that the disjointness of the various components appearing in the decomposition of Theorem 1 has been investigated by Lagier [5].

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Towards hyperbolic Kac–Moody geometry: groups, symmetric spaces, buildings and beyond

Walter Freyn

1. Kac–Moody geometry: Introduction and state of the art

The classical theory of Kac–Moody algebras emerged in the 1960th independently in the work of V. G. Kac [Kac68], R. V. Moody [Moo67, Moo69], I. L. Kantor [Kan70] and D.-N. Verma (unpublished). These first constructions of Kac–Moody algebras, and preliminary investigation of their associated group structures, were algebraic and combinatorial in nature. Similar to the construction of a finite dimensional simple Lie algebra from a Cartan matrix, Kac–Moody algebras were constructed from a finite set of generators subject to certain relations derived from generalized Cartan matrices [Kac90, MP95]. Hence from a structural point of view, Kac–Moody algebras first appeared as the direct infinite dimensional generalization of simple Lie algebras. This raised natural questions about the geometric properties of these structures.

Associated to Kac–Moody algebras, Jacques Tits introduced Kac–Moody groups as the generalization of algebraic groups (resp. Lie groups) to the framework of Kac–Moody theory. Similar to the theory of algebraic groups, he defined a group functor on the category of rings and showed the resulting (algebraic) Kac–Moody groups to allow for the construction of a twin BN-pair. This is the necessary algebraic structure for the existence of a twin building associated to the Kac–Moody group in a similar way as spherical buildings are associated to simple Lie groups via the BN-pair structure [Tit84]. This construction gives a first hint to a geometric world hidden behind the algebraic definitions of Kac–Moody theory.

Affine Kac–Moody algebras, a distinguished subclass noted already by V. G. Kac, R. V. Moody and I. L. Kantor, opened the door to the next steps towards Kac–Moody geometry. This subclass allows for an explicit description in terms of extensions of loop algebras. A self-suggesting completion of the loop algebras with respect to various norms opens the way to the use of functional analytic methods [Tit89, PS86]. Following this path E. Heintze, R. Palais, C.-L. Terng and G. Thorbergsson [HPTT95] discovered even closer links between affine Kac–Moody algebras and infinite dimensional differential geometry around 1980. C.-L. Terng proved that certain isoparametric submanifolds in Hilbert spaces and polar representations on Hilbert spaces can be described using completions of Kac–Moody algebras [Ter89, Ter95]. Furthermore she conjectured the existence of Kac–Moody symmetric spaces, but pointed out several severe problems that block the way towards their construction [Ter95].

In [Fre09], building on the work of C.-L. Terng and E. Heintze, the author solved this longstanding open problem, constructing affine Kac–Moody symmetric spaces as the global differential geometric objects conjoined to affine Kac–Moody groups. Thus, associated to affine Kac–Moody groups, there exist three classes of differential geometric objects: symmetric spaces, polar actions and isoparametric submanifold. Their construction parallels the classical finite dimensional theory, where the symmetries of polar representations, homogeneous isoparametric submanifolds and Riemannian symmetric spaces are described by semisimple Lie groups. A conceptual analysis of these constructions reveals that the restriction to spherical or affine Kac–Moody groups seems to be not inherent to the geometry but rather imposed by the techniques used and natural by the classes of examples studied.

Those developments hint to the conjecture, that there is a rich infinite dimensional geometry lurking behind all Kac–Moody algebras and groups, which parallels the finite dimensional resp. affine theory. The finite dimensional blueprint serves as a guide in exploring this universe. Thus let us sketch the finite dimensional theory:

Symmetric spaces are coset spaces G/K where G is a semisimple Lie group and K an open subgroup of the fixed point set of an involution of G. Polar representations are representations $G : V \longrightarrow V$ of a Lie group G on a vector space V such that there is a subspace Σ, called a section, meeting each orbit orthogonally [BCO03]. The spherical buildings we are interested in are certain simplicial complexes, whose simplices are in bijection to the parabolic subgroups of some simple Lie group [Ji06, AB08]. Isoparametric submanifolds are submanifold with flat normal bundle and constant principal curvatures [PT88]. Those classes of objects are related as follows. Let us start with a symmetric space $M = G/K$. The isotropy representation of a symmetric space is a polar representation, the converse being proven by J. Dadok [Dad85]. Principal orbits of polar representations are isoparametric submanifolds. Conversely a result of G. Thorbergsson [Tho91] shows any full irreducible isoparametric submanifold of $\mathbb{R}^n$ of rank at least three to be principal orbit of some isotropy representation [BCO03] and references therein. The boundary of a symmetric space of non-compact type can be identified with a building. Furthermore the building can be embedded into the unit sphere of the representation space of the isotropy representation and hence be seen geometrically in the tangent space of the corresponding symmetric spaces [Ebe96].

Important structural equivalences between those four classes of objects permeate the whole theory: For example, all four classes exhibit collections of "flat" subspaces, equipped with a structure-preserving action of a finite, discrete reflection group, the Weyl group. For buildings those "flat" subspaces are subcomplexes, called apartments; for polar actions they appear as sections, for symmetric spaces they are maximal flats (subspaces isometric to some Euclidean space) and for isoparametric submanifolds they appear as the normal spaces. For example via the isotropy representation of a symmetric space, flats are identified with sections of the (polar) isotropy representation. Similarly, via the embedding of a building into the boundary of a noncompact symmetric space, flats of the symmetric space are identified with apartments of the building.

By work done during the last 20 years, a similar infinite dimensional picture is now in central parts well established for the most special class of infinite dimensional Kac–Moody groups, namely the affine Kac–Moody groups. In this setting, the fundamental differential geometric objects are affine Kac–Moody symmetric spaces [Fre07, Fre09, Fre11b]. Their isotropy representations are polar representations on Hilbert spaces. Principal orbits of polar representations are proper Fredholm isoparametric submanifolds in Hilbert space [Ter89]. By a result of E. Heintze and X. Liu [HL99] the converse is true if the codimension is not 1. Furthermore there are twin cities as the appropriate generalization of buildings [Fre09, Fre11a].

The structural equivalences well-known from the finite dimensional theory carry over. The "flat" subspaces are still finite dimensional but now equipped with the action of an affine Weyl group. For example, flats in Kac–Moody symmetric spaces correspond via the isotropy representation to sections of the polar representation. Chambers in twin cities correspond to points in isoparametric submanifolds and the twin city can be embedded equivariantly into the tangent space of a Kac–Moody symmetric space, thus showing a correspondence between flats and apartments.

For more general Kac–Moody groups, the theory is in progress. While there is a well-developed algebraic and combinatorial theory of Kac–Moody algebras, Kac–Moody groups and twin buildings, there are only few recent results about functional analytic completions or manifold structures. We have constructed those completions for general Kac–Moody groups. Furthermore, we have some evidence for the existence of hyperbolic Kac–Moody geometry, having established some relations similar to the finite dimensional blueprint.

2. The need for hyperbolic Kac–Moody geometry

After Kac–Moody algebras of finite type and affine Kac–Moody algebras, hyperbolic Kac–Moody algebras form a third important subclass of Kac–Moody algebras. The Weyl groups of hyperbolic Kac–Moody algebras are hyperbolic reflection groups [Fei80, FKN09, FF83]. Hyperbolic Kac–Moody algebras are completely classified: There are infinitely many hyperbolic Kac–Moody algebras of rank 2 and a finite number of hyperbolic Kac–Moody algebras of higher rank, with the maximal possible rank being 10 [CCC$^+$10]. Recent interest in hyperbolic Kac–Moody algebras arouse because of the conjectured appearance of hyperbolic Kac–Moody groups and hyperbolic Kac–Moody symmetric spaces in M-theory and supergravity. Let us sketch two examples:

(1) **The Cremmer-Julia symmetry groups:** The central idea is to start with a supersymmetric 11-dimensional theory of gravitation and then investigate effects of reduction of dimension. In this way reduction of dimension on a d-dimensional torus gives a theory in $11-d$ dimensions. The scalar fields of this new theory exhibit symmetries of types $E_{n(n)}, n = 1, \dots, 11$, hence leading naturally to non-spherical Kac–Moody groups.
(2) Coset spaces of hyperbolic Kac–Moody groups G/K arise in various conjectures that aim to characterize the symmetries of M-theory by studying

the symmetries of 11-dimensional supergravity [DH01], [DHN02], [Wes01]. These may be summarized briefly as follows.

- E_{11} **conjecture** [Wes01] The maximal supergravity theory in 11 dimensions has E_{11} symmetry, which occurs as a nonlinear realization, that is, as a group coset space $E_{11}/K^*(E_{11})$ together with a spacetime dependence [Wes01].
- E_{10} **conjecture**[DKN07] The values of all fields of 11-dimensional supergravity and their spatial gradients can be mapped to a null geodesic motion in the infinite dimensional coset space $E_{10}/K(E_{10})$.

3. Affine Kac–Moody symmetric spaces

The state of the art of affine Kac–Moody geometry around 2006 is summarized by E. Heintze [Hei06]. The need for Kac–Moody symmetric spaces is described as one of the central open problems of the theory. In [Fre09] we developed the theory of Kac–Moody symmetric spaces. Kac–Moody symmetric spaces are infinite dimensional tame Fréchet Lorentz symmetric spaces. Their structure theory and their classification parallels the one of finite dimensional Riemann symmetric spaces. To state our main results about the geometry of Kac–Moody symmetric spaces, let us fix some notation: we denote by $G_{\mathbb{C}}$ a complex semisimple Lie group and by G a compact real form of $G_{\mathbb{C}}$. Furthermore let σ be a diagram automorphism of order n for $\mathfrak{g}_{\mathbb{C}}$ ($n = 1$ is allowed) and $\omega := e^{\frac{2\pi i}{n}}$. We define the holomorphic loop spaces

$$MG^{\sigma}_{\mathbb{C}} := \{f : \mathbb{C}^* \to G_{\mathbb{C}} | f \text{ is holomorphic and } \sigma \circ f(z) = f(\omega z)\}$$

and

$$MG^{\sigma}_{\mathbb{R}} := \{f : \mathbb{C}^* \to G_{\mathbb{C}} | f(S^1) \subset G, \text{f is holomorphic and } \sigma \circ f(z) = f(\omega z)\} .$$

Complex Kac–Moody groups $\widehat{MG}^{\sigma}_{\mathbb{C}}$ are now constructed as certain $(\mathbb{C}^*)^2$-bundles over $MG^{\sigma}_{\mathbb{C}}$. To simplify notation we omit the superscript σ whenever possible. Let ρ_* denote a suitable involution of the second kind [HG09].

Theorem 1 (affine Kac–Moody symmetric spaces of the "compact" type)**.** *Both the Kac–Moody group $\widehat{MG}^{\sigma}_{\mathbb{R}}$ equipped with its Ad-invariant metric, and the quotient space $X = \widehat{MG}^{\sigma}_{\mathbb{R}}/Fix(\rho_*)$ equipped with its $Ad(Fix(\rho_*))$-invariant metric are tame Fréchet symmetric spaces of the "compact" type with respect to their natural Ad-invariant metric. Their curvatures satisfy*

$$\langle R(X,Y)X, Y\rangle \geq 0 .$$

Theorem 2 (affine Kac–Moody symmetric spaces of the "non-compact" type)**.** *Both quotient spaces* $X = \widehat{MG}^{\sigma}_{\mathbb{C}}/\widehat{MG}^{\sigma}_{\mathbb{R}}$ *and* $X = H/Fix(\rho_*)$*, where* H *is a non-compact real form of* $\widehat{MG}^{\sigma}_{\mathbb{C}}$ *equipped with their Ad-invariant metric, are tame Fréchet symmetric spaces of the "non-compact" type. Their curvatures satisfy*

$$\langle R(X,Y)X,Y\rangle \leq 0\,.$$

Furthermore Kac–Moody symmetric spaces of the non-compact type are diffeomorphic to a vector space.

Similarly to finite dimensional Riemannian symmetric spaces Kac–Moody symmetric spaces appear in pairs, related by a duality relation. This duality relation can be defined similar as for finite dimensional Riemann symmetric spaces, yielding a duality between affine Kac–Moody symmetric spaces of the compact type and affine Kac–Moody symmetric spaces of the non-compact type. Similar to the finite dimensional case, the classification can be reduced to orthogonal symmetric affine Kac–Moody algebras (OSAKAS).

4. Twin cities

In Tits' definition of Kac–Moody groups, he constructed not one group, but groups at various "levels" depending on the level of completion of the Kac–Moody algebra. While the construction of affine twin buildings for "minimal" affine Kac–Moody groups is well known by work of Jacques Tits and several other researchers [AB08], any completion of the Kac–Moody group destroys its twin BN-pair structure — hence this theory fails for all completions. This raises an obstruction to the development of a full geometric theory associated to the completion.

It was not until 2009 that this problem was addressed in the author's work [Fre09], where this obstruction is removed by introducing a new structure, called twin cities. A twin city consists of two (usually uncountable) families of buildings, denoted $\mathfrak{B}^+$ and $\mathfrak{B}^-$, in which each building from one family is twinned with each building from the other family. The family of buildings becomes richer and richer, the weaker the regularity assumptions used. Thus twin cities also reflect the regularity of the corresponding affine Kac–Moody group. For minimal Kac–Moody groups, the two families just reduce to one building each. More precisely, we have the following result:

Theorem 3 (Twin cities)**.** *For each analytic Kac–Moody group* $\mathcal{G}$ *there exists an associated twin city* $\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-$*, such that*

(i) *Each connected component* $\Delta^{\pm}$ *in* $\mathfrak{B}^{\pm}$ *is an affine building.*
(ii) *Each pair* $(\Delta^+, \Delta^-) \in \mathfrak{B}^+ \cup \mathfrak{B}^-$*, consisting of a building* Δ^+ *in* $\mathfrak{B}^+$ *("positive" building) and a building* Δ^- *in* $\mathfrak{B}^-$ *("negative" building), is an affine twin building.*
(iii) $\mathfrak{B} = \mathfrak{B}^+ \cup \mathfrak{B}^-$ *has a spherical building at infinity* $\mathfrak{B}^{\infty}$*.*
(iv) $\mathcal{G}$ *acts on its twin city* $\mathfrak{B}$*.*

(v) *"Small" twin cities, associated to Kac–Moody groups, defined by stronger regularity conditions, embed into "big" twin cities, associated to Kac–Moody groups, defined by weaker regularity conditions.*

Like the spherical buildings of compact symmetric spaces, one can realize the twin cities of an affine Kac–Moody symmetric space geometrically in its tangent space (that is, in the group case in the Kac–Moody algebra). Denote by c a central element and by d a derivation. We have the following result:

Theorem 4 (Embedding of cities). *Denote by $\mathfrak{H}_{\ell,r}$ the intersection of the sphere of radius $l \in \mathbb{R}$ of a real affine Kac–Moody algebra $\widehat{L}(\mathfrak{g},\sigma)$ with the horospheres $r_d = \pm r \neq 0$, where $r_d \in \mathbb{R}$ is the coefficient of d in $\widehat{L}(\mathfrak{g},\sigma)$. There is a 2-parameter family $\varphi_{\ell,r}, (l,r) \in \mathbb{R} \times \mathbb{R}^+$ of $\widehat{L}(G,\sigma)$-equivariant immersions of the twin city $\mathfrak{B}^+ \cup \mathfrak{B}^-$ into $\widehat{L}(\mathfrak{g},\sigma)$. It embeds the geometric realization of $\mathfrak{B}$ into $\mathfrak{H}_{\ell,r}$. The two parts of the city $\mathfrak{B}^+$ and $\mathfrak{B}^-$ are immersed into the two sheets of $H_{\ell,r}$ described by $r_d < 0$ resp. $r_d > 0$ of the space $\mathfrak{H}_{\ell,r}$.*

As a third description, we construct cities in terms of periodic flags in Hilbert spaces. This approach uses parts of the representation theory of loop groups on Hilbert spaces, as developed in [PS86]. The cities are described by chains of subspaces in a generalization of the Sato Grassmannian.

Results along these lines hold also in more general situations: In joint work of L. Carbone, A. Feingold and the author we study embeddings of buildings of hyperbolic type into the s-representations of hyperbolic Kac–Moody algebras. This is one connection between symmetric spaces and buildings, which we conjecture to be true also in the general case. In the case of a compact real form of a hyperbolic Kac–Moody algebra, the interior of the lightcone in each Cartan subalgebras corresponds exactly to the Tits cone. Hence an embedding of the building has its image inside the lightcone of a hyperbolic Kac–Moody algebras with the boundary corresponding to the light cone. This gives a 1-parameter family of embeddings equivariant with respect to the action of the compact real form of the hyperbolic Kac–Moody group.

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Geometry of nilpotent cones in exceptional Lie algebras

Paul Levy

Let G be a simple, simply connected algebraic group over $\mathbb{C}$ and let $\mathfrak{g} = \mathrm{Lie}(G)$. We denote by $\mathcal{N}(\mathfrak{g})$ (or just $\mathcal{N}$ if there is no confusion) the set of nilpotent elements in $\mathfrak{g}$. It is well-known that $\mathcal{N}$ is an irreducible Zariski closed G-stable subset of $\mathfrak{g}$ of dimension $\dim G - r$, where r is the rank of G. The coordinate ring of $\mathcal{N}$ can be described as $\mathbb{C}[\mathfrak{g}]/I$, where I is an ideal generated by r algebraically independent homogeneous G-invariant polynomials. To set the scene, we consider a few examples:

Example 1: $\mathcal{N}(\mathfrak{sl}(2))$

The set of nilpotent elements of $\mathfrak{sl}(2)$ is the set of matrices of the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2+bc = 0$. This is an irreducible surface which is isomorphic to $\mathbb{C}^2/\{\pm 1\}$: to see this, we note that the subring of even degree polynomials in $\mathbb{C}[s,t]$ is generated by s^2, st, and $-t^2$, which satisfy the equation $a^2 + bc = 0$.

Example 2: the minimal nilpotent orbit in $\mathfrak{sp}(4)$

Here we consider not the whole nilpotent cone, but the subset X of symplectic matrices of rank ≤ 1. It can be shown that any element of X can be written in the form

$$\begin{pmatrix} sv & tv & -uv & -v^2 \\ su & tu & -u^2 & -uv \\ st & t^2 & -tu & -tv \\ s^2 & st & -su & -sv \end{pmatrix}, \quad s,t,u,v \in \mathbb{C}.$$

In this way we can identify the coordinate ring of X with the subring of even degree polynomials in $\mathbb{C}[s,t,u,v]$, which is precisely the coordinate ring of $\mathbb{C}^4/\{\pm 1\}$. More generally, the set of rank ≤ 1 symplectic $2n \times 2n$ matrices is isomorphic to $\mathbb{C}^{2n}/\{\pm 1\}$.

Example 3: $\mathcal{N}$ **in a neighbourhood of a subregular nilpotent element in** $\mathfrak{sl}(3)$

Consider the $\mathfrak{sl}(2)$-triple $\{h,e,f\}$ in $\mathfrak{sl}(3)$, where

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since the geometric structure of the whole nilpotent cone of $\mathfrak{sl}(3)$ is quite complicated, we can consider its geometry in a neighbourhood of f by restricting to the transverse slice

$$f + \mathfrak{g}^e = \left\{ \begin{pmatrix} a & b & d \\ 0 & -2a & c \\ 1 & 0 & a \end{pmatrix} : a,b,c,d \in \mathbb{C} \right\}$$

It is straightforward to show that the intersection with $\mathcal{N}(\mathfrak{sl}(3))$ is the set of such elements satisfying $d = -3a^2$ and $bc = 8a^3$. It follows that the coordinate ring of the intersection is isomorphic to $\mathbb{C}[s^3, st, t^3]$, which is the fixed point subring of $\mathbb{C}[s,t]$ under a group of order 3.

These examples give something of a taste for this topic, but they may be misleading. In particular, all three of the constructions here produce quotient varieties; this will not be true of all of the singularities in which we are interested.

1. Generalities about the geometry of $\mathcal{N}$

In this section we state some well-known facts about $\mathcal{N}$. The set of G-orbits in $\mathcal{N}$ is a finite partially ordered set, where $\mathcal{O} \geq \mathcal{O}'$ if and only if $\overline{\mathcal{O}} \supseteq \mathcal{O}'$. There is a unique maximal element in this partially ordered set: the G-orbit of regular nilpotent elements, which we denote $\mathcal{O}_{reg}$. Moreover, there is a unique maximal element in $(\mathcal{N}\backslash\mathcal{O}_{reg})/G$, the *subregular* nilpotent orbit $\mathcal{O}_{subreg}$. At the other end of the poset $\mathcal{N}/G$, there is an obvious minimal element, the zero orbit. Furthermore, there is a unique minimal element in $(\mathcal{N} \setminus \{0\})/G$, the *minimal nilpotent orbit*, which is the orbit of a highest root element.

The orbit structure of $\mathcal{N}$ is intimately related to its geometry. The smooth points of $\mathcal{N}$ are precisely those of $\mathcal{O}_{reg}$. More generally, the smooth points of $\overline{\mathcal{O}}$ are those of the orbit $\mathcal{O}$. We can therefore think of the orbits in terms of a stratification of $\mathcal{N}$: we have the smooth locus $\mathcal{O}_{reg}$ of $\mathcal{N}$, then the smooth locus $\mathcal{O}_{subreg}$ of $\mathcal{N} \setminus \mathcal{O}_{reg}$, and so on. At each stage, the smooth locus of the set that remains is a disjoint union of orbits.

2. Transverse slices

The geometry of $\mathcal{N}$ as a whole (or equivalently, in a neighbourhood of zero) is in general too complicated to determine in detail. A more manageable question is to describe the geometry of a nilpotent orbit closure in the neighbourhood of a point on its boundary. We therefore consider pairs $(\mathcal{O}', \mathcal{O})$ of orbits with $\overline{\mathcal{O}'} \supsetneq \mathcal{O}$, which are called **degenerations**; if there is no $\mathcal{O}''$ with $\mathcal{O}' > \mathcal{O}'' > \mathcal{O}$ then $(\mathcal{O}', \mathcal{O})$ is a **minimal** degeneration.

One means of studying geometry in a neighbourhood of a point is via transverse slices. If $f \in \mathcal{O}$, then a transverse slice to $\mathcal{O}$ at f is an affine linear subvariety $f + \mathfrak{v}$ of $\mathfrak{g}$ such that $\mathfrak{v} \oplus [\mathfrak{g}, f] = \mathfrak{g}$. The transversality ensures that $\mathcal{N}$ is locally analytically isomorphic at f to $G \cdot f \times ((f + \mathfrak{v}) \cap \mathcal{N})$, and similarly for a nilpotent orbit closure $\overline{\mathcal{O}'}$ containing f. Thus, questions about the singular structure of $\mathcal{N}$ or $\overline{\mathcal{O}'}$ near f can be reduced to questions about the intersection with $\mathfrak{v}$. This set-up can be interpreted more abstractly in terms of an equivalence relation on singularities.

Let $f \in \mathcal{O}$ be included in an $\mathfrak{sl}(2)$-triple $\{h, e, f\}$. Then the **Slodowy slice** $S_f := f + \mathfrak{g}^e$ is a transverse slice to $\mathcal{O}$ at f. For instance, in our Example 3 in the introduction, $\mathcal{N}$ is locally analytically isomorphic at f to a product of a smooth variety and a quotient of $\mathbb{C}^2/\Gamma$, where Γ is a cyclic subgroup of $\mathrm{SL}(2, \mathbb{C})$ of order 3.

There are two important special cases associated to orbits at either end of the poset $\mathcal{N}/G$:

The degeneration $(\mathcal{O}_{reg}, \mathcal{O}_{subreg})$

Let $f \in \mathcal{O}_{subreg}$, and consider the intersection $\mathcal{N} \cap S_f = \overline{\mathcal{O}_{reg}} \cap S_f$. By a famous result of Brieskorn [1] (for simply-laced $\mathfrak{g}$) and Slodowy [4] (for all $\mathfrak{g}$), the intersection $\mathcal{N} \cap S_f$ is isomorphic to a simple, or Kleinian singularity. (Recall that a Kleinian singularity is a quotient of $\mathbb{C}^2$ by the action of a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$; such groups are classified, and up to conjugacy they correspond to the simply-laced Dynkin diagrams.) Our Examples 1 and 3 from the introduction are therefore special cases of Brieskorn's theorem.

The degeneration $(\mathcal{O}_{min}, 0)$

Here we take $f = 0$, so that S_f is all of $\mathfrak{g}$. Then the singularity $\overline{\mathcal{O}_{min}} \cap S_f = \overline{\mathcal{O}_{min}}$ is a **minimal singularity**. For instance, our Example 2 is a minimal singularity of type C_2. In general, a minimal singularity of type C_n is isomorphic to $\mathbb{C}^{2n}/\{\pm 1\}$.

3. Minimal degenerations in classical Lie algebras

In two papers in the early 1980s [2, 3], Kraft and Procesi established a set of combinatorial rules for determining the (equivalence class of the) singularity associated to a degeneration $(\mathcal{O}, \mathcal{O}')$ of nilpotent orbits in a classical Lie algebra. These combinatorial rules can be expressed in a striking way in terms of Young diagrams. A consequence of their work is that the singularity associated to a minimal degeneration of nilpotent orbits in classical $\mathfrak{g}$ is one of:

- a Kleinian singularity;
- a minimal singularity;
- a union $X \cup X$ of two Kleinian singularities X, meeting transversally at the common singular point.

Moreover, if $\overline{\mathcal{O}}$ is non-normal then it is non-normal in codimension 2, and the non-normality is *branched*. (For the very even classes in $\mathfrak{so}(2n)$, this last result requires Sommers' solution to the normality question.)

4. Minimal degenerations in exceptional Lie algebras

In joint work with Fu, Juteau and Sommers (to appear), we have studied the singularities associated to minimal degenerations in exceptional Lie algebras. Due to the absence of combinatorial information classifying the nilpotent orbits, our approach has been largely case-by-case. In addition to the Kleinian and minimal singularities, we find that with one as-yet unsolved exception, the singularity associated to a minimal degeneration is one of the following:

- a union of $m \leq 10$ Kleinian singularities of the same type, meeting (pairwise) transversally at the common singular point;
- a union of 2 minimal singularities of the same type ($\neq A_1$), meeting transversally at the common singular point;
- $\mathbb{C}^4/\Gamma$, where Γ is a cyclic subgroup of $\mathrm{SL}_4(\mathbb{C})$ of order 3, with no fixed points;

- Spec($\mathbb{C}[s^2, st, t^2, s^3, s^2t, st^2, t^3]$), or equivalently, the image of the diagonal embedding $\mathbb{C}^2 \to \mathbb{C}^2/\mu_2 \times \mathbb{C}^2/\mu_3$, a non-normal surface with normalization isomorphic to $\mathbb{C}^2$;

- the image of the diagonal embedding $\mathbb{C}^4 \to \mathbb{C}^4/\mu_2 \times \mathbb{C}^4/\mu_3$, a non-normal variety with normalization isomorphic to $\mathbb{C}^4$.

Moreover, we find that non-normality of an orbit closure $\overline{\mathcal{O}}$ may be branched or unibranch (i.e. the normalization map is a bijection); and $\overline{\mathcal{O}}$ may be normal in codimension 2, or at a point of any $\mathcal{O}'$ such that $(\mathcal{O}, \mathcal{O}')$ is a minimal degeneration.

In addition, we find that if $\mathcal{O}$ is special and $\mathcal{O}'$ is the unique minimal orbit in its special piece, then the singularity associated to $(\mathcal{O}, \mathcal{O}')$ is a quotient $\mathbb{C}^{2m}/\Gamma$ where Γ is Lusztig's canonical quotient of the component group of $e \in \mathcal{O}$. This can be thought of as a solution to a local version of a conjecture of Lusztig.

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Computations with θ-groups

Willem de Graaf

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with adjoint group G. We consider gradings of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i,$$

where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. By convention, $m = \infty$ means that we are dealing with a $\mathbb{Z}$-grading. The subalgebra $\mathfrak{g}_0$ is reductive and we let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Then G_0 acting on $\mathfrak{g}_1$ is called a θ-group. One interesting question is to describe the G_0-orbits in $\mathfrak{g}_1$. This has been extensively studied by Vinberg in the 70's ([5], [6]). The aim of this talk is to give an overview of recent work aiming at developing algorithms and computer programs to deal with the nilpotent orbits of G_0. A basic result about these orbits is the following:

- Let $e \in \mathfrak{g}_1$ be nilpotent, then e lies in a homogeneous $\mathfrak{sl}_2$-triple (h, e, f), meaning that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, and $h \in \mathfrak{g}_0$, $e \in \mathfrak{g}_1$, $f \in \mathfrak{g}_{-1}$.
- Let (h, e, f), (h', e', f') be two homogeneous $\mathfrak{sl}_2$-triples. Then e, e' are G_0-conjugate if and only if (h, e, f), (h', e', f') are G_0-conjugate if and only if h, h' are G_0-conjugate.

Now one approach for classifying nilpotent orbits runs as follows (here $\mathfrak{h}_0$ is a fixed Cartan subalgebra of $\mathfrak{g}_0$):

(1) From the classification of the nilpotent G-orbits in $\mathfrak{g}$ we get a finite set $\mathcal{H} \subset \mathfrak{h}_0$ containing all $h \in \mathfrak{h}_0$ that lie in an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.
(2) Get rid if the G_0-conjugates in $\mathcal{H}$. (This is achieved by acting with the Weyl group).
(3) For each remaining element of $\mathcal{H}$ decide whether it lies in a homogeneous $\mathfrak{sl}_2$-triple. For the elements that do, compute such a triple.

A second approach is based on Vinberg's theory of carrier algebras. This associates to a nilpotent orbit $G_0 \cdot e$ a regular semisimple $\mathbb{Z}$-graded flat and complete subalgebra of $\mathfrak{g}$. Conversely, a subalgebra with these properties yields a nilpotent orbit. This reduces the problem of classifying the nilpotent orbits to a combinatorial problem involving root systems, and their conjugacy under the Weyl group.

For more details on both these approaches see [3]. They have been implemented in the GAP4 ([1]) package SLA ([2]).

Now let $\mathfrak{g}$ be a real simple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and adjoint group G. We can apply the above algorithms to the problem of listing the nilpotent G-orbits in $\mathfrak{g}$. This is an ongoing project with Heiko Dietrich. The idea is to use the Kostant-Sekiguchi correspondence which provides a bijection

$$\{\text{nilpotent } G\text{-orbits in } \mathfrak{g}\} \leftrightarrow \{\text{nilpotent } K^{\mathbb{C}}\text{-orbits in } \mathfrak{p}^{\mathbb{C}}\}.$$

Here $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$ and so on. Using the previously indicated algorithms we can list the nilpotent $K^{\mathbb{C}}$-orbits in $\mathfrak{p}^{\mathbb{C}}$. In order to use the bijection above, for each nilpotent $K^{\mathbb{C}}$-orbit in $\mathfrak{p}^{\mathbb{C}}$ we need a complex Cayley triple. That is a homogeneous $\mathfrak{sl}_2$-triple (h, e, f) such that $\sigma(e) = f$, where $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is the conjugation with respect to $\mathfrak{g}$. Once such a triple is found, the element $ih + e + f$ is a representative of the real orbit. Finding a Cayley triple in the end boils down to solving a system of polynomial equations. Currently we have performed this method for all real simple Lie algebras of ranks up to 8.

In a joint work with Vinberg and Yakimova ([4]) we have considered the problem of deciding whether $O' \subset \overline{O}$, where O', O are two nilpotent orbits of a θ-group. Let (h', e', f'), (h, e, f) be homogeneous $\mathfrak{sl}_2$-triples corresponding to these orbits, with $h, h' \in \mathfrak{h}_0$. Set $V = \mathfrak{g}_1$ and

$$V_k(h) = \{v \in V \mid [h, v] = kv\} \text{ and } V_{\geq l}(h) = \bigoplus_{k \geq l} V_k(h).$$

Let W_0 be the Weyl group of $\mathfrak{g}_0$ (with respect to $\mathfrak{h}_0$). For $w \in W_0$ set $U(w) = V_2(h') \cap V_{\geq 2}(wh)$. Then:

- $O' \subset \overline{O}$ if and only if there is a $w \in W_0$ such that $U(w)$ contains a point of O'.
- Moreover, in that case, $U(w) \cap O'$ is dense in $U(w)$.

This gives a straightforward probabilistic method to decide whether $O' \subset \overline{O}$. Two main problems remain (and are discussed in the mentioned reference): prove that $U(w) \cap O'$ is empty, and loop over W_0.

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Orbits of parabolic k-subgroups on symmetric k-varieties

Gerard F. Helminck

(joint work with Aloysius G. Helminck)

Symmetric k-varieties are homogeneous spaces $X_k := G_k^\theta \backslash G_k$. Here G is a reductive connected algebraic group defined over a field k, $\text{char}(k) \neq 2$, and G_k denotes the group of k-rational points of G. Further, θ is an automorphism of order two of G, defined over k, and G^θ is its fixed point group. The k-points of G^θ are denoted similarly by G_k^θ and the same notation will be used for any algebraic group H defined over k.

Orbits of parabolic k-subgroups on these varieties occur in various situations. Our motivating example comes from representation theory where we choose k to be a local field. A central issue there is the decomposition of natural representations of G_k related to X_k into irreducible ones. Here one can think of the right regular representation of G_k on $L^2(X_k, dx)$. The building blocks of these decompositions are families of intertwining operators from the C^∞-vectors of induced representations from the k-points P_k of a parabolic k-subgroup to $C^\infty(X_k)$, see [1]. In order that, these families are nontrivial on open $G_k^\theta \times P_k$-orbits, one can show, see [2], that P has to be θ-split, i.e. $P \cap \theta(P)$ is the Levi-component of P and $\theta(P)$. Such a P contains then also a (θ, k)-split torus A, i.e. A is k-split and $\theta(a) = a^{-1}$ for all $a \in A$, such that the Levi-component L of P is equal to the centralizer $Z_G(A)$ of A in G.

We have a number of structural results for this class of parabolic k-subgroups that can be used at the actual construction of these intertwining operators. First of all, we note that minimal θ-split parabolic k-subgroups are described by the maximal (θ, k)-split tori, as the following result shows:

Proposition 1. *Let P be a θ-split parabolic k-subgroup of G and A a θ-stable maximal k-split torus of P. Then the following conditions are equivalent:*

(*i*) *P is a minimal θ-split parabolic k-subgroup of G.*

(ii) $P \cap \theta(P)$ *has no proper* θ*-split parabolic* k*-subgroups.*
(iii) θ *is trivial on the isotropic factor of* $P \cap \theta(P)$ *over* k.
(iv) A^- *maximal* (θ, k)*-split torus of* G *and* $Z_G(A^-) = P \cap \theta(P)$.

In relation to the orbit structure, we found that minimal θ-split parabolic subgroups are conjugate under G^θ:

Lemma 2. *Let* P *be a minimal* θ*-split parabolic* k*-subgroup of* G *and* P_0 *a minimal parabolic* k*-subgroup of* G *contained in* P*. Then we have the following conditions:*
(i) $(G^\theta)^0 P = (G^\theta)^0 P_0$.
(ii) $(G^\theta)^0 P_0$ *is open in* G.

The orbits of G_k^θ on G_k/P_k contained in the open orbit $G^\theta P$ can be described by

Theorem 3. *Let* $\{A_i \mid i \in I\}$ *be representatives of the* G_k^θ*-conjugacy classes of maximal* (θ, k)*-split tori of* G*. There is a one to one correspondence between the* $G_k^\theta \times P_k$*-orbits on* G_k *contained in* $(G^\theta P)_k$ *and* $\cup_{i \in I} W(A_i)/W_{G_k^\theta}(A_i)$*. In particular the orbits are characterized by* $G_k^\theta n P_k$ *with* n *a representative for* $W(A_i)/W_{G_k^\theta}(A_i)$ *in* $N_{G_k}(A_i)$ $(i \in I)$.

We have for general parabolic k-subgroups P a description of the orbits $G_k^\theta \backslash G_k/P_k$. Details can be found in [3]. For simplicity, we describe the result for k a local field:

Theorem 4. *Let* k *be a local field and* P *a parabolic* k*-subgroup of* G*. The space* $G_k^\theta \backslash G_k/P_k$ *can be described by multiplets* $(Q', Q(1), \cdots, Q(r), T_1, \cdots, T_r)$*, where* Q' *is a parabolic* k*-subgroup, conjugate to* P *under* G_k*, each* $Q(i)$ *is a minimal parabolic* k*-subgroup in* Q' *containing the maximal* θ*-stable* k*-split torus* T_i *such that* $Q'_k \cap G_k^\theta Q(i)_k$ *is open in* Q'_k *and moreover the* $Q(i)$ *are not* $Q'_k \cap G_k^\theta$ *conjugate.*

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Contractions of Lie algebras

Oksana Yakimova

Let $\mathfrak{g} = \operatorname{Lie} G$ be a simple (non-Abelian) Lie algebra defined over a field $\mathbb{K}$ of characteristic zero. For safety, assume that $\mathbb{K}$ is algebraically closed and G is connected. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a symmetric decomposition of $\mathfrak{g}$ induced by an involution σ. We can contract $\mathfrak{g}$ to a new (non-reductive) Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ becomes an Abelian ideal. Set $G_0 = G^\sigma$ and $\tilde{G} = G_0 \ltimes \exp(\mathfrak{g}_1)$, here $\operatorname{Lie} \tilde{G} = \tilde{\mathfrak{g}}$. It was conjectured by D. Panyushev [3] that $\mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{G}}$ is a polynomial algebra in ℓ variables, where $\ell = \operatorname{rk} \mathfrak{g}$.

The decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ induces a bi-grading on the symmetric algebra $\mathcal{S}(\mathfrak{g})$. For each $H \in \mathcal{S}(\mathfrak{g})$, let $H^\bullet$ be its highest $\mathfrak{g}_1$-component, i.e., a bi-homogeneous summand with the highest degree in $\mathfrak{g}_1$. As can be easily seen, if $H \in \mathcal{S}(\mathfrak{g})^G$, then $H^\bullet \in \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{g}}$. It is known that $\operatorname{tr.deg} \mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{G}} = \ell$ and if $H_i^\bullet$ are algebraically independent for a set of homogeneous generators $H_1, \ldots, H_\ell$ of $\mathcal{S}(\mathfrak{g})^G$, then these highest components generate $\mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{G}}$ [3]. For many involutions, such sets of generators were constructed in the same paper of Panyushev. However, there are 4 cases, where $\mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{G}}$ cannot be generated by the highest components.

Note that $\mathcal{S}(\mathfrak{g}_1)^{G_0} \subset \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{G}}$. Let $\varphi : \mathbb{K}[\mathfrak{g}]^G \to \mathbb{K}[\mathfrak{g}_1]^{G_0}$ be the restriction homomorphism. Then $\varphi(H)$ is either zero or $H^\bullet$, the latter if and only if $H^\bullet$ is an element of $\mathcal{S}(\mathfrak{g}_1)$. According to [1], φ is not surjective for the following 4 symmetric pairs $(\mathfrak{g}, \mathfrak{g}_0)$:

$$(E_6, F_4), (E_7, E_6 \oplus \mathbb{K}), (E_8, E_7 \oplus \mathfrak{sl}_2), (E_6, \mathfrak{so}_{10} \oplus \mathfrak{so}_2)$$

and only for them. Here we concentrate on the other "surjective" pairs.

Theorem 1. *Suppose that* $\sum_{i=1}^{\ell} \deg_{\mathfrak{g}_1} H_i \le \dim \mathfrak{g}_1$*, then* $H_i^\bullet$ *are algebraically independent.*

Combining Theorem 1 and results of Panyushev [3] we get our main result.

Theorem 2. *Suppose that the restriction homomorphism* φ *is surjective. Then there are generators* $H_i \in \mathcal{S}(\mathfrak{g})^G$ *such that* $H_i^\bullet$ *generate* $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{G}}$.

The proof of Theorem 1 relies on a certain equality involving the Poisson tensor π of $\mathfrak{g}$. Suppose that $\dim \mathfrak{g} = n$ and choose a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$. Let Ω be a graded skew-symmetric algebra of the differential forms on $\mathfrak{g}^*$ with polynomial coefficients. It is also a free $\mathcal{S}(\mathfrak{g})$-module. Let also $\mathcal{W}$ be the dual skew-symmetric algebra, generated by the partial derivatives $\delta_i = \delta_{x_i}$. We have $\mathcal{W}^k \cong (\Omega^k)^*$. Let also $\omega = dx_1 \wedge \ldots \wedge dx_n$ be the volume form. Then there is an $\mathcal{S}(\mathfrak{g})$-linear map

$$\frac{1}{\omega} : \Omega^k \to (\Omega^{n-k})^* \cong \mathcal{W}^{n-k}$$

such that for $f \in \Omega^k$ and $g \in \Omega^{n-k}$, $(f/\omega)(g) = a$, where $f \wedge g = a\omega$ (with $a \in \mathcal{A}$). We have $\pi \in \mathcal{W}^2$ and $\pi = \sum_{i<j} [x_i, x_j] \partial_i \wedge \partial_j$. Given $k \in \mathbb{N}$, let

$$\Lambda^k \pi := \underbrace{\pi \wedge \pi \wedge \ldots \wedge \pi}_{k \text{ factors}},$$

be an element of $\mathcal{W}^{2k}$.

The Kostant regularity criterion [2, Theorem 9], can be expressed as follows. For any set of homogeneous generators $H_1, \ldots, H_\ell \in \mathcal{S}(\mathfrak{g})^G$, holds

$$\frac{dH_1 \wedge \ldots \wedge dH_\ell}{\omega} = c \Lambda^{(n-\ell)/2} \pi \,, \tag{1}$$

where c is a non-zero constant.

Let next ψ_t with $t \in \mathbb{K}^\times$ be an automorphism of $\mathfrak{g}$ (as a vector space) such that ψ_t is identity on $\mathfrak{g}_0$ and $t$id on $\mathfrak{g}_1$. One can extend ψ_t to $\mathcal{S}(\mathfrak{g})$, Ω, and $\mathcal{W}$. Set $\pi_t = \psi_t^{-1}(\pi)$. Then for the Poisson tensor $\tilde{\pi}$ of $\tilde{\mathfrak{g}}$ holds: $\tilde{\pi} = \lim_{t\to 0} \pi_t$. In other words, $\tilde{\mathfrak{g}}$ is an Inönü-Wigner contraction of $\mathfrak{g}$. It turns out that if $\sum \deg_{\mathfrak{g}_1} H_i \le \dim \mathfrak{g}_1$, then actually $\sum \deg_{\mathfrak{g}_1} H_i = \dim \mathfrak{g}_1$ and we can contract both sides of Equation (1) obtaining

$$\frac{dH_1^\bullet \wedge \ldots \wedge dH_\ell^\bullet}{\omega} = c\Lambda^{(n-\ell)/2}\tilde{\pi} \neq 0\,,$$

the Kostant regularity criterion for $\tilde{\mathfrak{g}}$.

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Quantum symmetric Kac–Moody pairs

Stefan Kolb

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0. Following Drinfeld and Jimbo there exists a quantized enveloping algebra $U_q(\mathfrak{g})$ which is a Hopf algebra deformation of the enveloping algebra $U(\mathfrak{g})$, depending on a parameter q such that $U_q(\mathfrak{g})$ specializes to $U(\mathfrak{g})$ if q tends to 1. Let, moreover, $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. In this case $U(\mathfrak{k})$ is a Hopf subalgebra of $U(\mathfrak{g})$. However, even if both $\mathfrak{g}$ and $\mathfrak{k}$ are finite dimensional simple Lie algebras, the Hopf algebra $U_q(\mathfrak{k})$ does generally not embed into $U_q(\mathfrak{g})$.

For finite dimensional $\mathfrak{g}$ this problem was resolved by G. Letzter in her theory of quantum symmetric pairs [Let99], [Let02]. She constructed subalgebras $U_q'(\mathfrak{k})$ of $U_q(\mathfrak{g})$ which specialize to $U(\mathfrak{k})$ as q tends to 1. However, $U_q'(\mathfrak{k})$ is not a Hopf subalgebra of $U_q(\mathfrak{g})$ but only satisfies the weaker (right) coideal property

$$\varDelta(U_q'(\mathfrak{k})) \subset U_q'(\mathfrak{k}) \otimes U_q(\mathfrak{g})$$

where $\varDelta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ denotes the coproduct of $U_q(\mathfrak{g})$. The coideal property appears very naturally if one bears in mind that $U_q(\mathfrak{g})$ is obtained from the Hopf algebra $U(\mathfrak{g})$ essentially by deforming the coproduct.

Recently, various examples appeared in the literature of coideal subalgebras of $U_q(\mathfrak{g})$ which are deformations of $U(\mathfrak{k})$ for suitable involutions on infinite dimensional $\mathfrak{g}$. Let $\mathfrak{g}' = \overset{\circ}{\mathfrak{g}} \otimes \mathbb{K}[t, t^{-1}] \oplus \mathbb{K}c$ be the derived Lie algebra of an untwisted affine Kac-Moody algebra. Let $\overset{\circ}{\theta} : \overset{\circ}{\mathfrak{g}} \to \overset{\circ}{\mathfrak{g}}$ be an involutive automorphism of the finite dimensional Lie algebra $\overset{\circ}{\mathfrak{g}}$. Consider the involutive automorphism $\theta' : \mathfrak{g}' \to \mathfrak{g}'$ given by $\theta'(x \otimes t^n) = \theta(x) \otimes t^{-n}$ and $\theta'(c) = -c$. In [MRS03] A. Molev, E. Ragoucy, and

P. Sorba introduced twisted q-Yangians which are quantum analogs of $U(\mathfrak{k})$ for the involution θ'. If $\overset{\circ}{\theta}$ is the Chevalley involution of $\mathfrak{sl}_2(\mathbb{K})$ the corresponding quantum algebra appeared under the name q-Onsager algebra in the physics literature [BK05]. It had previously appeared in Terwilliger's investigation of tridiagonal pairs and polynomial association schemes [Ter01]. Another example class are the quantized GIM Lie algebras introduced by Y. Tan [Tan05].

The aim of this talk is to extend G. Letzter's theory of quantum symmetric pairs to the setting of symmetrizable Kac-Moody algebras in such a way that it contains all the example classes listed above. Let $\mathfrak{b}^+$ denote the standard Borel subalgebra of $\mathfrak{g}$. An involutive automorphisms $\theta : \mathfrak{g} \to \mathfrak{g}$ is said to be of the second kind if $\dim(\theta(\mathfrak{b}^+) \cap \mathfrak{b}^+) < \infty$. It was shown in [KW92] that involutions of the second kind admit a classification in terms of Satake diagrams in a way analogous to the classification in the finite dimensional case in [Ara62]. In the finite case, this classification is at the heart of Letzter's constructions which we extend to the following result:

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra and $\theta : \mathfrak{g} \to \mathfrak{g}$ an involutive automorphism of the second kind. Then there exists an algebra automorphism $\theta_q : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ and a right coideal subalgebra $U_q'(\mathfrak{k})$ of $U_q(\mathfrak{g})$ which are deformations of θ and $U(\mathfrak{k})$, respectively. Moreover, there exists a rich structure theory for $U_q'(\mathfrak{k})$.

We call the pair $(U_q(\mathfrak{g}), U_q'(\mathfrak{k}))$ a quantum symmetric Kac-Moody pair. The structure theory in the above statement consists of Iwasawa decompositions for $U_q(\mathfrak{g})$ and triangular decompositions for $U_q'(\mathfrak{k})$. Moreover, the algebra $U_q'(\mathfrak{k})$ can be written explicitly in terms of generators and relations. It is straightforward to see that the q-Onsager algebra and the quantized GIM Lie algebras appear as examples of quantum symmetric Kac-Moody pairs. Moreover, the algebra $U_q'(\mathfrak{k})$ satisfies a maximality condition which should make it possible to identify twisted q-Yangians as quantum symmetric Kac-Moody pairs. In the finite case, all these structural properties are contained in Letzter's papers [Let99], [Let02], [Let03], [Let04]. A more rigorous and extended version of this talk will appear as [Kol].

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The classification of homogeneous compact geometries

Linus Kramer
(joint work with Alexander Lytchak)

In my talk I explained the following two results.

Theorem A (Grundöfer-Knarr-Kramer 1995/1998)
Suppose that $\widetilde{\Delta}$ is a homogeneous compact connected building whose Coxeter diagram is spherical, connected and of rank at least 2. Then $\widetilde{\Delta}$ is the building associated to a simple noncompact Lie group S. Topologically, it may be identified with the boundary at infinity of the Riemannian symmetric space $X = S/K$, where $K \subseteq S$ is a maximal compact subgroup.

By a *Tits geometry* we mean a residually connected thick geometry of type M, where M is a Coxeter diagram, as in Tits [5]. We call such a geometry Δ *compact* if the set of vertices $Vert(\Delta)$ carries a compact Hausdorff topology, such that the flag varieties are closed subsets of $Vert(\Delta)^k$. A 2-*covering* between two Tits geometries is a simplicial surjection which is bijective on all codimension 2 links.

Theorem B (Kramer-Lytchak 2011/12)
Suppose that Δ is a compact connected Tits geometry whose Coxeter diagram is spherical, connected, and of rank at least 2. Suppose that there is a compact group acting transitively on the maximal flags of Δ. Then there exists a compact homogeneous building $\widetilde{\Delta}$ as in Theorem A and an equivariant 2-covering $\widetilde{\Delta} \to \Delta$, or Δ is a unique compact connected homogeneous geometry of type C_3 which cannot be 2-covered by any building.

The exceptional C_3-geometry that arises here comes from a polar action of cohomogeneity 2 of the group $SU(3) \times SU(3)$ on the Cayley plane [6].

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A boundary approach towards 2-spherical Kac–Moody geometry

Tobias Hartnick
(joint work with Ralf Köhl)

The theory of compact connected spherical topological buildings provides an axiomatic framework for the study of boundaries of symmetric spaces. More precisely, let (W, S) be an irreducible spherical Coxeter system of rank ≥ 2, Δ a building of type (W, S). Δ is called *homogeneous* if its automorphism group acts transitively on the set of chambers of Δ. Given such a building we denote by $\mathcal{O}$ the set of pairs of opposite chambers in Δ and given $(c_1, c_2) \in \mathcal{O}$ and $s \in S$ we denote by $p_s(c_1, c_2)$ the projection of c_2 onto the s-panel of c_1. A Hausdorff topology τ on the set of chambers of Δ is called a *connected building topology* if it satisfies the following two axioms:

(TB1) The projection maps p_s are continuous on $\mathcal{O}$ with respect to the topology induced by τ.
(TB2) Panels are compact and connected.

In this case, τ itself is compact and connected, and there is the following classification result, which links topological buildings to the subject of the mini-workshop:

Theorem 1 (Grundhöfer-Knarr-Kramer, [1])**.** *If Δ is a homogeneous spherical building of type (W, S) and τ a connected building topology on the set of chambers of Δ, then the geometric realization $|\Delta|$ of (Δ, τ) is isomorphic to the boundary of a symmetric space of the non-compact type.*

In this talk we presented an extension of this result to the case, where the underlying Coxeter system (W, S) is no longer assumed to be spherical, but only two-spherical (and irreducible of rank ≥ 2). For this generalization we consider homogeneous twin buildings $\Delta = (\Delta^+, \Delta^-, \delta^*)$ of type (W, S). Using the twinning δ^* the set $\mathcal{O}$ of opposite pairs of chambers and the projection maps p_s can still be defined. In complete analogy with the spherical case, a Hausdorff topology τ on the set of chambers of Δ will be called a *connected twin building topology* if it satisfies axioms (TB1) and (TB2) above.

A major obstruction against the classification of such topologies is provided by the fact that they are in general *not* determined by their (combinatorially) local shape. To make this precise, let us fix base chambers $c_\pm$ in the two halves and denote by $E^\pm_{\leq w}$ the *Schubert varieties* of radius w around $c_\pm$, i.e. the collections of chambers of Weyl group distance at most w. Then, at least for certain Δ, their exist twin building topologies τ, τ' such that

$$\forall w \in W : (E^{\pm}_{\leq w}, \tau|_{E^{\pm}_{\leq w}}) \cong (E^{\pm}_{\leq w}, \tau'|_{E^{\pm}_{\leq w}}), \tag{1}$$

but $\tau \neq \tau'$. Let us call two topologies satisfying (1) *Schubert equivalent.* If we do not distinguish between Schubert equivalent twin building topologies, then a classification becomes feasible:

Theorem 2 (Hartnick-Köhl-Mars, [2])**.** *If Δ is a homogeneous twin building of type (W, S) and τ a connected twin building topology on the set of chambers of Δ, then Δ is homogeneous under a group G from one of the following classes:*

- *($\mathbb{C}$) a complex split Kac–Moody group;*
- *($\mathbb{R}_1$) a real split Kac–Moody group;*
- *($\mathbb{R}_2$) a real quasi-split Kac–Moody group, which arises as the fixed point subgroup of a complex split Kac–Moody group with respect to an involution of the first kind in the sense of* [3]*.*

If we equip G with the final group topology with respect to the Lie group topologies on its rank one subgroups, then the induced quotient topology is a Schubert equivalent refinement of τ.

Given a group G of class ($\mathbb{C}$), ($\mathbb{R}$1) or ($\mathbb{R}$2) we also construct a G-homogeneous k_ω-space X_G, which we refer to as the *Kac–Moody symmetric space of the non-compact type* associated with G. This space comes equipped with a distinguished set of geodesics, which allows us in particular to define the boundary ∂X_G. Using this notion we obtain the following boundary interpretation of topological twin buildings, which links our classification to the classical result of Grundhöfer, Knarr and Kramer quoted above:

Theorem 3 (Hartnick-Köhl)**.** *If Δ is a homogeneous twin building of type (W, S) and τ a connected twin building topology on the set of chambers of Δ, then there exists a Schubert equivalent refinement τ' of τ such that the geometric realization $|\Delta|$ of (Δ, τ') is isomorphic to the boundary of a Kac–Moody symmetric space of the non-compact type.*

The Kac–Moody symmetric spaces appearing in Theorem 3 are not to be confused with the (completed) Kac–Moody symmetric spaces appearing in work of W. Freyn. In fact, the Kac–Moody symmetric spaces referred to above are homogeneous spaces of Kac–Moody groups, while Freyn considers spaces, which are homogeneous under certain functional-analytic completions of such groups. The precise relation between the two notions is not yet fully understood and deserves further investigation. First steps in this direction were undertaken during the present mini-workshop.

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Canonical real structure on spherical varieties

Stéphanie Cupit-Foutou
(joint work with Dmitri Akhiezer)

This report deals with a work obtained jointly with Dmitri Akhiezer ([ACF]).

The ground field is the field of complex numbers $\mathbb{C}$. Let G be a connected reductive algebraic group. An algebraic G-variety is called spherical if it contains a dense (open) orbit for a Borel subgroup B of G, equivalently if it has finitely many B-orbits. Flag varieties, symmetric spaces G/H with $G^\tau \subset G \subset N_G(G^\tau)$ with τ an involution of G and nilpotent orbits $G.e$ with $(\mathrm{ad}e)^4 = 0$ are spherical varieties.

A real structure on a complex manifold X is an anti-holomorphic involution of X. Let $\sigma : G \to G$ be be the involution defining the split real form of G. A real structure μ is called σ-equivariant if it satisfies the following property

$$\mu(g \cdot x) = \sigma(g) \cdot \mu(x) \quad \text{for all } g \in G \text{ and all } x \in X.$$

0.1. There always exists a σ-equivariant real structure on an affine smooth spherical G-variety ([A]). However, in the non-spherical case, even if X is G-homogeneous, there may not exist any real structure on X; see [AP] for an example.

Theorem 1. *Let $X = G/H$ be spherical with H being self-normalizing in G. Then there exists a unique (up to G-automorphism) σ-equivariant real structure on X.*

This theorem follows essentially from the following statement.

Proposition 2. *Under the assumptions of Theorem 1, the assignment $gH \mapsto \sigma(g)\sigma(H)$ defines a real structure on G/H.*

Proof. The main step consists in proving that the subgroups H and $\sigma(H)$ of G are conjugate. This is achieved by the uniqueness part of the classification of spherical homogeneous spaces proved by Losev in [Lo]. □

0.2. A G-variety is called wonderful if it is smooth, complete, with r smooth prime G-divisors $D_1, \dots, D_r$ with normal crossings, such that the G-orbit closures are exactly all the intersections $\cap_{i\in I} D_i$, for any subset I of $\{1, \dots, r\}$.

Note that a wonderful variety has a unique closed G-orbit, that given by taking the intersection of all the D_i's.

Wonderful varieties are spherical; they encompass the flag varieties and the De Concini-Procesi compactifications of symmetric spaces. By a result of Knop ([K]), any spherical homogeneous space G/H with $H = N_G(H)$ has a wonderful compactification. Further, such a compactification is unique.

Thanks to nice properties of the automorphism group of a wonderful variety, we can prove:

Theorem 3. *There are finitely many real structures on a wonderful variety X (up to an automorphism of X).*

As a consequence of Theorem 1, we can obtain the following statement.

Theorem 4. *There exists a unique canonical real structure on the wonderful compactification of a spherical homogeneous space G/H with H self-normalizing.*

The real structure obtained in the above theorem for a wonderful variety X will be referred in the following as the canonical real structure of X.

0.3. Given a wonderful G-variety X equipped with a canonical real structure μ, define its real part as the set

$$\mathbb{R}X = \{x \in X : \mu(x) = x\}.$$

Note that in general, even if a complex manifold has a real structure, it may have an empty related real part.

Let G_0^σ denote the identity-component of the fixed point set G^σ.

Proposition 5. (i) *$\mathbb{R}X$ has finitely many G_0^σ-orbits.*

(ii) $\mathbb{R}X \cap G \cdot x \neq \emptyset, \forall x \in X$.

(iii) *The real part of the closed G-orbit of X is G^σ-homogeneous and is the unique closed G^σ-orbit of $\mathbb{R}X$.*

(iv) *$\mathbb{R}X$ is connected.*

Applying the so-called Local Structure Theorem ([BLV]), we obtain an estimate of the G_0^σ-orbits of $\mathbb{R}X$ in terms of the rank of X, that is the number r of the D_i's.

0.4. The natural question of determining exactly the number of G_0^σ-orbits of $\mathbb{R}X$ arises. Borel and Ji solve this problem in the case of the DeConcini-Procesi compactification ([BJ]).

Another problem one may address is that of considering more general involutions than that defining the split real form of G. In the case of the Cartan involution, one can prove that there is no related equivariant real structure on a projective space.

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Stable boundary cohomology of locally symmetric spaces

Tobias Hartnick
(joint work with Andreas Ott)

Let X be a symmetric space of the non-compact type with isometry group G and $\Gamma < G$ a lattice. Then Γ acts by isometries on X (leading to a locally symmetric space $X_0 := \Gamma\backslash X$) and this action extends to the Furstenberg boundary $\partial_F X$ of X. The action of Γ on $\partial_F X$ is doubly ergodic and amenable; basic invariants of this *boundary dynamical system* are encoded in the cohomology ring $H^\bullet_{L^\infty}(\Gamma, \partial_F X; \mathbb{R})$, which is defined as the cohomology of the cocomplex

$$\mathcal{C}^n := L^\infty(\partial_F X^{n+1}; \mathbb{R})^\Gamma, \quad (n \geq 0),$$

with differential given by

$$df(\xi_0, \ldots, \xi_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_{n+1}).$$

Gromov's theory of bounded cohomology provides a natural comparison map

$$c^\bullet : H^\bullet_{L^\infty}(\Gamma, \partial_F X; \mathbb{R}) \to H^\bullet(X_0; \mathbb{R})$$

relating the boundary dynamical system to the homotopy type of the locally symmetric space X_0. This talk was concerned with the question under which conditions this map is an isomorphism, i.e. to which extend the dynamical information coming from the boundery action coincides with the topological information extracted from the quotient space. We focused on the following result:

Theorem 1. *Assume that the real rank of G coincides with the rank of its maximal compact subgroup. Then $c^n : H^n_{L^\infty}(\Gamma, \partial_F X; \mathbb{R}) \to H^n(X_0; \mathbb{R})$ is surjective, whenever n is sufficiently small compared to the rank of X.*

The theorem applies most notably whenever the symmetric space X admits an invariant complex structure (Hermitian case), but also more generally when G is of Hodge type (e.g. $G = \mathrm{SO}(2p, q)$). In view of classical stability results in cohomology, and corresponding stability results in bounded cohomology (due to Monod [4]), the theorem is a consequence of the following more general statement:

Theorem 2 (Hartnick-Ott, [3])**.** *Assume that the real rank of G coincides with the rank of its maximal compact subgroup. Then the natural comparison map yields a surjection*

$$H^\bullet_{cb}(G; \mathbb{R}) \to H^\bullet_c(G; \mathbb{R}),$$

where $H^\bullet_{cb}(G; \mathbb{R})$, respectively $H^\bullet_c(G; \mathbb{R})$, denotes the continuous bounded, respectively continuous cohomology ring of G.

Theorem 2 provides a positive answer to an old question of Dupont [1] in the case where the real rank of G coincides with the rank of its maximal compact subgroup. The general case remains widely open. The proof of Theorem 2, as presented at the mini-workshop, proceeds in the following steps:

- Step 1: Observe that if $G^\bullet$ denotes the nerve (simplicial space) underlying G (considered as a topological category), then $H_c^\bullet(G;\mathbb{R})$ can be interpreted as the simplicial sheaf cohomology of $G^\bullet$ with coefficients in the sheaf C of continuous functions. On the other hand, the cohomology of the classifying space BG coincides with the simplicial sheaf cohomology of $G^\bullet$ with coefficients in the sheaf of locally constant functions. Thus inclusion of sheaves induces a natural transformation

$$\iota_G^\bullet : H^\bullet(BG;\mathbb{R}) \to H_c^\bullet(G;\mathbb{R}).$$

- Step 2: The image of ι_G consists of bounded classes. This is a consequence of Gromov's boundedness theorem for primary characteristic classes of flat bundles [2], combined with standard transfer arguments in continuous (bounded) cohomology.
- Step 3: In order to identify the image of ι_G, we utilize the following main lemma:

  Lemma. *Let G_c denote the compact dual group of G, $K < G$ a maximal compact subgroup and denote by $f_{G_c} : G_c/K \to BK$ the classifying map of the K-bundle $G_c \to G_c/K$. Then under the isomorphisms $H_c^n(G;\mathbb{R}) \cong H^n(G_c/K;\mathbb{R})$ and $H^n(BG;\mathbb{R}) \cong H^n(BK;\mathbb{R})$ the map ι_G^n gets intertwined with $(-1)^{n/2} f_{G_c}^*$.*

- Step 4: Combining Steps 1–3 we see that preimages of characteristic classes of the K-bundle $G_c \to G_c/K$ under the isomorphism $H_c^n(G;\mathbb{R}) \cong H^n(G_c/K;\mathbb{R})$ are bounded. Now the equal rank assumptions of Theorem 2 ensures that $H^n(G_c/K;\mathbb{R})$ is generated by these characteristic classes. This finishes the proof of Theorem 2.

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Double coset decompositions of Kac–Moody groups

Max Horn

1. Introduction

Let $\mathbb{F}$ be a field of characteristic different from 2. Let G be a Kac–Moody group over $\mathbb{F}$ with saturated twin BN-pair (B_+, B_-, N) of irreducible type (W, S), $|S| > 1$. In particular:

- (W, S) is a Coxeter system;
- (B_+, N) and (B_-, N) are BN-pairs of type (W, S) of G;
- $A := B_+ \cap B_-$ satisfies $A = N \cap B_+ = N \cap B_-$;
- $A \trianglelefteq N$ and $W \cong N/A$.

If W is finite, then G is a reductive algebraic group. This is also called the **spherical case**. B_+ and B_- are conjugate if and only if G is of spherical type.

Let $\sigma \in \mathrm{Aut}(\mathbb{F})$ with $\sigma^2 = \mathrm{id}_{\mathbb{F}}$. The **norm map** of $\mathbb{F}$ with respect to σ is

$$N_\sigma : \mathbb{F} \to \mathrm{Fix}_{\mathbb{F}}(\sigma) : x \mapsto xx^\sigma.$$

The composition θ of the Chevalley involution of G with σ is called **σ-twisted Chevalley involution**. Let K be the subgroup of G fixed by θ; we call this **unitary form** of G with respect to θ.

Our goal is to better understand the coset space $K\backslash G$. In the spherical case, this is a symmetric space (over the reals or complex numbers), respectively a symmetric variety. A natural question is to ask what happens in the non-spherical case. In analogy, one might call this a **Kac–Moody symmetric space**. The hope is that such spaces carry a sufficiently rich structure to warrant this name. In this presentation, we present some results on orbit structures on $K\backslash G$, with the hope that this will lead towards a better understanding of $K\backslash G$.

2. B-orbits on $K\backslash G$

Let B be a **Borel subgroup** of G, i.e. a subgroup conjugate to B_+ or B_-. A **maximal torus** of G is any conjugate of A.

Parameterizing the B-orbits on $K\backslash G$ is equivalent to studying double coset space $K\backslash G/B$. This is in turn equivalent to studying K-orbits on the building G/B. By exploiting the rich geometric structure of buildings, and the fact that θ interchanges the two buildings G/B_+ and G/B_-, one obtains the following:

Theorem 1 ([2])**.** *Let $\{A_i \mid i \in I\}$ be representatives of the K-conjugacy classes of θ-stable tori of G, and let B be a Borel subgroup of G. Then*

$$K\backslash G/B \cong \bigcup_{i\in I} W_K(A_i)\backslash W_G(A_i).$$

In [3], a similar result was shown for spherical G, over arbitrary fields (of characteristic not 2), while in [7] it was shown for split Kac–Moody groups over an algebraically closed field of characteristic zero.

Note that with similar methods as were used to prove the Theorem above, one can also study and parametrize orbits of parabolic subgroups on $K\backslash G$.

3. Iwasawa decomposition

We now focus on the special case when there is only one B-orbit on $K\backslash G$, i.e. $G = KB$.

Definition 2. A group G with a twin BN-pair admits an **Iwasawa decomposition**, if there exist an involution $\theta \in \mathrm{Aut}(G)$ such that $\theta(B_+) = B_-$ and $G = KB_+$ where $K := \mathrm{Fix}_G(\theta)$.

Theorem 3 ([1])**.** *Let G be a split semisimple algebraic group or a split Kac–Moody group over $\mathbb{F}$. Then G admits an Iwasawa decomposition if and only if there is $\sigma \in \mathrm{Aut}(\mathbb{F})$ with $\sigma^2 = \mathrm{id}_{\mathbb{F}}$ such that*

(1) -1 is not a norm,
(2) sums of norms are norms,
(3) G admits a σ-twisted Chevalley involution.

Question 4. What happens if one relaxes the condition $\theta(B_+) = B_-$?

4. Cartan and polar decompositions

Now that we have some understanding of K-B double cosets on B, the next natural question is to study K-K double cosets, respectively K-orbits on $K\backslash G$. In particular, we would like to find parameterizations for them.

To simplify things, we will from now on assume that θ is the σ-twisted Chevalley involution on G. Moreover, we will assume $G = KB$. Hence -1 is not a norm and sums of norms are again norms.

In the spherical settings, for real or complex algebraic groups, one has the Cartan decomposition $G = KAK$, which is closely related to the polar decomposition $G = \tau(G)K$, where $\tau(g) := g\theta(g^{-1})$. Note that τ induces a bijection between G/K and $\tau(G)$; unfortunately, this is not a group homomorphism.

In the non-spherical case, one can prove the following (for the spherical case, see [3]):

Proposition 5. *Suppose G is spherical, $G = KB$, and $N_\sigma^2(\mathbb{F}) = N_\sigma(\mathbb{F})$. Then $G = KAK$ holds if and only if $G = \tau(G)K$ and the elements of $\tau(G)$ are semi-simple (i.e. contained in a conjugate of A).*

Thus if $G = KAK$ holds, then every element g of G can be written as a product of an element of K, and a semi-simple element in $\tau(G)$. Indeed, $\tau(g)$ itself must be semi-simple. It seems unlikely that this is possible in the non-spherical case. The following example gives some evidence for this.

Example 6. Consider $G = \mathrm{SL}_n(\mathbb{F}[t, t^{-1}])$ for any $n \geq 2$, a split Kac–Moody group of type $\tilde{A}_{n-1}$. For θ we take the Chevalley involution. Then $G = KB$ holds.

On the matrix level, θ amounts to transposing followed by inversion followed by interchanging t and t^{-1}. Then the matrix $\left(\begin{smallmatrix} 1 & t \\ t & 1+t^2 \end{smallmatrix}\right)$ (suitably extended to a $(n \times n)$-matrix) satisfies

$$\tau(\left(\begin{smallmatrix} 1 & t \\ t & 1+t^2 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 1 & t \\ t & 1+t^2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t^{-1} \\ t^{-1} & 1+t^{-2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 2t^{-1}+t \\ t^{-1}+2t & 3+t^{-2}+t^2 \end{smallmatrix}\right)$$

but the eigenvalues of this matrix are

$$\lambda_{1,2} = \frac{1}{2}\left(2 + t^2 \pm t\sqrt{4+t^2}\right)$$

which shows that the matrix is not diagonalizable over $\mathbb{F}[t, t^{-1}]$. In fact the semi-simple elements of G are conjugates of the diagonal matrices in $\mathrm{SL}_2(\mathbb{F})$, while the eigenvalues given above are clearly not in $\mathbb{F}$.

More generally, one can ask the following:

Question 7. If $G = KAK$ holds, does it follow that G is spherical?

Note that the requirements for a polar decomposition are weaker than for a Cartan decomposition.

Question 8. When do we have a polar decomposition? Does $\mathrm{SL}_n(\mathbb{F}[t, t^{-1}])$ admit a polar decomposition?

5. Kostant decomposition

Let U be the unipotent radical of a Borel subgroup B of G. In the spherical case (at least over the reals and complex numbers) one has $G = KUK$; indeed, in [5] on sees that $G = KUK$, $G = KAK$, $G = KB$ and $G = \tau(G)K$ are essentially equivalent in the real case.

On the other hand, in the non-spherical case (even over the reals or complex numbers), $G = KB$ holds, while $G = KAK$ in general does not hold. Still, one may ask:

Question 9. When does $G = KUK$ hold in the non-spherical case?

As a first step, it would already be interesting to know the following:

Question 10. When does $G = KUK$ hold for $\mathrm{SL}_n(\mathbb{F}[t, t^{-1}])$?

The proof in [5] relies heavily on so-called **Kostant convexity**. For the non-spherical real and complex case, this convexity result was generalized in [6]. For another different generalization to the affine case, see [4].

Hence it would be interesting whether one can generalize this further to arbitrary non-spherical Kac–Moody groups, ideally replacing Lie algebra methods by building theoretical methods.

In closing, note that even with a good generalization of Kostant convexity, some serious obstacles remain in (dis)proving $G = KUK$; on the other hand, it is conceivable that one can prove $G = KUK$ without using Kostant convexity.

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Nilpotent Gelfand pairs and spherical functions on them

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(joint work with V. Fischer, F. Ricci)

Let G be a real Lie group, $K \subset G$ a compact subgroup, and assume that $X = G/K$ is connected. Then (G, K) is said to be a *Gelfand pair* if the algebra $\mathcal{D}(X)^G$ of G-invariant differential operators on X is commutative. Equivalently one can say that $L^1(K\backslash G/K)$ is commutative with respect to the convolution or that the action of G on $L^2(X)$ has a simple spectrum. If G/G^σ is a symmetric Riemannian homogeneous space in the sense of Élie Cartan, then (G, G^σ) is a Gelfand pair. In case G is compact, this was proved by Cartan himself on a case-by-case basis. Thirty years later Gelfand gave a conceptual proof that works for non-compact G as well.

A Gelfand pair is said to be nilpotent if $G = K \ltimes N$, where N is the unipotent radical, in other words, K is a maximal reductive subgroup of G. Together with the reductive pairs (meaning that G is reductive), they provide building blocks for all Gelfand pairs. All Gelfand pairs, nilpotent and not, are classified [3].

Since the algebra $\mathcal{D}(X)^G$ is commutative, one can consider its common eigenvectors. Common eigenfunctions that are K-invariant and normalised by the condition $\varphi(eK) = 1$ are called *spherical* functions on X. The name comes from the case $X = \mathrm{SO}_{n+1}/\mathrm{SO}_n$, where $\mathcal{D}(X)$ is generated by the Laplacian and its eigenfunctions are known as spherical harmonics.

Suppose that $X = N$ arises from a nilpotent Gelfand pair (G, K). Then $\mathcal{D}(N)^G \cong \mathbf{U}(\mathfrak{n})^K$, where $\mathfrak{n} = \mathrm{Lie}\, N$. This algebra is finitely generated, let us say by the operators $D_1, \ldots, D_d$. In this case they can be chosen to be formally self-adjoint, meaning that $D_i\varphi = \lambda_i(\varphi)\varphi$ with $\lambda_i(\varphi) \in \mathbb{R}$ for any bounded spherical function φ. Sending each φ to a d-tuple $(\lambda_1(\varphi), \ldots, \lambda_d(\varphi))$ we can identify the set of bounded spherical functions with a subset Σ_d in $\mathbb{R}^d$, which is called the Gelfand spectrum of (N, K). The identification is known to be an isomorphism of topological spaces.

Let $\mathcal{S}(N)$ be the set of Schwartz functions on N. A *Gelfand transform* of $F \in \mathcal{S}(N)^K$ is a function on Σ_d defined by: $\mathbf{G}F(\varphi) = \int_N F(x)\varphi(x^{-1})dx$. Fulvio Ricci has conjectured that $\mathbf{G}$ provides an isomorphism between $\mathcal{S}(N)^K$ and $\mathcal{S}(\Sigma_d) := \{f|_{\Sigma_d} \mid f \in \mathcal{S}(\mathbb{R}^d)\}$. It is known that $\mathbf{G}$ is injective and that $\mathcal{S}(\Sigma_d)$ is contained in its image. The problem is to show that the image of $\mathbf{G}$ is not larger, that any $\mathbf{G}F$ extends to a Schwartz function on $\mathbb{R}^d$.

If N is a Heisenberg Lie group, then the conjecture is known to be true. Our immediate goal, which is almost achieved, is to establish the conjecture for the ≪Vinberg's list≫ [2], a certain list of 12 Gelfand pairs, which are crucial for the future considerations. Here already we found some interesting results, for example, in all the cases the algebra $\mathcal{D}(N)^G$ is free and can be described explicitly [1]. Surprisingly, a lot of representation theory technique is used in this analytic problem, for example, the proof that goes by induction on $\dim \mathfrak{n}$ uses Luna's slice theorem.

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Vinberg's θ-groups in positive characteristic

Paul Levy

Let G be a reductive algebraic group over $\mathbb{C}$, let $\mathfrak{g} = \mathrm{Lie}(G)$ and let θ be an automorphism of G of order m. Then $d\theta$ is a linear automorphism of $\mathfrak{g}$ with eigenvalues $\subset \{1, \zeta, \ldots, \zeta^{m-1}\}$ where $\zeta = e^{\frac{2\pi i}{m}}$. Hence there is a direct sum decomposition:

$$\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \ldots \oplus \mathfrak{g}(m-1)$$

where $\mathfrak{g}(j) = \{x \in \mathfrak{g} : d\theta(x) = \zeta^j x\}$.

Moreover, this is a $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$, that is, $[\mathfrak{g}(j), \mathfrak{g}(k)] \subset \mathfrak{g}(j+k)$ for $j, k \in \mathbb{Z}/m\mathbb{Z}$. Let $G(0) = (G^\theta)^\circ$. Then $G(0)$ is reductive, $\mathfrak{g}(0) = \mathrm{Lie}(G(0))$ and $\mathrm{Ad}\, G(0)$ stabilizes each of the subspaces $\mathfrak{g}(j)$.

Kostant and Rallis [1] showed that for an involution (i.e. for $m = 2$), the action of $K = (G^\theta)^\circ$ on $\mathfrak{p}$ has invariant-theoretic properties which generalize known properties of the adjoint representation. This was extended to the case of arbitrary m, and the action of $G(0)$ on $\mathfrak{g}(1)$, by Vinberg in [4].

We will discuss Vinberg's results and their extension to other base fields; we recount Kac's classification of the periodic automorphisms of a simple Lie algebra; we discuss the classification of the so-called *positive rank gradings* of a simple Lie algebra; and we briefly touch on an application to the representation theory of reductive groups over p-adic fields.

1. Summary of Vinberg's results

The $\mathrm{Ad}\, G(0)$-orbit of an element $x \in \mathfrak{g}(1)$ is closed if and only if x is semisimple; it is unstable (i.e. its closure contains 0) if and only if x is nilpotent. A *Cartan subspace* is a maximal commutative subspace of $\mathfrak{g}(1)$ consisting of semisimple elements. Then any two Cartan subspaces are $G(0)$-conjugate; any semisimple element of $\mathfrak{g}(1)$ is contained in some Cartan subspace; and given a Cartan subspace $\mathfrak{c}$, any semisimple element of $\mathfrak{g}(1)$ is conjugate to some element of $\mathfrak{c}$. A consequence of the above facts is the following form of the Chevalley restriction theorem:

Theorem 1 (Vinberg)**.** *Let* $\mathfrak{c}$ *be a Cartan subspace of* $\mathfrak{g}(1)$*. Then restricting from* $\mathfrak{g}(1)$ *to* $\mathfrak{c}$ *induces an isomorphism*

$$\mathbb{C}[\mathfrak{g}(1)]^{G(0)} \to \mathbb{C}[\mathfrak{c}]^{W_\mathfrak{c}},$$

where $W_\mathfrak{c} = N_{G(0)}(\mathfrak{c})/Z_{G(0)}(\mathfrak{c})$*.*

Moreover, the group $W_\mathfrak{c}$ *is generated by complex reflections, hence* $\mathbb{C}[\mathfrak{c}]^{W_\mathfrak{c}}$ *is a polynomial ring.*

We make a couple of remarks about this theorem. Firstly, a complex reflection is a linear automorphism of a vector space, of finite order, such that the fixed point subspace is of codimension one. According to the celebrated theorem of Shephard-Todd, the ring of invariants for a finite group Γ acting on a vector space is a polynomial ring if and only if Γ is generated by complex reflections. The Shephard-Todd theorem is true if the characteristic of the ground field is zero, or coprime to the order of Γ; if the characteristic divides $|\Gamma|$ then one has the forward, but not the reverse implication. A fascinating feature of the theory of θ-groups is that one obtains many (though not all) of the "exceptional" complex reflection groups in the Shephard-Todd classification.

Secondly, while the ring of invariants $\mathbb{C}[\mathfrak{c}]^{W_\mathfrak{c}}$ is naturally the coordinate ring of the set of orbits $\mathfrak{c}/W_\mathfrak{c}$ (that is, the set-theoretic quotient by $W_\mathfrak{c}$), there is no natural way to give the set of $G(0)$-orbits on $\mathfrak{g}(1)$ a structure of variety (or scheme). However, the ring of invariants $\mathbb{C}[\mathfrak{g}(1)]^{G(0)}$ does determine a 'categorical quotient', i.e. a quotient in the category of varieties (or schemes), usually denoted $\mathfrak{g}(1)/\!\!/G(0)$. The points of $\mathbb{C}[\mathfrak{g}(1)]^{G(0)}$ are in correspondence with the *closed* $G(0)$-orbits in $\mathfrak{g}(1)$, and hence with the orbits of semisimple elements of $\mathfrak{g}(1)$.

2. Extension to other ground fields

There were three main obstacles to be overcome in generalizing Vinberg's theory to other base fields. The first problem is the proof of the first part of the Chevalley restriction theorem: standard arguments establish that the embedding $\mathfrak{c} \to \mathfrak{g}(1)$ determines a *bijection* $\mathfrak{c}/W_\mathfrak{c} \to \mathfrak{g}(1)/\!\!/G(0)$; but to prove that it is an isomorphism one needs to know that this morphism is separable. Separability can be established, under the assumption that $\mathrm{char}\, k \neq 2$, using some technical arguments broadly based on a proof by Richardson.

Another problem relates to Vinberg's argument to show that $W_\mathfrak{c}$ is generated by complex reflections, which used (for $\dim \mathfrak{c} \geq 2$) the simply-connectedness of $\mathfrak{c} \setminus \{0\}$

and its quotient by $W_{\mathfrak{c}}$. Panyushev showed that one can use étale cohomology this dependence on simply-connectedness, under the assumption that the characteristic is zero or is coprime to $|W_{\mathfrak{c}}|$. Hence Panyushev's theorem extended Vinberg theory to other (algebraically closed) fields of characteristic zero.

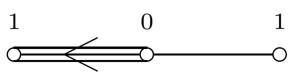
The last main obstacle is the failure of the Shephard-Todd theorem in positive characteristic. However, in many cases (for example, for exceptional type groups in good positive characteristic) it can be shown that the order of $W_{\mathfrak{c}}$ is not divisible by p, and hence the Shephard-Todd theorem applies. A case-by-case analysis, in combination with Panyushev's theorem, can then be used to establish Vinberg's results, under mild conditions on G and k [2].

3. Kac's theorem

Historically one of the first applications of Kac-Moody Lie algebras was Kac's classification of the automorphisms of a simple Lie algebra. We give an account of the classification for inner automorphisms. Fix a primitive m-th root of unity $\zeta \in \mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of simple roots for $\mathfrak{g}$ over $\mathfrak{h}$. Let $\hat{\alpha} = m_1\alpha_1 + \ldots + m_n\alpha_n$ be the highest root relative to Δ. For any sequence $a_0, \ldots, a_n$ of non-negative integers such that $a_0 + a_1 m_1 + \ldots + a_n m_n = m$, there is an inner automorphism θ of $\mathfrak{g}$ satisfying $\theta(e_{\alpha_i}) = \zeta^{a_i} e_{\alpha_i}$ for $1 \leq i \leq n$; moreover, by construction one then has $\theta(f_{\hat{\alpha}}) = \zeta^{a_0} f_{\hat{\alpha}}$.

According to Kac's theorem, any inner automorphism of $\mathfrak{g}$ of order m is conjugate to one of this form, and one then obtains a complete classification of the periodic automorphisms up to conjugacy. Moreover, one can read off the structure of $\mathfrak{g}(0)$ and its module $\mathfrak{g}(1)$ from the 'Kac diagram', which is the affine Dynkin diagram with the coordinates $a_0, \ldots, a_n$ attached to the nodes: $\mathfrak{g}(0)$ is the pseudo-Levi subalgebra with basis of simple roots given by the α_i such that $m_i = 0$, and $\mathfrak{g}(1)$ is a sum of highest weight modules for $\mathfrak{g}(0)'$, where the highest weights are given by the nodes on the diagram such that $a_i = 1$.

Example Let $\mathfrak{g}$ be of type G_2, and consider the automorphism θ of $\mathfrak{g}$ with Kac coordinates given by the following diagram, where the node on the right is the "affine vertex":



Then θ has order $3 + 1 = 4$ and $\mathfrak{g}(0)$ is isomorphic to $\mathfrak{gl}(2)$; moreover, $\mathfrak{g}(1)$ is isomorphic as a $\mathfrak{g}(0)' = \mathfrak{sl}(2)$-module to $V(3) \oplus V(1)$, i.e. to a sum of four-dimensional and two-dimensional irreducible modules.

4. Positive rank gradings and Popov's conjecture

The **rank** of a grading $\mathfrak{g} = \sum_{i \in \mathbb{Z}/(m)} \mathfrak{g}(i)$ is the dimension of a Cartan subspace of $\mathfrak{g}(1)$. For given $\mathfrak{g}$, there are finitely many positive rank automorphisms up to conjugacy. For G of classical type and $k = \mathbb{C}$, Vinberg classified the positive rank automorphisms in terms of eigenvalues of θ (for inner automorphisms) or of θ^2 (for outer automorphisms) [4]. This classification was extended to positive characteristic in [2]. The author was able to apply this classification to prove the following conjecture of Popov, for $\mathfrak{g}$ of classical type over k of characteristic $\neq 2$:

Conjecture: For any θ, there exists an affine linear subspace $\mathfrak{v}$ of $\mathfrak{g}(1)$ such that the restriction of the quotient morphism to $\mathfrak{v}$ is an isomorphism.

Such an affine linear subspace is called a *Kostant-Weierstrass slice*, or *KW-section*. The conjecture is trivially true if θ is of rank zero. For exceptional types, the classification of positive rank gradings is more ad hoc. In [3], we determine the positive rank gradings of exceptional Lie algebras in zero or good characteristic; we relate these gradings to representatives of certain elements of the Weyl group of $\mathfrak{g}$; and we determine the little Weyl group $W_{\mathfrak{c}}$ in each case. It is then straightforward to deduce the existence of a KW-section for any grading of an exceptional Lie algebra.

5. An application to representation theory of reductive groups over p-adic fields

Let $\mathcal{G}$ be a reductive group over the non-archimedean local field K and let k be the residue field of a maximal unramified extension of K. Kac's theorem can be interpreted as a statement about points in the affine building of a simple Lie algebra, and so one has a natural description in terms of points of the Bruhat-Tits building of $\mathcal{G}$. To any such point one associates the Moy-Prasad filtration of $\mathcal{G}$; the quotients in this filtration are the summands $\mathfrak{g}_i$ in the corresponding grading of the Lie algebra over k. The classification of the positive rank gradings of $\mathfrak{g}$ can be used to deduce to the classification of non-degenerate K-types, a long-standing problem in the representation theory of $\mathcal{G}$.

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