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Geometrie

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Abstract. During the meeting a wide range of topics in geometry was discussed. There were 18 one hour talks and four half an hour talks which took place after dinner.

Mathematics Subject Classification (2000): 53-xx.

Introduction by the Organisers

The schedule of 18 regular and 4 after dinner talks left plenty of room for discussions among the 53 participants. The after dinner talks (Monday – Thursday) were given by PhD students and very recent PhD's. They were nearly equally well attended and at least the organizers only heard positive comments. Apart from the good cake the speakers were the main contributors to the good atmosphere at the workshop since they all did an excellent job.

Fernando Codá Marques gave two very nice talks on his joint work with André Neves on the proof of the Willmore conjecture via min-max principles.

Karl-Theodor Sturm showed that the class of metric measure spaces endowed with the L^2-Gromov Hausdorff distance is an Alexandrov space. As a consequence every semiconvex function on this infinite dimensional Alexandrov space then gives rise to a flow on the space of metric measure spaces.

Of the 22 talks 7 (including 3 of the after dinner talks) were on or related to Ricci flow. Richard Bamler gave a very nice overview on the analytic open problems in the long term analysis of 3 dimensional Ricci flow and gave interesting partial answers to some of the questions. Esther Cabezas–Rivas established structure results for the Ricci flow on open manifolds with nonnegative complex curvature.

Takumi Yokota talked on a gap phenomenon for noncompact gradient Ricci solitions related to the reduced volume. Gregor Giesen addressed the uniqueness question for the instantaneously complete Ricci flow on surfaces with incomplete initial data. Zhenlei Zhang and Robert Haslhofer considered the moduli space of gradient Ricci solitons and possible degenerations of Ricci solitons. Hans-Joachim Hein established so-called ε-regularity of the Ricci flow, which means that curvature bounds in a parabolic neighbourhood of a point can essentially be solely deduced from a (sufficiently good) lower bound on Perelman's pointed entropy.

Sebastian Hoelzel gave a simple criterion which allows to decide whether the class of manifolds satisfying a certain curvature condition is invariant under surgery with codimension k.

Olivier Biquard and Claude LeBrun talked on four dimensional Einstein manifolds. The former gave a criterion that allows to desingularize certain four dimensional Einstein orbifolds. The latter established an analogue of the Hitchin-Thorpe inequality for Einstein four-manifolds with edge cone singularities. Characteristic numbers were also at the core of Ursula Hamenstädt's talk, who showed that for any surface bundle over a surface the Euler characteristic is bounded below by three times the signature.

Frank Pacard provided a surprising doubling construction for CMC hypersurfaces in Riemannian manifolds. Felix Schulze proved uniqueness of certain singularity models for the mean curvature flow. Peter Topping introduced a new flow for immersed surfaces, which has some advantages over the mean curvature flow. Sergei Ivanov considered the minimal surface problem in finite dimensional Banach spaces. He could show that 2-dimensional unit discs (in linear subspaces) are total minimizers among all surfaces with the same boundary.

Karsten Grove used work of Tits on buildings to classify isometric polar actions on positively curved manifolds. Marcus Khuri gave a very interesting talk on general relativity.

Marc Bourdon and Anton Petrunin introduced new constructions in geometric group theory. The former studied the question for which p the l_p cohomology of the boundary of Gromov hyperbolic groups is nontrivial. The latter gave a rather surprising construction which shows for example that any finitely presented group can be realized as the fundamental group of the underlying topological space of a compact 3-dimensional orbifold.

Workshop: Geometrie

Table of Contents

Fernando Codá Marques (joint with André Neves, Imperial College - UK)
Min-max theory and the Willmore conjecture: parts I and II 1643

Marc Bourdon (joint with Bruce Kleiner)
Some applications of ℓ_p-cohomology to boundaries of Gromov hyperbolic spaces 1644

Claude LeBrun (joint with Michael Atiyah)
Curvature, Cones, and Characteristic Numbers 1646

Esther Cabezas-Rivas (joint with Burkhard Wilking)
How to produce a Ricci flow via Cheeger-Gromoll exhaustion 1648

Takumi Yokota
A refinement of a gap theorem for gradient shrinking Ricci solitons 1651

Richard Bamler
Long-time analysis of 3 dimensional Ricci flow 1652

Ursula Hamenstädt
Signatures of surface bundles and Milnor Wood inequalities 1655

Karl-Theodor Sturm
The space of metric measure spaces is an Alexandrov space 1656

Gregor Giesen (joint with Peter M. Topping)
Instantaneously complete Ricci flows on surfaces 1658

Olivier Biquard
Desingularisation of Einstein orbifolds 1660

Anton Petrunin (joint with Dmitry Panov)
Telescopic actions 1662

Marcus A. Khuri (joint with Marcelo M. Disconzi)
Compactness and Non-compactness for the Yamabe Problem on Manifolds With Boundary 1662

Felix Schulze
Uniqueness of compact tangent flows in Mean Curvature Flow 1665

Robert Haslhofer (joint with Reto Müller)
Singularities in 4d Ricci flow 1668

Sergei Ivanov (joint with Dmitri Burago)
Convexity and semi-ellipticity of Busemann–Hausdorff surface area 1671

Hans-Joachim Hein (joint with Aaron Naber)
An ε-regularity theorem for the Ricci flow 1672

Frank Pacard (joint with Taoniu Sun)
Doubling construction for CMC hypersurfaces in Riemannian manifolds 1675

Sebastian Hoelzel
On surgery stable curvature conditions 1677

Zhenlei Zhang
Degeneration of shrinking Ricci solitons 1678

Peter Miles Topping (joint with Melanie Rupflin)
Flowing maps to minimal surfaces 1680

Karsten Grove (joint with Fuquan Fang and Gudlaugur Thorbergsson)
Tits Geometry and Positive Curvature 1681

Abstracts

Min-max theory and the Willmore conjecture: parts I and II

Fernando Codá Marques
(joint work with André Neves, Imperial College - UK)

In 1965, T. J. Willmore conjectured that the integral of the square of the mean curvature of any torus immersed in Euclidean three-space is at least $2\pi^2$. In these talks we will describe a proof of this conjecture that uses the min-max theory of minimal surfaces.

The Willmore conjecture can be reformulated as a question about surfaces in the three-sphere because if $\pi : S^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ denotes the stereographic projection and $\Sigma \subset S^3 \setminus \{(0,0,0,1)\}$ is a closed surface, then

$$\int_{\tilde{\Sigma}} \tilde{H}^2 d\tilde{\Sigma} = \int_{\Sigma} (1 + H^2) d\Sigma.$$

Here H and $\tilde{H}$ are the mean curvature functions of $\Sigma \subset S^3$ and $\tilde{\Sigma} = \pi(\Sigma) \subset \mathbb{R}^3$, respectively. The quantity $\mathcal{W}(\Sigma) = \int_\Sigma (1 + H^2)\, d\Sigma$ is then defined to be the *Willmore energy* of $\Sigma \subset S^3$. This energy is specially interesting because it has the remarkable property of being invariant under conformal transformations of S^3. This fact was already known to Blaschke and Thomsen in the 1920s. The Willmore Conjecture has received the attention of many mathematicians since the late 1960s.

Our Main Theorem is:

Theorem A. *Let $\Sigma \subset S^3$ be an embedded closed surface of genus $g \geq 1$. Then*

$$\mathcal{W}(\Sigma) \geq 2\pi^2,$$

and the equality holds if and only if Σ is the Clifford torus up to conformal transformations of S^3.

To each closed surface $\Sigma \subset S^3$, we associate a canonical 5-dimensional family of surfaces in S^3 with area bounded above by the Willmore energy of Σ. This area estimate follows from a calculation of A. Ros, based on the Heintze-Karcher inequality. The family is parametrized by the 5-cube I^5, and maps the boundary ∂I^5 into the space of geodesic spheres in a topologically nontrivial way if genus$(\Sigma) \geq 1$. The whole proof revolves around the idea of showing that the Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset S^3$ can be produced by applying min-max theory for the area functional to the homotopy class of this family. One key point is that, by a result of F. Urbano, the Clifford torus is the only non-totally geodesic minimal surface in S^3 with Morse index at most 5. After ruling out great spheres by a topological argument, the proof of Theorem A then reduces to the following statement about minimal surfaces in the three-sphere, also proven using min-max methods:

Theorem B. *Let $\Sigma \subset S^3$ be an embedded closed minimal surface of genus $g \geq 1$. Then* $\text{area}(\Sigma) \geq 2\pi^2$*, and* $\text{area}(\Sigma) = 2\pi^2$ *if and only if Σ is the Clifford torus up to isometries of S^3.*

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Some applications of ℓ_p-cohomology to boundaries of Gromov hyperbolic spaces

Marc Bourdon
(joint work with Bruce Kleiner)

In this talk we are interested in quasi-isometry invariant structure in Gromov hyperbolic spaces, primarily structure which is reflected in the boundary. For some hyperbolic groups Γ, the topological structure of the boundary $\partial\Gamma$ alone contains substantial information: witness the JSJ decomposition encoded in the local cut point structure of the boundary [1], and many situations where one can detect boundaries of certain subgroups $H \subset \Gamma$ by means of topological criteria. However, in many cases, for instance for generic hyperbolic groups, the topology reveals little of the structure of the group and is completely inadequate for addressing rigidity questions, since the homeomorphism group of the boundary is highly transitive. In these cases it is necessary to use the finer quasi-Mobius structure of the boundary and analytical invariants attached to it, such as modulus (Pansu, metric-measure, or combinatorial), ℓ_p-cohomology, and closely related quantities like the conformal dimension. The seminal work of Heinonen-Koskela [7] indicates that when $\partial\Gamma$ is quasi-Mobius homeomorphic to a Loewner space (an Ahlfors Q-regular Q-Loewner space in the sense of [7]), there should be a rigidity theory resembling that of lattices in rank 1 Lie groups.

In this talk we restrict our attention to proper Gromov hyperbolic spaces which satisfy the following two additional conditions:

- (Bounded geometry) Every R-ball can be covered by at most $N = N(R, r)$ balls of radius $r \leq R$.
- (Nondegeneracy) There is a $C \in [0, \infty)$ such that every point x lies within distance at most C from all three sides of some ideal geodesic triangle Δ_x.

The visual boundary ∂X of such a space X is a compact, doubling, uniformly perfect metric space, which is determined up to quasi-Mobius homeomorphism by the quasi-isometry class of X. Conversely, every compact, doubling, uniformly perfect metric space is the visual boundary of a unique hyperbolic metric space as above, up to quasi-isometry.

To simplify the discussion of homological properties, we will impose (without loss of generality) the additional standing assumption that X is a simply connected metric simplicial complex with links of uniformly bounded complexity, and with all simplices isometric to regular Euclidean simplices with unit length edges.

We recall [8, 6, 5] that for $p \in (1,\infty)$, the continuous (first) ℓ_p-cohomology $\ell_p H^1_{\mathrm{cont}}(X)$ is canonically isomorphic to $A_p(\partial X)/\mathbb{R}$, where $A_p(\partial X)$ is the space of continuous functions $u : \partial X \to \mathbb{R}$ which have a continuous extension $f : X^{(0)} \cup \partial X \to \mathbb{R}$ with p-summable coboundary:

$$\|df\|^p_{\ell_p} = \sum_{[vw]\in X^{(1)}} |f(v) - f(w)|^p < \infty ,$$

and where $\mathbb{R}$ denotes the subspace of constant functions. Associated with the continuous ℓ_p-cohomology are several other quasi-isometry invariants:

(1) The ℓ_p-equivalence relation $\sim_p$ on ∂X, where $x \sim_p y$ iff $u(x) = u(y)$ for every $u \in A_p(\partial X)$.
(2) The infimal p such that $\ell_p H^1_{\mathrm{cont}}(X) \simeq A_p(\partial X)/\mathbb{R}$ is nontrivial. We will denote this by $p_{\neq 0}(X)$. Equivalently $p_{\neq 0}(X)$ is the infimal p such that $\sim_p$ has more than one coset.
(3) The infimum $p_{\mathrm{sep}}(X)$ of the p such that $A_p(\partial X)$ separates points in ∂X, or equivalently, $p_{\mathrm{sep}}(X)$ is the infimal p such that all cosets of $\sim_p$ are points.

These invariants were exploited in [4, 5, 2] due to their connection with conformal dimension and the Combinatorial Loewner Property. Specifically, when ∂X is approximately self-similar (e.g. if ∂X is the visual boundary of a hyperbolic group) then $p_{\mathrm{sep}}(X)$ coincides with the conformal dimension of ∂X ; and if ∂X has the Combinatorial Loewner Property then the two critical exponents $p_{\neq 0}(X)$ and $p_{\mathrm{sep}}(X)$ coincide, i.e. for every $p \in (1,\infty)$, the function space $A_p(\partial X)$ separates points iff it is nontrivial.

The key idea presented here is new constructions of nontrivial elements in the ℓ_p-cohomology. The general approach for the construction is inspired by [5], and may be described as follows. Inside the Gromov hyperbolic complex X, we identify a subcomplex Y such that the relative cohomology of the pair $(X, X \setminus Y)$ reduces – essentially by excision – to the cohomology of Y relative to its frontier in X. Then we prove that the latter contains an abundance of nontrivial classes. This yields nontrivial classes in $\ell_p H^1_{\mathrm{cont}}(X)$ with additional control, allowing us to make deductions about the cosets of the ℓ_p-equivalence relation.

As an illustration of these ideas we obtain the following results, see [3] for more details.

Theorem 1. *Suppose A, B are hyperbolic groups, and we are given malnormal quasiconvex embeddings $C \hookrightarrow A$, $C \hookrightarrow B$. Suppose that there is a decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite index subgroups of A such that $\cap_{n\in\mathbb{N}} A_n = C$, and set $\Gamma_n := A_n \star_C B$.*

(1) If $p_{sep}(A) < p_{sep}(B)$ then, for all $p \in (p_{sep}(A), p_{sep}(B)]$ and every n large enough, the ℓ_p-equivalence relation on $\partial\Gamma_n$ possesses a coset different from a point and the whole $\partial\Gamma$. In particular for n large enough, $\partial\Gamma_n$ does not admit the CLP.

(2) If $p_{sep}(A) < p_{\neq 0}(B)$ then, for $p \in (p_{sep}(A), p_{\neq 0}(B))$ and every n large enough, the cosets of the ℓ_p-equivalence relation on $\partial\Gamma_n$ are single points and the boundaries of cosets gB, for $g \in \Gamma_n$. In particular, for large n, any quasi-isometry of Γ_n permutes the cosets gB, for $g \in \Gamma_n$.

Theorem 2. *Let X be a simply connected hyperbolic 2-complex. Assume that X is a union of 2-cells, where 2-cells intersect pairwise in at most a vertex or edge. If the perimeter of every 2-cell is at least $n \geq 7$ and the thickness of every edge lies in $[2, k]$, then one has*

$$p_{sep}(X) \leq 1 + \frac{\log(k-1)}{\log(n-5)}.$$

We notice that examples of such 2-complexes are provided by the Davis complexes of Coxeter groups with all exponents $m_{ij} \geq 4$.

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Curvature, Cones, and Characteristic Numbers

Claude LeBrun
(joint work with Michael Atiyah)

Recall that a Riemannian manifold (M, g) is said [2] to be *Einstein* if it has constant Ricci curvature. This is equivalent to requiring that the Ricci tensor r of g satisfy $r = \lambda g$ for some real number λ, called the Einstein constant of g. While one usually requires g to be a smooth metric on M, it is sometimes interesting to consider generalizations where g is allowed to have mild singularities. In the Kähler case, beautiful results [3, 4, 7] have recently been obtained regarding the situation in which g has specific conical singularities along a submanifold of real codimension two. Interesting self-dual 4-dimensional examples have also been constructed by Hitchin [6] and Abreu [1], and global quotient orbifolds with singular set purely of codimension 2 give us many more. Einstein manifolds with such *edge-cone singularities* were the focus of my lecture.

Let M be a smooth n-manifold, and let $\Sigma \subset M$ be a smoothly embedded $(n-2)$-manifold. Near any point $p \in \Sigma$, we can thus find local coordinates $(y^1, y^2, x^1, \ldots, x^{n-2})$ in which Σ is given by $y^1 = y^2 = 0$. Given any such adapted coordinate system, we then introduce an associated *transversal polar coordinate* system $(\rho, \theta, x^1, \ldots, x^{n-2})$ by setting $y^1 = \rho \cos\theta$ and $y^2 = \rho \sin\theta$. We define a Riemannian edge-cone metric g of cone angle $2\pi\beta$ on (M, Σ) to be a smooth Riemannian metric on $M - \Sigma$ which, for some $\varepsilon > 0$, can be expressed as

$$g = \bar{g} + \rho^{1+\varepsilon} h \tag{1}$$

near any point of Σ, where the symmetric tensor field h on M has infinite conormal regularity along Σ, and where

$$\bar{g} = d\rho^2 + \beta^2 \rho^2 (d\theta + u_j dx^j)^2 + w_{jk} dx^j dx^k \tag{2}$$

in suitable transversal polar coordinate systems; here $w_{jk}(x) dx^j dx^k$ and $u_j(x) dx^j$ are a smooth metric and a smooth 1-form on Σ. (Our conormal regularity hypothesis means that the components of h in (y, x) coordinates have infinitely many continuous derivatives with respect to $\partial/\partial x^j$, $\partial/\partial\theta$, and $\rho\,\partial/\partial\rho$.) Thus, an edge-cone metric g behaves like a smooth metric in directions parallel to Σ, but is modelled on a 2-dimensional cone in the transverse directions. If an edge-cone metric on (M, Σ) is Einstein on $M - \Sigma$, we will call it an Einstein edge-cone metric.

The Hitchin-Thorpe inequality [2, 5, 8] provides an important obstruction to the existence of Einstein metrics on 4-manifolds. If M is a smooth compact oriented 4-manifold which admits a smooth Einstein metric g, then the Euler characteristic χ and signature τ of M must satisfy the two inequalities

$$(2\chi \pm 3\tau)(M) \geq 0$$

because both expressions are represented by Gauss-Bonnet-type integrals for which the integrands become non-negative in the Einstein case. This hinges on several peculiar features of 4-dimensional Riemannian geometry; no analogous obstruction to the existence of Einstein metrics is currently known in any other dimension. Our main result is a generalization of the Hitchin-Thorpe inequality which is adapted to the edge-cone setting:

Theorem 1. *Let (M, Σ) be a pair consisting of a smooth compact 4-manifold and a fixed smoothly embedded compact oriented surface. If (M, Σ) admits an Einstein edge-cone metric with cone angle $2\pi\beta$ along Σ, then (M, Σ) must satisfy the two inequalities*

$$(2\chi \pm 3\tau)(M) \geq (1-\beta)\left[2\chi(\Sigma) \pm (1+\beta)[\Sigma]^2\right] .$$

As a consequence, whenever $\Sigma \subset M$ has non-zero self-intersection $[\Sigma]^2$, the existence of Einstein edge-cone metrics is obstructed for all sufficiently large β. It is also easy to find examples where existence is obstructed for all $\beta \in (0, \varepsilon)$.

Theorem 1 follows from modified Gauss-Bonnet and signature formulas that hold for arbitrary edge-cone metrics, and so are of independent interest:

Theorem 2. *Let M be a smooth compact oriented 4-manifold, and let $\Sigma \subset M$ be a smooth compact oriented embedded surface. Then, for any edge-cone metric g on (M, Σ) with cone angle $2\pi\beta$,*

$$\begin{aligned}\chi(M) &= (1-\beta)\chi(\Sigma) + \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu \\ \tau(M) &= \frac{1}{3}(1-\beta^2)[\Sigma]^2 + \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu\end{aligned}$$

where s, $\mathring{r}$, W_+, and W_- respectively denote the scalar, trace-free Ricci, self-dual Weyl, and anti-self-dual Weyl curvatures of g, $W = W_+ + W_-$, and $d\mu$ is the metric volume form.

Many gravitational instantons arise as $\beta \to 0$ limits of edge-cone metrics, and Theorem 2 has some interesting consequences in this context.

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How to produce a Ricci flow via Cheeger-Gromoll exhaustion

Esther Cabezas-Rivas
(joint work with Burkhard Wilking)

We give a detailed analysis of the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) \tag{1}$$

on open (i.e. complete and non-compact) n-manifolds (M, g) with $K^{\mathbb{C}} \geq 0$:

Definition 1. *If we extend the Riemannian curvature tensor* Rm *and the metric g at $p \in M$ to $\mathbb{C}$-multilinear maps on $T^{\mathbb{C}}M := TM \otimes \mathbb{C}$, then the complex sectional curvature of a 2-plane σ of $T^{\mathbb{C}}_p M$ is defined by*

$$K^{\mathbb{C}}(\sigma) = \mathrm{Rm}(u, v, \bar{v}, \bar{u}) = g(\mathrm{Rm}(u \wedge v), \overline{u \wedge v}),$$

where $\{u, v\}$ *is any unitary basis for* σ*, i.e.* $g(u, \bar{u}) = g(v, \bar{v}) = 1$ *and* $g(u, \bar{v}) = 0$*. We say* M *has nonnegative complex sectional curvature if* $K^{\mathbb{C}} \geq 0$*.*

Note that this is weaker than $\mathrm{Rm} \geq 0$ and implies nonnegative sectional curvature ($K \geq 0$). Unlike the latter, $K^{\mathbb{C}} \geq 0$ has the advantage to be invariant under (1).

In a broader scheme of things, Ricci flows on open manifolds arise naturally as singularity models for (1) on closed (i.e. compact and without boundary) manifolds. The condition $K^{\mathbb{C}} \geq 0$ is also relevant for singularity analysis prospects: for $n = 3$ it is well known by Hamilton-Ivey's estimates that singularity models satisfy $K^{\mathbb{C}} \geq 0$; even for higher dimensions we have strong indications that various Ricci flow invariant curvature conditions should pinch towards $K^{\mathbb{C}} \geq 0$.

Historically, the first question was to ensure that (1) admits a solution for a short time. This was settled for closed manifolds by Hamilton in [7]. But it seems hopeless to expect similar results for the open case; e.g. it is difficult to imagine how to construct a solution to (1) starting at a complete n-manifold ($n \geq 3$) built by attaching in a smooth way long spherical cylinders with radii converging to zero. The natural way to prevent such situations is to add extra conditions on the curvature; indeed, W. X. Shi proved in [9] that (1) starting on an open manifold with bounded curvature (i.e. $\sup_M |\mathrm{Rm}_g| \leq k_0 < \infty$) admits a solution on $[0, T(n, k_0)]$ also with bounded curvature.

Later on M. Simon (cf. [11]), assuming further that the manifold has $\mathrm{Rm}_g \geq 0$ and is non-collapsing ($\inf_M \mathrm{vol}_g (B_g(\cdot, 1)) \geq v_0 > 0$), was able to extend Shi's solution for a time interval $[0, T(n, v_0)]$, with $|\mathrm{Rm}_{g(t)}| \leq \frac{c(n,v_0)}{t}$ for $t > 0$. Although $T(n, v_0)$ does not depend on an upper curvature bound, such a bound is still needed to guarantee short time existence.

Our first result in [3] manages to remove any restriction on upper curvature bounds for open manifolds with $K^{\mathbb{C}} \geq 0$ which, by Cheeger, Gromoll and Meyer [4, 6], admit an exhaustion by convex sets C_ℓ. We are able to construct a Ricci flow with $K^{\mathbb{C}} \geq 0$ on the closed manifold obtained by gluing two copies of C_ℓ along the common boundary, and whose 'initial metric' is the natural singular metric on the double. By passing to a limit we obtain

Theorem 2. *Let* (M^n, g) *be an open manifold with nonnegative (and possibly unbounded) complex sectional curvature. Then there exists a constant* $\mathcal{T} = \mathcal{T}(n, g)$ *such that* (1) *has a smooth solution on* $[0, \mathcal{T}]$*, with* $g(0) = g$ *and* $K^{\mathbb{C}}_{g(t)} \geq 0$*.*

Using that by Brendle [1] the trace Harnack inequality holds for the closed case, it follows that the above solution on the open manifold satisfies the trace Harnack estimate as well. This solves an open question posed in [5, Problem 10.45]. The proof of Theorem 2 is considerably easier if $K^{\mathbb{C}}_g > 0$ since e.g. then, by Gromoll and Meyer [6], M is diffeomorphic to $\mathbb{R}^n$. We overcome the lack of such a property in the general case by proving (see [8] for a version asking $\mathrm{Rm}_g \geq 0$):

Theorem 3. *Let* (M^n, g) *be an open, simply connected Riemannian manifold with nonnegative complex sectional curvature. Then* M *splits isometrically as* $\Sigma \times F$*, where* Σ *is the* k*-dimensional soul of* M *and* F *is diffeomorphic to* $\mathbb{R}^{n-k}$*.*

In the nonsimply connected case M is diffeomorphic to a flat Euclidean vector bundle over the soul. Thus combining with the knowledge from [2] of the compact case, we extend the same classification of [2] for open manifolds with $K^{\mathbb{C}} \geq 0$; more precisely, any such a manifold admits a complete nonnegatively curved locally symmetric metric $\hat{g}$, i.e. $K_{\hat{g}} \geq 0$, $\nabla R_{\hat{g}} \equiv 0$.

It is not hard to see that, given any open manifold (M, g) with bounded curvature and $K_g^{\mathbb{C}} > 0$, for any closed discrete countable subset $S \subset M$ one can find a deformation $\bar{g}$ of g in an arbitrary small neighborhood U of S such that $\bar{g}$ and g are C^1-close, $(M, \bar{g})$ has unbounded curvature and $K_{\bar{g}}^{\mathbb{C}} > 0$. The following result, which is very much in spirit of [11], shows that this sort of local deformations will be smoothed out instantaneously by our Ricci flow.

Corollary 4. *Let (M^n, g) be an open manifold with $K_g^{\mathbb{C}} \geq 0$. If*

$$\inf\{\mathrm{vol}_g(B_g(p,1)) : p \in M\} = v_0 > 0, \tag{2}$$

then the curvature of $(M, g(t))$ is bounded above by $\frac{c(n,v_0)}{t}$ for $t \in (0, \mathcal{T}(n, v_0)]$.

Any nonnegatively curved surface satisfies (2), so can be deformed by (1) to one with bounded curvature. However, (2) is essential for $n \geq 3$:

Theorem 5. *a) There is an immortal 3-dimensional nonnegatively curved complete Ricci flow $(M, g(t))_{t\in[0,\infty)}$ with unbounded curvature for each t.*

b) There is an immortal 4-dimensional complete Ricci flow $(M, g(t))_{t\in[0,\infty)}$ with positive curvature operator such that $\mathrm{Rm}_{g(t)}$ is bounded if and only if $t \in [0, 1)$.

Part b) shows that even if $\mathrm{Rm}_{g(0)}$ is bounded one can run into metrics with unbounded curvature. Our next result gives a precise lower bound on the existence time for (1) in terms of supremum of the volume of balls, instead of infimum as in Corollary 4 and [11]. We stress that this is new even for $g(0)$ of bounded curvature.

Corollary 6. *For each complete (M^n, g) with $K_g^{\mathbb{C}} \geq 0$ we find $\varepsilon(n) > 0$ so that if*

$$\mathcal{T} := \varepsilon(n) \cdot \sup\left\{ \frac{\mathrm{vol}_g(B_g(p,r))}{r^{n-2}} \mid p \in M, r > 0 \right\} \in (0, \infty],$$

then any complete maximal Ricci flow $(M, g(t))_{t\in[0,T)}$ with $K_{g(t)}^{\mathbb{C}} \geq 0$ and $g(0) = g$ satisfies $\mathcal{T} \leq T$.

If M has a volume growth larger than r^{n-2}, this ensures $T = \infty$. Previously (cf. [10]) long time existence was only known in the case of Euclidean volume growth (EVG) under stronger curvature assumptions. We highlight that our volume growth condition cannot be further improved: indeed, as the Ricci flow on the metric product $\mathbb{S}^2 \times \mathbb{R}^{n-2}$ exists only for a finite time, the power $n-2$ is optimal. For $n = 3$ we can even determine exactly the extinction time:

Corollary 7. *Let (M^3, g) be open with $K_g \geq 0$ and soul Σ. Then a maximal complete Ricci flow $(M, g(t))_{t\in[0,T)}$ with $g(0) = g$ and $K_{g(t)} \geq 0$ has (a) $T = \frac{\mathrm{area}(\Sigma)}{4\pi\chi(\Sigma)}$ if $\dim \Sigma = 2$, (b) $T = \infty$ if $\dim \Sigma = 1$, and (c) $T = \frac{1}{8\pi} \lim_{r\to\infty} \frac{\mathrm{vol}_g(B_g(p,r))}{r}$ if $\Sigma = \{p_0\}$. If $T < \infty$ in (c), then (M, g) is asymptotically cylindrical and $\mathrm{Rm}_{g(t)}$ is bounded for $t > 0$.*

Finally, we analyze the long time behaviour of an immortal Ricci flow: For an initial metric with EVG we remark that a result in [10] can be adjusted to see that a suitable rescaled Ricci flow subconverges to an expanding soliton. Moreover,

Theorem 8. *Let $(M^n, g(t))$ be a non flat immortal Ricci flow with $K^{\mathbb{C}} \geq 0$ satisfying the trace Harnack inequality. If $(M, g(0))$ does not have EVG, then for $p_0 \in M$ we find $t_k \to \infty$ and $Q_k > 0$ so that $(M, Q_k g(t_k + t/Q_k), p_0)$ converges in the Cheeger-Gromov sense to a steady soliton which is not isometric to $\mathbb{R}^n$.*

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A refinement of a gap theorem for gradient shrinking Ricci solitons

Takumi Yokota

In [Yo], we dealt with ancient solutions to the Ricci flow and gradient shrinking Ricci solitons. A triple (M, g, f) consisting of a Riemannian manifold (M, g) with $f \in C^\infty(M)$ is called a *gradient shrinking Ricci soliton* if it satisfies the following equation with some positive constant $\lambda > 0$:

$$\mathrm{Ric}(g) + \mathrm{Hess}\, f = \frac{1}{2\lambda} g.$$

The integral $\Theta(M) := \int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g$ is sometimes called the *Gaussian density* of (M^n, g, f) with the potential function f being normalized so that

$$\lambda(R + |\nabla f|^2) = f \text{ on } M.$$

Here R and $d\mu_g$ denote the scalar curvature and the volume element of g, respectively.

Gradient shrinking Ricci solitons appear as ancient solutions to the Ricci flow in its analysis of finite time singularities. A Ricci flow $(M, g(t))$ is called an *ancient solution* if it exists for all $\tau := -t \in [0, \infty)$.

The main theorem of the addendum [Yo2] to [Yo] is the following gap theorem.

Theorem 1 ([Yo2])**.** *For any $n \geq 2$, there exists a constant $\epsilon_n > 0$ satisfying the following: Any complete n-dimensional gradient shrinking Ricci soliton (M^n, g, f) with $\Theta(M) > 1 - \epsilon_n$ is, up to scaling, the Gaussian soliton $(\mathbb{R}^n, g_E, |\cdot|^2/4)$.*

In [Yo], Theorem 1 was obtained as a corollary of the following gap theorem under the additional assumption on the Ricci curvature.

Theorem 2 ([Yo])**.** *For any $n \geq 2$, there exists a constant $\epsilon_n > 0$ satisfying the following: Any complete n-dimensional ancient solution $(M^n, g(\tau)), \tau \in [0, \infty)$ to the Ricci flow with Ricci curvature bounded below and $\lim_{\tau\to\infty} \tilde{V}_{(p,0)}(\tau) > 1 - \epsilon_n$ for some $p \in M$ is the Gaussian soliton, i.e., the trivial solution on the Euclidean space $(\mathbb{R}^n, g_E)$ for all $\tau \in [0, \infty)$.*

In the above statement, $\tilde{V}_{(p,0)}(\tau)$ denotes the *reduced volume* which was introduced and shown to be nonincreasing in $\tau > 0$ by G. Perelman [Pe], and we use the convention that $\tilde{V}_{(p,0)}(\tau) \leq 1$ for any Ricci flow. We assume that the Ricci curvature is bounded below only to ensure that the reduced volume is well-defined.

In [Yo2], Theorem 1 is proved by observing that the reduced volume is well-defined for ancient solutions generated by any complete gradient shrinking Ricci solitons and applying Theorem 2. Finally, we comment that our Theorem 1 is intimately related to the conjecture of Carrillo–Ni [CN].

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Long-time analysis of 3 dimensional Ricci flow

Richard Bamler

It is still an open problem how Perelman's Ricci flow with surgeries behaves for large times. For example, it is unknown whether surgeries eventually stop to occur and whether the full geometric decomposition of the underlying manifold is exhibited by the flow as $t \to \infty$.

In this talk, I present new tools to treat this question. In particular, I discuss the case in which the initial manifold only has hyperbolic or non-ashpheric components in its geometric decomposition (i.e. prime and torus-decomposition). Note that this is a purely topological condition. It turns out that in this case, surgeries do in fact stop to occur after some time and that the curvature is globally bounded by Ct^{-1}. Finally, I explain how to treat more general cases.

A Ricci flow with surgery $\mathcal{M}$ is a solution to the Ricci flow equation $\partial_t g_t = -2\,\mathrm{Ric}_t$ on a sequence of time intervals $[0, T_1), [T_1, T_2), \ldots$ such that the metric at the initial time of each time interval arises from the metric at the final time of the previous time interval through a certain surgery process which removes areas of high curvature and controlled topology. Perelman has used the Ricci flow with surgery to prove the Poincaré and Geometrization Conjecture ([2], [3], [4]). More precisely, given any initial metric on a closed 3-manifold, Perelman managed to construct a specific Ricci flow with surgery $\mathcal{M}$ on a maximal time interval $[0, T)$ and showed that the surgery times $T_1, T_2, \ldots$ do not accumulate on $[0, T)$. This means that on every finite time interval there are only finitely many surgeries. Furthermore, he could prove that the maximal time interval $[0, T)$ is finite if and only if the initial manifold is a connected sum of spherical space forms and copies of $S^1 \times S^2$. In this case it follows immediately that the number of surgeries performed on $[0, T)$ is finite and it is possible to give an accurate asymptotic characterization of the metric g_t as $t \nearrow T$.

In the more generic case $[0, T) = [0, \infty)$, it is an interesting question whether $\mathcal{M}$ always has finitely many surgeries, i.e. whether surgeries stop to occur after a finite time, and how the metric g_t behaves as $t \to \infty$. Observe that although the Ricci flow with surgery was used to solve such difficult problems, these basic questions were never answered, because once a few properties of the metric g_t for large t were determined, topological arguments could be used to finish the proof of the Geometrization Conjecture.

The results presented in this talk, answer some of these open questions in the case $[0, T) = [0, \infty)$ when the topology of the initial manifold is of a certain type. Consider the following classes of topological 3-manifolds

$$\mathcal{T}_1' = \{M \text{ prime, closed 3-manifold} \; : \; M \text{ has only hyperbolic components in its geometric decomposition}\}$$

$$\mathcal{T}_2' = \{M \text{ prime, closed 3-manifold} \; : \; M \text{ “has enough incompressible surfaces”}\}$$

The exact definition of $\mathcal{T}_2'$ is quite long and is omitted here. I remark that $\mathcal{T}_2'$ includes large classes of prime 3-manifolds but not all prime 3-manifolds. For example $\mathcal{T}_1' \subset \mathcal{T}_2'$, $T^3 \in \mathcal{T}_2'$. Also, manifolds whose geometric pieces are glued together in a certain way are of type $\mathcal{T}_2'$. However, the Heisenberg manifold, for instance, is not of type $\mathcal{T}_2'$. We now define for $i = 1, 2$

$$\mathcal{T}_i = \{M_1 \# \ldots \# M_m \; : \; M_i \in \mathcal{T}_i' \quad \text{or} \quad M_i \approx S^3/\Gamma,\; S^1 \times S^2\}.$$

Then the results presented in this talk are

Theorem 1 (cf [1])**.** *Assume $M \in \mathcal{T}_1$ and let g be an arbitrary Riemannian metric on M. Then there is a Ricci flow with surgery $\mathcal{M}$ starting from (M, g) on a maximal time interval $[0, T)$ which only has finitely many surgeries. Moreover*

$$|\mathrm{Rm}| < \frac{C}{t} \qquad \textit{for all} \qquad t \gg 1.$$

Theorem 2 (to appear)**.** *The same holds when $M \in \mathcal{T}_2$.*

Observe that the conditions in the Theorems above are purely topological. We mention a few consequences which do not use the somewhat unhandy notion of Ricci flows with surgery:

Corollary 1. *Assume $M \in \mathcal{T}_1$ and let $(g_t)_{t\in[0,\infty)}$ be a long-time existent Ricci flow (without surgeries) on M. Then (a posteriori $M \in \mathcal{T}_1'$ and)*

$$|\mathrm{Rm}| < \frac{C}{t} \qquad \text{for all} \qquad t > 0.$$

Corollary 2. *The same holds when $M \in \mathcal{T}_2$.*

Corollaries 1 and 2 are direct consequence of Theorems 1 and 2. The following Corollary asserts that there are indeed examples in which the conditions of these Corollaries are satisified.

Corollary 3. *If $M \in \mathcal{T}_2'$, then there is a long-time existent Ricci flow $(g_t)_{t\in[0,\infty)}$ on M.*

Corollary 3 follows from Theorem 2 in the following way: Consider any Ricci flow with surgery $\mathcal{M}$ starting from an arbitrary metric g on M. Since M is prime and not a spherical space form or $S^1 \times S^2$, we know that $\mathcal{M}$ is defined on the time interval $[0,\infty)$ and that the topology of the manifold does not change throughout time. Moreover, Theorem 2 implies that there is a time $T' > 0$ after which no surgeries occur. Then $\mathcal{M}$ restricted to the time interval $[T',\infty)$ is a long-time existent Ricci flow without surgeries.

In order to understand the proofs of Theorems 1 and 2, it suffices to prove Corollaries 1 and 2 on their own. In fact, the existence of surgeries would only add some technical details to these proofs. On the other hand, the curvature estimates achieved in those Corollaries, rule out the existence of surgeries after a certain time, since the curvature blows up around every surgery.

A major ingredient for the proof of Corollaries 1 and 2 is a generalization of a theorem of Perelman [3, 7.3]. This theorem asserts that if at a late time t in a Ricci flow (with surgery) the volume of the $\rho(x,t)$-ball $B(x,t,\rho(x,t))$ around a point x is bounded from below by wr^3, where $w > 0$ is an arbitrary constant and $\rho(x,t)$ is the maximum of all numbers $r > 0$ such that the sectional curvatures on $B(x,t,r)$ are bounded from below by $-r^{-2}$, then we have curvature control of the form $|\mathrm{Rm}| < K(w)t^{-1}$ on $B(x,t,\overline{\rho}(w)\sqrt{t})$. Here $\overline{\rho}(w) > 0, K(w) < \infty$ are constants depending only on w. This theorem can be generalized, by passing to the universal cover of M (Perelman's theorem also holds in the non-compact case under certain assumptions). Then, the volume bound of $B(x,t,\rho(x,t))$ in Perelman's theorem generalizes to a bound on the volume of the $\rho(x,t)$-ball in the universal cover of M (which is always greater or equal to the volume of $B(x,t,\rho(x,t))$).

This generalization gives us the opportunity to extend the known long-time curvature bounds to regions in the manifold which are collapsed along incompressible S^1 or T^2 fibers. For the proof of Corollary 1, a closer look at the geometry of the collapsed part of the manifold along with a minimal surface argument yields that these regions are large enough to conclude a curvature bound on the whole manifold. The proof of Corollary 2 is far more complex and it makes it necessary to deduce different types of curvature estimates similar to the generalization of Perelman's theorem, which also hold in regions that are collapsed along compressible fibers.

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Signatures of surface bundles and Milnor Wood inequalities

Ursula Hamenstädt

A surface bundle over a surface is a smooth closed 4-manifold E which admits a surjective projection $E \to B$ onto a closed surface B. The preimage of a point $x \in B$ is a closed surface S. Assume that the genus h of B and the genus g of S are at least 2. Such a surface bundle is determined up to diffeomorphism by its monodromy which is a homomorphism $\pi_1(B) \to \mathrm{Mod}(S)$. Here $\mathrm{Mod}(S)$ denotes the group of isotopy classes of diffeomorphisms of S.

The Bogomolov-Miyaoka-Yau inequality states that the Chern numbers of a complex surface M of general type are related by the inequality $c_1^2 \leq 3c_2$. It implies the relation

$$3|\sigma(M)| \leq |\chi(M)|$$

between the signature σ and the Euler characteristic χ.

Kotschick [3] showed that the Miyaoka inequality holds true for surface bundles $E \to B$ over surfaces which admit a complex structure or an Einstein metric. For general surface bundles over surfaces he used Seiberg Witten theory to derive the weaker inequality $2|\sigma(E)| \leq \chi(E)$.

Baykur [1] constructed for large enough fibre and base genus infinitely many surface bundles over surfaces which do not admit a complex structure. Obstructions to the existence of an Einstein metric in this setting are unknown.

In this talk we discuss the following

Theorem $3|\sigma(E)| \leq \chi(E)$ *for any surface bundle* E *over a surface.*

The proof of this result consists in three steps. First we identify an explicit cycle in E which is Poincaré dual to the first Chern class of the cotangent bundle of the fibres of E. This cycle can be viewed as a multi-section of $E \to B$ with singular points. Pulling back the cotangent bundle of the fibres with this multi-section and taking tensor products yields a complex line bundle over the base whose first

Chern class equals $3\sigma(E)$. This first Chern class can then be estimated with a Milnor Wood-type inequality.

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The space of metric measure spaces is an Alexandrov space

Karl-Theodor Sturm

We study the space $\mathbb{X}$ of equivalence classes of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ with finite L^2-size

$$\mathrm{size}(X, \mathsf{d}, \mathfrak{m}) = \left(\int_X \int_X \mathsf{d}^2(x,y)\, d\mathfrak{m}(x)\, d\mathfrak{m}(y) \right)^{1/2}.$$

The space $\mathbb{X}$ will be equipped with the L^2-distortion distance $\mathbb{\Delta}$ defined as

$$\mathbb{\Delta}\big((X_0, \mathsf{d}_0, \mathfrak{m}_0), (X_1, \mathsf{d}_1, \mathfrak{m}_1) \big) =$$
$$\inf_{\bar{\mathfrak{m}} \in \mathrm{Cpl}(\mathfrak{m}_0, \mathfrak{m}_1)} \left(\int_{X_0 \times X_1} \int_{X_0 \times X_1} |\mathsf{d}_0(x_0, y_0) - \mathsf{d}_1(x_1, y_1)|^2\, d\bar{\mathfrak{m}}(x_0, x_1) d\bar{\mathfrak{m}}(y_0, y_1) \right)^{\frac{1}{2}}$$

where $\mathrm{Cpl}(\mathfrak{m}_0, \mathfrak{m}_1)$ denotes the set of all probability measures $\bar{\mathfrak{m}}$ on $X_0 \times X_1$ with $(\pi_0)_* \bar{\mathfrak{m}} = \mathfrak{m}_0$ and $(\pi_1)_* \bar{\mathfrak{m}} = \mathfrak{m}_1$. Restricted to subsets of $\mathbb{X}$ with uniformly bounded diameters the $\mathbb{\Delta}$-topology coincides with the topology induced by Gromov's box distance $\underline{\square}_\lambda$ as well as that induced by the author's L^2-transportation distance $\mathbb{D}$ ([2], [4]).

The metric $\mathbb{\Delta}$, however, is not complete. The $\mathbb{\Delta}$-completion $\bar{\mathbb{X}}$ is the space of equivalence classes of pseudo metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ where X is a Polish space, $\mathfrak{m}$ a Borel measure and d a symmetric, measurable function on X^2 which satisfies the triangle inequality almost everywhere.

Theorem 1. $\big(\bar{\mathbb{X}}, \mathbb{\Delta}\big)$ *is a complete geodesic space of nonnegative curvature in the sense of Alexandrov.*
It is a cone over its unit sphere $\{(X, \mathsf{d}, \mathfrak{m}) \in \bar{\mathbb{X}} : \ \mathrm{size}(X, \mathsf{d}, \mathfrak{m}) = 1\}$ *which itself is an Alexandrov space of curvature* ≥ 1.

A rich class of functions $\mathcal{U} : \bar{\mathbb{X}} \to \mathbb{R}$ is given by the so-called polynomials of order $n \in \mathbb{N}$:

$$\mathcal{U}\big((X, \mathsf{d}, \mathfrak{m})\big) = \int_{X^n} u \left(\big(\mathsf{d}(x^i, x^j)\big)_{1 \leq i,j \leq n} \right) d\mathfrak{m}^n(x^1, \ldots, x^n)$$

for $u : \mathbb{R}^{n \times n} \to \mathbb{R}$ bounded and continuous.

Theorem 2. *Let* $\mathcal{U} : \bar{\mathbb{X}} \to \mathbb{R}$ *be defined as above.*

(i) *If* u *is* λ*-continuous on* $\mathbb{R}^{n\times n}$ *then* $\mathcal{U}$ *is* $\lambda n(n-1)$*-continuous on* $\bar{\mathbb{X}}$.
(ii) *If* u *is* κ*-convex on* $\mathbb{R}^{n\times n}$ *then* $\mathcal{U}$ *is* $\kappa n(n-1)$*-convex on* $\bar{\mathbb{X}}$.

Polynomials of order 4 can be used to characterize spaces with given upper or lower curvature bounds in the sense of Alexandrov.

Another important functional $\mathcal{F} : \bar{\mathbb{X}} \to \mathbb{R}_+$ measures the deviation of the volume growth from that of a given model space. Let $(X^*, \mathsf{d}^*, \mathfrak{m}^*)$ be a mm-space with volume growth

$$v_r^* = \mathfrak{m}^*\left(B_r^*(x)\right)$$

independent of $x \in X^*$. For each $(X, \mathsf{d}, \mathfrak{m}) \in \bar{\mathbb{X}}$ put

$$v_r(x) = \mathfrak{m}(B_r(x)), \quad w_t(x) = \int_0^t \int_0^s v_r(x) dr ds, \quad w_t^* = \int_0^t \int_0^s v_r^* dr ds \quad \text{and}$$

$$\mathcal{F}\big((X, \mathsf{d}, \mathfrak{m})\big) = \int_0^\infty (w_t(x) - w_t^*)^2 \rho_t dt.$$

Theorem 3. (i) *For suitable choice of the weight function* $\rho : \mathbb{R}_+ \to (0, \infty)$, *the function* $\mathcal{F} : \bar{\mathbb{X}} \to \mathbb{R}_+$ *is* κ*-convex and* λ*-Lipschitz continuous with* κ *and* λ *given explicitly.*
(ii) *For each* $(X_0, \mathsf{d}_0, \mathfrak{m}_0) \in \bar{\mathbb{X}}$ *there exists a unique gradient flow curve* $\big((X_t, \mathsf{d}_t, \mathfrak{m}_t)\big)_{t\geq 0}$ *for* $\mathcal{F}$.
(iii) $\mathcal{F}\big((X, \mathsf{d}, \mathfrak{m})\big) = 0 \iff \forall r > 0, \forall x \in X : v_r(x) = v_r^*$.

Remark 4 ([3])**.** *Assume that* X^* *is the* n*-dimensional sphere* S^n *and that* X *is a* k*-dimensional Riemannian manifold (both with normalized volume). Then if* $n \leq 3$

$$\mathcal{F}\big((X, \mathsf{d}, \mathfrak{m})\big) = 0 \iff X = S^n.$$

For further details, see [1].

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Instantaneously complete Ricci flows on surfaces

Gregor Giesen
(joint work with Peter M. Topping)

We consider the well-posedness of the initial value problem of the *Ricci flow*, i.e. of solutions $\big(g(t)\big)_{t\in[0,T]}$ on a connected n-dimensional Riemannian manifold $\big(\mathcal{M}^n, g_0\big)$ without boundary to

$$\text{(RF)} \qquad \begin{cases} \dfrac{\partial}{\partial t} g(t) = -2\mathrm{Rc}_{g(t)} \\ g(0) = g_0. \end{cases}$$

While classical theory provides existence and uniqueness in the case of compact manifolds (Hamilton [6]) or non-compact but complete manifolds with bounded curvature (Shi [8] and Chen-Zhu [2]), very little is known about the Ricci flow of incomplete Riemannian manifolds with unbounded curvature. As this turns out to be very difficult in higher dimensions, we restrict our attention to the two-dimensional situation, where the initial value problem (RF) simplifies to a scalar equation due to its conformal invariance in this very dimension: Writing locally $g_0 = \mathrm{e}^{2u_0}|\mathrm{d}z|^2$ for some (complex) isothermal coordinate $z = x + \mathrm{i}y$ on $\mathcal{U} \subseteq \mathcal{M}$ and some smooth function $u_0 \in \mathrm{C}^\infty\big(\mathcal{U}\big)$, the system (RF) is locally equivalent to

$$\text{(RF}_2\text{)} \qquad \begin{cases} \dfrac{\partial}{\partial t} u(t) = \mathrm{e}^{-2u(t)} \Delta u(t) \\ u(0) = u_0, \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. However, already simple examples reveal a vast non-uniqueness: Taking the flat metric on the open unit disc in the plane as the initial surface $\big(\mathbb{D}, |\mathrm{d}z|^2\big)$, for any prescribed Dirichlet boundary data on $[0,T]\times\partial\mathbb{D}$, standard parabolic theory provides divergent solutions to (RF$_2$) all starting from $u(0) \equiv 0$.

As a 'geometrical' replacement for these boundary conditions, Topping started to investigate those solutions of the Ricci flow which become *instantaneously complete*, i.e. they are complete for all positive time [9]. Along with a very general short-time existence result he also conjectured that this *instantaneous completeness* is the right condition in order to gain uniqueness in this class. In [4] we have generalised Topping's existence result to the most general case including long-time existence and removing any curvature assumptions:

Theorem 1 (1st Part of [4, Theorem 1.3])**.** *Let $\big(\mathcal{M}^2, g_0\big)$ be a smooth Riemannian surface which need not be complete, and could have unbounded curvature. Depending on the conformal class, we define $T \in (0,\infty]$ by*

$$T := \begin{cases} \dfrac{\mathrm{vol}_{g_0}\mathcal{M}}{4\pi\,\chi(\mathcal{M})} & \text{if } \big(\mathcal{M}, g_0\big) \overset{\text{conf}}{=} \mathbb{S}^2 \text{ or } \mathbb{R}P^2 \text{ or } \mathbb{C}, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a smooth Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ such that

(i) $g(0) = g_0$*;*
(ii) $\big(g(t)\big)_{t\in[0,T)}$ *is instantaneously complete;*
(iii) $\big(g(t)\big)_{t\in[0,T)}$ *is maximally stretched*[1]*,*

and this flow is unique in the sense that if $\big(g_2(t)\big)_{t\in[0,T_2)}$ *is any other Ricci flow on* $\mathcal{M}$ *satisfying (i) and (iii), then* $T_2 \le T$ *and* $g_2(t) = g(t)$ *for all* $t \in [0, T_2)$*.*

If $T < \infty$*, then we have*

$$(*)\qquad \operatorname{vol}_{g(t)} \mathcal{M} = 4\pi\,\chi(\mathcal{M})\,(T-t) \quad\longrightarrow\quad 0 \quad \text{as } t \nearrow T,$$

and in particular, T is the maximal existence time.

In contrast to the classical theory where the blow-up of the curvature would mean the solution to end (usually forming a singularity), here the only obstacle to continuing the flow is the entire consumption of the surface's area ($*$). Furthermore, Theorem 1 allows us to construct such examples where the curvature is unbounded for all time:

Theorem 2 ([5, Theorem 1.2])**.** *On every non-compact Riemann surface* $\mathcal{M}^2$ *there exists a complete immortal Ricci flow* $\big(g(t)\big)_{t\in[0,\infty)}$ *with unbounded curvature* $\sup_{\mathcal{M}} \mathrm{K}_{g(t)} = \infty$ *for all* $t \in [0,\infty)$*.*

The solutions of Theorem 1 are unique in the class of maximally stretched Ricci flows. This property of being maximally stretched turns out to be quite useful in applications, e.g. for constructing barriers. Another consequence is that these maximally stretched solutions are not somehow exotic; moreover, if they are complete and have bounded curvature, they coincide with the classical solutions obtained by Hamilton and Shi as long as both exist. The issue of uniqueness in the class of instantaneously complete solutions (as conjectured by Topping) is still partly open: We have shown uniqueness of an instantaneously complete Ricci flow starting at a surface which does *not* admit a complete hyperbolic metric [4, Theorem 1.6], while in [3] we have proved uniqueness, if the initial surface has uniformly negative curvature. The latter result has been slightly improved since then [4, Theorem 1.7].

The estimates we proved in order to establish the existence result (Theorem 1) allows us to describe the asymptotical behaviour in most cases:

Theorem 3 (2nd Part of [4, Theorem 1.3])**.** *Suppose* $\big(g(t)\big)_{t\in[0,\infty)}$ *is an instantaneously complete Ricci flow on a surface* $\mathcal{M}^2$ *such that* $\big(\mathcal{M}^2, g(0)\big)$ *admits a complete hyperbolic metric* g_{hyp} *of curvature* -1*, then we have convergence of the rescaled flow*

$$\frac{1}{2t}\, g(t) \longrightarrow g_{\mathrm{hyp}} \qquad \text{smoothly locally as } t \nearrow \infty.$$

[1] A Ricci flow $\big(g(t)\big)_{t\in[0,T]}$ on $\mathcal{M}^2$ is **maximally stretched**, if for *any* Ricci flow $\big(g_2(t)\big)_{t\in[0,T_2]}$ on $\mathcal{M}^2$ with $g_2(0) \le g(0)$, we have $g_2(t) \le g(t)$ for all $t \in \big[0, \min\{T, T_2\}\big]$.

If additionally there exists a constant $M > 0$ such that $g(0) \leq M g_{\text{hyp}}$, then the convergence is global: For any $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\eta \in (0,1)$ there exists a constant $C = C(k, \eta, g(0)) > 0$ such that

$$\left\| \frac{1}{2t} g(t) - g_{\text{hyp}} \right\|_{C^k(\mathcal{M}, g_{\text{hyp}})} \leq \frac{C}{t^{1-\eta}} \qquad \text{for all } t \in [1, \infty).$$

This generalises the known capability of the Ricci flow to uniformise compact surfaces ([7], [1]) by the very large class of non-compact surfaces which admit a complete hyperbolic metric.

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Desingularisation of Einstein orbifolds

Olivier Biquard

This work is devoted to the study of the compactification of the moduli space of Einstein 4-manifolds. Frequently the study of a compactification of a moduli space of solutions of a geometric partial differential equations (instantons in gauge theory, pseudo-holomorphic curves, etc.) has two parts:

- a *compactness theorem* giving the possible singular limits of the objects;
- a *gluing theorem* which shows that a singular limit is indeed at the boundary of the moduli space, that is one can reconstruct points in the moduli space near the singular limit.

The aim of this work is to understand a part of this picture in the case of Einstein 4-manifolds. There is a good compactness theorem [And89, BKN89]: if one has a sequence of compact Einstein 4-manifolds (M_i, g_i), with Einstein constant ± 1 or 0, with bounded diameter and volume bounded from below, and bounded L^2-norm of the curvature (which reduces to the control of the Euler characteristic via Chern-Weil formulas), then a subsequence must Gromov-Hausdorff converge to a compact Einstein 4-orbifold (M_0, g_0). Moreover, the compactness is smooth outside the singular points, the singular points are of type $\mathbb{R}^4/G$ where $\Gamma \subset SO_4$ is a finite subgroup with isolated fixed point at the origin, and at each singular point there is a sequence $t_i \to 0$ such that $\frac{g_i}{t_i} \to g_{ALE}$, where g_{ALE} is a Ricci flat ALE (Asymptotically Locally Euclidean) metric, which means that it has an end asymptotic to $\mathbb{R}^4/\Gamma$ where $\Gamma \subset SO_4$ is a finite subgroup.

The known possible 'bubbles', that is the Ricci flat ALE 4-manifolds, are Kronheimer's gravitational instantons [Kro89a]: these are hyperKähler 4-manifolds, which are deformations of the minimal resolution of $\mathbb{C}^2/\Gamma$ for a finite subgroup $\Gamma \subset SU_2$. Some finite quotients can also occur. It is still an open important question whether other Ricci flat ALE 4-manifolds exist.

One of the most typical examples is the singular Kummer surface $\mathbb{T}/\mathbb{Z}_2$ — an orbifold with sixteen singular points of type $\mathbb{C}^2/\mathbb{Z}_2$ —, which can be desingularized into a K3 surface, by gluing an Eguchi-Hanson metric (this is a U_2-invariant ALE hyperKähler metric on $T^*\mathbb{C}P^1$) at each singular point.

In this work we precisely study the possibe desingularisation of an Einstein 4-orbifold (M_0, g_0) with singular points of type $\mathbb{C}^2/\mathbb{Z}_2$, by gluing Eguchi-Hanson instantons at the singularities. It turns out that a local obstruction appears: remind that in dimension 4 the Riemannian curvature of an Einstein manifold decomposes into two pieces: $R = R_+ + R_-$, where $R_\pm$ is a symmetric endomorphism of $\pm$-selfdual 2-forms. Then we prove (see the precise statement in [Biq11]):

Theorem. *Let (M_0, g_0) be an Einstein orbifold, with singular points of type $\mathbb{C}^2/\mathbb{Z}_2$. Let (g_i) be a sequence of metrics on the (topological) desingularisation of M_0, which converge to g_0 so that they are close enough to the grafting of Eguchi-Hanson instantons to g_0. Then at each singular point $p \in M_0$,*

$$\det R_+^{g_0}(p) = 0. \qquad (\dagger)$$

Conversely, if (M_0, g_0) is a non compact, Asymptotically Hyperbolic Einstein manifold, and the condition (†) is satisfied at each singular point, then one can desingularise (M_0, g_0).

See [Biq11] for the precise definition of an Asymptotically Hyperbolic manifold: these are manifolds with boundary, with a complete metric in the interior, which behaves roughly like a real hyperbolic metric near the boundary. Of course, the prototype is real hyperbolic space $\mathbb{R}H^4$ itself.

An interesting point is that the obstruction (†) is never satisfied by a (nonzero) constant curvature metric, so real hyperbolic orbifolds or quotients of the sphere are not expected to be desingularisable by an Einstein 4-manifold. In particular the quotient $S^4/\mathbb{Z}_2$ of the 4-sphere with 2 fixed points, or the quotient $\mathbb{R}H^4/\mathbb{Z}_2$, are not expected to be desingularisable.

There are also some interesting consequences of the result on a wall crossing phenomenon on the Dirichlet problem at infinity for Einstein metrics as posed by Anderson [And05], again see the details in [Biq11].

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Telescopic actions

Anton Petrunin

(joint work with Dmitry Panov)

An isometric co-compact properly discontinuous group action H on X is called *telescopic* if for any finitely presented group G, there exists a subgroup H' of finite index in H such that G is isomorphic to the fundamental group of X/H'.

We construct examples of telescopic actions on some CAT[-1] spaces, in particular on 3 and 4-dimensional hyperbolic spaces. As applications we give new proofs of the following statements.

Aitchison's theorem. *Every finitely presented group G can appear as the fundamental group of M/J, where M is a closed 3-manifold and J is an involution which has only isolated fixed points.*

Taubes' theorem. *Every finitely presented group G can appear as the fundamental group of a compact complex 3-manifold.*

Compactness and Non-compactness for the Yamabe Problem on Manifolds With Boundary

Marcus A. Khuri

(joint work with Marcelo M. Disconzi)

The Yamabe problem consists of finding a constant scalar curvature metric $\tilde{g}$ which is pointwise conformal to a given metric g on an n-dimensional ($n \geq 3$) compact Riemannian manifold M without boundary. This is equivalent to producing a positive solution to the following semilinear elliptic equation

$$L_g u + K u^{\frac{n+2}{n-2}} = 0, \text{ on } M, \tag{1}$$

where K is a constant, $L_g = \Delta_g - c(n)R_g$ is the conformal Laplacian for g with scalar curvature R_g, and $c(n) = \frac{n-2}{4(n-1)}$. If $u > 0$ is a solution of (1) then the new metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $c(n)^{-1}K$. This problem was solved in the affirmative through the combined works of Yamabe [12], Trudinger [11], Aubin [1] and Schoen [10] (see also [8] for a complete overview).

It is natural to ask what can be said about the full set of solutions to (1) in the case of positive Yamabe invariant. While this set is noncompact in the C^2 topology when the underlying manifold is S^n with the round metric, when M is not conformally equivalent to the round sphere compactness was established in various cases. However in a surprising turn of events, counterexamples to compactness were found by Brendle [2] when $n \geq 52$, and subsequently by Brendle and Marque [3] for $25 \leq n \leq 51$. Finally, Khuri, Marques and Schoen [7] proved that compactness does hold in all remaining cases, that is, for $n \leq 24$.

An obvious extension of such problems is to consider manifolds with boundary. In this case one would like to conformally deform a given metric to one which has not only constant scalar curvature but constant mean curvature as well. This problem is equivalent to showing the existence of a positive solution to the boundary value problem

$$\begin{cases} L_g u + K u^{\frac{n+2}{n-2}} = 0, & \text{in } M, \\ B_g u = \partial_{\nu_g} u + \frac{n-2}{2}\kappa_g u = \frac{n-2}{2} c u^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \tag{2}$$

where ν_g is the unit outer normal and κ_g is the mean curvature. If such a solution exists then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $c(n)^{-1}K$ and the boundary has mean curvature c. This Yamabe problem on manifolds with boundary was initially investigated by Escobar [4, 5], who solved the problem affirmatively in several cases. With contributions from several authors most of the cases have now been solved.

Consider subcritical approximations to equation (2), where a priori estimates are readily available. Thus we define

$$\Phi_p = \Big\{ u > 0 \;\Big|\; L_g u + K u^p = 0 \text{ in } M,\; B_g u = 0 \text{ on } \partial M \Big\},$$

for $p \in [1, \frac{n+2}{n-2}]$. Furthermore, as the case $K < 0$ has already been treated in [6], we will assume from now on that $K > 0$. Then our main result may be stated as follows.

Theorem 1. *(Compactness) Let (M^n, g) be a smooth compact Riemannian manifold of dimension $3 \leq n \leq 24$ with umbilic boundary, and which is not conformally equivalent to the standard hemisphere (S^n_+, g_0). Then for any $\varepsilon > 0$ there exists a constant $C > 0$ depending only on g and ε such that*

$$C^{-1} \leq u \leq C \text{ and } \| u \|_{C^{2,\alpha}(M)} \leq C$$

for all $u \in \cup_{1+\varepsilon \leq p \leq \frac{n+2}{n-2}} \Phi_p$, where $0 < \alpha < 1$.

This theorem is established by a fine analysis of blow-up behavior at boundary points; such a fine analysis was carried out for interior blow-up points in [7]. The entire problem is reduced to showing the positivity of a certain quadratic form on a finite dimensional vector space, which may be analyzed in a similar manner as is done in the appendix of [7]. Of course this theorem also relies on the Positive Mass Theorem of General Relativity, in its usual form. That is, although we are concerned with manifolds having boundary, we are still able to use the standard Positive Mass Theorem by employing a doubling procedure.

Another key feature of our approach is to employ a version of conformal normal coordinates adapted to the boundary, which elucidates the dependence of various geometric quantities on the conformally invariant umbilicity tensor and Weyl tensor. This coordinate system can be thought of as a good compromise between traditional conformal normal coordinates [8] and the so-called conformal Fermi coordinates [9]. This is because although the latter has been shown to be a powerful tool to study the Yamabe problem on manifolds with boundary, a critical part of the compactness result in [7] is the proof of the positivity of the quadratic form mentioned earlier. This proof makes substantial use of the the radial symmetry coming from normal coordinates and we would like to preserve as much as possible of that original argument.

In general, it is expected that wherever blow-up occurs, these conformally invariant quantities will vanish to high order because, up to a conformal change, the geometry of the manifold resembles that of a sphere near the blow-up. As we are assuming that the boundary is umbilic here, we focus on the Weyl tensor.

Theorem 2. *(Weyl vanishing) Let g be a smooth Riemannian metric defined in the unit half n-ball B_1^+, $6 \leq n \leq 24$. Suppose that there is a sequence of positive solutions $\{u_i\}$ of*

$$\begin{cases} L_g u_i + K u_i^{p_i} = 0, & \text{in } B_1^+, \\ B_g u_i = 0, & \text{on } \overline{B_1^+} \cap \mathbb{R}^{n-1}, \end{cases}$$

$p_i \in (1, \frac{n+2}{n-2}]$, such that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that $\sup_{B_1^+ \setminus B_\varepsilon^+} u_i \leq C(\varepsilon)$ and $\lim_{i\to\infty}(\sup_{B_1^+} u_i) = \infty$. Assume also that $\overline{B_1^+} \cap \mathbb{R}^{n-1}$ is umbilic. Then the Weyl tensor W_g satisfies

$$|W_g|(x) \leq C|x|^l$$

for some integer $l > \frac{n-6}{2}$.

Remark. It may appear that since the boundary is umbilic, the proofs of Theorems 1 and 2 should follow directly from [7] by applying a reflection argument. However, the techniques employed in [7] require a higher degree of regularity than what is typically available from a simple reflection of the metric.

In analogy to the case without boundary, one wonders if Theorem 1 is false when $n \geq 25$. We have also been able to answer this question.

Theorem 3. *Assume that $n \geq 25$. Then there exists a smooth Riemannian metric g on the hemisphere S_+^n and a sequence of positive functions $u_i \in C^\infty(S_+^n)$, such that:*

(i) *g is not conformally flat (so in particular (S_+^n, g) is not conformally equivalent to (S_+^n, g_0), where g_0 is the round metric),*

(ii) *∂S_+^n is umbilic in the metric g,*

(iii) *for each i, u_i is a positive solution of the boundary value problem*

$$\begin{cases} L_g u_i + K u_i^{\frac{n+2}{n-2}} = 0, & in\ S^n_+, \\ B_g u_i = 0, & on\ \partial S^n_+, \end{cases}$$

where K is a positive constant,

(iv) $\sup_{S^n_+} u_i \to \infty$ *as* $i \to \infty$.

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Uniqueness of compact tangent flows in Mean Curvature Flow

Felix Schulze

In this work we study Mean Curvature Flow (MCF) of n-surfaces of codimension $k \geq 1$ in $\mathbb{R}^{n+k}$, which are close to self-similarly shrinking solutions. In the smooth case we consider a family of embeddings $F : M^n \times (t_1, t_2) \to \mathbb{R}^{n+k}$, for M^n closed, such that

$$\frac{d}{dt} F(p,t) = \vec{H}(p,t) \ ,$$

where $\vec{H}(p,t)$ is the mean curvature vector of $M_t := F(M,t)$ at $F(p,t)$. We denote with $\mathcal{M} = \bigcup_{t\in(t_1,t_2)}(M_t \times \{t\}) \subset \mathbb{R}^{n+k} \times \mathbb{R}$ its space-time track.

In the following, let Σ^n be a smooth, closed, embedded n-surface in $\mathbb{R}^{n+k}$ where the mean curvature vector satisfies

$$\vec{H} = -\frac{x^\perp}{2}.$$

Here x is the position vector at a point on Σ and $^\perp$ the projection to the normal space of Σ at that point. Such a surface gives rise to a self-similarly shrinking solution $\mathcal{M}_\Sigma$, where the evolving surfaces are given by

$$\Sigma_t = \sqrt{-t} \cdot \Sigma, \quad t \in (-\infty, 0).$$

We denote its space-time track by $\mathcal{M}_\Sigma$.

We also want to study the case that the flow is allowed to be non-smooth. Following [5], we say that a family of Radon measures $(\mu_t)_{t\in[t_1,t_2)}$ on $\mathbb{R}^{n+k}$ is an integral n-Brakke flow, if for almost every t the measure μ_t comes from a n-rectifiable varifold with integer densities. Furthermore, we require that given any $\varphi \in C^2_c(\mathbb{R}^{n+k}; \mathbb{R}^+)$ the following inequality holds for every $t > 0$

$$\bar{D}_t \mu_t(\varphi) \leq \int -\varphi|\vec{H}|^2 + \langle \nabla\varphi, \vec{H}\rangle \, d\mu_t, \tag{1}$$

where $\bar{D}_t$ denotes the upper derivative at time t and we take the left hand side to be $-\infty$, if μ_t is not n-rectifiable, or does not carry a weak mean curvature. Note that if M_t is moving smoothly by mean curvature flow, then $\bar{D}_t$ is just the usual derivative and we have equality in (1).

We restrict to integral n-Brakke flows which are close to a smooth self-similarly shrinking solution. The assumption that the Brakke flow is close in measure to a smooth solution with multiplicity one actually yields that the Brakke flow has unit density. This implies that for almost all t the corresponding Radon measures can be written as

$$\mu_t = \mathcal{H}^n \llcorner M_t \, .$$

Here M_t is a n-rectifiable subset of $\mathbb{R}^{n+k}$ and $\mathcal{H}^n$ is the n-dimensional Hausdorff-measure on $\mathbb{R}^{n+k}$. If the flow is (locally) smooth, then M_t can be (locally) represented by a smooth n-surface evolving by MCF. Conversely, if M_t moves smoothly by MCF, then $\mu_t := \mathcal{H}^n \llcorner M_t$ defines a unit density n-Brakke flow.

Theorem 1. *Let $\mathcal{M} = (\mu_t)_{t\in(t_1,0)}$ with $t_1 < 0$ be an integral n-Brakke flow such that*

i) $(\mu_t)_{t\in(t_1,t_2)}$ is sufficiently close in measure to $\mathcal{M}_\Sigma$ for some $t_1 < t_2 < 0$.

ii) $\Theta_{(0,0)}(\mathcal{M}) \geq \Theta_{(0,0)}(\mathcal{M}_\Sigma)$, where $\Theta_{(0,0)}(\cdot)$ is the respective Gaussian density at the point $(0,0)$ in space-time.

Then $\mathcal{M}$ is a smooth flow for $t \in [(t_1+t_2)/2, 0)$, and the rescaled surfaces $\tilde{M}_t := (-t)^{-1/2} \cdot M_t$ can be written as normal graphs over Σ, given by smooth sections $v(t)$ of the normal bundle $T^\perp\Sigma$, with $|v(t)|_{C^m(T^\perp\Sigma)}$ uniformly bounded for all $t \in [(t_1+t_2)/2, 0)$ and all $m \in \mathbb{N}$. Furthermore, there exists a self-similarly shrinking surface Σ' with

$$\Sigma' = graph_\Sigma(v')$$

and

$$|v(t) - v'|_{C^m} \leq c_m(\log(-1/t))^{-\alpha_m}$$

for some constants $c_m > 0$ and exponents $\alpha_m > 0$ for all $m \in \mathbb{N}$.

The above theorem implies uniqueness of compact tangent flows as follows. Let the parabolic rescaling with a factor $\lambda > 0$ be given by

$$\mathcal{D}_\lambda : \mathbb{R}^{n+k} \times \mathbb{R} \to \mathbb{R}^{n+k} \times \mathbb{R}, (x,t) \mapsto (\lambda x, \lambda^2 t)\ .$$

Note that any Brakke flow $\mathcal{M}$ (smooth MCF) is mapped to a Brakke flow (smooth MCF), i.e. $\mathcal{D}_\lambda(\mathcal{M})$ is again a Brakke flow (smooth MCF).

Let (x_0, t_0) be a point in space-time and $(\lambda_i)_{i\in\mathbb{N}}, \lambda_i \to \infty$, be a sequence of positive numbers. If $\mathcal{M}$ is a Brakke flow with bounded area ratios, then the compactness theorem for Brakke flows (see [5, 7.1]) ensures that

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - (x_0, t_0)) \to \mathcal{M}'\ , \tag{2}$$

where $\mathcal{M}'$ is again a Brakke flow. Such a flow is called a *tangent flow* of $\mathcal{M}$ at (x_0, t_0). Huisken's monotonicity formula ensures that $\mathcal{M}'$ is self-similarly shrinking, i.e. it is invariant under parabolic rescaling.

Corollary 2. *Let $\mathcal{M}$ be an integral n-Brakke flow with bounded area ratios, and assume that at $(x_0, t) \in \mathbb{R}^{m+k} \times \mathbb{R}$ a tangent flow of $\mathcal{M}$ is $\mathcal{M}_\Sigma$. Then this tangent flow is unique, i.e. for any sequence $(\lambda_i)_{i\in\mathbb{N}}$ of positive numbers, $\lambda_i \to \infty$ it holds*

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - (x_0, t_0)) \to \mathcal{M}_\Sigma\,.$$

Until recently, other than the shrinking sphere and the Angenent torus [2] no further examples of compact self-similarly shrinking solutions in codimension one were known. However, several numerical solutions of D. Chopp [3] suggest that there are a whole variety of such solutions. In a recent preprint [7], N. Møller shows that it is possible to desingularize the intersection lines of a self-similarly shrinking sphere and the Angenent torus to obtain a new, compact, smoothly embedded self-similarly shrinking solution. In higher codimensions this class of solutions should be even bigger.

In a recent work of Kapouleas/Kleene/Møller [6] and X.H. Nguyen [8] non-trivial, non-compact, self-similarly shrinking solutions were constructed. In [4], G. Huisken showed that, under the assumption that the second fundamental form is bounded, the only solutions in the mean convex case are shrinking spheres and cylinders.

The analogous problem for minimal surfaces is the uniqueness of tangent cones. This was studied in [11, 1, 12], and, in the case of multiplicity one tangent cones with isolated singularities, completely settled by L. Simon in [9]. One of the main tools in the analysis therein is the generalisation of an inequality due to Łojasiewicz for real analytic functions to the infinite dimensional setting.

Also in the present work, this Simon-Łojasiewicz inequality for "convex" energy functionals on closed surfaces, plays a central role. We adapt several ideas from [9, 10]. We prove a smooth extension lemma for Brakke flows close to $\mathcal{M}_\Sigma$ and introduce the rescaled flow. Furthermore, we treat the Gaussian integral of

Huisken's monotonicity formula for the rescaled flow as an appropriate "energy functional" on Σ and use the Simon-Łojasiewicz inequality to prove a closeness lemma. This lemma and the extension lemma are then applied to prove the main theorem and its corollary.

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Singularities in 4d Ricci flow

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(joint work with Reto Müller)

Hamilton's Ricci flow [1], $\partial_t g = -2\mathrm{Rc}_g$, has been very successful in many cases. Highlights include Perelman's spectacular proof of the geometrization conjecture [2, 3], the Brendle-Schoen proof of the differentiable sphere theorem [4], and many deep results on the Kähler Ricci flow. However, relatively little is known in the general higher-dimensional case without strong positivity assumptions on the curvatures. In this general case extremely complicated singularities can form and a key problem is to study the nature of these singularities. The aim here is to report on our work on the compactness properties of the space of singularity models, for full details please see [5].

By Perelman's monotonicity formula the singularity models for the Ricci flow are the gradient shrinking Ricci solitons (shrinkers), given by a manifold M, a metric g and a function f such that the following equation holds:

$$\mathrm{Rc}_g + \mathrm{Hess}_g f = \tfrac{1}{2} g\,. \tag{1}$$

Solutions of (1) correspond to selfsimilar solutions of the Ricci flow, moving only by homotheties and diffeomorphisms. Shrinkers always come equipped with a natural basepoint $p \in M$, a minimum point for the potential f, in fact $f(x) \sim \frac{1}{4} d(x,p)^2$. After imposing the normalization $\int_M (4\pi)^{-n/2} e^{-f} dV = 1$, shrinkers also have a well-defined Perelman entropy,

$$\mu(g) = \int_M \big(R + |\nabla f|^2 + f - n\big)(4\pi)^{-n/2} e^{-f} dV\,. \tag{2}$$

Our first main result says that the space of shrinkers with bounded entropy and locally bounded energy is orbifold-compact in arbitrary dimensions:

Theorem 1. *For every sequence of shrinkers (M_i^n, g_i, f_i) satisfying the entropy and local energy assumptions,*

$$\mu(g_i) \geq \underline{\mu} > -\infty, \qquad \int_{B_r(p_i)} |Rm_{g_i}|^{n/2} dV_{g_i} \leq C(r) < \infty\,, \tag{3}$$

there exists a subsequence that converges to an orbifold shrinker in the pointed orbifold Cheeger-Gromov sense.

Here, the limit can have a discrete set of orbifold points modeled on finite quotients $\mathbb{R}^n/\Gamma$ ($\Gamma \subset \mathrm{O}(n)$). Away from these points the convergence is smooth. See also [6, 7, 8, 9] for related compactness results for Ricci solitons, and [10, 11, 12] for the fundamental results in the Einstein case. The strength of our Theorem 1 is that it works for noncompact manifolds and that we do not require any other assumptions, in particular no volume, diameter or pointwise curvature bounds. In fact, most interesting singularity models for the Ricci flow are noncompact, the cylinder being the most basic example. Also, assuming a lower bound for the entropy is very natural, since it is nondecreasing along the Ricci flow by Perelman's celebrated monotonicity formula [2]. In dimension four, a delicate localized Gauss-Bonnet argument even allows us to drop the assumption on energy in favor of essentially an upper bound for the Euler characteristic:

Theorem 2. *For four-dimensional shrinkers (M^4, g, f) we have the weighted L^2-estimate*

$$\int_M |Rm|^2 e^{-f} dV \leq C(\underline{\mu}, \overline{\chi}, C_{tech}) < \infty\,, \tag{4}$$

depending only on a lower bound $\underline{\mu}$ for the entropy, an upper bound $\overline{\chi}$ for the Euler characteristic, and a technical constant C_{tech} such that

$$|\nabla f|(x) \geq 1/C_{tech} \qquad \textit{whenever} \qquad d(x,p) \geq C_{tech}\,. \tag{5}$$

Actually, we believe that the technical condition (5) is always satisfied. It remains an interersting open problem to prove that this is indeed the case.

Outline of the proofs. We first sketch the main steps to prove Theorem 1: Volume comparison implies the existence of a pointed Gromov-Hausdorff limit $(M_\infty, d_\infty, p_\infty)$. Using the lower bound for the entropy and the fact that the scalar curvature is locally bounded on shrinkers we obtain a lower bound for the volume of small balls (noncollapsing). The shrinker equation and the Bianchi identity yield an elliptic equation of the schematic form

$$\Delta \mathrm{Rm} = \nabla f * \nabla \mathrm{Rm} + \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}\,. \tag{6}$$

We then prove the following ε-regularity estimate:

$$||\mathrm{Rm}||_{L^{n/2}(B_\delta(x))} \leq \varepsilon(r) \Rightarrow ||\nabla^k \mathrm{Rm}||_{L^\infty(B_{\delta/2}(x))} \leq \frac{C_k(r)}{\delta^{2+k}} ||\mathrm{Rm}||_{L^{n/2}(B_\delta(x))}\,. \tag{7}$$

A key step here is an uniform estimate for the local Sobolev constant. Putting things together we can pass to a smooth Cheeger-Gromov limit away from a discrete set of singular points. Finally, the singular points are of C^∞-orbifold type.

To get across the idea of the proof of Theorem 2, recall that the Gauss-Bonnet formula for 4-manifolds with boundary has the schematic form

$$\chi(B) = \int_B \left(|\mathrm{Rm}|^2 - |\mathrm{Rc}|^2 + R^2\right) dV + \int_{\partial B} \left(I\!I * \mathrm{Rm} + I\!I * I\!I * I\!I\right) dA\,. \tag{8}$$

We choose (essentially) e^{-f} as a weight function on M, use the coarea formula and apply (8). The goal is then to estimate $\int_M |\mathrm{Rm}|^2 e^{-f} dV$, by controlling all the other terms. The hardest term has the form $\int_M |\mathrm{Rc}|^3 e^{-f} dV$ and comes from the boundary term cubic in the second fundamental form (very roughly $I\!I \sim \nabla^2 f \sim \mathrm{Rc}$). At first sight, it seems impossible to control the cubic Ricci term by the bulk terms which are only quadratic. However, we have the following weighted L^3-estimate for shrinkers:

$$\int_M |\mathrm{Rc}|^3 e^{-f} dV \leq \varepsilon \int_M |\mathrm{Rm}|^2 e^{-f} dV + C(\varepsilon, \underline{\mu})\,. \tag{9}$$

Our proof of the key estimate (9) is based on a delicate use of partial integrations and soliton identities. The proof of Theorem 2 can then be finished by estimating and absorbing all the remaining terms.

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Convexity and semi-ellipticity of Busemann–Hausdorff surface area

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(joint work with Dmitri Burago)

Let $X = (X, \|\cdot\|)$ be a finite-dimensional normed space. By the area of a two-dimensional surface in X we mean the two-dimensional Hausdorff measure, cf. [4]. In this lecture I present the following results of a recent work [3] solving the two-dimensional case of an old problem posed by Busemann, see e.g. [5, page 180].

Theorem 1. *Let $D \subset X$ be a two-dimensional disc contained in a two-dimensional affine subspace. Then D minimizes the area among all surfaces with the same boundary. That is, for every surface S in X such that $\partial S = \partial D$, one has* $\mathrm{Area}(S) \geq \mathrm{Area}(D)$.

Theorem 2. *The two-dimensional area density in X, regarded as a function on the set of simple bi-vectors (i.e., on the two-dimensinal Grassmannian cone), admits a convex extension to $\Lambda^2 X$.*

In the language of geometric measure theory, Theorem 1 says that the area density is semi-elliptic over $\mathbb{Z}$ and $\mathbb{Z}_2$, and Theorem 2 is equivalent to semi-ellipticity of the area over $\mathbb{R}$ (see [2]). Since the area is semi-elliptic for every norm, it is elliptic in the case when the norm is smooth and strictly convex. By a classic result of Almgren, the ellipticity implies the existence and regularity almost everywhere of solutions of the Plateau problem in X.

By the results of [1], Theorem 1 has the following corollaries.

Corollary 1. *Let d be a metric on $\mathbb{R}^2$ invariant under the action of $|mathbbR^2$ by translations. Let B_R denote the metric ball in $(\mathbb{R}^2, d)$ of radius R centered at the origin. Then*

$$\liminf_{R\to\infty} \frac{\mathcal{H}^2(B_R, d)}{R^2} \geq \pi$$

where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure.

Corollary 2. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$, B its unit ball, and d a metric on B such that $d(x,y) \geq \|x-y\|$ for all $x, y \in \partial B$. Then $\mathcal{H}^2(B,d) \geq \pi$.*

The proof of the theorems is based on an almost elementary inequality for the areas of symmetric convex polygons in the plane. This inequality is used to construct a calibrating 2-form (with constant coefficients) for every two-dimensional plane in X. The existence of such a calibrating form implies the convexity and semi-ellipticity of the area.

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An ε-regularity theorem for the Ricci flow

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(joint work with Aaron Naber)

As a motivating example, consider the regularity theory at infinity of complete Ricci-flat manifolds of Euclidean volume growth: Bishop-Gromov tells us that the ratio $\mathrm{Vol}(B_r(x_0))/\omega_n r^n$ is nonincreasing in r, and constant (only) on metric cones. This allows one to prove that blowdowns exist in the Gromov-Hausdorff sense and (after a great amount of work [3]) are, in fact, metric cones. Such tangent cones at infinity are then very likely unique (proved in special cases [4], expected in general). Singular cones do appear in examples (the asymptotic cone of Kronheimer's hyper-Kähler metric on the regular semisimple orbit in $\mathfrak{sl}(3,\mathbb{C})$ is the nilpotent variety of $\mathfrak{sl}(3,\mathbb{C})$, i.e. a $\mathbb{Z}_3$-quotient of a smooth cone [6]). However, the singularities of the tangent cones have codimension ≥ 2, and ≥ 4 in the Kähler case. A basic input in proving regularity results of this kind is *ε-regularity*: the ability to detect smooth points x by smallness of the local density $\mathcal{V}_x(r) = \log(\mathrm{Vol}(B_r(x))/\omega_n r^n)$.

Virtually none of the ingredients in the above theory are at present available for the Ricci flow (given the understanding that blowdowns $\leftrightarrow$ tangent flows, metric cones $\leftrightarrow$ gradient shrinking solitons). We make a start by proving ε-regularity.

Let $(M^n, g(t))$ be a smooth (compact) Ricci flow, parametrized by $t \in [-T, 0]$. To state our main result [5], we need to recall Perelman's pointed entropy [10],

$$\mathcal{W}_x(s) = \int_M (|s|(|\nabla f_x|^2 + R) + f_x - n)\, d\nu_x, \tag{1}$$

where $d\nu_x = H_x\, d\mathrm{vol}$ for the backward heat kernel H_x with pole at $(x, 0)$, and

$$H_x(y,s) = H(x,0\,|\,y,s) = (4\pi|s|)^{-\frac{n}{2}}\, e^{-f_x(y,s)}. \tag{2}$$

We also recall Perelman's global entropy functional $\mu(g,\tau)$.

Theorem 1. *For all n, C there exists $\varepsilon = \varepsilon(n, C)$ such that if we have*

$$R(g(s)) \geq -\frac{C}{|s|}, \quad \inf_{\tau\in(0,2|s|)} \mu(g(s),\tau) \geq -C, \tag{3}$$

for some $s \in [-T, 0)$, then, for all $x_0 \in M$ in the final time slice,

$$\mathcal{W}_{x_0}(s) \geq -\varepsilon \implies |\mathrm{Rm}(x)| \leq (\varepsilon r)^{-2} \text{ for all } x \in P_{\varepsilon r}(x_0, 0), \tag{4}$$

where $r^2 = |s|$ and, in general, $P_r(x,t) = B_r(x,t) \times [t - r^2, t]$.

The assumptions (3) are natural and minimal: Since both lower scalar curvature and μ-entropy bounds are propagated by the flow, (3) can be derived from bounds on the geometry of the initial time slice $(M, g(-T))$, which we view as given.

One might think that Theorem 1 is obvious by contradiction. Indeed there exist well-known proofs of ε-regularity theorems along these lines, e.g. Anderson [1] (for Einstein manifolds) and White [11] (for minimal surfaces and the mean curvature flow). However, these proofs all involve passing from a basepoint of small density to a nearby one that locally minimizes the regularity scale. One therefore needs to know that the density is small in a whole definite neighborhood of x_0.

Continuity of the density is obvious (in an effective Lipschitz fashion) from the monotonicity formulas for Einstein manifolds and minimal surfaces. For the mean curvature flow we at least have continuity in a form that survives to singular limits. However, effective continuity in x is certainly not obvious for $\mathcal{W}_x(s)$, and for all we know might in fact fail. This is the whole difficulty of the proof.

The **first ingredient** to fix this is to regularize the $\mathcal{W}$-entropy by averaging in time. This leads to our introducing what we call the *Nash entropy*; see [9]:

$$\mathcal{N}_x(s) := \frac{1}{|s|}\int_s^0 \mathcal{W}_x(t)\, dt \overset{!}{=} \int_M f_x\, d\nu_x - \frac{n}{2}. \tag{5}$$

This is a straightforward but very useful computation. It quantifies the smoothness gained by averaging in that it shows that $\mathcal{N}_x$ no longer depends on any derivatives of the heat kernel. Moreover, we don't lose too much information because

$$\mathcal{N}_s(x) = -\int_s^0 2|t|\left(1 - \frac{t}{s}\right)\int_M \left|\mathrm{Ric} + \nabla^2 f_x - \frac{g}{2|t|}\right|^2 d\nu_x\, dt. \tag{6}$$

We are then able to prove the following, which leads to a proof of Theorem 1, and in fact of a strengthening of Theorem 1 in which $\mathcal{W}_{x_0}$ gets replaced by $\mathcal{N}_{x_0}$.

Theorem 2. *The Nash entropy is effectively Lipschitz in x. Precisely,*

$$|\mathcal{N}_{x_1}(s) - \mathcal{N}_{x_2}(s)| \leq C'(n,C)|s|^{-\frac{1}{2}} d_{g(0)}(x_1,x_2) \tag{7}$$

for all $x_1, x_2 \in M$ in the final time slice.

In order to prove Theorem 2, we need to estimate $\nabla_x \mathcal{N}_x(s)$. This leads to the **second ingredient**, which we learned about from the work of Qi Zhang [12]: The conjugate heat kernel $H(x,t\,|\,y,s)$ is also the fundamental solution of the ordinary *forward* heat operator $\square_{x,t} = \partial_t - \Delta_{g(t),x}$ with pole at (y,s). Thus, if we want to bound $\nabla_x f_x$, then we need to think about gradient estimates for the forward heat equation. Luckily we have the following parabolic Bochner formula:

$$\square \frac{1}{2}|\nabla u|^2 = -|\nabla^2 u|^2 + \langle \nabla \square u, \nabla u\rangle, \tag{8}$$

which nicely generalizes the usual elliptic Bochner formula, the centerpiece of the theory of spaces with lower Ricci bounds. The final result of Zhang's work in this direction allows us to deduce that, for some constant $C' = C'(n,C)$,

$$|\nabla_x f_x|^2 \leq \frac{C'}{|s|}(C' + f_x). \tag{9}$$

If we substitute this into the integral computing $\nabla_x \mathcal{N}_x$, then we quickly see that it now suffices to estimate the 4th moment of the heat kernel, $\int |f_x|^2\, d\nu_x$. This calls for a Poincaré inequality for the measure $d\nu_x$ because the expression $\int |\nabla_y f_x|^2\, d\nu_x$ featuring on the right-hand side would then be controlled by the μ-entropy.

In actual fact, and this is then the **third ingredient**, we are even able to prove a logarithmic Sobolev inequality for $d\nu_x$ whose optimal constant is universal.

Theorem 3. *The following functional inequality holds along every Ricci flow:*

$$\varphi > 0,\ \int_M \varphi^2\, d\nu_x = 1 \implies \int_M \varphi^2 \log \varphi^2\, d\nu_x \leq 4|s| \int_M |\nabla \varphi|^2\, d\nu_x. \tag{10}$$

Moreover, we have equality if and only if either $\varphi \equiv 1$, or if the flow isometrically splits off an $\mathbb{R}$-factor.

This follows by adapting the classical Bakry-Émery and Bakry-Ledoux proofs for shrinking Ricci solitons and static manifolds with $\mathrm{Ric} \geq 0$, see [2], to the case of the conjugate heat kernel measure coupled to an arbitrary Ricci flow. The reason why this works is ultimately once again the Bochner formula (8). Let us also point out that the applications of the LSI for $d\nu_x$ in (10), and of Perelman's LSI for the Riemannian measure $d\mathrm{vol}$ from [10], seem to be essentially disjoint.

Final remark: An LSI as in (10) implies Gaussian concentration of the measure; see [7]. Unfortunately, in the case of the Ricci flow, this does not give us pointwise Gaussian upper heat kernel bounds in any reasonable generality. However, we do obtain Gaussian upper bounds assuming that the Ricci flow is of type I. Somewhat surprisingly, these were not known before. See [8] for a recent application.

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Doubling construction for CMC hypersurfaces in Riemannian manifolds

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(joint work with Taoniu Sun)

Assume that $n \geq 2$ and that we are given a compact $(n+1)$-dimensional Riemannian manifold (M, g) and a compact n-dimensional manifold Λ. We define $\mathbb{M}(M, g, \Lambda)$ to be the set of immersed hypersurfaces in M which are diffeomorphic to Λ and have their mean curvature function which is constant (it is customary to distinguish minimal hypersurfaces whose mean curvature vanishes identically from constant mean curvature hypersurfaces whose mean curvature is constant not equal to 0).

Adapting the result of White [5], one can prove that, for a generic choice of the metric g on the ambient manifold M, the set $\mathbb{M}(M, g, \Lambda)$ is a smooth one dimensional manifold (possibly empty) which might have infinitely many (compact or non compact) connected components. Understanding the possible degeneration of sequences of constant mean curvature surfaces with fixed topology will certainly give some information about $\mathbb{M}(M, g, \Lambda)$ and this will also provide a partial answer to the existence problem.

Under mild assumptions, we prove that a minimal hypersurface Λ immersed in a Riemannian manifold (M, g) is the multiplicity 2 limit of a family of constant mean curvature hypersurfaces whose topology degenerates as their mean curvature tends to 0. The constant mean curvature hypersurfaces we construct have small mean curvature and are obtained by performing the connected sum between two *copies* of Λ at finitely many carefully chosen points.

Assume that Λ is a smooth, compact orientable, minimal hypersurface immersed in a $(n+1)$-dimensional Riemannian manifold (M, g). The Jacobi operator about

Λ appears in the expression of the second variation of the area functional and is defined by

$$J_\Lambda := \Delta_\Lambda + |A_\Lambda|^2 + \mathrm{Ric}_g(N,N), \tag{1}$$

where Δ_Λ is the Laplace-Beltrami operator on Λ, A_Λ is the second fundamental form, $|A_\Lambda|^2$ is the square of the norm of A_Λ (i.e. the sum of the square of the principal curvatures of Λ). Finally, Ric_g denotes the Ricci tensor on (M,g) and N denotes a unit normal vector field on Λ. Recall that :

Definition 1. *A minimal hypersurface Λ is said to be nondegenerate if*

$$J_\Lambda : C^{2,\alpha}(\Lambda) \longrightarrow C^{0,\alpha}(\Lambda),$$

is injective.

If Λ is nondegenerate, the implicit function theorem guaranties the existence of $\varepsilon_0 > 0$ and a smooth one parameter family of immersed constant mean curvature hypersurfaces Λ_ε, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, whose mean curvature is constant equal to ε. Moreover, Λ_ε is a normal geodesic graph over Λ for some function whose $\mathcal{C}^{2,\alpha}$ norm is bounded by a constant times ε.

We assume that Λ is a nondegenerate, compact, orientable minimal hypersurface which is immersed in M and we define φ_0 to be the (unique) solution of

$$J_\Lambda \, \varphi_0 = 1. \tag{2}$$

Our main result reads :

Theorem 1. *Assume that $n \geq 2$ and that $p \in \Lambda$ is a nondegenerate critical point of φ_0. Further assume that $\varphi_0(p) \neq 0$. Then, there exist $\varepsilon_0 > 0$ and a one parameter family of compact, connected constant mean curvature hypersurfaces $\hat{\Lambda}_\varepsilon$, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, which are immersed in M and satisfy the following properties :*

- *(i) The mean curvature $\hat{\Lambda}_\varepsilon$ is constant equal to ε ;*
- *(ii) Away from any given neighborhood of p in M, the hypersurface $\hat{\Lambda}_\varepsilon$ is, for ε small enough, a normal geodesic graph over a subset of the disjoint union $\Lambda_\varepsilon \sqcup \Lambda_{-\varepsilon}$;*
- *(iii) The hypersurface $\hat{\Lambda}_\varepsilon$ is the connected sum of Λ_ε and $\Lambda_{-\varepsilon}$ at points in Λ_ε and $\Lambda_{-\varepsilon}$ which are close to p.*

The proof of this result is based on a perturbation argument, hence, if Λ is not embedded, $\hat{\Lambda}_\varepsilon$ will not be embedded either. However, when Λ is embedded, the question of the embeddedness of the hypersurfaces $\hat{\Lambda}_\varepsilon$ is addressed in the following :

Corollary 2. *Assume that Λ is embedded and further assume that the function φ_0 does not change sign. Then, for $\varepsilon > 0$ small enough, $\hat{\Lambda}_\varepsilon$ is embedded. If Λ is embedded and φ_0 changes sign, the hypersurfaces $\hat{\Lambda}_\varepsilon$ are not embedded anymore for any $\varepsilon > 0$ small.*

This result is very much influenced by the result of Kapouleas and Yang [3], Butscher and Pacard [1], [2] and by the result of Ritoré [4] where similar doubling constructions are considered when the ambient manifold is either the unit sphere S^{n+1} or a quotient of $\mathbb{R}^{n+1}$.

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On surgery stable curvature conditions

Sebastian Hoelzel

A p-surgery of an n-dimensional manifold M is performed by deleting an open region diffeomorphic to $S^{n-p-1} \times B^{p+1}$ and gluing in a new region diffeomorphic to $B^{n-p} \times S^p$ by identifying along the common boundary. This yields a new manifold

$$\chi_p(M) := \left[M \backslash \left(S^{n-p-1} \times B^{p+1}\right)\right] \cup_{S^{n-p-1} \times S^p} \left[B^{n-p} \times S^p\right],$$

which generally will depend on the region removed.

It is natural to ask whether or not a specific curvature condition survives this surgery operation, i.e. suppose M admits a metric of, say, positive scalar curvature, does so $\chi_p(M)$? That latter question was answered affirmately in [2] and, independently, [5], so positive scalar curvature is indeed surgery stable (for $p = 0, \ldots, n-3$). The surgery stability of quite other curvature conditions - such as positive isotropic curvature, positive p-curvature, among others - was investigated in [4], [3], [7].

Now, in order to give a general answer, one first formulates a convenient notion of a curvature condition which covers all pointwise ones. Consider the space $S^2_B(\mathfrak{so}(n))$ of n-dimensional algebraic curvature operators and let $C \subset S^2_B(\mathfrak{so}(n))$ be a subset. If C is invariant under the natural representation of $O(n)$ on this vector space, it is meaningful to say that a Riemannian manifold (M^n, g) satisfies C, if for all $p \in M$ the pullback $\iota^*\mathfrak{R}(p)$ of the curvature operator at the point p is contained in the interior of C, where $\iota : \mathbb{R}^n \to T_pM$ is an isometry.

Given this notion, one can state the following surgery theorem which is going to be part of the author's Ph.D. thesis:

Theorem 1. *Let $C \subset S^2_B(\mathfrak{so}(n))$ be a $O(n)$-invariant closed convex cone. Suppose $S^{n-k-1} \times \mathbb{R}^{k+1}$ equipped with its standard product metric satisfies C. Then the curvature condition C is stable under p-surgeries for $p = 0, \ldots, k$.*

This theorem subsumes all curvature conditions mentioned above. Furthermore, it should be remarked that actually a stronger version of the theorem holds which requires less by relaxing the requirement on the shape of C - essentially some kind of inner cone condition suffices - and delivers more because it is possible to apply the surgery process along arbitrary compact submanifolds of dimension at most k. Moreover there exists corresponding versions in the equivariant class as well as in the class of conformally flat manifolds.

As an application of the above theorem, consider a closed simply connected manifold M. It is well known that if M admits a metric with positive curvature operator, then it is actually diffeomorphic to the sphere (see [1]). If we require the curvature operator only to be non-negative, then the manifold is diffeomorphic to a (compact) symmetric space. In the nonsymmetric case, one now can employ methods put forward in [2] and [6] in addition to the above surgery theorem to construct metrics with pointwise almost non-negative curvature operators:

Theorem 2. *Let M^n be a closed simply-connected manifold, $n \geq 5$. If M admits a metric of positive scalar curvature, then for any $\epsilon > 0$ there exists a metric g_ϵ on M such that*

$$\mathfrak{R}(g_\epsilon) \geq -\epsilon \left\|\mathfrak{R}(g_\epsilon)\right\|_{g_\epsilon},$$

where $\mathfrak{R}(g_\epsilon)$ denotes the curvature operator of the metric g_ϵ.

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Degeneration of shrinking Ricci solitons

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An important problem in Riemannian geometry is to study the geometric structure of Gromov-Hausdorff limits of Riemannian manifolds: Let (M_i, g_i) be a sequence of Riemannian manifolds and (X, d) a Gromov-Hausdorff limit of (M_i, g_i). What can we say about this metric space X?

By Gromov's compactness theorem, when $\dim_{M_i} = n$ and

$$Ric_{M_i} \geq -(n-1), \tag{1}$$

the limit space is a length metric space. Further structural results are obtained by Cheeger and Colding in their serial papers [2]-[4], and Cheeger, Colding and Tian's paper [5]; see also [1] for a survey of the results in this topic.

There is a generalization of Cheeger, Colding and Tian's theory to the case of bounded Bakry-Émery Ricci curvature, a kind of modified Ricci curvature by a potential function:

$$Ric_f =: Ric_M + \mathrm{Hess}(f). \tag{2}$$

In the very special case of constant Bakry-Émery Ricci curvature, which is known as Ricci solitons by Hamilton, we can state some precise convergent results; see [7] and [6] for details. As Ricci solitons are steady solutions to the Ricci flow, we hope the results here are useful in the study of Ricci flow.

For simplicity we restrict to the case of closed shrinking Ricci soliton. Same results for other kinds of Ricci solitons, compact or not, can be deduced bin the same way. A triple (M, g, f) is called a shrinking Ricci soliton if (M, g) is a Riemannian manifold such that

$$Ric_M + \mathrm{Hess}(f) = g. \tag{3}$$

Let (M_i, g_i, f_i) be a sequence of shrinking Ricci solitons. By the volume comparison of Bakry-Émetry Ricci curvature, there always exist a Gromov-Hausdorff limit (X, d) of (M_i, g_i). However, to obtain more geometric information of X, we need some estimate of f_i. This is guaranteed by a Shur type lemma to the scalar curvature.

The following is the result that we want to show:

Let (M_i, g_i, f_i) be a sequence of shrinking Ricci solitons such that

$$\mathrm{vol}_{M_i} \geq v, \quad \mathrm{diam}_{M_i} \leq D \tag{4}$$

for some positive constants v, D independent of i, then, passing a subsequence if possible, (M_i, g_i, f_i) converges to a triple (X, d, f_∞) where (X, d) is a compact length metric space and f_∞ is a Lipschitz function on X. The lime X has a closed singular locus $\mathcal{S}$ of (Hausdorff) codimension ≥ 2; off the singular locus f_∞ is smooth and there is a smooth Riemannian metric g_∞ there which satisfies shrinking Ricci soliton equation (3) with potential f_∞. Moreover, g_∞ induced the same metric structure as d on $X\backslash\mathcal{S}$ and $g_i \to g_\infty$ smoothly on $X\backslash\mathcal{S}$.

If M_i are assumed to be Kähler, then g_∞ is still Kähler with respect to a complex structure J_∞ and the codimension of $\mathcal{S}$ is ≥ 4.

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Flowing maps to minimal surfaces

Peter Miles Topping
(joint work with Melanie Rupflin)

We introduce a new flow which is related to the mean curvature flow and the harmonic map flow. Our flow is designed to look for minimal surfaces, but has a more tractable singularity structure than the mean curvature flow.

The flow arises by taking a gradient flow for the energy

$$E(u) = E(u, g) = \frac{1}{2} \int_M |du|_g^2 d\mu_g$$

of a map $u : M \to N$ from a closed, orientable Riemannian surface (M, g) to a closed n-dimensional Riemannian manifold N. The gradient flow for E, varying u, would (with respect to the L^2 inner product) be the harmonic map flow of Eells and Sampson. However, we change the situation by allowing both the map u and the metric g to vary. This we need to set up with great care – exploiting as many symmetries as possible – in order to arrive at a system of equations that are as simple as possible, and for which solutions will exist. We end up allowing the domain metric to flow within the space of constant curvature, constant area metrics. With respect to an appropriate inner product, the equations are then

$$\partial_t u = \tau_g(u)$$
$$\partial_t g = Re(P_g(\Phi(u, g)))$$

where $\tau_g(u)$ is the tension field of u (i.e. its Laplacian), $\Phi(u, g)$ is the Hopf differential of u with respect to g, and P_g represents the projection onto the holomorphic quadratic differentials.

In reasonable situations, the output of our flow at infinite time is then a pair (u, g) where u is a harmonic map that is also (weakly) conformal with respect to g. It is well-known that such a harmonic map (when nonconstant) represents a minimal immersion with (according to Gulliver, Osserman and Royden) branching at isolated singularities.

In the talk I sketched what the flow is, and what it does. I discussed the new types of singularity that can occur, and indicated the new estimates that can be used to analyse them, or even rule them out. I also showed how the nature of the flow depends on the genus of the domain. In the genus zero case, our flow is easily seen to coincide with the classical harmonic map flow. In the genus one case, the flow can be rewritten to coincide with a flow of Ding-Li-Liu. In the higher genus case, the behaviour of the domain metric is a lot more complicated owing to the non-integrability of the distribution of 'horizontal' tangent vectors in the space of constant curvature metrics.

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Tits Geometry and Positive Curvature

Karsten Grove
(joint work with Fuquan Fang and Gudlaugur Thorbergsson)

There is a well known link between (maximal) irreducible polar representations and isotropy representations of irreducible symmetric spaces provided by Dadok [Da]. Moreover, the theory by Tits [Ti1] and Burns - Spatzier [BSp] provides a link between irreducible symmetric spaces of non-compact type of rank at least three and compact topological spherical irreducible buildings of rank at least three.

We discover and exploit a rich structure of a (connected) chamber system of finite (Coxeter) type M associated with any polar action of cohomogeneity at least two on any simply connected (closed) positively curved manifold. Although this chamber system is typically not a (Tits) geometry of type M, we prove that in all cases but one that its universal (Tits) cover indeed is a building. We construct a topology on this universal cover making it into a compact topological building in the sense of Burns and Spatzier using also the extension of [GKMW] .

Our work shows that the exception indeed provides a new example (also discovered by Lytchak [Ly]) of a C_3 geometry whose universal cover is not a building.

We use this structure to prove the following rigidity theorem:

Any polar action of cohomogeneity at least two on a simply connected positively curved manifold is smoothly equivalent to a polar action on a rank one symmetric space.

The analysis and methods used in the reducible case (including the case of fixed points), the case of cohomogeneity two (cf. also independent work of Kramer and Lytchak [KL]), and the general irreducible case in cohomogeneity at least three are quite different from one another. Throughout the local approach to buildings by Tits [Ti2] plays a significant role. The present work and different independent work of Lytchak [Ly] on foliations on symmetric spaces are the first instances where this approach has been used in differential geometry.

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