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Arithmetic Geometry

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Abstract. The focus of the workshop was the connection between algebraic geometry and arithmetic. Most lectures were on p-adic topics, underlining the importance of Fontaine's theory in the field, namely it gives a relation between "coherent" and "étale" invariants. Lectures on other topics ranged from anabelian geometry to general algebraic geometry (although with number theoretic applications) and to results on global Shimura varieties.

Mathematics Subject Classification (2000): 11G99.

Introduction by the Organisers

The meeting dealt with topics in the intersection of number theory and algebraic geometry. In many talks Fontaine's comparison between crystalline and étale cohomology played a prominent role. As usual the hospitality of the staff at Oberwolfach provided for a pleasant meeting.
Traditional number theory was represented by the talks of Yuan who proved that there are infinitely many congruent numbers with a given number of prime factors, and Stix who described various conditions on Galois sections related to rational points on hyperbolic curves. Also Wintenberger dealt with Iwasawa theory, constructing one representation which almost (up to one unknown property) allows to prove Leopoldt's conjecture. Finally Poonen dealt with the possible asymptotics of isomorphism classes of Selmer groups, and Zhang gave formulas of Gross-Zagier type for Heegner cycles on Shimura varieties.
On the other hand pure algebraic geometry played prominently in talks by Olsson (on independence "of l" of localized Chern classes), of Pop (on liftings of coverings of curves) and of Rydh (on eliminating ramification in stacks by Kummer and Artin-Schreier extensions). Also Gabber's contribution on pseudoreductive

groups fell into this category, although an important motivation lies in the theory of Galois representations.
Concerning p-divisible groups Scholze proved that the Dieudonné-module is fully faithful over semiperfect rings in characteristic p, and Vasiu investigated to which extend a p-divisible group is determinant by its truncation at level n.
In the theory of p-adic moduli spaces Hellman defined admissible (ϕ, Γ)-modules (belonging to Galois-representations) and investigated their moduli space. Viehmann determined the connected components of Deligne-Lusztig varieties.
About the theory of automorphic Galois representations Harris constructed them for automorphic forms which are not polarised (that is the automorphic representations Π is not selfdual). The construction proceeds via Eisenstein cohomology in a bigger group, and showing that there are congruences to cuspidal cohomology, by relating it to coherent cohomology. Also Kisin proved a variant of the Breuil-Mézard conjecture which gives the multiplicity of the base ring of the universal representation with given Hodge-numbers.
Last not least Bhatt described the new approach to p-adic comparison theorems via the h-topology.

Workshop: Arithmetic Geometry

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Abstracts

Birational Galois sections with local conditions for hyperbolic curves

Jakob Stix

The section conjecture in anabelian geometry describes rational points of anabelian varieties in terms of profinite groups. We report on progress made in [Sx12].

1. The section conjecture of anabelian geometry

1.1. **The conjecture.** Let k be a number field and $\mathrm{Gal}_k = \mathrm{Gal}(\bar{k}/k)$ its absolute Galois group. A rational point $a \in X(k)$ of a geometrically connected variety X/k yields by functoriality a section

$$s_a : \mathrm{Gal}_k = \pi_1(\mathrm{Spec}(k)) \to \pi_1(X)$$

of the projection map $\mathrm{pr}_* : \pi_1(X) \to \mathrm{Gal}_k$, which, due to neglecting base points, is only well defined up to conjugation by elements of $\pi_1(X_{\bar{k}}) \subset \pi_1(X)$.

A section s is **cuspidal** if X has a smooth completion $X \subset \overline{X}$ and a k-rational point $a \in (\overline{X} \setminus X)(k)$ such that s factors over the corresponding decomposition subgroup $D_a \subset \pi_1(X)$, well defined up to $\pi_1(X_{\bar{k}})$-conjugacy.

Conjecture 1 (Grothendieck [Gr83]). *Let k be a number field and X/k a smooth, geometrically connected curve with non-abelian $\pi_1(X_{\bar{k}})$. Then the map $a \mapsto s_a$*

$$X(k) \to \mathscr{S}_{\pi_1(X/k)} = \{\text{sections } s : \mathrm{Gal}_k \to \pi_1(X) \text{ of } \mathrm{pr}_*\} / \pi_1(X_{\bar{k}})\text{-}\textit{conjugacy}$$

is a bijection onto the complement of the set of cuspidal sections $\mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)}$.

2. Local conditions for Galois sections

2.1. **A hierarchy of sections.** We introduce local conditions on sections that are shared by sections s_a coming from rational points. For a place v of the number field k we denote by $k \hookrightarrow k_v$ its completion and consider $\mathrm{Gal}_{k_v} \subset \mathrm{Gal}_k$ as a subgroup by fixing a choice of a prolongation of v to $\bar{k}$.

A **Selmer** section is a section s that locally comes from a point, i.e., such that for all v we have $a_v \in \overline{X}(k_v)$ and

$$s|_{\mathrm{Gal}_{k_v}} = s_{a_v} : \mathrm{Gal}_{k_v} \to \pi_1(X_{k_v}) \subset \pi_1(X).$$

Since the map $a \mapsto s_a$ is injective over number fields as well as over local fields when X is a curve, we obtain a well defined map: the **associated adele**

$$\underline{a} : \mathscr{S}^{\mathrm{Selmer}}_{\pi_1(X/k)} = \{s \in \mathscr{S}_{\pi_1(X/k)} \ ; \ \text{Selmer section}\} \to \overline{X}(\mathbb{A}_k)_\bullet$$

where $\mathbb{A}_k$ denotes the ring of k-adeles and $(-)_\bullet$ means that we have replaced the archimedian components by their connected components.

Let K be the function field of X/k. Then a **birationally liftable** section is a section s that lifts to a section $\tilde{s} : \mathrm{Gal}_k \to \mathrm{Gal}_K$ along the natural surjection $\mathrm{Gal}_K \twoheadrightarrow \pi_1(X)$. It follows from Koenigsmann's lemma [Ko05] §2.4, see also [Sx12] Prop. 1, that birationally liftable sections are Selmer sections.

An **adelic** section is a Selmer section $s : \mathrm{Gal}_k \to \pi_1(X)$ such that $\underline{a}(s) \in X(\mathbb{A}_k)_\bullet$ is even an adelic point of X. Finally, a **birationally adelic** section is a section s that admits a lift $\tilde{s} : \mathrm{Gal}_k \to \mathrm{Gal}_K$ such that for all open $U \subset X$ the induced section

$$s_U : \mathrm{Gal}_k \xrightarrow{\tilde{s}} \mathrm{Gal}_K \twoheadrightarrow \pi_1(U)$$

yields an adelic or cuspidal section of U. Clearly we obtain a hierarchy of sections:

$$X(k) \amalg \mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)} \subseteq \{s \in \mathscr{S}_{\pi_1(X/k)} \ ; \ \text{ birationally adelic}\} \subseteq \mathscr{S}^{\mathrm{Selmer}}_{\pi_1(X/k)}.$$

2.2. **The support.** The **support** of a Selmer section $s : \mathrm{Gal}_k \to \pi_1(X)$ with adele $\underline{a}(s) = (a_v(s))_v$ is the Zariski closure

$$Z(s) = \overline{\bigcup_v \mathrm{im}(a_v(s) : \mathrm{Spec}(k_v) \to \overline{X})} \subseteq \overline{X}.$$

We say that a Selmer section s has **finite support** if $Z(s)$ is finite over k. If $Z = Z(s)$ is finite and the genus of $\overline{X}$ is ≥ 1, then by Stoll [St07] Theorem 8.2,

$$Z(k) = \big\{(a_v) \in \overline{X}(\mathbb{A}_k)^{\mathrm{f-desc}}_\bullet \ ; \ a_v \in Z(k_v) \text{ for a set of places } v \text{ of density } 1\big\}$$

where $(-)^{\mathrm{f-desc}}$ means that we require (a_v) to survive all descent obstructions imposed by torsors over $\overline{X}$ under finite groups G/k.

Since for a Selmer section s the adele $\underline{a}(s)$ survives all finite descent obstructions, we conclude by the usual limit argument with the tower of all neighbourhoods of s, that the image of the map

$$X(k) \amalg \mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)} \hookrightarrow \mathscr{S}^{\mathrm{Selmer}}_{\pi_1(X/k)}$$

consists precisely of the Selmer sections with finite support.

Theorem 2. *Let k be a totally real number field or an imaginary quadratic number field. Then we have*

$$X(k) \amalg \mathscr{S}^{\mathrm{cusp}}_{\pi_1(X/k)} = \{s \in \mathscr{S}_{\pi_1(X/k)} \ ; \ \textit{birationally adelic}\}$$

for X/k a smooth, geometrically connected curve with non-abelian $\pi_1(X_{\bar{k}})$.

Proof: It suffices to find an open $U \subseteq X$ and a quasi-finite map $f : U \to \mathbb{T}$ to a torus $\mathbb{T}$ such that all adelic sections $\mathrm{Gal}_k \to \pi_1(\mathbb{T})$ come from rational points, because if $\pi_1(f) \circ s = s_t$ for $t \in \mathbb{T}(k)$, then $Z(s) \subseteq f^{-1}(t)$ will be finite. If $k/\mathbb{Q}$ is imaginary quadratic, then $\mathbb{T} = \mathbb{G}_\mathrm{m}$ suffices since $\mathfrak{o}_k^*$ is finite. When k is totally real, we can use for $\mathbb{T}$ the norm 1 torus of a totally imaginary quadratic extension of k. For details we refer to [Sx12] §3+4. □

3. Almost compatible systems of ℓ-adic representations

3.1. **The representations.** Let $f : E \to X$ be a family of elliptic curves. Any section $s : \mathrm{Gal}_k \to \pi_1(X)$ leads to a system of ℓ-adic representations

$$\rho_s = (\rho_{s,\ell}) = \Big(\rho_{E/X,s,\ell} : \mathrm{Gal}_k \xrightarrow{s} \pi_1(X, \bar{x}) \xrightarrow{\rho_{E/X,\ell}} \mathrm{GL}_2(\mathbb{Z}_\ell)\Big)_\ell$$

where $\rho_{E/X,\ell}$ is the monodromy representation on the fibre $\mathrm{T}_\ell(E_{\bar{x}}) \cong \mathbb{Z}_\ell^2$ corresponding to $\mathrm{R}^1 f_* \mathbb{Z}_\ell(1)$. If s is a Selmer section, and E/X has bad semistable reduction, then

(i) $\det(\rho_s) = \varepsilon$ is the cyclotomic character,

(ii) for all finite places v of k the local representation $\rho_{s,\ell}|_{\mathrm{Gal}_{k_v}}$ for $v \nmid \ell$ has a semisimplification $\rho_{s,v,\ell}^{\mathrm{ss}}$ with

$$\det(\mathbf{1} - \mathrm{Frob}_v \cdot T | \rho_{s,v,\ell}^{\mathrm{ss}}) = 1 - a_v(\rho)T + N(v)T^2 \in \mathbb{Z}[T]$$

and the trace of Frobenius $a_v(\rho)$ is independent of ℓ,

(iii) moreover, the semisimplification $\rho_{s,v,\ell}^{\mathrm{ss}}$ has weight -1 or weights 0 and -2.

3.2. **Integrality.** Let $G_\ell \subseteq \mathrm{GL}_2(\mathbb{F}_\ell)$ be the image of $\rho_{s,\ell}$ mod ℓ, and let $M_\ell \subseteq G_\ell$ be the subset of elements with one eigenvalue ± 1. Then either $\rho_{s,\ell}$ is reducible for all $\ell \gg 0$ or otherwise $\# M_\ell / \# G_\ell \to 0$ when $\ell \to \infty$. Combining this argument with Chebotarev's density theorem and the description of monodromy of the Legendre family of elliptic curves allows to prove the following, see [Sx12] §5+6.

Theorem 3. *Let $s : \mathrm{Gal}_k \to \pi_1(X)$ be a birationally liftable section of a smooth, geometrically connected curve X/k with non-abelian $\pi_1(X_{\bar{k}})$ over a number field k and with smooth completion $\overline{X}$. Then the associated adele $\underline{a}(s) \in \overline{X}(\mathbb{A}_k)_\bullet$ is either integral for a set of places v of Dirichlet density 1, or the section s is cuspidal.*

Note that the statement on integrality in Theorem 3 is well defined although integrality depends on the chosen model of X over $\mathrm{Spec}(\mathfrak{o}_k)$.

3.3. **Modularity.** The following result, see [Sx12] §7, requires $k = \mathbb{Q}$ since it relies on the arithmetic of the Eisenstein quotient of the modular jacobians $J_0(\ell)$ and on Serre's modularity conjecture proven by Khare and Wintenberger.

Theorem 4. *Let $X/\mathbb{Q}$ be a smooth, geometrically connected curve with non-abelian $\pi_1(X_{\bar{\mathbb{Q}}})$. A section $s : \mathrm{Gal}_{\mathbb{Q}} \to \pi_1(X)$ comes from a rational point or is cuspidal, if and only if s is birationally liftable (to say $\tilde{s}$) and for every open $U \subseteq X$ and every family E/U of elliptic curves the associated family of ℓ-adic representations $\rho_{E/U,\tilde{s}}$ has one of the following properties:*

*(i) **Finite conductor:** There exists a finite set of places S independent of ℓ such that $\rho_{E/U,\tilde{s},\ell}$ is unramified outside ℓ and the places in S.*

*(ii) **Reducible:** There is a character $\delta : \mathrm{Gal}_{\mathbb{Q}} \to \{\pm 1\}$ such that for all ℓ we have an exact sequence $0 \to \delta\varepsilon \to \rho_{E/U,\tilde{s},\ell} \to \delta \to 0$, where ε is the ℓ-adic cyclotomic character.*

The two cases in Theorem 4 reflect the dichotomy of the section s being associated to a rational point $a \in U(k)$ or to being already cuspidal for U/k.

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Tame and wild ramification via stacks

David Rydh

We report on work on ramification via stacks whose main goals are:

- removing ramification ("étalification") using **stacky blow-ups**; and
- the existence of compactifications of Deligne–Mumford stacks.

Ramification is the obstruction for a generically étale morphism to be étale everywhere. It can be argued that ramification in this sense is succinctly described by stacks and, more explicitly, by **stacky blow-ups**. In characteristic zero, stacky blow-ups are combinations of blow-ups and root stacks (related to Kummer theory). Remarkably, in positive characteristic it is sufficient to consider Kummer and Artin–Schreier theory to design stacky blow-ups that deal with all wild ramification—although the results are somewhat weaker. Before going into details about ramification, let us recall a similar picture involving flatness where the key result is Raynaud–Gruson's celebrated **flatification** theorem. (All schemes and stacks are quasi-compact and quasi-separated.)

Theorem 1 (Flatification [RG71])**.** *Let S be a scheme, let $U \subset S$ be an open subscheme and let $f\colon X \to S$ be a morphism of finite type such that $f|_U$ is flat and of finite presentation. Then, there exists a U-admissible blow-up $\pi\colon \widetilde{S} \to S$ such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{S}$ is flat and of finitepresentation.*

An important consequence of this theorem is that U-admissible blow-ups are cofinal in the directed set of U-admissiblemodifications. To remove ramification, there is an analogous **étalification** result.

Theorem 2 (Étalification)**.** *Let S be a scheme of equal characteristic, let $U \subset S$ be an open subscheme and let $f\colon X \to S$ be a morphism of finite type such that $f|_U$ is étale. Then, there exists a U-admissible **stacky blow-up** $\pi\colon \widetilde{S} \to S$ and a $\pi^{-1}(U)$-admissible **blow-up** $\psi\colon \widetilde{X} \to X \times_S \widetilde{S}$ such that $\widetilde{X} \to \widetilde{S}$ is étale.*

Stacky blow-ups, defined below, are certain special and explicit stacky modifications. Here, by stacky modification, we signify a proper and birational morphism of stacks. Every blow-up is a stacky blow-up and compositions ofstacky blow-ups are stacky blow-ups. Since étale stacky modifications are isomorphisms, it follows from the étalification theorem that U-admissible stacky blow-ups are cofinal in the directed category of U-admissible stacky modifications. For tamely ramified morphisms, tame stacky blow-ups are enough and the theorem in this case (including mixed characteristic) is proved in [Ryd11a].

Ramification of curves. Let $f\colon C' \to C$ be a finite flat morphism of smooth curves. To study the ramification, we may work étale-locally and thus assume that $C = \mathrm{Spec}(V)$ and $C' = \mathrm{Spec}(V')$ are spectra of strictly henselian discrete valuation rings. If f is tamely ramified, then f is determined by the ramification index e. Explicitly $V' = V[z]/z^e - t$, where t is a uniformizing parameter of V. The action of the cyclic group $\mathbb{Z}/e\mathbb{Z}$ on C', given by $k.z = \xi^k z$, is generically free and transitive so that the stack quotient $[C'/(\mathbb{Z}/e\mathbb{Z})] \to C$ is proper and birational—a stacky modification. The stack quotient is identified as the eth **root stack** of the Cartier divisor D defined by $t = 0$. In particular, up to étale morphisms, C' is equivalent to a stacky modification of C. More generally we have the following generalization of Abhyankar's lemma:

Lemma. *Let V be any valuation ring (not necessarily discrete), and let $X \to S = \mathrm{Spec}(V)$ be a quasi-finite flat and tamely ramified morphism (this notion is defined via valuations on X). Then there exists divisors $D_1, D_2, \dots, D_n$ on S and integers $e_1, e_2, \dots, e_n$ such that if $S' = S_{(D_i, e_i)}$ is the e_ith root stack of the D_is then $X'^{\mathrm{norm}} = (X \times_S S')^{\mathrm{norm}}$ is étale over S'^{norm}.*

Root stacks. Let $D \hookrightarrow X$ be an effective Cartier divisor. Then the simple cyclic covering of degree r of X ramified along D exists if and only if $\mathcal{L} = \mathcal{O}_X(D)$ has an rth root $\mathcal{L}^{1/r}$. If $\mathcal{L} = \mathcal{O}_X$ and D is defined by the section s, then this covering is given by $\mathrm{Spec}_X(\mathcal{O}_X[z]/z^r - s)$. The rth root stack of D is the quotient of this covering by its Galois group:

$$X_{(D,r)} = [\mathrm{Spec}_X(\mathcal{O}_X[z]/z^r - s)/\boldsymbol{\mu}_r].$$

In general, there is no cyclic covering ramified along D. Using the added flexibility given by stacks, it is on the other hand possible to glue theconstruction above so that the the root stack exists for any Cartier divisor D [Cad07].

Definition. *Let S be an algebraic stack, let $Z \hookrightarrow S$ be a closed substack (of finite presentation) and let r be a positive integer. We let $\mathrm{Bl}_{Z,r}(S)$ denote the algebraic stack $\mathrm{Bl}_Z(S)_{(E,r)}$ where E is the exceptional divisor. That is, first we blow up S along Z and then we take the rth root stack along the exceptional divisor E. We say that $\pi\colon X \to Y$ is a* ***Kummer blow-up*** *if there is a closed substack $Z \hookrightarrow Y$ and a positive integer r, invertible along Z, such that $X = \mathrm{Bl}_{Z,r}(Y)$. We say that $\pi\colon X \to Y$ is a* ***tame stacky blow-up*** *if $\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_{n-1} \circ \pi_n$ where each π_i is a Kummer blow-up.*

Proof of the étalification theorem. For tamely ramified morphisms one uses:

- an extension result for tame stacky blow-ups along étale morphisms that follows from étale devissage [Ryd11b];
- Abhyankar's lemma, from which the theorem quite easily follows when the base is the spectrum of a valuation ring;
- Riemann–Zariski spaces of valuations: used to pass from the valuative-local case to the global case (here non-discrete valuations are necessary).

Wild ramification. The proof in the wild case is similar but more involved. Wild stacky blow-ups are built up from three operations: blow-ups, root stacks and **Artin–Schreier stacks**. Recall that an Artin–Schreier covering of smooth curves $C' \to C$ in characteristic p is given by a birational equation

$$K(C') = K(C)[z]/z^p - z - f, \quad f \in K(C).$$

The covering is ramified over the poles of f and the order of the pole determines the unique jump in the filtration of higher ramification groups. The corresponding stack, given by taking the stack quotient of C' by the Galois group $\mathbb{Z}/p\mathbb{Z}$, is a wild Deligne–Mumford stack and the archetypal example of an **elementary Artin–Schreier stack**.

Artin–Schreier stacks. General Artin–Schreier stacks are more complicated:

(1) to accommodate ramification from covers with arbitrary Galois groups, it is necessary to include Weil restrictions of elementary Artin–Schreier stacks along finite étale morphisms (over the branch locus);
(2) to be able to construct global Artin–Schreier stacks in dimension two and higher, it seems necessary to allow for certain $\boldsymbol{\mu}_p$ and $\boldsymbol{\alpha}_p$ stabilizers.

Generalizing the étalification theorem, allowing for X and S to be Deligne–Mumford stacks, is straight-forward. In characteristic p, it is also crucial to pass beyond Deligne–Mumford stacks since Artin–Schreier stacks can have non-reduced stabilizer groups due to (2) above. Although the proof of the étalification theorem employs étale methods, it is possible to extend the theorem to stacks that are “almostDeligne–Mumford” and this is sufficient for the following application.

Theorem 3 (Compactification of Deligne–Mumford stacks)**.** *Let $f\colon X \to S$ be a separated morphism of finite type between Deligne–Mumford stack. Then f can be compactified, that is, there exists a proper algebraic stack $\overline{f}\colon \overline{X} \to S$ with finite stabilizers together with an open immersion $X \to \overline{X}$ over S.If f has tame stabilizers, then f has a tame compactification, that is, there is a compactification such that $\overline{X}$ is tame and Deligne–Mumford.*

Examples indicate that wild Deligne–Mumford stacks do not always admit Deligne–Mumford compactifications. Further applications include stacky semi-stable reduction, and, for tame stacks, a refined Chow lemma and abelianification of stabilizers.

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Moduli of p-divisible groups

Peter Scholze
(joint work with Jared Weinstein)

In this talk, we explained some results about moduli spaces of p-divisible groups. Specifically, we talked about the following results.

A semiperfect ring is by definition a ring R of characteristic p such that the Frobenius morphism $\Phi : R \to R$ is surjective. If R is a semiperfect ring, a construction of Fontaine defines a p-adically complete PD thickening $A \to R$. Indeed, let $W = W(\varprojlim_\Phi R)$, which surjects onto R. Let $W_{\mathrm{PD}} \subset W[p^{-1}]$ be generated by all divided powers in the kernel of the surjection $W \to R$, and let A be the p-adic completion of W_{PD}. There is a Frobenius morphism ϕ on A.

Theorem 1. *Let R be a semiperfect ring. Then the Dieudonné module functor from p-divisible groups over R to finite projective A-modules M equipped with a Frobenius morphism F and a Verschiebung morphism V, $FV = VF = p$, is fully faithful.*

The proof of this theorem proceeds in two steps. First, by an explicit calculation, we handle the case of morphisms from $\mathbb{Q}_p/\mathbb{Z}_p$ to μ_{p^∞}. This involves the integrality of the Artin-Hasse exponential

$$\exp(t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \ldots) \in \mathbb{Z}_p[[t]] \ .$$

Secondly, we reduce the case of morphisms from G to H, G and H general, to this special case by working over the ring S representing morphisms from $\mathbb{Q}_p/\mathbb{Z}_p$ to G, and from H to μ_{p^∞}. This ring S is still semiperfect.

The other result is a classification of p-divisible groups over $\mathcal{O}_C$, where C is an algebraically closed and complete extension of $\mathbb{Q}_p$. For this, recall the following result of Fargues, [3].

Theorem 2. *Let G be a p-divisible group over $\mathcal{O}_C$. Then there is a natural short exact sequence*

$$0 \to \mathrm{Lie}\ G \otimes C(1) \to T(G) \otimes C \to (\mathrm{Lie}\ G^*)^* \otimes C \to 0 \ .$$

Here, $T(G)$ denotes the Tate module of G. Using this filtration on $T(G) \otimes C$, we get the following analogue of Riemann's classification of abelian varieties over the complex numbers.

Theorem 3. *The category of p-divisible groups over $\mathcal{O}_C$ is equivalent to the category of pairs (Λ, W), where Λ is a finite free $\mathbb{Z}_p$-module, and $W \subset \Lambda \otimes C$ is a subvectorspace. The functor is given by sending G to $\Lambda = T(G)$, and $W =$ Lie $G \otimes C(1) \subset T(G) \otimes C$. Moreover, there is a p-divisible group H over $\bar{\mathbb{F}}_p$ such that $G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p$ is isogenous to $H \otimes_{\bar{\mathbb{F}}_p} \mathcal{O}_C/p$.*

The last part says that G is isotrivial in the sense of Fargues, [3]. This follows from our first theorem, and an isotriviality result for Dieudonné modules proved by Fargues – Fontaine, [5].

The fully faithfulness part in this theorem was already observed by Fargues, [6]. Essentially, one shows that the generic fibre of the p-divisible group in the sense of rigid-analytic geometry is (a union of) open balls, and can be defined directly in terms of (Λ, W). In order to prove essential surjectivity, we handle first the case where C is spherically complete and the norm map $|\cdot| : C \to \mathbb{R}_{\geq 0}$ is surjective. In that case, one uses that an increasing union of closed balls can be easily shown to be an open ball.

As an application, we can reprove Faltings's result on the image of the Rapoport-Zink period morphism, [2]. Take any p-divisible group H over $\bar{\mathbb{F}}_p$, of dimension d and height h. One gets the Rapoport-Zink space $\mathcal{M}^{\mathrm{RZ}}$, and on its generic fibre $\mathcal{M}_\eta^{\mathrm{RZ}}$, one has the étale period morphism

$$\pi : \mathcal{M}_\eta^{\mathrm{RZ}} \to \mathcal{F} = \mathrm{Grass}_{d,h} \ ,$$

considered as a morphism of adic spaces, cf. [4].

Theorem 4. *The image of the period morphism is as conjectured.*

It is enough to check this on C-valued points, where C is such a big field for which we already know the classification result for p-divisible groups. In that case, one uses Theorem 3 to construct the desired p-divisible group. In general, one uses the description of the image of the period morphism to deduce Theorem 3 for general C.

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Extensions of Iwasawa modules

Jean-Pierre Wintenberger
(joint work with Chandrashekar Khare)

In a first part of the talk, we discussed the "Reciprocity Conjecture" that we formulated by analogy of a standard property of Jacobian of curves and its links with Leopoldt Conjecture. Reciprocity conjecture is now a theorem of Romyar Sharifi ([2]). It allows to prove that Leopoldt conjecture for a totally real number field F and a prime number $p \neq 2$ is equivalent to the splitting up to isogeny of an exact sequence of Iwasawa modules. This splitting is equivalent to the existence of a $\mathbb{Z}_p$-extension L_Q of the cyclotomic extension $\mathcal{F}_\infty = F(\mu_{p^\infty})$, which is Galois over F, is such that the action of $\mathrm{Gal}(\mathcal{F}_\infty/F)$ on $\mathrm{Gal}(L_Q/\mathcal{F}_\infty)$ is via the cyclotomic character, and which satisfies suitable properties of ramification. In a second part of the talk we discussed the possibility to construct L_Q by modular methods as developed by Andrew Wiles to prove Main Conjecture of Iwasawa theory ([3]).

Let us suppose for simplicity that $[F(\mu_p) : F] = 2$.

Let us state the Reciprocity Conjecture. Let F_∞ be the cyclotomic $\mathbb{Z}_p$-extension of F. Let M_∞ be the maximal abelian pro-p extension of F_∞ which is unramified outside the primes of F above p. Let Q be a finite set of primes q of F that are not above p. We suppose for simplicity that Q has 2 elements q_1 and q_2 and that the prime ideals q_1 and q_2 are inert in F_∞. Let Y_∞ (resp. Y'_∞) be the Galois groups $\mathrm{Gal}(M_\infty/F_\infty)$ (resp. $\mathrm{Gal}(M_\infty/F)$). We have the exact sequence :

$$(0) \to Y_\infty \to Y'_\infty \to \Gamma \to (0)$$

with $\Gamma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$. Let γ be a generator of Γ. The subfield N_∞ of M_∞ fixed by $(\gamma - \mathrm{id})(Y_\infty)$ is the maximal abelian pro-p extension of F which is unramified outside p. Hence, for q prime ideal of F prime to p, the Frobenius $\mathrm{Frob}_q \in Y'_\infty/(\gamma - \mathrm{id})(Y_\infty)$ is well defined (and one of the equivalent statements of Leopoldt Conjecture for F and p is that N_∞ is a finite extension of F_∞). We let $M'(Q)$ be the $\mathbb{Z}_p$-submodule of $Y'_\infty/(\gamma - \mathrm{id})(Y_\infty)$ generated by Frob_{q_1} and Frob_{q_2}. We define $M(Q)$ as the $\mathbb{Z}_p$-submodule of elements of $M(Q)'$ with trivial image in Γ. The module $M(Q)$ is a cyclic $\mathbb{Z}_p$-submodule of $Y_\infty/(\gamma - \mathrm{id})(Y_\infty)$. We identify $Y_\infty/(\gamma - \mathrm{id})(Y_\infty)$ with $H^1(\Gamma, Y_\infty)$.

Let $\mathcal{A}_\infty$ (resp. $\mathcal{A}_{\infty,Q}$) be the inductive limit of the p-Sylow subgroups of the class groups of $F(\mu_{p^n})$ (resp. class groups of conductor Q_n the product of primes over Q in $F(\mu_{p^n})$). Let us denote by $\mathcal{A}^-$ be the part of $\mathcal{A}$ acted by -1 by the complex conjugation. Class Field Theory gives an exact sequence :

$$(0) \to \mu_{p^\infty} \to \mathcal{A}^-_{\infty,Q} \to \mathcal{A}^-_\infty \to (0).$$

The cohomology class c_Q of this extension leaves in $H^1(\Gamma, \mathrm{Hom}(\mathcal{A}^-_\infty, \mu_{p^\infty}))$. Iwasawa proved that $\mathcal{A}^-_\infty$ and Y_∞ are naturally Pontryagin duals. This allows to see c_Q as an element of $H^1(\Gamma, Y_\infty)$.

The reciprocity conjecture says that c_Q generate $M(Q)$. Romyar Sharifi proved a slightly refined form of the conjecture, in particular specifying a generator of $M(Q)$ which identifies with c_Q.

It is not difficult to deduce from this statement that Leopoldt Conjecture is equivalent to the finiteness of the order of c_Q for all Q (or for a single "generic" Q). This is equivalent to the splitting up to isogeny (i.e. splitting after inverting p) of the exact sequence :

$$(0) \to \mathbb{Z}_p(1) \to \mathcal{X}^-_{\infty,Q} \to \mathcal{X}^-_\infty \to (0),$$

where $\mathcal{X}^-_\infty$ (reps. $\mathcal{X}_{\infty,Q}$) is the Galois group over $\mathcal{F}_\infty$ of the maximal pro-p abelian extension of $\mathcal{F}_\infty$ that is unramified everywhere (resp. unramified at the primes over q). This is equivalent to the existence of a $\mathbb{Z}_p$-extension L_Q as above, the ramification conditions is that it is unramified outside Q but is ramified at Q.

(In fact there are proofs of the equivalence of the Leopoldt Conjecture and the existence of L_Q that do not use Reciprocity Conjecture ([1])).

The modular methods of Wiles may be adapted to take into account ramification at auxiliary primes Q. One has to consider critical stabilization at q of Λ-adic Eisenstein series i.e. the eigenvalues of the Hecke operator U_q is the eigenvalue which is not equal to 1. This allows to construct a extension L'_Q which satisfy the criteria except that it might be unramified at q_1 (or, it is the same, at Q). The extension L'_Q is obtained from an Hida family by specialization at weight 0.

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p-adic derived de Rham cohomology

Bhargav Bhatt

Our talk explained a simple proof of the crystalline conjecture of Fontaine, and the semistable conjecture of Fontaine-Jannsen; these were proven earlier and independently by Faltings, Tsuji, and Niziol. For simplicity, we only discuss the former in this exposition. Our proof is heavily inspired by Beilinson's recent proof [1] of Fontaine's de Rham comparison conjecture; the main difference is that Beilinson works with a certain completion of derived de Rham cohomology, while we must work in the non-completed context. We first fix the relevant notation:

Notation 0.1. *Fix a mixed characteristic* $(0,p)$ *local field* K *with perfect residue field* k*, and an algebraic closure* $K \subset \overline{K}$*. We write* $\mathcal{O}_K$ *for the ring of integers,* $W = W(k)$ *for the ring of Witt vectors, and* $\mathcal{O}_{\overline{K}}$ *for the integers of* $\overline{K}$*. For an* $\mathcal{O}_K$*-scheme* $\overline{X}$ *with generic fibre* X*, we write* $\mathrm{R}\Gamma_{\mathrm{crys}}(\overline{X})$ *for the crystalline cohomology of* $\overline{X} \to \mathrm{Spec}(W)$*, and* $\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}})$ *for the* p*-adic étale cohomology of* $X_{\overline{K}}$*. Let* A_{crys} *be the crystalline period ring of Fontaine, and let* $t \in A_{\mathrm{crys}}$ *be the usual element.*

The crystalline conjecture comes from the last phrase in the following theorem, together with Hyodo-Kato theory (which we do not discuss):

Theorem 0.2. *Let* $\overline{X}$ *be a proper smooth* $\mathcal{O}_K$*-scheme. There is a natural map*

$$c_{\mathrm{crys}} : \mathrm{R}\Gamma_{\mathrm{crys}}(\overline{X}) \otimes A_{\mathrm{crys}} \to \mathrm{R}\Gamma_{\text{ét}}(X_{\overline{K}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{crys}}$$

preserving Chern classes, Frobenius and Galois actions, and Hodge filtrations; here the tensor product on the left takes place over $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(\mathcal{O}_K))$*. Moreover,* $c_{\mathrm{crys}}[1/t]$ *is an isomorphism.*

Theorem 0.2 (as well as the de Rham comparison conjecture) can be viewed as a p-adic analogue of de Rham's classical comparison over $\mathbf{C}$. The semistable conjecture, which is not discussed in this text, is a version of Theorem 0.2 for $\mathcal{O}_K$-schemes with semistable special fibres. Our proof of Theorem 0.2 relies on Illusie's derived de Rham cohomology [2, §VIII], which is discussed next.

1. Derived de Rham cohomology

Derived de Rham cohomology is a replacement for usual de Rham cohomology that works better for singular varieties; the difference roughly is the replacement of the cotangent sheaf with the cotangent complex. More precisely:

Definition 1.1. *For a map $f : X \to S$ of schemes, let $\mathrm{dR}_{X/S} \in D(f^{-1}\mathcal{O}_S)$ be the homotopy-colimit of the simplicial cochain complex $\Omega^*_{P/f^{-1}\mathcal{O}_S}$ where $P \to \mathcal{O}_X$ is the canonical polynomial algebra resolution of $\mathcal{O}_X$ as an $f^{-1}\mathcal{O}_S$-algebra.*

Note that the above definition does *not* give the Hodge-completed theory; this is crucial for the application to Theorem 0.2. As de Rham cohomology of a polynomial algebra in characteristic 0 is trivial, one can show that $\mathrm{dR}_{X/S} \simeq f^{-1}\mathcal{O}_S$ if S is a $\mathbf{Q}$-scheme, i.e., this non-Hodge-completed theory is meaningless in characteristic 0. However, modulo p, the theory turns out to be quite interesting:

Theorem 1.2 (Derived Cartier isomorphism). *Let $f : X \to S$ be a map of $\mathbf{F}_p$-schemes. Then $\mathrm{dR}_{X/S}$ admits an increasing bounded below exhaustive filtration $\mathrm{Fil}^{\mathrm{conj}}_{\bullet}$ with graded pieces given by*

$$\mathrm{Cartier}_i : \mathrm{gr}^{\mathrm{conj}}_i(\mathrm{dR}_{X/S}) \simeq \wedge^i L_{X^{(1)}/S}[-i].$$

Here $X^{(1)}$ is the derived Frobenius twist of f.

For smooth maps, this is the usual Cartier isomorphism. The general statment leads to a tight connection between derived de Rham and crystalline cohomology for lci singularities, generalising Berthelot's work for smooth maps:

Theorem 1.3. *Let $f : X \to S$ be an lci morphism of flat $\mathbb{Z}/p^n$-schemes. Then there is a functorial isomorphism*

$$\mathrm{R}f_*\mathrm{dR}_{X/S} \simeq \mathrm{R}f_*\mathcal{O}_{X/S,\mathrm{crys}}.$$

A p-adic consequence of this connection and Fontaine's description of A_{crys} is:

Corollary 1.4. *There is a natural isomorphism $\widehat{\mathrm{dR}_{\mathcal{O}_{\overline{K}}/\mathbb{Z}_p}} \simeq A_{\mathrm{crys}}$, where the completion on the left is p-adic and derived.*

The definitions and results of this section have logarithmic analogues using Gabber's log cotangent complex [3, §8] instead of Illusie's classical one; this topic will not be discussed further here, but it is implicit in the sequel.

2. The comparison map

We will realise c_{crys} as a sheafification map on the following site:

Definition 2.1. *Let $\mathcal{P}_K$ denote the category of pairs $(X, \overline{X})$ where X is K-variety, and $\overline{X}$ is a reduced proper flat $\mathcal{O}_K$-scheme containing X as a dense open subscheme; morphisms are defined in the obvious way. We say that a family $\{f_i : (X_i, \overline{X_i}) \to (X, \overline{X})\}$ is an h-cover if the family $\{X_i \to X\}$ is an h-cover; recall that the h-topology is the one generated by étale covers and proper surjective maps. We write $\mathcal{P}_{\overline{K}}$ for the analogous category of pairs over $\overline{K}$.*

The h-topology is relevant for us thanks to cohomological descent [4]:

Theorem 2.2 (Deligne). *Given $(X, \overline{X}) \in \{\mathcal{P}_K, \mathcal{P}_{\overline{K}}\}$ and a torsion abelian group A, pullback induces an isomorphism*

$$\mathrm{R}\Gamma_h((X, \overline{X}), A) \simeq \mathrm{R}\Gamma_{\text{ét}}(X, A).$$

Theorem 2.2 makes étale cohomology of the generic fibre accessible to mixed characteristic data. The relation with crystalline cohomology uses period sheaves:

Definition 2.3. *We define (Zariski) presheaves $\mathfrak{a}^c_{\mathrm{crys}}$ and $\mathfrak{a}_{\mathrm{crys}}$ on $\mathcal{P}_K$ or $\mathcal{P}_{\overline{K}}$ via*

$$\mathfrak{a}^c_{\mathrm{crys}}(X, \overline{X}) = \mathrm{dR}_{(\mathcal{O}(\overline{X}),\mathrm{can})/W} \quad \text{and} \quad \mathfrak{a}_{\mathrm{crys}}(X, \overline{X}) = \mathrm{R}\Gamma(\overline{X}, \mathrm{dR}_{(\overline{X},\mathrm{can})/W}).$$

Here $(\overline{X}, \mathrm{can})$ is the log scheme defined by the open subset $X \subset \overline{X}$, while $(\mathcal{O}(\overline{X}), \mathrm{can})$ is the log structure on $\mathcal{O}(\overline{X})$ defined by the generic fibre. Let $\mathcal{A}^c_{\mathrm{crys}}$ and $\mathcal{A}_{\mathrm{crys}}$ denote the h-sheafifications of $\mathfrak{a}^c_{\mathrm{crys}}$ and $\mathfrak{a}_{\mathrm{crys}}$ respectively (as complexes). Pullback of forms induces a natural map $\eta : \mathcal{A}^c_{\mathrm{crys}} \to \mathcal{A}_{\mathrm{crys}}$.

The sheaf $\mathcal{A}^c_{\mathrm{crys}}$ is easily described, at least modulo p^n, by combining Deligne's theorem with Corollary 1.4:

Proposition 2.4. *For any $(X, \overline{X}) \in \{\mathcal{P}_K, \mathcal{P}_{\overline{K}}\}$, one has*

$$\mathrm{R}\Gamma_h((X, \overline{X}), \mathcal{A}^c_{\mathrm{crys}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n) \simeq \mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} A_{\mathrm{crys}}/p^n.$$

The map η thus provides a map from étale cohomology to the h-sheafification of derived de Rham cohomology. The identification of the latter comes from the following p-adic Poincare lemma, and is the key to the construction of c_{crys}:

Theorem 2.5. *The map η induces isomorphisms $\mathcal{A}^c_{\mathrm{crys}} \otimes \mathbb{Z}/p^n \to \mathcal{A}_{\mathrm{crys}} \otimes \mathbb{Z}/p^n$.*

Theorem 2.5 essentially asserts that the cokernel of the map on derived de Rham cohomology modulo p^n induced by $(X, \overline{X}) \to (\mathrm{Spec}(K), \mathrm{Spec}(\mathcal{O}_K))$ is h-locally trivial, i.e., higher derived de Rham cohomology is highly divisible by p after passage to an h-cover. Thus, Theorem 2.5 follows from the following:

Theorem 2.6. *For a proper $\mathcal{O}_K$-scheme X, there exists a proper surjection $\pi : Y \to X$ with $\pi^* : H^i(X, \Omega^j_{X/\mathcal{O}_K}) \to H^i(Y, \Omega^j_{Y/\mathcal{O}_K})$ divisible by p for $i + j > 0$.*

Theorem 2.6 is the main geometric ingredient behind the construction of c_{crys}, and is proven using de Jong's curve fibration results (together with a trick from geometric class field theory). Putting these results together lets us finish:

Sketch of proof of Theorem 0.2. Let $\overline{X}$ be a proper smooth $\mathcal{O}_K$-scheme, and let $X = \overline{X}[1/p]$. Then one has a map

$$c : \mathrm{R}\Gamma_{\mathrm{crys}}(\overline{X})/p^n \simeq \mathfrak{a}_{\mathrm{crys}}(X, \overline{X})/p^n \to \mathfrak{a}_{\mathrm{crys}}((X_{\overline{K}}, \overline{X}_{\mathcal{O}_{\overline{K}}}))/p^n \to \mathcal{A}_{\mathrm{crys}}((X_{\overline{K}}, \overline{X}_{\mathcal{O}_{\overline{K}}}))/p^n$$

where the first equivalence comes ultimately from Theorem 1.3, the second map by base change, the third is the sheafification adjunction, and reductions modulo p^n are derived. The target of c is identified with $\mathrm{R}\Gamma_{\text{ét}}(X) \otimes_{\mathbb{Z}_p} A_{\mathrm{crys}}/p^n$ by Theorem 2.5 and Proposition 2.4. Taking p-adic limits and linearising over A_{crys} then gives c_{crys}. One can then check that c_{crys} preserves Chern classes and the multiplicative structures; a formal argument using Poincare duality then implies the rest. □

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On the Breuil-Mézard conjecture

Vytautas Paškūnas

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, L a finite extension of $\mathbb{Q}_p$ with the ring integers $\mathcal{O}$, uniformizer ϖ and residue field k. Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a continuous representation of the absolute Galois group of $\mathbb{Q}_p$ such that $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho) = k$. Let R_ρ be the universal deformation ring of ρ and $\rho^{\mathrm{un}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R_\rho)$ the universal deformation. If $\mathfrak{n} \in \text{m-Spec}\, R_\rho[1/p]$, where m-Spec denote the maximal ideals, then the residue field $\kappa(\mathfrak{n})$ is a finite extension of L. Thus specializing the universal deformation at $\mathfrak{n}$, we obtain a continuous representation $\rho^{\mathrm{un}}_{\mathfrak{n}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathcal{O}_{\kappa(\mathfrak{n})})$, which reduces to ρ modulo the maximal ideal of $\mathcal{O}_{\kappa(\mathfrak{n})}$. Since $\kappa(\mathfrak{n})$ is a finite extension of L, $\rho^{\mathrm{un}}_{\mathfrak{n}}$ lives in Fontaine's world of p-adic Hodge theory. In particular, we may fix some p-adic Hodge theoretic data and investigate the locus of points in $\mathrm{Spec}\, R_\rho$ for which $\rho^{\mathrm{un}}_{\mathfrak{n}}$ has this data. For simplicity of the exposition we discuss only the crystalline case here. Then we fix a pair of integers $\mathbf{w} = (a, b)$ with $b > a$ and a continuous character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ congruent to $\det \rho$ modulo ϖ and such that $\psi\varepsilon^{-(a+b)}$ is trivial on the inertia subgroup $I_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$, where ε is the cyclotomic character.

We let $\sigma(\mathbf{w}) := \mathrm{Sym}^{b-a-1} L^2 \otimes \det^a$ and let $\overline{\sigma(\mathbf{w})}$ be the semi-simplification of a reduction modulo ϖ of a $K := \mathrm{GL}_2(\mathbb{Z}_p)$-invariant $\mathcal{O}$-lattice in $\sigma(\mathbf{w})$. One may show that $\overline{\sigma(\mathbf{w})}$ does not depend on the choice of a lattice. For each irreducible k-representation σ of K we let $m_\sigma(\mathbf{w})$ be the multiplicity with which σ occurs in $\overline{\sigma(\mathbf{w})}$.

Recall that the group of d-dimensional cycles $\mathcal{Z}_d(R)$ of a noetherian ring R is a free abelian group generated by $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = d$. If M is a finitely generated R-module of dimension at most d then $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module of finite length, which we denote by $\ell_{R_\mathfrak{p}}(M_\mathfrak{p})$, for all $\mathfrak{p}$ with $\dim R/\mathfrak{p} = d$. We let $z_d(M) := \sum_\mathfrak{p} \ell_{R_\mathfrak{p}}(M_\mathfrak{p})\mathfrak{p}$, where the sum is taken over all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} = d$.

Theorem 1. *Assume that $p \geq 5$. There exists a reduced, two dimensional, $\mathcal{O}$-torsion free quotient $R^\psi_\rho(\mathbf{w})$ of R_ρ such that for all $\mathfrak{n} \in \text{m-Spec}\, R_\rho[1/p]$, $\mathfrak{n}$ lies in $\text{m-Spec}\, R^\psi_\rho(\mathbf{w})$ if and only if $\rho^{\text{un}}_\mathfrak{n}$ is crystalline with Hodge-Tate weights (a, b) and determinant ψ.*

Moreover, for each irreducible k-representation σ of K there exists a one dimensional cycle $z(\sigma, \rho)$ of R_ρ, not depending on $\mathbf{w}$, such that for all $\mathbf{w}$ we have an equality of one dimensional cycles:

$$z_1(R^\psi_\rho(\mathbf{w})/(\varpi)) = \sum_\sigma m_\sigma(\mathbf{w}) z(\sigma, \rho),$$

where the sum is taken over the set of isomorphism classes of smooth irreducible k-representations of K.

Further, $z(\sigma, \rho)$ is non-zero if and only if $\operatorname{Hom}_K(\sigma, \beta) \neq 0$, where β is a smooth k-rerpresentation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to ρ by Colmez in [3] *(the so called atome automorphe). In which case the Hilbert-Samuel multiplicity of $z(\sigma, \rho)$ is one, with one exception when $\rho \cong \left(\begin{smallmatrix} \bar\varepsilon & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$ and $*$ is peu ramifié in the sense of* [9]*, in which case the multiplicity is* 2.

The result is new only in the case when $\rho \cong \left(\begin{smallmatrix} \bar\varepsilon & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$. The other cases have been treated by Kisin in [6]. Different parts of Kisin's arguments have been improved by Breuil-Mézard [2] and Emerton-Gee [5]. In particular, the formulation of the problem in terms of the language of cycles is due to Emerton-Gee. In all these papers, there is a local part of the argument using the p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$, see [3], [1] and a global part of the argument, which uses Taylor-Wiles-Kisin patching argument. Our proof is purely local and uses the results of [8] instead of the global input.

Let us sketch the argument, at least in the case, when $\rho \not\cong \left(\begin{smallmatrix} 1 & * \\ 0 & \bar\varepsilon \end{smallmatrix}\right) \otimes \chi$ for any character χ. Via the local class field theory we consider $\zeta := \psi\varepsilon^{-1}$ as a character of the center Z of $\mathrm{GL}_2(\mathbb{Q}_p)$. Using [6] and Colmez's Montreal functor one may construct a deformation N of $\beta^\vee$ to R^ψ_ρ on which Z acts by ζ^{-1}, here $\vee$ denotes the Pontryagin dual and R^ψ_ρ is the quotient of R_ρ parameterizing all the deformations with determinant ψ. One may show that N is the universal deformation of $\beta^\vee$ with a fixed central character ζ^{-1}. We show using [8] that N is a finitely generated and projective $\mathcal{O}[\![K]\!]$-module in the category of compact $\mathcal{O}[\![K]\!]$-modules with central character ζ^{-1}. This implies that $\lambda \mapsto M(\lambda) := \operatorname{Hom}^{\text{cont}}_K(N, \lambda^\vee)^\vee$ is an exact and covariant functor from the category of smooth finite length K-representations on $\mathcal{O}$-torsion modules with central character ζ to the category of finitely generated R^ψ_ρ-modules. Moreover, if we choose a K-invariant $\mathcal{O}$-lattice Θ in $\sigma(\mathbf{w})$ and let

$M(\Theta) := \mathrm{Hom}_K^{\mathrm{cont}}(N, \Theta^d)^d$, where $(*)^d = \mathrm{Hom}_{\mathcal{O}}(*, \mathcal{O})$, then $M(\Theta)$ is a finitely generated, $\mathcal{O}$-torsion free R_ρ^ψ-module, and we have $M(\Theta)/(\varpi) \cong M(\Theta/(\varpi))$. If d is the Krull dimension of $M(\Theta)$ then putting the two equalities together we obtain $z_{d-1}(M(\Theta)/(\varpi)) = \sum_\sigma m_\sigma(\mathbf{w}) z_{d-1}(M(\sigma))$. The cycles $z(\sigma, \rho)$ in the Theorem are precisely the cycles $z_{d-1}(M(\sigma))$.

Let m be an $R_\rho^\psi[1/p]$-module of finite length, we may choose a finitely generated R_ρ^ψ-submodule m^0 of m, which is an $\mathcal{O}$-lattice in m. Since N is compact one may show that $\Pi(\mathrm{m}) := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\mathrm{m}^0 \otimes_{R_\rho^\psi} N, L)$ equipped with the supremum norm is an admissible unitary L-Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. One may further show that $\mathrm{Hom}_K(\sigma(\mathbf{w}), \Pi(\mathrm{m})) \cong \mathrm{m} \otimes_{R_\rho^\psi} N$ by undoing all the duals. This equation enables us to transfer representation theoretic information to commutative algebra information about $R_\rho^\psi/\mathfrak{a}$, where $\mathfrak{a}$ is the annihilator of $M(\Theta)$. For example taking $\mathrm{m} = \kappa(\mathfrak{n})$ for $\mathfrak{n} \in \text{m-Spec}\, R_\rho^\psi[1/p]$, we deduce that $\mathfrak{n}$ lies in the support of $M(\Theta)$ if and only if $\mathrm{Hom}_K(\sigma(\mathbf{w}), \Pi(\kappa(\mathfrak{n})))$ is non-zero. Since $\sigma(\mathbf{w})$ is a (locally) algebraic representation of K, the image of such homomorphism lands in the subspace of locally algebraic vectors $\Pi(\kappa(\mathfrak{n}))^{\mathrm{alg}}$ of $\Pi(\kappa(\mathfrak{n}))$. These have been computed by Colmez in [3] and Emerton in [4] and shown to encode the classical Langlands correspondence. From this we deduce that $\mathrm{Hom}_K(\sigma(\mathbf{w}), \Pi(\kappa(\mathfrak{n})))$ is non-zero if and only if $\rho_{\mathfrak{n}}^{\mathrm{un}}$ is crystalline with Hodge-Tate weights (a, b), in which case $\dim_{\kappa(\mathfrak{n})} \mathrm{Hom}_K(\sigma(\mathbf{w}), \Pi(\kappa(\mathfrak{n}))) = 1$. From this one deduces that $M(\Theta)$ is a generically free $R_\rho^\psi/\mathfrak{a}$-module of rank 1, which implies that they have same d-dimensional cycles and hence their reductions modulo ϖ have the same $(d-1)$-dimensional cycles, as ϖ is regular on both of them, and the existence of the ring $R_\rho^\psi(\mathbf{w})$, which is isomorphic to $R_\rho^\psi/\sqrt{\mathfrak{a}}$. To finish the proof we show that $\mathfrak{a} = \sqrt{\mathfrak{a}}$ by showing that for all $\mathfrak{n} \in \text{m-Spec}\, R_\rho^\psi[1/p]$, which lie in the support of $M(\Theta)$, the ring R_ρ^ψ localized at $\mathfrak{n}$ is regular, by relating the embedding dimension to a representation theoretic computation, which we can do in the crystalline case discussed here (or more generally crystabeline case) and semi-stable non-crystalline case, using the explicit description of locally algebraic and locally analytic vectors in $\Pi(\kappa(\mathfrak{n}))$ and which is done in general by Dospinescu in a forthcoming paper.

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Heegner points and congruent numbers

Xinyi Yuan

This talk introduces the recent result of Ye Tian [1] on congruent numbers.

1. Congruent numbers

Definition 1. *A square-free positive integer* n *is called a congruent number if it satisfies one of the following three equivalent conditions:*

(1) *There is a rational number* β *such that both* $\beta^2 - n$ *and* $\beta^2 + n$ *are squares of rational numbers.*
(2) *There is a right-angled triangle with rational sides and with area* n*.*
(3) *The Mordell–Weil group of the elliptic curve* $E^{(n)} : ny^2 = x^3 - x$ *contains an element of infinite order.*

In 973, the Arabs introduced congruent numbers in the manner of (1), and raised the question to determine whether a given number is a congruent number. The equivalence of the definitions is elementary and roughly as follows.

(1) ⇒ (2). Assume $\alpha^2 = \beta^2 - n$ and $\gamma^2 = \beta^2 - n$. Then

$$n = \frac{1}{2}(\gamma^2 - \alpha^2) = \frac{1}{2}(\gamma + \alpha)(\gamma - \alpha)$$

is the area of the right-angled triangle given by

$$(\gamma + \alpha)^2 + (\gamma - \alpha)^2 = (2\beta)^2.$$

(2) ⇒ (3). Assume that $n = \frac{1}{2}ab$ with $a^2 + b^2 = c^2$. Up to transposition between a and b, we can write

$$(a, b, c) = d(u^2 - v^2, 2uv, u^2 + v^2), \quad d \in \mathbb{Q}_{>0}, u, v \in \mathbb{Z}.$$

It follows that $n = d^2uv(u^2 - v^2)$. It yields that

$$ny^2 = x^3 - x, \quad y = \frac{1}{dv^2}, \; x = \frac{u}{v}.$$

It gives a non-torsion point on the elliptic curve $E^{(n)}$.

There are many conjectures concerning congruent numbers. The simplest one seems to be the following statement.

Conjecture 2. *Every square-free positive integer congruent to* $5, 6, 7$ *modulo* 8 *is a congruent number.*

The conjecture is a consequence of the Birch and Swinnerton-Dyer conjecture, or just the parity conjecture. In fact, the global root number of the L-function $L(s, E^{(n)})$ is -1 if n is congruent to $5, 6, 7$ modulo 8. Thus the analytic rank must be odd. The parity conjecture implies that the Mordell–Weil rank is also odd, and thus it is nonzero.

On the other hand, the global root number of the L-function $L(s, E^{(n)})$ is 1 if n is congruent to $1, 2, 3$ modulo 8. Following a conjecture of Goldfeld, the probability of such n to be congruent is expected to be 0.

It is worth noting that, assuming the Birch and Swinnerton-Dyer conjecture, Tunnell has an effective algorithm to determine whether a given number is congruent.

2. The result of Tian

Ye Tian proves that a large class of integers are congruent numbers. For example, he confirms the following result.

Theorem 3. *Let $r \in \{5, 6, 7\}$. For each positive integer k, there are infinitely many square-free congruent numbers, congruent to r modulo 8, and with exactly k odd prime factors.*

When $k = 1$, the result was due to the seminal work of Heegner, completed by Stephens and Birch. In that case, they prove that every such integer is a congruent number. When $k = 2$, the result is due to Gross and Monsky.

Tian obtains the above result as a corollary of his main result. In the case $r = 5$, the main result is as follows.

Theorem 4. *Let $n = p_0 p_1 \cdots p_k$ be a product of distinct primes $p_0, \cdots, p_k$ with*

$$p_0 \equiv 5 \mod 8, \quad p_i \equiv 1 \mod 8, \quad i = 1, \cdots, k.$$

Assume that the class group of $K = \mathbb{Q}(\sqrt{-2n})$ does not contain any element of order 4. Then n is a congruent number.

There is a simple criterion for the assumption on the class group. Consider the simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$, where the vertex set $\mathcal{V} = \{p_0, \cdots, p_k\}$, and the edge set $\mathcal{E}$ contains the edge $p_i p_j$ if and only if the quadratic symbol $\left(\frac{p_i}{p_j}\right) = -1$. Then the assumption is equivalent to the statement that the number of maximal sub-trees of Γ is odd. In particular, the assumption is satisfied if Γ is a tree.

3. Sketch of proof

Like the previous cases, Tian's proof still confirms some rational point, constructed by the theory of complex multiplication, gives a non-torsion rational point on $E^{(n)}$. Such a construction originated in Heegner's work, so such a point is now called a Heegner point.

In the following, we describe the construction of the point. We first replace the elliptic curve to the related modular curve.

Note that the elliptic curve $E^{(1)} : y^2 = x^3 - x$ has conductor 32. It happens that $X_0(32)$ has genus one, and thus $E_0 = (X_0(32), \infty)$ is already an elliptic curve over $\mathbb{Q}$. It has the Weierstrass equation $y^2 = x^3 + 4x$. A modular parametrization $E_0 \to E^{(1)}$ can be obtained as an isogeny of degree two. It suffices to prove that the Mordell–Weil group of the quadratic twist $E_0^{(n)}$ contains an element of infinite order.

Consider the point $\tau = \frac{1}{8}\sqrt{-2n}$ in the upper half plane $\mathcal{H}$. Via the quotient, it gives a point P in $X_0(32)(\mathbb{C})$. It is easy to have $P \in E_0(H')$, where $H' \subset K^{\mathrm{ab}}$ is a cyclic extension of degree 4 over the Hilbert class field H of K. One checks that $2P \in E_0(H)$. The Heegner point which plays a major role here is

$$Q = \mathrm{tr}_{H/K(\sqrt{n})}(2P) = \sum_{\sigma \in \mathrm{Gal}(H/K(\sqrt{n}))} \sigma(2P).$$

One need to modify the definition by a rational torsion point of order 4 if $k = 0$. This point will essentially gives a non-torsion rational point.

By definition, $Q \in E_0(K(\sqrt{n}))$. Considering the action of the complex conjugation easily gives $Q \in E_0(\mathbb{Q}(\sqrt{n}))$. One further shows that Q actually lies in

$$E_0(\mathbb{Q}(\sqrt{n})^-) = \{\alpha \in E_0(\mathbb{Q}(\sqrt{n})) : \iota(\alpha) = -\alpha\},$$

where ι denotes the unique non-trivial automorphism of $\mathbb{Q}(\sqrt{n})$. This is perfect since there is a natural isomorphism $E_0(\mathbb{Q}(\sqrt{n})^-) = E_0^{(n)}(\mathbb{Q})$.

Tian's conclusion for the Heegner point is as follows.

Theorem 5. *Let $n = p_0 p_1 \cdots p_k$ be as above. Then*

(1) $Q \in 2^k\ E_0(\mathbb{Q}(\sqrt{n})^-) + E_0(\mathbb{Q}(\sqrt{n})^-)_{\mathrm{tors}}$,
(2) $Q \notin 2^{k+1}\ E_0(\mathbb{Q}(\sqrt{n})^-) + E_0(\mathbb{Q}(\sqrt{n})^-)_{\mathrm{tors}}$.

The second part of the theorem implies that Q is non-torsion, but the theorem actually gives the 2-divisibility of Q. The proof of the theorem uses induction on k. In the induction process from $k-1$ to k, one need to compare two different Heegner points on E_0. The comparison is obtained by computing the canonical heights of these points using the Gross–Zagier formula proved by Yuan–Zhang–Zhang [2].

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Construction of p-adic Galois representations using Eisenstein cohomology

Michael Harris
(joint work with Kai-Wen Lan, Richard Taylor, Jack Thorne)

This is a report on the joint paper [HLTT] which constructs n-dimensional Galois representations weakly associated to certain cuspidal automorphic representations of $GL(n)$ over a totally real or CM field. Let me explain what is meant by "weakly" "associated" and "certain". Say E is a totally real or CM field, E^+ its maximal totally real subfield, and let $c \in Gal(E/E^+)$ denote complex conjugation. Let Π be a cuspidal automorphic representation of $GL(n, E)$, and let S be the set of finite places of E at which Π is ramified. Suppose the archimedean component Π_∞ is a representation with non-trivial cohomology; this is what is meant by "certain. Equivalently, the infinitesimal character of Π_∞ coincides with the infinitesimal character of a finite-dimensional irreducible representation of $GL(n, E \otimes_{\mathbb{Q}} \mathbb{R})$. Let p be a prime number. Then there exists a unique semisimple n-dimensional representation

$$\rho_\Pi : Gal(\bar{E}/E) \to GL(n, \overline{\mathbb{Q}}_p)$$

such that, for all primes v of E that are neither in S nor divide p, the local Euler factors $L(s, \Pi_v \otimes |\det|^{\frac{1-n}{2}})$ and $L(s, \rho_\Pi|_{Gal(\bar{E}_v/E_v)})$ coincide. This coincidence of local Euler factors is what we mean by "weakly associated"; the representations would be called "strongly associated" if we knew that the local representations at all primes w of E are associated to the local component Π_w by the local Langlands correspondence.

Suppose Π is *polarized* in the sense that $\Pi^c \simeq \Pi^\vee$. Then the existence of ρ_Π has been proved, in increasing generality, in a series of papers beginning with Clozel's 1991 article [C91], and culminating in recent work involving many people. In this case, ρ_Π is itself polarized:

$$\rho_\Pi^c \xrightarrow{\sim} \rho_\Pi^\vee \otimes \overline{\mathbb{Q}}_p(1-n).$$

The construction of ρ_Π when Π is not polarized also makes use of Shimura varieties, but the route is much more indirect. Assume F is a CM field and V_n is a maximally isotropic hermitian space over F of dimension $2n$. Let $G_n \subset GL(V_n)$ be the group of similitudes of the hermitian form on V_n with $\mathbb{Q}$-rational similitude factor (the precise definition is given below), and let $X = X_n$ be the corresponding Shimura variety. The group $R_{F/\mathbb{Q}}GL(n)_F \times \mathbb{G}_m$ arises as the Levi factor of a maximal parabolic subgroup $P_n \subset G_n$. By inflation, Π defines an irreducible adelic representation of P_n, again denoted Π, and it has been known for some time that $Ind_{P_n}^{G_n}\Pi$ contributes non-trivially to the Eisenstein cohomology of X. More than 20 years ago, Clozel suggested it might be possible to use these Eisenstein cohomology classes to construct ρ_Π. The natural representation on p-adic Eisenstein cohomology is not associated to Π, however; in fact, it is easy to see that one instead obtains a sum of powers of abelian characters. Clozel's observation was revived about ten years ago by Skinner, who suggested that one construct p-adic congruences between Eisenstein cohomology and cuspidal cohomology. The cuspidal cohomology of X_n breaks up as a sum of polarized $2n$-dimensional representations, by the results mentioned above. Using the eigenvariety, these $2n$-dimensional representations, or more precisely their corresponding *pseudorepresentations*, in the sense of Taylor, fit into a p-adic family; using the congruences Skinner expected to recover the hypothetical $\rho_\Pi \oplus \rho_\Pi^c$ (up to abelian twists) as a specialization of this family of pseudorepresentations.

The eigenvariety approach has not yet been carried out in detail, though it remains a viable option for the future. The present approach starts with Eisenstein cohomology, as in Clozel's proposal, and uses congruences to cusp forms, as suggested by Skinner, but it is based on a different construction of p-adic automorphic forms, one closer to older ideas of Katz and Coleman. Let $U \subset G_n(\mathbb{A}_f)$ be an open compact subgroup that is hyperspecial at p (though this hypothesis can be relaxed), and let $X_U = X/U$ denote the corresponding finite level Shimura variety. We use the canonical integral models $\mathfrak{X}_U$ of X_U, and of their compactifications, both the minimal (Baily-Borel-Satake) compactification $\mathfrak{X}_U^*$ and the various toroidal compactifications $\mathfrak{X}_U^{tor}$, constructed in the thesis of Lan; these are flat reduced irreducible schemes over $Spec(\mathbb{Z}_p)$, and the compactiications are projective.

An important observation is that the relevant Eisenstein cohomology classes can be realized geometrically in the weight 0 subspace of *rigid cohomology* of the ordinary locus of the special fiber of $\mathfrak{X}_U$, with compact supports in the direction of the toroidal boundary. This can in turn be calculated by a spectral sequence whose $E_1^{r,s}$ terms are given by coherent cohomology of automorphic vector bundles $\mathfrak{X}_U$, extended in a certain way to $\mathfrak{X}_U^{tor}$, and then to $\mathfrak{X}_U^*$. Using the fact that the ordinary locus in the special fiber of $\mathfrak{X}_U^*$ is affine, the higher coherent cohomology all vanishes, which implies that the Eisenstein classes can be approximated modulo arbitrarily high powers of p by (holomorphic) cusp forms. Standard techniques due to Taylor and others then show that the systems of Hecke eigenvalues on the Eisenstein classes are approximated in a similar way by cuspidal Hecke eigenvalues. This gives a first construction of the pseudorepresentations predicted by Skinner, and by refining this construction one obtains the desired n-dimensional p-adic Galois representation.

Many questions remain open; the most intriguing is whether this technique can be extended to attach Galois representations to *torsion* cohomology of the locally symmetric spaces attached to $GL(n)$.

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Connected components of minuscule affine Deligne-Lusztig varieties

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(joint work with Miaofen Chen, Mark Kisin)

1. Introduction

Let k be a finite field with $q = p^r$ elements and let $\overline{k}$ be an algebraic closure of k. We consider the two cases $F = k((t))$ and $F = W(k)[1/p]$. Let accordingly $L = \overline{k}((t))$ or $L = W(\overline{k})[1/p]$. Let $\mathcal{O}_F$ and $\mathcal{O}_L$ be the valuation rings. We denote by ϵ the uniformizer t or p. Let $\sigma : x \mapsto x^q$ be the Frobenius of $\overline{k}$ over k and also the induced Frobenius of L over F.

Let G be a connected reductive group over $\mathcal{O}_F$. We denote by G_F the generic fibre of G. Let $B \subset G$ be a Borel subgroup and $T \subset B$ the centralizer of a maximal split torus in B. We denote by $X_*(T)$ the set of cocharacters of T, defined over $\mathcal{O}_L$. Write $K = G(\mathcal{O}_L)$. The Cartan decomposition gives an isomorphism

$$X_*(T)/W \cong K\backslash G(L)/K, \quad \mu \mapsto \mu(\epsilon). \tag{1.0.1}$$

For $b \in G(L)$ and a dominant coweight $\mu \in X_*(T)$ the affine Deligne-Lusztig variety $X^G_\mu(b) = X_\mu(b)$ is defined as

$$X_\mu(b)(\overline{k}) = \{g \in G(L)/K \mid g^{-1}b\sigma(g) \in K\mu(\epsilon)K\}.$$

A priori, they are just sets of points, and do not have the structure of an algebraic variety. However, there is still a meaningful notion of a set of connected components $\pi_0(X_\mu(b))$. In some particular arithmetic cases for minuscule μ the set $X_\mu(b)$ is the set of $\overline{k}$-valued points of a moduli space of p-divisible groups as defined by Rapoport and Zink [4]. If F is a function field, it is the set of $\overline{k}$-valued points of a locally closed subscheme locally of finite type of the affine Grassmannian LG/K where LG denotes the loop group of G (compare [3], [1]). It is closed if μ is minuscule. From now on we assume (in both cases) that μ is minuscule.

For the image $\bar{b} \in G^{\mathrm{ad}}(L)$ of b, we define

$$J_{\bar{b}}(F) := \{g \in G(L) : \sigma(g) = \bar{b}^{-1}g\bar{b}\}.$$

This group acts by left multiplication on $X_\mu(b)$. Our first main result is

Theorem 1. *$J(F)$ acts transitively on $\pi_0(X_\mu(b))$.*

The second aim of our work is to prove the following theorem which completely determines the set of connected components of $X_\mu(b)$. For the moment we can show this result for groups whose Dynkin diagram is not of type E_6. Consider the exact sequence

$$0 \to \pi_1(G)^\Gamma \to \pi_1(G) \overset{\alpha}{\to} \pi_1(G) \to \pi_1(G)_\Gamma \to 0$$

where the first and last maps are the inclusion and projection, where $\alpha(x) = \sigma(x) - x$, and where $\pi_1(G)$ is the quotient of $X_*(T)$ by the coroot lattice of G. In [2], Kottwitz defines a homomorphism $w_G : G(L) \to \pi_1(G)$. For unramified groups G it has the following easy description. Let $b \in G(L)$ and let μ_b be the dominant representative of its image under the inverse of the Cartan isomorphism above. Then $w_G(b)$ is the image of μ_b under the canonical projection from $X_*(T)$ to $\pi_1(G)$. The images of $\kappa_G(b)$ and of μ in $\pi_1(G)_\Gamma$ agree as soon as $X_\mu(b) \neq \emptyset$. Then there is an element $c_{b,\mu} \in \pi_1(G)$ with $\alpha(c_{b,\mu}) = \kappa_G(b) - \mu$. Its $\pi_1(G)^\Gamma$-coset is uniquely determined by this equation.

Theorem 2 (in progress)**.** *Let G, μ, and b be as above and indecomposable with respect to the Hodge-Newton decomposition. Assume that G is simple and that $X_\mu(b) \neq \emptyset$.*

(1) Either $\kappa_M(b) \neq \mu$ for all proper standard parabolic subgroups P of G with $b \in M$ or $[b] = [\mu(\epsilon)]$ with μ central.
(2) In the first case, w_G induces a bijection $\pi_0(X_\mu(b)) \cong c_{b,\mu}\pi_1(G)^\Gamma$.
(3) In the second case, $X_\mu(b) \cong J/(J \cap K) \cong G(F)/G(\mathcal{O}_F)$ is discrete.

Our results are used by M. Chen to study the representations on the sets of geometrically connected components of generic fibers of Rapoport-Zink spaces in terms of local Langlands correspondences. Furthermore, M. Kisin uses the above results for his work on mod p points of Shimura varieties of Hodge type.

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Families of trianguline representations and finite slope spaces

Eugen Hellmann

The idea of a *finite slope space* is due to Kisin [5] who aims at giving a description of the Coleman-Mazur eigencurve in terms of p-adic Galois representations instead of p-adic modular forms. Roughly speaking he promotes an equality[1]

$$\left\{ (f,\lambda) \;\middle|\; \begin{array}{c} f \text{ an overconvergent } p\text{-adic eigenform (of some fixed tame level)}, \\ \lambda \in \bar{\mathbb{Q}}_p^\times \text{ such that } U_p f = \lambda f \end{array} \right\}$$

$$= \left\{ (\rho,\lambda) \;\middle|\; \begin{array}{c} \rho : G_{\mathbb{Q},S} \to \mathrm{GL}(V_\rho) \text{ a continuous representation} \\ \text{on a 2-dimensional } \bar{\mathbb{Q}}_p\text{-vector space } V_\rho, \\ \lambda \in \bar{\mathbb{Q}}_p^\times \text{ such that } \exists \text{ nontrivial } G_p\text{-equivariant crystalline period} \\ V_\rho \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p)^{\varphi=\lambda} \end{array} \right\}$$

on the level of rigid analytic spaces. Here $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ is the Galois group of the maximal extension $\mathbb{Q}_S$ of $\mathbb{Q}$ that is unramified outside a fixed finite set S of

[1] Actually in loc. cit. Kisin only constructs a map from one side to the other. It is shown in [6] that this map is an equality.

primes not containing p (corresponding to the tame level[2]) and $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the local Galois group at p. The above bijection can be seen as an overconvergent variant of the Fontaine-Mazur conjecture.

In this talk we outlined a new construction of the finite slope space which directly generalizes beyond the 2-dimensional case and apply this construction to eigenvarieties for certain definite unitary groups.

It was remarked by Colmez [4] that the existence of a crystalline period on a 2-dimensional G_p-representation V can be rephrased as follows: Let $D = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ the (φ,Γ)-module over the Robba ring $\mathcal{R}$ associated to a p-adic G_p-representation V. Then V admits (up to twist) a crystalline period if and only if V is trianguline, i.e. D is a successive extension of (φ,Γ)-modules of rank 1. Using local class field theory one easily finds that the (φ,Γ)-modules of rank 1 are parametrized by the continuous characters of $\mathbb{Q}_p^\times$.

In order to describe families of these objects, we use a relative version of the Robba ring, that is, given a rigid analytic space X, we contruct a sheaf of rings $\mathcal{R}_X = \mathcal{R}\hat{\otimes}_{\mathbb{Q}_p}\mathcal{O}_X$ on X. Using work of Berger and Colmez [2] we find that the (φ,Γ)-module functor generalizes to families. In order to construct families of Galois representations from families of (φ,Γ)-modules we embed the category of rigid spaces in the category of adic spaces locally of finite type over $\mathbb{Q}_p$.

Theorem 1. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let D be a family of (φ,Γ)-modules over X. Then there exists a natural maximal open subspace $X^{\mathrm{adm}} \subset X$ and a family of G_p-representations $\mathcal{V}$ on X such that there is a natural isomorphism*

$$\mathbf{D}^{\dagger}_{\mathrm{rig}}(\mathcal{V}) \cong D|_{X^{\mathrm{adm}}}.$$

Let $d > 0$ and let $\mathcal{T} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^\times, \mathbb{G}_m(-))$ denote the rigid space parametrizing continuous characters of $\mathbb{Q}_p^\times$. The space is product $\mathcal{T} = \mathbb{G}_m \times \mathcal{W}$, where $\mathcal{W}$ is the space of continuous characters of $\mathbb{Z}_p^\times$. For a rigid space X and an X-valued point $\delta \in \mathcal{T}(X)$ we write $\mathcal{R}_X(\delta)$ for the rank one (φ,Γ)-module on X on which φ acts via multiplication with $\delta(p)$ and $\Gamma = \mathbb{Z}_p^\times$ acts via multiplication by $\delta|_{\mathbb{Z}_p^\times}$. For technical reasons we further need a Zariski-open subspace $\mathcal{T}_d^{\mathrm{reg}} \subset \mathcal{T}^d$ of d-tuples of characters satisfying a certain regularity condition, see [3, Def. 2.28, 3.1].

Closely following the work of Chenevier [3] we find that the functor $\mathcal{S}_d$ that assigns to a rigid space X the set of isomorphism classes of a (φ,Γ)-module D on X that is (locally on X) a successive nowhere split extension of the rank one objects $\mathcal{R}_X(\delta_i)$ for $(\delta_1,\dots,\delta_d) \in \mathcal{T}_d^{\mathrm{reg}}$ is representable[3]. The forgetful morphism $\mathcal{S}_d \to \mathcal{T}_d^{\mathrm{reg}}$ is smooth and proper.

In what follows, we assume $p > d$. Let E be a finite extension of $\mathbb{Q}$ such that p completely splits in E and denote by E_S the maximal extension of E that is

[2]Strictly speaking one should impose some condition on the conductor of ρ which corresponds to the choice of the tame level.

[3]Strictly speaking, one needs to add an additional rigidification to ensure representability.

unramified outside the primes of E lying above the primes in S. We write $\mathfrak{X}_{\bar\tau}$ resp. $\mathfrak{X}_{\bar\tau_p}$ for the generic fiber of the formal spectrum of the universal deformation ring of a d-dimensional pseudo-representation $\bar\tau : G_{E,S} \to \bar{\mathbb{F}}_p$ resp. its restriction $\bar\tau_p$ to G_p.

By Theorem 1 there exists a natural open subspace $\mathcal{S}_d^{\mathrm{adm}} \subset \mathcal{S}_d$ such that the restriction of the universal trianguline (φ,Γ)-module to $\mathcal{S}_d^{\mathrm{adm}}$ comes from a family $\rho : G_p \to \mathrm{GL}(\mathcal{V})$ of Galois-representation on some vector bundle $\mathcal{V}$. The trace

$$T = \mathrm{tr}\,\rho : G_p \longrightarrow \mathcal{O}_{\mathcal{S}_d^{\mathrm{adm}}}$$

factors over the sheaf of integral elements, as the group G_p is compact. We denote by $\mathcal{S}(\bar\tau_p)$ the open and closed subspace of $\mathcal{S}_d^{\mathrm{adm}}$, where the reduction of T modulo the ideal of topologically nilpotnet elements equals $\bar\tau_p$.

Theorem 2. *The above construction yields a finite and injective map*

$$\pi_{\bar\tau_p} : \mathcal{S}(\bar\tau_p) \longrightarrow \mathfrak{X}_{\bar\tau_p} \times \mathcal{T}_d^{\mathrm{reg}}.$$

Using the properness of $\pi_{\bar\tau_p}$ we can define finite slope spaces as follows.

(i) The *regular subset of the local finite slope space* is the Zariski-closed subspace $X(\bar\tau_p)^{\mathrm{reg}} = \pi_{\bar\tau_p}(\mathcal{S}(\bar\tau_p)) \subset \mathfrak{X}_{\bar\tau_p} \times \mathcal{T}_d^{\mathrm{reg}}$.

(ii) The *local finite slope space* is the Zariski-closure $X(\bar\tau_p)$ of $X(\bar\tau_p)^{\mathrm{reg}}$ in $\mathfrak{X}_{\bar\tau_p} \times \mathcal{T}^d$.

(iii) The *global finite slope space* is the fiber product $X(\bar\tau) = \mathfrak{X}_{\bar\tau} \times_{\mathfrak{X}_{\bar\tau_p}} X(\bar\tau_p)$.

Finally we draw some applications. Let G be a unitary group over $\mathbb{Q}$ with corresponding imaginary quadratic field E. We assume that p splits in E, that G is split at p and that G is compact at infinity[4] .

Let Z be a certain set of automorphic representations $\pi = \pi_\infty \otimes \pi_f$ of $G(\mathbb{A})$ that are unramified at p. Let $\mathcal{H}$ be a Hecke-algebra (see [1, 7.2.1]) associated with a fixed tame level of the representations $\pi \in Z$. Roughly an *eigenvariety* associated with Z is a rigid analytic space Y containing Z as a Zariski-dense accumulation subset together with

(i) a morphism $\psi : \mathcal{H} \to \Gamma(Y, \mathcal{O}_Y)$ interpolating the Hecke-characters attached to the automorphic representations $\pi \in Z$,

(ii) a morphism $\omega : Y \to \mathcal{W}^d$ interpolating the highest weights[5] of π_∞ at the points $\pi \in Z$.

See [1, Chapter 7] for a more precise definition. Under certain assumption on the set Z and the group G, see [1, 7.5], these eigenvarieties exist and there exist $\bar{\mathbb{Q}}_p$-valued continuous representations ρ_π of $G_{E,S}$ attached to the representations $\pi \in Z$. In this case there exists a continuous pseudo-representation $T : G_{E,S} \to \Gamma(Y, \mathcal{O}_Y)$ interpolating the traces $\mathrm{tr}\,\rho_\pi$ of the Galois representations at the points $\pi \in Z$.

[4]We will need additional technical assumptions on G, however these assumptions are essential for our approach.

[5]We regard $\mathbb{Z}$ as a subset of $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{G}_m(-))$ by means of $k \mapsto (z \mapsto z^k)$.

Theorem 3. *Under the above assumptions there exists a closed embedding of the eigenvariety*

$$Y \longrightarrow \coprod_{\bar{\tau}} X(\bar{\tau})$$

into the disjoint union of the finite slope spaces indexed by the continous $\bar{\mathbb{F}}_p$*-valued pseudo-representations of* $G_{E,S}$.

As in the 2-dimensional case one could conjecturally describe the image of this map by posing a duality condition on the Galois representations ρ parametrized by the finite slope space. This again is an overconvergent variant of the Fontaine-Mazur conjecture. Finally this theorem has the following consequence.

Corollary 4. *The Galois representations attached to overconvergent automorphic forms* (*of finite slope*) *satisfying a certain regularity assumption are trianguline when restricted to* G_p.

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Subtle invariants for p-divisible groups and Traverso's conjectures

Adrian Vasiu

(joint work with Ofer Gabber, Eike Lau, Marc-Hubert Nicole)

1. Notations

We report on the joint works [1] and [2] aimed towards the classification of p-divisible groups over an algebraically closed field k of positive characteristic p.

Let $W(k)$ be the p-typical Witt ring of k. Let σ be the Frobenius automorphism of $W(k)$. Let D be a p-divisible group over k of positive height r and let (M, ϕ, ϑ) be its (contravariant) Dieudonné module. Thus M is a free $W(k)$-module of rank r, $\phi : M \to M$ is a σ-linear endomorphism, and $\vartheta : M \to M$ is a σ^{-1}-linear endomorphism such that we have $\phi \circ \vartheta = \vartheta \circ \phi = p1_M$.

Let $d = \dim_k(M/\phi(M))$ and $c = \dim_k(M/\vartheta(M))$ be the dimension and the codimension (respectively) of D. We have $c + d = r$. Let $t := \min\{c, d\}$.

2. What one would like to achieve?

• Classify all D's. This means: (a) list all isomorphism classes with c and d fixed; (b) decide when another p-divisible group E over k specializes to D in the sense below; (c) understand the abstract groups $\mathrm{Hom}(E[p^m], D[p^m])$ for all $m \in \mathbb{N}^*$; and (d) identify good invariants of D (which go up or down under specializations).

• Refine the Newton polygon stratifications associated to p-divisible groups over $\mathbb{F}_p$-schemes using good invariants.

• Generalize to quadruples of the form (M, ϕ, ϑ, G), where G is an integral, closed subgroup subscheme of $\boldsymbol{GL}_M$ subject to the axioms of [1], Definition 2. [Here we will concentrate on the case $G = \boldsymbol{GL}_M$ which corresponds simply to D.]

Definition 1. We say that E specializes to D if there exists a p-divisible group $\mathcal{D}$ over $k[[x]]$ such that $\mathcal{D}_k = D$ and $\mathcal{D}_{\overline{k((x))}}$ is isomorphic to $E_{\overline{k((x))}}$.

3. Classical invariants

It is well known that we have a direct sum decomposition of F-isocrystals

$$(M[\frac{1}{p}], \phi) = \oplus_{\beta \in [0,1] \cap \mathbb{Q}} (I_\beta, \phi_\beta)^{m_D(\beta)},$$

where each (I_β, ϕ_β) is simple of Newton polygon slope β and each $m_D(\beta) \in \mathbb{N}$ is a multiplicity. With these multiplicities one builds up the Newton polygon of D: it is an increasing, concave up, piecewise linear, continuous function $\nu_D : [0, r] \to [0, d]$. Note that $\nu_D(c) \leq \frac{cd}{r}$. Let $a_D := \dim(\boldsymbol{Hom}(\boldsymbol{\alpha}_p, D[p]))$. Thus $\boldsymbol{\alpha}_p^{a_D} \oplus (\mathbb{Z}/p\mathbb{Z})^{m_D(0)} \oplus \boldsymbol{\mu}_p^{m(1)} \subset D[p]$. For $m \in \mathbb{N}$ let $\gamma_D(m) := \dim(\boldsymbol{Aut}(D[p^m])) = \dim(\boldsymbol{End}(D[p^m]))$.

4. Subtle invariants

In this section we introduce six geometric invariants of D.

(a) Isomorphism number. It is the smallest $n_D \in \mathbb{N}^*$ such that the isomorphism class of D is uniquely determined by $D[p^{n_D}]$. We have $n_D \leq cd + 1$, cf. [5].

Conjecture 1 (Traverso [6]**).** *If $cd > 0$, then $n_D \leq t$.*

(b) Isogeny cutoff. It is the smallest $b_D \in \mathbb{N}^*$ such that the isogeny class (i.e., the Newton polygon ν_D) of D is uniquely determined by $D[p^{b_D}]$.

Conjecture 2 (Traverso [6]**).** If $cd > 0$, then $b_D \leq \lceil \frac{cd}{r} \rceil$.

Remark 1. Conjecture 2 is proved in [3]. Conjecture 1 is incorrect. Below we will provide corrected, refined, and optimal versions of these two conjectures.

(c) Minimal height. There exists a unique (up to isomorphism) p-divisible group D_0 over k such that $n_{D_0} = 1$ and $\nu_{D_0} = \nu_D$; it is called the minimal p-divisible group of Newton polygon ν_D. The minimal height of D is the smallest $q_D \in \mathbb{N}$ such that there exists an isogeny $D_0 \to D$ whose kernel is annihilated by p^{q_D}.

(d) Level torsion. We denote also by ϕ the σ-linear automorphism $\mathrm{End}(M[\frac{1}{p}]) \to \mathrm{End}(M[\frac{1}{p}])$ given by $x \mapsto \phi \circ x \circ \phi^{-1}$. We have a direct sum decomposition

$(\mathrm{End}(M[\frac{1}{p}]),\phi) = (L_+,\phi) \oplus (L_0,\phi), \oplus(L_-,\phi)$ such that all Newton polygon slopes of (L_+,ϕ) are positive, all Newton polygon slopes of (L_0,ϕ) are zero, and all Newton polygon slopes of (L_-,ϕ) are negative. Let O_+ (resp. O_0 and O_-) be the largest $W(k)$-submodule of $\mathrm{End}(M)$ which is contained in L_+ (resp. L_0 and L_-) and for which we have $\phi(O_+) \subset O_+$ (resp. $\phi(O_0) = O_0$ and $\phi^{-1}(O_-) \subset O_-$). Thus $O := O_+ \oplus O_0 \oplus O_-$ is a $W(k)$-lattice of $\mathrm{End}(M[\frac{1}{p}])$ contained in $\mathrm{End}(M)$. The level torsion l_D of D is the smallest $l_D \in \mathbb{N}$ such that $p^{l_D}\mathrm{End}(M) \subset O \subset \mathrm{End}(M)$.

(e) Endomorphism number. Let $e_D \in \mathbb{N}$ be such that for all positive integers $m \geq s$, the images of the restriction homomorphisms $\tau_{\infty,s} : \mathrm{End}(D) \to \mathrm{End}(D[p^s])$ and $\tau_{m,s} : \mathrm{End}(D[p^m]) \to \mathrm{End}(D[p^s])$ are equal if and only if $m \geq s + e_D$.

(f) Coarse endomorphism number. Let $f_D \in \mathbb{N}$ be such that for all positive integers $m \geq s$, $\tau_{m,s}$ has finite image if and only if $m \geq s + f_D$.

Remark 2. We have natural variants for pairs $n_{D,E}$, $l_{D,E}$, $e_{D,E}$, and $f_{D,E}$. For instance, $n_{D,E} \in \mathbb{N}^*$ is the smallest number such that the natural restriction homomorphism $\mathrm{Ext}^1(D,E) \to \mathrm{Ext}^1(D[p^{n_{D,E}}], E[p^{n_{D,E}}])$ is injective. The equality $n_D = n_{D,D}$ provides a cohomological interpretation of n_D.

5. Our results

Theorem A ([2]). **(a)** *Let $j(\nu_D)$ be $\nu_D(c)+1$ if $(c,\nu_D(c))$ is a breakpoint of ν_D and be $\lceil \nu_D(c) \rceil$ otherwise. We have $b_D \leq j(\nu_D)$ and the equality holds if $a_D \leq 1$.*

(b) *If D is not ordinary, then $n_D \leq \lfloor 2\nu_D(c) \rfloor \leq \lfloor \frac{2cd}{r} \rfloor$; mooreover, the equality $n_D = \lfloor \frac{2cd}{r} \rfloor$ does hold for certain isoclinic p-divisible groups D with $a_D = 1$.*

(c) *We have $q_D \leq \lfloor \nu_D(c) \rfloor$ with equality if $a_D \leq 1$.*

Remark 3. In general, if $a_D = 1$ then n_D can vary. If $t > 0$ is fixed, then $\lfloor \frac{2cd}{r} \rfloor$ can be any integer in the interval $[t, 2t-1]$. Thus Conjecture 1 is in essence off by a factor 2. If $|c-d| \leq 2$, then $\lfloor \frac{2cd}{r} \rfloor = t$ and the Conjecture 1 holds. The simplest case when Conjecture 1 fails is when $\{c,d\} = \{2,6\}$.

Theorem B ([2]). *If D is not ordinary, then $n_D = l_D = e_D = f_D$.*

Remark 4. The inequalities $e_D \leq l_D \leq f_D \leq e_D$ are proved in [2] (the first one is easy, the second one is hard, and the third one is trivial), equality $n_D = f_D$ is a consequence of the next theorem, while in [7] it was first proved that $n_D \leq l_D$.

Theorem C ([1]). For all $s \in \mathbb{N}^*$, the sequence $(\gamma_S(s+i) - \gamma_D(i))_{i \in \mathbb{N}}$ is decreasing. Moreover, if $cd > 0$, then we have $a_D^2 \leq \gamma_D(1) < \cdots < \gamma_D(n_D) = \gamma_D(n_d+s) \leq cd$.

Theorem D ([2]). *If $\mathcal{D}$ is a p-divisible group of constant Newton polygon over an $\mathbb{F}_p$-scheme S, then for all $m \in \mathbb{N}$ and $\square \in \{b,n,q\}$, the set $\{s \in S | \square_{\mathcal{D}_s} \leq m\}$ is closed in S. Thus for each $\triangle \in \{b,n,q,bn,bq,nq,bnq\}$ we get a natural $\triangle$-stratification of S in a finite number of reduced, locally closed subschemes.*

Theorem E ([1]). *For $m \in \{1,2,\ldots,n_D-1\}$ there exist an infinite number of truncated Barsotti–Tate groups of level $m+1$ over k which are pairwise non-isomorphic and lift $D[p^m]$.*

Remark 5. The case $m = 1$ of Theorem E is a stronger form of [4], Theorem 4.

Theorem F ([1]). *Let $\Gamma_D(s)$ be the reduced group of the identity component of $\boldsymbol{Aut}(D[p^s])$. If $s \geq 2n_D$, then the unipotent group $\Gamma_D(s)$ is commutative.*

Example. We assume that $c = d > 0$; thus D could be the p-divisible group of an abelian variety of dimension d over k. (i) Then $n_D \leq d$. (ii) For $m \in \mathbb{N}^*$, an endomorphism of $D[p^m]$ lifts to an endomorphism of D if and only if it lifts to an endomorphism of $D[p^{m+d}]$ (or $D[p^{m+\lfloor 2\nu_D(c)\rfloor}]$). (iii) The group $\Gamma_D(2d)$ (or $\Gamma_D(2\lfloor 2\nu_D(c)\rfloor)$ is commutative. (iv) E specializes to D if and only if $E[p^d]$ specializes to $D[p^d]$; the same holds with d replaced by $\max\{\lfloor 2\nu_D(c)\rfloor, \lfloor 2\nu_E(c)\rfloor\}$.

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The Oort Conjecture on Lifting Covers of Curves

Florian Pop

The *lifting problem for covers of curves* is a wide generalizationof the lifting problem for curves. The contexts is as follows: Let k be an algebraically closed field of characteristic $p > 0$. Let $W(k)$ be the ring of Witt vectors over k, and $W(k) \hookrightarrow R$ be a finite extension of DVRs. It is well known that given a complete smooth k-curve X, there exist "many" *liftings* of X to R, i.e., complete smooth R-curves X_R with special fiber X (Deuring, Grothendieck, Deligne–Mumford, Popp, etc.). The lifting problem for covers of curves asks about the liftability of (possibly ramified) generically Galois covers of complete smooth k-curves as follows:

The Lifting Problem. Let G be a finite group, and $Y \to X$ be a (possibly ramified) G-cover of complete smooth k-curves. Under which hypothesis is there a finite extension $W(k) \hookrightarrow R$ and a G-cover of complete smooth R-curves $Y_R \to X_R$ with special fiber the given G-cover $Y \to X$?

The lifting problem for covers of curves has a positive answer in the case $Y \to X$ is étale and/or tamely ramified (Grothendieck, SGA I). Moreover, after fixing X_R, and specifying the ramification divisor, the cover $Y_R \to X_R$ is canonical. (This is Grothendieck's Specialization Theorem for the tame fundamental group.)

The liftability of non-tame G-covers $Y \to X$ is not always possible, one reason being the *Hurwitz bound* $84(g-1)$ on the order of the automorphism group of a curve of genus $g > 1$ in characteristic zero. In positive characteristic the corresponding bounds are non-linear in the genus g (Deuring, Roquette, Nakajima, Henn, Stichtenoth, etc.). Thus taking any complete smooth k-curve Y of genus $g_Y > 1$ such that $|\mathrm{Aut}_k(Y)| > 84(g_Y - 1)$, and setting $G := \mathrm{Aut}_k(Y)$ and $X = G\backslash Y$, one gets: The G-cover $Y \to X$ has no liftings.

The Oort Conjecture (1987). *Let $Y \to X$ be a G-cover of complete smooth k-curves with* cyclic *inertia groups. Then $Y \to X$ has a lifting.*

Going beyond Grothendieck's result on lifting of tame G-covers mentioned above, the first result which involved typical wild ramification is Oort–Sekiguchi–Suwa which tackled the case of cyclic $\mathbb{Z}/p$-covers. Work by Raynaud, Kato, Garuti, Green–Matignon, Saidi, etc., showed that actually the global lifting problem has a positive solution over some given R iff the corresponding *local lifting problems* have solutions over R. This lead to the formulation of the *local Oort Conjecture,* which became the focus of the research after the late 1990's. Using the local-global principle for the lifting and inspired by the so called Sechiguchi–Suwa theory, Green–Matignon gave in 1998 a positive answer to the Oort Conjecture in the case of inertia groups of the form $\mathbb{Z}/mp^e$ with $(p,m) = 1$ and $e \leq 2$. Yet another approach to the Oort Conjecture was initiated by Bertin–Mézard, who studied the deformation theory for covers, whereas Chinburg–Guralnick–Harbater initiated the study of the so called *Oort groups,* and showed that the class of Oort groups is quite restrictive. Finally, in their very recent work, Obus–Wewers [5] developed a new approach to tackle the Oort Conjecture. It relies on the fact proved by Garuti that the *birational* lifting is always possible, and the geometry of the lifted G-cover of (not necessarily smooth) R-curves is well understood. Garuti's results were extended by Saidi, who remarked that in special cases a birational lifting is actually a lifting. Obus–Wewers refined this idea and found much more general sufficient conditions for a birational lifting to be a lifting. They use in a very technical –and ingenious– way Raynaud's differential lifting data, and reduce the lifting problem to solving an over-determined linear system of equations. In particular, Obus–Wewers prove the local Oort Conjecture for $\mathbb{Z}/mp^e$ covers, where $(p,m) = 1$ and $e \leq 3$. **Much more importantly**, they prove the local Oort Conjecture for general cyclic covers, provided the ramification is subject to some explicit (strong) restrictions. For the discussion of the above results see the survey [4]. Finally, we announce the main result presented, which is:

Theorem (P, 2012). *The Oort Conjecture on lifting covers of curves holds.*

The proof is based on three main facts. The first one is a deformation result in characteristic p, the so called *characteristic p Oort Conjecture,* showing that a G-cover $Y \to X$ with cyclic inertia groups has a deformation over $k[[\varpi]]$ whose generic fiber has no *essential ramification.* The second fact, following from Obus–Wewers's special result, is that G-covers without essential ramification have liftings. The two facts above imply that letting $\boldsymbol{\kappa}$ be the algebraic closure of $k((\varpi))$, the following holds: For every G-cover $Y \to X$ as in the Oort conjecture there is a closely related G-cover $Y_{\boldsymbol{\kappa}} \to X_{\boldsymbol{\kappa}}$ which has a lifting over some finite extension $\mathcal{R}$ of $W(\boldsymbol{\kappa})$. Finally, the third step in the proof of the Oort Conjecture uses de Jong's alteration techniques and shows that every Harbater–Katz–Gabber cyclic cover $Y \to \mathbb{P}^1_k$ has a lifting over some finite (explicit) extension R of $W(k)$. This completes the proof by the local-global principle mentioned above.

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The Breuil-Mézard conjecture for potentially Barsotti-Tate representations

Mark Kisin

(joint work with Toby Gee)

1. Potentially semi-stable representations.

Let $p > 2$ be a prime and $K/\mathbb{Q}_p$ a finite extension. We write $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ for the absolute Galois group of K. Let $E \subset \bar{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$, $\mathcal{O} = \mathcal{O}_E$ the ring of integers of E, and $\pi_E \in \mathcal{O}$ a uniformizer. We will assume that E is large enough, for example so that it contains all the embeddings of K in $\bar{\mathbb{Q}}_p$.

Let V be a finite dimensional E-vector space equipped with a continuous action of G_K. Supose that V is potentially semi-stable. Fontaine's functor D_{dR} attaches to V a finite free $E \otimes_{\mathbb{Q}_p} K$-module, equipped with a filtration by projective submodules. Such a filtration is given by a collection of tuples of integers.

$$\lambda = (\lambda_{\sigma,1}, \lambda_{\sigma,2}, \dots, \lambda_{\sigma,d})_{\sigma: K \hookrightarrow \bar{\mathbb{Q}}_p},$$

where $d = \dim_E V$. We may also think of the filtration as given by a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to \operatorname{Res}_{K/\mathbb{Q}_p} \operatorname{GL}_d$. We call λ *the p-adic Hodge type* of V.

Also attached to V is a representation $\tau(V)$ of the Weil-Deligne group WD_K. If V is potentially crystalline, this amounts to a representation of G_K whose restriction to the inertia subgroup $I_K \subset G_K$ has finite image. The restriction $\tau(V)|_{I_K}$ is called the *Galois type* of V. Now fix a p-adic Hodge type λ, and a representation $\tau : I_K \to \mathrm{GL}_d(E)$ with open kernel. Let $\mathbb{F}/\mathbb{F}_p$ be a finite field containing the residue field of E, and let $\bar\rho : G_K \to \mathrm{GL}_d(\mathbb{F})$ be a continuous representation. We suppose, for simplicity, that $\operatorname{End}_{G_K} \bar\rho = \mathbb{F}$, when $\bar\rho$ admits a universal deformation $R_{\bar\rho}$. One has the following [7]

Theorem 1. *There exists a quotient $R_{\lambda,\tau}$ of $R_{\bar\rho} \otimes_{W(\mathbb{F})} \mathcal{O}$ such that for any finite extension E'/E a morphism $x : R_{\bar\rho} \to E'$ factors through $R_{\lambda,\tau}$ if and only if the corresponding E'-representation of G_K is potentially crystalline of type (λ, τ).*

Moreover $R_{\lambda,\tau}[1/p]$ is formally smooth over E, and equidimensional, with the dimension depending only on λ.

2. The Breuil-Mézard conjecture.

The Breuil-Mézard conjecture [3] predicts the Hilbert-Samuel multiplicity $e(R_{\lambda,\tau}/\pi_E)$ in terms of representation theory when λ is regular. We will explain this for $\mathrm{GL}_{2/K}$. Let $\lambda = \{\lambda_{\sigma,1}, \lambda_{\sigma,2}\}_{\sigma:K\hookrightarrow\bar{\mathbb{Q}}_p}$, with $\lambda_{\sigma,1} > \lambda_{\sigma,2}$ and set (for E large enough)

$$W(\lambda) = \otimes_{\sigma:K\hookrightarrow E} \operatorname{Sym}^{\lambda_{\sigma,1}-\lambda_{\sigma,2}-1} E^2 \otimes \det{}^{\lambda_{\sigma,2}}.$$

Attached to $\tau : I_K \to \mathrm{GL}_2(E)$ there is a finite dimensional, smooth E-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathcal{O}_K)$, with the following property: If π is an infinite dimensional, smooth, irreducible representation of $\mathrm{GL}_2(K)$, then $\pi|_{\mathrm{GL}_2(\mathcal{O}_K)}$ contains $\sigma(\tau)$ if and only if $\mathrm{LL}(\pi)|_{I_K} \sim \tau$ and $N = 0$ on $LL(\pi)$. Here $LL(\pi)$ is the WD_K-representation attached to π by the (suitably normalized) local Langlands correspondence.

Let $L_{\lambda,\tau} \subset W(\lambda) \otimes_E \sigma(\tau)$ be a $\mathrm{GL}_2(\mathcal{O}_K)$-stable $\mathcal{O}_E$-lattice. Then

$$(L_{\lambda,\tau}/\pi_E L_{\lambda,\tau})^{\mathrm{ss}} \sim \oplus_a a^{n(a)}$$

where a runs over the isomorphism classes of irreducible mod p representations of $\mathrm{GL}_2(\mathcal{O}_K)$. Thus, over $\bar{\mathbb{F}}_p$, each a has the form

$$a \sim \otimes_{\sigma:\mathcal{O}_K/\pi_K\hookrightarrow\bar{\mathbb{F}}_p} \operatorname{Sym}^{a_\sigma} \bar{\mathbb{F}}_p^2 \otimes \det{}^{b_\sigma}$$

where $0 \le a_\sigma \le p-1$, and $0 \le b_\sigma \le p-2$.

Conjecture 2. *For $a, \bar\rho$ as above, there exist non-negative integers $\mu_a(\bar\rho)$ such that for all λ, τ*

$$e(R_{\lambda,\tau}/\pi_E R_{\lambda,\tau}) = \sum_a n(a)\mu_a(\bar\rho).$$

Remarks.

(1) The conjecture can be seen as an infinite sequence of equations (parameterized by all possible choices of (λ, τ)) in the finitely many unknowns $\mu_a(\bar{\rho})$ where $\bar{\rho}$ is fixed. The right hand side is easy to compute but it is not known, in general, how to compute the left hand side directly.
(2) We can choose τ trivial, and $\lambda = \lambda_a$ such that $L_{\lambda_a,\mathbf{1}}/\pi_E L_{\lambda_a,\mathbf{1}} \sim a$. This then gives a formula for $\mu_a(\bar{\rho})$ in terms of the left hand side. This shows that if a solution exists it is unique.
(3) If $K/\mathbb{Q}_p$ is unramified one can compute $\mu_a(\bar{\rho})$ for most a, and for almost all a when $K = \mathbb{Q}_p$. This gives a more explicit form of the conjecture which is how it was originally formulated by Breuil-Mézard.
(4) When $K = \mathbb{Q}_p$ the result is essentially known [9]. The proof uses the p-adic local Langlands correspodence of Colmez and Berger-Breuil, [5], [4] (which is available only for $K = \mathbb{Q}_p$) and a global argument.
(5) The conjecture is closely related to modularity lifting theorems and the Fontaine-Mazur conjecture. That is, to the statement that global representations which are locally potentially semi-stable of type (λ, τ) are modular.

3. Main results.

Let $\lambda_0 = (1, 0)_{\sigma:K\to\bar{\mathbb{Q}}_p}$. Then V is of type (λ_0, τ) for some τ if it is potentially Barsotti-Tate (with regular Hodge-Tate weights).

Theorem 3. *(Toby Gee, M.K) There exist unique non-negative integers $\mu_a(\bar{\rho})$ such that for all τ as above*

$$e(R_{\lambda_0,\tau}/\pi_E R_{\lambda_0,\tau}) = \sum_a n(a)\mu_a(\bar{\rho}),$$

where $n(a)$ is as before.

Moreover, if $K/\mathbb{Q}_p$ is unramified then $\mu_a(\bar{\rho}) \neq 0$ if and only if $\bar{\rho}$ has a crystalline lift of type $(\lambda_a, 1)$, where λ_a is as in (2) of the remarks above.

The proof uses the modularity lifting theorems for potentially Barsotti-Tate representations proved in [8] and [6] and the relation between the Breuil-Mézard conjecture and such modularity lifting theorems, mentioned above.

One can use this result to prove the *Buzzard-Diamond-Jarvis* conjecture. Let F be a totally real field which is unramified over p, G_F the absolute Galois group of F, and $\bar{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ a modular Galois representation. For $v|p$ a prime of F, fix an irreducible mod p representation σ_v of $\mathrm{GL}_2(\kappa(v))$, where $\kappa(v)$ is the residue field of v, and set $\sigma = \otimes_{v|p}\sigma_v$.

Let D be a quaternion algebra over F, ramified at all but one infinite place, and unramified at all $v|p$. We say $\bar{r}$ is modular for D of weight σ is $\bar{r}$ appears in the cohomology of the Shimura curve attached to D, with coefficients in the local system corresponding to σ. The BDJ conjecture is an explicit prediction in terms of $\bar{r}|_{G_{F_v}}$ and σ_v, of the σ for which there exists a D such that $\bar{r}$ is modular for D of weight σ.

Corollary 4. *(Toby Gee, M.K) Supose that $\bar{r}$ is modular, that $\bar{r}|_{G_{F(\zeta_p)}}$ is irreducible, and if $p = 5$, assume further that the projective image of $\bar{r}$ is not isomorphic to either* $\mathrm{PGL}_2(\mathbb{F}_5)$ *or* $\mathrm{PSL}_2(\mathbb{F}_5)$.

There exists a D as above such that $\bar{r}$ is modular for D of weight σ, if and only if σ is a Serre weight predicted by the BDJ conjecture..

The proof uses the analogous results about Serre weights for unitary groups proved in [2]. The relation between the Breuil-Mézard conjecture and modularity lifting theorems allows one to relate this problem to the purely local quantities $e(R_{\lambda,\tau}/\pi_E R_{\lambda,\tau})$, and thereby infer the result for quaternion algebras from that for unitary groups.

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On pseudo-reductive groups and compactification theorems

Ofer Gabber

For a field k with separable and algebraic closures $k \subset k_s \subset \bar{k}$ consider a smooth connected affine k-group G. The geometric unipotent radical $\mathcal{R}_u(G_{\bar{k}})$ is defined over the perfect closure of k but not necessarily over k. Let $\mathcal{R}_{us,k}(G)$ (resp. $\mathcal{R}_{u,k}(G)$, resp. $\mathcal{R}_k(G)$) be the largest smooth connected normal subgroup of G which is split unipotent (resp. unipotent, resp. solvable). The group G is called pseudo-reductive (resp. quasi-reductive) if $\mathcal{R}_{u,k}(G)$ (resp. $\mathcal{R}_{us,k}(G)$) is trivial. If k' is the field of definition of $\mathcal{R}_u(G_{\bar{k}})$ and $G^{\mathrm{red}}_{k'}$ is the largest reductive quotient of $G_{k'}$, we have a natural homomorphism $i_G : G \to \mathrm{R}_{k'/k}(G^{\mathrm{red}}_{k'})$. The group scheme $\mathrm{Ker}(i_G)$ is unipotent, and G is pseudo-reductive if and only if $\mathrm{Ker}(i_G)(k_s)$ is finite. If N is a k-group of finite type such that $(N_{\ bark})^0_{\mathrm{red}}$ is unipotent and $N(k_s)$ is finite and if $\pi : P \to X$ is a torsor under N over an integral geometrically normal k-scheme X, then every rational section of π extends to a section; without assuming that $N(k_s)$ is finite each rational section of $P/N' \to X$ extends to a section. We say that a pseudo-reductive G is mpr (or of minimal type) if i_G is a monomorphism on Cartan subgroups of G.

Next we define various subgroups for G a general k-group of finite type. Let G' be the largest smooth subgroup of G ([2], Lemma C.4.1), $L_{\bar{k}}\,G$ the largest smooth affine connected subgroup of $G_{\bar{k}}$, $\bar{L}G$ the smallest subgroup of G such that $(\bar{L}G)_{\bar{k}} \supset L_{\bar{k}}\,G$, $L'G$ the largest smooth connected affine subgroup of G. Thus $L'G$ is normal in G' and $G'^0/L'G$ is a pseudo-abelian variety in the sense of Totaro [3]. Put $\mathcal{R}_*(G) := \mathcal{R}_*(L'G)$ and let $\mathrm{psz}(G)/\mathcal{R}_{u,k}(G)$ (resp. $\mathrm{lpsz}(G)/\mathcal{R}_{u,k}(G)$) be the center of $G'^0/\mathcal{R}_{u,k}(G)$ (resp. of $L'G/\mathcal{R}_{u,k}(G)$).

Any pseudo-reductive G is a central extension of its mpr quotient $G/(\mathrm{Ker}(i_G)\cap$ Cartan subgroup) that will be discussed in the second edition of [2]. For G mpr, i_G is an embedding except when the characteristic is 2 and the geometric root system is non reduced.

A split mpr datum over k consists of a split k-torus T, a finite purely inseparable extension k'/k, a reductive k'-group G' with a maximal torus identified with $T_{k'}$, a smooth connected subgroup C of $\mathrm{R}_{k'/k}(T_{k'})$ containing T, and for every root α of G' with respect to $T_{k'}$ a homomorphism of smooth connected groups endowed with C-actions: $f_\alpha : V_\alpha \to \mathrm{R}_{k'/k}(U_\alpha\, G')$, such that for every α, f_α is injective on k_s-points, α is a weight of T on $\mathrm{Lie}(V_\alpha)$, and for every non-zero elements $u, v \in V_\alpha(k_s)$, $m(f_\alpha(u)) \cdot m(f_\alpha(v)) \in C(k_s)$ (where $m(-)$ is the associated Weyl element as in [2], Proposition 3.4.2) and conjugation by $m(f_\alpha(u))$ lifts to maps $(V_\beta)_{k_s} \to (V_{s_\alpha(\beta)})_{k_s}$.

If G is mpr with split maximal torus T and geometric unipotent radical defined over k' one has an associated split mpr datum, and conversely from a split mpr datum one can reconstruct the corresponding pseudo-reductive group as the functor on smooth k-algebras associating to A the subgroup of $G'(A \underset{k}{\otimes} k')$ consisting of those elements g such that for every $\varphi : A \to L$ with L a separable field extension of k, $\varphi_*(g)$ lies in the subgroup of $G'(L \underset{k}{\otimes} k')$ generated by $C(L)$ and the images of the $V_\alpha(L)$.

For a perfect smooth connected affine group there is a universal central extension. For k of characteristic 2 and a non-zero finite dimensional k-subspace $V \subset k^{1/2}$, if K is the purely inseparable extension of k generated by V, then the algebraic subgroup H_V of $\mathrm{R}_{K/k}(\mathrm{SL}_2)$ generated by $\begin{pmatrix} 1 & \underline{V} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \underline{V} & 1 \end{pmatrix}$ (cf.[2], Proposition 9.1.6) is perfect mpr, and its universal central extension $\tilde{H}_V$ is the corresponding subgroup of $\mathrm{R}_{C(V)/k}(\mathrm{SL}_2)$ with $C(V)$ the Clifford algebra of V with respect to the squaring map $V \to k$. In general, $\tilde{H}_V$ is only quasi-reductive. This generalizes in characteristic 2 to a construction of "Clifford" central extensions of perfect pseudo-reductive groups which map to all pseudo-reductive central extensions.

For a homomorphism $f : G \to G'$ between smooth k-groups, $\mathrm{char}(k) = p > 0$, one associates a p-linear $\mathrm{obs}(f) : \mathrm{Ker}(\mathrm{Lie}(f)) \to \mathrm{Coker}(\mathrm{Lie}(f))$ which measures the obstruction to extending morphisms $\mathrm{Spec}\, k[t]/(t^p) \to \mathrm{Ker}(f)$ to $\mathrm{Spec}\, k[t]/(t^{p+1})$.

Let $\underline{\mathrm{obs}}(f)$ be the associated map of functors on k-algebras. For $\mathrm{char}(k) = 2$ and G mpr, $\mathrm{Ker}(i_G)$ is commutative with trivial Verschiebung and there is a unique G-isomorphism $\mathrm{Ker}(i_G) \simeq \mathrm{Ker}(\underline{\mathrm{obs}}(i_G))$ that induces the canonical identification on the Lie algebras.

A compactification (resp. partial compactification) of a k-scheme of finite type X is a schematically dense open immersion $X \hookrightarrow \bar{X}$ with $\bar{X}$ a proper (resp. separated of finite type) scheme over k. If G is a k-group and G' a subgroup we say that a left equivariant (partial) compactification $G \hookrightarrow \bar{G}$ is compatible with G' if for the schematic closure $\bar{G}'$ of G' in $\bar{G}$, the morphism from the contracted product $G \overset{G'}{\times} \bar{G}' \to \bar{G}$ is an open immersion. Similarly one defines compatibility with G' for an equivariant compactification of G/H where H is a subgroup scheme of G'.

If G is mpr with trivial central torus, for a choice of a split maximal torus T and a positive system of roots one defines a partial compactification $\bar{C}$ of $C = Z_G(T)$ in the Weil restriction of a toric variety, and closing the equivalence relation on $G \times C \times G$ defined by the product map to G gives a smooth equivalence relation on $G \times \bar{C} \times G$ which defines the "wonderful partial compactification" $\mathrm{wpc}(G)$ (which is independent of the choices). In characteristic 2, $\mathrm{wpc}(G) \to \mathrm{wpc}(i_G(G))$ has the structure of a torsor under the kernel of a 2-linear map between vector bundles, and similarly for $G/P \to i_G(G)/i_G(P)$ if P is a pseudo-parabolic subgroup of G.

We say that a k-group of finite type G satisfies condition $(*)$ if all tori of $G_{\bar{k}}$ are in $(G')_{\bar{k}}$. We use ideas from [1], Section 10.2, in particular the compactification theorem for commutative group schemes ([1], Section 10.2, Theorem 7) inspired our Theorem B below and we use a dévissage procedure as in [1], Section 10.2, Lemma 15 to get suitable compactifications of a k-group G from compactifications of a subgroup H and of G/H, but with H not necessarily normal in G. We use the compactification of $Y = \mathrm{R}_{k'/k}(X)$ (for X projective over k') in $\mathrm{Hilb}_{[k':k]}(X)$; this is applied to wonderful compacifications, flag varieties and artinian homogeneous spaces $X = G_{k'}/(G_{k'})_{\mathrm{red}}$ where k' is the field of definition of $(G_{\bar{k}})_{\mathrm{red}}$; in the last case condition $(*)$ ensures that the G-orbit of the unique k-point of Y is closed in Y.

For an action of a k-group G on a k-scheme X and an extension K/k, a K-orbit of the action is a non-empty locally closed subscheme of X_K which is an fpqc homogeneous space under G_K.

Theorem A. *Let G be a k-group of finite type and P a pseudo-parabolic subgroup of $L'G$. Then G/P has an equivariant projective compactification, compatible with G'^0, $\bar{L}G$, $L'G$, with a G-linearized line bundle relatively ample for $G/P \to G/\bar{L}G$, such that the boundary has no separable point and if $(*)$ holds there is no k_s-orbit contained set-theoretically in the boundary.*

Theorem B. *Let G be a k-group of finite type. Then G admits a projective compactification $G \hookrightarrow \bar{G}$ with a left action of G extending left translation, a right action of G', compatibility with $\mathcal{R}_{us,k}(G)$, $\mathcal{R}_{u,k}(G)$, $\mathcal{R}_k(G)$, $\mathrm{psz}(G)^0$, $\mathrm{lpsz}(G)^0$,*

G'^0, $\bar{L}G$, $L'G$, an invariant ample effective divisor with support $\bar{G} - G$ if G is affine, such that

(a) *For every separable extension K of k the following are equivalent*
   (i) *G_K has a subgroup isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$*
   (ii) $\bar{G}(K) \neq G(K)$
   (iii) $\overline{L'G}(K) \neq L'G(K)$
   (iv) *There is a K-orbit of the left action of G on $\bar{G}$ admitting a separable point and contained in $\bar{G}_K - G_K$.*

(b) *For every separably closed separable extension K of k, $\bar{G}(K) = \bar{G}'(K) = G(K)\overline{L'G}(K)$, and if $(*)$ holds every K-orbit of the left action of G on $\bar{G}$ has a K-point.*

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The p-adic Simpson correspondence

Ahmed Abbes

(joint work with Michel Gros)

In 1965, extending an earlier result of Weil, Narasimhan and Seshadri [11] have established a bijective correspondence between the set of equivalence classes of unitary irreducible representations of the fundamental group of a compact Riemann surface X of genus ≥ 2 and the set of isomorphism classes of stable vector bundles of degree 0 on X. The correspondence has been later extended to any complex smooth projective variety by Donaldson [5]. The analogue for general linear representations is due to Simpson ; to obtain a Narasimhan and Seshadri type correspondence, one needs to add to the vector bundle an additional structure. It's the notion of Higgs bundle that was first introduced by Hitchin for algebraic curves. If X is a scheme, smooth, separated and of finite type over a field K, a *Higgs bundle* on X is a couple (M, θ) formed by a locally free $\mathscr{O}_X$-module of finite type M and an $\mathscr{O}_X$-linear morphism $\theta\colon M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/K}$ such that $\theta \wedge \theta = 0$ (we need in our work twisted and rigid variants of this notion). Simpson's main result [13, 14, 15, 16] establishes an equivalence of categories between the category of complex finite dimensional linear representations of the fundamental group of a complex smooth projective variety and the category of semi-stable Higgs bundles with vanishing Chern classes (cf. [10]).

Simpson's results and subsequent developments have led since few years to the question of a p-adic analog. The first examples of such a construction (even if it was not formulated in terms of Higgs modules) can be traced back to Hyodo's work [9], who considered the important case of p-adic variations of Hodge structures, that he called *Hodge-Tate local systems*. The most advanced approach to date is due Faltings [8]. It generalises previous results of Tate, Sen and Fontaine. It relies on his theory of almost-tale extensions [7], and extends his previous work in p-adic Hodge theory. Once achieved, the *p-adic Simpson correspondence* should provide the best Hodge-Tate type statements in p-adic Hodge theory. But so far, Faltings's construction seems satisfactory only for curves and even in this case, many fundamental questions remain open.

In a joint work in progress with Michel Gros [1, 2, 3], we develop a new approach of this p-adic Simpson correspondence, closely related to Faltings's approach, and inspired by the work of Ogus and Vologodsky [12] on an analogue in characteristic p of the complex Simpson correspondence. The need to resume and develop Faltings's construction has been felt because of the number of results sketched in a relatively short and extremely dense article and of the quite restrictive conditions that limit its potential applications. Indeed, the most simple Hodge-Tate local systems such as the higher direct images of the constant sheaf by a smooth and proper morphism over a base of dimension ≥ 2 rarely enter in the set-up of Faltings's construction. Our approach includes them. More precisely, the correspondence we develop generalises simultaneously Faltings and Hyodo's constructions. We should mention that Tsuji has developed another approach for the p-adic Simpson correspondence [19]. Deninger and Werner [4] have also developed a partial analogue of Narasimhan and Seshadri's theory for p-adic curves, that corresponds to Higgs bundles with trivial Higgs field in the p-adic Simpson correspondence.

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$, $\overline{K}$ an algebraic closure of K, C the completion of $\overline{K}$ for the absolute value induced by the valuation of K. Let X be a smooth and separated K-scheme of finite type, $\overline{x}$ a geometric point of $X_{\overline{K}}$. We would like to construct a functor from the category of p-adic representations of the geometric fundamental group $\pi_1(X_{\overline{K}}, \overline{x})$ (*i.e.*, continuous, finite dimensional linear $\mathbb{Q}_p$-representations of $\pi_1(X_{\overline{K}}, \overline{x})$) to a category of Higgs modules on X_C (or more precisely on the associated rigid space X_C^{an}). Following Faltings's strategy, which is only partially achieved at this stage, this functor should extend to a strictly larger category than that of p-adic representations of $\pi_1(X_{\overline{K}}, \overline{x})$, called the category of *generalised representations* of $\pi_1(X_{\overline{K}}, \overline{x})$. It should then be an *equivalence of categories* between this new category and the category of Higgs modules. The main motivation of our work is to construct such an equivalence of categories. When X is a smooth and proper curve over K, Faltings showed that Higgs modules associated to "true" p-adic representations of $\pi_1(X_{\overline{K}}, \overline{x})$ are semi-stable of slope zero, and expressed the hope that all semi-stable Higgs modules of slope zero are obtained in this way. This statement would correspond to the difficult part of Simpson's result in the complex case.

The notion of a generalised representation is due to Faltings. Roughly speaking, they are p-adic *semi-linear* continuous representations of $\pi_1(X_{\overline{K}}, \overline{x})$ on modules over a certain p-adic ring equipped with a continuous action of $\pi_1(X_{\overline{K}}, \overline{x})$. Faltings' approach to construct a functor $\mathscr{H}$ from the category of these generalised representations to a category of Higgs modules is divided in two steps. He first defines $\mathscr{H}$ for the so-called *small* generalised representations, a technical notion that does not seem to be of a geometric nature (as the notion of Hodge-Tate). This first step is achieved in any dimension. In the second step, realised only for curves, he extends the functor $\mathscr{H}$ to all generalised representations by descent. Indeed, any generalised representation becomes small on a finite tale covering of $X_{\overline{K}}$. Our new approach, valid in any dimension, allows to define the functor $\mathscr{H}$ directly on a category of generalised representations of $\pi_1(X_{\overline{K}}, \overline{x})$, that we call *Dolbeault* generalised representations, strictly larger than the category of small generalised representations and containing generalised representations coming from Hodge-Tate representations. This approach seems to be well suited for descent and gives a hope to extend the construction in higher dimensions beyond the case of Dolbeault representations.

I gave in my lecture an overview of our approach. I explained first the construction for an affine scheme of a particular type [1], also referred as *small* by Faltings. Then I developed some global aspects of the theory [2]. The general construction is obtained from the affine case by a gluing technique that turns out to contain unexpected difficulties. For this purpose, we use *Faltings topos*, a fibered variant of the notion of Deligne's *covanishing topos* that we develop in [3].

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Heights of Special Cyces of Shimura Varieties

Shou-Wu Zhang
(joint work with Xinyi Yuan, Wei Zhang)

In this talk, I have reported some recent generalizations of Gross–Zagier formula.

First I described a complete Gross–Zagier formula on Shimura curves over totally real fields. This conceptional and representation theoretical formula is described purely in terms of the incoherent quaternion algebras and the spaces of modular parametrizations. More precisely, for a totally real number field F and a finite set Σ of places of F of odd cardinality including all archimedean places, there is a well-define projective system of Shimura curves X over F of type Σ. For any parametrization $f : X \to A$ of a simple abelian variety A, any CM point P on X by an imaginary quadratic extension E, and any character $\chi : Gal(\bar{E}/E) \to L^\times$ where L is an extension of the (commutative) field $M := End^0(A)$ we may formulate a CM-point

$$P_\chi(f) = \int_{Gal(\bar{E}/E)} f(P)^\sigma \otimes \chi(\sigma) d\sigma \in A(\bar{E})_{\mathbb{Q}} \otimes_M L.$$

Our formula gives an explicit formula in $L \otimes \mathbb{C}$:

$$(P_\chi(f), P_\chi(f)) = L'(\rho_{A_E} \otimes \chi, 1)\alpha(f, f)$$

where the left hand is the $M \otimes \mathbb{R}$-valued Neron–Tate height pairing respect to a polarization λ of A, ρ_A is a Galois representation attached to A with coefficients in M_ℓ, and $\alpha(f, f)$ is an explicit integration of the matrix coefficients of f with respect to the polarization λ. This formula has removed all ramification assumptions imposed in Gross–Zagier's original formula and the various generalizations by myself and has been used by Ye Tian on his recent breakthrough on the ancient congeruent number problem.

Then I describe a formula which relates the Neron–Tate heights of the image of a Shimura curve in the product of three parametrized abelian varieties and the central derivatives of the triple product L-series. More precisely, let $f_i : X \to A_i$ be the parametrizations of three simple abelian varieties as above. Let $f : X \to A := A_1 \times A_2 \times A_3$ be the their product. Let M be the tensor product of $End^0(A_i)$ over $\mathbb{Q}$. Then we can define Neron–Tate height of $f_*(X)$ with value in $M \otimes \mathbb{R}$ respect to the polarizations as

$$(f_*(X), f_*(X)) = \hat{c}_1(\mathcal{P}_1) \cdot \hat{c}_1(\mathcal{P}_2) \cdot \hat{c}_1(\mathcal{P}_3) \cdot [f_*(X) \times f_*(X)]$$

here the right hand side is certain arithmetic intersection of Poincare bundles with respect to the polarizations. Our formula gives an identity in $M \otimes \mathbb{C}$:

$$(f_*(X), f_*(X)) = L'(\rho_{A_1} \otimes \rho_{A_2} \otimes \rho_{A_3}, 2) \cdot \beta(f, f).$$

Again (f,f) is the integration of the matrix coefficients of f with respect to the polarizations. This formula has some interesting applications, including a new construction of rational points on an elliptic curve E using another elliptic curve.

Finally, I describe a generalization to higher dimensional varieties conjectured by Gan–Gross–Prasad for diagonal cycles on Shimura varieties defined by $U(n-1,1)\times U(n,1))$ over totally real number fields. More precisely, for a totally real field F, type Σ, and a positive integer n, there is a well-defined projective system of unitary Shimura varieties X_n of dimension $n-1$ defined by incoherent unitary group $\mathbb{G}_n$ of type Σ in n variables. One considers the diagonal embedding

$$Y := X_n \to X := X_n \times X_{n+1}.$$

For an automorphic and cuspidal representation π of $\mathbb{G} := \mathbb{G}_n \times \mathbb{G}_{n+1}$ with trivial archimedean components, we can define the component Y_π in $Ch^n(X)\otimes\mathbb{C}$ and show that it is cogomologically trivial under Grothendieck's standard conjecture. Thus we can define the (conjectured) Beilinson–Bloch height of Y_π and its Hecke translations. For any Hecke operator $f\in\mathcal{H}(\mathbb{G})$, we have a refined arithmetic Gan–Gross–Prasad conjecture:

$$(f_*Y_\pi, f_*Y_\pi) = L'(n+1,\rho_\pi)\gamma(f,f)$$

where ρ_π is the Galois representation of $Gal(\bar{F}/F)$ attached to π, and again $\gamma(f,f)$ is the integration of the matrix coefficients. Wei Zhang has proposed an approach to handle this conjecture using relative trace formula. Especially he has proposed an arithmetic fundamental lemma as follows for unramifield quadratic extension E/F of p-adic fields.

Let $\mathcal{N}_n$ be the Rapoport–Zink space over E^{ur} parametrizing formal $\mathcal{O}_E$-modules of signature $(n-1,1)$. Then we have a diagonal embedding

$$\mathcal{Y} := \mathcal{N}_n \to \mathcal{X} := \mathcal{N}_n \times \mathcal{N}_{n+1}.$$

The arithmetic fundamental lemma says:

$$(\mathcal{Y}, (1\times\delta)^*\mathcal{Y}) = orb'(\gamma, 1_{S_{n+1}}(\mathcal{O}_F))$$

where $\delta\in U(n+1)$ and $\gamma\in GL_{n+1}$ are conjugatate under $GL_n(E)$, and S_{n+1} is the subvariety of $Res_{E/F}GL_n$ defined by $g\bar{g}=1$, and

$$Orb'(\gamma, 1_{S_{n+1}(\mathcal{O}_F)}) = \int_{GL_n(F)} 1_{S_{n+1}(\mathcal{O}_F)}(h\gamma h^{-1})(-1)^{ord(deth)}\log|h|dh.$$

When $n=1,2$, the formula have been proved by Wei Zhang himself. When $n\geq 3$, formula for some very special pairs (γ,δ) has been proved by Rapoport, Tersteige, and Wei Zhang.

The following are some references related to my talk, plus some papers of Yifeng Liu on arithmetic inner product formula which I have no time to report in the talk.

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Modeling the distribution of Selmer groups, Shafarevich–Tate groups, and ranks of elliptic curves

Bjorn Poonen
(joint work with Manjul Bhargava, Daniel Kane, Hendrik Lenstra, Eric Rains)

1. Selmer groups

Let k be a global field. Let Ω be the set of places of k. Let $\mathbf{A}$ be the adele ring of k. Let E be an elliptic curve over k. Let $n \geq 1$. Then there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \dfrac{E(k)}{nE(k)} & \longrightarrow & \mathrm{H}^1(k, E[n]) & \longrightarrow & \mathrm{H}^1(k, E) \\ & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & \dfrac{E(\mathbf{A})}{nE(\mathbf{A})} & \xrightarrow{\alpha} & \mathrm{H}^1(\mathbf{A}, E[n]) & \longrightarrow & \mathrm{H}^1(\mathbf{A}, E) \end{array} \tag{1.0.1}$$

with exact rows. Define $\mathrm{Sel}_n E := \beta^{-1}(\operatorname{im} \alpha)$ and $Ш = Ш(E) := \ker \gamma$. (These definitions are equivalent to the ones involving maps to direct products over v.) Then (1.0.1) yields a short exact sequence

$$0 \longrightarrow \frac{E(k)}{nE(k)} \longrightarrow \mathrm{Sel}_n E \longrightarrow Ш[n] \longrightarrow 0.$$

Fix a prime p, set $n = p^e$, and take the direct limit over e to obtain

$$0 \longrightarrow E(k) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \longrightarrow \mathrm{Sel}_{p^\infty} E \longrightarrow Ш[p^\infty] \longrightarrow 0. \tag{Seq_E}$$

Let $\mathscr{E}$ be the set of elliptic curves over k, or more precisely, a set containing one representative of each isomorphism class. Order $\mathscr{E}$ by height.

Question 1.1. *Given an (abstract) short exact sequence $\mathscr{S}$ of $\mathbb{Z}_p$-modules, what is* $\mathrm{Prob}_{E \in \mathscr{E}}(\mathrm{Seq}_E \simeq \mathscr{S})$*?*

(Here "probability" is to be interpreted as density, i.e., the limit as $X \to \infty$ of the fraction of $E \in \mathscr{E}$ of height up to X having the property.)

2. Orthogonal Grassmanian

Fix $n \geq 1$. For each commutative ring A, define the standard hyperbolic quadratic form $Q\colon A^{2n} \to A$ by $Q(x_1, \dots, x_n, y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_n$, and let $\mathrm{OGr}_n(A)$ be the set of locally free rank n A-submodules $Z \leq A^{2n}$ such that Z is a direct summand and $Q|_Z = 0$. It is well known that the functor OGr_n is represented by a smooth projective scheme over $\mathbb{Z}$ with two connected components. If k is a field, then $Z, Z' \in \mathrm{OGr}_n(k)$ lie in the same component if and only if $\dim(Z \cap Z') \equiv n \pmod 2$. The space $\mathrm{OGr}_n(\mathbb{Z}_p)$ carries a probability measure compatible with the uniform measure on the finite set $\mathrm{OGr}_n(\mathbb{Z}/p^e\mathbb{Z})$ for each e.

For $0 \leq r \leq n$, define $\mathcal{S}_{n,r}$ as the closure of the locus of $Z \in \mathrm{OGr}_n$ where $\mathrm{rk}(Z \cap W) = r$. One can define a probability measure on $\mathcal{S}_{n,r}(\mathbb{Z}_p)$ as well.

3. Model

Let $V := \mathbb{Z}_p^{2n}$. Let $W := \mathbb{Z}_p^n \times \{0\}$ and choose $Z \in \mathrm{OGr}_n(\mathbb{Z}_p)$ at random (or choose both W and Z at random). Define

$$R := (Z \cap W) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \qquad \text{“}R\text{ational points”, or “}R\text{ank”}$$

$$S := \left(Z \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right) \cap \left(W \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right) \qquad \text{“}S\text{elmer group”}$$

$$T := S/R \qquad \text{“Shafarevich–}T\text{ate group”.}$$

Theorem 3.1. *The limit as $n \to \infty$ of the distribution of $0 \to R \to S \to T \to 0$ exists.*

Conjecture 3.2. *The limit distribution equals the distribution of Seq_E for $E \in \mathscr{E}$.*

To motivate this conjecture, we will observe that it leads to predictions that are compatible with other conjectures and theorems regarding ranks, Selmer groups, and Shafarevich–Tate groups.

4. Consequences

4.1. **Rank.** We have $R \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$ where

$$r := \mathrm{rk}_{\mathbb{Z}_p}(Z \cap W) = \begin{cases} 0, & \text{for } Z \text{ in one component of } \mathrm{OGr}_n \\ 1, & \text{for } Z \text{ in the other component of } \mathrm{OGr}_n, \end{cases}$$

ignoring a lower-dimensional locus of measure 0. Thus Conjecture 3.2 would imply that asymptotically 50% of elliptic curves over k have rank 0, and 50% have rank 1.

4.2. **Shafarevich–Tate group.** The group R is always the maximal divisible subgroup of S, and T is always finite, so the exact sequence $0 \to R \to S \to T \to 0$ splits. Because T is finite, Conjecture 3.2 would imply that $Ш[p^\infty]$ is finite for 100% of $E \in \mathscr{E}$, as is conjectured for all E.

For any $E \in \mathscr{E}$, the Cassels–Tate pairing on Ш restricts to an alternating pairing

$$Ш[p^\infty] \times Ш[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p$$

that is nondegenerate if $Ш[p^\infty]$ is finite. Analogously, given $x, y \in T$, lift x to $z_x \in Z \otimes \mathbb{Q}_p$ and y to $w_y \in W \otimes \mathbb{Q}_p$, and define $[x, y] := Q(z_x - w_y) \mod \mathbb{Z}_p$; one checks that this makes T a symplectic p-group, by which we mean a finite abelian p-group equipped with a nondegenerate alternating pairing $[\ ,\ ]\colon T \times T \to \mathbb{Q}_p/\mathbb{Z}_p$. In particular, $\#T$ is a square.

We now model $Ш[p^\infty]$ for an elliptic curve of rank r by any of the distributions in the following theorem:

Theorem 4.1. *The following distributions on symplectic p-groups coincide:*

(i) *The limit as $n \to \infty$ of the distribution of T when Z is sampled from $\mathcal{S}_{n,r}(\mathbb{Z}_p)$ (using $\mathcal{S}_{n,r}$ instead of OGr_n corresponds to restricting to rank r elliptic curves).*

(ii) *The limit as $n \to \infty$ of the distribution of $(\operatorname{coker} A)_{\mathrm{tors}}$, where $A\colon \mathbb{Z}_p^{2n+r} \to \mathbb{Z}_p^{2n+r}$ is chosen at random from the space of matrices in $M_{2n+r}(\mathbb{Z}_p)$ such that $A^T = -A$ and $\operatorname{rk}(\operatorname{coker} A) = r$; it turns out that $(\operatorname{coker} A)_{\mathrm{tors}}$ carries the structure of symplectic p-group.*

(iii) *The distribution for which the probability of a given symplectic p-group $(G, [\ ,\ ])$ equals*

$$\frac{\#G^{1-r}}{\operatorname{Aut}(G, [\ ,\ ])} \prod_{i=r+1}^{\infty} (1 - p^{1-2i}).$$

(This is a corrected version of a model for $Ш[p^\infty]$ proposed in [1] *in analogy with the Cohen–Lenstra heuristic for class groups.)*

4.3. Sel_{p^e}. For 100% of elliptic curves E over k, the torsion subgroup $E(k)_{\mathrm{tors}}$ is trivial. In this case, $\mathrm{Sel}_{p^e} E = (\mathrm{Sel}_{p^\infty} E)[p^e]$. So Conjecture 3.2 implies that the distribution of $\mathrm{Sel}_{p^e} E$ should equal the limit as $n \to \infty$ of the distribution of $Z \cap W$ for random $Z, W \in \mathrm{OGr}_n(\mathbb{Z}/p^e\mathbb{Z})$. There is a strong arithmetic justification for the latter distribution being a model of $\mathrm{Sel}_{p^e} E$:

Theorem 4.2 (Theorem 4.14 of [2])**.**

(a) *The locally compact abelian group $\mathrm{H}^1(\mathbf{A}, E[n])$ carries a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form (arising from Mumford's theory of the Heisenberg group).*

(b) *The images of the two maps*

$$\begin{array}{ccc} & & \mathrm{H}^1(k, E[n]) \\ & & \big\downarrow \beta \\ \dfrac{E(\mathbf{A})}{nE(\mathbf{A})} & \xrightarrow{\ \alpha\ } & \mathrm{H}^1(\mathbf{A}, E[n]) \end{array}$$

are maximal isotropic subgroups.

As further justification, we prove also:

(i) The map β is injective for asymptotically 100% of elliptic curves E, at least when $\operatorname{char} k \nmid n$; in this case, $\mathrm{Sel}_n E$ is isomorphic to $(\operatorname{im} \alpha) \cap (\operatorname{im} \beta)$.

(ii) The group $\operatorname{im}\alpha$ is a direct summand.

Is also $\operatorname{im}\beta$ a direct summand, at least for 100% of $E \in \mathscr{E}$? We know that the answer is yes at least when $E[n] \subseteq E(k)$.

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Localized chern classes and independence of ℓ

Martin Olsson

Let k be an algebraically closed field, let ℓ be a prime different from the characteristic of k, and let $c = (c_1, c_2) : C \to X \times X$ be a correspondence over k, with C and X separated finite type schemes over k. An *action* of c on $\mathbb{Q}_\ell$ is a map $u : c_1^*\mathbb{Q}_\ell \to c_2^!\mathbb{Q}_\ell$, or equivalently a global section $u \in H^0(C, c_2^!\mathbb{Q}_\ell)$. From such an action u we obtain a class $\mathrm{Tr}_c(u) \in H^0(F, \Omega_F)$, where $F := C \times_{c, X\times X, \Delta_X} X$ denotes the fixed point scheme of c and Ω_F denotes the dualizing complex. In particular, if $\Gamma \subset F$ is a proper open and closed subscheme then we can consider $\mathrm{lt}_\Gamma(u) := \int \mathrm{Tr}_c(u) \in \mathbb{Q}_\ell$, which is called the *local term of u along* Γ. We prove rationality and independence of ℓ of these local terms in three different contexts.

1. Local terms for smooth X

If X is smooth of some dimension d, then we have $c_2^!\mathbb{Q}_\ell \simeq \Omega_C(-d)[-2d]$ so an action of c on $\mathbb{Q}_\ell$ is given by a class in $H^{-2d}(C, \Omega_C(-d))$. This group $H^{-2d}(C, \Omega_C(-d))$ is up to a Tate twist the *Borel-Moore homology* of C.

For a finite type separated k-scheme X and integer i, the *i-th Borel-Moore homology group of* X, denoted $H_i(X)$, is defined to be $H^{-i}(X, \Omega_X)$, where Ω_X is the dualizing complex of X. These groups were considered already by Grothendieck and Laumon in [1]. The most important feature of these groups in our context is that there is a cycle class map

$$\mathrm{cl}_X^s : A_s(X) \to H_{2s}(X)(-s).$$

In the case when X is smooth of dimension d we have $\Omega_X = \mathbb{Q}_\ell(d)[2d]$ and this map reduces to the usual cycle class map $A_s(X) \to H^{2d-2s}(X, \mathbb{Q}_\ell(d-s))$.

Using Gabber's localized cycle classes, we show that for any cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f'} & V \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y, \end{array} \qquad (1.0.1)$$

where f is a regular embedding of codimension c, there is a homological Gysin homomorphism

$$f^!_{\mathrm{hom}} : H_i(V) \to H_{i-2c}(W)(c).$$

In the case of the cartesian square

$$\begin{array}{ccc} \mathrm{Fix}(c) & \longrightarrow & C \\ \downarrow & & \downarrow c \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

coming from a correspondence with X and C smooth, the resulting map

$$\Delta^!_{\mathrm{hom}} : H^{2(d_C - d_X)}(C, \mathbb{Q}_\ell(d_C - d_X)) \to H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$$

sends a section u to the corresponding class $\mathrm{Tr}_c(u) \in H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$.

A slightly weakened form of our main result about Borel-Moore homology is the following:

Theorem 1.1. *For a cartesian square (1.0.1) of quasi-projective schemes with f a regular embedding of codimension c, the diagram*

$$\begin{array}{ccc} A_s(V) & \xrightarrow{f^!} & A_{s-c}(W) \\ \downarrow \mathrm{cl}_V & & \downarrow \mathrm{cl}_W \\ H_{2s}(V)(-s) & \xrightarrow{f^!_{\mathrm{hom}}} & H_{2(s-c)}(W)(c-s) \end{array}$$

commutes for all s.

This has the following consequence for local terms:

Corollary 1.2. *Let $c : C \to X \times X$ be a correspondence of quasi-projective schemes with X smooth, and let d be the dimension of X. Let $\alpha \in A_d(C)$ be a d-cycle and let $u_\alpha : c_1^*\mathbb{Q}_{\ell,X} \to c_2^!\mathbb{Q}_{\ell,X}$ be the resulting action. Then for any proper connected component $\Gamma \subset \mathrm{Fix}(c)$ the local term $\mathrm{lt}_\Gamma(u_\alpha)$ is equal to the degree of the restriction of the refined intersection product $\alpha.\Delta_X \in A_0(\mathrm{Fix}(c))$ to Γ. In particular, this local term is in $\mathbb{Z}$ and independent of ℓ.*

2. Localized chern classes and local terms

Next we consider the case when X and C are possibly singular schemes over k. In this case one can obtain classes in $H^i(C, c_2^!\mathbb{Q}_{\ell,X})$ from certain complexes on C. Recall that a complex of coherent sheaves $K^\cdot$ on C is called *c_2-perfect* if there exists a factorization of c_2

$$C \overset{i}{\hookrightarrow} P \xrightarrow{p} X$$

with i a closed embedding and p smooth, such that the complex $i_*K^\cdot$ on P is quasi-isomorphic to a bounded complex of locally free sheaves of finite rank on P.

One can define the Grothendieck group of c_2-perfect complexes, which we denote by $K(c_2 : C \to X)$. For notational convenience set

$$\widehat{H}^*(c_2 : C \to X) := \oplus_i H^{2i}(C, c_2^! \mathbb{Q}_\ell(i)).$$

We show that there is a natural map

$$\tau_X^C : K(c_2 : C \to X) \to \widehat{H}^*(c_2 : C \to X)$$

enjoying many good properties akin to the usual Riemann-Roch transformation in intersection theory. In particular, for an object $K^\cdot \in K(c_2 : C \to X)$ we get a class $u_K \in H^0(C, c_2^! \mathbb{Q}_\ell)$ by taking the degree 0 part of $\tau_X^C(K)$.

Theorem 2.1. *For every proper connected component* Γ *of* $\mathrm{Fix}(c)$*, the local term* $\mathrm{lt}_\Gamma(u_K)$ *is in* $\mathbb{Q}$ *and independent of* ℓ*.*

3. Quasi-finite correspondences and local terms

Finally we consider the setting when c_2 is quasi-finite. In this case there is a clear notion of what it should mean for an element $u \in H^0(C, c_2^! \mathbb{Q}_{\ell,X})$ to be rational. Indeed, if I denotes the set of irreducible components of C which dominate X, then for each $i \in I$ with corresponding component C_i, there exists a dense open subset $U \subset C_i$ such that $c_2^! \mathbb{Q}_\ell|_U$ is canonically isomorphic to $\mathbb{Q}_{\ell,U}$. We therefore get a map

$$H^0(C, c_2^! \mathbb{Q}_\ell) \to \mathbb{Q}_\ell^I,$$

which is injective. For $u \in H^0(C, c_2^! \mathbb{Q}_\ell)$ we write $w(u) \in \mathbb{Q}_\ell^I$ for the image of u under this map. The vector $w(u)$ is called the *weight vector* of u. The main result about quasi-finite correspondences is the following:

Theorem 3.1. *(i) Let* $u \in H^0(C, c_2^! \mathbb{Q}_\ell)$ *be an element with weight vector* $w(u) \in \mathbb{Q}^I \subset \mathbb{Q}_\ell^I$*. Then for every proper component* $\Gamma \subset \mathrm{Fix}(c)$ *the local term* $\mathrm{lt}_\Gamma(u)$ *is in* $\mathbb{Q}$*.*

(ii) Let ℓ *and* ℓ' *be two primes different from the characteristic of* k*, and let* $u \in H^0(C, c_2^! \mathbb{Q}_{\ell,X})$ *and* $u' \in H^0(C, c_2^! \mathbb{Q}_{\ell',X})$ *be two actions with weight vectors* $w(u)$ *and* $w(u')$ *in* $\mathbb{Q}^I$ *and equal. Then for every proper component* $\Gamma \subset \mathrm{Fix}(c)$ *we have*

$$\mathrm{lt}_\Gamma(\mathbb{Q}_{\ell,X}, u) = \mathrm{lt}_\Gamma(\mathbb{Q}_{\ell',X}, u').$$

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