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Scaling Limits in Models of Statistical Mechanics

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Abstract. This has been the third workshop around Statistical Mechanics organized in the last 6 years. The main topic consisted of spatial random processes and their connections to statistical mechanics. The common underlying theme of the subjects discussed at the meeting is the existence of a *scaling limit*, i.e., a continuum object that approximates the discrete one under study at sufficiently large spatial scales. The specific topics that have been discussed included two-dimensional and high-dimensional critical models, random graphs and various random geometric problems, such as random interlacements, polymers, etc. The workshop bolstered interactions between groups of researchers in these areas and led to interesting and fruitful exchanges of ideas.

Mathematics Subject Classification (2000): 60xx, 82xx.

Introduction by the Organisers

The workshop was devoted to a rapidly developing area at the interface of probability, mathematical physics, analysis and combinatorics. Three themes naturally dominated the meeting were: two-dimensional critical statistical mechanics, high-dimensional and random-graph problems, random walks in random environments and random interlacements. There have been also several talks that do not quite fall into any of these categories, but were inspiring in their own right. We will now comment on each of these themes separately.

Two-dimensional critical models: The critical behavior in models on planar graphs has been one of the very hot subjects in the last decade, with its connections to

Stochastic or Schramm Loewner Evolution (SLE). Several new developments have been presented in this area: Dmitry Chelkak presented a recent calculation of asymptotic scaling for multi-point correlation functions in critical two-dimensional Ising model, Gábor Pete discussed results and conjectures concerning the time needed to break a crossing in two-dimensional dynamical percolation, Wendelin Werner presented an interesting fragmentation process related to CLE_4. Further, Greg Lawler talked about parametrization of SLE by Minkowski content and Ioan Manolescu explained an interesting transformation that relates percolation on various radial planar graphs, and shows that they fall in the same universality class.

In addition, David Wilson discussed interesting numerical coincidences in probabilities that a two-dimensional spanning tree contains a given edge and Bálint Vetö analyzed (and showed fantastic pictures) of non-colliding two-dimensional Brownian bridges. Although not for a critical model, naturally included to this area is the lecture by Hugo Duminil-Copin, who gave a full classification of the infinite-volume Gibbs states for the supercritical Potts model on the square lattice, resolving an old conjecture in this field.

Problems in random geometry: A good number of lectures have been devoted to problems that deal with various *geometric* (and geometry-related) properties of random spatial structures. Specifically, Oren Louidor described the asymptotic shape of isoperimetric sets on two-dimensional supercritical percolation cluster, Hubert Lacoin talked about counting self-avoiding paths in a similar context, while Noam Berger outlined a proof of absence of percolation of frustrated edges in the ground-state of a two-dimensional spin glass. Renato dos Santos talked about random walks on the supercritical (and time-evolving) contact process and Nina Gantert described recent progress in the understanding of the random walk on an oriented percolation cluster. Marcelo Hilario discussed percolation of cylinders in three dimensions. Allan Sly presented his recent work with R. Basu on compatibility of binary sequences, Lipshitz embeddings and rough isometries of one dimensional Bernoulli and Poisson sequences, which uses novel multiscale approach. This is a remarkable achievement and marks a breakthrough point in more than two decades history of these questions. Frank den Hollander discussed the scaling limit in the hierarchical Cannings model, which shows extremely rich behavior that is described in terms of Möbius transformations.

High-dimensional critical models: The meeting also featured a number of talks in the area of high-dimensional critical models; an area driven by a perturbation technique called the *lace expansion*. Akira Sakai gave a very thoughtful presentation of the lace expansion for the φ^4-model, Mark Holmes explained the scaling limit of high-dimensional lattice trees, Markus Heydenreich described the backbone scaling limit in high-dimensional percolation and, finally, Tom Hulshof addressed the random walk on the incipient infinite cluster above the upper critical dimension.

Random graph problems: Related to the above topic is the subject of *random graphs* and various problems thereupon. Nicolas Broutin studied scaling limits of random graph models, and Louigi Addario-Berry and Christina Goldschmidt discussed the relations between critical Erdős-Rényi random graphs and minimum

spanning trees on the complete graph. Asaf Nachmias presented results concerning recurrence of planar graph limits.

Random interlacements: A rather new area that was represented at the meeting is that of *random interlacements*. We have been fortunate to have two of major players of this field in the audience. Artëm Sapozhnikov gave us a chance to understand the geometry of the vacant set — the complement of the interlacement set. Jiří Černý in turn discussed the behavior of the chemical (or graph-theoretical) distance of the random interlacement set.

Polymer problems and front propagation: Another area of traditional interest has been that of *polymer models.* Two talks were devoted to this topic, one by Francis Comets who talked about recent results on polymer localization, and the other by Tom Alberts who discussed scaling limits of polymer models. Of a different, but related, nature has been the lecture of Alejandro Ramirez, who addressed fluctuations of a moving front in a one-dimensional model of infection spread.

Summary: It has been quite remarkable to see how the community has developed in the 6 years since our first workshop in this thematic series. There is a large number of new, highly motivated and capable young researchers and there are several subjects that barely saw the light of day 6 years ago. The meeting demonstrated this very clearly in its open exchange of ideas for current and future work. A considerable amount of time was left to spontaneous discussions and interactions among the participants which, in many ways, is at least as important as the formal presentation of completed works. The appreciating and warm atmosphere in which the meeting took place was extremely helpful to foster such an exchange of information and ideas.

We would like to hereby thank the 'Mathematisches Forschungsinstitut Oberwolfach' for help in running the workshop, and for providing us with a friendly and stimulating environment throughout the entire meeting.

Workshop: Scaling Limits in Models of Statistical Mechanics

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Gibbs states of the non-critical planar Potts model

Hugo Duminil-Copin
(joint work with L. Coquille, D. Ioffe and Y. Velenik)

Motivation. Since the seminal works of Dobrushin and Lanford-Ruelle [6, 9], the equilibrium states of a lattice model of statistical mechanics in the thermodynamic limit — the so-called Gibbs states — have been central objects of interest in statistical physics. Under very weak assumptions, it can be shown that the set $\mathcal{G}$ of all Gibbs states is a non-empty simplex. The analysis of $\mathcal{G}$ is thus reduced to determining its extremal elements. In general, this is a very hard problem which remains essentially completely open in dimensions 3 and higher, for any nontrivial model, even in perturbative regimes.

The problem of determining all extremal Gibbs states amounts to understanding all possible local behaviors of the system. Pirogov-Sinaĭ's theory often allows, at very low temperatures, to determine the pure phases of the model, i.e., the extremal, translation invariant (or periodic) Gibbs states, as perturbations of the corresponding ground states. However, it might be the case that suitable boundary conditions induce interfaces resulting in the *local* coexistence of different thermodynamic phases. That such a phenomenon can occur was first proved for the nearest-neighbor ferromagnetic (n.n.f.) Ising model on $\mathbb{Z}^3$ by Dobrushin, by considering the model in a cubic box with + spins on the top half boundary of the box and − spins on the bottom half.

In two dimensions, the situation is very different. In the late 1970s, Aizenman [1] and Higuchi [8] proved that $\mathcal{G} = \{\alpha\mu^+ + (1-\alpha)\mu^- \,:\, 0 \le \alpha \le 1\}$ for the n.n.f. Ising model on $\mathbb{Z}^2$. A much more general result, restricted to very low temperatures, was established by Dobrushin and Shlosman [7]. Their approach deals with finite systems and was based on a rough theory of fluctuations for interfaces in the system. Recently, [4] provided an alternative proof of the Aizenman-Higuchi theorem by developing a comprehensive theory of fluctuations. In the present work, we extend the approach of [4] to nearest neighbor ferromagnetic Potts models on $\mathbb{Z}^2$.

Statement of the results. Let $\Omega = \{1, \dots, q\}^{\mathbb{Z}^2}$ be the space of configurations. Let Λ be a finite subset of $\mathbb{Z}^2$, and $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$ be its complement. The *finite-volume Gibbs measure* in Λ for the q-state Potts model with boundary conditions $\sigma \in \Omega$ and at inverse-temperature $\beta > 0$ is the probability measure on Ω (with the associated product σ-algebra) defined by

$$\mathbb{P}^{\sigma}_{\beta,\Lambda}(\eta) = \begin{cases} \frac{1}{Z^{\sigma}_{\beta,\Lambda}} \mathrm{e}^{-\beta H_\Lambda(\eta)} & \text{if } \eta_i = \sigma_i \text{ for all } i \in \Lambda^c \\ 0 & \text{otherwise,} \end{cases}$$

where the normalization constant $Z^{\sigma}_{\beta,\Lambda}$ is the partition function. The Hamiltonian in Λ is given by

$$H_\Lambda(\eta) = \sum_{\substack{i\sim j \\ \{i,j\}\cap\Lambda\neq\emptyset}} \delta_{\eta_i,\eta_j}$$

where $i \sim j$ if i and j are nearest neighbors in $\mathbb{Z}^2$. A probability measure $\mathbb{P}$ on Ω is an *infinite-volume Gibbs measure* for the q-state Potts model at inverse temperature β if and only if it satisfies the following DLR condition:

$$\mathbb{P}(\cdot|\mathcal{F}_{\Lambda^c})(\sigma) \;=\; \mathbb{P}^{\sigma}_{\beta,\Lambda} \qquad \text{for } \mathbb{P}\text{-a.e. } \sigma, \text{ and all finite subsets } \Lambda \text{ of } \mathbb{Z}^2.$$

Let $\mathcal{G}_{q,\beta}$ be the space of infinite-volume q-state Potts measures.

In the case of pure boundary condition $i \in \{1,\dots,q\}$, meaning that $\sigma_x = i$ for every $x \in \Lambda^c$, we denote the measure by $\mathbb{P}^{(i)}_{\beta,\Lambda}$. For $i \in \{1,\dots,q\}$, $(\mathbb{P}^{(i)}_{\beta,\Lambda})_\Lambda$ converges when $\Lambda \nearrow \mathbb{Z}^2$ and the limit measure in infinite-volume is a Gibbs measure denoted $\mathbb{P}^{(i)}_{\beta}$.

Theorem 1 ([5]). *For any $q \geq 2$ and $\beta > \beta_c(q)$,*

$$\mathcal{G}_{q,\beta} = \Big\{ \sum_{i=1}^{q} \alpha_i \mathbb{P}^{(i)}_{\beta}, \text{ where } \alpha_i \geq 0, \forall i \in \{1,\dots,q\} \text{ and } \sum_{i=1}^{q} \alpha_i = 1 \Big\}.$$

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Conformal invariance of spin correlations in the planar Ising model

Dmitry Chelkak
(joint work with Clément Hongler and Konstantin Izyurov)

We rigorously prove existence and conformal covariance of scaling limits of spin correlations in the critical Ising model (defined on square grid approximations of a bounded planar domain Ω). This solves a number of conjectures coming from physical and mathematical literatures. The proof is based on convergence results and delicate local analysis near singularities for discrete holomorphic spinor observables which allow us to compute the logarithmic derivatives of those correlations with respect to positions of points, and relate the correlations for various boundary conditions to each other.

Namely, in [1] we prove the following

Theorem 1. Let $\Omega \subset \mathbb{C}$ be a bounded multiply connected domain with two marked points a, b on the outer boundary component $\gamma_0 \subset \partial\Omega$, and $\gamma_1, ..., \gamma_m$ be *some* of the inner components of $\partial\Omega = \gamma_0 \sqcup \gamma_1 \sqcup \cdots \sqcup \gamma_n$, $n \geq m$. If discrete domains Ω^δ approximate Ω as $\delta \to 0$, then

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\dots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_{+}[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\dots\sigma(\gamma_m^\delta)\,]} \;\to\; \vartheta_{ab}^{(\Omega)}(\gamma_1, \dots, \gamma_m)\,,$$

and this limit is a conformal invariant of the configuration $(\Omega; a, b)$ independent of single-point boundary components γ_j with $j > m$.

Remark. (i) Here $\mathbb{E}_{a^\delta b^\delta}$ denotes the expectation corresponding to the critical Ising model in Ω^δ with Dobrushin boundary conditions ("–" on the boundary arc $a^\delta b^\delta$ and "+" on the boundary arc $b^\delta a^\delta$), while $\mathbb{E}_+$ stands for "+" boundary conditions everywhere on γ_0^δ. We assume the so-called *monochromatic* boundary conditions on *all* inner boundary components $\gamma_1^\delta, ..., \gamma_n^\delta$: the spins are constant along each of γ_j^δ but their (random) values $\sigma(\gamma_j^\delta)$ are unknown.

(ii) If all (or some) inner boundary components degenerate to single points ($\gamma_j = \{w_j\}$ and $\Omega = \Omega_0 \setminus \{w_1, ..., w_k\}$, then one arrives at the ratios of *spin-spin expectations* for the critical Ising model defined in a bounded domain Ω_0 (which can be simply or multiply connected). Moreover, it does not matter whether γ_j^δ are single faces approximating w_j or small boundary components shrinking to w_j.

The similar result holds for any number of "+/–" changes on γ_0:

Corollary 2. For any $l \geq 0$ and $2l+2$ boundary points $a_0, ..., a_{2l+1} \in \gamma_0$ one has

$$\frac{\mathbb{E}_{a_0^\delta \dots a_{2l+1}^\delta}[\,\sigma(\gamma_1^\delta)\dots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_{+}[\sigma(\gamma_1^\delta)\dots\sigma(\gamma_m^\delta)\,]} \;\to\; \frac{\mathrm{Pf}\,[\,\zeta^{-1}_{a_j a_k}\vartheta^{(\Omega)}_{a_j a_k}(\gamma_1, \dots, \gamma_m)\,]_{0\leq j<k\leq 2l+1}}{\mathrm{Pf}\,[\,\zeta^{-1}_{a_j a_k}\,]_{0\leq j<k\leq 2l+1}}\,,$$

where $\zeta_{ab}^\Omega = \zeta_{ba}^\Omega$ are conformal invariants of $(\Omega; a, b)$ independent of single-point inner components. In particular, $\zeta_{ab}^{\mathbb{C}_+\setminus\{w_1,..,w_k\}} = |b-a|$.

Thus, in order to prove convergence of spin-spin expectations with "+/–/+/–" conditions on the outer boundary component it is sufficient to analyze the scaling

limits of spin-spin expectations for "+" boundary conditions. This is done in [2]. Namely, we prove the following

Theorem 3. Let Ω be a bounded (simply connected) domain and Ω_δ be discretizations of Ω by the 45° rotated square grids $\delta(1+i)\mathbb{Z}^2$ of diagonal mesh size 2δ. Then, for any $\epsilon > 0$ and any $k = 0, 1, \ldots$, we have

$$\varrho(\delta)^{-\frac{k+1}{2}} \cdot \mathbb{E}^+_{\Omega_\delta}\left[\sigma_{w_0}\sigma_{w_1}\ldots\sigma_{w_k}\right] \quad \to \quad \langle\sigma_{w_0}\sigma_{w_1}\ldots\sigma_{w_k}\rangle^+_\Omega$$

as $\delta \to 0$, uniformly over all $w_0, w_1, \ldots, w_k \in \Omega$ at distance at least ϵ from $\partial\Omega$ and from each other, and the functions $\langle\sigma_{w_0}\sigma_{w_1}\ldots\sigma_{w_k}\rangle^+_\Omega$ have the following covariance under conformal mappings $\varphi : \Omega \to \Omega'$:

$$\langle\sigma_{w_0}\sigma_{w_1}\ldots\sigma_{w_k}\rangle^+_\Omega = \langle\sigma_{\varphi(w_0)}\sigma_{\varphi(w_1)}\ldots\sigma_{\varphi(w_k)}\rangle^+_{\Omega'} \cdot \prod_{j=0}^k |\varphi'(w_j)|^{\frac{1}{8}}.$$

Remark. (i) The normalizing factors $\varrho(\delta)$ are given by the full-plane correlations $\varrho(\delta) := \mathbb{E}_{\mathbb{C}_\delta}[\sigma_{0_\delta}\sigma_{1_\delta}]$. It is a celebrated result of T. T. Wu that $\varrho(\delta)$ can be calculated explicitly and have the asymptotics $\varrho(\delta) \sim \text{const} \cdot \delta^{\frac{1}{4}}$ as $\delta^{\frac{1}{4}}$. Note that those explicit formulae can be re-proved by our method as well.

(ii) We derive the scaling limits of spin-spin expectations in Ω_δ from the following asymptotic behavior of their "discrete logarithmic derivatives" with respect to positions of points:

$$\frac{1}{2\delta}\left(\frac{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{w+2\delta}\sigma_{w_1}\ldots\sigma_{w_k}\right]}{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_w\sigma_{w_1}\ldots\sigma_{w_k}\right]} - 1\right) \quad \to \quad \mathfrak{Re}\mathcal{A}_\Omega(w; w_1, \ldots, w_k),$$

$$\frac{1}{2\delta}\left(\frac{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_{w+2i\delta}\sigma_{w_1}\ldots\sigma_{w_k}\right]}{\mathbb{E}^+_{\Omega_\delta}\left[\sigma_w\sigma_{w_1}\ldots\sigma_{w_k}\right]} - 1\right) \quad \to \quad -\mathfrak{Im}\mathcal{A}_\Omega(w; w_1, \ldots, w_k)$$

The function $\mathcal{A}_\Omega(w; w_1, \ldots, w_k)$ is defined explicitly via solution to some special interpolation-type problem for analytic functions given below:

> Let $f_{[\Omega,w;w_1,\ldots,w_k]}$ be the unique (double-valued) holomorphic function in $\Omega\backslash\{w, w_1, \ldots, w_k\}$ which has a square-root-type branching around each of $w, w_1, ..., w_k$ and satisfies the following conditions:
>
> $$\begin{aligned} \mathfrak{Im}\left[f_{[\Omega,w;w_1,\ldots,w_k]}(z)\sqrt{\nu_{\text{out}}(z)}\right] &= 0,\ z \in \partial\Omega; \\ \lim_{z\to w_j}\sqrt{z-w_j}\cdot f_{[\Omega,w;w_1,\ldots,w_k]}(z) &\in i\mathbb{R}; \\ \lim_{z\to w}\sqrt{z-w}\cdot f_{[\Omega,w;w_1,\ldots,w_k]}(z) &= 1. \end{aligned}$$
>
> where $\nu_{\text{out}}(z)$ denotes the outer normal to $\partial\Omega$ at z (in order to define $f_{[\Omega,w;w_1,\ldots,w_k]}$ for nonsmooth domains Ω one should use a proper conformal mapping). Then we set

$$f_{[\Omega,w;w_1,\ldots,w_k]}(z) = \frac{1}{\sqrt{z-w}} + 2\mathcal{A}_\Omega(w; w_1, \ldots, w_k)\cdot\sqrt{z-w} + O(|z-w|^{\frac{3}{2}}).$$

(iii) It is easy to check that $f_{[\Omega,w;w_1,\dots,w_k]}(z) = f_{[\Omega',\varphi(w);\varphi(w_1),\dots,\varphi(w_k)]}(\varphi(z))(\varphi'(z))^{\frac{1}{2}}$ and so one has the following covariance rule under conformal mappings $\varphi : \Omega \to \Omega'$:

$$\mathcal{A}_\Omega(w; w_1, \dots, w_k) = \mathcal{A}_{\Omega'}(\varphi(w); \varphi(w_1), \dots, \varphi(w_k)) \cdot \varphi'(w) + \frac{1}{8}\frac{\varphi''(w)}{\varphi'(w)}.$$

This gives a new interpretation of the famous scaling exponent $\frac{1}{8}$ (firstly derived by Onsager in 1944) for spin-spin expectations in the critical 2D Ising model.

(iv) The analogue of Theorem 3 holds true for multiply connected domains (with *monochromatic* boundary conditions on inner boundary components), thus allowing one to compute the scaling limits of spin-spin correlations in those domains without complicated PDE analysis (which is usual for CFT methods). Moreover, one can treat *fixed* (and, further, "+/−/+/−") boundary conditions on inner components as well: indeed, "integrating" discrete logarithmic derivatives of $\mathbb{E}_+[\sigma(w)]$, $\mathbb{E}_+[\sigma(w)\sigma(w_1)]$ etc along a curve connecting the outer boundary γ_0 with one of the macroscopic inner components (say, γ_1), one arrives at expectations $\mathbb{E}_+[\sigma(\gamma_1)]$, $\mathbb{E}_+[\sigma(\gamma_1)\sigma(w_1)]$ etc., which allows one to compute all conditional expectations of the form $\mathbb{E}_+[\sigma(w_1)|\sigma(\gamma_1) = \pm]$. This can be done for any number of spins and inner boundary components and further generalized to "+/−" boundary conditions via the same "Pfaffian" route as Corollary 2.

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How long may it take to lose a crossing in dynamical percolation?

Gábor Pete

(joint work with Alan Hammond, Elchanan Mossel and Oded Schramm)

The well-known **expander hitting lemma** of Ajtai-Komlós-Szemerédi [1] states that if $(\omega_t)_{t\in\mathbb{R}}$ is a stationary reversible Markov process on a state space X, with a spectral gap $1 - \lambda_2 > c > 0$ (e.g., simple random walk on an expander graph), and $A \subset X$ is an event with stationary measure $\pi(A) < 1 - \epsilon$, then

$$\mathbf{P}\big[\omega_s \in A \text{ for all } s \in [0,t]\big] < \exp(-\alpha t) \tag{1}$$

for some $\alpha = \alpha(c, \epsilon) > 0$. This is very useful in a variety of discrete problems, such as derandomization of algorithms [7].

However, when studying scaling limits of Markov processes, some discrete microscopic information about the process is always lost, and the macroscopic events that define the scaling limit might evolve on a much shorter time scale than the mixing time of the entire process. In such a case, one would naturally be interested in exit time tails on this shorter time scale, but the spectral gap of the entire discrete process and hence the estimate (1) would be irrelevant.

A typical example of this is **planar dynamical percolation**. Consider a quad $\mathcal{Q}$, i.e., a piecewise smooth Jordan domain with four marked points $a, b, c, d \in \partial\mathcal{Q}$, and critical dynamical site percolation on the triangular grid of mesh $1/n$, with i.i.d. Poisson clocks at each site switching between open and closed, at rate $1/|\mathsf{Piv}_{\mathcal{Q},n}| = n^{-3/4+o(1)}$, where $\mathsf{Piv}_{\mathcal{Q},n}$ is the set of pivotal sites for the left-right crossing event $\mathcal{C}_{\mathcal{Q},n} = \{ab \leftrightarrow cd \text{ inside } \mathcal{Q}\}$. This is the relevant time scaling in the sense that now the expected number of switches of the left-right crossing event in unit time is of unit order. In fact, it is proved in [3, 4] that the scaling limit of this process exists as a Markov process. On the other hand, the following decorrelation result was proved in [2] for the discrete dynamical percolation process $(\omega_t^n)_{t\in\mathbb{R}}$:

$$\mathbf{P}\big[\omega_0^n, \omega_t^n \in \mathcal{C}_{\mathcal{Q},n}\big] - \mathbf{P}\big[\omega_0^n \in \mathcal{C}_{\mathcal{Q},n}\big]^2 \asymp_{\mathcal{Q}} t^{-2/3}, \qquad \text{as } t \to \infty\,, \tag{2}$$

uniformly in n. Note that this is a much faster decorrelation than that of majority, say: we are on the time scale $n^{-3/4} + o(1)$ instead of order 1. But the point is that the scaling limit of critical percolation cares only about open connections, but not about the majority of the bits being open or closed.

The above discussion motivates the following result of [5]. Consider a continuous or discrete time Markov process, $(\omega_t)_{t\in\mathbb{R}}$ or $(\omega_t)_{t\in\mathbb{Z}}$, on some state space X, transition operator P_t, with some (not necessarily unique) stationary probability measure π; we will always consider the process run in stationarity, i.e., with $\omega_0 \sim \pi$. Let $\mathcal{C} \subset X$ be an event with $\pi(\mathcal{C}) = \mathbf{P}\big[\omega_0 \in \mathcal{C}\big] = p$, and let $f = \mathbb{1}_{\mathcal{C}}$. The decay of correlations of f in time is often quantified by the function $d : (0,\infty) \to [0,\infty)$ in one of the following two inequalities:

$$\mathbf{P}\big[\omega_0, \omega_t \in \mathcal{C}\big] - \mathbf{P}[\omega_0 \in \mathcal{C}]^2 = (f, P_t f) - (\mathbf{E}_\pi f)^2 \le d(t)\,\mathrm{Var}[f] \tag{3}$$

and

$$\mathrm{Var}[P_t f] = (P_t f, P_t f) - (\mathbf{E} f)^2 \le d(2t)\,\mathrm{Var}[f]\,, \tag{4}$$

for all $t \in [0,\infty)$. Of course, for reversible Markov processes, (3) is equivalent to (4). Now, Theorem 1.1 of [5] says that, assuming (4), we have that

$$\mathbf{P}\Big[\omega_s \in \mathcal{C}\ \forall s \in [0,t]\Big] \le \min_{k\in\mathbb{N}^+} \left\{ \left(\frac{p+1}{2}\right)^k + \frac{16p}{(1-p)^2} d\left(\frac{2t}{k}\right) \right\}, \tag{5}$$

and therefore

$$\mathbf{P}\Big[\omega_s \in \mathcal{C}\ \forall s \in [0,t]\Big] \le \begin{cases} t^{-\alpha+o(1)} & \text{if } d(t) = \Theta(t^{-\alpha}), \\ \exp\big(-t^{\frac{\alpha}{1+\alpha}+o(1)}\big) & \text{if } d(t) = \exp(-\Theta(t^\alpha))\,, \end{cases} \tag{6}$$

as $t \to \infty$, where the $o(1)$ terms depend only on p, α and the constant factors implicit in the $\Theta(\cdot)$ notation.

Plugging the pairwise decorrelation estimate (2) into (6) as an input, we immediately get an upper bound $t^{-2/3+o(1)}$ for the probability of continual $[0,t]$ left-right connection $\mathcal{C}_{\mathcal{Q},n}$ in the rescaled dynamical percolation process $(\omega_t^n)_{t\in\mathbb{R}}$, or in the scaling limit process $(\omega_t)_{t\in\mathbb{R}}$. However, this can be improved easily: cut

$\mathcal{Q}$ vertically into L slabs, note that a left-right connection in $\mathcal{Q}$ implies a left-right connection in all of them, and use the upper bound $t^{-2/3+o(1)}$ in all of them independently, arriving at the superpolynomial upper bound in

$$(1/4)^{\lceil 2t \rceil} \leq \mathbf{P}\big[\omega_s \in \mathcal{C}_{\mathcal{Q}} \ \forall\, s \in [0,t]\big] \leq C_K t^{-K} \quad \text{for all } K > 0\,, \tag{7}$$

both for the discrete process ω_t^n, uniformly in n, and the scaling limit ω_t. The lower bound follows from the dynamical FKG-inequality of [10, Corollary II.2.21].

In the talk I presented a heuristic argument, relying on a strong conjectural form of **universality** and **renormalization** in critical percolation (see [9] and the discussions around Question 1.7 and Conjecture 1.11 in [11]), that the tail probability in (7) is at most $\exp(-t^{2/3+o(1)})$. This may well be the true decay, although I would not bet a large amount on this.

Finally, let me mention that (1) is generalized in [5] in another direction, as well: for the case of continuous time dynamics where $\pi(A^c)$ might be very small, but where $\mathbf{P}\big[\exists t \in [0,1] : \omega_t \in A^c\big]$ is already bounded away from zero. This generalization implies that, for critical dynamical percolation on the infinite triangular lattice or $\mathbb{Z}^2$, we have

$$\exp(-c_1\, t) \leq \mathbf{P}\big[[0,t] \cap \mathcal{E} = \emptyset\big] \leq \exp(-c_2\, t)\,, \quad \text{for some } c_1, c_2 > 0\,, \tag{8}$$

where $\mathcal{E}$ is the set of **exceptional times** when the cluster of the origin is infinite, known to be nonempty for $\mathbb{Z}^2$ and having Hausdorff dimension 31/36 on the triangular lattice by [2]. This result is used in [6] to prove that although the configuration at a well-defined "typical" exceptional time has the distribution of Kesten's **Incipient Infinite Cluster** [8], the configuration at the very first exceptional time is different.

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Non-colliding Brownian bridges and the asymmetric tacnode process

Bálint Vető

(joint work with Patrik Ferrari)

We consider non-colliding Brownian bridges starting from two points and returning to the same position. These positions are chosen such that, in the limit of large number of bridges, the two families of bridges just touch each other forming a tacnode. We obtain the limiting process at the tacnode, the *(asymmetric) tacnode process*. It is a determinantal point process with correlation kernel given by two parameters: (1) the curvature's ratio $\lambda > 0$ of the limit shapes of the two families of bridges, (2) a parameter $\sigma \in \mathbb{R}$ controlling the interaction on the fluctuation scale. This generalizes the result for the symmetric tacnode process ($\lambda = 1$ case).

Systems of non-colliding Brownian motions have been much studied recently. They arise in random matrix theory (see e.g. [11]) and also as limit processes of random walk or discrete growth models, see e.g. [9, 14].

Considering non-colliding Brownian bridges (as well as discrete analogues), various kinds of determinantal processes appear naturally. Assume that the starting and ending points are chosen such that in the limit of large number of bridges occupy a region bordered by a deterministic limit shape (see Figure 1 for an illustration). Then, inside the limit shape (in the bulk) one observes the process with the sine kernel, see e.g. [12]. At the edge of the limit shape, the last bridge is described asymptotically by the Airy_2 process [14]. Whenever there is a cusp in the limit shape, then the process around the cusp is the Pearcey process [15, 13]. All these process are quite robust, in the sense that by moving the initial and/or ending points of the bridges, the only changes are geometric (e.g., the position and direction of the edge/cusp changes and numerical coefficients in the scaling), but the processes are the same without free parameter.

The case of the tacnode is more delicate and the limit process is described by two parameter. In [1], Adler, Ferrari and van Moerbeke derived the symmetric tacnode process as a limit of non-intersecting random walks. Delvaux, Kuijlaars and Zhang gave a solution in term of a 4×4 Riemann-Hilbert problem [5]. In the most recent solution by Johansson [10], the asymptotic analysis for the system of non-colliding Brownian bridges was restricted to the symmetric tacnode. In [7], we analyze the general case starting with the formula on two sets of Brownian bridges of [10] and we report about this work here. More recently, the tacnode process was described in a Double Aztec diamond model [2], for non-intersecting

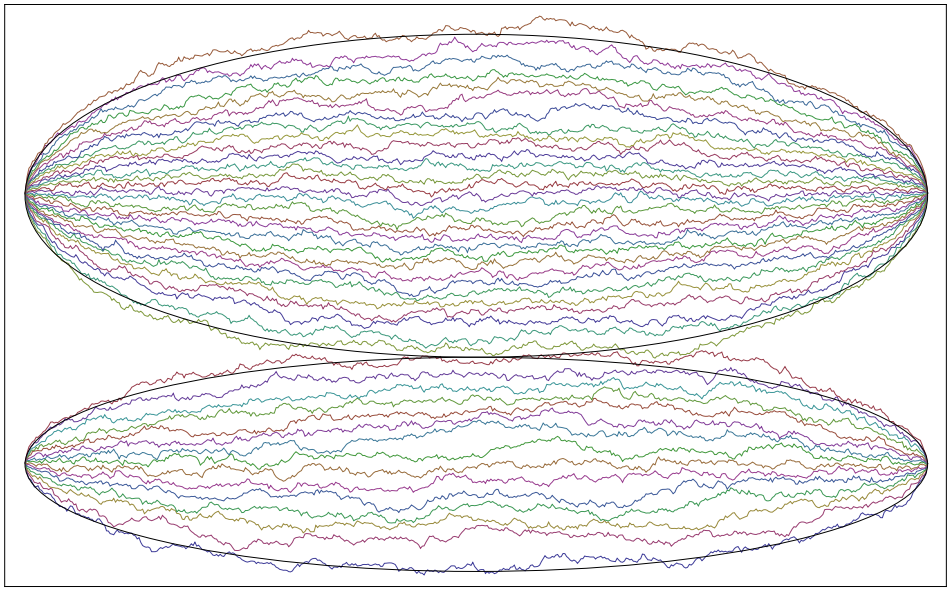

FIGURE 1. The asymmetric system of non-colliding Brownian motions with 15 respectively 30 paths in the two groups, i.e. $n = 15, \lambda = 2$.

Bessel processes [6] and the Pearcey process was shown to arise as a certain limit of the tacnode process [8].

In this report, we consider $(1 + \lambda)n$ non-colliding standard Brownian motions with two starting points and two endpoints where $\lambda > 0$ is a fixed parameter. More precisely, n of the Brownian motions start at a_1 at time 0 and arrive at a_1 at time 1, the remaining λn Brownian particles have starting and ending points at a_2 at time 0 and 1 respectively with $a_1 < a_2$. For finite times $t_1, \ldots, t_k$, the positions of the particles at these times form an extended determinantal point process. For a fixed integer n and a fixed $\lambda > 0$, let us denote by $\mathcal{L}_{n,\lambda n}(s, u, t, v)$ the kernel of this determinantal point process with $s, t \in (0, 1)$ and $u, v \in \mathbb{R}$. The kernel $\mathcal{L}_{n,\lambda n}(s, u, t, v)$ was obtained in [10]. We take the $n \to \infty$ limit in the model described above. The global picture is that the two systems of non-colliding Brownian motions form two ellipses touching each other at a tacnode (see Figure 1 for an illustration). Under proper rescaling, we obtain a limiting determinantal point process in the neighbourhood of the point of tangency.

In the general case, two parameters modulate the limit process. One of them is the strength of interaction, called σ, the other one is a measure of asymmetry, called λ which we have chosen to be the ratio of curvatures of the two ellipses at the point of tangency. For $\lambda = 1$, we get back to the symmetric case treated in [10].

The scaling of the starting and ending points is

$$a_1 = -\left(\sqrt{n} + \frac{\sigma}{2} n^{-1/6}\right), \qquad a_2 = \sqrt{\lambda}\left(\sqrt{n} + \frac{\sigma}{2} n^{-1/6}\right).$$

In this setting, the tacnode is at $(1/2, 0)$, so that the space-time scaling we need to consider are

$$s = \frac{1}{2}\left(1 + \tau_1 n^{-1/3}\right), \quad t = \frac{1}{2}\left(1 + \tau_2 n^{-1/3}\right), \quad u = \frac{1}{2}\xi_1 n^{-1/6}, \quad v = \frac{1}{2}\xi_2 n^{-1/6}.$$

The limiting kernel takes the form

$$\begin{aligned}(1)\quad \mathcal{L}^{\lambda,\sigma}_{\mathrm{tac}}(\tau_1,\xi_1,\tau_2,\xi_2) &= -\frac{1}{\sqrt{4\pi(\tau_2-\tau_1)}}\exp\left(-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}\right)\mathbb{1}(\tau_1<\tau_2)\\ &+ L^{\lambda,\sigma}_{\mathrm{tac}}(\tau_1,\xi_1,\tau_2,\xi_2) + \lambda^{1/6} L^{\lambda^{-1},\lambda^{2/3}\sigma}_{\mathrm{tac}}\left(\lambda^{1/3}\tau_1, -\lambda^{1/6}\xi_1, \lambda^{1/3}\tau_2, -\lambda^{1/6}\xi_2\right).\end{aligned}$$

For the formula of L_{tac}, for more precize description of the result and for further details, see [7].

The (asymmetric) tacnode process has an intrinsic symmetry under the reflection on the horizontal axis that is inherited from the finite system of Brownian motions. This corresponds to the following transformation of the variables:

$$\lambda \longrightarrow \lambda^{-1}, \quad n \longrightarrow \lambda n, \quad \tau_i \longrightarrow \lambda^{1/3}\tau_i, \quad \xi_i \longrightarrow -\lambda^{1/6}\xi_i, \quad \sigma \longrightarrow \lambda^{2/3}\sigma.$$

This explains the presence of the last term on the right-hand side of (1).

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Supercritical phase of the vacant set of random interlacements

Artem Sapozhnikov
(joint work with Alexander Drewitz, Balázs Ráth)

1. Model

The model of random interlacements was recently introduced by Sznitman in [6] in order to describe the local picture left by the trajectory of a random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$ when it runs up to times of order N^d, or on the discrete cylinder $(\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}$, $d \geq 2$, when it runs up to times of order N^{2d}, see [4], [9]. Informally, the random interlacement Poisson point process consists of a countable collection of doubly infinite trajectories on $\mathbb{Z}^d$, and the trace left by these trajectories on a finite subset of $\mathbb{Z}^d$ "looks like" the trace of the above mentioned random walks.

The set of vertices visited by at least one of these trajectories is the random interlacement at level u, denoted by $\mathcal{I}^u$, and the complement of this set is the vacant set at level u, denoted by $\mathcal{V}^u$. These are one parameter families of translation invariant, ergodic, long-range correlated random subsets of $\mathbb{Z}^d$, see [6]. While $\mathcal{I}^u$ induces a connected subgraph of $\mathbb{Z}^d$ for all levels u, the graph induced by $\mathcal{V}^u$ has a non-trivial percolation phase transition in u, as shown in [6] and [3]. We study the *connectivity structure* of $\mathcal{V}^u$ in the *supercritical phase*.

Before we present the results, we define $\mathcal{V}^u$ formally. For $x \in \mathbb{Z}^d$, $d \geq 3$, let P_x be the law of a simple random walk X on $\mathbb{Z}^d$ with $X(0) = x$. Let K be a finite subset of $\mathbb{Z}^d$. The equilibrium measure of K is defined by

$$e_K(x) = P_x\left[X(t) \notin K \text{ for all } t \geq 1\right], \quad \text{for } x \in K,$$

and $e_K(x) = 0$ for $x \notin K$. The capacity of K is the total mass of the equilibrium measure of K:

$$\operatorname{cap}(K) = \sum_x e_K(x).$$

Since $d \geq 3$, for any finite set $K \subset \mathbb{Z}^d$, the capacity of K is positive. Therefore, we can define the normalized equilibrium measure by

$$\widetilde{e}_K(x) = e_K(x)/\operatorname{cap}(K).$$

Definition 1. *Let $u > 0$ and $K \subset\subset \mathbb{Z}^d$. Let N_K be a Poisson random variable with parameter $u \cdot \operatorname{cap}(K)$. Let X_i be independent simple random walks (independent from N_K) with the distribution $P_{\widetilde{e}_K}$. We define the random subset $\mathcal{V}^u_K$ of K by*

$$\mathcal{V}^u_K = K \setminus \bigcup_{i=1}^{N_K} \operatorname{range}(X_i),$$

i.e., $\mathcal{V}^u_K$ is the set of vertices in K which are not visited by any of the random walks X_i, $1 \le i \le N_K$. By construction, for any $u > 0$ and $K \subseteq K' \subset\subset \mathbb{Z}^d$,

$$\mathcal{V}^u_K \overset{\mathcal{D}}{=} \mathcal{V}^u_{K'} \cap K.$$

Therefore, for each $u > 0$, there exists a unique random set $\mathcal{V}^u \subset \mathbb{Z}^d$ such that for all $K \subset\subset \mathbb{Z}^d$,

$$\mathcal{V}^u \cap K \overset{\mathcal{D}}{=} \mathcal{V}^u_K.$$

The set $\mathcal{V}^u$ is called the vacant set of random interlacements at level u. (The complement of $\mathcal{V}^u$ is called the random interlacements at level u.) One can easily see from the definition that for any $u > 0$,

$$\mathbb{P}[\mathcal{V}^u \supset K] = e^{-u \cdot \mathrm{cap}(K)}, \quad \text{for all } K \subset\subset \mathbb{Z}^d.$$

This family of equations uniquely defines the law of $\mathcal{V}^u$, and can be taken as the definition of $\mathcal{V}^u$.

The vacant set exhibits a non-trivial percolation phase transition in u, i.e., there exists $u_* \in (0, \infty)$ such that (i) for any $u > u_*$, almost surely, all connected components of $\mathcal{V}^u$ are finite, and (ii) for any $u < u_*$, almost surely, $\mathcal{V}^u$ contains an infinite connected component. In particular, the finiteness of u_* for $d \ge 3$ and the positivity of u_* for $d \ge 7$ were proved in [6], and the latter result was extended to all dimensions $d \ge 3$ in [3]. It is also known that $\mathcal{V}^u$ contains at most one infinite connected component (see [7]); in particular, for any $u < u_*$, the infinite connected component is almost surely unique, and we denote it by $\mathcal{V}^u_\infty$.

2. Results

We study connectivity structure of $\mathcal{V}^u$ in the supercritical regime $u < u_*$. Even though, we believe that all our results are valid for any $u < u_*$, we can so far only prove them for any $u < \overline{u}$, where $\overline{u}$ is defined as follows.

Definition 2. *Let $d \ge 3$. Let $\overline{u} = \overline{u}(d)$ be the supremum over all u' such that for each $u \in (0, u')$, there exist constants $c = c(d, u) > 0$ and $C = C(d, u) < \infty$ with the property that for all $R \ge 1$, we have*

$$\mathbb{P}\left[\mathcal{V}^u_\infty \cap \mathrm{B}(0, R) \neq \emptyset\right] \ge 1 - Ce^{-R^c},$$

and

$$\mathbb{P}\left[\begin{array}{c}\text{any two connected subsets of } \mathcal{V}^u \cap \mathrm{B}(0,R) \text{ with} \\ \text{diameter} \ge R/10 \text{ are connected in } \mathcal{V}^u \cap \mathrm{B}(0,2R)\end{array}\right] \ge 1 - Ce^{-R^c}.$$

Heuristically, for any $u < \overline{u}$, the vacant set has a unique macroscopic component in large boxes. As in models such as Bernoulli percolation, we expect that $\overline{u} = u_*$, but we cannot prove this conjecture so far. (Note that the bound $\overline{u} \le u_*$ is straightforward from the definition.)

Theorem 1. *(*[1, Theorem 1.1]*) For any $d \ge 3$, $\overline{u} > 0$.*

For $d \geq 5$, this theorem follows from the stronger statement of [8, Theorem 3.2]. Our novelty here is twofold. Firstly, our result is new for $d \in \{3, 4\}$. Secondly, our proof is conceptually different from that of [8], and applies to all dimensions $d \geq 3$. An immediate corollary from the above theorem is that for any $d \geq 3$, there exist $c = c(d) > 0$ and $C = C(d) < \infty$ such that for all $u \leq \overline{u}$, we have

$$\mathbb{P}\left[n \leq |\mathcal{C}^u(0)| < \infty\right] \leq Ce^{-n^c},$$

where $C^u(0)$ is the connected component of the origin in $\mathcal{V}^u$.

For $x, y \in \mathcal{V}^u_\infty$, let $\rho_u(x, y)$ be the graph distance in $\mathcal{V}^u_\infty$ between x and y, and $\mathrm{B}_{\rho_u}(x, r)$ the ball in $\mathcal{V}^u_\infty$ with center x and radius $\lfloor r \rfloor$. Our next result is the shape theorem for $\mathcal{V}^u_\infty$, which states that large balls in $\mathcal{V}^u_\infty$ with respect to the metric ρ_u after rescaling have an asymptotic deterministic shape.

Theorem 2 (Shape theorem, [2]). *For any $u \in (0, \overline{u})$, there exists a convex compact subset $D_u \subset \mathbb{R}^d$ such that for each $\varepsilon \in (0, 1)$, there exists a $\mathbb{P}[\cdot \,|\, 0 \in \mathcal{V}^u_\infty]$-almost surely finite random variable $\tilde{R}_{\varepsilon,u}$ satisfying*

$$(1) \qquad \forall\, R \geq \tilde{R}_{\varepsilon,u} \;:\; \mathcal{V}^u_\infty \cap (1-\varepsilon)R \cdot D_u \subseteq \mathrm{B}_{\rho_u}(0, R) \subseteq \mathcal{V}^u_\infty \cap (1+\varepsilon)R \cdot D_u.$$

Moreover, D^u is invariant with respect to the isometries of $\mathbb{R}^d$ which preserve 0 and $\mathbb{Z}^d$, is non-decreasing in u, and tends to L_1-ball of $\mathbb{R}^d$ as $u \to 0$.

The main ingredient in the proof of the shape theorem is the following theorem, which roughly speaking states that with high probability the ρ_u-distance between vertices in $\mathcal{V}^u_\infty$ is comparable to the graph distance in $\mathbb{Z}^d$ between these vertices.

Theorem 3 (Chemical distance, [2]). *For any $d \geq 3$ and $u \in (0, \overline{u})$, there exists $C = C(d, u) < \infty$ such that for all $R \geq 1$,*

$$\mathbb{P}\left[\text{for all } x, y \in \mathcal{V}^u_\infty \cap \mathrm{B}(0, R),\ \rho_u(x, y) \leq CR\right] \geq 1 - C \cdot e^{-(\log R)^2}.$$

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Spanning trees of graphs on surfaces and the intensity of loop-erased random walk

David B. Wilson
(joint work with Richard W. Kenyon)

A spanning tree of a graph is a collection of edges which connects all the vertices and has no cycles. Spanning trees were first investigated by Kirchhoff in his study of electrical resistor networks [Kir90]; in particular he showed that the determinant of the combinatorial Laplacian counts spanning trees.

The uniform random spanning tree (UST) is a well studied probability model, related to several other probability models. For example, the loop-erased random walk of Lawler (see [Law91, Law99, LL10]) was shown by Pemantle [Pem91] to have the same distribution as the paths connecting vertices in the uniform spanning tree. The abelian sandpile model of self-organized criticality was shown by Majumdar and Dhar [MD92] to be closely related to spanning trees (recurrent states in the sandpile model are in bijection with spanning trees). Lawler, Schramm, and Werner [LSW04] showed that the branches of the spanning tree converge in the scaling limit to SLE_2 and the "peano curve" winding between the spanning tree and its dual converges to SLE_8.

We show here how to compute the probabilities of various connection topologies for uniform random spanning trees on graphs embedded in surfaces. As an application, we show how to compute the "intensity" of the loop-erased random walk in $\mathbb{Z}^2$, that is, the probability that the walk from $(0,0)$ to ∞ passes through a given vertex or edge. For example, the probability that it passes through $(1,0)$ is $5/16$; this confirms a 15-year old conjecture about the stationary sandpile density on $\mathbb{Z}^2$. We do the analogous computation for the triangular lattice, honeycomb lattice and $\mathbb{Z} \times \mathbb{R}$, for which the probabilities are $5/18$, $13/36$, and $1/4 - 1/\pi^2$ respectively.

Our techniques involve applying the vector bundle Laplacian [Ken11] and asymptotics of the "Green's function derivative" for planar graphs, together with a generalization of the grove counting techniques of [KW11] to graphs on annuli.

Response matrices and groves. Let $\mathcal{G}$ be a graph and $c\colon \mathcal{G} \to \mathbb{R}_{>0}$ a positive conductance on each edge. Let $\mathcal{N}$ be a subset of its vertices. The triple $(\mathcal{G}, c, \mathcal{N})$ is a **resistor network**. Associated to this data is the Dirichlet-to-Neumann matrix (also called response matrix) L, defined as follows. Given a function $f : \mathcal{N} \to \mathbb{R}$ find its harmonic extension h on $\mathcal{G}$, that is a function on the vertices of $\mathcal{G}$ that is harmonic on $\mathcal{G} \setminus \mathcal{N}$ and has values f on $\mathcal{N}$. Then $L(f) = -\Delta(h)|_{\mathcal{N}}$ is a linear function of f, where Δ is the (positive semidefinite) graph Laplacian. In electrical terms, $L(f)$ gives the current flow into the nodes $\mathcal{N}$ when they are held at f volts. While it is not obvious from this definition, L is a symmetric negative semidefinite matrix.

Planar resistor networks where $\mathcal{N}$ is a subset of the outer face were called **circular planar networks** in [CIM98], and were studied in [CdV94, CdVGV96, CIM98]. These authors classified which matrices occur as response matrices: they

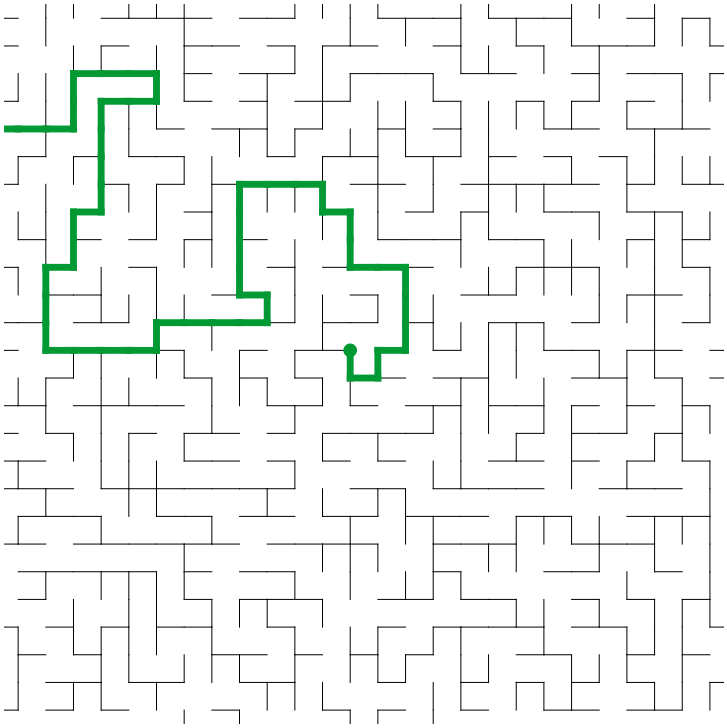

FIGURE 1. A portion of the uniform spanning tree on $\mathbb{Z}^2$, with the path from $(0,0)$ to ∞ shown in bold. The uniform spanning tree on $\mathbb{Z}^2$ can be constructed as a weak limit of uniform spanning trees on large boxes. It is known that the limiting measure exists, is unique, and is supported on trees of $\mathbb{Z}^2$ [Pem91]. Almost surely, within the uniform spanning tree of $\mathbb{Z}^2$, each vertex has a unique infinite path starting from it [BLPS01] (see also [LwP12]). The path to infinity is a loop-erased random walk (LERW) [Pem91] (see also [Wil96]).

are precisely the matrices whose "non-interlaced" minors are nonnegative. (A non-interlaced minor is one in which there are no 4 indices $a < b < c < d$ for which a and c are rows and b and d are columns or vice versa.) Furthermore they showed how to construct a graph having a given matrix L.

In a resistor network a **grove** is a spanning forest (set of edges with no cycles) in which every component contains at least one vertex in $\mathcal{N}$. (Our term grove is a generalization, to arbitrary graphs and arbitrary connections, of the groves defined by Carroll and Speyer in [CS04].) In [KW11] we studied the natural probability measure on groves (where each grove occurs with probability proportional to the product of its edge weights), showing for circular planar graphs how to compute the probability that a random grove has a given connection topology in terms of the entries in L.

Graphs on surfaces. We study here the same problem for a graph $\mathcal{G}$ embedded on a surface Σ. Here the usual notion of response matrix is not rich enough to

extract information about the underlying topological structure of a grove. If the surface has nodes on multiple faces, then these faces can be joined by cutting the surface along one or more zippers. Each such zipper can be given a "phase" z (where $z \in \mathbb{C}^*$ or $z \in \mathrm{SL}_2(\mathbb{C})$) so that paths crossing the zipper in one direction acquire the phase factor z, and paths crossing in the other direction acquire a phase factor of z^{-1}. The usual response matrix entries $L_{i,j}$ then generalize to elements $\mathscr{L}_{i,j}$ of $\mathbb{C}$ or $\mathrm{GL}_2(\mathbb{C})$ in which the phase encodes topological information. We show here how $\mathscr{L}$ can be used to compute connection probabilities of (certain types of) groves on $\mathcal{G}$.

The question of characterizing which matrices $\mathscr{L}$ occur as a function of the topology of Σ remains open. See Lam and Pylyavskyy [LP12] for related work in the case when the surface Σ is an annulus.

We give special attention to the case where the surface Σ is an annulus; this is the easiest case beyond the planar one (but already quite involved) and also has applications to the study of spanning trees on planar graphs.

Applications to planar graphs. Using these techniques one can in principle compute the probability that the path of the uniform spanning tree from a to b in a planar graph passes through a given set of edges or vertices (as in 1). We carry out this computation for $\mathbb{Z}^2$ for a single edge or vertex (see 2); our method shows that the answer is in $\mathbb{Q}[\frac{1}{\pi}]$. For the triangular and hexagonal lattices, these probabilities are in $\mathbb{Q}[\frac{\sqrt{3}}{\pi}]$.

For example, we show that the probability that the loop-erased walk in $\mathbb{Z}^2$ from $(0,0)$ to ∞ contains the point $(1,0)$ is $5/16$. (See 1.) This number was predicted by Levine and Peres [LP11] and by Poghosyan and Priezzhev [PP11], by relating this probability to the average stationary density of the abelian sandpile model.

The connection between the spanning trees and the abelian sandpile model was discovered by Majumdar and Dhar [MD92], and Priezzhev [Pri94] used this connection to compute the height distribution of the abelian sandpile model, in terms of two integrals that could not be evaluated in closed form. Grassberger evaluated these integrals numerically, and conjectured that the stationary density of the sandpile on $\mathbb{Z}^2$ is $17/8$. Later Jeng, Piroux, and Ruelle [JPR06] showed how to express one of these two integrals in terms of the other, and determined the sandpile height distribution in closed form, under the assumption that the remaining integral, which numerically is 0.5 ± 10^{-12}, is exactly $1/2$. Our derivation of this probability that LERW passes through $(1,0)$ confirms these conjectures (although our methods are different), and shows that this aforementioned integral is exactly $1/2$.

While we were writing up our results, Poghosyan, Priezzhev, and Ruelle independently found another proof that the probability of visiting $(1,0)$ is $5/16$ [PPR11]. (They also asked about the probability about visiting other points, and remarked that the probability that the LERW visits $(1,1)$ is numerically close to $2/9$. This differs from the true value by about 10^{-3}.)

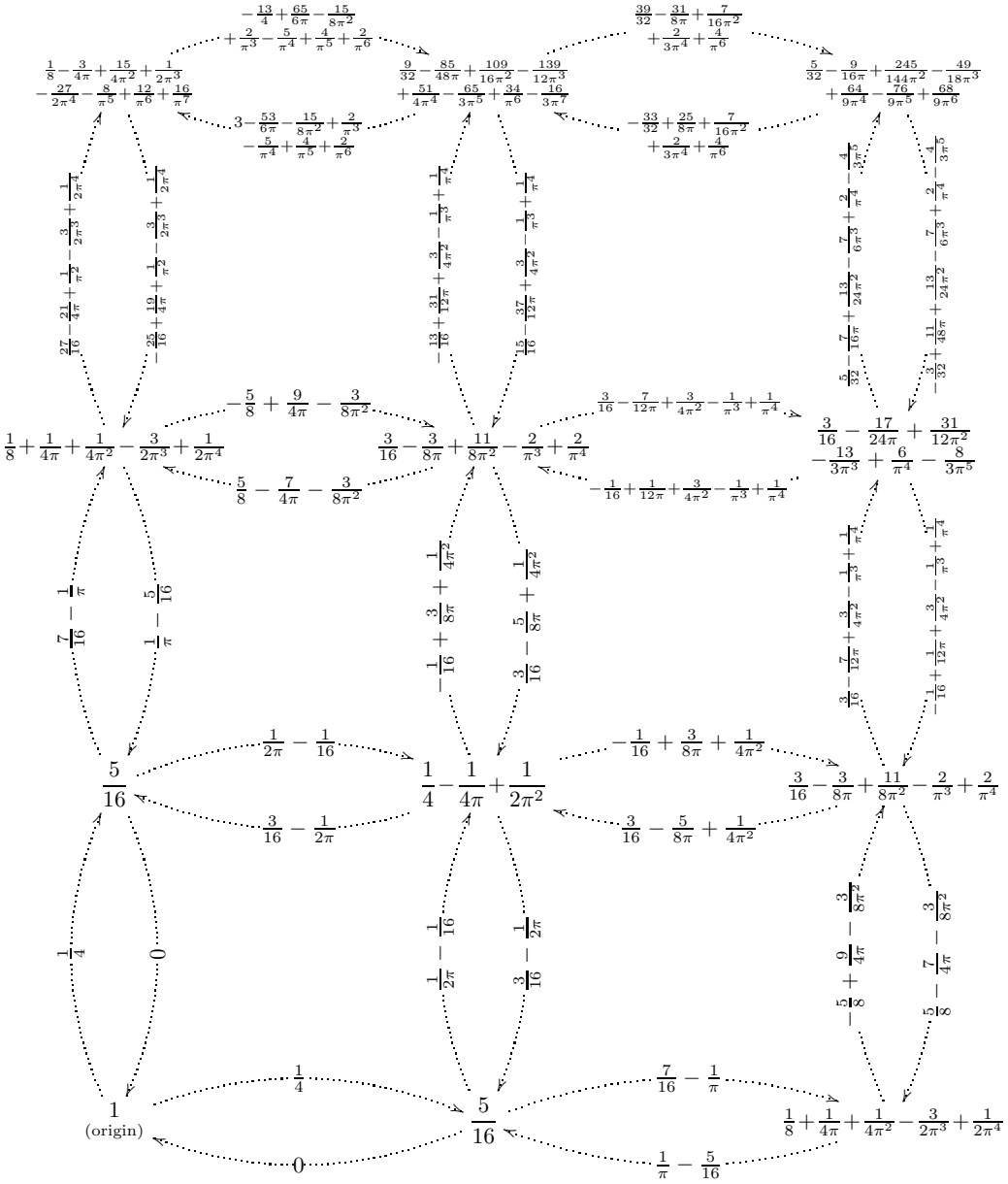

FIGURE 2. Intensity of loop-erased random walk on $\mathbb{Z}^2$. The origin is at the lower-left, and directed edge-intensities as well as vertex-intensities of the LERW are shown.

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Lack of percolation for unsatisfied edges in two dimensional spin glass

Noam Berger
(joint work with Ran Tessler)

We consider a shift-covariant Gibbs measure for the spin-glass in two dimensions. This is defined as follows: A probability measure μ on the space $\mathbb{R}^{\mathbb{E}^2} \times \{+1,-1\}^{\mathbb{Z}^2}$ (We use $\mathbb{E}^2$ to denote the set of edges of the lattice $\mathbb{Z}^2$) is called a shift-covariant Gibbs at inverse temperature β measure if it satisfies the following three requirement:

(1) The $\mathbb{R}^{\mathbb{E}^2}$ marginal is i.i.d. normal zero-one variables.
(2) Conditioned on the values (J_e) at the edges, μ is a Gibbs measure w.r.t. spin glass with interactions (J_e).
(3) μ is shift invariant.

We then say that an edge $e = (x, y)$ is satisfied if $\omega(x)\omega(y)J_e > 0$ and is unsatisfied if $\omega(x)\omega(y)J_e < 0$.

The main (and only) result in the talk is that if β is large enough (i.e. when the temperature is low enough), then for every such μ, the dual unsatisfied edges span only finite connected components.

The proof of the result is combinatorial in nature, and is done by contradiction. We first extract a shift invariant forest out of the graph spanned by the unsatisfied edges. By the Burton-Keane Theorem, it cannot contain 3-or-more-ended trees. We then extract the backbones of the trees in the forest. The backbone of a 2-ended tree can be canonically defined, while the definition of the backbone of a one-ended tree is an ad hoc definition.

We then show that these backbones are typically very long, and the energy calculation leads to the desired contradiction.

Application of the lace expansion to the φ^4 model

Akira Sakai

The φ^4 model is a standard model in scaler field theory. Roughly speaking, it is Gaussian free field combined with quartic self-interaction. It is known to exhibit a phase transition and critical behavior similarly to the Ising model. There were intensive researches in the 1980's when Aizenman [1] and Fröhlich [2] succeeded in showing mean-field behavior above 4 dimensions under the assumption of reflection-positivity and Gawędzki and Kupiainen [4] and Hara and Tasaki [5, 6] succeeded in showing mean-field type behavior for the weakly coupled nearest-neighbor model in 4 dimensions using a rigorous renormalization-group method. The goal of my research [9] is to investigate the critical two-point function above the upper-critical dimension without assuming reflection-positivity.

Let $\Lambda \subset \mathbb{Z}^d$ and define the Hamiltonian for the spin configuration $\varphi = \{\varphi_x\}_{x\in\Lambda}$ as

$$H_\Lambda(\varphi) = \frac{1}{2}\sum_{\{u,v\}\subset\Lambda} J_{u,v}(\varphi_u - \varphi_v)^2 + \sum_{v\in\Lambda}\left(\frac{\mu}{2}\varphi_v^2 + \frac{\lambda}{4!}\varphi_v^4\right),$$

where $\lambda \geq 0$ is the intensity of self-interaction and $\mu \in \mathbb{R}$ plays the role of the temperature. We assume that the spin-spin coupling $J_{u,v}$ is ferromagnetic (i.e., nonnegative), translation-invariant, $\mathbb{Z}^d$-symmetric and finite-range. For example, $J_{o,x} = \delta_{|x|,1}$ for the nearest-neighbor model. Let

$$\langle f\rangle_\mu = \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{\int_{\mathbb{R}^\Lambda} f(\varphi)\, e^{-H_\Lambda(\varphi)}\, \mathrm{d}^\Lambda\varphi}{\int_{\mathbb{R}^\Lambda} e^{-H_\Lambda(\varphi)}\, \mathrm{d}^\Lambda\varphi}.$$

Using the Lebowitz inequality [7], we can show that there is a $\mu_c = \mu_c(d, J, \lambda)$ such that the susceptibility $\chi_\mu \equiv \sum_{x\in\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_\mu$ is finite if and only if $\mu > \mu_c$ and diverges as $\mu \downarrow \mu_c$. Moreover, if the bubble condition [1]

$$\sum_{x\in\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_{\mu_c}^2 < \infty$$

holds, then we can identify the speed of divergence as $\chi_\mu \asymp (\mu - \mu_c)^{-1}$ (i.e., the ratio of the LHS to the RHS is bounded away from zero and infinity). For a special class of models that satisfy reflection-positivity [3], the Fourier transform $\hat{G}_\mu(k) \equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}\langle\varphi_o\varphi_x\rangle_\mu$, $k \in [-\pi,\pi]^d$, obeys the Gaussian infrared bound

$$0 \leq \hat{G}_\mu(k) \leq O(|k|^{-2}) \quad \text{uniformly in } \mu > \mu_c,$$

which implies that the bubble condition holds for $d > 4$, hence $\chi_\mu \asymp (\mu - \mu_c)^{-1}$. Although the nearest-neighbor model satisfies reflection-positivity, it is often hard to verify that condition.

In [9], we prove asymptotic behavior of the critical two-point function $\langle\varphi_o\varphi_x\rangle_{\mu_c}$, without assuming reflection-positivity:

Theorem 1. *Let $\rho = 2(d-4) > 0$ and $0 < \lambda \ll 1$ (depending on d and J). Then there is a $\Phi_\mu(x) = \langle\varphi_o^2\rangle_\mu\delta_{o,x} + O(\lambda)\,(|x| \vee 1)^{-(d+2+\rho)}$, where $O(\lambda)$ is uniform in $\mu \geq \mu_c$, such that*

$$\mu_c = -\frac{\lambda}{2}\sum_{x\in\mathbb{Z}^d}\Phi_{\mu_c}(x), \qquad \langle\varphi_o\varphi_x\rangle_{\mu_c} \underset{|x|\uparrow\infty}{\sim} \frac{\frac{d}{2}\Gamma(\frac{d-2}{2})\pi^{-d/2}}{\sum_{y\in\mathbb{Z}^d}|y|^2\big(J_{o,y} - \frac{\lambda}{2}\Phi_{\mu_c}(y)\big)}|x|^{2-d}.$$

In my talk, the following three key steps are explained:

(1) The Griffiths-Simon construction [10] to approximate the φ^4 model on Λ to some Ising model on $\Lambda \times \{1, 2, \ldots, N\}$.
(2) The lace expansion for the Ising two-point function [8].
(3) Detail estimates on the expansion coefficients in terms of N [9].

These steps yield a linearized version of the Schwinger-Dyson equation, which further yields the aforementioned asymptotic expression for $\langle\varphi_o\varphi_x\rangle_{\mu_c}$.

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High-dimensional lattice trees converge to super-Brownian motion.

Mark Holmes

(joint work with Remco van der Hofstad, Edwin Perkins)

A standard result due to Kolmogorov is that critical Galton-Watson branching processes survive until time n with probability decreasing like c/n for large n.

A well known result in the super-processes literature is that rescaled critical branching random walk converges to super-Brownian motion. In one formulation of this statement, starting with a single initial particle, and under appropriate scaling, the limiting "law" is a σ-finite measure $\mathbb{N}_0$ on the space of cadlag measure-valued paths $\{X_t\}_{t\geq 0}$ (here each X_t is a finite measure on $\mathbb{R}^d$). If the spatial dimension d is larger than 8, then outside a set of measure zero under this measure, the supports of the random measures $\int_r^s X_t(\cdot)dt$ and $\int_u^v X_t(\cdot)dt$ are disjoint if $r < s < u < v$. One can interpret this as saying that the measure-valued process X_t is *self-avoiding* under $\mathbb{N}_0$, when d is larger than 8.

A lattice tree is a finite connected set of lattice bonds, containing no cycles. We take a spread-out model, where all sites within a box of radius L of any site x are considered to be neighbours of x. We choose a finite lattice tree at random according to a critically weighted measure of the form e.g.

$$\mathbb{P}(T) = \frac{z_c^{|T|}}{\sum_{T' \ni o} z_c^{|T'|}}.$$

Each vertex in the lattice tree has a time/generation component given by the tree distance from the root, so in particular the origin is the only vertex of generation 0. The survival/extinction time of the tree is the first generation that is empty.

In [1] the analogue to Kolmogorov's survival asymptotics is proved for (sufficiently spread out) lattice trees in dimensions $d > 8$. This together with asymptotic formulae for the Fourier transforms of the r-point functions is sufficient to prove convergence of lattice trees to super-Brownian motion in the sense of finite dimensional distributions. Tightness is required to establish pathwise convergence.

In the paper [2] we prove a set of sufficient conditions for tightness of the sequence of measure-valued processes, for convergence to super-Brownian motion. The main condition is one that involves fourth moments of increments of the measure-valued processes integrated with respect to a rich class of test functions (the Fourier transforms). These moments can be represented as differences of the rescaled Fourier transforms of the 5-point functions. The bound required is stronger than one has available from existing estimates on these quantities, so one needs to improve the estimates of the differences. Sufficient estimates are currently being verified for suffiently spread-out models of lattice trees in dimensions $d > 8$ via the lace expansion.

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A new lace expansion for percolation & the backbone scaling limit

Markus Heydenreich

(joint work with Remco van der Hofstad, Tim Hulshof, Grégory Miermont)

We consider *percolation* on the hypercubic lattice $\mathbb{Z}^d$, where every bond is made *open* with probability $p \in [0,1]$, independently of each other. By $\{0 \leftrightarrow x\}$ we denote the event that there is a path of open bonds between the origin 0 and x, and we write $\{0 \leftrightarrow \infty\}$ for the event that 0 is in an infinite cluster. Our interest is the critical percolation threshold $p_c = \inf\{p \in [0,1] \colon \mathbb{P}_p(0 \leftrightarrow \infty) > 0\}$.

In a seminal paper, Hara and Slade [2] use the *lace expansion* to analyze critical percolation when the dimension d is sufficiently large. The heart of their approach is an inclusion-exclusion expansion of the probability of $\{0 \leftrightarrow x\}$. Arguably, the lace expansion is the key ingredient for any result about critical percolation in high dimensions so far.

We present a new lace expansion, based on an algebraic expansion using lace graphs, which resembles the original lace expansion by Brydges and Spencer [1]. This new expansion allows us to identify path properties of large critical cluster, as we describe now.

To this end, we say that a bond b is *pivotal* for the event $\{0 \leftrightarrow x\}$ if $0 \leftrightarrow x$ on the (possibly modified) configuration where b is made occupied, while $0 \nleftrightarrow x$ on the (possibly modified) configuration where b is made vacant. For every $n \geq 1$,

we define a probability measure on marked configurations, i.e. on the set of pairs (ω, x) where ω is a percolation configuration and $x \in \mathbb{Z}^d$, by

$$\mathbb{E}^*_{p_c,n}[F] = \frac{\mathbb{E}_{p_c}\Big[\sum_{x\in\mathbb{Z}^d} F(\cdot,x)\,\mathbb{1}(0 \leftrightarrow x \text{ with } n-1 \text{ pivotal bonds})\Big]}{\mathbb{E}_{p_c}\Big[\sum_{x\in\mathbb{Z}^d} \mathbb{1}(0 \leftrightarrow x \text{ with } n-1 \text{ pivotal bonds})\Big]} \tag{1}$$

for every non-negative measurable function $F = F(\omega, x)$, and we denote by x_* the distinguished vertex under $\mathbb{P}^*_{p_c,n}$. The construction in (1) allows us to condition on a cluster being large in the sense of pivotal distance. Under this measure, we let $S'_0 = 0$, and $S'_1, S'_2, \ldots, S'_{n-1}$ be the top of the ith pivotal bond for the event $\{0 \leftrightarrow x_*\}$, and $S_n = x_*$, hence defining a random process $\big(S'_i\big)_{i=1}^n$.

In the following statement, we let $(B_t, t \geq 0)$ be a standard d-dimensional Brownian motion.

Theorem 1 ([6]). *Consider the (nearest-neighbor) percolation in sufficiently high dimension. There exist a constant $\sigma > 0$ such that the following convergence holds as $n \to \infty$ in the space of right-continuous functions with left limits, $D([0,1], \mathbb{R}^d)$, endowed with the Skorohod J_1 topology:*

$$\frac{1}{\sqrt{n}} S'_{\lceil nt \rceil} \Rightarrow \sigma B_t. \tag{2}$$

We now present three extensions of the theorem; for details we refer to [6].

Theorem 1 indicates that the set of pivotal bonds is close to the image of a Brownian motion when properly renormalized. It is natural to ask whether the geometry of the backbone is well-captured by the pivotal bonds, where we define x to be in the backbone whenever $\{0 \leftrightarrow x\} \circ \{x \leftrightarrow x^*\}$ (i.e., the two events occur disjointly). To this end, define

$$\mathcal{B}_n := \{0 \leftrightarrow x \text{ with at most } n \text{ pivotal bonds}\} \circ \{x \leftrightarrow x^*\}$$

to be the set of vertices of $\mathbb{Z}^d$ that are in the backbone, and are separated from the origin by at most n pivotal bonds. Indeed, under the conditions of Theorem 1, the following convergence in distribution holds in the space of non-empty compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric:

$$\frac{1}{\sqrt{n}} \mathcal{B}_{\lfloor nT \rfloor} \Rightarrow \{\sigma B_t : 0 \leq t \leq T\}^{\mathrm{cl}}, \qquad T \in [0,1], \tag{3}$$

where A^{cl} is the closure of $A \subseteq \mathbb{R}^d$.

The statement of Theorem 1 applies also to the *incipient infinite cluster* (IIC). The IIC is a critical cluster conditioned to be infinite through a suitable limiting scheme. The first such construction on the two-dimensional lattice has been carried out by Kesten [8], and constructions for the high-dimensional setting are contained in [5, 7]. Under the IIC measure, the event $\{0 \leftrightarrow \infty\}$ happens almost surely, and we denote by $S'_1, S'_2, S'_3, \ldots$ the endpoints of the first, second, third,... pivotal bond for this event. Then the convergence in (2) holds, and also backbone convergence in the sense of (3). Hence, the backbone of a (high-dimensional) incipient infinite cluster converges to a Brownian motion path. It is an open problem to identify

the scaling limit of the entire IIC cluster. Hara and Slade [3, 4] conjectured that a properly rescaled cluster converges to integrated Superbrownian excursion (ISE), and proved partial results.

Indeed, Theorem 1 does not only apply to nearest-neighbor percolation, but also to a large class of *spread-out models* in dimension $d > 6$, which is the conjectured upper critical dimension for percolation. Finally, the theorem also holds for *long-range percolation*, but then convergence is not to a Brownian motion path, but rather to an α-stable motion (with α determined by the occupation probabilities of the bonds).

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Random walk on the high-dimensional IIC

Tim Hulshof

(joint work with Makus Heydenreich, Remco van der Hofstad, Grégory Miermont)

The incipient infinite cluster (IIC) is a critical percolation conditioned to have an infinite cluster at the origin. It is believed that an approach to analyzing the structure of critical percolation clusters can be made by studying the behavior of simple random walk on the incipient infinite cluster. I will discuss the main results of three recent papers (one of which is still in preparation) [5] [6] and [7].

All results concern high-dimensional (high-d) percolation models. High dimension typically means that the dimension is high enough so that the 'triangle diagram' is finite, i.e., that d is large enough so that

$$\sum_{x,y} \mathbb{P}_{p_c}(0 \longleftrightarrow x)\mathbb{P}_{p_c}(x \longleftrightarrow y)\mathbb{P}_{p_c}(y \longleftrightarrow 0) < \infty.$$

Finiteness of the triangle diagram is a well known indicator of 'mean-field' behavior of the model, and it was proved in [2] that it holds in high d for finite-range

percolation models ($d \geq 19$ is the best estimate for nearest-neighbor percolation, finiteness for other finite-range models has been proved for dimensions $d > 6$). Finiteness has also been proved for long-range models with spread-out parameter α when $d > 3(2 \wedge \alpha)$ in [3].

Summarizing, these results that will be discussed are:

(1) That Kesten's construction for 2-dimensional models [8] also works for constructing the high-dimensional IIC;
(2) That the backbone of the IIC has Brownian motion as the scaling limit;
(3) That random walk on the IIC is strongly subdiffusive;
(4) That random walk behaves significantly different on finite-range IIC clusters and on long-range IIC clusters.

Result (1) deals with another aspect of the construction of the IIC. Kesten proved the following construction of the IIC in two dimensions [8]: let F be a cylinder event, and let $Q_r = [-r, r]^d$,

$$\mathbb{P}_{\mathsf{IIC}}(F) = \lim_{r\to\infty} \mathbb{P}_{p_c}(F \mid 0 \longleftrightarrow Q_r^c),$$

where $\mathbb{P}_{p_c}$ is the measure of critical percolation. Using the result from [10] that $\mathbb{P}_{p_c}(0 \longleftrightarrow Q_r^c) \asymp 1/r^2$, we prove that this limit also yields the IIC for high-d models (and that this limit is the same as the other known high-d constructions give, cf. [4]).

Result (2) concerns the scaling limit of the 'backbone' of the IIC. The backbone of the IIC is the subgraph of the IIC that is induced by the set of vertices in the IIC that are connected to 0 and ∞ by paths that are mutually edge-disjoint. Indeed, we prove that the backbone of the IIC of finite-range percolation in sufficiently high d converges to the path of a Brownian motion in the Hausdorff sense [7].

The main emphasis of the talk will be on the result (3), as well as other properties of random walk on high-d IICs and properties of the random walk on the backbone of the IIC. Define the intrinsic ball B_r in the IIC as the set

$$B_r(0) = \{x \in \mathbb{Z}^d : d_{\mathsf{IIC}}(0, x) \leq r\}$$

where d_{IIC} is the graph-metric on the IIC. Kozma and Nachmias showed that the exit time of a simple random walk started at 0 from B_r, τ_{B_r}, satisfies $\mathbb{E}_{\mathsf{IIC}}[\tau_{B_r}] \asymp r^3$ [9] for finite-range IICs in high d. We prove that the same relation also holds for long-range percolation in high d [6].

We also prove that finite-range models in high d the exit time of a random walk from a the Euclidean ball Q_r obeys the asymptotic relation $\mathbb{E}_{\mathsf{IIC}}[\tau_{Q_r}] \asymp r^6$ [6]. The proofs of these asymptotic relations are inspired by [1] and [11].

The final result, (4), concerns random walk on the IIC of long-range percolation, where the edge-probabilities decay as a power-law of the length of the edge, with power $d + \alpha$ for some $\alpha \in (0, \infty)$. When $\alpha < 4$ the random walk typically leaves the ball Q_r significantly faster than a random walker on a finite-range IIC would. Indeed, we prove that for random walk on the long-range IIC, the exit time τ_{Q_r} is with high probability not bigger than $Cr^{3(4\wedge\alpha)/2}$, i.e., we prove that uniformly in

r,

$$\int P^0_\omega(\tau_{Q_r} > \theta r^{3(4\wedge\alpha)/2})\mathbb{P}_{\mathsf{IIC}}(\mathrm{d}\,\omega) \qquad \text{as} \qquad \theta \to \infty,$$

where ω is a realization of the IIC, and P^0_ω is the probability measure of random walk started at 0 on the configuration ω [6].

The main tools for proving results (3) and (4) are bounds on the volume and effective resistance of the IIC intersected with B_r and Q_r.

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Lipschitz Embeddings of Random Sequences

Allan Sly

(joint work with Riddhipratim Basu)

With his compatible sequences and clairvoyant demon scheduling problems Winkler introduced a fascinating class of dependent or "co-ordinate" percolation type problems (see e.g. [1, 6, 12]). These, and other problems in this class, can be interpreted either as embedding one sequence into another according to certain rules or as oriented percolation problems in $\mathbb{Z}^2$ where the sites are open or closed according to random variables on the co-ordinate axes. A natural question of this class posed by Grimmett, Liggett and Richthammer [8] asks whether there exists a Lipshitz embedding of one Bernoulli sequence into another. The following theorem answers the main question of [8].

Theorem 1. *Let $\{X_i\}_{i\in\mathbb{Z}}$ and $\{Y_i\}_{i\in\mathbb{Z}}$ be independent sequences of independent identically distributed Ber($\frac{1}{2}$) random variables. For sufficiently large M almost surely there exists a strictly increasing function $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $X_i = Y_{\phi(i)}$ and $1 \leq \phi(i) - \phi(i-1) \leq M$ for all i.*

The original question of [8] was slightly different asking for a positive probability on the natural numbers with the condition $\phi(0) = 0$, which is implied by our theorem (and is equivalent by ergodic theory considerations). Among other results they showed that Theorem 1 fails in the case of $M = 2$. Despite impressive progress on a range of similar questions (see e.g. [3, 6, 7, 1, 2, 9]) the question of embedding one random sequence into another remained open. The difficulty lies in the presence of long strings of ones and zeros on all scales in both sequences which must be paired together.

In a similar vein is the question of a rough, (or quasi-), isometry of two independent Poisson processes. Informally, two metric spaces are roughly isometric if their metrics are equivalent up to multiplicative and additive constants. The formal definition, introduced by Gromov in the case of groups and more generally by Kanai, is as follows.

Definition 2. *We say two metric spaces X and Y are roughly isometric with parameters (M, D, C) if there exists a mapping $T : X \to Y$ such that for any $x_1, x_2 \in X$,*

$$\frac{1}{M} d_X(x_1, x_2) - D \leq d_Y(T(x_1), T(x_2)) \leq M d_X(x_1, x_2) + D,$$

and for all $y \in Y$ there exists $x \in X$ such that $d_Y(T(x), y) \leq C$.

Originally Abért asked whether two independent infinite components of bond percolation on a Cayley graph are roughly isometric. Szegedy asked the problem when these sets are independent Poisson process in $\mathbb{R}$ (see [10] for a fuller description of the history of the problem). The most important progress on this question is by Peled [10] who showed that Poisson processes on $[0, n]$ are roughly isometric with parameter $M = \sqrt{\log n}$. The question of whether two independent Poisson processes on $\mathbb{R}$ are roughly isometric for fixed (M, D, C) was the main open question of [10]. We prove that this is indeed the case.

Theorem 3. *Let X and Y be independent Poisson processes on $\mathbb{R}$ viewed as metric spaces. There exists (M, D, C) such that almost surely X and Y are (M, D, C)-roughly isometric.*

Again the challenge is to find a good matching on all scales, in this case to the long gaps in the each point processes with ones of proportional length in the other. Both results follows from an abstract theorem which applies to range of different models. Like essentially all results in this area, our approach is multi-scale using renormalization. The novelty of our approach is that, as far as possible, we ignore the anatomy of what makes different configurations difficult to embed and instead consider simply the probability that they can be embedded into a random block proving recursive power-law estimates for these quantities.

Independent Results Two other researchers have also solved some of these problems independently. Vladas Sidoravicius [11] solved the same set of problems and described his approach to us which is quite different from ours and Peter Gács sent us a draft of his paper [5] solving Theorem 1 extending his work on the scheduling problem [4].

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Cylinders Percolation in three dimensions

Marcelo Hilário

(joint work with Augusto Teixeira and Vladas Sidoravicius)

We study percolation on the subset obtained by removing from $\mathbb{R}^3$ a Poissonian ensemble of infinite cylinders of radius one. Before presenting our model, let us give some motivation to study it.

Perhaps the simplest model for a random environment in $\mathbb{R}^d$ is the so-called "continuum" (or Boolean) percolation, in which a Poissonian ensemble of unit balls is placed in $\mathbb{R}^d$. Each one of the balls can be thought as being an obstacle. Letting $\mathcal{V}$ stand for the vacant set, *i.e.* the complement of this random set of obstacles, the primary question one can ask is whether $\mathcal{V}$ contains or not an unbounded connected component with positive probability. If so, one says that the vacant set $\mathcal{V}$ percolates. Due to the uniformly boundedness of the obstacles, a number of techniques developed in the study of Bernoulli site percolation can be adapted to this continuum case. However, for other models containing large obstacles, the

induced random environment may feature long-range dependencies, often leading to some intriguing behavior and challenging problems.

The model we consider is governed by a Poisson point process, on the space $\mathbb{L}$ of lines in $\mathbb{R}^d$, having intensity measure $u\mu$ where u is a positive real parameter and μ is, up to a multiplicative constant, the unique Haar measure in $\mathbb{L}$ which is invariant with respect to isometries of $\mathbb{R}^d$.

Having specified the intensity measure $u\mu$, a corresponding Poisson point process can be easily constructed in an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P}_u)$. Each element $\omega \in \Omega$ is a point measure, i.e.

$$\omega = \sum_{i \geq 0} \delta_{l_i}, \text{ where } l_i \text{ runs over a countable collection of lines in } \mathbb{R}^d. \tag{1}$$

We are mainly interested in the set

$$\mathcal{L}(\omega) = \bigcup_{l \in \mathrm{supp}(\omega)} C(l), \tag{2}$$

where $C(l)$ stands for the cylinder of radius one around l. As well as its complement

$$\mathcal{V}(\omega) = \mathbb{R}^d \setminus \mathcal{L}(\omega), \tag{3}$$

the so-called 'vacant set'. Intuitively speaking, the set $\mathcal{V}$ represents what is left after we drill through all the lines in the support of ω. As for the parameter u, it controls the amount of cylinders to be removed from $\mathbb{R}^3$: as u increases, more and more cylinders are drilled, making it increasingly harder for $\mathcal{V}$ to be well connected.

Our main contribution is to prove that

$$\begin{gathered}\text{For } d = 3 \text{ and } u \text{ small enough, the vacant set } \mathcal{V} \text{ contains} \\ \text{almost surely an unbounded connected component.}\end{gathered} \tag{4}$$

If one defines the critical parameter by

$$u_* = \inf\{u \geq 0;\ \mathbb{P}_u[\mathcal{V} \text{ has an unbounded connected component}] = 0\}, \tag{5}$$

then our result means that u_* is strictly positive.

The model was introduced by I. Benjamini and first studied by J. Tykesson and D. Windisch in [1], where among other results they established the existence of a phase transition for the vacant set left by these cylinders in $\mathbb{R}^d$ when $d \geq 4$.

$$u_* < \infty, \text{ for every } d \geq 3 \text{ and} \tag{6}$$

$$u_* > 0, \text{ for every } d \geq 4. \tag{7}$$

In order to explain why the three dimensional case is qualitatively different from the others, let us briefly describe how the case $d \geq 4$ was handled in [1]. In that work, the authors restricted their attention to the intersection between $\mathcal{V}$ and $\mathbb{R}^2$ (naturally embedded in $\mathbb{R}^d$). They proved that for $d \geq 4$ and for u small enough there exists $\mathbb{P}_u$-a.s. an unbounded connected component in $\mathcal{V} \cap \mathbb{R}^2$, yielding (7). However, as they also observed, this strategy is destined to fail in three dimensions, as

$$\begin{gathered}\text{for } d = 3, \text{ for every } u > 0, \text{ the set } \mathcal{V} \cap \mathbb{R}^2 \text{ contains} \\ \mathbb{P}_u\text{-a.s. no unbounded connected component.}\end{gathered} \tag{8}$$

In view of (8), in order to establish Theorem 4 we have to search for connections outside the plane $\mathbb{R}^2$. But, first of all, why would someone be interested in restricting the set $\mathcal{V}$ to $\mathbb{R}^2$? This is done in order to use the so-called 'path duality' of the plane, which roughly speaking, states that

(9) if the connected component of $\mathcal{V} \cap \mathbb{R}^2$ containing the origin is bounded, then there exists a circuit surrounding the origin in $\mathcal{L} \cap \mathbb{R}^2$.

The above statement reduces the task of proving percolation to showing that typical paths in $\mathcal{L} \cap \mathbb{R}^2$ are small. In our case, we also make use of a statement similar to (9), however, instead of $\mathbb{R}^2$, we will intersect $\mathcal{V}$ with a periodic surface H. This surface is contained in the slab $\mathbb{R}^2 \times [0, 1000]$ and has two important properties. First, H is homeomorphic to $\mathbb{R}^2$, which allows us to use duality on H in an indirect way. Moreover, H is 'rough', meaning that its intersection with any fixed cylinder gives rise to small connected components only.

The proof that $\mathcal{V} \cap H$ percolates follows a renormalization argument that keeps track of crossing probabilities (by paths in $\mathcal{L} \cap H$) at various different scales. We show that the probabilities of some mild crossing events fulfill a contracting recursion relation. For values of u that are sufficiently small, the recursion relations can be triggered using the rough shape of the surface H (which would not be possible if we have considered $\mathbb{R}^2$ instead of H). This enables us to get the desired result by proving that those crossing probabilities decay fast enough as the scale is increased, meaning that there exist typically no long dual circuits.

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Chemical distance on random interlacements

Jiří Černý
(joint work with S. Popov)

The random interlacements model was introduced by A.-S. Sznitman in [6] in order to describe the microscopic structure in the bulk which arises when studying the disconnection time of a discrete cylinder or the vacant set of random walk on a discrete torus. It can be informally described as a dependent site percolation on $\mathbb{Z}^d$, $d \geq 3$, which is 'generated' by a Poisson cloud of independent simple random walks whose intensity is driven by a non-negative multiplicative parameter u. The set covered by these random walks is called the *interlacement set at level* u and is denoted by $\mathcal{I}^u$.

It is known that for every $u > 0$, the set $\mathcal{I}^u$ is 'supercritical', more precisely $\mathcal{I}^u$ is an infinite connected subset of $\mathbb{Z}^d$. It is thus natural to ask how $\mathcal{I}^u$ resembles to $\mathbb{Z}^d$, on large scales. Recently, many results appeared going in this direction: In [3], it is shown that the interlacement set percolates in thick enough slabs, [4] shows that random walk on $\mathcal{I}^u$ is transient.

The question we want to answer is how the internal (or chemical) distance ρ_u on $\mathcal{I}^u$, defined by

$$\rho_u(x,y) = \min\{n : \text{there exist } x_0, x_1, \dots, x_n \in \mathcal{I}^u \text{ such that } x_0 = x, x_n = y, \\ \text{and } \|x_k - x_{k-1}\|_1 = 1 \text{ for all } k = 1, \dots, n\},$$

compares with the usual Euclidean distance. Our first main result is the following theorem

Theorem 1. *Let $\mathbb{P}_0^u = \mathbb{P}[\cdot \,|\, 0 \in \mathcal{I}^u]$ be the conditional distribution given that the origin is in the interlacement set $\mathcal{I}^u$, and denote by $\Lambda^u(x,n) = \{y \in \mathcal{I}^u : \rho^u(x,y) \le n\}$ the ball centred at x with radius n in the internal distance. Then for every $u > 0$ and $d \ge 3$ there exists a compact convex set $D_u \subset \mathbb{R}^d$ such that for any $\epsilon > 0$, $\mathbb{P}_0^u$-a.s. for all n large enough*

$$\big((1-\epsilon)nD_u \cap \mathcal{I}^u\big) \subset \Lambda^u(0,n) \subset (1+\epsilon)nD_u.$$

The key technical step in the proof of Theorem 1 is the following 'large deviation estimate' (resembling [2, Theorem 1.1] valid for the Bernoulli percolation).

Theorem 2. *For every $u > 0$ and $d \ge 3$ there exist constants $C, C' < \infty$ and $\delta \in (0,1)$ such that*

$$\mathbb{P}_0^u[\text{there exists } x \in \mathcal{I}^u \cap [-n,n]^d \text{ such that } \rho_u(0,x) > Cn] \le C' e^{-n^\delta}.$$

Theorem 2 can also be used to answer a related question: 'How much the range of the random walk on the torus resembles the torus?' To this end we consider $(X_k)_{k\in\mathbb{N}}$ to be a simple random walk on the discrete d-dimensional torus of size N, $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, and let $\mathcal{I}_N^u$ be the range of the random walk up to time uN^d, $\mathcal{I}_N^u = \{X_0, \dots, X_{uN^d}\}$. Finally, let ρ_N^u be the internal distance on $\mathcal{I}_N^u$.

Theorem 3. *For large enough $\bar{C}$ and γ, we have*

$$P^N\left[\rho_N^u(x,y) \le \bar{C} d_N(x,y) \text{ for all } x,y \in \mathcal{I}_N^u \text{ such that } d_N(x,y) \ge \ln^\gamma N\right] \xrightarrow{N\to\infty} 1.$$

This improves the result of Shellef [5], where a similar claim was proved for $\bar{C} = \log \dots \log N$ being arbitrarily many times iterated logarithm.

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Recurrence of planar graph limits

Asaf Nachmias
(joint work with Ori Gurel-Gurevich)

A *distributional limit* of finite graphs G_n is a random rooted infinite graph (U, ρ) with the property that neighborhoods of G_n around a random vertex converge in distribution to neighborhoods of U around ρ. This limit was defined by Benjamini and Schramm [2]. Their motivation was the study of infinite random planar maps, a widely studied model in the probability, combinatorics and statistical physics communities for generic two-dimensional geometries and quantum gravity. The canonical example of such a limit is Angel and Schramm's [1] *uniform infinite planar triangulation* (UIPT) and is obtained by taking the distributional limit of a uniform random triangulation on n vertices. Here a triangulation is a simple planar graph in which every face has 3 edges.

These authors conjectured that the UIPT is almost surely recurrent (see [1, Conjecture 1.12] and [2, Page 3]). It is shown in [2] that a distributional limit of uniformly bounded degree finite planar graphs is almost surely recurrent. However, the degrees of random planar maps and the UIPT are unbounded so one cannot appeal to this result. In this paper we prove this conjecture.

Theorem 1 ([3])**.** *Let (U, ρ) be a distributional limit of planar graphs such that the degree of ρ has an exponential tail. Then U is almost surely recurrent.*

Corollary 1. *The UIPT is almost surely recurrent.*

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Random walk on the supercritical contact process

Renato S. dos Santos
(joint work with Frank den Hollander)

Let

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \big(\xi_t(x)\big)_{x\in\mathbb{Z}} \tag{1}$$

be a càdlàg Markov process with state space $\Omega = \{0,1\}^{\mathbb{Z}}$. We say that at time t the site x is *occupied by a particle* if $\xi_t(x) = 1$ and is *vacant* or, alternatively, *occupied by a hole*, if $\xi_t(x) = 0$. Having fixed a realization of ξ, the random walk in dynamic random environment (RWDRE)

$$W = (W_t)_{t\geq 0} \tag{2}$$

is the time-inhomogeneous Markov process that starts from 0 and has local transition rates

$$\begin{aligned} &x \to x+1 \quad \text{at rate} \quad \alpha_1\,\xi_t(x) + \alpha_0\,[1-\xi_t(x)], \\ &x \to x-1 \quad \text{at rate} \quad \beta_1\,\xi_t(x) + \beta_0\,[1-\xi_t(x)], \end{aligned} \tag{3}$$

where

$$\alpha_0, \alpha_1, \beta_0, \beta_1 \in (0,\infty), \tag{4}$$

i.e., on occupied (resp. vacant) sites the random walk jumps to the right at rate α_1 and to the left at rate β_1 (resp. α_0 and β_0).

The literature on RWDRE is still modest (for a recent overview, see Avena [2], Chapter 1). Several results are available are available when the random environment exhibits uniform and fast enough mixing. For example, in Avena, den Hollander and Redig [3] a strong law of large numbers (SLLN) was proved under a mixing condition called *cone-mixing*. Roughly speaking, this is the requirement that for every $m > 0$ all states inside the space-time cone (see Fig. 1)

$$\mathrm{CONE}_t := \big\{(x,s) \in \mathbb{Z}^d \times [t,\infty)\colon\ \|x\| \le m(s-t)\big\}, \tag{5}$$

are asymptotically independent of the states at time zero in the limit as $t \to \infty$. The proof of the SLLN in [3] uses a *regeneration-time* argument.

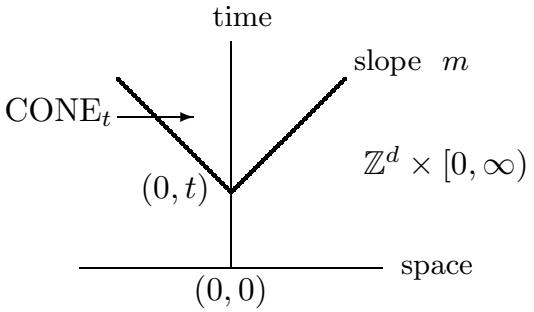

FIGURE 1. The cone defined in (5).

Another example is Redig and Völlering [7], where a LLN and a CLT are proved under more restrictive mixing conditions, requiring in particular uniform polynomial mixing with a high enough degree.

Many interacting particle systems exhibit uniform and fast mixing (e.g. cone-mixing), including spin-flip systems with spin-flip rates that are weakly dependent on the configuration, e.g. the stochastic Ising model above the critical temperature. However, also many interacting particle systems are not cone-mixing, including independent simple random walks, the exclusion process, the supercritical contact process and the voter model. Some attempts have been made to analyze scaling properties of RWDRE for such environments, with partial success. Examples include: independent simple random walks (den Hollander, Kesten and Sidoravicius [5]) and the exclusion process (Avena, dos Santos and Völlering [4]).

Here we consider the *contact process* in the supercritical phase. The latter is the interacting particle system $\xi = (\xi_t)_{t\geq 0}$ on $\Omega := \{0,1\}^{\mathbb{Z}}$ with local transition rates given by

$$\eta \to \eta^x \text{ with rate } \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda\{\eta(x-1) + \eta(x+1)\} & \text{if } \eta(x) = 0, \end{cases} \tag{6}$$

where $\lambda \in (0,\infty)$ and η^x is defined by $\eta^x(y) := \eta(y)$ for $y \neq x$, $\eta^x(x) := 1 - \eta(x)$. We call a site *infected* when its state is 1, and *healthy* when its state is 0. See Liggett [6], Chapter VI, for proper definitions.

The empty configuration $\mathbf{0} \in \Omega$, given by $\mathbf{0}(x) = 0$ for all $x \in \mathbb{Z}$, is an absorbing state for ξ, while the full configuration $\mathbf{1} \in \Omega$, given by $\mathbf{1}(x) = 1$ for all $x \in \mathbb{Z}$, evolves towards an equilibrium measure ν_λ that is stationary and ergodic under space-shifts. It is known that there is a critical threshold $\lambda_c \in (0,\infty)$ such that: (1) for $\lambda \in (0,\lambda_c]$, $\nu_\lambda = \delta_{\mathbf{0}}$; (2) for $\lambda \in (\lambda_c,\infty)$, $\rho_\lambda := \nu_\lambda(\eta(0) = 1) > 0$. We say that the system is *supercritical* if $\lambda > \lambda_c$.

In the following, we will assume that

$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1 =: \gamma, \tag{7}$$

and that

$$v_1 > v_0 \text{ with } v_1 := \alpha_1 - \beta_1 \text{ and } v_0 := \alpha_0 - \beta_0, \tag{8}$$

i.e., the jump rate is constant and equal to γ everywhere, while the drift to the right is larger on infected sites than on healthy sites.

Let $\mathbb{P}_{\nu_\lambda}$ denote the joint law of W and ξ when the latter is started from ν_λ. Our results read as follows.

Theorem 1. *Suppose that* (7–8) *hold.*
(a) For every $\lambda \in (\lambda_c,\infty)$ there exists a $v(\lambda) \in [v_0,v_1]$ such that

$$\lim_{t\to\infty} t^{-1} W_t = v(\lambda) \quad \mathbb{P}_{\nu_\lambda}\text{-a.s. and in } L^p,\ p \geq 1. \tag{9}$$

(b) The function $\lambda \mapsto v(\lambda)$ is non-decreasing and right-continuous on (λ_c,∞), with $v(\lambda) \in (v_0,v_1)$ for all $\lambda \in (\lambda_c,\infty)$ and $\lim_{\lambda\to\infty} v(\lambda) = v_1$.

A FCLT and an LDP hold under an additional restriction, namely, $\lambda \in (\lambda_W,\infty)$ with

$$\lambda_W := \inf\big\{\lambda \in (\lambda_c,\infty)\colon\ |v_0| \vee |v_1| < \iota(\lambda)\big\}. \tag{10}$$

Here, $\lambda \mapsto \iota(\lambda)$ is the infection propagation speed (see Liggett [6]), which is known to be continuous, strictly positive and strictly increasing on (λ_c,∞), with $\lim_{\lambda\downarrow\lambda_c} \iota(\lambda) = 0$ and $\lim_{\lambda\to\infty} \iota(\lambda) = \infty$.

Theorem 2. *Suppose that* (7–8) *hold.*
(a) For every $\lambda \in (\lambda_W,\infty)$ there exists a $\sigma(\lambda) \in (0,\infty)$ such that, under $\mathbb{P}_{\nu_\lambda}$,

$$\left(\frac{W_{nt} - v(\lambda)nt}{\sigma(\lambda)\sqrt{n}}\right)_{t\geq 0} \Longrightarrow (B_t)_{t\geq 0} \qquad \text{as } n \to \infty, \tag{11}$$

where B is standard Brownian motion and $\Longrightarrow$ denotes weak convergence in path space.
(b) The functions $\lambda \mapsto v(\lambda)$ and $\lambda \mapsto \sigma(\lambda)$ are continuous on (λ_W, ∞).
(c) For every $\lambda \in (\lambda_W, \infty)$, $(t^{-1}W_t)_{t>0}$ under $\mathbb{P}_{\nu_\lambda}$ satisfies the large deviation principle on $\mathbb{R}$ with a finite and convex rate function that has a unique zero at $v(\lambda)$.

The main ideas in the proofs of these theorems go as follows. For Theorem 1, monotonicity properties of both ξ and W allow us to implement a subadditivity argument to show the LLN for W under $\mathbb{P}_{\mathbf{1}}$. This LLN is then transferred, with the same speed, to W under $\mathbb{P}_{\nu_\lambda}$ by a coupling argument. Theorem 2 is proved with the help of *regeneration times*. These are, roughly speaking, times at which the random walk is "captured" inside an infection cluster generated by a single infection. The main issue is to show that there are enough times at which the regeneration attempt can be made; this is done via a stochastic domination result for the states of ξ to the left of certain random infection paths.

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Directed random walk on an oriented percolation cluster

Nina Gantert

(joint work with Matthias Birkner, Jiri Cerny, Andrej Depperschmidt)

The motivation for the results we report here came from the following question about ancestral processes in mathematical population genetics. Consider a discrete-time variant of the contact process: a $\{0,1\}^{\mathbb{Z}^d}$-valued Markov chain $(\eta_n)_n$ (see below for precise definitions) where $\eta_n(x) = 1$ is interpreted as the event that the site $x \in \mathbb{Z}^d$ is inhabited by a particle in generation n. We can view this contact process as a "toy example" of a spatial stochastic population model with fluctuating population sizes and local dispersal: Sites x can have *carrying capacity* 0 or 1 in a given generation n, and in order for a particle at site x to be present not only must the corresponding site be *open* (i.e. have the carrying capacity 1)

but there must also have been a particle in the neighbourhood of x in the previous generation $n-1$ who put her *offspring* there. If there was more than one particle in the neighbourhood of x in generation $n-1$, we think of randomly assigning one of them to put an offspring to site x. Note that this implicitly models a density-dependent population regulation because particles in sparsely-populated regions will now have a higher chance of actually leaving an offspring.

We will let the carrying capacities be i.i.d. Bernoulli random variables, and consider the process η in the stationary regime. In this regime every living particle at generation 0, say, has an infinite line of ancestors. The question of interest is the distribution of the spatial location of distant-in-time ancestors.

By reversing the time direction, the problem has the following equivalent description. We consider an infinite cluster $\mathcal{C}$ of the oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}$, and a simple directed random walk on this cluster. The "time-slices" of the cluster $\mathcal{C}$ can be seen to be equal in distribution to the time-reversal of the (non-trivial) stationary discrete-time contact process $(\eta_n)_n$, and the directed walk on $\mathcal{C}$ can be interpreted as the spatial embedding of the ancestral lineage of one individual drawn from the equilibrium population. The question posed in the previous paragraph thus amounts to understanding the long time behaviour of this random walk.

In this formulation, the model is of independent interest in the context of the theory of random media: The directed random walk on an oriented percolation cluster can be viewed as a random walk in a Markovian dynamical random environment. The investigation of such random walks is an active research area with a lot of recent progress. The random walk we consider however does not satisfy the usual independence or mixing conditions that appear in the literature; see Remark 3 below. In fact, in our case the evolution of the environment as a process in time is rather complicated.

On the other hand, as a random walk on a random cluster, the model is very natural. The investigation of random walks on percolation clusters is a very active research area as well. An important difference to our model is that usually, the walk can move in all (open) directions, whereas we consider a *directed* random walk.

We now give a precise definition of the model. Let $\omega = \{\omega(x,n) : (x,n) \in \mathbb{Z}^d \times \mathbb{Z}\}$ be a family of independent Bernoulli random variables (representing the carrying capacities) with parameter $p > 0$ on some probability space $(\Omega, \mathring{A}, \mathbb{P})$. We call a site (x,n) *open* if $\omega(x,n) = 1$ and *closed* if $\omega(x,n) = 0$. We say that there is an *open path* from (y,m) to (x,n) for $m \le n$ if there is a sequence $x_m, \dots, x_n$ such that $x_m = y$, $x_n = x$, $||x_k - x_{k-1}|| \le 1$ for $k = m+1, \dots, n$ and $\omega(x_k, k) = 1$ for all $k = m, \dots, n$. In this case we write $(x,m) \to (y,n)$.

Given a set $A \subset \mathbb{Z}^d$ we define the *discrete time contact process* $(\eta_n^A)_{n \ge m}$ starting at time $m \in \mathbb{Z}$ from the set A as such that $\eta_n^A(y) = 1$ if and only if there is an open path from (x,m) to (y,n) for some $x \in A$. Often we will identify the configuration

η_n^A with the set $\{x \in \mathbb{Z}^d : \eta_n^A(x) = 1\}$. Taking $m = 0$, we set

$$\tau^A = \inf\{n \geq 0 : \eta_n^A = \emptyset\}, \tag{1}$$

and in the case $A = \{0\}$ we write τ^0.

It is well known [8] that there is a critical value $p_c \in (0,1)$ such that $\mathbb{P}(\tau^0 = \infty) = 0$ for $p \leq p_c$ and $\mathbb{P}(\tau^0 = \infty) > 0$ for $p > p_c$. In the following we consider only the supercritical case $p > p_c$. In this case the law of $\eta_n^{\mathbb{Z}^d}$ converges weakly to the upper invariant measure which is the unique non-trivial extremal invariant measure of the discrete-time contact process. By taking $m \to -\infty$ while keeping $A = \mathbb{Z}^d$ one obtains the stationary process

$$\eta = (\eta_n)_{n\in\mathbb{Z}} = (\eta_n^{\mathbb{Z}^d})_{n\in\mathbb{Z}}. \tag{2}$$

We interpret the process η as a population process, where $\eta_n(x) = 1$ means that the position x is occupied by an individual in generation n. We are interested in the behaviour of the "ancestral lines" of individuals. Note that because of the discrete time, there can in principle be several individuals alive in the previous generation that could be ancestors of a given individual at site y, namely all those at some y' with $||y' - y|| \leq 1$. In that case, we stipulate that one of these potential ancestors is chosen uniformly at random to be the actual ancestor, independently of everything else in the model.

Due to time stationarity, we can focus on ancestral lines of individuals living at time 0. It will be notationally convenient to time-reverse the stationary process η and consider the process $\xi = (\xi_n)_{n\in\mathbb{Z}}$ defined by $\xi_n(x) = 1$ if $(x,n) \to \infty$ (i.e. there is an infinite directed open path starting at (x,n)) and $\xi_n(x) = 0$ otherwise. Note that indeed $\mathcal{L}((\xi_n)_{n\in\mathbb{Z}}) = \mathcal{L}((\eta_{-n})_{n\in\mathbb{Z}})$. We will from now on consider the forwards evolution of ξ as the "positive" time direction.

On the event $B_0 = \{\xi_0(\mathbf{0}) = 1\}$ there is an infinite path starting at $(\mathbf{0}, 0)$. We define the oriented cluster by

$$\mathcal{C} = \{(x,n) \in \mathbb{Z}^d \times \mathbb{Z} : \xi_n(x) = 1\}$$

and let

$$U(x,n) = \{(y, n+1) : ||x - y|| \leq 1\} \tag{3}$$

be the neighbourhood of the site (x,n) in the next generation. On the event B_0 we may define a $\mathbb{Z}^d$-valued random walk $X = (X_n)_{n\geq 0}$ starting from $X_0 = \mathbf{0}$ with transition probabilities

$$\mathbb{P}(X_{n+1} = y | X_n = x, \xi) = \begin{cases} |U(x,n) \cap \mathcal{C}|^{-1}, & \text{when } (y, n+1) \in U(x,n), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Note that $(X_n, n)_{n\geq 0}$ is a directed random walk on the percolation cluster $\mathcal{C}$, and X can be also viewed as a random walk in a (dynamical) random environment, where the environment is given by the process ξ. As the environment ξ is the time-reversal of the stationary Markov process η, it is itself Markovian and stationary, the invariant measure being the upper invariant measure of the discrete-time contact process η. While the evolution of η is easy to describe forwards in

time by local rules, η is not reversible, and the time evolution of its reversal ξ seems complicated. The transition probabilities for ξ cannot be described by local functions: For example, when viewed as a function of $a = (a_x)_{x\in\mathbb{Z}^d} \in \{0,1\}^{\mathbb{Z}^d}$, $f(a) = \mathbb{P}(\xi_{n+1}(0) = 1|\xi_n = a)$, there will be no finite set $K \subset \mathbb{Z}^d$ such that f depends only on $(a_x)_{x\in K}$ (this can for example be seen by considering various a's that have a "sea of 0's around the origin"). Presently it is not at all clear to us how to describe the forwards in time evolution of ξ in a more tangible way. Note however, that the process ξ does form a finite-range Markov field in the larger space $\mathbb{Z}^d \times \mathbb{Z}$ because this is true for η, but it is unclear at the moment what use we could make of that fact.

The complicated nature of ξ disallows checking many of the usual conditions that appear in the literature on random walks in dynamical random environment. Some of such conditions (like e.g. ellipticity) are obviously violated by our model. To our best knowledge, the random walk in a (dynamic) random environment that we consider here is not contained in one of the classes studied in the literature so far; cf Remark 3 below.

Our first result shows the law of large numbers, and a central limit theorem for X when averaging over both the randomness of the walk's steps and the environment ω. We write P_ω for the conditional law of $\mathbb{P}$, given ω, and E_ω for the corresponding expectation. With this notation we have $\mathbb{P}(X_{n+1} = y|X_n = x, \xi) = P_\omega(X_{n+1} = y|X_n = x)$.

Theorem 1 (LLN & annealed CLT)**.** *For any $d \geq 1$ we have*

$$P_\omega\Big(\frac{1}{n}X_n \to 0\Big) = 1 \quad \text{for } \mathbb{P}(\,\cdot\,|B_0)\text{-a.a. } \omega, \tag{5}$$

and for any $f \in C_b(\mathbb{R}^d)$

$$\mathbb{E}\Big[f\left(X_n/\sqrt{n}\right)\,\Big|\,B_0\Big] \xrightarrow{n\to\infty} \Phi(f), \tag{6}$$

where $\Phi(f) = \int f(x)\,\Phi(dx)$ with Φ a non-trivial centred isotropic d-dimensional normal law.

We prove this theorem by exhibiting a regeneration structure for X and ξ, and then showing that the second moments of temporary and spatial increments of X at regeneration times are finite (in fact we will prove existence of exponential moments).

It is natural to study also two (or even more) walkers on the same cluster. On the one hand, this allows to obtain information on the long-time behaviour in a multi-type situation. [15] and more recently [20] employed this for the (continuous-time) contact process. It is also very natural from the modelling of ancestral lineages point of view, where it corresponds to jointly describing the space-time embedding of the ancestry of a sample of size two (or more) individuals when the walks start from different sites. On the other hand, good control of the behaviour of two or more "replicas" of X on the same cluster allows us to strengthen the annealed CLT (6) to the quenched version.

Theorem 2 (Quenched CLT). *For any $d \geq 1$ and $f \in C_b(\mathbb{R}^d)$*

$$E_\omega\left[f\left(X_n/\sqrt{n}\right)\right] \xrightarrow{n\to\infty} \Phi(f) \quad \text{for } \mathbb{P}(\,\cdot\,|B_0)\text{-a.a. } \omega, \tag{7}$$

where Φ is the same non-trivial centred isotropic d-dimensional normal law as in (6).

Let us conclude by mentioning some generalisations of the random walk that we consider here.

Remark 1 (More general neighbourhoods). *For simplicity, we defined $U(x,n)$ as in* (3)*, but the proofs go through for any finite, symmetric neighbourhood (by "symmetric" we mean that $y \in U(x,n)$ if and only if $-y \in U(x,n)$). In this case the resulting law Φ will in general not be isotropic, see the end of the proof of Theorem 1.*

Remark 2 (Contact process with fluctuating population size). *Let $K(x,n)$, $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}$ be i.i.d. $\mathbb{N} = \{1,2,\dots\}$ valued random variables and let us define the* discrete time contact process with fluctuating population size, *$\widehat{\eta} = (\widehat{\eta}_n)_{n\in\mathbb{Z}}$, by*

$$\widehat{\eta}_n(x) = \eta_n(x)K(x,n), \tag{8}$$

and its reversal $\widehat{\xi} = (\widehat{\xi}_n)_{n\in\mathbb{Z}}$ by

$$\widehat{\xi}_n(x) = \xi_n(x)K(x,n). \tag{9}$$

One can interpret $K(x,n)$ as a random "carrying capacity" of the site (x,n). Now conditioned on $\widehat{\xi}_0(\mathbf{0}) \geq 1$ the ancestral random walk, we call it X as before, can be defined by $X_0 = \mathbf{0}$ and

(10)

$$\mathbb{P}\big(X_{n+1} = y | X_n = x, \widehat{\xi}\big) = \begin{cases} \dfrac{\widehat{\xi}_{n+1}(y)}{\sum_{(y',n+1)\in U(x,n)} \widehat{\xi}_{n+1}(y')} & \text{if } (y,n+1) \in U(x,n), \\ 0 & \text{otherwise.} \end{cases}$$

For such random walks in random environments our arguments can also be adapted and the same results as above can be obtained.

Remark 3 (Relation to other approaches to RWDRE in the literature). *Random walks in dynamic random environments (RWDRE) are currently an active area of research (they can of course by explicitly including the "time" coordinate in principle be expressed in the context of random walk in random (non-dynamic) environments, but the often more complicated structure of the law of the environment does make them somewhat special inside this general class). To the best of our knowledge, the walk* (4) *and our results in Theorems 1, 2 are not covered by approaches in the literature. Here is a list of corresponding results so far together with a very brief discussion:*

- [4] *obtain a CLT under "abstract" conditions on the environment process, that appear very hard to verify explicitly for ξ, in particular the absolute continuity condition for ξ viewed relative to the walk w.r.t. the a priori law on ξ.*
- [12] *consider environments that are "refreshed in each step" (i.e. time-slices are i.i.d.), this does not apply to (ξ_n).*
- *Individual coordinates $(\xi_n(x))_{n\in\mathbb{N}}$ with $x \in \mathbb{Z}^d$ fixed are not (independent) Markov chains, in contrast to the set-up in* [5].
- *(ξ_n) does not fulfil the required uniform coupling conditions employed in* [17].
- *(ξ_n) does not fulfil the cone mixing condition considered in* [1].
- [11] *weaken the cone-mixing condition from* [1] *to so-called conditional cone-mixing and obtain a LLN for a class of (continuous-time) random walks in dynamic random environments with this (and some further technical) assumptions. Further research is required to investigate whether a similar condition can be established for ξ but note that presently, the approach in* [11] *does not yield a CLT.*

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Counting self-avoiding path on a supercritical percolation cluster

Hubert Lacoin

The self-avoiding walk on $\mathbb{Z}^d$ has been introduced by Flory and Ott as a natural model for polymers. In spite of the apparent simplicity of the model, mathematicians understanding is very far of being complete, in particular in low dimension (d=2,3,4).

For this reason the disordered version of the model: Self-avoiding walk in a random potential has not received much attention from the mathematical community. On the contrary some conjecture on the model are present in the Physics literature (e.g. [1, 2]), in which some conjectures where formulated.

Our aim is to approach the problem by studying the asymptotic of the partition function. A particular case of interest is the one where the environment is given by supercritical Bernouilli percolation.

Let $(\omega_e)_{e\in E_d}$ be a fixed realization of IID Bernouilli variables indexed by the edges of $\mathbb{Z}^d$, taking value in $\{0,1\}$ and such that

$$\mathbb{P}(\omega_e = 1) = p$$

with $p > p_c$ the percolation threshold. We say that edges are open if $\omega_e = 1$ and that a lattice path is open if all the edges in it are open. Set

$$\mathcal{S}_N := \{(S_n)_{n\in[0,1]} \mid S_0 = 0,\ \forall n \in [0, N-1]\ |S_{n+1} - S_n| = 1, \quad \forall i \neq j \in [0, N], S_i \neq S_j\}. \tag{1}$$

We are interested in the asymptotic growth of

$$Z_N := \sum_{S\in\mathcal{S}_N} \mathbf{1}_{\{S \text{ is open }\}}. \tag{2}$$

The expectation of Z_N is equal to

$$\mathbb{E}[Z_N] := |\mathcal{S}_N| p^N. \tag{3}$$

Thus

$$\limsup_{N\to\infty} (\mathbb{E}[Z_N])^{1/N} = p\mu_d. \tag{4}$$

where μ_d is the connectivity constant of $\mathbb{Z}^d$.

Our result is that on $\mathbb{Z}^2$, Z_N grows exponentially slower than its expectation, indicating localization phenomenon for the corresponding measure.

Theorem 1. *For any d, conditioned on the event that there exists an infinite open path from 0 to infinity,*

$$\limsup_{N\to\infty} (Z_N)^{1/N} := \mu_d(p) \tag{5}$$

exists almost surely and is non-random. Moreover, when $d = 2$ one has for all $p < 1$

$$\mu_2(p) < p\mu(1). \tag{6}$$

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Stability of the Greedy Algorithm on the Circle

Leonardo Trivellato Rolla
(joint work with Vladas Sidoravicius)

Summary. We consider a single-server system with service stations in each point of the circle. Customers arrive after exponential times, at uniformly-distributed locations. The server moves at finite speed and adopts a greedy routing mechanism. It was conjectured by Coffman and Gilbert in 1987 that the service rate exceeding the arrival rate is a sufficient condition for the system to be positive recurrent, for any value of the speed. In this talk we show that the conjecture holds true.

Model and results. In this talk we study a greedy single-server system on the unit-length circle $\mathbb{R}/\mathbb{Z}$. Customers arrive following a Poisson process with rate λ. Each arriving customer chooses a position on $\mathbb{R}/\mathbb{Z}$ uniformly at random and waits for service. If there are no customers in the system, the server stands still. Otherwise, the server chooses the nearest waiting customer and travels in that direction at speed $v > 0$, ignoring any new arrivals. Upon reaching the position of such customer, the server stays there until service completion, which takes a random time T that is independent of the past configurations and has expectation μ^{-1}.

The above system was introduced by Coffman and Gilbert in 1987 [CG87], and since then became a paradigm example of a routing mechanism that depends on the system state. This is the so-called *greedy server*, due to the simple strategy of targeting the nearest customer.

Continuous-space models provide natural approximations for systems with a large number of service stations embedded in a spacial structure, and their description is usually more transparent than the discrete-space formulation, mostly because the latter often is obscured by combinatorial aspects. However, systems with greedy routing strategies in the continuum are extremely sensitive to microscopic perturbations, and their rigorous study represents a mathematical challenge.

It was conjectured in [CG87] that the greedy server on the circle should be a stable system when $\lambda < \mu$, for any $v > 0$. Since then, a number of related models have been proposed and studied. Stability was verified only under light-traffic assumptions, i.e., for λ and μ fixed and v large enough, and for the greedy server on a discrete ring $\mathbb{Z}/n\mathbb{Z}$. However, these approximations were unable to identify and tackle the main difficulty of this system, which is is due to the interplay between the server's motion and the environment of waiting customers that surround it. This interplay is given by the interaction resulting from the choice of the next customer and the removal of those who have been served. In this talk we prove stability for the greedy server.

Theorem 1. *Suppose that the distribution of the service time T is geometric, exponential, or deterministic. For any $\lambda < \mu$ and any $v > 0$, the greedy server on the circle is stable.*

For the proof of Theorem 1 we consider a representation for the conditional distribution of the set of waiting customers in terms of a stochastic evolution of profiles. In this framework, the server learns only the information that is necessary and sufficient to determine the next movement, and the positions of further waiting customers remain unknown. In [FRS11] it was shown that the greedy server on the real line is transient, which is an important ingredient in our proof of stability.

In the sequel we review some known results on the greedy server and related models, discuss the problem of self-interaction, describe the approach based on a stochastic evolution of profiles, present a heuristic discussion in order to highlight the main ideas of the proof.

Previous results on the greedy server and related models. Stability was verified for the greedy server on $\mathbb{R}/\mathbb{Z}$ under light-traffic assumptions [KS96], and

for the greedy server on a discrete ring $\mathbb{Z}/n\mathbb{Z}$ [FL96, FL98, MQ99], see below. It was also shown for several related models, including a class of non-greedy policies [KS94], a gated-greedy variant on convex spaces [AL94], and random non-greedy servers on general spaces [AF97]. See [RNFK11] and references therein for a recent review.

The *light-traffic* regime is given by

$$\lambda\left(\frac{1}{\mu}+\frac{1}{2v}\right)<1.$$

This regime was studied in [KS96], particularly the limit $\lambda \to 0$ for which the first terms of Taylor expansions of some performance measurements were computed. A simple coupling argument works for proving stability under light-traffic assumption. In this case $\frac{1}{2v}$ gives an upper bound for the travel time between two consecutive services, since $\frac{1}{2}$ is the largest distance within the unit-length circle. Adding this bound to the service time allows a comparison between the greedy server on $\mathbb{R}/\mathbb{Z}$ and a stable $M/G/1$ system, which proves that the former is stable.

This kind of argument could be pushed down to smaller values of v than the above inequality allows, by obtaining a stochastic upper bound on the distance to the nearest customer better than $\frac{1}{2}$. However, it may not be extended to general $v > 0$, because the presence of the server interferes severely with the conditional distribution of the locations of remaining customers.

On the other hand, stability under the general condition $\lambda < \mu$ is known to hold for the *polling server* on $\mathbb{R}/\mathbb{Z}$, i.e., the server whose strategy is to always travel in the same direction. In [KS92] this fact was proven using a decomposition of the set of waiting customers into a collection of Galton-Watson trees that turn out to be subcritical for $\lambda < \mu$. This decomposition provides a detailed description of the busy cycles (sequence of configurations observed between two consecutive regeneration times) and the stationary state, but if one only wants to prove stability, there is a simple and robust argument. Take $\epsilon < 1 - \frac{\lambda}{\mu}$ and K such that

$$\frac{K}{\mu}+\frac{1}{v}<(1-\epsilon)\frac{K}{\lambda}.$$

The above inequality says that, whenever the number of waiting customers N is larger than K, the time it will typically take to serve the first N customers, including service and travel time, is less than the time it will typically take for the next $N - \epsilon N$ arrivals, resulting in a net decrease by ϵN on the number of waiting customers.

Simulations indicate that under heavy traffic conditions the greedy server dynamics resembles that of the polling server [CG87]. This suggests that a possible strategy for proving stability of the greedy server might be to adapt the above argument. In this case the first step would be to understand its local behavior, and a natural approach is to consider a system on an infinite line. A model on $\mathbb{Z}$ was studied in [KM97], where it is shown that the server is eventually going to move in a fixed random direction.

Some direct attempts also include the study of a greedy server model on the finite ring $\mathbb{Z}/n\mathbb{Z}$, which was shown to be stable in [FL96, FL98, MQ99]. Each of these references provides different arguments under a variety of general assumptions.

Yet, discrete models have not been able to grasp the microscopic nature of the greedy mechanism in continuous space, neither on $\mathbb{Z}$ nor on $\mathbb{Z}/n\mathbb{Z}$, and there are major obstacles in extrapolating any approach based on a discrete approximation. This difficulty is due to the *self-interaction* of the server's path at the *microscopic level*, which takes place because the server's trajectory influences the set of waiting customers and at the same time is determined by the latter.

Self-interacting motions. The main difficulty in studying greedy server systems in continuous spaces is due to the interplay between the server's motion and the environment of waiting customers that surround it. This interplay is given by the interaction resulting from the greedy choice of the next customer and the removal of those who have been served. The server's path is *self-repelling*, since the removal of already served customers makes it less likely for the greedy server to take the next step back into the recently visited regions.

In some well-known examples of self-repelling motions, the self-interaction comes from an explicit prescription of the distribution of next step in terms of the past occupation times. For the excited random walks [BW03], perturbed Brownian motions [CPY98, CD99, Dav96, Dav99, PW97], and excited Brownian motions [RS11], whenever there is a drift, it is pushing the motion in a certain fixed direction. For the random walk avoiding its past convex hull [ABV03, Zer05] and the prudent walk [BFV10, BM10], there is a growing forbidden region containing the previous trajectory, which strongly pushes the motion outwards.

For our greedy server, and also for the true self-avoiding walk [Tót95, Tót99], the true self-repelling motion [TW98], and the Brownian motion with repulsion [MT08], there is a mixture of information, and "self-repulsion" does not immediately imply "repulsion towards ∞", since the particle is allowed to cross its past path, receiving contradictory signals from its left and right-hand sides. In fact, some of the latter models are recurrent and some are transient.

It was clear since these models were introduced that they could not be treated via standard methods and tools. Still nowadays, a lot remains to be understood even in dimension $d = 1$, and, despite the existence of a few disconnected techniques that have proved useful in particular situations, this rich field of study lacks a systematic basis.[1]

The greedy server model has two particular features. Unlike most of the above models, here there is no direct prescription of how the past trajectory influences the future in terms of occupation times. Moreover, this evolution is time-inhomogeneous in the sense that customers keep accumulating, which yields an

[1]Except for the family of universality classes given by the Schramm-Löwner Evolutions [Sch00], which include 2-dimensional loop-erased random walk [LSW04a] and several other models [LSW04b, Smi01, Smi10].

increasing bias towards least recently explored regions with decreasing traveled inter-distances after each service.

Stochastic evolution of profiles. To address the issues mentioned in the above subsection, we consider a representation of the customers environment which reflects its randomness as perceived by the server.

More precisely, we only want to learn the information that is necessary and sufficient to determine the next movement, and the positions of further waiting customers should remain unknown. Each time the server has to scan the system state to determine the position of the next target, we acquire exactly two pieces of information: the presence of a customer at that position and the absence of any other customer at smaller distances.

The arrivals are represented by a space-time Poisson Point Process $\nu \subseteq (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, and in this approach one is ignoring the points of ν that have not yet influenced the server's trajectory. One can think of this scheme as re-sampling the set of waiting customers at each departure time, according to the appropriate conditional distribution. The latter is given in terms of the space-time region where the configuration ν has not been revealed. In this setting the state of the system is given by the positions of the server and the current customer, plus the profile corresponding to the boundary of this region where ν is unknown. The knowledge of this triplet determines the distribution of its future without the need of any further information from the past, yielding a Markovian evolution.

Heuristics. If the server is busy most of the time, the system must be stable, since in average the service time is smaller than the inter-arrival time. The fundamental problem in showing stability is therefore the possibility that the server spend a long time zigzagging on regions with low density of customers, due to a trapping configuration produced by the stochastic dynamics.

For the analogous model on the real line this cannot be the case: the server may zigzag for a finite period of time, but it is bound to eventually choose a direction and head that way [FRS11].

On the same grounds, since the greedy routing mechanism is *local*, this can neither be the case on the circle – at least *until the server realizes that it is not operating on the infinite line.*

Suppose we are given a configuration where the circle is crowded of waiting customers, and, from this point on, our goal is to alleviate this situation. We would like to say that, with high probability, after a short time the server will choose a direction and then cope with its workload as the polling server would.

There are two situations where the server may feel that it is on the circle rather than on the line. First, if it arrives at a given point x for the second time after performing a whole turn on the circle, it will encounter an environment that has been affected by its previous visit. This is not a serious problem, because if it happens it will imply that all the customers which were initially present will have then been served, and typically the server will have served more customers than new ones will have arrived.

The second difference is what poses a real issue. The server has a tendency to go into regions that have been *least recently visited*, since in these regions the average interdistance between customers is smaller, and they have bigger chance to attract the server via its greedy mechanism. This is indeed how transience is proved on $\mathbb{R}$. Let us call the *age* of a point in space the measurement in time units of how recently it was visited by the server in the past. On the line, the age is minimal at the server's position, and *increases as we go further away from the server*. The new regions encountered thus become older and older, and the server surrenders to the fact that the cleared regions it is leaving behind cannot compete with the old regions ahead.

However, this is not true on the circle: the age profile cannot increase indefinitely. This gives rise to the possibility of the following tricky scenario. Imagine that on a tiny region around some point x the system is much older than on any close neighborhood. When the server enters this region, it will take a very long time to finish with all the waiting customers. After finishing with all these customers tightly packed in space, there will no longer be a strong difference between the ages ahead and behind the server, who may end up going back to the region that has just been cleared, invalidating the argument.

We deal with this difficulty by making two key observations. First, the age of the points on the circle is monotone in some sense: there is only one local minimum, located at the server's position, and one local maximum x, and the age increases as we move from the server towards x. Second, if the above scenario effectively happens and the server changes direction, the new configuration may become worse in terms of the number of waiting customers, but will be better in the sense that this sharp peak in the age profile has been flattened. In order to say that the new configuration is "better" in this situation, we need to quantify "badness" taking into account a trade-off between diminishing the overall workload and leveling this singular region with excessively high concentration. This is achieved by considering a Lyapunov functional that combines the total number of customers and the maximum local density.

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Renormalisation of hierarchically interacting Cannings processes

Frank den Hollander

(joint work with Andreas Greven, Sandra Kliem and Anton Klimovsky)

In order to analyse universal patterns in the large space-time behaviour of interacting multi-type stochastic populations on countable geographic spaces, a key approach has been to carry out a renormalisation analysis in the hierarchical mean-field limit. This has provided considerable insight into the structure of interacting systems of finite-dimensional *diffusions*, such as Fisher-Wright or Feller diffusions, and their infinite-dimensional analogues, such as Fleming-Viot or Dawson-Watanabe superdiffusions.

The present talks brings a new class of interacting *jump processes* into focus. We start from a single-colony C^{Λ}-process, which arises as the continuum-mass limit of a Λ-Cannings individual-based population model, where Λ is a finite non-negative measure that describes the offspring mechanism, i.e., how individuals in a single colony are replaced via resampling. The key feature of the Λ-Cannings individual-based population model is that the offspring of a single individual can be a positive fraction of the total population. After that we introduce a system of *hierarchically interacting* C^{Λ}-processes, where the interaction comes from migration and reshuffling-resampling on *all* hierarchical space-time scales *simultaneously*. More precisely, individuals live in colonies labelled by the hierarchical group Ω_N of order N, and are subject to *migration* based on a sequence of migration coeffients $\underline{c} = (c_k)_{k\in\mathbb{N}_0}$ and to *reshuffling-resampling* based on a sequence of resampling measures $\underline{\Lambda} = (\Lambda_k)_{k\in\mathbb{N}_0}$, both acting in k-blocks for all $k \in \mathbb{N}_0$. The reshuffling is linked to the resampling: before resampling in a block takes place all individuals in that block are relocated uniformly, i.e., resampling is done in a locally "panmictic" manner. We refer to this system as the $C_N^{\underline{c},\underline{\Lambda}}$-process. The dual process of the C^{Λ}-process is the Λ-coalescent, whereas the dual process of the $C_N^{\underline{c},\underline{\Lambda}}$-process is a spatial coalescent with multi-level block coalescence.

For the above system we carry out a *full renormalisation analysis* in the *hierarchical mean-field limit* $N \to \infty$. Our main result is that, in the limit as $N \to \infty$, on each hierarchical scale $k \in \mathbb{N}_0$ the k-block averages of the $C_N^{\underline{c},\underline{\Lambda}}$-process converge to a random process that is a superposition of a C^{Λ_k}-process and a Fleming-Viot process, the latter with a volatility d_k and with a drift of strength c_k towards the limiting $(k+1)$-block average. It turns out that d_k is a function of c_l and Λ_l for all $0 \le l < k$. Thus, it is through the volatility that the renormalisation manifests itself. We investigate how d_k scales as $k \to \infty$, which requires an analysis of compositions of certain Möbius-transformations, and leads to four different *regimes*.

We discuss the implications of the scaling of d_k for the behaviour on large space-time scales of the $C_N^{\underline{c},\underline{\Lambda}}$-process. We compare the outcome with what is known from the renormalisation analysis of hierarchically interacting Fleming-Viot diffusions, pointing out several new features. In particular, we obtain a *new classification* for when the process exhibits *clustering* (= develops spatially expanding mono-type regions), respectively, exhibits *local coexistence* (= allows for different types to live next to each other with positive probability). Here, the simple dichotomy of recurrent versus transient migration for hierarchically interacting Fleming-Viot diffusions, namely, $\sum_{k\in\mathbb{N}_0}(1/c_k) = \infty$ versus $< \infty$, is replaced by a dichotomy that expresses a trade-off between migration and reshuffling-resampling, namely, $\sum_{k\in\mathbb{N}_0}(1/c_k)\sum_{l=0}^{k}\Lambda_l([0,1]) = \infty$ versus $< \infty$. Thus, while recurrent migrations still only give rise to clustering, there now are transient migrations that do the same when the block resampling is strong enough, namely, $\sum_{l\in\mathbb{N}_0}\Lambda_l([0,1]) = \infty$. Moreover, in the clustering regime we find a richer scenario for the *cluster formation* than for Fleming-Viot diffusions. In particular, there are four universality classes, in which superdiffusive, diffusive and subdiffusive clustering occur. Finally, we show that for finite N the same dichotomy between clustering and local coexistence holds as for $N \to \infty$, even though we lack proper control on the cluster formation, respectively, on the distribution of the types that survive.

A truly pathwise approach to polymer localization

Francis Comets

Directed polymers are Gibbs measure on path space for d-dimensional Random Walk or Brownian Motion, interacting with a random environment: the path picks up rewards along its way, which are random, space-time dependent. They are positive temperature version of (oriented) last passage percolation. As such, they are expected to belong to the same universality class. In dimension $1 + d = 1 + 1$, this is the KPZ universality class, on which important results where obtained recently. Localization for polymers is reminiscent of random geodesics in first passage percolation. In joint works Mike Cranston and Nobuo Yoshida. we can analyse it for two models: the Parabolic Anderson Model, and Brownian Motion in Poisson Medium.

Parabolic Anderson Model: one starts with the law P_κ of the Simple Random Walk $X = (X(t), t \geq 0)$ in $\mathbb{Z}^d$ with jump rate $\kappa > 0$. Random environment consists in independent standard Brownian motions $(W_x(\cdot))_{x\in\mathbb{Z}^d}$ Anderson polymer model. For a given environment, the Gibbs measure is the probability measure on the path space defined by

$$\mu_T(f) = \frac{1}{Z_T} E_\kappa \left[f(X) \exp\left\{ \beta \int_0^T dW_{X(s)}(s) \right\} \right]$$

for $f : \mathcal{D}_T \to \mathbb{R}$. ($T > 0$= time horizon.)

For fixed environment W, define the favourite path y_T^* for the polymer with time horizon T as the function

$$y_T^*(t) = \arg\max \left\{ \mu_T(X(t) = y); y \in \mathbb{Z}^d \right\}$$

We prove that, in the parameter region where the quenched Lyapunov exponent $\lim_{T\to\infty} T^{-1}\mathbb{E} \ln Z_T$ is strictly smaller than the annealed one $\lim_{T\to\infty} T^{-1} \ln \mathbb{E} Z_T$, and when the difference is strictly increasing in absolute value, then

$$\liminf_{T\to\infty} \mathbb{E}\mu_{\kappa,\beta,T} \left(\frac{1}{T} \int_0^T \delta_0(X(t) - y_T^*(t))dt \right) \geq C > 0.$$

In words, the polymer path has a positive contact fraction with the so-called favourite path.

As β^2/κ diverges, we obtain that the lim inf in the above left-hand side tends to 1. This is called complete localization, since the contact fraction achieves its maximal value.

Brownian motion in Poissonian medium: In this model, X is a Brownian motion in $\mathbb{R}^d$ and the random environment is given by a space and time Poisson field η with intensity $\nu > 0$ times Lebesgue measure in dimension $d+1$. Now the polymer measure is

$$d\mu_t = (Z_T)^{-1} \exp\left(\beta\eta\{(s,x) : s \in [0,T], |x - X(s)| \leq 1\}\right) dP,$$

Similar results are obtained, with complete localization as $\beta^2\nu$ diverges with bounded β.

For the models the starting point is a control of the overlap between two random polymers which are independent except for they share the same environment. Such an estimate is obtained via an integration by parts formula.

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Scaling Limits of Directed Polymer Models

Tom Alberts

(joint work with Konstantin Khanin, Jeremy Quastel)

Directed polymers in a random environment are models of motion through an inhomogeneous random medium. The model is motivated by polymer formation in chemical solutions, where the charge of the solution affects the polymer geometry in a complicated way. The random environment is meant to model the charge distribution in a statistically homogeneous way.

In a purely mathematical sense, a directed polymer can be viewed as a random variable taking values in a product space of environments and paths. Let (ω, S)

denote such an environment-path pair. At the discrete level, the environment is a functions $\omega : \mathbb{N} \times \mathbb{Z}^d \to \mathbb{R}$ where the $\mathbb{N}$ coordinate is time and the $\mathbb{Z}^d$ coordinate is space. The path is a continuous function from $[0, \infty)$ (or some subset) to $\mathbb{R}$. We restrict our attention to the case when S is a path of random walk started from the origin. At inverse temperature $\beta > 0$ the pair is chosen according to the measure

$$\frac{1}{Z_n^\omega(\beta)} \exp\left\{\beta \sum_{i=1}^{n} \omega(i, S_i)\right\} dQ(\omega)\, dP(S), \tag{1}$$

where under Q the field $\{\omega(i,x)\}_{i\in\mathbb{Z}, x\in\mathbb{Z}^d}$ are i.i.d. random variables, and P is the simple random walk measure on paths. The partition function $Z_n^\omega(\beta)$ normalizes the quenched path measure

$$dP_{n,\beta}^\omega(S) := \frac{1}{Z_n^\omega(\beta)} \exp\left\{\beta \sum_{i=1}^{n} \omega(i, S_i)\right\} dP(S)$$

to have total mass one. Under this joint measure the marginal distribution of the environment is an i.i.d. collection of random variables, and the conditional distribution given the environment is simply $P_{n,\beta}^\omega$.

Polymer Scaling Limits

A scaling limit of a polymer measure can be thought of as applying an n-dependent scaling transformation to the random element (ω, S) to produce a new pair (ω_n, S_n), and then taking a limit of the law of (ω_n, S_n). The limit should be a random element (W, X) in the space of continuum environments on $[0,1] \times \mathbb{R}$ and continuous paths on $[0,1]$. This can be viewed as both an analogue and an extension to the standard Donsker's theorem, where the transform is a diffusive scaling of space and time and the resulting S_n has Brownian motion as a distributional limit. In the polymer case, however, one must also keep track of the scaling limit of the environment, and this requires more work. One must decide in what space the environment variables is taking values, and then put an appropriate topology on that space so as to guarantee convergence. Both the choice of the space and the topology require important consideration.

For $d = 1$ and $\beta > 0$ recent work [ACQ10] suggests that it might be possible to take distributional limits. If they exist, they would be new processes of central importance in probability theory. The limit would be an analogue of Brownian motion, is that it would be a continuous time diffusion taking values in continuum space. It would likely be a sort of universal scaling limit that is very different from Brownian motion, and its properties would likely reveal many properties of the discrete polymer.

Intermediate Disorder Regime

Recent work [AKQ12], in conjunction with [AKQ12a], describes one situation in which it is possible to take as scaling limit of the pair (ω_n, S_n). The pair is

defined by

$$\omega_n(t,x) := n^{-3/4}\omega(nt, x\sqrt{n}), \quad S_n(t) := n^{-1/2}S(nt),$$

which corresponds to a diffusive scaling of space and time with the extra $n^{-3/4}$ scaling on the environment. In the topology of the space of distributions, the field ω_n converges to a space-time white noise on $[0,1]\times\mathbb{R}$, whose properties are well known. The scaled path S_n converges to a continuous diffusion $X : [0,1]\times\mathbb{R}$, whose law depends on β and the white noise. This diffusion is called the continuum random polymer, and it in fact gives a one-parameter family of laws. Under this law the paths are continuous, with Holder exponent $1/2-\epsilon$ for every ϵ, and accumulate quadratic variation at rate 1. However they are actually singular with respect to Brownian motion. As random objects they should belong to the KPZ [KPZ86] universality class. As β varies they are conjectured to act as a bridge between the Brownian motion process and what is conjectured to be the scaling limit of last passage percolation.

The Convergence Diagram

The diagram indicates the current status of the scaling limit problem for 1+1-dimensional directed polymers. The top level $\mathbb{P}_{n,\beta}$ represents the law of a pair (ω, S) distributed according to (1). Scaling limits are found by taking $n\to\infty$, in conjunction with applying an n-dependent scaling transformation of space and time. The downward arrows on the left and right correspond to taking $n\to\infty$ with β fixed at $\beta = 0$ or $\beta = \infty$, respectively. It remains a big open problem to determine if there is a scaling limit on the right. The remaining downward arrows correspond to taking scalings with β varying with the size of n, the so-called near-critical scalings. With $\beta_n = \beta n^{-1/4}$ one accesses the intermediate disorder regime, which gives a one-parameter family of measures that are conjectured to act as a crossover (or a bridge) between Brownian motion and the limit on the right, if it exists. Studying the properties of this bridge is another interesting open problem.

The left sloping diagonal arrow is the study of environment scalings with multiplicative factors that are faster than $n^{-1/4}$, and it is known as a result of [AKQ12] that under these scalings the pair (ω_n, S_n) scales to regular Brownian motion. For environment scalings slower than $n^{-1/4}$ it remains an open question to determine if a distribution limit of the pair exists, and if it does whether it is the same as the limit for last passage percolation.

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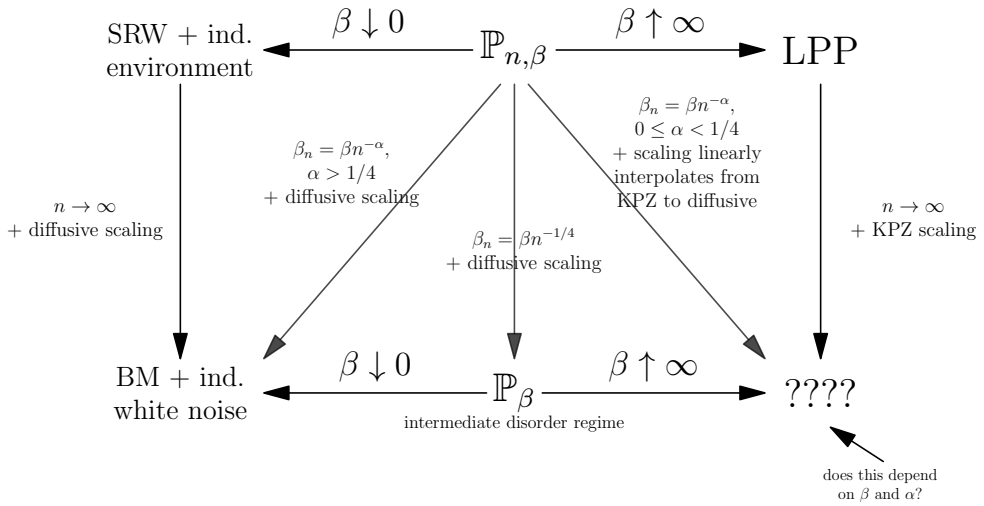

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The local limit of the minimum weight spanning tree of the complete graph

Louigi Addario-Berry

1. Introduction

In this abstract, we outline the main results a longer work, currently in preparation, which proves the existence of the local weak limit of the minimum weight spanning tree of a randomly-weighted complete graph. We briefly provide a more detailed description of the object of study, before proceeding to the statement of our results.

Let K_n = be the complete graph on vertices $\{1, \dots, n\}$. Let $W_n = \{U_e, e \in E(K_n)\}$ be independent Uniform$[0,1]$ edge weights, and let $\mathcal{T}_n$ be the resulting *minimum weight spanning tree* of K_n: this is the connected graph G on $\{1, \dots, n\}$ minimizing the total weight $\sum_{e \in E(G)} U_e$. The random finite tree $\mathcal{T}_n$, and its $n \to \infty$ local limit $\mathcal{T}$ (for which we show existence and give an explicit distributional construction), are the main objects of interest in this abstract.

1.1. **Prim's algorithm.** We briefly recall *Prim's algorithm*, a standard algorithm for constructing the minimum spanning tree of a finite connected graph $G = (V, E)$ with distinct, non-negative edge weights $w = (w_e, e \in E)$

Prim's algorithm.
Fix a starting vertex $v \in V$. At each step, consider all edges joining the component currently containing v with its complement, and from among these add the unique edge of smallest weight. Stop when all vertices are connected.

Both of these procedures build the (unique) minimum spanning tree $T = (V, E')$ of G.

1.2. **Local weak convergence.** In this section we briefly recall the notion of local weak convergence. For unweighted graphs, such convergence was introduced by Benjamini and Schramm [6]. The formulation we use here appears in Aldous and Steele [3)]. A *rooted weighted graph* (RWG) is a triple (G, r, w), where G is a graph with countable vertex degrees, $v \in V(G)$ is a vertex of G, and $w : e(G) \to [0, \infty]$ is a weighting of the edges of G. We say two RWGs (G, v, w) and (G', v', w') are *isomorphic* if there exists an isomorphism $\phi : V(G) \to V(G')$ between G and G' such that $\phi(v) = v'$ and such that for all edges $xy \in e(G)$, $w(xy) = w'(\phi(x)\phi(y))$. If (G, v, w) and (G', v', w') are isomorphic we write $(G, v, w) \simeq (G', v', w')$

We say an RWG (G, v, w) is *locally finite* if, viewing the edge weights w as lengths, for all $u \in V(G)$ and all $r \in [0, \infty)$, the set of vertices of G at distance at most r from u is finite. We next define a semi-metric d_0 on the space of locally finite RWGs by setting

$$d_0((G, v, w), (G', v', w')) = \frac{1}{2^n},$$

where

$$n = \sup \left\{ k : (B_G(v, k), v, w|_{B_G(v,k)}) \simeq (B_{G'}(v', k), v', w'|_{B_{G'}(v',k)}) \right\} .$$

Now let d_{LWC} be the push-forward of d_0 to the set $\mathcal{G}^*$ of isomorphism-equivalence classes of locally finite RWGs. It is straightforward to verify that $(\mathcal{G}^*, d_{\mathrm{LWC}})$ forms a complete, separable metric space, and we refer to convergence in distribution in this metric space as local weak convergence, or sometimes simply as weak convergence.

1.3. **The local weak limit of the early stages Prim's algorithm.** It turns out that, from the perspective of Prim's algorithm, it is natural to view the local structure of the weak limit of $\mathcal{T}_n$ as appearing in two distinct phases. The first phase corresponds to the early steps of Prim's algorithm – the first $s(n)$ steps, say, for some function $s(n)$ tending to infinity arbitrarily slowly. The second phase then corresponds to the last $n(1 - 1/s(n))$ steps of Prim's algorithm. (The preceding description implies that the structure added by Prim's algorithm between steps $s(n)$ and $n(1-1/s(n))$ does not contribute to the structure of the local weak limit of $\mathcal{T}_n$, a fact which we prove in the extended version of this abstract.) Write for $1 \le k \le n$ write $\mathcal{T}_n(k)$ for the subtree of $\mathcal{T}_n$ containing 1 and built by the first k steps of Prim's algorithm on $\mathcal{T}_n$, so in particular $\mathcal{T}_n(1)$ is a single vertex and $\mathcal{T}_n(n) = \mathcal{T}_n$. The paper [1] describes the weak limit of the early steps of Prim's algorithm, and we now summarize this description.

As is the case for many random combinatorial optimization problems on the complete graph, the local weak limit is naturally described in terms of the *Poisson–weighted infinite tree* (PWIT) of Aldous and Steele [3)]. The PWIT is the following random RWG. Let $\mathcal{U}$ be the Ulam–Harris tree; this is the tree with vertices $V = \bigcup_{n \ge 0} \mathbb{N}^n$ (write $\mathbb{N}^0 = \{\emptyset\}$), and for eack $k \ge 1$ and each vertex $v = (n_1, \ldots, n_k) \in$

$\mathbb{N}^k$, an edge between v and its parent $(n_1, \ldots, n_{k-1})$. (If $k = 1$ then the parent of v is the root vertex $\emptyset$.) Independently for each $v = (n_1, \ldots, n_k) \in V$, let $(w_i, i \geq 1)$ be the atoms of a homogenous rate one Poisson process on $[0, \infty)$, and for each $i \geq 1$ give the edge from v to its child $v' = (n_1, \ldots, n_k, i)$ the weight $W_{\{v,v'\}} = w_i$. Writing $W = \{W_e, e \in E\}$, where E is the edge set of $\mathcal{U}$, the Poisson–weighted infinite tree is (a random RWG with the distribution of) the triple $(\mathcal{U}, \emptyset, W)$.

For $k \geq 1$, write $\mathcal{T}(k)$ for the subtree of $\mathcal{U}$ built by the first k steps of Prim's algorithm on $(\mathcal{U}, \emptyset, W)$, started at $\emptyset$, and write $\mathcal{T} = \lim_{k\to\infty} \mathcal{T}(k)$. The following observation appears in [1] (see the remark just after Theorem 27 of [1], together with the remark in Section 1.1 of the same paper)

Proposition 1.1 ([1]). *Fix any function $f(n) : \mathbb{N} \to \mathbb{N}$ such that $f(n) \to \infty$ and $f(n) = o(n^{1/2})$. Then for each n we may couple $(\mathcal{T}_n(f(n)), 1, W_n)$ and $(\mathcal{T}(f(n)), \emptyset, W)$ so that*

$$\mathbf{P}\left\{(\mathcal{T}_n(f(n)), 1, W_n) \neq (\mathcal{T}(f(n)), \emptyset, W)\right\} \to 0$$

as $n \to \infty$. In particular, $(\mathcal{T}_n(f(n)), 1, W_n) \xrightarrow{\mathrm{d}} (\mathcal{T}, \emptyset, W)$ in the local weak sense.

The fact that the first assertion of the proposition implies distributional convergence is because, since $(\mathcal{T}, \emptyset, W)$ is almost surely locally finite, necessarily $(\mathcal{T}(k), \emptyset, W) \xrightarrow{\text{a.s.}} (\mathcal{T}, \emptyset, W)$ in the local weak sense, as $k \to \infty$. We will in fact require the same result holds as long as $f(n) = o(n)$, and prove this the extended version of this abstract.

The following explicit description of the distribution of $(\mathcal{T}, \emptyset, W)$ is given by Theorem 27 of [1]. Say that an edge $e \in E(\mathcal{T})$ is a *forward maximal edge* if the removal of e separates $\emptyset$ from infinity and if, for any other edge $e' \in E(\mathcal{T})$, if the path from e' to $\emptyset$ contains e then $W(e') < W(e)$. Write $S_1 = \emptyset$, and list the forward maximal edges of $\mathcal{T}$ in increasing order of distance from $\emptyset$ as $(\{R_i, S_{i+1}\}, i \geq 1)$. The removal of all forward maximal edges separates $\mathcal{T}$ into an infinite sequence of random trees $(T_i, i \geq 1)$, where for each $i \geq 1$, R_i and S_i are vertices of T_i.

For each $i \geq 1$, let $Z_i = |V(T_i)|$ and let $X_i = W(\{R_i, S_{i+1}\})$. The sequence $((X_i, Z_i), i \geq 1)$ turns out to be a Markov process, whose distribution we now explain.

For $\lambda \geq 0$, write $\mathrm{PGW}(\lambda)$ to denote a Galton-Watson tree with offspring distribution Poisson(λ). We recall that for $0 < \lambda \leq 1$, the random variable $|\,\mathrm{PGW}(\lambda)|$ has the *Borel–Tanner*(λ) distribution, given by

$$\mathbf{P}\left\{|\,\mathrm{PGW}(\lambda)| = m\right\} = \frac{1}{m}\mathbf{P}\left\{\mathrm{Poisson}(\lambda m) = m - 1\right\} = \frac{1}{m}\frac{e^{-\lambda m}(\lambda m)^{m-1}}{(m-1)!}\,. \tag{1}$$

For $\lambda \geq 1$, write $\theta(\lambda) = \mathbf{P}\left\{|\,\mathrm{PGW}(\lambda)| = \infty\right\}$; the function θ is strictly increasing, infinitely differentiable for $\lambda > 1$, and concave.[1] For $\lambda > 1$ we also write B_λ for a random variable whose distribution is a "truncated, size-biased" analogue of the

[1]The concavity of θ can be verified using the implicit formula $\theta(\lambda) = e^{\lambda(\theta(\lambda)-1)}$.

Borel–Tanner distribution: the distribution of B_λ is given by

$$\mathbf{P}\{B_\lambda = m\} = \frac{\theta(\lambda)}{\theta'(\lambda)} \frac{e^{-\lambda m}(\lambda m)^{m-1}}{(m-1)!}\,. \tag{2}$$

The fact that $\sum_{m\geq 1} \frac{\theta(\lambda)}{\theta'(\lambda)} \frac{e^{-\lambda m}(\lambda m)^{m-1}}{(m-1)!} = 1$, so that the preceding equation indeed defines a probability distribution, is proved in Addario-Berry, Griffiths and Kang, addario12prim, Corollary 29. The justification for the epithets 'truncated' and 'size-biased' for the random variable B_λ, can be found in [1].

By [1], $((X_i, Z_i), i \geq 1)$ is a Markov process taking values in $(1, \infty) \times \mathbb{N}$, with transition kernel κ given by

$$\kappa((x,\ell),(y,m)) = \frac{\theta(y)}{\theta(x)} \frac{e^{-ym}(ym)^{m-1}}{(m-1)!} \mathbf{1}_{[y<x]}\,.$$

In other words, for $1 < y < x \leq \infty$ and $1 \leq \ell, m < \infty$ we have

$$\mathbf{P}\{X_{n+1} \in dy, Z_{n+1} = m \mid (X_n, Z_n) = (x,\ell)\} = \frac{\theta(y)dy}{\theta(x)} \frac{e^{-ym}(ym)^{m-1}}{(m-1)!}\,.$$

Equivalently (see [1], Lemma 28 and Corollary 29), we have

$$\mathbf{P}\{X_{n+1} \in dy \mid X_n = x\} = \frac{\theta'(y)dy}{\theta(x)}\,, \tag{3}$$

and, conditional on X_{n+1}, the random variable Z_{n+1} has distribution $B_{X_{n+1}}$ and is (conditionally) independent of $((X_i, Z_i), i \leq n)$. Furthermore, $X_1 \stackrel{\mathrm{d}}{=} \theta^{-1}(U)$, where U is Uniform$[0,1]$, and, conditional on X_1, the random variable Z_1 is distributed as B_{X_1}. Together with the above transition kernel, this initial distribution specifies the distribution of the whole process $((X_i, Z_i), i \geq 1)$. We remark that for each $i \geq 1$, the distribution of X_{i+1} given that $X_i = x$ is the same as the distribution of X_1 given that $X_1 \leq x$.

Conditional on the sequence $(Z_i, i \geq 1)$, independently for each $i \geq 1$, T_i is distributed as a uniformly random labelled tree with Z_i vertices. (Equivalently, T_i is distributed as PGW(λ) conditioned to have Z_i vertices – this conditional distribution does not depend on $\lambda > 0$; see, e.g., (ref lyons peres book).) Finally, for each $i \geq 1$, conditional on T_i, the vertices R_i and S_i are independent, uniformly random elements of $V(T_i)$, and the weights $\{W(e) : e \in E(T_i)\}$ are independent and uniform on $[0, X_i]$ (recall that $W(\{R_i, S_{i+1}) = X_i$.

The process $(X_i, i \geq 1)$ is sometimes dubbed the *forward maximal process* (for invasion percolation on the PWIT in [1] and for invasion percolation on regular trees in [4]). In this abstract, we instead use this term for the process $((X_i, Z_i), i \geq 1)$ as a matter of convenience.

One of the results we prove shows that $\mathcal{T}$ has almost sure quadratic volume growth.

Theorem 1.2. *As $r \to \infty$,*

$$\frac{\log |B_{\mathcal{T}}(\emptyset, r)|}{\log r} \xrightarrow{\text{a.s.}} 2\,.$$

This result should be compared with [4]. Theorem 1.6, which shows that when $\mathcal{T}$ is the local weak limit of invasion percolation on an infinite d-ary tree, $|B_{\mathcal{T}}(\emptyset, r)|/(dr^2)$ converges in distribution as $r \to \infty$ (and explicitly describe the Laplace transform of the limit). It is also shown in [4] that

$$\limsup_{r\to\infty} \mathbf{E}\left[r^2/|B_{\mathcal{T}}(\emptyset, r)|\right] < \infty.$$

However, we do not see how to deduce Theorem 1.5 from the results of [4].

1.4. **The Poisson Galton–Watson aggregation model.** In this section we describe a stochastic process we term the *Poisson Galton–Watson aggregation model*, which we will use together with the tree $(\mathcal{T}, \emptyset, W)$ of the preceding subsection to describe the local weak limit of $\mathcal{T}_n$.

Fix a finite or infinite, locally finite graph $G = (V, E)$, non-negative weights $x = (x_v, v \in V)$, a non-negative function $f \in \mathcal{L}^1((1, \infty))$, and a measurable function $m : (1, \infty) \to (0, 1)$. We now describe the Poisson Galton–Watson aggregation model with base graph G, start times x, rate f and mean m. Informally, the way to picture this is as follows. At time $t \geq 1$, there are some set of *active* vertices (those vertices with $x_v \leq t$). Once a vertex v is active, Poisson Galton–Watson trees begin to attach themselves at v at rate f; this process happens independently for each active vertex. A tree attaching to v at time s is distributed as $\mathrm{PGW}(m(s))$. When a tree attaches to v, all its vertices immediately become active.

More formally, this is a process $((G(t), x(t)), t \geq 1)$, with $G(t) = (V(t), E(t))$ and $x(t) = (x_v(t), v \in V(t))$. The first coordinate $(G(t), t \geq 1)$ will form an increasing graph process, and the weight vectors $(x(t), t \geq 1)$ will be consistent in the sense that for each $1 \leq s < t$, $x(t)|_{V(s)} \equiv x(s)$.

Consider a vertex v added at some time $t > 1$ – in other words, for which $v \in V(t)$ but $v \notin V(t-) = \bigcup_{t'<t} V(t')$. For each $s \geq t$, we set $x_v(s) = t$. Let $s_v = (s_v^i, i \geq 1)$ be the atoms of a Poisson process on (t, ∞) with rate $f(t)$ at time t. Independently for each $i \geq 1$, let T_v^i be distributed as $\mathrm{PGW}(m(s_v^i))$. Then, at time s_v^i, add an edge from v to the root of T_v^i. In particular, T_v^i is contained in $G(s)$ for each $s \geq s_v^i$. We set $x_w(s) = s_v^i$ for each vertex w of T_v^i and each $s \geq s_v^i$.

Since the graphs $(G(t), t \geq 1)$ are increasing and the vectors $(x(t), t \geq 1)$ are consistent, we may define the limit $G(\infty) = (V(\infty), E(\infty)) = \lim_{t\to\infty} G(t)$, and we define $x(\infty) = (x_v(\infty), v \in V(\infty))$ by setting $x_v = x_v(\infty) = \lim_{t\to\infty} x_v(t)$. For a vertex $v \in V(\infty)$, if $v \notin V(1)$ then the value $x_v = x_v(\infty)$ is the time at which v was added, so $v \in V(x_v)$ but $v \notin V(x_v-)$. If $v \notin V(1)$ then since a random tree attached at time x_v has distribution $\mathrm{PGW}(m(x_v))$, the degree $\deg_{G(x_v)}(v)$ has distribution $\mathrm{Poisson}(m(x_v))+1$, where the 1 accounts for the parent of v. It follows that for any $v \in V(\infty)$, the degree $\deg_{G(\infty)}(v)$ of v in $G(\infty)$ has distribution

$$\deg_{G(\infty)}(v) \stackrel{\mathrm{d}}{=} \begin{cases} \deg_{G(1)}(v) + \mathrm{Poisson}(\int_{x_v}^{\infty} f(t)\mathrm{d}t) & \text{if } v \in V(1) \\ 1 + \mathrm{Poisson}(m(x_v) + \int_{x_v}^{\infty} f(t)\mathrm{d}t) & \text{if } v \notin V(1). \end{cases}$$

Since $f \in \mathcal{L}^1$, it follows that for all $v \in G(1)$, the $\deg_{G(\infty)}(v)$ is finite in expectation and so almost surely; from this it is easily seen that $G(\infty)$ is almost surely locally

finite, and is almost surely finite if $G(1)$ is finite. For the careful reader, an explicit construction of the process $((G(t), x(t)), t \geq 1)$, in the case when G is a tree is straightforward to describe, via the formalism of the Ulam–Harris tree; this will be done in the extended version of the current abstract.

If the initial graph $G = (V, E)$ possesses edge weights $w = (w_e, e \in E)$ in addition to the vertex weights x, we may view the final graph $G(\infty)$ as possessing edge weights $w(\infty) = (w_e(\infty), e \in E(\infty))$ by applying the following rules, independently for each edge. Conditional on the process $(G(t), t \geq 1)$, consider an edge $e = \{u, v\} \in G(\infty)$. If $u \in V(1)$ and $v \in V(1)$ then e already has an edge weight. Otherwise, if $x_u = x_v$ then both u and v lie within a tree T that was attached at time x_v, and we let $w_e(\infty)$ be uniformly distributed on $[0, x_v]$. If $x_u \neq x_v$ (say $x_u < x_v$) then v is a.s. the root of a tree T that was attached at time x_v, and u is its parent; in this case we let $w_e(\infty) = x_v$.

1.5. **The local weak limit of $\mathcal{M}_n$.** We are now prepared for the statements of our main results. Let $(\mathcal{T}, \emptyset, W)$ be as in Section 1.3. Recall that $\mathcal{T}$ is comprised of a collection $(T_i, i \geq 1)$ of trees by adding an edge $R_i S_{i+1}$ from T_i to T_{i+1} for each $i \geq 1$, and that the edge $R_i S_{i+1}$ has weight X_i. We define vertex weights $x = (x_v, v \in V(\mathcal{T}))$ by setting $x_v = X_i$ for each $v \in V(T_i)$.

Next, recall the definition of the "percolation probability" $\theta(x)$ from that Section 1.3. Given $u > 1$, we also define the Poisson Galton–Watson "dual parameter u^*, which is the unique value $\lambda < 1$ for which $ue^{-u} = \lambda e^{-\lambda}$. (Another identity for u^* which we will use later is that $u *^{=} u(1 - \theta(u))$.) It is straightforward to verify that for $u > 1$, if T is has distribution $\mathrm{PGW}(u)$ then the conditional distribution of T, given that t is finite, is $\mathrm{PGW}(u^*)$.

For $t \in (1, \infty)$ let $f(t) = 1 - \theta(t)$, and let $\mathcal{M} = G(\infty)$ be the graph obtained from the Poisson Galton–Watson aggregation model with base graph $(\mathcal{T}, W)$, start times x, rate f and mean m, and let $G(\infty)$ have edge weights $\mathcal{W} = w(\infty)$ as described in the last paragraph of Section 1.4. Note that $\emptyset \in V(\infty)$ since $\emptyset \in V(\mathcal{T}) = V(1)$. The first main result is that $(\mathcal{M}, \emptyset, \mathcal{W})$ is the local weak limit of the minimum spanning tree of $(K_n, 1, W_n)$.

Theorem 1.3. *As $n \to \infty$, we have $(\mathcal{T}_n, 1, W_n) \xrightarrow{\mathrm{d}} (\mathcal{M}, \emptyset, \mathcal{W})$, in the local weak sense.*

We briefly mention some consequences of this result. Theorem 1.3 in particular extends a result of Aldous (ref rsa, theorem 1). There is a.s a unique edge $e \in E(\mathcal{T})$ incident to $\emptyset$ whose removal separates $\emptyset$ from ∞; write $\mathcal{F}$ for the component containing $\emptyset$ after e is removed. Correspondingly, let e_n be the edge of $\mathcal{T}_n$ whose removal minimizes the size of the resulting component containing 1, and write $\mathcal{F}_n$ for this component. Aldous proved that $(\mathcal{F}_n, 1)$ converges in distribution in the local weak sense, to an almost surely finite limit, which in our setting has the distribution of $\mathcal{F}$. This convergence also follows from Theorem 1.3.

The characterization given by Theorem 1.3 is straightforward enough to yield an explicit, though somewhat complicated, description of the distribution of the degree of the root vertex $\emptyset$. First, as above, let U be Uniform$[0, 1]$, let $X_1 =$

$\theta^{-1}(U)$, and given the value of X_1, let Z_1 have be distributed as B_{X_1}, whose distribution is given in (2).

For integer $m \geq 1$, let ν_m be the distribution of the degree of node 1 in a uniformly random labelled tree with nodes $\{1, \ldots, m\}$; this distribution is explicitly given by

$$\nu_m(k) = k \cdot \frac{(m-k)^{m-k-1}}{m^{m-1}} \cdot \binom{m-2}{k-1}.$$

Let D_1 have distribution $\nu_{Z_1}(k)$; in other words, given that $Z_1 = m$, D_1 has distribution ν_m. Also, let D_2 have distribution Bernoulli$(1/Z_1)$, with D_2 conditionally independent of D_1 given (X_1, Z_1). Finally, let D_3 be Poisson, with parameter

$$\int_{X_1}^{\infty} (1 - \theta(t)) \mathrm{d}t \,,$$

and with D_3 conditionally independent of D_1 and of D_2 given (X_1, Z_1).

Corollary 1.4. *The random variable* $\deg_{\mathcal{M}}(\emptyset)$ *is distributed as* $D_1 + D_2 + D_3$.

It is interesting to contrast this result with Proposition 2 of [2], which expresses $\deg_{\mathcal{M}}(\emptyset)$ as a mixture of Poisson random variables:

$$\mathbf{P}\left\{\deg_{\mathcal{M}}(\emptyset) = i + 1\right\} = \int_0^1 \mathbf{P}\left\{\text{Poisson}(\Phi(u)) = i\right\} \mathrm{d}u \,,$$

where for $0 \leq u < 1$ we set $\phi(u) = \int_0^u \log(1/x)/(1-x)\mathrm{d}x$. By a classic result of Frieze [8], we have that

$$\mathbf{E}\left[\sum_{e \in E(\mathcal{M}) : \emptyset \in e} \mathcal{W}(e)\right] = 2\zeta(3) \,.$$

Combining this fact with Theorem 1.3 and the results of [11], it follows that for $U \stackrel{\mathrm{d}}{=} \text{Uniform}[0,1]$, we have

$$\mathbf{E}\left[\int_{\theta^{-1}(U)}^{\infty} t(1 - \theta(t)) \mathrm{d}t\right] = 2\zeta(3) - \zeta(2) \,,$$

which is perhaps not obvious.

Our second main result is to show that the limit $\mathcal{M}$ has cubic volume growth.

Theorem 1.5. *As* $r \to \infty$,

$$\frac{\log |B_{\mathcal{M}}(\emptyset, r)|}{\log r} \xrightarrow{\text{a.s.}} 3 \,.$$

This volume growth agrees with predictions from [9], and, as shown by Theorem 1.2, stands in contrast to the behaviour of the weak limit $\mathcal{T}$ of the early stages of Prim's algorithm. A parallel dichotomy of dimensionality appears when studying the rescaled, measured metric space (Gromov–Hausdoff–Prokhorov) limit of the minimum spanning tree; see the forthcoming manuscript, described elsewhere in this collection in the abstract of Christina Goldschmidt, for some details.

Broadly speaking, the structure $\mathcal{T}$ built in the early stages determines the global metric structure, and the second stage, where $\mathcal{M}$ is built from $\mathcal{T}$, is responsible for adding mass.

In proving Theorem 1.2, we in fact provide stronger bounds, showing that the multiplicative fluctuations of $|B_{\mathcal{T}}(\emptyset, r)|/r^2$ are at most polylogarithmic in r. In fact, by optimizing our proofs, it would be possible to show that $|B_{\mathcal{T}}(\emptyset, r)|/r^2$ is almost surely eventually dominated by some polynomial in $\log\log r$. It would be interesting to pin down precisely the almost sure fluctuations of $|B_{\mathcal{T}}(\emptyset, r)|/r^2$. The work of Duquesne and Le Gall [7] on the exact Hausdorff measure function for the Brownian CRT heuristically suggests that such fluctuations may be of order exactly $\log\log r$, but the connection between the two settings is rather tenuous. At any rate, our method of bounding fluctuations in the negative direction is too weak to prove bound that are polynomial in $\log\log r$, so we have not made a serious effort at optimizing the bounds in our proofs.

It would be interesting to understand the spectral and diffusive properties of $\mathcal{M}$. Results of Barlow et. al. [5] and of Kumagai and Mizumi [10] suggest that for simple random walk $(x_n, n \geq 1)$ on $\mathcal{M}$, started from the root $\emptyset$, we should have $\mathbf{P}\{x_n = \emptyset\} = n^{-3/4+o(1)}$ and $d_{\mathcal{M}}(\emptyset, x_n) = n^{1/4+o_p(1)}$. However, the bounds we obtain in this work are too weak to prove such a result.

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The scaling limit of critical random graphs

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(joint work with Louigi Addario-Berry and Christina Goldschmidt)

Random graphs and the phase transition

In the Erdős–Rényi [9] $G(n,p)$ model, a graph on $\{1,2,\dots,n\}$ is chosen randomly by joining any two vertices by an edge with probability p, independently for different pairs of vertices [12, 8].

The phase transition. This model exhibits a radical change in structure (or *phase transition*) for large n when the average degree approaches one, that is for $p = p(n) \sim 1/n$. For $p \sim c/n$ with $c < 1$, the largest connected component has size (number of vertices) $O(\log n)$. On the other hand, when $c > 1$, there is a connected component containing a positive proportion of the vertices (the *giant component*). The cases $c < 1$ and $c > 1$ are called *subcritical* and *supercritical* respectively. This phase transition was discovered by Erdős and Rényi in their seminal paper [9]; they further observed that in the *critical* case, when $p = 1/n$, the largest components of $G(n,p)$ have sizes of order $n^{2/3}$. For this reason, the phase transition in random graphs is sometimes dubbed the *double jump*.

The critical window. The apparent double jump is actually only an artefact of the parametrization which is much too crude, and understanding the critical random graph (when $p = p(n) \sim 1/n$) requires a different and finer scaling: the natural parametrization turns out to be of the form $p = p(n) = 1/n + \lambda n^{-4/3}$, for $\lambda = o(n^{1/3})$ [7, 15, 16]. We will restrict our attention to $\lambda \in \mathbb{R}$; this parameter range is then usually called the *critical window*. One of the most significant results about random graphs in the critical regime was proved by Aldous [6]. He observed that one could encode various aspects of the structure of the random graph (specifically, the sizes and surpluses of the components) using stochastic processes. Fix $\lambda \in \mathbb{R}$, set $p = 1/n + \lambda n^{-4/3}$ and write $|\mathcal{C}_i^n|$ and $s(\mathcal{C}_i^n)$ for the size and surplus (that is, the number of edges which would need to be removed in order to obtain a tree) of $\mathcal{C}_i^n$, the i-th largest component of $G(n,p)$.

Theorem 1 (Aldous [6])**.** *As $n \to \infty$.*

$$(n^{-2/3}|\mathcal{C}_i^n|, s(\mathcal{C}_i^n)))_{i\geq 1} \xrightarrow{d} ((|\gamma_i|, s(\gamma_i)))_{i\geq 1}.$$

Here, the convergence of the first co-ordinate takes place in $\ell^2_{\searrow}$, the set of infinite sequences $(x_1, x_2, \dots)$ with $x_1 \geq x_2 \geq \cdots \geq 0$ and $\sum_{i\geq 1} x_i^2 < \infty$. (See also [16, 13].) The limit is described in terms of a Brownian motion with parabolic drift, $(W^\lambda(t), t \geq 0)$, where

$$W^\lambda(t) := W(t) + t\lambda - \frac{t^2}{2}$$

and $(W(t), t \geq 0)$ is a standard Brownian motion. The limit $(|\gamma_i|, i \geq 1)$ has the distribution of the ordered sequence of lengths of excursions of the reflected process $B^\lambda(t) := W^\lambda(t) - \min_{0\leq s\leq t} W^\lambda(s)$ above 0, while $(s(\gamma_i), i \geq 1)$ is the sequence of numbers of points of a Poisson point process with rate one in $\mathbb{R}^+ \times \mathbb{R}^+$ lying under the corresponding excursions.

The scaling limit

We are interested in the scaling limit of the sequence of connected components of $G(n,p)$, for $p = 1/n + \lambda n^{-4/3}$, viewed as metric spaces equipped with the graph distance. The distance we use to compare metric spaces is the Gromov–Hausdorff distance $\mathrm{d_{GH}}$ [11], that is the infimum of the Hausdorff distance over isometric embeddings of the two compact metric spaces into a single larger metric space. For two sequences of compact metric spaces $\mathbf{A} = (A_i, i \geq 1)$ and $\mathbf{B} = (B_i, i \geq 1)$, we define

$$\mathrm{d}^{\infty,4}_{\mathrm{GH}}(\mathbf{A}, \mathbf{B}) = \left(\sum_{i\geq 1} \mathrm{d_{GH}}(A_i, B_i)^4 \right)^{1/4}.$$

Theorem 2. *Suppose that $p = 1/n + \lambda n^{-4/3}$, for $\lambda \in \mathbb{R}$. Then, there exists a sequence of non-trivial random compact metric spaces $(\mathcal{C}_i, i \geq 1)$ such that, as $n \to \infty$,*

$$n^{-1/3}(\mathcal{C}_i^n, i \geq 1) \to (\mathcal{C}_i, i \geq 1)$$

in distribution for the distance $\mathrm{d}^{\infty,4}_{\mathrm{GH}}$.

It is possible to describe a limit connected component using a *canonical subtree* in which one makes *point identifications* (the equivalent of the surplus edges). The first step is to explain how to construct a *real tree*, a tree-like path-connected metric space, from a continuous excursion.

Trees encoded by excursions. The construction is classical and we follow [5, 10, 14]. For $\sigma > 0$, consider a continuous function $f : [0, \sigma] \to [0, \infty)$ such that $f(0) = f(\sigma) = 0$ and $f(s) > 0$ for all $s \in (0, \sigma)$. Define $d_f : [0, \sigma]^2 \to [0, \infty)$ by

$$d_f(x, y) = f(x) + f(y) - 2\inf\{f(s) : x \wedge y \leq s \leq x \vee y\}.$$

One easily verifies that d_f is a pseudo-metric on $[0, \sigma]$. Let $x \sim y$ if $d_f(x, y) = 0$. Write $\mathcal{T}_f$ for the the quotient $[0, \sigma]/\sim$; then $(\mathcal{T}_f, d_f)$ is a real tree (with a slight abuse of notation since d_f is not really defined on $\mathcal{T}_f$).

A limit connected component. The limit sequence is built from independent copies of the connected components with the right sizes (given by Theorem 1),

and it suffices to describe one limit connected component $\mathcal{C}^{(\sigma)}$ with a fixed size σ ($\sigma n^{2/3}$ in the discrete picture). One first picks an excursion $\tilde{\mathbf{e}}^{(\sigma)}$ of length σ with the following law

$$\mathbf{P}\big(\tilde{\mathbf{e}}^{(\sigma)} \in \mathcal{B}\big) = \frac{\mathbf{E}\big[1_{\{\mathbf{e}^{(\sigma)} \in \mathcal{B}\}} \cdot \exp(\int_0^\sigma \mathbf{e}^{(\sigma)}(x)dx)\big]}{\mathbf{E}\big[\exp(\int_0^\sigma \mathbf{e}^{(\sigma)}(x)dx)\big]}, \tag{1}$$

where $\mathbf{e}^{(\sigma)}$ is a Brownian excursion of length σ. The excursion $\tilde{\mathbf{e}}^{(\sigma)}$ encodes a real tree $\mathcal{T}_{2\tilde{\mathbf{e}}^{(\sigma)}}$ via the construction we have just described. (This is similar to Aldous' continuum random tree $\mathcal{T}_{2\mathbf{e}}$, where $\mathbf{e}$ is a standard Brownian excursion of length one [4, 3, 5]) Note that $\tilde{\mathbf{e}}^{(\sigma)}$ is precisely distributed as an excursion of B^λ, conditioned to have length σ (regardless of its starting point, as it should be from the discrete $G(n,p)$ picture). The real tree $\mathcal{T}_{2\tilde{\mathbf{e}}^{(\sigma)}}$ is our canonical tree in which we want to *identify* some points to create some cycles.

The point identifications come from a Poisson point process $\mathcal{P}$ of unit intensity in $\mathcal{A}(\tilde{\mathbf{e}}^{(\sigma)}) := \{(x,y) : 0 \le x \le \sigma, 0 \le y \le 2\tilde{\mathbf{e}}^{(\sigma)}(x)\}$. In general, for a continuous excursion h, for every point $(x,y) \in \mathcal{A}(h)$ let

$$\ell(x,y) = \sup\{x' \le x : y = h(x')\}.$$

The point (x,y) identifies two points in the tree $\mathcal{T}_h$: the images of x and $\ell(x,y)$ in the canonical projection π_h from $[0,1]$ onto $\mathcal{T}_h$. Given the almost surely finite pointset $\mathcal{P}$, for every $(x,y) \in \mathcal{P}$, we define $\pi(x) \sim_{\mathcal{P}} \pi(\ell(x,y))$ and define the metric space as the quotient space $\mathcal{C}^{(\sigma)} := \mathcal{T}_{2\tilde{\mathbf{e}}^{(\sigma)}(x)}/\sim_{\mathcal{P}}$. The metric space $\mathcal{C}^{(\sigma)}$ is the scaling limit of one connected component appearing in Theorem 2 conditioned on the length of the corresponding excursion being σ.

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The scaling limit of the minimum spanning tree of the complete graph

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(joint work with L. Addario-Berry, N. Broutin, G. Miermont)

The minimum spanning tree. Consider the complete graph, K_n, on vertices labelled by $\{1, 2, \ldots, n\}$. Put independent random weights on the edges which are uniformly distributed on $[0, 1]$ and find the spanning tree M_n of smallest total weight; this is the so-called *minimum spanning tree (MST)*. Now think of M_n as a metric space by taking the metric to be the graph distance divided by $n^{1/3}$. We also endow M_n with a probability measure by placing mass $1/n$ on each vertex. Our main result is the following theorem.

Theorem 1. *There exists a random compact measured metric space $\mathcal{M}$ such that, as $n \to \infty$,*

$$M_n \to \mathcal{M}$$

in distribution.

The convergence here is in the sense of the Gromov–Hausdorff–Prokhorov distance, which we now define.

The Gromov–Hausdorff–Prokhorov distance. Let (X, d, μ) and (X', d', μ') be measured metric spaces. A *correspondence* between X and X' is defined to be a measurable subset R of $X \times X'$ such that for every $x \in X$ there exists $x' \in X'$ such that $(x, x') \in R$ and vice versa. A *partial coupling* of μ and μ' is a finite Borel measure on $X \times X'$ such that $p_*\pi \leq \mu$ and $p'_*\pi \leq \mu'$, where $p : X \times X' \to X$ and $p' : X \times X' \to X'$ are the canonical projections. The *distortion* of R is

$$\sup\{|d(x, y) - d'(x', y')| : (x, x') \in R, (y, y') \in R\}$$

and the *discrepancy* of π is

$$\max\{(\mu - p_*\pi)(X), (\mu' - p'_*\pi)(X')\}.$$

The Gromov–Hausdorff–Prokhorov distance, $\mathrm{d_{GHP}}((X, d, \mu), (X', d', \mu'))$, between (X, d, μ) and (X', d', μ') is then the infimum of the values $\epsilon > 0$ such that there exist a correspondence R and a partial coupling π such that the distortion of R and the discrepancy of π are both strictly smaller than ϵ and, furthermore, $\pi(R^c) < \epsilon$.

Write $\mathscr{M}$ for the set of measured isometry-equivalence classes of compact separable measured metric spaces. Then $(\mathscr{M}, \mathrm{d_{GHP}})$ is a complete separable metric space.

Scaling limits of random discrete trees. Theorem 1 is very much in the spirit of recent work on the scaling limits of a wide variety of random discrete trees, which began with the seminal work of Aldous [4, 5, 6] in the early 1990's. The prototypical result [4, 8] is that the uniform random tree on vertices labelled by $\{1, 2, \ldots, n\}$, considered as a measured metric space, converges in distribution to the *Brownian continuum random tree (CRT)*. The Brownian CRT is an example of a *random* $\mathbb{R}$*-tree* (see Le Gall [9]).

Kruskal's algorithm and the Erdős–Rényi random graph. The proof of Theorem 1 relies on *Kruskal's algorithm* for building the MST.

- Start from a forest of isolated vertices. List the edges of the graph as $e_1, e_2, \ldots, e_{\binom{n}{2}}$ in increasing order of weight.
- At step i, add edge e_i as long as it does not create a cycle.
- Stop when all vertices are connected.

Now consider the *Erdős–Rényi random graph process*, a natural coupling of the different Erdős–Rényi random graphs (see, for example, [7] and the references therein). This process may be obtained as follows: for a fixed parameter value $p \in [0, 1]$, keep all edges whose weight is at most p. By using the same edge-weights, we can easily couple the random graph process and a continuous-time version of Kruskal's algorithm so that, at a fixed time p, the components of the two processes have the same vertex-sets (and, indeed, so that the components of the Kruskal process are the MST's of the components of the Erdős–Rényi process). In particular, it is straightforward to see that the Kruskal process also undergoes the Erdős–Rényi phase transition at $p = 1/n$. In particular, for $p = (1 + \epsilon)/n$, there is a unique *giant component* containing a positive proportion of the vertices. It turns out that the metric structure of the MST has already essentially been built by this point (although the vast majority of the mass still lies outside the giant component).

In order to gain a finer understanding of the metric structure, we need to look earlier in the evolution of the process, within the *critical window* i.e. when $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ for some $\lambda \in \mathbb{R}$. Results from [1, 2] (described by Nicolas Broutin elsewhere in this report) tell us that for fixed λ, the Erdős–Rényi graph possesses a scaling limit when the graph distance is rescaled by $n^{-1/3}$ within each component. This scaling limit is a collection of random $\mathbb{R}$-trees in each of which a finite number of points have been identified to create cycles. By breaking these cycles appropriately, we are able to obtain that, at a fixed point λ in the critical window, the Kruskal process also has a scaling limit as a sequence of $\mathbb{R}$-trees. Starting from a large enough value of λ we are then able to control the way that the mass and metric structure evolve as the smaller trees gradually attach to the largest tree, in order to obtain our scaling limit.

The limit metric space. The limit metric space $\mathcal{M}$ is a random measured $\mathbb{R}$-tree, which is almost surely binary and whose mass measure is concentrated on its leaves. It shares all of these properties with the Brownian CRT, $\mathcal{T}$. However,

they are not the same object. Suppose that $N(\epsilon)$ is the number of balls of radius $\epsilon > 0$ needed to cover $\mathcal{M}$. Then

$$\frac{N(\epsilon)}{\log(1/\epsilon)} \to 3$$

in probability, as $\epsilon \to 0$. Since $\mathcal{T}$ has box-counting dimension 2, it follows that the laws of $\mathcal{M}$ and $\mathcal{T}$ are mutually singular.

The results presented here are the subject of the paper [3] which is currently in preparation.

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Asymptotics for the Isoperimetry in 2D Supercritical Percolation

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(joint work with Marek Biskup, Eviatar Procaccia, Ron Rosenthal)

1. Motivation

Isoperimetry is a subject that lies at the heart of geometric measure theory. It provides a fundamental link between metric structures and measures on the underlying space. Isoperimetric inequalities have proved to be an essential ingredient for many analytical results. Indeed, they have found applications in areas such as concentration of measure, heat-kernel estimates, mixing time of diffusions/random walks, etc.

Isoperimetric problems were originally formulated in continuum spaces but recently found their way into discrete mathematics as well. For a finite graph $G = (V, E)$, isoperimetry is often characterized by means of the *Cheeger constant* Φ_G, defined as

$$\Phi_G := \inf\left\{ \frac{|\partial_G U|}{|U|} : U \subset V,\, 0 < |U| \le \frac{1}{2}|V| \right\}. \tag{1}$$

Here, $\partial_G U$ is the edge-boundary of U in G, that is the set of edges in E with one and only one of its points in U. The quantities $|\partial_G U|$ and $|U|$ denote the sizes of the respective sets.

If G is infinite, the above definition does not make sense. A number of objects can be substituted instead; for our purposes the most interesting is the *anchored isoperimetric profile*. Given a vertex $0 \in V$, which we shall refer to as the *anchor*, the isoperimetric profile of G anchored at 0 is the non-increasing function $\Phi_{G,0} : \mathbb{N} \to \mathbb{R}_+$ given by

$$\Phi_{G,0}(n) := \inf\left\{ \frac{|\partial_G U|}{|U|} : 0 \in U \subset V\,, G(U) \text{ connected}, 0 < |U| \le n \right\}. \tag{2}$$

Here $G(U)$ is the sub-graph of G whose vertex set is U and the edge set contains all edges wity at least one vertex in U. In the graphs we consider, the choice of an anchor will not affect the $n \to \infty$ asymptotics of the isoperimetric profile and, consequently, we shall often omit it from the notation. (In general graphs, the choice of an anchor does not impact the asymptotic order of the profile, but it may change the leading constant.)

The aim of the present note is to study the asymptotic of the above quantities in random graphs that are generated by supercritical Bernoulli bond percolation on $\mathbb{Z}^d$. To be more precise, regard $\mathbb{Z}^d$ as the graph with edge set $\mathcal{E}^d$ given by all (unordered) nearest neighbor pairs. Let $\mathbb{P}_p$ denote the product (Bernoulli) measure on $\{0,1\}^{\mathcal{E}^d}$ with the density of 1's to be $p \in [0,1]$. A sample from $\mathbb{P}_p$, to be called generically ω, can be identified with a subgraph of $\mathbb{Z}^d$, with edge set composed only of edges e where $\omega(e) = 1$. These edges will be referred to as open (or occupied); those with $\omega(e) = 0$ will be called closed (or vacant). The restriction of ω to the box $\mathrm{B}(n) := [-n, n]^d \cap \mathbb{Z}^d$ will be denoted ω^n.

It is well known (cf Grimmett [Gri99]) that, for $d \ge 2$, there is $p_c(d) \in (0,1)$ such whenever $p > p_c(d)$, the graph associated with ω contains a $\mathbb{P}_p$-almost surely unique infinite connected component. This component, usually referred to as the *infinite cluster*, will be denoted by $\mathbf{C}^\infty$. The asymptotic density of $\mathbf{C}^\infty$ in $\mathbb{Z}^d$ is

$$\theta_p := \mathbb{P}(0 \in \mathbf{C}^\infty) \in (0,1]\,. \tag{3}$$

Similarly, for $p > p_c(d)$, with $\mathbb{P}_p$-probability tending rapidly to 1 as $n \to \infty$, there exists a unique connected component of ω^n whose size is linear in $|\mathrm{B}(n)|$; all other components have size at most poly-logarithmic in $|\mathrm{B}(n)|$. We shall denote this *giant component* by $\mathbf{C}^n$.

The objective of this work is to study the asymptotics of the anchored isoperimetric profile of $\mathbf{C}^\infty$ and the Cheeger constant of $\mathbf{C}^n$. It is not hard to surmise the leading order of these quantities by invoking an analogy with the full lattice:

$$\Phi_{\mathbf{C}^\infty}(n) \asymp n^{-1/d} \quad \text{and} \quad \Phi_{\mathbf{C}^n} \asymp n^{-1}\,. \tag{4}$$

And, indeed, thanks to sophisticated facts from percolation theory, one can establish these bounds with probability tending to 1 as $n \to \infty$ (Benjamini and Mossel [BM03], Mathieu and Remy [MR04], Berger, Biskup, Hoffman and Kozma

[BBHK08] and Pete [Pet08]). This led Itai Benjamini to formulate the following natural conjecture:

Conjecture 1.1. *The limit* $\lim_{n\to\infty} n\Phi_{\mathbf{C}^n}$ *exists* $\mathbb{P}_p$*-a.s.*

As a step towards this goal Procaccia and Rosenthal [PR11] have recently shown that $\mathrm{Var}(n\Phi_{\mathbf{C}^n}) \le Cn^{2-d}$. This implies concentration of $n\Phi_{\mathbf{C}^n}$ around its mean for dimensions ≥ 3.

2. Results

We will now proceed to state the main conclusions of the present note. We begin with the isoperimetric profile where the result is easiest to formulate:

Theorem 2.1 (Isoperimetric profile)**.** *Let* $d = 2$ *and* $p > p_c(\mathbb{Z}^2)$*. Then there exists a constant* $\varphi_p \in (0,\infty)$ *such that* $\mathbb{P}_p(\cdot|0 \in \mathbf{C}^\infty)$*-almost surely,*

$$\lim_{n\to\infty} n^{1/2}\,\Phi_{\mathbf{C}^\infty}(n) = \theta_p^{-1/2}\varphi_p\,, \tag{5}$$

where θ_p *is defined in* (3)*.*

Next we will address the Cheeger constant for the giant component $\mathbf{C}^n$. As it turns out, it is more natural to look at the quantity $\Phi_{\mathbf{C}^n;\mathbf{C}^\infty}$, where for a finite subgraph $G = (V,E)$ of a (possibly infinite) graph H, we define

$$\Phi_{G;H} := \inf\biggl\{\frac{|\partial_H U|}{|U|}\colon U \subset V,\, 0 < |U| \le \frac{1}{2}|V|\biggr\}, \tag{6}$$

i.e., with edges "sticking out" of G still included in the boundary of U. The rationale for this modification is that one thus avoids various boundary effects. Writing $\widetilde{\Phi}_{\mathbf{C}^n} := \Phi_{\mathbf{C}^n;\mathbf{C}^\infty}$, we then settle a version of Conjecture 1.1:

Theorem 2.2 (Cheeger constant)**.** *Let* $d = 2$ *and* $p > p_c(\mathbb{Z}^2)$ *and let* φ_p *be as in Theorem 2.1 and* θ_p *as in* (3)*. Then,* $\mathbb{P}$*-almost surely,*

$$\lim_{n\to\infty} n\widetilde{\Phi}_{\mathbf{C}^n} = \sqrt{2}\theta_p^{-1}\varphi_p\,. \tag{7}$$

Having established the existence of a limit value, the next natural question is its characterization. Not too surprisingly, φ_p admits the representation as the isoperimetric constant for a specific isoperimetric problem on $\mathbb{R}^2$. More precisely:

Theorem 2.3 (Limit value)**.** *Let* $d = 2$ *and* $p > p_c(\mathbb{Z}^2)$*. There exists a norm* β_p *on* $\mathbb{R}^2$*, which is symmetric with respect to the reflections through the axes and diagonals of* $\mathbb{Z}^2$*, such that* φ_p *from Theorems 2.1 and 2.2 satisfies*

$$\varphi_p = \inf\bigl\{\mathrm{len}_{\beta_p}(\gamma)\colon \gamma \text{ is a Jordan curve in } \mathbb{R}^2,\, \mathrm{Leb}(\mathrm{int}(\gamma)) = 1\bigr\}\,. \tag{8}$$

Here Leb *stands for the Lebesgue measure on* $\mathbb{R}^2$*,* $\mathrm{int}(\gamma)$ *denotes the interior of* γ *and* len_{β_p} *is the length of* γ *under the norm* β_p*.*

The isoperimetric problems in $\mathbb{R}^n$ have a long history and much is known about them. In particular, it is known that the infimum in (8) is achieved by a unique (up to reparametrizations) curve $\hat{\gamma}_p$ which is the boundary of the normalized *Wulff shape*: $\widehat{W}_p$ for the norm β_p. More explicitly, define

$$W_p := \bigcap_{\hat{n}\in\mathbb{S}^1} \{x : (\hat{n}, x) \le \beta_p(\hat{n})\} \quad ; \qquad \widehat{W}_p = W_p/\sqrt{\mathrm{Leb}(W_p)} \tag{9}$$

and set

$$\hat{\gamma}_p := \partial \hat{W}_p. \tag{10}$$

Then

$$\varphi_p = \mathrm{len}_{\beta_p}(\hat{\gamma}_p). \tag{11}$$

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Fluctuations of the front in a one dimensional model for the spread of an infection

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(joint work with Jean Bérard)

Consider the following microscopic model of infection or epidemic reaction on the integer lattice $\mathbb{Z}$: There are two types of particles; X particles, which move as independent, continuous–time, symmetric, nearest neighbor random walks of total jump rate D_X; and Y particles, which also move as independent, continuous–time, symmetric, nearest neighbor random walks of total jump rate D_Y, independently of the X particles. Initially, at each site $z \in \{1, 2, \ldots\}$ there are $\eta_Y(z) \ge 0$

particles, and none at the sites $\{0,-1,\ldots\}$; at each site $z \in \{0,-1,\ldots\}$ there are $\eta_X(z) \geq 0$ particles, and none at the sites $\{1,2,\ldots\}$. When an X particle jumps to a site where there are Y particles, all of them immediately become X particles Furthermore, when a Y particle jumps to a site where there are X particles, it immediately becomes an X particle.

We are interested in the asymptotic behavior of the rightmost site r_t visited by the X particles up to time t, which we call the *front*. Let $\eta(t,x)$ be the total number of X and Y particles at $x \in \mathbb{Z}$ at time $t \geq 0$. Note that if we call $\eta_X(t,x)$ the number of X particles and $\eta_Y(t,x)$ the number of Y particles at $x \in \mathbb{Z}$ at time $t \geq 0$, then $\eta_X(t,x) = \eta(t,x)$ for $x \leq r_t$ and $\eta_X(t,x) = 0$ for $x > r_t$, while $\eta_Y(t,x) = 0$ for $x \leq r_t$ and $\eta_Y(t,x) = \eta(t,x)$ for $x > r_t$.

Define

$$\mathbb{T} := \{\eta : 0 < \sum_{x\in\mathbb{Z}} e^{-\theta|x|}\eta(x) < \infty\}.$$

The state space of our process will be

$$\mathbb{S} = \{(r,\eta) \ : \ r \in \mathbb{Z}, \eta \in \mathbb{T}\}, \tag{1}$$

where $\theta > 0$ is chosen sufficiently small. Let us call $P_{\eta,0}$ the law of this interacting particle system with initial state $(\eta,0) \in \mathbb{S}$. Let us call $\mathcal{P}(\mathbb{T})$ the set of probability measure on $\mathbb{N}^{\mathbb{Z}}$ supported on $\mathbb{T}$. For each probability measure $\mu \in \mathcal{P}(\mathbb{T})$, we will let $P_{\mu,0} := \int P_{\eta,0} d\mu$. Furthermore, for each $\rho > 0$, we call μ_ρ the product Poisson probability measure on $\mathbb{N}^{\mathbb{Z}}$ of parameter ρ, which is easy to see that is supported on $\mathbb{T}$. In [2] it is shown that whenever the initial law of η is product Poisson, there exists $v \in (0,\infty)$, such that a.s.,

$$\lim_{t\to\infty} r_t/t = v.$$

A shape theorem was proved also in higher dimensions in [2].

Here we are interested in the fluctuations of r_t. The main result presented here is the following.

Theorem 1. *Assume $D_X = D_Y > 0$. Let $\rho > 0$ and $\mu \in M_\rho$. Then, there exists $\sigma^2 \in (0,\infty)$, non-random, and independent of μ such that*

$$B_t^\epsilon := \epsilon^{1/2}\left(r_{\epsilon^{-1}t} - \epsilon^{-1}vt\right), \qquad t \geq 0, \tag{2}$$

under the law $P_{\mu,0}$ converges in law as $\epsilon \to 0$ to Brownian motion with variance σ^2.

As in [1], where the case $D_X > 0$ and $D_Y = 0$ was studied, the method is based on a renewal structure. There exists a sequence of regeneration times $\{\kappa_n : n \geq 1\}$, with bounded second moments, with the property that for each $n \geq 1$ the particles $\eta(x,\kappa_n)$, $x < r_{\kappa_n}$ do not affect the behavior of r_t for $t \geq \kappa_n$. As a side benefit we are able to study the invariant measure of the process as observed from the front.

Theorem 2. *Let $\rho > 0$ and assume that that $\mu \in M_\rho$. Consider the process as seen from the front, $\tau_{-r_t}\eta(t)$ and denote its distribution $\mu(t)$ under the law $P_{\mu,0}$. There is a unique invariant measure $\mu(\infty)$ and $\mu(t) \Rightarrow \mu(\infty)$.*

A key idea in the construction of the regeneration time sequence for this process, is to define them as a time where both a particular behavior occurs after it (as in [1]), and a certain kind of decoupling occurred in the past, between the Y particles ahead of the front at the regeneration time, and the X particles behind the front at the same time: no interaction occurred between them in the infinite past. To do this, it is necessary to time-reverse the dynamics, making use of the fact that the full process is reversible under any Poisson product measure in the case $D_X = D_Y$.

Theses results have extensions to the case $D_X > D_Y > 0$, where the dynamics is modified so that a Y particle is infected whenever it visits any site previously visited by some X particle.

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The conformally invariant fragmentation movie

Wendelin Werner

Building on the definition and main properties of the simple Conformal Loop Ensembles, as constructed and charaterized in joint work with Scott Sheffield [2], we describe the non-rooted Markovian construction of CLE_4, where loops appear uniformly at random in a Poissonian way on the boundary of the already explored domains. This process $\Omega = (\Omega_t, t \geq 0)$ constructed and described in joint work with Hao Wu [2] has the following properties.

- If one starts at time 0 with the unit disk, then at each positive time t, it almost surely consists of a countable family Ω_t of disjoint open subsets of the unit disk $\Omega_t := (O^j(t), j \in J(t))$ such that the complement of the union of these sets has zero Lebesgue measure.
- It is a fragmentation process: For any $s \leq t$ and $j \in J(t)$, there exists almost surely $i \in J(s)$ such that $O^j(t) \subset O^i(s)$.
- This process is conformally invariant (without any time-change) i.e. for any Moebius transformation Φ of the unit disk, the laws of the two processes $(\Phi(\Omega_t), t \geq 0)$ and $(\Omega_t, t \geq 0)$ are identical.
- For any given s and any $O^i(s)$, the future evolution of the fragmentation of $O^i(s)$ into subpieces evolves like the image of an independent copy of Ω under a conformal map from the unit disk onto $O^i(s)$ – and the evolutions in the different components of Ω_s are independent (conditionally on Ω_s).

We finally describe further features of this process, how it allows to construct a new interesting coupling of the Gaussian Free Field and the CLE_4 (this is ongoing work by Scott Sheffield, Sam Watson and Hao Wu [1]) as well as some related open questions.

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Minkowski content and natural parametrization for the Schramm-Loewner evolution (SLE)

Gregory F. Lawler
(joint work with M. Rezaei)

The (chordal) Schramm-Loewner evolution (SLE) is a measure on paths in a simply connected domain D connecting distinct boundary points. The definition as given by Schramm [4] is on curves modulo parametrization. By choosing a parametrization by capacity, he was able to define the process in terms of an SDE driving by a Brownian motion. However, this parametrization is not the same as the natural fractal parametrization which would be obtained as the scaling limit of the length of discrete curves. Indeed, it is singular with respect to it.

S. Sheffield and I [1] defined a natural parametrization or natural length for $\kappa < 8$ using a definition motivated by the following conjecture. Let $\gamma_{s,t} = \gamma[s,t]$ be a piece of the curve γ. Then the amount of time to traverse $\gamma_{s,t}$ should be given (up to multiplicative constant) by the d-dimensional Minkowski content

$$\mathrm{Cont}_d(\gamma_{s,t}) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathrm{Area}\{z : \mathrm{dist}(z, \gamma_{s,t}) \le \epsilon\}. \tag{1}$$

Here $d = 1 + \frac{\kappa}{8}$ is the Hausdorff dimension of the paths. The definition did not require showing that the limit above exists, but rather assumed that it did and gave a property that the natural length should have. Then the natural length was defined using a Doob-Meyer decomposition. The original paper proved existence of the process for $\kappa < 5.0\cdots$, but it was later shown to be valid for $\kappa < 8$ by W. Zhou and me [2]. The nature of the construction made it hard to establish that the length was a function of the curve only. In particular, one would like it to be independent of the domain in which the curve lies. M. Rezaei and I gave a new version of the construction that shows this independence.

We announce that we can show that the Minkowski content exists, that is, the limit (1) exists, is finite, and nonzero, and is the same (up to multiplicative constant) as the natural parametrization previously constructed. It follows immediately that the natural length is independent of domain and is reversible.

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Entropy-driven phase transitions in low-temperature antiferromagnetic Potts models

Roman Kotecký
(joint work with Alan Sokal, Jan Swart)

We consider Gibbs states for the three-state Potts antiferromagnet on a graph G that belongs to a class of quasi-transitive plane bipartite quadrangulations. Namely, let $G = (V, E)$ be a quadrangulation of the plane, and let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be its sublattices, connected through bonds along the diagonals of quadrilaterals. We are assuming that G_0 is a locally finite, 3-connected, quasi-transitive triangulation with one end. A particular case of the considered graphs is the diced lattice [1] whose two mutually dual sublattices are the triangular and hexagonal lattice, respectively.

We prove [2] that the set $\mathcal{G}_{3,\beta}(G)$ of Gibbs states, on any of the considered graphs G and at sufficiently low temperature $1/\beta$, contains (at least) three distinct infinite-volume Gibbs measures, which exhibit spontaneous magnetisation in the sense that vertices in the sublattice V_0 have a higher probability to be in one state than in either of the other two states. For zero temperature, this amounts to the claim that Gibbs states obtained as limits of uniform distributions on the set of perfect 3-colourings on finite subgraphs of G with boundary conditions with a fixed colour on the sublattice V_0 differ in dependence on which fixed colour is specified in the boundary conditions.

The proof is based on a suitable bound on the strength of the entropic penalty for switching colours on V_0. This is encoded in a probability of "entropic" contours– at zero temperature these are cycles on the subgraph G_1 separating the areas of fixed colour on V_0. The bound is sufficiently sharp (when compared with the number of possible cycles) only for large contours. Nevertheless, already this allows to construct distinct Gibbs states yielding different probabilities to the events $\mathcal{I}_{\Delta,m}$ consisting of all those configurations for which a large finite set $\Delta \subset V_0$ is mono-colour with a fixed colour m.

To turn this into a characterisation by a specified single site magnetisation uses a suitably chosen random cluster representation for the model in a large finite volume $\Lambda \subset V$, $\Delta \subset \Lambda$, with a fixed boundary condition on V_0, say 1. Showing that the difference of probabilities that Δ is uniformly coloured in the colour 1 or in the colour 2 equals the probability that there is a cluster connection between Δ and the boundary of Λ, easily yields (by finite-energy bound) a lower bound on the probability that there is a cluster connection between a fixed lattice site and the boundary of Λ, which, in its turn, equals the magnetisation.

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Percolation on isoradial graphs

Ioan Manolescu

(joint work with Geoffrey Grimmett)

The main goal of this work is the study of percolation on isoradial graphs, and, more precisely, to show criticality and universality of arm exponents for these models. It presents the results of [2].

An *isoradial graph* G is a planar graph embedded in the plane in such a way that every face is inscribed in a circle of radius 1. To each edge e we attach a parameter $p(e) \in [0,1]$, which is an explicit function of the length of e. We associate to G a canonical percolation model $\mathbb{P}_G$, under which each edge e is taken open with probability $p(e)$ and closed with probability $1-p(e)$, independently of other edges. Thus, isoradial graphs provide a large class of planar percolation models expected to be critical and to belong to the same universality class. These models include the critical homogeneous bond percolation on the square, triangular and hexagonal lattices. More generally, isoradial graphs have proved to be a particularly convenient setting for the study of various statistical mechanics models.

We will focus on two features of percolation models. The *box crossing property* (or RSW property) states that the probability of crossing rectangular domains of given aspect ratio is bounded away from 0 and 1, uniformly in the size of the domain. It is standard (with todays tools) that if a percolation model $(G, \mathbb{P})$ has the box crossing property, then it is critical. More precisely, under $\mathbb{P}$ there exists a.s. no infinite cluster, but as soon as the intensities are increased, a unique infinite cluster appears.

The *arm exponents* are constants that describe the asymptotic behaviour of certain unlikely events, such as that the cluster of a given vertex has large radius. These are believed to exists for most critical percolation planar models, and in addition not to depend on the model.

Our first theorem states that, for G in a large class of isoradial graphs $\mathcal{G}$, the afore-mentioned percolation measure $\mathbb{P}_G$ has the box crossing property, and thus is critical. The second result states that the one-arm exponent and the $2j$-alternating arms exponents are universal across $\mathcal{G}$. More precisely, if such an exponent exists for a model in $\mathcal{G}$, then it exist for all and its value is independent of the model.

The class $\mathcal{G}$ is defined in terms of two conditions, the bounded angles property and the square grid property. It contains, but is not limited to, all periodic isoradial graphs.

A central ingredient of the proof of both results is the star–triangle transformation. Isoradial graphs, and the measures associated, are such that whenever

we encounter a triangle in such a graph, it may be replaced by a star, without modifying connection probabilities. The converse is also true. This transformation is called the star–triangle (or Yang-Baxter) transformation.

The star–triangle transformation allows us to gradually modify the regular square lattice $\mathbb{Z}^2$ into any graph of $\mathcal{G}$. This technique is inspired by work by Baxter [1] and later Kenyon [3]. In applying these transformations certain properties, especially those related to connections, are preserved. Amongst these is the box crossing property. It is a well known fact that homogeneous percolation on $\mathbb{Z}^2$ with intensity $\frac{1}{2}$ satisfies the box crossing property, and our first theorem follows. For the second we use the same transformations, along with the newly acquired box crossing property, to obtain certain inequalities between probabilities of arm events in different isoradial graphs. The result follows easily.

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