Mathematisches Forschungsinstitut Oberwolfach

Report No. 06/2014

DOI: 10.4171/OWR/2014/06

Dirichlet Series and Function Theory in Polydiscs

Organised by
Andreas Defant, Oldenburg
Manuel Maestre, Burjassot
Eero Saksman, Helsinki
Kristian Seip, Trondheim

2 February – 8 February 2014

Abstract. The interaction between Dirichlet series and function theory in polydiscs dates back to a fundamental insight of Harald Bohr and the subsequent groundbreaking work on multilinear forms and polarization by Bohnenblust and Hille. Since around 1997, there has been a revival of interest in the research area opened up by these early contributions. The workshop reflected the status of the field and led to fruitful discussions on problems of current interest and future research directions.

Mathematics Subject Classification (2010): 30B50, 32A05, 32A07, 46B07, 46B09, 46G20, 58C10.

Introduction by the Organisers

We have in recent years seen a remarkable growth of interest in certain functional analytic aspects of the theory of ordinary Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s}.$$

Contemporary research in this field owes much to the following fundamental observation of H. Bohr [3]: By the transformation $z_j = p_j^{-s}$ (here p_j is the j-th prime number) and the fundamental theorem of arithmetic, an ordinary Dirichlet series may be thought of as a function of infinitely many complex variables $z_1, z_2, \ldots$. By a classical approximation theorem of Kronecker, this is more than just a formal transformation: If, say, only a finite number of the coefficients a_n are nonzero (so that questions about convergence of the series are avoided), the supremum of the Dirichlet polymonial $\sum a_n n^{-s}$ in the half-plane $\mathrm{Re}\, s > 0$ equals the supremum of the corresponding polynomial on the infinite-dimensional polydisc $\mathbb{D}^\infty$. In a groundbreaking work of Bohnenblust and Hille [2], it was later shown that homogeneous polynomials—the basic building blocks of functions analytic on polydiscs—may, via the method of polarization, be transformed into symmetric multilinear forms. Bohnenblust and Hille used this revolutionary insight to solve a long-standing problem in the field: Bohr had shown that the width of the strip on which a Dirichlet series converges uniformly but not absolutely is $\leq 1/2$, but Bohnenblust and Hille were able to prove that this upper estimate is even optimal.

In retrospect, one may in the work of Bohr and Bohnenblust–Hille see the seeds of a theory of Hardy H^p spaces of Dirichlet series. However, this research took place before the modern interplay between function theory and functional analysis, as well as the advent of the field of several complex variables, and the area was in many ways dormant until the late 1990's. One of the main goals of the 1997 paper of Hedenmalm, Lindqvist, and Seip [4] was to initiate a systematic study of Dirichlet series from the point of view of modern operator-related function theory and harmonic analysis. Independently, at the same time, a paper of Boas and Khavinson [1] attracted renewed attention, in the context of several complex variables, to the original work of Bohr.

The main object of study in [4] is the Hilbert space of Dirichlet series $\sum_n a_n n^{-s}$ with square summable coefficients a_n. This Hilbert space $\mathcal{H}^2$ consists of functions analytic in the half-plane $\operatorname{Re} s > 1/2$. Its reproducing kernel at s is $k_s(w) = \zeta(\overline{s} + w)$, where ζ is the Riemann zeta function. By Bohr's tranformation, $\mathcal{H}^2$ may be thought of as the Hardy space H^2 on the infinite-dimensional torus $\mathbb{T}^\infty$. One of the main results of [4] is the characterization of the space of multipliers for $\mathcal{H}^2$ as the space $\mathcal{H}^\infty$ consisting of Dirichlet series representing bounded analytic functions in $\operatorname{Re} s > 0$. This result is in line with classical results for Hardy spaces on the disc or the polydisc. It was used to solve a problem about Riesz bases of integer dilations of a single function in $L^2(0,1)$ via a transformation introduced independently by Wintner and Beurling. The paper stated and explored a number of other problems as well and revealed connections with ergodic theory and several complex variables.

The classical theory of Hardy spaces and the operators that act on them serves as an important source of incitement for the field of Dirichlet series that has evolved after 1997. Two distinct features should however be noted. First, a number of new phenomena, typically crossing existing disciplines, appear that are not present in the classical situation. Second, many of the classical objects change radically and require new viewpoints and methods in order to be properly understood and analyzed. It was recognized by the organizers that further progress requires novel and unconventional combinations of expertise from harmonic, functional, and complex analysis as well analytic number theory. The aim of the workshop was to nurture the exchange of ideas needed to accomplish such interaction across different sub-disciplines.

There were 26 participants and 18 talks given. The talks covered most of the recent developments, for instance on the Bohnenblust–Hille inequality both in the polynomial and multi-linear case, Bohr radii, Bohr's absolute convergence problem, monomial expansions of H^p functions in infinitely many variables, and Fatou-type theorems, and a survey talk on connections with the Riemann hypothesis. There were also a number of talks with a more marginal connection with the main topic of the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

References

[1] H. P. Boas and D. Khavinson. Bohr's power series theorem in several variables. *Proc. Amer. Math. Soc.*, 125(10):2975–2979, 1997.

[2] H. F. Bohnenblust and E. Hille. On the absolute convergence of Dirichlet series. *Ann. of Math. (2)*, 32(3):600–622, 1931.

[3] H. Bohr. Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der *Dirichlet*–schen Reihen $\sum \frac{a_n}{n^s}$. *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.*, 4:441–488, 1913.

[4] H. Hedenmalm, P. Lindqvist, and K. Seip. A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$. *Duke Math. J.*, 86(1):1–37, 1997.

Workshop: Dirichlet Series and Function Theory in Polydiscs

Table of Contents

Hervé Queffélec (joint with Kristian Seip)
Approximation numbers of composition operators on a Hilbert space of Dirichlet series 341

Joaquim Ortega-Cerdà (joint with Robert J. Berman)
Marcinkiewicz-Zygmund sequences on real algebraic varieties 344

Fréderic Bayart (joint with Daniel Pellegrino, Juan B. Seoane-Sepúlveda)
Asymptotic value of the Bohr radius of the n-dimensional polydisk 346

Leonhard Frerick (joint with F. Bayart, A. Defant, M. Maestre and P. Sevilla-Peris)
Monomial expansions of H_p-functions 347

Michel Balazard
Nyman's and Báez-Duarte's criteria for the Riemann hypothesis: survey and open problems 348

Pablo Sevilla (joint with Bayart-Defant-Frerick-Maestre, Carando-Defant)
Convergence in H_p-spaces of Dirichlet series 352

Michel J. G. Weber
On convergence of series of dilated functions 355

István Berkes
GCD sums, dilated series and polydisc theory 358

Håkan Hedenmalm
On Hörmander's solution of the $\bar{\partial}$-equation 361

Michael T. Lacey (joint with Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero and Brett D. Wick)
Two Weight Inequalities for Hilbert and Cauchy Transforms 361

Domingo García (joint with D. Carando, A. Defant, M. Maestre, D. Pérez-García and P. Sevilla-Peris)
Bohr's absolute convergence problem in Banach spaces 364

John E. McCarthy (joint with Jim Agler and N. J. Young)
Matrix Monotone Functions in Several Variables 367

Sunke Schlüters
The unconditional basis constant in a certain space of polynomials 369

Juan B. Seoane-Sepulveda (joint with N. Albuquerque, F. Bayart, and D. Pellegrino)
Sharp generalizations of the multilinear Bohnenblust-Hille inequality ... 370

Mieczysław Mastyło
On the fundamental problem of Fourier analysis ... 373

Christoph Thiele
The Hilbert transform along vector fields constant on Lipschitz curves .. 375

Jan-Fredrik Olsen (joint with Alexandru Aleman and Eero Saksman)
Hardy spaces in infinitely many variables ... 378

Dmitry Khavinson
Problems for Polynomials of One and several Variables ... 381

Problem Session: Dirichlet Series and Function Theory in Polydiscs ... 383

Abstracts

Approximation numbers of composition operators on a Hilbert space of Dirichlet series

Hervé Queffélec
(joint work with Kristian Seip)

Recall that the n-th approximation number $a_n(T)$, $n = 1, 2, \dots$ of an operator $T : X \to Y$ between Banach spaces is the distance, for the operator-norm, of T to the operators of rank $< n$, and that T is compact if $a_n(T) \downarrow 0$, the converse being true if for example Y has a Schauder basis. The rate of decay of those approximation numbers is, among other approaches, a quantitative way to measure the degree of compactness of T.

We will deal here with symbolic operators $T = T_\varphi$, the function φ being called the symbol of T. This study was achieved (1995) for Hankel operators by Megretskii, Peller and Treil ([6]). The authors showed that, for such operators, the sequence $(a_n(T))$ is non-increasing and otherwise arbitrary. A similar study was undertaken for composition operators C_φ on Hardy, Bergman, or Dirichlet spaces by Lefèvre, Li, Queffélec, and Rodríguez-Piazza (2012-2013) ([5], [4]). The authors proved that $a_n(C_\varphi) \geq \delta r^n$ for some $r > 0$ and that, for a compact C_φ, the sequence $(a_n(C_\varphi))$ can have arbitrarily slow decay. A prominent role is played by interpolation sequences with respect to H^∞, the multiplier space of the Hardy space H^2.

The results exposed and discussed here are taken from a recent joint work ([8]) with K.Seip, where we began a similar study for composition operators on the space $\mathcal{H}^2$ formed by Dirichlet series $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ with square-summable coefficients. This is a Hilbert space of analytic functions on the half-plane $\mathbb{C}_{1/2} = \{s : \Re s > 1/2\}$, introduced by Beurling (about 1945), and revisited by Hedenmalm, Lindqvist, and Seip ([3]), to study completeness or Riesznessproblems. The situation is here significantly different in several respects:

(1) There are few composition operators on $\mathcal{H}^2$, exactly described by a theorem of Gordon and Hedenmalm ([2]), and depending on a non-negative integer c_0. They are those of the form

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} := c_0 s + \psi(s)$$

where c_0 is a non-negative integer and the Dirichlet series ψ converges in some half-plane. We show, more precisely, that ψ converges uniformly in each half-plane $\mathbb{C}_\varepsilon = \{s : \Re s > \varepsilon\}$ where $\varepsilon > 0$.

(2) The space $\mathcal{H}^\infty$ of multipliers of $\mathcal{H}^2$ does not live on the same half-plane $\mathbb{C}_{1/2}$ as $\mathcal{H}^2$, but on the half-plane $\mathbb{C}_0 = \{s : \Re s > 0\}$. We circumvent this difficulty by avoiding the appeal to $\mathcal{H}^\infty$ and by using weighted interpolation on the space $\mathcal{H}^2$ itself.

(3) The study of composition operators on $\mathcal{H}^2$ quite often leads, through the Bohr point of view, to the study of composition operators in several variables, namely on the polydisk of $\mathbb{C}^d$, $d \leq \infty$, a study still full of mysteries especially when $d = \infty$. Here, we need to use *two* Hilbert spaces.

In this presentation of the work with K.Seip ([8]), I will focus on our main tools, which can roughly be divided in two parts:
a rather soft part coming from Functional Analysis (alternative definitions of approximation numbers, Riesz upper and lower sequences in Hilbert spaces, spectrum and Weyl's inequalities, etc).
A rather hard part coming from Function Theory (Carleson measures, interpolation sequences, d-bar corrections, etc), the two parts being connected by a simple but useful "mapping equation" using the reproducing kernel of the Hilbert space involved. *Here are our main results:*

(1) Suppose that c_0 is a nonnegative integer and that $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ generates a compact composition operator C_φ on $\mathcal{H}^2$.
   (a) If $c_0 = 0$, then $a_n(C_\varphi) \gg r^n$ for some $0 < r < 1$.
   (b) If $c_0 = 1$, then $a_n(C_\varphi) \gg n^{-\Re c_1 - \varepsilon}$ for every $\varepsilon > 0$.
   (c) If $c_0 > 1$, then $a_n(C_\varphi) \gg n^{-A}$ for some $A > 0$.

   In particular, if $c_0 \geq 1$, the approximation numbers decay quite slowly. The three estimates are optimal. In particular, we can have $a_n(C_\varphi) \leq Cr^n$ with $r < 1$ for symbols of the form
$$\varphi(s) = c_1 + c_2 2^{-s} \text{ with } \Re c_1 > |c_2| + \frac{1}{2}.$$
   The proof relies on a general lower bound, which is the following:

   Suppose that $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ determines a bounded composition operator C_φ on $\mathcal{H}^2$. Let $S = (s_j)$ and $S' = (s'_j)$ be finite sets in $\mathbb{C}_{1/2}$, both of of cardinality n, such that $\varphi(s'_j) = s_j$ for every j. Then
$$a_n(C_\varphi) \geq [M_{\mathcal{H}^2}(S)]^{-1} \|\mu_{S',\mathcal{H}^2}\|_{\mathcal{C},\mathcal{H}^2}^{-1/2} \inf_j \left(\frac{\zeta(2\Re s_j)}{\zeta(2\Re s'_j)} \right)^{1/2}.$$
   Here, $M_{\mathcal{H}^2}(S)$ and $\|\mu_{S',\mathcal{H}^2}\|_{\mathcal{C},\mathcal{H}^2}$ denote the interpolation constant and the Carleson constant of S and S', and ζ the Riemann zeta function. The issue is to estimate sharply each of the the three terms appearing in the RHS. This is mainly done by passing to the more familiar Hardy space of the half-plane $\mathbb{C}_{1/2}$ and then by comparing the interpolation and Carleson sequences of both spaces through a d-bar correction argument recently introduced by the second-named author ([9]).

(2) We give a general lower and upper estimate, in terms of Blaschke products, Carleson measures, interpolation sequences.

(3) As a consequence, we obtain a rather sharp estimate (up to a logarithmic factor) of those approximation numbers for polynomials symbols, in terms of the so-called "dimension" of the symbol. A polynomial symbol is one of the form $\varphi(s) = c_1 + \sum_{j=1}^{d} c_{q_j} q_j^{-s}$ where the positive integers $q_1, \ldots, q_d$

are "independent " and where we are in the limiting case

$$\kappa(\varphi) := \Re c_1 - \sum_{j=1}^{d} |c_{q_j}| = 1/2.$$

Our estimate then is

$$a\left(\frac{1}{n}\right)^{(d-1)/2} \leq a_n(C_\varphi) \leq b\left(\frac{\log n}{n}\right)^{(d-1)/2}$$

where a, b are positive constants. This latter result is a nearly optimal improvement of a previous result of the author with C.Finet and S.Volberg ([1]). In particular, if $d = 2$, we recover the fact that C_φ is compact, not Hilbert-Schmidt (not in the Schatten class S_2) and in the Schatten class S_4 ([1]), but moreover obtain a much sharper result: C_φ belongs to the Schatten class S_p for each $p > 2$.

(4) The fact that, for compact operators C_φ on $\mathcal{H}^2$, the approximation numbers can decay quite slowly. Namely, thanks to a transference principle, and exploiting results in the H^2-case (the Hardy space of the disk) proved in a paper of the first named author with D.Li and L.Rodriguez-Piazza ([5]), or in a sharpened form in a more recent paper of the two authors of this work ([7]), one can show the following:

if $\varepsilon_n \downarrow 0$, there exists a compact operator C_φ on $\mathcal{H}^2$ such that

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

References

[1] C. Finet, H. Queffélec, and A. Volberg, Compactness of composition operators on a Hilbert space of Dirichlet series, *J. Funct. Anal.* **211** (2004), 271–287.

[2] J. Gordon and H. Hedenmalm, The composition operators on the space of Dirichlet series with square-summable coefficients, *Michigan Math. J.* **46** (1999), 313–329.

[3] H. Hedenmalm, P. Lindqvist, and K. Seip, A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$, *Duke Math. J.* **86** (1997), 1–37.

[4] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Approximation numbers of composition operators on the Dirichlet space, *Ark. Mat.*, to appear.

[5] D. Li, H. Queffélec, L. Rodríguez-Piazza, On approximation numbers of composition operators *J. Approx. Theory* **164** (4) (2012), 431–459.

[6] V. Megretskii,V.Peller, S.Treil, The inverse problem for self-adjoint Hankel operators, *Acta Mathematica* **174** (1995), 241–309.

[7] H. Queffélec, K. Seip, Decay rates for approximation numbers of composition operators, *J. Anal. Math.*, to appear.

[8] H. Queffélec, K. Seip, Approximation numbers of composition operators on the H^2 space of Dirichlet series, *arXiv.org*, 2013.

[9] K. Seip, Zeros of functions in Hilbert spaces of Dirichlet series, *Math. Z.,* **274**, (3-4) (2013), 1327-1339.

Marcinkiewicz-Zygmund sequences on real algebraic varieties

Joaquim Ortega-Cerdà
(joint work with Robert J. Berman)

The main goal of the work presented is to generalize the Landau type necessary density conditions for sampling in the Paley-Wiener space to a general setting where the Hilbert space of functions $H_k(M)$ consist of polynomials of degree at most k restricted to a smooth affine real algebraic variety M. We equip the polynomials with the L^2 norm

$$\|p_k\|^2_{L^2(\mu)} := \int_M |p_k|^2 \, d\mu$$

defined by a volume form μ on M. With this scalar product $H_k(M)$ becomes a reproducing kernel Hilbert space. We denote by $K_k(x,y)$ the corresponding reproducing kernel.

A sequence Λ_k of points on M is said to be *Marcinkiewicz-Zygmund* for $H_k(M)$ if the family of normalized functions

$$\kappa_{x_i} := K_k(\cdot, x_i^{(k)})/\|K_k(\cdot, x_i^{(k)})\|$$

for $x_i^{(k)} \in \Lambda_k$, form a *frame* in the Hilbert space $H_k(M)$, in the sense of Duffin-Schaeffer, i.e.:

$$\frac{1}{C}\|f\|^2 \le \sum_{x_i\in\Lambda_k} |\langle f, \kappa_{x_i}\rangle|^2 \le C\|f\|^2,$$

which is equivalent to the sampling inequalities:

$$\frac{1}{C}\|f\|^2 \le \sum_{x_i\in\Lambda_k} \frac{|f(x_i)|^2}{K_k(x_i,x_i)} \le C\|f\|^2 \tag{1}$$

where we will assume that C can be taken to be independent of k.

Condition (1) may seem a bit abstract since we don't have an explicit expression for the reproducing kernel K_k but observe that we only need the asymptotics $K_k(x,x)$ as $k \to \infty$. In our setting one first result is that $K_k(x,x) \simeq k^n$ uniformly on M.

Our main result is the following:

Theorem. *Let M be an n-dimensional affine real algebraic variety, which is non-singular and compact, let μ be a volume form on M. A necessary condition for a sequence Λ_k of sets of points on X to be Marcinkiewicz-Zygmund for $H_k(M)$ is that the density of the points is at least equal to the density of the equilibrium measure μ_{eq} of M, as $k \to \infty$,*

$$\liminf_{k\to\infty} \frac{\#\{\Lambda_k \cap \Omega\}}{\#N_k} \ge \frac{\mu_{eq}(\Omega)}{\mu_{eq}(M)}$$

for any given smooth domain Ω in X.

The density is given in terms of the equilibrium measure μ_{eq} on M. In order to define μ_{eq} we need to consider "complexifications" X and $H_k(X)$ of the real variety M and the real vector space $H_k(M)$, respectively. More precisely, X is the complex algebraic variety in $\mathbb{C}^m$ defined by the common complex zeroes of the ideal defining M and $H_k(X)$ is the complex vector space consisting of restrictions to X of polynomials in $\mathbb{C}^m$ of total degree at most k. Then M is indeed the real part of X in the sense that it consists of all points in z in X such that $\bar{z} = z$ and real vector space $H_k(M)$ is the the real part of $H_k(X)$ in the sense that it consists of all p_k in $H_k(X)$ such that $\overline{p_k} = p_k$ (restricted to M).

The equilibrium measure μ_{eq} is defined as the Monge-Ampere measure of the Siciak extremal function v_M attached to the compact subset M of the affine complex algebraic variety X

Let M be an n-dimensional affine real algebraic variety, which is non-singular and compact. Then M is non-pluripolar and regular. Moreover, the equilibrium measure μ_M is absolutely continuous with respect to the Lebesgue measure dV_M on M and its density is bounded from above and below by:

$$\frac{1}{C} dV_M \leq \mu_{eq} \leq C dV_M$$

on M.

One of the key ingredients on the proof of the theorem comes from the convergence of the Bergman measure to the equilibrium measure in a very general context as proved in [1].

Some of the ideas for the proof of density of the Marcinkiewicz-Zygmund sequences arise from a new proof of Landau necessary conditions for sampling sequences in the context of the Paley-Wiener space that appear in [3] coupled with some elements of optimal transport theory that were used in [2].

References

[1] R. Berman; S. Boucksom and D. Witt Nyström, Fekete points and convergence towards equilibrium measures on complex manifolds, Acta Mathematica 207 (2011), 1–27.

[2] N. Lev, J. Ortega-Cerdà. Equidistribution estimates for Fekete points on complex manifolds, to appear in JEMS.

[3] S. Nitzan;A. Olevskii, Revisiting Landau's density theorems for Paley-Wiener spaces. C. R. Math. Acad. Sci. Paris 350 (2012), no. 9-10, 509-512.

Asymptotic value of the Bohr radius of the n-dimensional polydisk

Fréderic Bayart
(joint work with Daniel Pellegrino, Juan B. Seoane-Sepúlveda)

Following Boas and Khavinson [1], the Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z\in r\mathbb{D}^n}\sum_\alpha |a_\alpha z^\alpha| \le \sup_{z\in\mathbb{D}^n}\left|\sum_\alpha a_\alpha z^\alpha\right|.$$

The Bohr radius K_1 was studied and estimated by H. Bohr himself, and it was shown independently by M. Riesz, I. Schur and F. Wiener that $K_1 = 1/3$. For $n \ge 2$, exact values or K_n are unknown. However, in [1], the two inequalities

$$\frac{1}{3}\sqrt{\frac{1}{n}} \le K_n \le 2\sqrt{\frac{\log n}{n}} \tag{1}$$

were established.

The paper of Boas and Khavinson was a source of inspiration for many subsequent papers, linking the asymptotic behaviour of K_n to various problems in functional analysis (geometry of Banach spaces, unconditional basis constant of spaces of polynomials,...). Hence there was a big interest in recent years in determining the behaviour of K_n for large values of n.

In [2], the authors showed that

$$K_n = b_n\sqrt{\frac{\log n}{n}} \text{ with } \frac{1}{\sqrt{2}} + o(1) \le b_n \le 2.$$

We have been able to delete the unpleasant factor $\frac{1}{\sqrt{2}}$ to show that

$$K_n \sim_{+\infty} \sqrt{\frac{\log n}{n}}.$$

The main tool used to prove this is a substantial improvement of the polynomial Bohnenblust–Hille inequality which says the following: for any $m \ge 1$, there exists a constant $D_m \ge 1$ such that, for any $n \ge 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \le D_m\|P\|_\infty,$$

where $\|P\|_\infty = \sup_{z\in\mathbb{D}^n}|P(z)|$. The best constant D_m in this inequality will be denoted by $\mathrm{B}_m^{\mathrm{pol}}$. We have shown that, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\mathrm{B}_m^{\mathrm{pol}} \le C(\varepsilon)(1+\varepsilon)^m.$$

In turn, to prove this last result, we improve a class of inequalities which are very useful in Harmonic Analysis, the so-called Blei's inequalities. The proof of our variants of Blei's inequalities use interpolation, allowing a simpler and clearer proof of them.

References

[1] H. Boas, D. Khavinson, *Bohr's power series theorem in several variables*, Proc. Amer. Math. Soc. **125** (1997), 2975–2979.

[2] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, K. Seip, *The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive*, Annals of Math. **174** (2011), 485–497.

Monomial expansions of H_p-functions

Leonhard Frerick

(joint work with F. Bayart, A. Defant, M. Maestre and P. Sevilla-Peris)

Let $f \in H_\infty(B_{c_0})$, i.e.

$$f \sim \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha$$

is a bounded holomorphic function on B_{c_0} (the open unit ball of c_0). Here $\mathbb{N}_0^{(\mathbb{N})} = \{(\alpha_1, \dots, \alpha_n, 0, \dots) : \alpha_j \in \mathbb{N}_0\}$. We consider the problem for which $z \in B_{c_0}$ the above expansion converges, i.e. when

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |\frac{\partial^\alpha f(0)}{\alpha!} z^\alpha| < \infty.$$

By a result of Cole and Gamelin $H_\infty(B_{c_0}) = H_\infty(\mathbb{T}^{\mathbb{N}})$ ($\mathbb{T}$ the torus), so we can consider the more general problem replacing $H_\infty(B_{c_0})$ by $H_p(\mathbb{T}^{\mathbb{N}}) =: H_p$.

More precisely, for $F \subset H_1$ we define the set of monomial convergence of F as

$$\text{mon } F := \{z \in B_{c_0} : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |\frac{\partial^\alpha f(0)}{\alpha!} z^\alpha| < \infty \text{ for all } f \in F\}.$$

Besides the above mentioned H_p we can consider for F also spaces of $m-$homogeneous polynomials, i.e.

$$F = H_p^m := \{f \in H_p : f \text{ is a } m - \text{homogeneous polynomial}\}.$$

Let us recall some known results about mon F: Toeplitz showed that $\ell_{4+\varepsilon} \not\subset$ mon H_∞^2 and Bohr proved $\ell_2 \cap B_{c_0} \subset$ mon H_∞. Moreover, Bohnenblust and Hille proved $\ell_{\frac{2m}{m-1}} \subset$ mon H_∞^m. Recently, Defant Maestre and Prengel showed mon $H_\infty \subset \ell_{2+\varepsilon}$ for all $\varepsilon > 0$ and mon $H_\infty^m \subset \ell_{\frac{2m}{m-1},\infty}$. Our contributions to the problem of characterizing mon F are the following results:

- mon $H_p^m = \ell_2, 1 \le p < \infty$,
- mon $H_p = \ell_2 \cap B_{c_0}, 1 \le p < \infty$,
- mon $H_\infty^m = \ell_{\frac{2m}{m-1},\infty}$,
- $\{z \in B_{c_0} : \limsup_n \frac{1}{\log n} \sum_{j=1}^n (z_j^\star)^2 < 1\} \subset$ mon H_∞

$\subset \{z \in B_{c_0} : \limsup_n \frac{1}{\log n} \sum_{j=1}^n (z_j^\star)^2 \leq 1\}$.
Here $(z_1^\star, z_2^\star, \dots)$ is the decreasing rearrangement of $z = (z_1, z_2, \dots)$.
Hedenmalm, Lindquist, Seip and Bayart proved that the Bohr map is an isometric isomorphism between H_p and the corresponding space $\mathbb{H}_p$ of Dirichlet series. Using these results we can apply our descriptions of the domains of monomial convergence to multiplicative ℓ_1- multipliers for $\mathbb{H}_p$ and for $\mathbb{H}_p^m$:
If $b \in B_{c_0}$ is multiplicative, then

$$\sum_{n=1}^{\infty} |a_n b_n| < \infty \text{ for all } \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \in \mathbb{H}_p$$

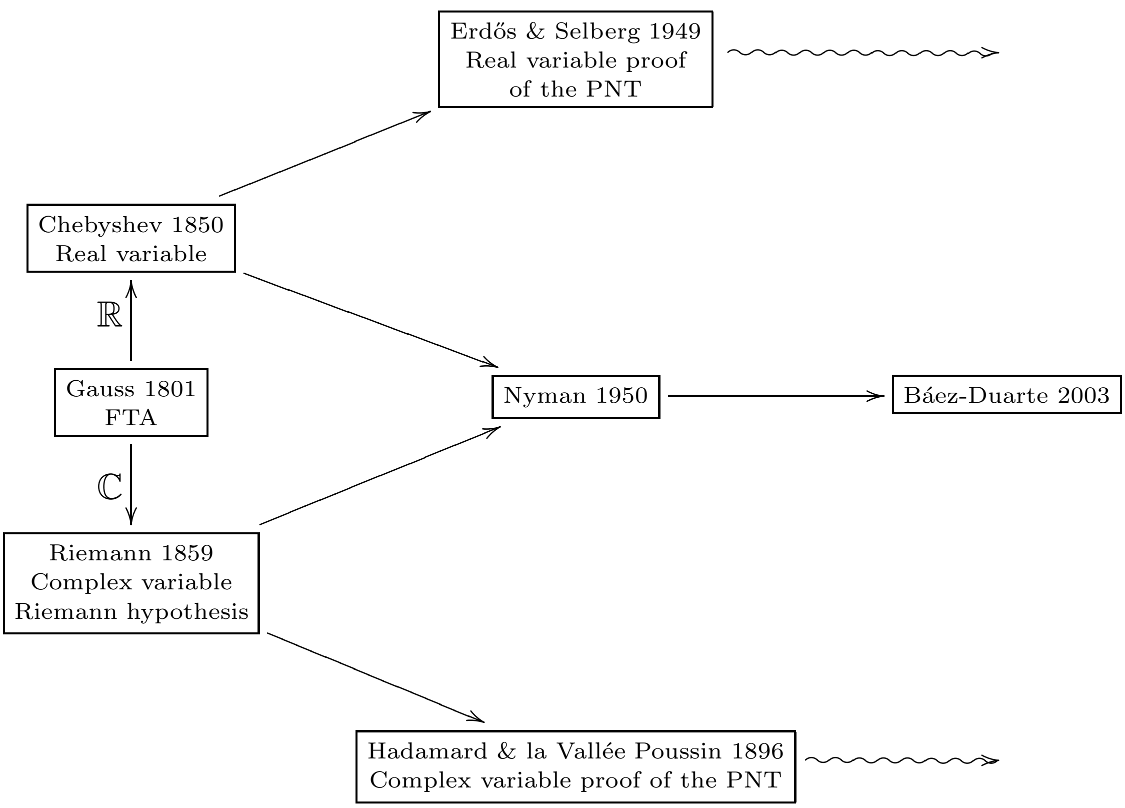
if and only if $(b_{p_n})_{n \in \mathbb{N}} \in$ mon H_p. Here $(p_1, p_2, \dots)$ is the sequence of prime numbers. Clearly, a corresponding result for $\mathbb{H}_p^m$ also holds.

Nyman's and Báez-Duarte's criteria for the Riemann hypothesis: survey and open problems

Michel Balazard

1. Nyman's and Báez-Duarte's criteria in historical perspective

The following diagram displays Nyman's and Báez-Duarte's contributions within the ongoing quest for understanding the distribution of prime numbers.



Knowing the fundamental theorem of arithmetic (FTA), one can view the problem of the distribution of prime numbers as an *inverse problem*: to deduce from the flawless regularity of the sequence of integers some necessary level of regularity in the sequence of its “multiplicative atoms”, the prime numbers. Two approaches to this inverse problem have been proposed: either by real or by complex analysis.

The real variable route was opened in 1850 by Chebyshev. The core of his contribution was to show the relevance of approximate formulas of the following type

$$\chi(x) \approx \sum_{\alpha \geqslant 1} c_\alpha \{x/\alpha\}, \tag{1}$$

where $\chi(x) = [x \geqslant 1]$ is the (multiplicative) Heaviside function, and $\{x\}$ is the *fractional part* function.

About one century later, Selberg found a new reformulation of the FTA as an identity between functions of a real variable. This was the starting point of an inversion process performed by himself and Erdős, and eventually leading in 1949 to real variable proofs of the prime number theorem (PNT)

$$\pi(x) = \sum_{p \leqslant x} 1 \sim \mathrm{li}(x) = \int_0^x \frac{dt}{\ln t} \quad (x \to \infty).$$

The upper undulated arrow of our diagram indicates that the real variable approach to the distribution of prime numbers is still being developed.

In 1859 Riemann introduced the complex variable in this theory and expressed $\pi(x)$ as a sum involving the zeros of the analytic continuation of

$$\zeta(s) = \sum_n n^{-s}.$$

He observed that the equality $\Re\rho = \frac{1}{2}$ held true very likely for all the complex zeros ρ of ζ : this is the famous Riemann hypothesis (RH), still unproved 155 years later.

Following Riemann's ideas, Hadamard and la Vallée Poussin gave the first proofs of the PNT in 1896. Moreover, RH is equivalent to a quantitative statement about the distribution of prime numbers : for every $\varepsilon > 0$,

$$\pi(x) = \mathrm{li}(x) + O(x^{1/2+\varepsilon}). \tag{2}$$

The lower undulated arrow of our diagram indicates the later development of the complex variable approach to the asymptotic study of $\pi(x)$.

In 1950 Nyman, a student of Beurling, gave in his thesis [12] an equivalent form of the Riemann hypothesis in terms of Chebyshev's approximations (1) : RH is true if, and only if for every positive δ the inequality

$$\int_0^\infty |\chi(t) - \sum_{\alpha \geqslant 1} c_\alpha e_\alpha(t)|^2 \frac{dt}{t^2} < \delta \tag{3}$$

can be achieved by a suitable choice of the coefficients c_α, where $e_\alpha(t) = \{t/\alpha\}$.

In 2003 Báez-Duarte strengthened Nyman's criterion by proving that in (3) one could restrict the α's to *integer* values (cf. [1]).

Our first open problem is methodological.

Question 1. *Is there a real variable proof of the equivalence of* (2) *and* (3)*?*

2. The autocorrelation function $A(\lambda)$

Trying to approximate χ in $L = L^2(0, \infty; t^{-2}dt)$ by linear combinations of the e_α leads to the consideration of the inner products

$$\langle e_\alpha, e_\beta \rangle = \int_0^\infty \{t/\alpha\}\{t/\beta\} \frac{dt}{t^2} = \frac{1}{\alpha} A(\alpha/\beta),$$

where

$$A(\lambda) = \int_0^\infty \{t\}\{\lambda t\} \frac{dt}{t^2},$$

is the *multiplicative autocorrelation function of the "fractional part" function.*

No useful reformulation of RH in terms of $A(\lambda)$ has been found yet. Nevertheless, this function is an interesting mathematical object in itself. The main results obtained so far are the following :

• Vasyunin's explicit formula for $A(\lambda)$ if $\lambda = h/k$ is rational, with h, k positive integers without common factor, in terms of

$$V(h,k) = \sum_{j=1}^{k-1} \{jh/k\} \cot(j\pi/k),$$

the *Vasyunin sum* (1995, cf. [15]). The Vasyunin sum, along with variants and generalizations, has been the object of recent work by Bettin and Conrey (2013, cf. [8], [9]), and by Rassias (2014, cf. [14]).

• A is a continuous function on $[0, \infty[$ with a strict local maximum at every positive rational (Báez-Duarte *et al* 2005, cf. [3]). Let us recall the following question from [3].

Question 2. *Does there exist a positive irrational where A has a strict local maximum?*

• A is differentiable at λ if, and only if λ is an irrational number such that the series

$$\sum_{j \geqslant 0} (-1)^j \frac{\log q_{j+1}}{q_j}$$

converges, where $(q_j)_{j \geqslant 0}$ denotes the increasing sequence of the denominators of the convergents to the continued fraction of λ (Balazard and Martin 2013, cf. [4], see also la Bretèche and Tenenbaum [10]).

3. The distance d_n

Let d_n denote the distance in L between χ and the vector space generated by $e_1, \dots, e_n$.Báez-Duarte's criterion states that RH is equivalent to the fact that d_n tends to 0 as $n \to \infty$. The main conjecture about d_n is the asymptotic formula

$$d_n^2 \sim \frac{2+\gamma-\ln 4\pi}{\ln n} \quad (n \to \infty) \tag{4}$$

(conjecture 2 of [2]). Burnol has shown that

$$d_n^2 \gtrsim \frac{2+\gamma-\ln 4\pi}{\ln n} \quad (n \to \infty)$$

and that (4) implies not only RH, but also the simplicity of the zeros of ζ (cf. [11][1]). In the opposite direction, Bettin, Conrey, Farmer (2013, cf. [7]) have shown that (4) is a consequence of RH and the following quantitative version of the simplicity hypothesis :

$$\sum_{|\Im\rho|\leqslant T} \frac{1}{|\zeta'(\rho)|} \ll T^c,$$

where $c < 3/2$.

4. A relevant Hilbert space of Dirichlet series

Define E^2 as the space of Dirichlet series $D(s) = \sum_n a_n n^{-s}$ such that

- $D(s)$ converges for some s ;
- $D(s)/s$ has an analytic continuation $F(s)$ to the half-plane $\Re s > 1/2$;
- F belongs to the Hardy space H^2 of the half-plane $\Re s > 1/2$.

One provides a Hilbert space structure on E^2 by putting $\|D\|_{E^2} = \|F\|_{H^2}$. Using Plancherel's theorem, one can prove that

$$\|D\|^2 = \sup_{\sigma>1/2} \int_{-\infty}^{\infty} |D(\sigma+i\tau)|^2 \frac{d\tau}{\sigma^2+\tau^2} = 2\pi \int_0^{\infty} |S(t)|^2 \frac{dt}{t^2},$$

where $S(t) = \sum_{n\leqslant t} a_n$ is the sum function of the sequence of coefficients of $D(s)$. Thus the map $D \mapsto (2\pi)^{1/2} S$ defines a Hilbert space isomorphism between E^2 and the subspace of L of functions which assume a constant value on each interval $(n, n+1)$ (n integer).

Now for each positive integer n define $T_n : E^2 \to E^2$ by $T_n D(s) = n^{-s} D(s)$. The set $\mathcal{T} = \{T_n, n\geqslant 1\}$ is a semi-group of continuous operators on E^2. A general problem is to describe the structure of those closed subspaces V of E^2 which are $\mathcal{T}$-*invariant*, meaning that $T_n V \subset V$ for each n.

Let us end with a specific question. Define V as the closure in E^2 of the set

$$\left\{ \zeta(s) \sum_{k=1}^{n} \frac{c_k}{k^s},\ n\geqslant 1,\ c_k \in \mathbb{C},\ \sum_{k=1}^{n} \frac{c_k}{k} = 0 \right\}.$$

The subspace V is $\mathcal{T}$-invariant.

[1] See also [13] for a new proof of the estimate $d_n^2 \gg (\ln n)^{-1}$.

Question 3. *Is it true that*

$$V = \{D \in E^2,\ D(s)/\zeta(s) \text{ is holomorphic for } \Re s > 1/2\}\ ? \tag{5}$$

We note that, if RH is true, the right-hand side of (5) is just E^2. In this case, one can deduce from Báez-Duarte's criterion that V also coincides with E^2. The point is to prove or disprove (5) unconditionally ; the analogous question in the context of Nyman's criterion is known to have a positive answer (cf. Bercovici & Foias [6], or [5]).

References

[1] L. Báez-Duarte – " A strengthening of the Nyman-Beurling criterion for the Riemann hypothesis", *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **14** (2003), p. 5–11.

[2] L. Báez-Duarte, M. Balazard, B. Landreau and E. Saias – " Notes sur la fonction ζ de Riemann, 3", *Adv. Math.* **149** (2000), p. 130–144.

[3] ———, " Étude de l'autocorrélation multiplicative de la fonction " partie fractionnaire"", *Ramanujan J.* **9** (2005), p. 215–240.

[4] M. Balazard and B. Martin – " Sur l'autocorrélation multiplicative de la fonction *partie fractionnaire* et une fonction définie par J. R. Wilton", (2013), hal-00823899

[5] M. Balazard and E. Saias – " The Nyman-Beurling equivalent form for the Riemann hypothesis", *Expo. Math.* **18** (2000), p. 131–138.

[6] H. Bercovici and C. Foias – " A real variable restatement of Riemann's hypothesis", *Israel J. Math.* **48** (1984), p. 57–68.

[7] S. Bettin, J. B. Conrey and D. W. Farmer – " An optimal choice of Dirichlet polynomials for the Nyman–Beurling criterion", *Proc. Steklov Inst. Math.* **280** Supp. 2 (2013), p. S30–S36.

[8] S. Bettin – " A generalization of Rademacher's reciprocity law", *Acta Arith.* **159** (2013), p. 363–374.

[9] S. Bettin and B. Conrey – " Period functions and cotangent sums", *Algebra Number Theory* **7** (2013), p. 215–242.

[10] R. de la Bretèche and G. Tenenbaum – " Dérivabilité ponctuelle d'une intégrale liée aux fonctions de Bernoulli", preprint.

[11] J.-F. Burnol – " A lower bound in an approximation problem involving the zeros of the Riemann zeta function", *Adv. Math.* **170** (2002), p. 56–70.

[12] B. Nyman – *On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces*, Thesis, University of Uppsala, 1950.

[13] M. Radziwill – " Limitations to mollyfying $\zeta(s)$", (2012), arxiv:1207.6583v3

[14] M. Rassias – " A cotangent sum related to the zeros of the Estermann zeta function", preprint.

[15] V. I. Vasyunin – " On a biorthogonal system associated with the Riemann hypothesis", *Algebra i Analiz* **7** (1995), p. 118–135.

Convergence in H_p-spaces of Dirichlet series

Pablo Sevilla

(joint work with Bayart-Defant-Frerick-Maestre, Carando-Defant)

A Dirichlet series is a formal series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$, with $a_n \in \mathbb{C}$ and $s \in \mathbb{C}$. It is a well know fact that the maximal domains of convergence of Dirichlet series are half-plains. We consider the abscissas that define the maximal half-plains on which a given Dirichlet series converges, converges uniformly and converges absolutely:

$$\sigma_c = \inf\{\sigma\colon \sum a_n n^{-s} \text{ converges on } [\mathrm{Re}s > \sigma]\}$$
$$\sigma_u = \inf\{\sigma\colon \sum a_n n^{-s} \text{ converges uniformly on } [\mathrm{Re}s > \sigma]\}$$
$$\sigma_a = \inf\{\sigma\colon \sum a_n n^{-s} \text{ converges absolutely on } [\mathrm{Re}s > \sigma]\}$$

It is easily seen that for every Dirichlet series $\sigma_a - \sigma_c \leq 1$ and that for $\sum_{n=1}^{\infty}(-1)^n n^{-s}$ we have $\sigma_a - \sigma_c = 1$.

Where it converges, the Dirichlet series defines a holomorphic function. We can then consider a fourth abscissa

$$\sigma_b = \inf\{\sigma\colon \sum a_n n^{-s} \text{ defines a bounded, holomorphic function on } [\mathrm{Re}s > \sigma]\}$$

Harald Bohr, in a series of papers (see e.g. [5, 6]) started the study of these abscissas. He proved that $\sigma_b = \sigma_u$ and considered the number

$$S = \sup\{\sigma_a - \sigma_u\colon \text{ Dirichlet series}\}$$

and showed that $S \leq 1/2$. The problem (sometimes called *Bohr's absolute convergence problem*) was open for over 15 years, until Bohnenblust and Hille proved in [4] that actually $S = 1/2$. A complete account on Bohr's problem and its solution by Bohnenblust and Hille can be found in [9].

Recently these results have been revisited using tools and techniques of functional analysis. If $\mathcal{H}_\infty$ denotes the Banach space of Dirichlet series that define a bounded, holomorphic function on $[\mathrm{Re}s > 0]$ (with the norm $\|\sum a_n n^{-s}\| = \sup_{[\mathrm{Re}s>0]} |\sum_{n=1}^{\infty} a_n \frac{1}{n^s}|$), then we can rewrite

$$\sigma_u = \inf\{\sigma\colon \sum \tfrac{a_n}{n^\sigma} n^{-s} \in \mathcal{H}_\infty\}$$

and $S = \sup_{\mathcal{H}_\infty} \sigma_a$. On the other hand we consider $H_\infty(B_{c_0})$, the Banach space of holomorphic (i.e. Fréchet differentiable), bounded functions on B_{c_0} (the open unit ball of c_0), with the norm $\|f\| = \sup_{z\in B_{c_0}} |f(z)|$. Then a very important result of Hedenmalm, Lindvist and Seip [10] (see also [5]) shows that $\mathcal{H}_\infty = H_\infty(B_{c_0})$ isometrically. The isometry is given in the following way: let us denote by $(p_n)_n$ the sequence of prime numbers, then if $n \in \mathbb{N}$ we take its prime decomposition $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k} = p^\alpha$ and we do

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha(f) z^\alpha \leftrightsquigarrow \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{with } a_n = a_{p^\alpha} = c_\alpha(f)\,,$$

here $c_\alpha(f)$ are the coefficients of f and are calculated as follows: if $\alpha = (\alpha_1, \dots, \alpha_k)$, take the restriction of f to $\mathbb{D}^k$ and compute the α-th coefficient using the Cauchy integral formula. The mapping defined in this way is sometimes called the *Bohr transform*. With this isometry, S equals the infimum over all σ such that

$$(1) \qquad \Big(\frac{1}{p_n^\sigma}\Big)_n \in \mathrm{mon}H_\infty(B_{c_0}) = \{z \in B_{c_0}\colon \sum |c_\alpha(f) z^\alpha| < \infty \;,\; \forall f \in H_\infty(B_{c_0})\}\,.$$

We consider now the infinite dimensional torus $\mathbb{T}^{\mathbb{N}}$ and $d\omega$ the Haar measure on it. We define for $1 \leq p < \infty$ the space $H_p(\mathbb{T}^{\mathbb{N}})$ consisting of those functions

$f \in L_p(\mathbb{T}^{\mathbb{N}})$ for which $\widehat{f}(\alpha) \neq 0$ (the α-th Fourier coefficient) only if $\alpha_j \geq 0$ for all j. Then the Hardy space of Dirichlet series $\mathcal{H}_p$ is defined to be the image via the Bohr transform of $H_p(\mathbb{T}^{\mathbb{N}})$ and we have a new abscissa

$$\sigma_p = \inf\{\sigma\colon \sum \tfrac{a_n}{n^\sigma} n^{-s} \in \mathcal{H}_p\} \,.$$

These spaces were introduced by Bayart [2], and he showed that

$$S_p = \sup\{\sigma_a - \sigma_p\colon \text{ Dirichlet series}\} = \sup_{\mathcal{H}_p} \sigma_a = \frac{1}{2} \,.$$

We look at these results from a different point of view, focusing on the side of power series in infinitely many variables. We define the set

$$\text{mon} H_p(\mathbb{T}^{\mathbb{N}}) = \{z \in B_{c_0}\colon \sum |\widehat{f}(\alpha) z^\alpha| < \infty \ , \forall f \in H_p(\mathbb{T}^{\mathbb{N}})\}$$

and show [3]

$$\text{mon} H_p(\mathbb{T}^{\mathbb{N}}) = \ell_2 \cap B_{c_0} \,.$$

From this, proceeding as in (1) we recover the fact that $S_p = 1/2$. We also recover results in [1] that give that there are Dirichlet series in $\mathcal{H}_p$ for which $\sum |\frac{a_n}{n^{1/2}}| = \infty$.

Finally we consider Dirichlet series $\sum a_n n^{-s}$ with $a_n \in X$ (a Banach space). We define the spaces $\mathcal{H}_p(X)$ for $1 \leq p \leq \infty$ and abscissas σ_a^X, σ_u^X and σ_p^X in an analogous way. The case $p = \infty$ was treated in [8], where it was shown that

$$S(X) = \sup \sigma_a^X - \sigma_u^X = 1 - \frac{1}{\cot X} \,,$$

where $\cot X$ is the optimal cotype of X. A natural question now is if, like in the scalar case, all these strips are equal and we show that, indeed, this is the case [7]:

$$S_p(X) = \sup \sigma_a^X - \sigma_p^X = \sup_{\mathcal{H}_p(X)} \sigma_a^X = 1 - \frac{1}{\cot X} \,.$$

References

[1] R. Balasubramanian, B. Calado, and H. Queffélec. The Bohr inequality for ordinary Dirichlet series. *Studia Math.*, 175(3):285–304, 2006.

[2] F. Bayart. Hardy spaces of Dirichlet series and their composition operators. *Monatsh. Math.*, 136(3):203–236, 2002.

[3] F. Bayart, A. Defant, L. Frerick, M. Maestre, and P. Sevilla-Peris. Monomial series expansion of h_p-functions and multipliers of $\mathcal{H}_p$-Dirichlet series. *preprint.*

[4] H. F. Bohnenblust and E. Hille. On the absolute convergence of Dirichlet series. *Ann. of Math. (2)*, 32(3):600–622, 1931.

[5] H. Bohr. Über die Bedeutung der Potenzreihen unendlich vieler Variablen in der Theorie der *Dirichlet*–schen Reihen $\sum \frac{a_n}{n^s}$. *Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl.*, pages 441–488, 1913.

[6] H. Bohr. Über die gleichmäßige Konvergenz Dirichletscher Reihen. *J. Reine Angew. Math.*, 143:203–211, 1913.

[7] D. Carando, A. Defant, and P. Sevilla-Peris. Bohr's absolute convergence problem for $\mathcal{H}_p$-dirichlet series in banach spaces. *Anal. PDE*, to appear.

[8] A. Defant, D. García, M. Maestre, and D. Pérez-García. Bohr's strip for vector valued Dirichlet series. *Math. Ann.*, 342(3):533–555, 2008.

[9] A. Defant and P. Sevilla-Peris. The BohnenblustHille cycle of ideas from a modern point of view. *Funct. Approx. Comment. Math.*, to appear.
[10] H. Hedenmalm, P. Lindqvist, and K. Seip. A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$. *Duke Math. J.*, 86(1):1–37, 1997.

On convergence of series of dilated functions

Michel J. G. Weber

One of the oldest and most central problems in the theory of systems of dilated sums is the study of the convergence in norm or almost everywhere of the series $\sum_{k=1}^{\infty} c_k f(n_k x)$, where f is a periodic function and $\mathcal{N} = \{n_k, k \geq 1\}$ a sequence of positive integers. Our main concern in this talk is the search of individual conditions (linking $f, \mathcal{N}$ and (c_k)) ensuring convergence, a barely investigated part of the theory. We develop an approach based on elementary Dirichlet convolution calculus and on a new decomposition of squared sums, in contrast with recents works.

We show that this approach is powerful enough to recover and even slightly improve a recent a.e. convergence result [1] (Theorem 3) in the case $\mathcal{N} = \mathbb{N}$ without using analysis on the polydisc. Denote $e(x) = e^{2i\pi x}$, $e_n(x) = e(nx)$, $n \geq 1$. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1[$. Let $f(x) \sim \sum_{j=1}^{\infty} a_j e_j(x)$. Let $f_n(x) = f(nx)$, $n \in \mathbb{N}$. We assume throughout that

$$f \in L^2(\mathbb{T}), \qquad \langle f, 1 \rangle = 0. \tag{1}$$

A key preliminary step naturally consists with the search of bounds of $\| \sum_{k \in K} c_k f_k \|_2$ integrating in their formulation the arithmetical structure of K. That question has received a satisfactory answer only for specific cases. We state two of our mean results. Let $d(n)$ be the divisor function, namely the number of divisors of n.

Theorem 1. *Assume that $\sum_{m=1}^{\infty} a_m^2 d(m) < \infty$. Then, for any finite set K of positive integers,*

$$\| \sum_{k \in K} c_k f_k \|_2^2 \;\leq\; \Big(\sum_{m=1}^{\infty} a_m^2 d(m) \Big) \sum_{k \in K} c_k^2 d(k^2).$$

Theorem 1 is deduced from a more general result. In [9], we recently showed a similar estimate, based on properties of Hooley's Delta function

$$\Delta(v) = \sup_{u \in \mathbb{R}} \sum_{\substack{d | v \\ u < d \leq eu}} 1,$$

however restricted to sets K such that $K \subset]e^r, e^{r+1}]$ for some integer r. Consider now the class of functions introduced in [7],

$$f^s(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi j x}{j^s} \tag{2}$$

where $s > 1/2$, and recall that

$$\langle f_k^s, f_\ell^s \rangle = \zeta(2s)\frac{(k,\ell)^{2s}}{k^s \ell^s}.$$

Let for $u \in \mathbb{R}$, $\sigma_u(k) = \sum_{d|k} d^u$. In particular $\sigma_0 = d$, $\sigma_1 = \sigma$ is the usual sum-divisor function and $\sigma_{-\alpha}(n) = n^{-\alpha}\sigma\alpha(n)$. We obtain the following mean estimate.

Proposition 2. [1]*Let $s > 0$ and $0 \le \tau \le 2s$. Then for any finite set K of integers, and any coefficients c_k,*

$$\sum_{k,\ell \in K} c_k c_\ell \frac{(k,\ell)^{2s}}{k^s \ell^s} \quad \le \quad M(K)\Big(\sum_{k\in K} |c_k|^2 \sigma_{\tau - 2s}(k)\Big),$$

where

$$M(K) = \sum_{k \in F(K)} \frac{1}{\sigma_\tau(k)},$$

and $F(K) = \{d \ge 1; \exists k \in K : d|k\}$.

This implies in particular, if $s > 1/2$, $0 < \varepsilon \le 2s - 1$

$$\zeta(2s)^{-1} \Big\| \sum_{k\in K} c_k f_k^s \Big\|_2^2 \quad \le \quad \frac{1+\varepsilon}{\varepsilon}\Big(\sum_{k \in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k)\Big).$$

And

$$\sum_{k,\ell \in K} c_k c_\ell \frac{(k,\ell)^2}{k\ell} \quad \le \quad \frac{\pi^2}{6}\Big(\sum_{k \in F(K)} \frac{\varphi(k)}{k^2}\Big)\Big(\sum_{k\in K} |c_k|^2 \sigma_{-1}(k)\Big),$$

where φ is Euler's totient function. Proposition 2 is used to establish the following almost everywhere convergence result.

Theorem 3. *Let $f \in \mathrm{BV}(\mathbb{T})$, $\langle f, 1\rangle = 0$. Assume that*

$$\sum_{k=1}^{\infty} c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2} < \infty. \tag{3}$$

Then the series $\sum_k c_k f_k$ converges almost everywhere.

This slightly improves Theorem 3 in [1] ($n_k = k$), where it was assumed that the series

$$\sum_{k=1}^{\infty} c_k^2 \,(\log\log k)^\gamma$$

converges for some $\gamma > 4$.

[1]This statement has been recently significantly improved in [8].

Finally turning to averages of dilated functions we show that an old result proved by Koksma in 1953 is optimal. Khinchin conjectured that for all $f \in L^1(\mathbb{T})$ with $\langle f, 1\rangle = 0$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(kx) = 0 \qquad uad\text{a.e.} \tag{4}$$

This conjecture remained open for nearly 50 years and was disproved by Mastrand [6]. Koksma [5] proved that (4) holds if the Fourier coefficients of f satisfy

$$\sum_{k=1}^{\infty} |a_k|^2 \sigma_{-1}(k) < \infty, \tag{5}$$

Recall that $\sigma_{-1}(k) = \mathcal{O}(\log\log k)$. We prove in [2] the following

Theorem 4. *Let $w(n) \geq 1$ be a function of a natural argument, which is sub-multiplicative and bounded in mean. Assume that*

$$w(n) = o(\log\log n).$$

Then there exists a function f satisfying (1) with

$$\sum_{k=1}^{\infty} |a_k|^2 w(k) < \infty$$

such that (4) is not valid.

A nonnegative function w of a natural argument is sub-multiplicative if $w(nm) \leq w(n)w(m)$ for all m, n. And w is said to be bounded in mean if $\limsup_{J\to\infty} \frac{1}{J}\sum_{j=1}^{J} w(j) < \infty$. As $\sigma_{-1}(n) = \prod_{i=1}^{r}\big(1+\frac{1}{p_i}+\dots\frac{1}{p_i^{\alpha_i}}\big)$, $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, it is obvious that σ_{-1} is submultiplicative, and is further bounded in mean by Wintner's theorem.

For instance, for any $0 < \varepsilon < 1$, the weights $w(n) = \sigma_{-1}(n)^{1-\varepsilon}$, do not generally imply (4). Towards the proof, we first establish a variant of Bourgain's entropy criterion (Proposition 1 in [4]) on the Sobolev space $L^2_w(\mathbb{T})$ over the circle consisting with functions f such that

$$\|f\|_w^2 := \sum_{n\in\mathbb{Z}} w_n a_n^2(f) < \infty.$$

Consider the dilation operators $T_j f(x) = f(jx)$. To $f \in L^2(\mathbb{T})$ we associate

$$F_{J,f} = \frac{1}{\sqrt{J}} \sum_{1\leq j\leq J} g_j T_j f, \qquad (J \geq 1)$$

where $g_1, g_2, \dots$ are i.i.d. standard Gaussian random variables.

Proposition 5. *Let $S_n\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$, $n = 1, 2, \dots$ be continuous operators commuting with T_j on $L^2(\mathbb{T})$, $S_nT_j = T_jS_n$ for all n and j. Assume that the following property is fulfilled:*

$$\sup_{n\geq 1} |S_n(f)| < \infty \quad \text{a.e.} \qquad (\forall f \in L^2_w(\mathbb{T}))$$

Then there exists a constant C depending on $\{S_n, n \geq 1\}$ only, such that

$$\sup_{\varepsilon>0} \varepsilon \sqrt{\log N_f(\varepsilon)} \leq C \limsup_{J\to\infty} \left(\mathbf{E}\, \|F_{J,f}\|_w^2\right)^{1/2}, \qquad (\forall f \in L^2_w(\mathbb{T}))$$

where $N_f(\varepsilon)$ is the entropy number associated with the set $C_f = \{S_n f, n \geq 1\}$, namely the minimal number of $L^2(\mathbb{T})$ open balls of radius ε, centered in C_f and enough to cover C_f.

References

[1] Aistleitner C., Berkes I., Seip. K. (2013) *GCD sums from Poisson integrals and systems of dilated functions. J. Eur. Math. Soc.*, to appear.

[2] Berkes I., Weber M. (2013) *On series $\sum c_k f(kx)$ and Khinchin's conjecture*, Israël J. Math, to appear.

[3] Berkes I., Weber M. (2012) *On series of dilated functions*, Quartely J. Math, to appear.

[4] Bourgain J. (1988) *Almost sure convergence and bounded entropy*, Israel J. Math. **63**, 79–95.

[5] Koksma J. F. (1953) *Estimations de fonctions à l'aide d'intégrales de Lebesgue*, Bull. Soc. Math. Belg. **6**, 4–13.

[6] Marstrand J. M. (1970) *On Khinchin's conjecture about strong uniform distribution*, Proc. London Math. Soc. **21**, 540–556.

[7] Lindqvist P., Seip K. (1998) *Note on some greatest common divisor matrices*, Acta Arith. **LXXXIV 2**, 149–154.

[8] Weber M. (2013) *An arithmetical approach to the convergence problem of series of dilated functions*, preprint.

[9] Weber M. (2011) *On systems of dilated functions*, C. R. Acad. Sci. Paris, Sec. 1 **349**, 1261–1263.

GCD sums, dilated series and polydisc theory

István Berkes

Greatest common divisor (GCD) sums are sums of the form

$$\sum_{k,\ell=1}^{N} \frac{(\gcd(n_k, n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \tag{1}$$

where $0 < \alpha \leq 1$ and $(n_k)_{k\geq 1}$ is a sequence of distinct positive integers. The study of such sums was initiated by Koksma who in the 1930's observed that such sums can be used to estimate integrals of the form

$$\int_0^1 \left(\sum_{k=1}^{N} \left(\mathbb{1}_{[a,b)}(\{n_k x\}) - (b-a) \right) \right)^2 dx$$

which, in turn, give important information on the distribution of the sequence $\{n_k x\}_{k\geq 1}$ for almost all $x \in (0,1)$. Here $\{\cdot\}$ denotes fractional part. In a profound paper Gál [6] proved that

$$\frac{1}{N} \sum_{k,\ell=1}^{N} \frac{(\gcd(n_k, n_\ell))^2}{n_k n_\ell} \leq c\, (\log\log N)^2,$$

and moreover that this bound is optimal up to the value of the absolute constant c. Dyer and Harman [5] proved that

$$\frac{1}{N}\sum_{k,\ell=1}^{N}\frac{\gcd(n_k,n_\ell)}{\sqrt{n_k n_\ell}} \le C\exp\left(\frac{c\log N}{\log\log N}\right)$$

for two absolute constants C and c, and they used this estimate to prove results in metric Diophantine approximation. Our main result is the following theorem yielding optimal upper bounds for the sum (1) when $0<\alpha<1$, $\alpha\neq 1/2$.

Theorem 6. *For every $\varepsilon>0$, there exists a positive constant C_ε such that the following holds. For $0<\alpha<1$ and an arbitrary N-tuple of distinct positive integers $n_1,n_2,...,n_N$, we have*

$$\frac{1}{N}\sum_{k,\ell=1}^{N}\frac{(\gcd(n_k,n_\ell))^{2\alpha}}{(n_k n_\ell)^\alpha} \le C_\varepsilon\exp\left((1+\varepsilon)g(\alpha,N)\right),$$

where

$$g(\alpha,N)=\begin{cases}\left(\frac{8}{1-\alpha}+\frac{16\cdot 2^{-\alpha}}{\sqrt{2\alpha-1}}\right)(\log N)^{1-\alpha}(\log\log N)^{-\alpha}+\frac{1}{1-\alpha}(\log N)^{(1-\alpha)/2}, & 1/2<\alpha<1\\ 50\alpha(\log N\log\log N)^{1/2}+(1-2\alpha)\log N, & 0<\alpha\le 1/2.\end{cases}$$

The proof of Theorem 6, given in [1], is based on identifying the GCD sum (1) as a certain Poisson integral on the infinite dimensional polydisc $\mathbb{D}^\infty$, combined with intricate combinatorial arguments found in Gál's work [6]. As a byproduct, estimates for the largest eigenvalues of the associated GCD matrices are also found.

One can show by examples that Theorem 6 is best possible (up to a constant factor in the exponent) when $0<\alpha<1/2$ and $1/2<\alpha<1$. The case $\alpha=1/2$ is considerably harder and a near optimal estimate will be given in [4].

In what follows, we apply Theorem 1 to get asymptotically precise results for the growth of sums

$$\sum_{k=1}^{N} f(n_k x)$$

and for the almost everywhere convergence of series

$$\sum_{k=1}^{\infty} c_k f(n_k x)$$

for some classes of measurable functions f satisfying

$$f(x+1)=f(x),\qquad \int_0^1 f(x)\,dx=0. \tag{2}$$

Such dilated sums arise in many problems of analytic number theory, Diophantine approximation, uniform distribution theory, harmonic analysis, ergodic theory and probability theory.

Theorem 7. *Let $(n_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers, let f be a function satisfying (2) and having bounded variation on $[0,1]$. Then for every $\varepsilon > 0$,*

$$\left|\sum_{k=1}^{N} f(n_k x)\right| = \mathcal{O}\left((N \log N)^{1/2} (\log\log N)^{5/2+\varepsilon}\right) \qquad \text{a.e.} \tag{3}$$

when $N \to \infty$.

Apart from the exponent of $\log\log N$, the estimate (3) is sharp, as shown by the following result of Berkes and Philipp [3, Theorem 1]: There exists an increasing sequence $(n_k)_{k\geq 1}$ such that

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^{N} \cos 2\pi n_k x\right|}{(N \log N \log\log N)^{1/2}} = \infty \qquad \text{a.e.}$$

Theorem 8. *Let f be a function satisfying (2) and having bounded variation on $[0,1]$. Let $(c_k)_{k\geq 1}$ be a real sequence satisfying*

$$\sum_{k=1}^{\infty} c_k^2 (\log\log k)^{\gamma} < \infty \tag{4}$$

for some $\gamma > 4$. Then for every increasing sequence $(n_k)_{k\geq 1}$ of positive integers the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges a.e. On the other hand, for every $0 < \gamma < 2$ there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers and a real sequence $(c_k)_{k\geq 1}$ such that (4) holds, but $\sum_{k=1}^{\infty} c_k f(n_k x)$ is a.e. divergent for $f(x) = x - [x] - 1/2$.

Let C_α, $\alpha > 0$ denote the class of functions f satisfying (2) such that the Fourier series

$$f(x) \sim \sum_{j=1}^{\infty} (a_j \cos 2\pi j x + b_j \cos 2\pi j x)$$

of f satisfies

$$|a_j| = \mathcal{O}(j^{-\alpha}), \quad |b_j| = \mathcal{O}(j^{-\alpha}).$$

Theorem 9. *Assume that $f \in C_\alpha$ for some $1/2 < \alpha < 1$. Then the series $\sum_{k=1}^{\infty} c_k f(kx)$ is convergent in L^2 norm and is almost everywhere convergent, provided*

$$\sum_{k=1}^{\infty} c_k^2 \exp\left(\frac{K(\log k)^{1-\alpha}}{\log\log k}\right) < \infty, \qquad \text{where } K = 7/(1-\alpha) + 9/\sqrt{2\alpha - 1}. \tag{5}$$

On the other hand, for any $1/2 < \alpha < 1$ there exist a function $f \in C_\alpha$, a sequence $(c_k)_{k\geq 1}$ and a constant $\hat{K} = \hat{K}(\alpha)$ such that (5) holds with K replaced by $\hat{K}$, but the series $\sum_{k=1}^{\infty} c_k f(kx)$ does not converge in L^2 norm.

For the proof of Theorems 2-4 we refer to [1], [2].

References

[1] C. Aistleitner, I. Berkes and K. Seip, *GCD sums from Poisson integrals and systems of dilated functions*, J. Eur. Math. Soc., to appear.
[2] C. Aistleitner, I. Berkes, K. Seip and M. Weber, *Convergence of dilated series, GCD sums and eigenvalues of GCD matrices*. Preprint.
[3] I. Berkes and W. Philipp, *The size of trigonometric and Walsh series and uniform distribution mod1*, J. London Math. Soc. **50** (1994), 454–464.
[4] A. Bondarenko and K. Seip, *GCD sums and complete sets of square-free numbers*, arXiv:1402.0249v2 [math.NT]
[5] T. Dyer and G. Harman, *Sums involving common divisors*, J. London Math. Soc. **34** (1986), 1–11.
[6] I. S. Gál, *A theorem concerning Diophantine approximations*, Nieuw Arch. Wiskunde **23** (1949), 13–38.

On Hörmander's solution of the $\bar{\partial}$-equation

Håkan Hedenmalm

We explain how Hörmander's classical solution of the $\bar{\partial}$-equation in the plane with a weight which permits growth near infinity carries over to the rather opposite situation when we ask for decay near infinity. Here, however, a natural condition on the datum (of moment type) needs to be imposed. The condition is not only natural but also necessary to have the result at least in the Fock weight case. The norm identity which leads to the estimate is related to general area-type results in the theory of conformal mappings.

Two Weight Inequalities for Hilbert and Cauchy Transforms

Michael T. Lacey
(joint work with Eric T. Sawyer, Chun-Yen Shen, Ignacio Uriarte-Tuero and Brett D. Wick)

1. Introduction

We characterize those pairs of weights w, σ on the real line for which the weighted Hilbert transform

$$H_\sigma f(x) := \int f(x-y)\frac{\sigma(dy)}{y}$$

is bounded, as a map from $L^2(\mathbb{R}, \sigma)$ to $L^2(\mathbb{R}, w)$. In addition, boundedness of the Cauchy transform is characterized when one weight is on $\mathbb{R}$ and the other is on $\mathbb{C}_+$, say. These results have deep connections to operator theory, spectral theory, and properties analytic function spaces.

The characterization, in both cases, is in terms of a Poisson variant of the A_2 condition, and testing inequalities. Indeed, the result for the Cauchy transform contains that of the Hilbert transform, so we state it.

$$\mathsf{C}_\sigma f(z) \equiv \int_{\mathbb{R}} \frac{f(w)}{w-z}\sigma(dw) \tag{1}$$

as a map between $L^2(\mathbb{R};\sigma)$ and $L^2(\mathbb{R}^2_+;\tau)$, where σ and τ are two arbitrary weights, i.e. locally finite positive Borel measures.

Define the Poisson averages by

$$P_\tau(I) := \int_{\mathbb{R}^2_+} \frac{|I|}{(|I| + \operatorname{dist}(x, Q_I))^2}\,\tau(dx)$$

where $I \subset \mathbb{R}$ is an interval, and $Q_I = I \times [0, |I|]$ is the Carleson box over I. Our characterization is as follows.

Theorem 10. *Let σ be a weight on $\mathbb{R}$, and τ a weight on the closed upper half-plane $\mathbb{R}^2_+$. The two weight inequality* (1) *holds if and only if these conditions hold uniformly over all intervals $I \subset \mathbb{R}$, and Carleson cubes $Q_I = I \times [0, |I|)$. An A_2 condition holds:*

$$P_{\tau \mathbf{1}_{\mathbb{R}^2 \setminus Q_I}}(I) \cdot \frac{\sigma(I)}{|I|} + \frac{\tau(Q_I)}{|I|} \cdot P_{\sigma \mathbf{1}_{\mathbb{R}\setminus I}}(I) \equiv \mathbf{A}_2 < \infty, \tag{2}$$

and, these testing inequalities hold: For a finite positive constant $\mathbf{T}$,

$$\int_{Q_I} |\mathsf{C}_\sigma \mathbf{1}_I(x)|^2\,\tau(dx) \le \mathbf{T}^2 \sigma(I), \tag{3}$$

$$\int_I |\mathsf{C}^*_\tau \mathbf{1}_{Q_I}(t)|^2\,\sigma(dt) \le \mathbf{T}^2 \tau(Q_I). \tag{4}$$

Moreover, if $\mathbf{T}$ is the best constant in these inequalities and $\mathbf{A}_2$ is the best constant in the A_2 condition, then setting $\mathbf{R} \equiv \mathbf{A}_2^{1/2} + \mathbf{T}$ we have $\mathbf{N} \simeq \mathbf{R}$.

Notice that if τ is supported on the real line, the Cauchy transform reduces to the Hilbert transform. The form of the Theorem above combines the main results of [5, 3], for the Hilbert transform, but with a mild additional constraint on the pair of weights, with Hytönen's 'Poisson A_2 with holes' condition of [2]. The result of the Cauchy is drawn from [6].

2. History, Motivations

The two weight inequality for the Hilbert transform was addressed as early as 1976 by Muckenhoupt and Wheeden [7]. But, it received much wider recognition as an important problem with the 1988 work of Sarason [12]. The latter was part of important sequence of investigations that identified de Branges spaces as an essential tool in operator theory. His question concerning the composition of Toeplitz operators, was raised therein, and advertised again in [11]. The solution of his question is equivalent to a two weight inequality for the Hilbert transform, with the weights on the circle arising from the modulus squared of two outer functions.

Sarason wrote that it was 'tempting' to conjecture that the full Poisson A_2 condition would be sufficient for the two weight inequality. In an important development, F. Nazarov [8] showed that this was not the case. His counterexample was used in the setting of Model Spaces and Spectral Theory as well.

Nazarov-Treil-Volberg were creating the field of non-homogeneous Harmonic Analysis, in a series of ground-breaking papers [9, 10]. Their work, and a revitalization of the perspective of Eric Sawyer from the 1980's, lead them to conjecture the characterization proved in this paper. Moreover, their influential proof strategy, [13], lead to a verification of the conjecture in the case that both weights were doubling. At the same time, their approach is generic, in that it applies to general Calderón-Zygmund operators.

The characterization for the Cauchy transform above requires a stringent assumption on the supports of the weights. Nevertheless, it yields, as easy corollaries, a characterization of Carleson measures for an arbitrary Model Space, a question posed by W. Cohn [1] in 1981. Another elementary corollary is a characterization of the norm of a composition operator as a map from a Model Space to any one of a range of Hardy-Bergman spaces. See the introduction to [6] for these results. Additional applications to these subjects should be forthcoming.

3. Method of Proof

The method of proof uses the fundamental tools of non-homogeneous harmonic analysis, especially random grids, and Haar functions adapted to the weights in question. It also incorporates particular properties of the Hilbert transform. One of this is crudely expressed as follows: Suppose that σ is a weight not supported on an interval I. Then, $H\sigma$ restricted to I is a smooth function, with minimum absolute value of the derivative being at least the Poisson-like integral

$$\int_{\mathbb{R}\setminus I} \frac{1}{|I|^2 + \operatorname{dist}(x,I)^2}\, d\sigma. \tag{5}$$

This observation, with the A_2 assumption, and testing inequalities, imply a subtle, and slightly technical, essential improvement in the A_2 assumption. This property is referred to as the *energy inequality* in [4].

This observation is used to introduce a sequence of stopping intervals, which smooths out certain irregularities of the weight. There are additional difficulties to overcome. One must also use the natural stopping intervals, which control the averages of the objective functions f and g.

Roughly speaking, this breaks the remaining difficulties into two parts: A 'global' part, which is controlled by a two weight inequality for the Poisson inequality, relative to a weight on the upper half-plane induced by the weights, and the stopping intervals. This is identified and proved in [5], under the mild additional assumption that the pair of weights w and σ do not share a common point mass. The latter assumption is removed by dyadic methods of Hytönen [2].

The last part is the 'local' part, in which one must bound a highly non-intrinsic bilinear form. This is done by a delicate multi-scale recursion, discovered in [3].

References

[1] William S. Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math., **103** (1982), no. 2, 347–364.

[2] Tuomas P. Hytönen, *The two-weight inequality for the Hilbert transform with general measures* (2013), available at http://www.arxiv.org/abs/1312.0843.

[3] Michael T. Lacey, *Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, II*, Duke, to appear (2013), available at http://www.arxiv.org/abs/1301.4663.

[4] Michael T. Lacey, Eric T. Sawyer, and Ignacio Uriarte-Tuero, *A Two Weight Inequality for the Hilbert transform Assuming an Energy Hypothesis*, J Funct Anal **263** (2012), 305–363.

[5] Michael T. Lacey, Eric T. Sawyer, Ignacio Uriarte-Tuero, and Chun-Yun Shen, *Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, Part I*, Duke, to appear, available at http://www.arxiv.org/abs/1201.4319.

[6] Michael T. Lacey, Eric T. Sawyer, Ignacio Uriarte-Tuero, Chun-Yun Shen, and Brett Wick, *Two Weight Inequalities for the Cauchy Transform from $\mathbb{R}$ to $\mathbb{C}_+$* (2013), available at http://arxiv.org/abs/1310.4820.

[7] Benjamin Muckenhoupt and Richard L. Wheeden, *Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*, Studia Math. **55** (1976), no. 3, 279–294.

[8] F. Nazarov, *A counterexample to Sarason's conjecture*, Preprint, MSU (1997), available at http://www.math.msu.edu/~fedja/prepr.html.

[9] F. Nazarov, S. Treil, and A. Volberg, *Accretive system Tb-theorems on nonhomogeneous spaces*, Duke Math. J. **113** (2002), no. 2, 259–312.

[10] F. Nazarov, S. Treil, and A. Volberg, *The Tb-theorem on non-homogeneous spaces*, Acta Math. **190** (2003), no. 2, 151–239.

[11] Donald Sarason, *Products of Toeplitz operators*, Linear and complex analysis. Problem book 3. Part I (V. P. Havin and N. K. Nikolski, eds.), Lecture Notes in Mathematics, vol. 1573, Springer-Verlag, Berlin, 1994, pp. 318–319.

[12] Donald Saraon, *Exposed points in H^1. I*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), Oper. Theory Adv. Appl., vol. 41, Birkhäuser, Basel, 1989, pp. 485–496.

[13] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, CBMS Regional Conference Series in Mathematics, vol. 100, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2003.

Bohr's absolute convergence problem in Banach spaces

Domingo García

(joint work with D. Carando, A. Defant, M. Maestre, D. Pérez-García and P. Sevilla-Peris)

In the first part of this talk we survey about monomial convergence domains, Dirichlet series and multiplicative ℓ_1-multipliers for Dirichlet series making the connections of all these topics. In the second part of the talk we present our results.

Maximal domains where Dirichlet series converge absolutely, uniformly or conditionally are half planes $[\operatorname{Re} s > \sigma]$ where $\sigma = \sigma_a, \sigma_u$ or σ_c defines the abscissa of absolute, uniform or conditional convergence, respectively; more precisely, $\sigma := \inf r$ is the infimum taken over all r such that on $[\operatorname{Re} s > r]$ we have convergence of the requested type.

Clearly, $\sigma_c \leq \sigma_u \leq \sigma_a$, and an easy exercise shows that $\sup \sigma_a - \sigma_c = 1$, the least upper bound taken over all Dirichlet series. But it is much more complicated to control the number

$$S := \sup \sigma_a - \sigma_u\,,$$

the width of the largest possible strip on which a Dirichlet series converges uniformly but not absolutely.

Bohr showed in 1913 that the width of the strip (in the complex plane) on which a given Dirichlet series $\sum a_n/n^s$, $s \in \mathbb{C}$, converges uniformly but not absolutely, is at most 1/2, and Bohnenblust-Hille in 1931 that this bound in general is optimal.

We study the width of Bohr's strip for *vector valued* Dirichlet series instead *scalar valued* ones, i.e., we want to relate the maximal width of uniform but non-absolute convergence for Dirichlet series $\sum \frac{a_n}{n^s}$ with coefficients a_n in some fixed complex Banach space X with the geometry of X.

The following theorem is our main result – it relates the width of Bohr's strip for Dirichlet series in a Banach space X with the optimal cotype of X.

Theorem 11. *For each Banach space X we have*

$$S(X) = 1 - \frac{1}{\mathrm{Cot}(X)}\,,$$

where

$$\mathrm{Cot}(X) := \inf\{2 \leq p \leq \infty \mid X \text{ has cotype } p\}\,;$$

From this result we obtain some consequences.

Recall that

$$\mathrm{Cot}(\ell_p) = \begin{cases} 2 & 1 \leq p \leq 2 \\ p & 2 \leq p \leq \infty\,, \end{cases}$$

which obviously leads to the following.

Corollary 12. *For each $1 \leq p \leq \infty$*

$$S(\ell_p) = \begin{cases} 1/2 & 1 \leq p \leq 2 \\ 1/p' & 2 \leq p \leq \infty\,. \end{cases}$$

Here we write p' for the conjugate index of p.

The next corollary is immediate.

Corollary 13. *For every $t \in [\frac{1}{2}, 1]$ there is a Banach space X for which $t = S(X)$.*

We also state a consequence which reflects the two extreme cases $S(X) = 1$ and $S(X) = 1/2$.

Corollary 14. *Let X be an infinite dimensional Banach space. Then $S(X) = 1$ if and only if X has no finite cotype (i.e., X contains all Banach spaces ℓ_∞^n uniformly), and $S(X) = 1/2$ if and only if X has cotype $2 + \varepsilon$ for every $\varepsilon > 0$.*

Finally, we characterize multiplicative ℓ_1-multipliers of vector-valued $\mathcal{H}_p$-Dirichlet series in terms of polynomial cotype.

A Banach space X has hypercontractive polynomial cotype q, $2 \leq q < \infty$, if there is a constant $C > 0$ such that for each $m, N \in \mathbb{N}$ and each m-homogeneous polynomial $P = \sum_{|\alpha|=m} c_\alpha z^\alpha : \ell_\infty^N \to X$ we have

$$\Big(\sum_{|\alpha|=m} \|c_\alpha\|^q \Big)^{1/q} \leq C^m \Big(\int_{\mathbb{T}^N} \|P(z)\|^2 dz \Big)^{1/2}.$$

Obviously, every Banach space with hypercontractive polynomial cotype q has cotype q, but conversely we know that e.g. every Banach lattice X with cotype q has hypercontractive polynomial cotype q (it seems in fact an open question whether the notion of cotype q and the notion of hypercontractive polynomial cotype q coincide). In the next theorem we obtain a characterization of multiplicative ℓ_1-multipliers of $\mathcal{H}_p(X)$.

Theorem 15. *Let $1 \leq p \leq \infty$, and X be an infinite dimensional Banach space with hypercontractive polynomial cotype* $\mathrm{Cot}(X)$. *Then for every multiplicative sequence $(b_n) \in B_{c_0}$ the following are equivalent:*

(1) *(b_n) is an ℓ_1-multiplier of $\mathcal{H}_p(X)$*
(2) *(b_n) is an ℓ_1-multiplier of $\mathcal{H}_p^m(X)$ for all $m \in \mathbb{N}$*
(3) *(b_n) is an ℓ_1-multiplier of $\mathcal{H}_p^m(X)$ for some $m \in \mathbb{N}$*
(4) *$(b_n) \in \ell_{\mathrm{Cot}(X)'}$*

To see an example note that ℓ_r has hypercontractive polynomial cotype $\mathrm{Cot}(\ell_r) = \max\{2, r\}$ which gives the following.

Corollary 16. *Let $1 \leq r, p \leq \infty$. Then for every multiplicative sequence $(b_n) \in B_{c_0}$ the following are equivalent:*

(1) *(b_n) is an ℓ_1-multiplier of $\mathcal{H}_p(\ell_r)$*
(2) *(b_n) is an ℓ_1-multiplier of $\mathcal{H}_p^m(\ell_r)$ for all $m \in \mathbb{N}$*
(3) *(b_n) is an ℓ_1-multiplier of $\mathcal{H}_p^m(\ell_r)$ for some $m \in \mathbb{N}$*
(4) *$(b_n) \in \ell_{\min\{2, r'\}}$*

References

[1] F. Bayart, A. Defant, L. Frerick, M. Maestre, P. Sevilla-Peris, Monomial expansions of H_p-functions in infinitely many variables, preprint.

[2] H.F. Bohnenblust, E. Hille, On the absolute convergence of Dirichlet series, *Ann. of Math.* (2) **32** (1934) 600-622.

[3] H. Bohr, Über die gleichmässige Konverenz Dirichletscher Reihen, *J. Reine Angew. Math.* **143** (1913) 203-211.

[4] H. Bohr, A theorem concerning power series, *Proc. London Math. Soc.* (2) **13** (1914) 1-5.

[5] D. Carando, A. Defant, D. García, M. Maestre, P. Sevilla-Peris, Multiplicative ℓ_1-multipliers of vector-valued $\mathcal{H}_p$-Dirichlet series, preprint.

[6] D. Carando, A. Defant, P. Sevilla-Peris, Bohr's absolute convergence problem for $\mathcal{H}_p$-Dirichlet series in Banach spaces, preprint.

[7] D. Carando, A. Defant, P. Sevilla-Peris, Homogeneous cotype of Banach spaces, preprint.

[8] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip, The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive, *Ann. of Math.* (2) **174** (2011), 1, 485–497.

[9] A. Defant, D. García, M. Maestre, Bohr's power series theorem and local Banach space theory, *J. Reine Angew. Math.* **557** (2003), 173–197.
[10] A. Defant, D. García, M. Maestre, New strips of convergence for Dirichlet series, *Publ. Mat.* **54** (2010), 369–388.
[11] A. Defant, D. García, M. Maestre, D. Pérez-García, Bohr's strip for vector valued Dirichlet series, *Math. Ann.* **342** (2008), 533–555.

Matrix Monotone Functions in Several Variables

John E. McCarthy
(joint work with Jim Agler and N. J. Young)

In 1934, K. Löwner published a very influential paper [2] studying functions on an open interval $E \subseteq \mathbb{R}$ that are matrix monotone, *i.e.* functions f with the property that whenever S and T are self-adjoint matrices whose spectra are in E then

$$S \leq T \qquad \Rightarrow \qquad f(S) \leq f(T). \tag{1}$$

This property is equivalent to being locally matrix monotone, *i.e.* if $S(t)$ is a C^1 arc of self-adjoint matrices with $\sigma(S(t)) \subset E$ then

$$S'(t) \geq 0 \qquad \Rightarrow \qquad \frac{d}{dt} f(S(t)) \geq 0. \tag{2}$$

The object of the talk is to extend the above notions to several variables. In particular, we want to study functions of d variables applied to d-tuples of commuting self-adjoint operators. Given two d-tuples $S = (S^1, \dots, S^d)$ and $T = (T^1, \dots, T^d)$, we shall say that $S \leq T$ if and only if $S^r \leq T^r$ for every $1 \leq r \leq d$. We want to study functions that satisfy (1) or (2) for d-tuples. The talk is based largely on the paper [1].

1. Dimension one

Let E be an open set in $\mathbb{R}$, and let $n \geq 2$ be a natural number. The Löwner class $\mathcal{L}_n^1(E)$ is the set of C^1 functions $f : E \to \mathbb{R}$ with the property that, whenever $\{x_1, \dots, x_n\}$ is a set of n distinct points in E, then the matrix A, defined by

$$A_{ij} = \begin{cases} \dfrac{f(x_j) - f(x_i)}{x_j - x_i} & \text{if} \quad i \neq j \\ f'(x_i) & \text{if} \quad i = j, \end{cases}$$

is positive semi-definite.

Theorem 17 (Löwner)**.** *Let $E \subseteq \mathbb{R}$ be open, and let $f \in C^1(E)$. Then f is locally n-matrix monotone on E if and only if f is in*
$mathcal L_n^1(E)$.

We shall use Π to denote the upper half-plane, $\{z \in \mathbb{C} : \Im z > 0\}$.

DEF: Let $E \subseteq \mathbb{R}$ be open. The Pick class on E, denoted $\mathcal{P}(E)$, is the set of real-valued functions f on E for which there exists an analytic function $F : \Pi \to \overline{\Pi}$ such that F extends analytically across E and

$$\lim_{y \searrow 0} F(x+iy) = f(x) \quad \forall\, x \in \ E.$$

Theorem 18 (Löwner)**.** *Let $E \subseteq \mathbb{R}$ be open, and let $f \in C^1(E)$. The following are equivalent:*

(i) The function f is locally operator monotone on E.

(ii) The function f is in $\mathcal{L}^1_n(E)$ for all n.

(iii) The function f is in $\mathcal{P}(E)$.

2. Dimension $d \geq 2$

We shall let $CSAM^d_n$ denote the set of d-tuples of commuting self-adjoint n-by-n matrices, and CSA^d be the set of d-tuples of commuting self-adjoint bounded operators. If S is a commuting d-tuple of self-adjoint operators acting on the Hilbert space $\mathcal{H}$, and f is a real-valued continuous (indeed, measurable) function on the joint spectrum of S in $\mathbb{R}^d$, then $f(S)$ is a well-defined self-adjoint operator on $\mathcal{H}$.

DEF: Let E be an open set in $\mathbb{R}^d$, and f be a real-valued C^1 function on E. We say f is locally M_n-monotone on E if, whenever S is in $CSAM^d_n$ with $\sigma(S) = \{x_1, \ldots, x_n\}$ consisting of n distinct points in E, and $S(t)$ is a C^1 curve in $CSAM^d_n$ with $S(0) = S$ and $\frac{d}{dt} S(t)|_{t=0} \geq 0$, then $\frac{d}{dt} f(S(t))|_{t=0}$ exists and is ≥ 0.

We define the Löwner classes in d variables, $\mathcal{L}^d_n(E)$, by:

DEF: Let E be an open subset of $\mathbb{R}^d$. The set $\mathcal{L}^d_n(E)$ consists of all real-valued C^1-functions on E that have the following property: whenever $\{x_1, \ldots, x_n\}$ are n distinct points in E, there exist positive semi-definite n-by-n matrices $A^1, \ldots, A^d$ so that

$$\begin{aligned} A^r(i,i) &= \left.\frac{\partial f}{\partial x^r}\right|_{x_i} \\ \text{and} \quad f(x_j) - f(x_i) &= \sum_{r=1}^{d} (x^r_j - x^r_i) A^r(i,j) \qquad \forall\, 1 \leq i,j \leq n. \end{aligned}$$

Here is our d-variable version of Theorem 17.

Theorem 19. *Let E be an open set in $\mathbb{R}^d$, and f a real-valued C^1 function on E. Then f is locally M_n-monotone if and only if f is in $\mathcal{L}^d_n(E)$.*

Using a multi-variable version of Nevanlinna's representation, we can give a complete characterization of the rational functions of two variables that are operator monotone on rectangles.

Theorem 20. *Let F be a rational function of two variables. Let Γ be the zero-set of the denominator of F. Assume F is real-valued on $\mathbb{R}^2 \setminus \Gamma$. Let E be an open* *rectangle in* $\mathbb{R}^2 \setminus \Gamma$. *Then* F *is globally operator monotone on* E *if and only if* F *is in* $\mathcal{L}(E)$*, that is if and only if* F *is the restriction to* E *of an analytic function from* Π^2 *to* $\overline{\Pi}$ *that extends analytically across* E.

References

[1] J. Agler, J.E. McCarthy, and N.J. Young. Operator monotone functions and Löwner functions of several variables. *Ann. of Math.*, 176(3):1783–1826, 2012.

[2] K. Löwner. Über monotone Matrixfunktionen. *Math. Z.*, 38:177–216, 1934.

The unconditional basis constant in a certain space of polynomials

Sunke Schlüters

In [4], [3], [1] and [2] is proven that for every $N \in \mathbb{N}$ and any choice of scalars $(a_n)_n \subset \mathbb{C}$ we have

$$\sum_{n\le N} |a_n| \le N^{\frac{1}{2}} e^{\left(-\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}} \sup_{t\in\mathbb{R}} \Big| \sum_{n\le N} a_n n^{-it} \Big|. \tag{1}$$

Furthermore this constant is optimal. Using Kronecker's theorem, this translates into an inequality about polynomials in several variables.

Let $\Lambda(N) = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \,|\, 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \cdots \le N\}$ denote the set of all multiindices corresponding to integers less or equal N. Then (1) is equivalent to the fact, that for any choice of scalars $(c_\alpha)_\alpha \subset \mathbb{C}$ we have

$$\sum_{\alpha\in\Lambda(N)} |c_\alpha| = \sup_{z\in B_{\ell_\infty}} \sum_{\alpha\in\Lambda(N)} |c_\alpha z^\alpha| \le N^{\frac{1}{2}} e^{\left(-\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}} \sup_{z\in B_{\ell_\infty}} \Big| \sum_{\alpha\in\Lambda(N)} c_\alpha z^\alpha \Big|.$$

Our main result is the following more general inequality: For any choice of scalars $(c_\alpha)_\alpha \subset \mathbb{C}$ and any $1 \le r \le \infty$,

$$\begin{aligned}\sup_{z\in B_{\ell_r}} &\sum_{\alpha\in\Lambda(N)} |c_\alpha z^\alpha| \\ &\le N^{1-\frac{1}{\min\{2,r\}}} e^{-\left(\sqrt{2}(1-\frac{1}{\min\{2,r\}})+o(1)\right)\sqrt{\log N \log\log N}} \sup_{z\in B_{\ell_r}} \Big| \sum_{\alpha\in\Lambda(N)} c_\alpha z^\alpha \Big|.\end{aligned}$$

The exponent of N in this inequality is shown to be optimal. However, the proof of the optimality of the other exponent remains open.

Using techniques introduced in [4] and [3], we reduce the problem to the estimation of the unconditional basis constant in spaces of homogeneous polynomials. The estimation of those unconditional basis constants further requires massive use of local Banach space theory and number theoretical results.

References

[1] de la Bretèche, Régis: Sur l'ordre de grandeur des polynômes de Dirichlet, in: *Acta Arithmetica*, **134** (2008), #2, pages 141–148, ISSN 0065-1036.
URL http://dx.doi.org/10.4064/aa134-2-5

[2] Defant, Andreas; Frerick, Leonhard; Ortega-Cerdà, Joaquim; Ounaïes, Myriam and Seip, Kristian: The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive, in: *Annals of Mathematics*, **174** (2011), #1, pages 485–497, ISSN 0003-486X.
URL http://annals.math.princeton.edu/2011/174-1/p13

[3] Konyagin, SV and Queffélec, H: The Translation 1/2 in the Theory of Dirichlet Series, in: *Real Analysis Exchange*, (2001), pages 155–176.
URL http://projecteuclid.org/euclid.rae/1212763958

[4] Queffélec, Hervé: H. Bohr's vision of ordinary Dirichlet series; old and new results, in: *J. Anal*, (1995).

Sharp generalizations of the multilinear Bohnenblust-Hille inequality

Juan B. Seoane-Sepulveda

(joint work with N. Albuquerque, F. Bayart, and D. Pellegrino)

The Bohnenblust-Hille inequality [3] asserts that for all positive integers $m \geq 2$ there exists a constant $C = C(m) \geq 1$ such that

$$\left(\sum_{i_1,\dots,i_m=1}^{\infty} \left| A(e_{i_1},\dots,e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C \left\| A \right\|$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). This inequality, and the behavior and growth of the constants involved in it, have important applications in various fields of Analysis and Mathematical Physics ([2, 5, 7]). The case $m = 2$ is the well-known Littlewood's 4/3 inequality, [6]. The Bohnenblust-Hille inequality is, actually, a very particular case of a large family of sharp inequalities. More precisely (see [1]),

Theorem 21 (Albuquerque, Bayart, Pellegrino, and Seoane). *Let $m \geq 1$, let $q_1,\dots,q_m \in [1,2]$. The following assertions are equivalent:*
(1) There is a constant $C_{q_1\dots q_m} \geq 1$ such that

$$\left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2=1}^{\infty} \left(\cdots \left(\sum_{i_{m-1}=1}^{\infty} \left(\sum_{i_m=1}^{\infty} \left| A\left(e_{i_1},\dots,e_{i_m}\right)\right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-2}}{q_{m-1}}} \cdots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{q_1\dots q_m} \left\| A \right\|$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$.
(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}$.

The Bohnenblust-Hille inequality is just the case $q_1 = \cdots = q_m = 2m/(m+1)$.
The strategy for the proof of (2) $\Rightarrow$ (1) in Theorem 21 is actually simpler than all previous known proofs of the Bohnenblust-Hille inequality. The starting point

is the generalized Littlewood mixed $(\ell^1,\ell^2)-$norm inequality, that is that Theorem 21 holds true when $(q_1,\dots,q_m)=(1,2,\dots,2)$. This property is well-known and it is a consequence of Khintchine's inequality. Using this, Minkowski's inequality, and interpolation theory, the general case can de inferred. Several generalizations of the Bohnenblust-Hille inequality have already been obtained. One of them is a very nice vector-valued version due to Defant and Sevilla-Peris (see [4]).

Theorem (Defant/Sevilla-Peris)**.** *Let $m\geq 1$, $1\leq s\leq q\leq 2$. Define*

$$\rho=\frac{2m}{m+2\left(\frac{1}{s}-\frac{1}{q}\right)}.$$

Then there exists a constant $C>0$ such that, for every continuous m-linear mapping $A:c_0\times\cdots\times c_0\to\ell_s$, then

$$\left(\sum_{i_1,\dots,i_m=1}^{+\infty}\|A(e_{i_1},\dots,e_{i_m})\|_q^\rho\right)^{1/\rho}\leq C\|A\|.$$

Second, and in the spirit of the Hardy-Littlewood generalization of Littlewood's 4/3 inequality, Praciano-Pereira studied in [8] the effect of replacing c_0 by ℓ_p in the Bohnenblust-Hille inequality. Denote $\mathbf{p}=(p_1,\dots,p_m)\in[1,+\infty]^m$ and let

$$\left|\frac{1}{\mathbf{p}}\right|=\frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

For $p\geq 1$, let also $X_p=\ell_p$ and define $X_\infty=c_0$.

Theorem (Praciano-Pereira)**.** *Let $m\geq 1$, $1\leq p_1,\dots,p_m\leq+\infty$ with $\left|\frac{1}{\mathbf{p}}\right|\leq\frac{1}{2}$. Define*

$$\rho=\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}.$$

Then there exists a constant $C>0$ such that, for every continuous m-linear mapping $A:X_{p_1}\times\cdots\times X_{p_m}\to\mathbb{C}$,

$$\left(\sum_{i_1,\dots,i_m=1}^{+\infty}|A(e_{i_1},\dots,e_{i_m})|^\rho\right)^{1/\rho}\leq C\|A\|.$$

Both approaches can be embedded into our framework, since we are also able to prove the following result (see [1]).

Theorem 22. *Let $m\geq 1$, let $1\leq s\leq q\leq 2$, let $\mathbf{p}=(p_1,\dots,p_m)\in[1,+\infty]^m$ such that*

$$\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|\geq 0.$$

Let also $q_1,\dots,q_m\in[\lambda,2]$, where

$$\lambda=\frac{1}{\frac{1}{2}+\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|}.$$

Then the following are equivalent:

(1) There is a constant $C_{q_1\ldots q_m} \geq 1$ such that

$$\left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2=1}^{\infty} \left(\cdots \left(\sum_{i_{m-1}=1}^{\infty} \left(\sum_{i_m=1}^{\infty} \| A\left(e_{i_1}, ..., e_{i_m}\right) \|_q^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-2}}{q_{m-1}}} \cdots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{q_1\ldots q_m} \|A\|$$

for all continuous m-linear forms $A : X_{p_1} \times \cdots \times X_{p_r} \to \ell_s$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m}{2} + \frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|$.

When all the q_i's are equal, we obtain the following corollary.

Corollary 23. *Let $m \geq 1$, let $1 \leq s \leq q \leq 2$, let $\mathbf{p} = (p_1, \ldots, p_m) \in [1, +\infty]^m$ such that*

$$\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right| \geq 0.$$

Let us define

$$\rho = \frac{2m}{m + 2\left(\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|\right)}.$$

Then there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to \ell_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|_q^{\rho} \right)^{1/\rho} \leq C\|A\|.$$

It is plain that this corollary extends Defant and Sevilla-Peris result to the ℓ_p-case (and we get the same result if we choose $p_1 = \cdots = p_m = \infty$). To show that Theorem 22 also implies Theorem 21 and the result of Praciano-Pereira, it suffices to choose $q = 2$, $s = 1$ and to consider only m-linear mappings which have their range in the span of the first basis vector, see [1].

References

[1] N. Albuquerque, F. Bayart, D. Pellegrino, and J.B. Seoane-Sepúlveda, Sharp generalizations of the multilinear Bohnenblust-Hille inequality, J. Funct. Anal., 266 (2014) 3726–3740.

[2] H. P. Boas, The football player and the infinite series, Notices Amer. Math. Soc. 44 (1997), no. 11, 1430–1435.

[3] H. F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. (2) 32 (1931), no. 3, 600-622.

[4] A. Defant and P. Sevilla-Peris, A new multilinear insight on Littlewood's 4/3-inequality, J. Funct. Anal. 256 (2009), no. 5, 1642–1664.

[5] D. Diniz, G. A. Muñoz-Fernández, D. Pellegrino, and J.B. Seoane-Sepúlveda, The asymptotic growth of the constants in the Bohnenblust-Hille inequality is optimal, J. Funct. Anal. 263 (2012), no. 2, 415–428.

[6] J. E. Littlewood, On bounded bilinear forms in an infinite number of variables, Quart. J. Math. Oxford 1 (1930), 164–174.

[7] A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics 53 (2012).
[8] T. Praciano-Pereira, On bounded multilinear forms on a class of lp spaces, J. Math. Anal. Appl. 81 (1981), no. 2, 561–568.

On the fundamental problem of Fourier analysis

Mieczysław Mastyło

One of the most fundamental open problems in the theory of Fourier series is to characterize the space of all integrable functions on the one-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for which the almost everywhere convergence of the Fourier series holds (we identify $\mathbb{T}$ with the unit interval $I = [0,1)$ with the Lebesgue measure m). As usual for a complex-valued function f in $L^1(\mathbb{T})$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ and $x \in \mathbb{T}$, we define the kth *Fourier coefficient* of f, and the *partial sum* $S_n f$ of the Fourier series of f, by

$$\widehat{f}(k) = \int_I f(x) e^{-2\pi i k x}\, dx, \qquad S_n f(x) = \sum_{|k|\le n} \widehat{f}(k) e^{2\pi i k x}\,.$$

The famous theorem of Kolmogorov [7] states that there exists a function $f \in L^1(\mathbb{T})$ such that the Fourier series $\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x}$ of f diverges almost everywhere (i.e., the sequence $(S_n f)$ diverges almost everywhere on $\mathbb{T}$).

This result motivated the mentioned problem in Fourier analysis to describe a subset of functions in $L^1(\mathbb{T})$ for which the Fourier series converges almost everywhere. There is a long history about this problem however we do not provide a comprehensive bibliography here; Luzin [8] announced in 1913 a conjecture on almost everywhere convergence of the Fourier series of continuous functions. The general study of this problem was initiated in the seminal work of Carleson [3] published in 1965. In this paper the answer to Luzin's conjecture was given; it was even proved that the Fourier series of square summable functions converge almost everywhere. Carleson's theorem was later extended, by Hunt [5] for the class of $L^p(\mathbb{T})$ functions for all $1 < p < \infty$. The key of Hunt's result is the following estimate of the non-increasing rearrangement $(S\chi_A)^*$ of $S\chi_A$ for any measurable set $A \subset I$,

$$(S\chi_A)^*(t) \le C\, \frac{m(A)}{t}\left(1 + \log^+ \frac{t}{m(A)}\right), \qquad 0 < t < 1,$$

where $C0$ is a positive constant. In 1996, Antonov [1] proved that the inequality can be extended to the set of functions $f \in L^1(\mathbb{T})$ such that $\|f\|_\infty \le 1$; namely,

$$(Sf)^*(t) \prec \frac{\|f\|_1}{t}\left(1 + \log^+ \frac{t}{\|f\|_1}\right), \qquad 0 < t < 1.$$

Using this inequality, he proved that the Carleson operator

$$S\colon L\log L \log\log\log L(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$$

is continuous, where for $\overline{\log}\ t := 1 + \log^+ t$, $L\log L\log\log\log L$ denotes the rearrangement invariant (r.i. for short) Banach space on I equipped with the norm

$$\|f\| := \int_0^1 f^*(t)\overline{\log}\,\frac{1}{t}\overline{\log}\,\overline{\log}\,\overline{\log}\frac{1}{t}\,dt < \infty$$

and as usual $L^{1,\infty}$ denotes the weak Marcinkiewicz space on I. From this result, the almost everywhere convergence of the Fourier series of functions from space $L\log L\log\log\log L(\mathbb{T})$ follows.

In 2002, Arias-de-Reyna [2] defined a quasi-Banach space $QA(\mathbb{T})$ and improved Antonov's result showing that $L\log L\log\log\log L(\mathbb{T}) \subset QA(\mathbb{T})$. This motivates the study of the space $QA(\mathbb{T})$, which is a non-locally convex space, has separating dual space and it is logconvex in the sense of Kalton (see [6]).

The structure of the space QA seems highly non-trivial and so a natural problem appeared to characterize some class of Banach spaces contained in QA which are more natural from the point of view of applications to Fourier analysis. In a joint work [4] with Maria J. Carro and Luis Rodríguez-Piazza, we study some properties of QA and show several applications to convergence of Fourier series. We recall that QA consists of all measurable function $f \in L^0(m)$ such that there exists a sequence $(f_n)_{n=1}^\infty$ with $f_n \in L^\infty$ such that $f = \sum_{n=1}^\infty f_n$ a.e. with

$$(*) \qquad \sum_{n=1}^{\infty}(1+\log n)\|f_n\|_\infty \psi\left(\frac{\|f_n\|_1}{\|f_n\|_\infty}\right) < \infty,$$

where $\psi(t) = t\log(e/t)$ for all $0 < t \le 1$ and $\psi(0) = 0$ (we put $0/0 := 0$ by convention). It is not difficult to verify that QA is a quasi-Banach lattice on (I, m) equipped with the quasi-norm

$$\|f\|_{QA} = \inf \sum_{n=1}^{\infty}(1+\log n)\|f_n\|_\infty \psi\left(\frac{\|f_n\|_1}{\|f_n\|_\infty}\right),$$

where the infimum is taken over of all representations $f = \sum_{n=1}^\infty f_n$ of f which satisfy the condition $(*)$.

In [4] we prove that QA is not a Banach space and the Banach envelope of QA is the Lorentz space $L\log L$. We also prove that there exists a Lorentz space Λ_φ strictly bigger than the Antonov space $L\log L\log\log\log L$ which is contained in QA. We also prove that the Lorentz space $L\log L\log\log\log\log L$ is not contained in QA To see this, we study a lattice constant $e_n(QA)$ and show that it is equivalent to $n(1+\log n)$. Using this result, we give a necessary condition for r.i. Banach spaces X to be contained in QA. In particular we show how to construct a family of Lorentz spaces Λ_φ on (I, m) such that $\Lambda_\varphi \subset QA$. We also prove that there is no r.i. Banach space X on (I, m) contained in QA with fundamental function φ_X ($\varphi_X(t) = \|\chi_{[0,t]}\|_X$, $t \in I$) satisfying

$$\lim_{t\to 0+}\frac{\varphi_X(t)}{\rho(t)} = 0,$$

where $\rho(t) = t\,\overline{\log}\,\frac{1}{t}\,\overline{\log}\,\overline{\log}\,\overline{\log}\,\frac{1}{t}$ for all $0 < t < 1$. This result seems interesting on its own since it shows in particular that under the mild assumption expressed in terms of the fundamental functions the Antonov space $L\log L\log\log\log L$ is the largest Lorentz space contained in the Arias-de-Reyna space QA. Based on some interpolation property of QA proved in [4], we show that in a large classes of Banach spaces there is no the largest space contained in QA. In particular this holds for the class of r.i. Banach spaces as well as for the class of Lorentz and Marcinkiewicz spaces.

References

[1] N. Y. Antonov, *Convergence of Fourier series*, East J. Approx. **2** (1996), 187–196.

[2] J. Arias-de-Reyna, *Pointwise Convergence of Fourier Series*, J. London Math. Soc. **65** (2002), 139–153.

[3] L. Carleson, *Convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.

[4] M. J. Carro, M. Mastyło and L. Rodríguez-Piazza, *Almost everywhere convergent Fourier series*, J. Fourier Anal. Appl. **18** (2012), no. 2, 266–286.

[5] R. A. Hunt, *On the convergence of Fourier series.*, 1968 Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967) pp. 235–255 Southern Illinois Univ. Press, Carbondale, Ill.

[6] N. J. Kalton, *Convexity, type and the three space problem. Studia Math.* **69** (1980/81), no. 3, 247–287.

[7] A.N. Kolmogorov, *Une série de Fourier-Lebesgue divergente presque partout*, Fund. Math. **4** (1923), 324–328.

[8] N. Luzin, *Sur la convergence des séries trigonométries de Fourier*, C. R. Acad. Sci. paris **156** (1913), 1655–1658.

The Hilbert transform along vector fields constant on Lipschitz curves

Christoph Thiele

Let $v : \mathbf{R}^2 \to \mathbf{R}^2$ be a vector field in the plane, given by

$$v(x) = (v_1(x_1, x_2), v_2(x_1, x_2)) \ .$$

Let f be a test function defined on the plane. The Hilbert transform of f along the given vector field is defined as

$$H_v f(x_1, x_2) := p.v. \int f(x_1 - v_1(x_1, x_2)t, x_2 - v_2(x_1, x_2)t)\, \frac{dt}{t} \ .$$

Geometrically, at a fixed point of the plane we consider the line through the point in direction of the vector field. We then apply the standard Hilbert transform, that is convolution with the principal value distribution $1/t$, to the restriction of f to this line. Finally, we evaluate this standard Hilbert transform at the given point in the plane.

Thanks to a symmetry under dilations of the Hilbert kernel, the length of the vector $v(x)$ is irrelevant for the value of the Hilbert transform, and thus we may for example normalize so that $v_1(x_1, x_2) = 1$, leaving the exceptional points with $v_1(x_1, x_2) = 0$ to a trivial separate discussion.

We are concerned with L^p bounds for this operation H_v given suitable assumptions on the vector field v. Questions of this nature trace back at least to A. Zygmund's consideration of the closely related Hardy Littlewood maximal operator along a vector field, and the related question of Lebesgue Differentiation along the vector field. It is well understood that on the regularity scale a Lipschitz assumption on the vector field is necessary to allow for any L^p bounds. Here the Lipschitz assumption is effective only if the large distance part of the Hilbert kernel or maximal operator is truncated away at a threshold compatible with the Lipschitz condition. Sufficiency of a Lipschitz condition for Lebesgue Differentiation in L^2 is Zygmund's conjecture, while raising the corresponding question for the Hilbert transform is attributed to Eli Stein. The best result known to date on the scale of regularity assumptions is boundedness of the maximal operator on a bounded subdomain of the plane under the assumption of real analyticity of the vector field (Bourgain [3]), witha corresponding result for the Hilbert transform by (Stein and Street [7], with an important special case by Christ, Nagel, Stein, and Wainger [4]).

We consider a different type of conditions on the vector field. An observation attributed to Coifman is that if the vector field depends only on one variable, that is $v_2(x_1, x_2) = u(x_1)$ for some measurable function u, where we still assume $v_1(x_1, x_2) = 1$, then L^2 boundedness of the Hilbert transform along the vector field follows from the L^2 case of the Carleson-Hunt theorem. The latter states boundedness in L^2 of the Carleson operator

$$C_N f(x) = p.v. \int f(x-t) e^{iN(x)t} \frac{dt}{t}$$

uniformly in the choice of a measurable function N. To recap Coifman's insight, we formally apply the Fourier inversion formula in the second variable:

$$\|p.v \int f(x_1 - t, x_2 - u(x_1)t) \frac{dt}{t} \|_2$$

$$= \| \int e^{2\pi i \eta x_2} [p.v. \int \widehat{f}(x_1 - t, \eta) e^{-2\pi i \eta u(x_1) t} \frac{dt}{t}] \, d\eta \|_2 \ .$$

An application of Plancherel's identity identifies this as

$$= \| \| p.v. \int \widehat{f}(x_1 - t, \eta) e^{-2\pi i \eta u(x_1) t} \frac{dt}{t} \|_{L^2(x)} \|_{L^2(\eta)} \ .$$

Applying for fixed η the Carleson-Hunt theorem on the inner norm estimates the above by $\| \| \widehat{f}(x_1, \eta) \|_{L^2(x)} \|_{L^2(\eta)}$, which by another application of Plancherel's identity proves boundedess of H_v in L^2. Conversely it is not hard to see that boundedness of the Hilbert transform along v in L^2 implies the L^2 case of the Carleson-Hunt theorem. It is a particular feature of this setup that no truncation of H_v is needed.

The author with Bateman [2] have extended Coifman's estimate to $p \neq 2$ as follows.

Theorem 24. *Assume $v(x_1, x_2) = (1, u(x_1))$ for some measurable function u. Then for each $\frac{3}{2} < p < \infty$ there is a constant C such that*

$$\|H_v f\|_p \leq C\|f\|_p$$

for all test functions f.

The proof is based on replacing the Fourier transform in Coifman's argument by a Littlewood Paley decomposition in the second variable, which is a suitable tool in L^p for $p \neq 2$. Lacey and Li [6] provide an adaption of the argument of Carleson and Hunt that yields an L^p bound for $p > 2$ of the Hilbert transform along the vector field, under the assumption that the function f is supported on a single frequency band. This result was extended by Bateman [1] using substantially the one variable assumption on the vector field to $p > 1$. Combining all Littlewood Paley bands into an estimate for an unconstrained function f is the main achievement of [2]. It is not known whether the endpoint 3/2 is sharp. Some of the employed techniques break down at 3/2, while a more substantial breakdown of the overall strategy happens at 4/3.

A natural perturbation of the above theorem concerns vector fields constant along Lipschitz perturbations of the straight vertical lines relevant in the theorem. The following theorem is due to to Shaoming Guo [5], where a version of the theorem appears in slightly larger generality. It is the first result in the context of Stein's or Zygmund's conjecture that is efficiently exploiting a Lipschitz condition.

Theorem 25. *Let h be an auxiliary Lipschitz function $h : \mathbf{R}^2 \to \mathbf{R}$ with gradient almost everywhere in $B = B_{1/2}(1, 0)$, that is*

$$\|\nabla h(x_1, x_2) - (1, 0)\| \leq 1/2$$

for almost every $(x_1, x_2) \in \mathbf{R}^2$. Assume further $v(x_1, x_2) = (1, u(h(x_1, x_2)) \in B$ for almost every (x_1, x_2) and some measurable function u. Then there is a constant C such that

$$\|H_v f\|_2 \leq C\|f\|_2$$

for all test functions f.

Note that the case $\nabla(h) = (1, 0)$ reduces to the L^2 case of the previous theorem. The main novelty of Guo's work consists of implementing a bi-parameter Littlewood Paley theory adapted to a family of Lipschitz curves, namely the level lines of the function h. This is reminiscent of but different from the one parameter Littlewood Paley theory adapted to a single Lipschitz curve that features prominently in the celebrated proof of boundeness of the Cauchy integral along a Lipschitz curve. It is conceivable that Guo's result can be extended to the same range of exponents as the first theorem, investigations to this extend are in progress.

References

[1] Michael Bateman, *Single annulus L^p estimates for Hilbert transform along vector fields*, Rev. Mat. Iberoam. **29** (2013), no. 3, 1021–1069.

[2] Michael Bateman and Christoph Thiele, *Lp estimates for the Hilbert transform along one-variable vector field*, Anal. PDE **6** (2013), no. 7, 1577–1600.

[3] J. Bourgain, *A remark on the maximal function associated to an analytic vector field*, Analysis at Urbana, Vol. I (Urbana, IL, 1986), London Math. Soc. Lecture Note Ser., vol. 137, Cambridge Univ. Press, Cambridge, 1989, pp. 111–132.
[4] Michael Christ, Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Signular and maximal Radon transforms: analysis and geometry*, Ann. of Math. (2) **150** (1999), no. 2, 489–577.
[5] Shaoming Guo, *Hilbert transform along measureable vector fields constant on Lipschitz curves: L^2 boundedness*, arXiv:1401:2890 (2014).
[6] Michael T. Lacey and Xioachun Li, *Maximal theorems for the directional Hilbert transform on the plane*, Trans. Amer. Math. Soc. **358** (2006), no. 9, 4099–4117 (electronic).
[7] Elias M. Stein and Brian Street, Multi-parameter singular Radon transforms, Math. Res. Lett. **18** (2011), no. 2, 257–277.

Hardy spaces in infinitely many variables

Jan-Fredrik Olsen
(joint work with Alexandru Aleman and Eero Saksman)

We report on an ongoing effort, joint with Alexandru Aleman and Eero Saksman, to better understand the Hardy spaces $H^p(\mathbb{T}^\infty)$, where $\mathbb{T}^\infty = \{(z_1, z_2, \dots) : |z_i| = 1\}$ denotes the infinite dimensional polydisc. Our interest in this space is motivated by the connection to the Dirichlet-Hardy spaces as introduced in [8] and [2]. For our previous work along these lines, see [1] and also [5, 9, 11, 10, 14].

One way to define these spaces is as follows. Since $\mathbb{T}^\infty$ is a compact abelian group with normalized Haar measure dm and dual $\mathbb{Z}_0^\infty$, i.e., sequences with finitely many non-zero entries, one can readily form the well-understood spaces

$$L^p(\mathbb{T}^\infty) = \Big\{ f(z_1, z_2, \dots) \sim \sum_{\nu \in \mathbb{Z}_0^\infty} a_\nu z^\nu : \int_{\mathbb{T}^\infty} |f(z)|^p \mathrm{d}m(z) < \infty \Big\},$$

where we use the multi-index notation $z^\nu = z_1^{\nu_1} z_2^{\nu_2} \dots$. (For more on the L^p-theory on groups, see [12].) In this notation, we denote by $H^p(\mathbb{T}^\infty)$ the subspace of $L^p(\mathbb{T}^\infty)$ with Fourier spectrum supported on the "narrow cone" $\mathbb{N}_0^\infty \subset \mathbb{Z}_0^\infty$. Note that in the literature, there are also other natural notions of Hardy spaces on $\mathbb{T}^\infty$. For instance, in [12, Chapter 8], Hardy spaces are defined on groups that have order relations. When considering spaces defined on $\mathbb{T}^\infty$, this leads to the spectrum being supported on roughly "half" of $\mathbb{Z}_0^\infty$, leading to Hardy spaces which are considerably larger than the ones we consider.

As mentioned above, a classical motivation for studying the spaces $H^p(\mathbb{T}^\infty)$ is the connection to Dirichlet series. The connection can be observed by restricting functions in $H^p(\mathbb{T}^\infty)$ to the complex path $(2^{-s}, 3^{-s}, 5^{-s}, \dots)$. In this way, one obtains, in a natural way, Dirichlet series. Formally,

$$f(z_1, z_2, \dots) = \sum_{\nu \in \mathbb{N}_0^\infty} a_\nu z^\nu \quad \Longrightarrow \quad f\big(2^{-s}, 3^{-s}, 5^{-s}, \dots\big) = \sum_{n=1}^\infty a_n n^{-s}.$$

In particular, when restricted to this path, the monomial z^ν becomes the "Dirichlet monomial" n^{-s} with $n = 2^{\nu_1} 3^{\nu_2} \cdots p_k^{\nu_k}$, where p_k denotes the k-th prime number.

Clearly, the above only makes sense through an abuse of notation. Namely, we let $f(2^{-s}, 3^{-s}, \ldots)$ denote the evaluation of f on a point in $\mathbb{D}^\infty$. To see that this actually makes sense, it suffices to recall that Cole and Gamelin [4] showed that functions in $H^p(\mathrm{T}^\infty)$ have analytic point evaluations exactly on $\mathbb{D}^\infty \cap \ell^2$ for $0 < p < \infty$ and $\mathbb{D}^\infty \cap c_0$ for $p = \infty$, respectively.

By keeping the $H^p(\mathbb{T}^\infty)$-norm, the above correspondence between functions in $H^p(\mathbb{T}^\infty)$ and their Dirichlet series counter-parts becomes the defining isometric and multiplicative isomorphism between these Hardy spaces and the Dirichlet-Hardy spaces $\mathcal{H}^p$ (see [2, 8]). The inverse of this correspondence is often refereed to as the "Bohr Lift" in honor of Harald Bohr, who was the first to note that Dirichlet series are restrictions of power series in infinitely many variables [3].

In addition to being motivational, for us the relevance of this connection is that it in part gives functions in $H^p(\mathbb{T}^\infty)$ a life on the interior of $\mathbb{T}^\infty$, but also tells us how to interpret this life in terms of one-variable analytic Dirichlet series functions. Namely, it was observed by Helson [6] that by a change of variables followed by Fubini, it holds for $f \in H^p(\mathbb{T}^\infty)$ that

$$+\infty > \int_{\mathbb{T}^\infty} |f(z_1, z_2, \cdots)|^p \mathrm{d}m(z) = \int_{\mathbb{T}^\infty} \left(\int_{\mathbb{R}} |f(2^{-\mathrm{i}t} z_1, 3^{-\mathrm{i}t} z_2, \ldots)|^p \frac{\mathrm{d}t}{\pi(1+t^2)} \right) \mathrm{d}m(z).$$

In other words, for almost every $z \in \mathbb{T}^\infty$, the function $f_z(\mathrm{i}t) := f(2^{-\mathrm{i}t} z_1, 3^{-\mathrm{i}t} z_2, \ldots)$ is p-integrable with respect to the probability measure $\mathrm{d}t/\pi(1+t^2)$. From this, and drawing upon the above notations, it is not hard to deduce that for almost every $z \in \mathbb{T}^\infty$ then $f_z(\mathrm{i}t)$ extends to a Dirichlet series $f_z(s) = \sum_{n \geq 1} a_n n^{-s} z^\nu$, where $s = \sigma + \mathrm{i}t$, and which belonging to the one-dimensional Hardy space $H^p\big(\mathrm{d}t/(1+t^2)\big)$ of functions analytic on $\operatorname{Re} s > 0$.

In [14], Saksman and Seip showed how a restricted Fatou theorem on the existence of radial boundary values almost everywhere for $H^\infty(\mathbb{T}^\infty)$ could easily be established from a one-variable counterpart using the observation due to Helson above. By the same ideas, it is also easily observed that this also holds for all $0 < p < \infty$. Moreover, in the same way, it follows from the one-dimensional counterpart that the maximal operator

$$Mf(z) = \sup_{\sigma > 0} \big| f_z(2^{-\sigma}, 3^{-\sigma}, \ldots) \big|$$

belongs to $L^p(\mathbb{T}^\infty)$ with exactly the same norm as its one-variable counter-part.

With the above techniques, we are able to deduce several classical results from the one-variable theory of Hardy spaces in the infinite dimensional setting. Most notably, the theorems of F. and M. Riesz on the integrability of $\log|f|$ for $f \in H^p(\mathbb{T}^\infty)$, and on the absolute continuity of analytic measures on $\mathbb{T}^\infty$. We remark that both these applications were previously known, even in a more general context, although relying on far deeper arguments (see [7] and [12, chapter 8]).

Although the one-dimensional complex path $(2^{-s}, 3^{-s}, \ldots)$ used by Helson leads to a beautiful connection with Dirichlet series, it does have some drawbacks. By the above-mentioned result due to Cole and Gamelin, for $0 < p < \infty$ the space $H^p(\mathbb{T}^\infty)$ only has analytic point evaluations on $\mathbb{D}^\infty \cap \ell^2$. This adds the restriction $\operatorname{Re} s > 1/2$, stopping us from reaching the boundary at $\operatorname{Re} s = 0$. Although this can be overcome by considering only "almost every z" in the Helson approach above, it does leave the problem of having to work on conditionally convergent Dirichlet series.

To be able to work with the more amenable absolutely convergent power series, we change the approach path to the point $z \in \mathbb{T}^\infty$ from $(2^{-s}z_1, 3^{-s}z_2, \ldots)$, as $s \to 1$ non-tangentially, to the "exponential path" $(sz_1, s^2z_2, s^3z_3, \ldots)$, as $s \to 1$ non-tangentially. At the price of disconnecting our approach from the theory of Dirichlet series, this leads to a much more robust set of tools, allowing further simplifications to our proofs of the above mentioned results by F. and M. Riesz.

In this description of our work in progress, we end by noting that the tools thus obtained, allow us, in a simple way, to also extend a result by Fatou saying that the Poisson extensions of singular measures have vanishing radial limits almost everywhere. This was previously established by Marcinkiewcz and Zygmund for finite dimensional polydiscs using quite elaborate arguments (see [13, Theorem 2.3.1]). By our arguments, the proof is reduced to essentially repeating a suitable one-variable argument word by word.

References

[1] Alexandru Aleman, Jan-Fredrik Olsen, and Eero Saksman, *Fourier multipliers of Hardy spaces of Dirichlet series*, to appear in the IMRN, 2013.

[2] Frédéric Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math. **136** (2002), no. 3, 203–236.

[3] Harald Bohr, *Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen* $\sum a_n/n^s$, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. (1913), 441–488.

[4] Brian J. Cole and T. W. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. London Math. Soc. **53** (1986), no. 3, 112–142.

[5] Håkan Hedenmalm and Eero Saksman, *Carleson's convergence theorem for Dirichlet series*, Pacific J. Math. **208** (2003), no. 1, 85–109.

[6] Henry Helson, *Compact groups and Dirichlet series*, Ark. Mat. **8** (1969), 139–143.

[7] Henry Helson and David Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165–202.

[8] Håkan Hedenmalm, Peter Lindqvist, and Kristian Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in* $L^2(0,1)$, Duke Math. J. **86** (1997), 1–37.

[9] Jan-Fredrik Olsen, *Local properties of Hilbert spaces of Dirichlet series*, J. Funct. Anal. **261** (2011), no. 9, 2669–2696.

[10] Jan-Fredrik Olsen and Eero Saksman, *On the boundary behaviour of the Hardy spaces of Dirichlet series and a frame bound estimate*, J. Reine Angew. Math. **663** (2012), 33–66.

[11] Jan-Fredrik Olsen and Kristian Seip, *Local interpolation in Hilbert spaces of Dirichlet series*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 203–212 (electronic).

[12] Walter Rudin, *Fourier Analysis on Groups*, Interscience Publishers (a division of John Wiley and Sons), New York-London, 1962.

[13] Walter Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[14] Eero Saksman and Kristian Seip, *Integral means and boundary limits of Dirichlet series*, Bull. Lond. Math. Soc. **41** (2009), no. 3, 411–422.

Problems for Polynomials of One and several Variables

Dmitry Khavinson

1. Walsh's Coincidence Theorem

Let $z_1, \ldots, z_N \in \mathbb{C}$ and $\prod_{k=1}^{k=N}(z + z_k) = \sum_{k=0}^{N} L_k(z_1, \ldots, z_N) z^{N-k}$, where L_k stand for standard linear symmetric forms of N variables of degree k. Any linear symmetric form of N variables can be written as

$$L(z_1, \ldots, z_N) = \sum_{0}^{N} A_K L_k(z_1, \ldots, z_N),\ A_k \in \mathbb{C}.$$

When $z_1 = \ldots = z_N = z$, we obtain

$$L(z, \ldots, z_N) = \sum_{0}^{N} \binom{N}{k} A_k z^k, \quad \text{a polynomial in}\, \mathbb{C}.$$

Theorem. *(**Walsh's coincidence theorem** - cf.* [9].*) If for $z_1, \ldots, z_N) \in \mathbb{D}^N$, the polydisk, a linear symmetric form $L(z_1, \ldots, z_N) = 0$, then $\exists z \in \overline{\mathbb{D}}$: $L(z, \ldots, z) = 0$.*

Question. *For which other polynomials P, symmetric with respect to all permutations of the variables, the statement remains true, i.e., if the zero set $\{P = 0\}$ is connected in the polydisk , then it intersects the main diagonal of the polydisk?*

As was noted by S. Shimorin [10], a direct generalization fails for $N \geq 3$ since, e.g., for $P := \sum_1^3 z_j^2 - \sum_{i \neq j} z_i z_j - c$, where $c \neq 0$ is small, the "zero" set of P is a connected "cylinder" in $\mathbb{D}^3$, symmetric with respect to all the permutations of the variables, that does not intersect the main diagonal. It would be interesting to produce a similar example in $\mathbb{D}^2$ of a polynomial $P = P(z_1, z_2)$, such that $P(z_1, z_2) = P(z_2, z_1)$ and the set $\{P = 0\}$ is connected in $\mathbb{D}^2$ but does not intersect the main diagonal. Of course, such an example is impossible in $\mathbb{R}^2$ in view of the intermediate value theorem for continuous functions.

2. Extending the Fundamental Theorem of Algebra

Theorem. *(**FTA**.) $\forall P(z) := \sum_0^N a_k z^k$, $a_N \neq 0$ has precisely N roots (zeros) in $\mathbb{C}$ counting multiplicities.*

Possible Extensions In late 1980s T. Sheil-Small initiated study of the FTA for harmonic polynomials $h := p(z) - \overline{q(z)}$, such that $n := \deg p > \deg q =: m$. (Such normalization of degrees yields $\sharp\{h = 0\} < \infty$ cf. [9]) Using classical Bezout's theorem, A. Wilmshurst [9] showed that $\sharp\{h = 0\} \leq n^2$ and produced an elegant example showing that n^2 is sharp for all n. m in Wilmshurst's example equals $n - 1$. Let $Z(m, n) := \sharp\{h = 0\}$. D.Khavinson and G. Swiatek [8] showed that $Z(1, n) \leq 3n - 2$, a result conjectured by Sheil-Small and Wilmshurst. L. Geyer [6] proved that this bound is sharp for all n. In 2013, P. Bleher, Y. Homma, L. Ji and R. Roeder [2] showed that the sets $SP_n(k)$ of polynomials p of degree n

such that $p - \overline{z}$ has precisely k zeros are all open sets in the space of polynomials of degree n for $n \leq k \leq 3n-2$, $k = n, n+2, \ldots, 3n-2$ thus, showing that all these valencies can be assumed with positive probabilities. A. Wilmshurst [9] conjectured that $Z(m,n) \leq m(m-1) + 3n - 2$. In the forthcoming paper S. Y. Lee, A. Lerario and E. Lundberg proved that this is not true for all $n \geq 4$ and $m = n - 3$.

Question. *Find the asymptotics of $Z(m,n)$ for large m, n. More generally, let $P(x,y) \geq 0$, $\deg P = n$ has only isolated zeros in $\mathbb{R}^2$. Then the maximal number of real zeros for P is $\leq n^2$ and, it is not hard to see using Wilmshurst's example, is $\geq [n^2/4]$. Can any number between these two bounds of real zeros of P occur?*

3. Bohr's radius for polynomials

Theorem. *(**H. Bohr**, [3]) Let $f(z) = \sum_0^\infty a_k z^k$, $\|f\|_\infty \leq 1$ in $\mathbb{D}$. Then, $R_{\mathcal{B}} := \sup\{|z| : \sum |a_k||z|^k \leq 1\} = 1/3$ and this bound is sharp.*

(Curiously, Bohr's paper was actually written by G. Hardy based on Bohr's letters, and the sharp bound 1/3 was due in fact, independently, to M. Riesz, I. Schur and F. Wiener.)

Furthermore, ([1],[4]), one has a simple asymptotic for the majorant series

$$M(r) := \sup\{\sum_0^\infty a_k r^k,\ f : \|f\|_\infty \leq 1 \,\text{in}\, \mathbb{D}\} \sim \frac{1}{\sqrt{1-r^2}}.$$

Fix $N \geq 1$ and denote by Π_N the set of polynomials $\{P_N(z) := \sum_0^N a_k z^k,\ a_N \neq 0,\ \|P\|_\infty = 1\}$. Define by analogy the n-th Bohr radius $R_N := \sup\{r < 1 : \sum_0^N |a_k| r^k \leq 1,\ \forall P \in \Pi_N\}$. Clearly, $R_N > 1/3$.

Question. *What is the rate of convergence of R_N to $1/3$ as $N \to \infty$? In other words, what is the asymptotic of $R_N - \frac{1}{3}$?*

Z. Guadarrama [11] showed that there exist constants C_1, C_2 such that

$$\frac{C_1}{3^{N/2}} \leq R_N - \frac{1}{3} \leq C_2 \frac{\log N}{N}.$$

R. Fournier [5] showed that R_N equals to the smallest root in $(0,1)$ of the determinant of a certain Toeplitz matrix and, based on some numerical evidence, cautiously suggested that

$$R_N - \frac{1}{3} \sim \frac{\pi^2}{3N^2} + O\left(\frac{1}{N^4}\right).$$

The difficulty of the problem lies with the fact that neither the extremal problem for $R_{\mathcal{B}}$ (D.Khavinson, cf. [11]), nor its counterpart for R_N (R. Fournier, [5]) admit nonconstant solutions.

The author gratefully acknowledges partial support from the National Science Foundation under the grant DMS-0855597 and the support from the MFO Foundation.

References

1. C. Bénéteau, A. Dahlner and D. Khavinson, Remarks on the Bohr phenomenon. Comput. Methods Funct. Theory 4 (2004), no. 1, 1 – 19.
2. P. Bleher, Y. Homma, L. Ji and R. Roeder, Counting zeros of rational harmonic functions and its application to gravitational lensing, to appear in Intern.Math. Res. Notes, 2013.
3. H. Bohr, A theorem concerning power series, Proc. London Math. Soc. (2) 13 (1914), 1–5.
4. E. Bombieri and J. Bourgain, A remark on Bohr's inequality. Int. Math. Res. Not. 2004, no. 80, 4307 - 4330.
5. R. Fournier, Asymptotics of the Bohr radius for polynomials of fixed degree. J. Math. Anal. Appl. 338 (2008), no. 2, 1100 - 1107.
6. L. Geyer, Sharp bounds for the valence of certain harmonic polynomials. Proc. Amer. Math. Soc. 136 (2008), no. 2, 549 - 555.
7. D. Khavinson and G. Neumann, From the fundamental theorem of algebra to astrophysics: a "harmonious" path. Notices Amer. Math. Soc. 55 (2008), no. 6, 666 - 675.
8. D. Khavinson and G. Swiatek, On the number of zeros of certain harmonic polynomials. Proc. Amer. Math. Soc. 131 (2003), no. 2, 409 - 414.
9. T. Sheil-Small, Complex Polynomials, Cambridge Studies in Advanced Mathematics, 75. Cambridge University Press, Cambridge, 2002. xx+428 pp. ISBN: 0-521-40068-6.
10. S. Shimorin, a written communication.
11. Z. Guadarrama, Bohr's radius for polynomials in one complex variable. Comput. Methods Funct. Theory 5 (2005), no. 1, 143 - 151.

Problem Session: Dirichlet Series and Function Theory in Polydiscs

There was an evening problem session chaired by Eero Saksman. The proposers presented their problems orally to Håkan Hedenmalm, who then presented them in real-time on the blackboard. There was general agreement that this way of organizing the problem session worked well. A broad range of interesting problems were presented and discussed.

1. C. Beneteau: Cyclic Polynomials in the Dirichlet Spaces of the Bidisk

Consider a scale of Hilbert spaces of holomorphic functions on the bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < 1\}$$

indexed by a parameter $\alpha \in (-\infty, \infty)$. We say that a holomorphic function $f\colon \mathbb{D}^2 \to \mathbb{C}$ belongs to the *Dirichlet-type space* $\mathfrak{D}_\alpha$ (see [3]) if its power series expansion

$$f(z_1, z_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$$

satisfies

$$\|f\|_\alpha^2 = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{k,l}|^2 < \infty.$$

The spaces $\mathfrak{D}_\alpha$ are a natural generalization to two variables of the classical Dirichlet-type spaces D_α, $-\infty < \alpha < \infty$, consisting of functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ that are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and satisfy

$$\|f\|_{D_\alpha}^2 = \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty.$$

A function is called *cyclic* if the closure (in the norm of the relevant Dirichlet space) of polynomial multiples of the function is the whole space. In the classical Hardy space of one variable, a function is cyclic if and only if it's outer. In the classical Dirichlet space of one variable (for $\alpha = 1$), Brown and Shields ([2]) proved that an outer function whose boundary zero set has positive logarithmic capacity cannot be cyclic. The famous Brown and Shields conjecture is that the converse is true. All polynomials of one variable that have no zeros in the unit disk are cyclic in all one variable Dirichlet spaces for $\alpha \leq 1$. In two variables, one of the surprising results is that polynomials with no zeros in the disk are not always cyclic in all Dirichlet spaces; for example, the polynomial $p(z_1, z_2) = 1 - z_1 z_2$ is cyclic in $\mathfrak{D}_\alpha$ if and only if $\alpha \leq 1/2$ ([1]). A preliminary study of this phenomenon including a natural definition of a relevant capacity is contained in [1]. An interesting question is thus the following:

Problem: Characterize cyclic polynomials in the Dirichlet spaces $\mathfrak{D}_\alpha$.

2. J. E. McCarthy: The Lindelöf Hypothesis and $\mathcal{BMO}$

Problem: Can one use functional analysis, specifically a BMO theory, to prove the Lindelöf hypothesis?

Motivation. For a finite Dirichlet series $f(s)$, define a norm for $1 \leq p < \infty$ by

$$\|f\|_{\mathcal{H}^p}^p = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p dt; \tag{1}$$

and let $\mathcal{H}^p$ be the completion of the finite Dirichlet series in this norm. We shall call the expression on the right-hand side of (1) a limit in the mean of order p. Let $\zeta_1(s) = \sum (-1)^n n^{-s}$ be the alternating zeta function. The Lindelöf hypothesis can be rephrased[1] [5, Thm. 13.2] as the following assertion:

$$\forall \frac{1}{2} < \sigma < 1, \forall p < \infty, \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta_1(\sigma + it)|^p dt < \infty.$$

It is known that $\zeta_1(\sigma + it)$ is unbounded on the imaginary axis, so in some sense you want to show that an unbounded function is in every $\mathcal{H}^p$ space for p finite. Can

[1]The advantage of working with ζ_1 rather than ζ is that it is analytic up to the imaginary axis; but as $\zeta_1(s) = (2^{1-s} - 1)\zeta(s)$, they have the same order of growth on any vertical line except $\sigma = 1$.

you do this by developing a BMO-type space, $\mathcal{BMO}$ say, which has the properties that

$$\text{(2)} \qquad \forall f \in \mathcal{BMO}, \quad \forall \quad \sigma > 0, \forall p < \infty, \ \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^p dt < \infty.$$

$$\text{(3)} \qquad \forall \quad \sigma > \frac{1}{2}, \ \zeta_1(\sigma+it) \in \mathcal{BMO}.$$

Discussion:

There are two obvious candidates, $\mathcal{BMO}_1$ and $\mathcal{BMO}_2$. The space $\mathcal{BMO}_1$ is the dual of $\mathcal{H}^1$; the space $\mathcal{BMO}_2$ is the dual of

$$\mathcal{H}^2 \odot \mathcal{H}^2 = \{\sum_{n=1}^{\infty} f_n g_n : f_n, g_n \in \mathcal{H}^2, \sum \|f_n\|^2 < \infty, \sum \|g_n\|^2 < \infty\}.$$

Ortega-Cerda and Seip have shown that $\mathcal{BMO}_1 \subsetneq \mathcal{BMO}_2$ [4].

The space $\mathcal{BMO}_2$ is interesting in its own right — it is the space of symbols of bounded Hankel forms on $\mathcal{H}^2$. To prove that a function h is in $\mathcal{BMO}_2$, it is enough to show that for any $f, g \in \mathcal{H}^2$, we have that $\langle fs, h\rangle$ is bounded by $\|f\|\|g\|$. It is easy to see that if $h(s) = \zeta(\sigma+s)$ and one does the formal calculation on the Dirichlet coefficients, then

$$\sum \widehat{fg}(k)\hat{h}(k) \ = \ f(\sigma)g(\sigma),$$

and the right-hand side is bounded by $\|f\|\|g\|$ if $\sigma > 1/2$. But $h(1+s)$ does not have limits in the mean for any $p \geq 1$, because of the pole.

The same argument shows that $\zeta(\sigma+s)$ is in $\mathcal{BMO}_1$ for every $\sigma > 1/2$.

So neither $\mathcal{BMO}_1$ nor $\mathcal{BMO}_2$ are the right space for the purpose of proving Lindelöf. Is there another, smaller, space sandwiched between them and $\mathcal{H}^\infty$ that satisfies both (2) (which $\mathcal{H}^\infty$ obviously does) and (3)?

3. H. Queffelec

3.1. **Question 1.** In the eighties, in the course of his studies on Sidon sets, Pisier coined a new norm (Pisier norm) on the set of trigonometric polynomials $f(t) = \sum a_n e^{int}$, by setting

$$[f] = \mathbb{E}\,\|f_\omega\|_\infty$$

where $f_\omega(t) = \sum \varepsilon_n(\omega) a_n e^{int}$ is the randomized version of f. Here, (ε_n) is a sequence of independent Rademacher variables (i.e. taking values ± 1 with probability 1/2). The motivation for considering this norm came from a result of D.Rider which can be rephrased as: if $E \subset \mathbb{Z}$ is a subset of the integers and if $\mathcal{P}_E$ denotes the set of trigonometric polynomials with spectrum in E, then the inequality

$$\|f\|_\infty \leq C[f] \text{ for all } f \in \mathcal{P}_E$$

characterizes Sidon sets. Pisier naturally considered the sets $E \subset \mathbb{Z}$ for which the reverse inequality holds:

$$[f] \leq C\|f\|_\infty \text{ for all } f \in \mathcal{P}_E$$

and called those new sets *stationary*. He showed that

(1) A stationary set cannot contain arbitrarily long arithmetic progressions
(2) A product, or sum, of Sidon sets is stationary.

The question is:

Is the set S of perfect squares stationary?

It is well-known that S is not Sidon and cannot contain arithmetic progressions of length bigger than 4. This seems a natural example to consider, since we know plenty on it: gaussian sums, ... The (conjectured) non-stationarity of S amounts to find amounts to find $f_N = \sum_{n=1}^{N} a_n e^{in^2 t} \in \mathcal{P}_S$ such that

$$|a_n| \approx 1 \text{ and } \|f_N\|_\infty = o(\sqrt{N \log N}).$$

3.2. **Question 2.** Let $A(\mathbb{D})$ be the disk algebra, W the set of absolutely convergent Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=0}^{\infty} |a_n| < \infty$ and H^2 the Hardy space of the disk. It is obvious that any $f \in W$ can be written as a rectangular convolution $f = g*h$ where $g, h \in H^2$. Using the Kahane-Katznelson-de Leeuw theorem (under the Kisliakov form), one can prove (Calado's thesis, Prop 3.1.6) that one of the functions g or h can be taken in $A(\mathbb{D})$. If f has a Hadamard lacunary Taylor series, it can be proved (Calado's thesis, Thm 3.2.4) that both g and h can be taken in $A(\mathbb{D})$. The question, motivated by results of Kahane's book "Some random series of functions" p.90 on the square factorization $f = g * g$, is the following:

Has any function $f \in A(\mathbb{D})$ a rectangular factorization $f = g * h$ with both g and h in $A(\mathbb{D})$?

4. K. Seip: Two Problems on Dirichlet Polynomials

For a Dirichlet polynomial $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ and $1 \le p < \infty$, we define

$$\|f\|_p^p := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p dt.$$

Problem (a) The first problem is Helson's conjecture [7]: Show that

$$\Big\| \sum_{n_1}^{N} n^{-s} \Big\|_1 = o(\sqrt{N}).$$

In fact, an improvement of any of the known bounds

$$\left(\sum_{n=1}^{N} \frac{1}{d(n)} \right)^{1/2} \le \Big\| \sum_{n_1}^{N} n^{-s} \Big\|_1 \le \sqrt{N}$$

would be of interest. (Here d denotes the divisor function.) The lower bound was found by Helson in [6], while the upper bound follows trivially from the inequality $\|f\|_1 \le \|f\|_2$. By a classical estimate of Wilson, the lower bound behaves asymptotically as a constant times $\sqrt{N}/(\log N)^{1/4}$.

Problem (b) Now fix a large N and let $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ be an arbitrary Dirichlet polynomial of degree N. By a well-known result of Montgomery [8, p. 50] (see also [9]), we have

$$\frac{1}{T}\int_0^T |f(it)|^2 dt = (1 + T^{-1}O(N))\|f\|_2^2.$$

This means in particular that if we choose $T = N$, then there is a universal constant C such that

$$\frac{1}{T}\int_0^T |f(it)|^2 dt \leq C\|f\|_2^2.$$

Now choose a constant $C > 1$ and define $T_p(N, C)$ as the smallest T such that

$$\frac{1}{T}\int_0^T |f(it)|^p dt \leq C\|f\|_p^p$$

for all Dirichlet polynomials f of degree at most N. How does $T_p(N, C)$ depend on p? In particular, do we have $T_p(N, C) = O(N)$ when $1 \leq p < 2$? Presumably, the value of $C > 1$ is not important.

5. C. Thiele: The triangle Hilbert integral

Let f_1, f_2, f_3 be three test functions (compactly supported and smooth) in the plane. Define

$$\Lambda(f_1, f_2, f_3) = p.v. \int\int\int f_1(x_2, x_3) f_2(x_3, x_1) f_3(x_1, x_2) \frac{1}{x_1 + x_2 + x_3}\, dx_1 dx_2 dx_3 \ .$$

Here the principle value symbol stands or a restriction of the integral to the region $|x_1 + x_2 + x_3| > \epsilon$ and subsequent limit $\epsilon \to 0$, in analogy to the classical Hilbert transform.

Conjecture. *There is a constant C such that*

$$\Lambda(f_1, f_2, f_3) \leq C\|f_1\|_3\|f_2\|_3\|f_3\|_3$$

for all triples of test functions f_1, f_2, f_3.

Here $\|f\|_3$ stands for the Lebesgue L^3 norm of f. A natural extension of this conjecture would allow three different exponents $2 < p_1, p_2, p_3 < \infty$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$. However, by symmetry and interpolation, the stated conjecture is evidently the easiest of the possible L^p estimates. A (likely only moderately) easier conjecture arises by replacing the L^3 norms by Lorentz norms ("weak type") $\|f\|_{3,\infty}$.

There are obvious variants of the above for other degrees of multilinearity. We note that the bilinear form

$$\Lambda_2(f_1, f_2) = p.v. \int\int f_1(x_2) f_2(x_1) \frac{1}{x_1 + x_2}\, dx_1 dx_2$$

is closely related to the classical Hilbert transform and is well known to satisfy

$$|\Lambda_2(f_1, f_2)| \leq C\|f_1\|_2\|f_2\|_2 \ .$$

It is cheap to conjecture an purely L^4 bound for the analoguous four-linear form. However, it is quite well understood that the complexity within this family of problems grows considerably with the degree of multilinearity.

Denoting by x the vector (x_1, x_2) and by β_1 , β_2, and β_3 three vectors in the plane spanning a non-degenerate triangle, we may by a change of variables and a reinterpretation of the functions f_i rewrite the trilinear form Λ as

$$\int\int p.v. \int f_1(x+\beta_1 t) f_2(x+\beta_2 t) f_3(x+\beta_3 t) \, \frac{dt}{t} \, dx_1 dx_2 \ .$$

The analoguous conjectural bound is then to hold uniformly in all choices of $\beta_1, \beta_2, \beta_3$. This shows a structural similarity with the bilinear Hilbert transform [13]. Indeed, by a limiting process, letting the tuple β_i tend to a degenerate triangle, one may deduce the L^3 bound of [13] from the above conjecture. One even obtains uniform bounds as in [11]. By a version of the method of rotations, one can deduce from the above conjecture bounds for certain two dimensional bilinear Hilbert transforms with odd and homogeneous kernels of the type studied in [10] and [12]. The range of exponents for which one knows bounds for the twisted paraproduct of [12] is a reason to be cautious about extending the above conjecture past the threshold $p_i > 2$. Finally, as elaborated on in [10], one may deduce the $p = 3$ case of the Carleson-Hunt theorem from the above conjecture.

6. A. Defant

Let X be a Banach sequence space in the sense that $\ell_1 \subset X \subset c_0$ (as sets) and the canonical sequences e_k form a 1-unconditional basis of X. For $n \in X$ define $X_n = \text{span}_{1 \le k \le n} e_k$, and denote its open unit ball by B_{X_n}. As usual we write $H_\infty(B_{X_n})$ for the Banach space of all bounded holomorphic functions on (the complete Reinhardt domain) B_{X_n}. Then the nth Bohr radius of B_{X_n} is given by

$$K(B_{X_n}) := \sup \left\{ 0 \le r \le 1 \,\Big|\, \forall\, f \in H_\infty(B_{X_n}) \,\forall\, z \in B_{X_n} : \sum_\alpha \Big| \frac{\partial f(0)}{\alpha!} z^\alpha \Big| \le \|f\|_\infty \right\}.$$

Bohr's famous power series theorem shows that $K(\mathbb{D}) = 1/3$ (where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$).

For a given Banach sequence space X, there is a tremendous amount of literature dealing with the asymptotic order of these Bohr radii. Improving an earlier result of Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip, it was recently proved by Bayart, Pellegrino and Seoane-Sepúlveda that

$$\limsup_{n \to \infty} \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log n}{n}}} = 1 \,,$$

and by Defant and Frerick (improving earlier results of Aizenberg, Boas, Defant, Dineen, Frerick, and Khavinson) that for $1 \le p \le \infty$ and every n we have (with constants independent of n)

$$K(B_{\ell_p^n}) \asymp \left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{p,2\}}}.$$

In particular, for all $1 < p \leq \infty$ the sequence of Bohr radii $K(B_{\ell_p^n})$ tends to zero, and only for $p = 1$ we have that $K(B_{\ell_1^n})$ stays away from zero: by a result of Aizenberg $K(B_{\ell_1^n}) \geq \frac{1}{3}e^{-1/3}$. Our problem is about the converse of Aizenberg's result.

PROBLEM

Let X be a Banach sequence space such that $\inf_n K(B_{X_n}) > 0$. Is it true that $X = \ell_1$ (as sets)?

The problem seems of probabilistic nature. By results of Bayart, Defant, Maestre and Prengel we know that each X with $\inf_n K(B_{X_n}) > 0$ is at least very close to ℓ_1: There is $C > 0$ such that for every $\varepsilon > 0$, every n and every choice of $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\frac{1}{C\,(\log\log n)^{\varepsilon}} \sum_{k=1}^{n} |\lambda_k| \;\leq\; \Big\| \sum_{k=1}^{n} \lambda_k e_k \Big\|_X \;\leq\; C \sum_{k=1}^{n} |\lambda_k|\,.$$

By an important result of Lempert every bounded and holomorphic function $f \in H_\infty(B_{\ell_1})$ has a monomial series representation $f(z) = \sum_\alpha \frac{\partial f(0)}{\alpha!} z^\alpha$ which converges in every $z \in B_{\ell_1}$. An equivalent formulation of the above problem is: **Is $X = \ell_1$ the unique Banach sequence space satisfying Lempert's theorem?**

7. J. Ortega-Cerdà: L^1-Bernstein inequality in high dimensions

This problem has been solved by R. Berman and J. Ortega-Cerdà and its solution will be included in an upcoming paper on Marcinkiewicz-Zygmund sequences for polynomials.

Let M be a smooth compact algebraic manifold in $\mathbb{R}^n$. Prove that there is a constant C such that for any polynomial p,

$$\int_M |\nabla_t p| \leq C \deg(p) \int_M |p|,$$

where ∇_t denotes the tangential gradient.

Remark: The corresponding L^∞ version, i.e.

$$\sup_M |\nabla_t p| \leq C \deg(p) \sup_M |p|$$

is true and it is know to be equivalent to the algebraicity of M, see L. P. Bos, N. Levenberg, P. D. Milman, and B. A. Taylor, Tangential Markov inequalities characterize algebraic submanifolds of $\mathbb{R}^n$, Indiana University Mathematics Journal, vol. 44, no. 1, pp. 115–138, 1995.

8. M. Balazard

The first problem is an old question of mine, already included as Problem 24 in Montgomery and Vorhauer's collection [14].

Question. *Does there exist an ordinary Dirichlet series converging in some open half-plane, and having there one zero, and only one?*

The second and third problem are related to Nyman's criterion for the Riemann hypothesis. Let

$$\begin{aligned} L &= L^2(0,\infty; t^{-2}dt) \\ \chi(t) &= [t \geqslant 1] \quad (t > 0) \\ e_\alpha(t) &= \{t/\alpha\} \quad (\alpha > 0, t > 0) \quad \text{where } \{\} \text{ is the fractional part} \\ D(\lambda) &= \operatorname{dist}_L\big(\chi, \operatorname{span}_L(e_\alpha, 1 \leqslant \alpha \leqslant \lambda)\big) \quad (\lambda \geqslant 1). \end{aligned}$$

Question. *Is $D(\lambda)$ a continuous function of λ?*

Question. *Is $D(\lambda)$ a strictly decreasing function of λ?*

References

[1] C. Bénéteau, A.A. Condori, C. Liaw, D. Seco, and A.A. Sola, Cyclicity in Dirichlet-type spaces and extremal polynomials II: Functions on the Bidisk, preprint.

[2] L. Brown and A.L. Shields, Cyclic vectors in the Dirichlet space, Trans. Amer. Math. Soc. **285** (1984), 269-304.

[3] D. Jupiter and D. Redett, Multipliers on Dirichlet type spaces, Acta Sci. Math. (Szeged) **72** (2006), 179-203.

[4] Joaquim Ortega-Cerdà and Kristian seip. A lower bound in Nehari's theorem on the polydisc. *J. Anal. Math.*, 118(1):339-243, 2012.

[5] E.C. Titchmarsh. *The theory of functions.* Oxford University Press, London, 1932.

[6] H. Helson, *Hankel forms and sums of random variables* Studia Math. **176** (2006), 85–92.

[7] H. Helson, *Hankel forms*, Studia Math. **198** (2010), 79–84.

[8] H. L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics **227**, Springer-Verlag, Berlin-New York, 1971.

[9] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. **8** (1974), 73–82.

[10] Ciprian Demeter and Christoph Thiele, *On the two-dimensional bilinear Hilbert transform*, Amer. J. Math. **132** (2010), no. 1, 201–256.

[11] Loukas Grafakos and Xiaochun Li, *Uniform bounds for the bilinear Hilbert transform. I*, Ann. of Math. (2) **159** (2004), no. 3, 899–933.

[12] Vjekoslav Kovač, *Boundedness of the twisted paraproduct*, Rev. Mat. Iberoam. **28** (2012), no. 4, 1143–1164.

[13] Michael Lacey and Christoph Thiele, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, Ann. of Math. (2)* ***146*** *(1997), no. 3, 693–724.*

[14] H. L. Mongomery and U. Vorhauer – "Some unsolved problems in harmonic analysis as found in analytic number theory", (2002), `http://nato-us.org/analysis2000/papers`.

Reporter: Ole Fredrik Brevig

Participants

Prof. Dr. Michel Balazard
Institut de Mathématiques de Luminy
Case 907
163 Avenue de Luminy
13288 Marseille Cedex 9
FRANCE

Prof. Dr. Frederic Bayart
Laboratoire de Mathématiques
UMR 6620 du CNRS
Université Blaise Pascal-Clermont II
63177 Aubiere Cedex
FRANCE

Prof. Dr. Catherine Beneteau
Department of Mathematics
University of South Florida
Tampa, FL 33620-5700
UNITED STATES

Dr. Istvan Berkes
Institute of Statistics
University of Technology
Münzgrabenstr. 11/III
8010 Graz
AUSTRIA

Prof. Dr. Jose Bonet
IUMPA
Universidad Politecnica de Valencia
c. de Vera
46022 Valencia
SPAIN

Ole Fredrik Brevig
Department of Mathematical Sciences
NTNU
7491 Trondheim
NORWAY

Prof. Dr. Andreas Defant
Institut für Mathematik
Carl von Ossietzky
Universität Oldenburg
Postfach 2503
26015 Oldenburg
GERMANY

Dr. Leonhard Frerick
Abteilung Mathematik
Fachbereich IV
Universität Trier
54286 Trier
GERMANY

Dr. Domingo Garcia
Facultad de Matematicas
Universidad de Valencia
Avda. Dr. Moliner 50
46100 Burjassot (Valencia)
SPAIN

Prof. Dr. Hakan Hedenmalm
Department of Mathematics
Royal Institute of Technology
10044 Stockholm
SWEDEN

Prof. Dr. Dmitry Khavinson
Department of Mathematics
University of South Florida
Tampa, FL 33620-5700
UNITED STATES

Prof. Dr. Michael T. Lacey
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Prof. Dr. Manuel Maestre
Facultad de Matematicas
Universidad de Valencia
Avda. Dr. Moliner 50
46100 Burjassot (Valencia)
SPAIN

Prof. Dr. Mieczyslaw Mastylo
Fac. of Mathematics & Computer Science
A. Mickiewicz University
ul. Umultowska 87
61-614 Poznan
POLAND

Prof. Dr. John E. McCarthy
Department of Mathematics
Washington University in St. Louis
Campus Box 1146
One Brookings Drive
St. Louis, MO 63130-4899
UNITED STATES

Dr. Jan-Fredrik Olsen
Department of Mathematics
University of Lund
Box 118
221 00 Lund
SWEDEN

Prof. Dr. Joaquim Ortega Cerda
Facultat de Matematiques
Universitat de Barcelona
Gran Via, 585
08071 Barcelona
SPAIN

Dr. Hervé Queffélec
Université Lille I
Laboratoire Paul Painleve
UFR de Mathématiques
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Eero Saksman
Dept. of Mathematics & Statistics
University of Helsinki
P.O.Box 68
00014 University of Helsinki
FINLAND

Sunke Schlüters
Institut für Mathematik
Carl von Ossietzky
Universität Oldenburg
Postfach 2503
26015 Oldenburg
GERMANY

Prof. Dr. Kristian Seip
Department of Mathematical Sciences
NTNU
7491 Trondheim
NORWAY

Prof. Dr. Juan Benigno Seoane-Sepúlveda
Depto. de Analisis Matematico
Facultad de Matematicas
Universidad Complutense de Madrid
28040 Madrid
SPAIN

Dr. Pablo Sevilla Peris
IUMPA
Universitat Politecnica de Valencia
46022 Valencia
SPAIN

Prof. Dr. Christoph Thiele
Hausdorff Center for Mathematics
Universität Bonn
Villa Maria
Endenicher Allee 62
53115 Bonn
GERMANY

Prof. Dr. Michel Weber
Institut de Mathématiques
Université de Strasbourg
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE