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# Mini-Workshop: Batalin-Vilkovisky Algebras, Operads, and Hopf Algebroids

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Abstract. This workshop brought together 17 researchers whose work involves Batalin–Vilkovisky algebras, operads, and related structures such as Gerstenhaber algebras and cyclic homology. It featured introductory lectures on some relevant topics followed by talks about recent research results.

*Mathematics Subject Classification (2010):* 18D10, 18D50, 18G30, 19D55, 55P48, 55P50, 83C47.

## Introduction by the Organisers

A lot of research in homological algebra comes in one of two flavours: some subjects of study have their origins in algebraic topology (model categories, simplicial objects, operads etc), the others in algebra and algebraic geometry (Tor and Ext, derived categories). The two are strongly interacting, and problems and solutions have often found their way from one community into the other. But there is still some cultural difference, and the wider aim of this workshop was to bring researchers from both backgrounds together. Batalin-Vilkovisky algebras have cropped up recently in various places in both algebra and algebraic topology. Thus, we have decided to make these the central topic of the workshop, as they were an ideal catalyst for discussions.

The two other topics of the workshop are promising directions for interaction, topics in which the two communities can learn from one another. Operads, originally invented by topologists, have been used for the purposes of homotopical algebra and deformation theory in a prominent way in the past two decades, but still are finding their place in the repertoire of methods of contemporary algebra.

Hopf algebroids undeservedly have not yet attracted full attention of experts in operad theory, and this workshop appeared a perfect opportunity to fix that.

In short, a Batalin-Vilkovisky (BV from now on) algebra is a graded commutative algebra $V$ equipped with a differential $\Delta$. This differential is not assumed to turn $V$ into a differential graded algebra. Instead, the failure of the (graded) Leibniz rule,

$$\{v, w\} := \Delta(vw) - \Delta(v)w - (-1)^{\deg v} v\Delta(w), \quad v, w \in V,$$

is assumed to turn $V$ into a Gerstenhaber algebra. Thus BV algebras are a special type of Gerstenhaber algebra in which the bracket is generated by a differential; hence they are alternatively referred to as exact Gerstenhaber algebras.

Historically, the notion was coined in the BRST formalism in quantum field theory but more recently it has become clear that cohomology rings of various mathematical objects tend to have a canonical BV algebra structure. For example, the paradigmatic example of a Gerstenhaber algebra is the Hochschild cohomology of an associative algebra, and this is BV whenever the algebra is a Calabi-Yau algebra [4]. An analogous result holds for Lie-Rinehart cohomology [5] and in fact the Ext-algebra of any Hopf algebroid [6]. In the operadic world, BV algebras arise form operads with multiplication [8].

In these references, Poincaré duality identifies the cohomology of the BV operator with a suitable variation of cyclic homology. Furthermore, Koszul-type dualities relate the BV operator in some cases to Frobenius algebra structures on Koszul dual objects [10]. However, this is so far only established for concrete cohomology theories and examples, the deeper reason behind these mechanisms seems not yet fully understood.

The original application in quantum field theory provides a homological interpretation of some physical equations. From a mathematical point of view, this can be seen as a special case of homological perturbation theory, or homotopy transfer formulae [9].

In a slightly different context, BV algebras have been applied in symplectic topology, see e.g. [1], and one of the aims of the workshop was to make the algebraic audience aware of these results.

However, the general aim was to report on the most up-to-date developments that the participants want to share. Hence we asked a few speakers to give introductory and survey lectures that explained some central notions and results, and afterwards the remining speakers gave research talks on whatever topic they felt was most relevant to the workshop.

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# Mini-Workshop: Batalin-Vilkovisky Algebras, Operads, and Hopf Algebroids

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# Abstracts

## Operads

Vladimir Dotsenko

### 1. Operads of endomorphisms and nonsymmetric operads

Associative algebras give an axiomatic framework to deal with linear transformations of a vector space under composition of maps. Instead of talking about properties of compositions we instead work with arbitrary associative algebras, which, by definition, is a vector space $A$ together with a linear map $\mu\colon A\otimes A\to A$ (that is, a bilinear map $A\times A\to A$) satisfying the associativity axiom $\mu(\mu(a,b),c)=\mu(a,\mu(b,c))$ for all $a,b,c\in A$.

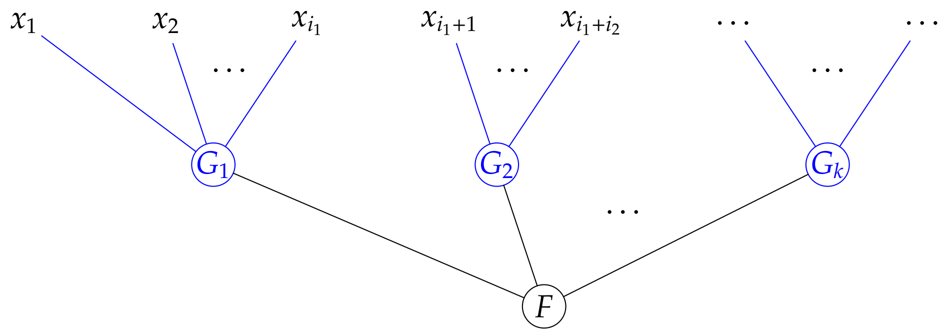
Similarly, if instead of the space of linear maps $\mathrm{Hom}(V,V)$ we work with the collection of all multilinear maps $\{\mathrm{Hom}(V^{\otimes n},V)\}_{n\geq 0}$, there are obvious ways to compose these, obtaining new multilinear maps. Basically, the role of compositions is now played by the maps

$$\gamma_{k,i_1,\ldots,i_k}\colon \mathrm{Hom}(V^{\otimes k},V)\otimes\mathrm{Hom}(V^{\otimes i_1},V)\otimes\cdots\otimes\mathrm{Hom}(V^{\otimes i_k},V)\to\mathrm{Hom}(V^{\otimes i_1+\cdots+i_k},V),$$

defined as

$$(\gamma_{k,i_1,\ldots,i_k}(F\otimes G_1\otimes G_2\otimes\cdots\otimes G_k))(v_1\otimes\cdots\otimes v_{i_1+\cdots+i_k})=$$
$$=F(G_1(x_1,\cdots,x_{i_1}),G_2(x_{i_1+1},\cdots,x_{i_1+i_2}),\ldots,G_k(x_{i_1+\cdots+i_{k-1}+1},\ldots,x_{i_1+\cdots+i_k})).$$

These maps satisfy obvious associativity-like properties that are evident from representing these compositions combinatorially as two-level trees




Namely, any three-level tree composition can be computed in two possible ways, and the results of these should be the same. Plus, there is a "unit" $\mathrm{id}\in\mathrm{Hom}(V,V)$, the identity map. Clearly,

$$\gamma_{k,1,1,\ldots,1}(f,\mathrm{id},\mathrm{id},\ldots,\mathrm{id})=f=\gamma_{1,k}(\mathrm{id},f).$$

By definition, a *nonsymmetric operad* is a collection of vector spaces $\{\mathcal{P}(n)\}_{n\geq 0}$ equipped with structure maps

$$\gamma_{k,i_1,\ldots,i_k}\colon\mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k)\to\mathcal{P}(i_1+\cdots+i_k),$$

and the "unit" $\mathrm{id}\in\mathcal{P}(1)$ which satisfy the same "three-level tree associativity" between them (in this case, it is a requirement on the structure maps, not a property that is satisfied automatically as for a specific choice of those in the case of multilinear maps), and the same unit axioms.

The collection of all multilinear maps discussed above is denoted $\mathrm{End}_V$ and called the endomorphism operad of $V$; by definition $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$.

The following examples show how known notions fit into operadic framework.

**Example 1.** First, let $A$ be an associative algebra. Let us define an operad $\mathcal{O}_A$ by the formula

$$\mathcal{O}_A(n) = \begin{cases} A, n = 1, \\ 0, n \neq 1 \end{cases} .$$

The only nontrivial structure map (that is, a map between nonzero vector spaces) that we have to define is $\gamma_{1,1} \colon \mathcal{O}_A(1) \otimes \mathcal{O}_A(1) \to \mathcal{O}_A(1)$, that is a map $A \otimes A \to A$. For such a map, we take the product in $A$. For the unit of $\mathcal{O}_A$, we shall take the unit in $A$. The operad axioms reduce then to axioms of an associative algebra.

**Example 2.** Next, let $A$ be an associative algebra, and let $M$ be a left $A$-module. Let us define an operad $\mathcal{O}_{A,M}$ by the formula

$$\mathcal{O}_{A,M}(n) = \begin{cases} M, n = 0, \\ A, n = 1, \\ 0, n > 1 \end{cases} .$$

The only nontrivial structure maps that we have to define are

$$\gamma_{1,1} \colon \mathcal{O}_{A,M}(1) \otimes \mathcal{O}_{A,M}(1) \to \mathcal{O}_{A,M}(1),$$

and $\gamma_{1,0} \colon \mathcal{O}_{A,M}(1) \otimes \mathcal{O}_{A,M}(0) \to \mathcal{O}_{A,M}(0)$, that is maps $A \otimes A \to A$ and $A \otimes M \to M$. For such maps, we take the product in $A$, and the action of $A$ on $M$. For the unit of $\mathcal{O}_{A,M}$, we shall take the unit in $A$. The operad axioms reduce then to axioms of an associative algebra and a module over it.

## 2. Symmetric operads

In fact, there is one extra bit of structure that the endomorphism operad possesses that is not visible while we only look at linear transformations. Namely, permuting arguments of an operation, that is the transformations

$$(\sigma(F))(x_1, \ldots, x_n) = F(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for each $\sigma \in S_n$, give each space $\mathrm{End}_V(n)$ a structure of a right $S_n$-module. This permutation action satisfies two kinds of compatibility with compositions. First, we can let $\sigma = (\sigma_1, \ldots, \sigma_k) \in S_{i_1} \times \cdots \times S_{i_k} \subset S_{i_1 + \cdots + i_k}$ act on $\gamma_{k,i_1,\ldots,i_k}(F \otimes G_1 \cdots \otimes G_k)$, or alternatively, we can let $\sigma_1$ act on $G_1$, ..., $\sigma_k$ act on $G_k$, and then apply the structure map $\gamma_{k,i_1,\ldots,i_k}$; the results must be the same, so that

$$(\sigma_1, \ldots, \sigma_k)(\gamma_{k,i_1,\ldots,i_k}(F \otimes G_1 \otimes \cdots \otimes G_k)) = \gamma_{k,i_1,\ldots,i_k}(F \otimes \sigma_1(G_1) \otimes \cdots \otimes \sigma_k(G_k)).$$

Also, we may either let $\tau \in S_k$ act on $F$, and then compute the structure map $\gamma_{k,i_1,\ldots,i_k}(\tau(F) \otimes G_1 \otimes \cdots \otimes G_k)$, or alternatively we can consider the permutation $\widetilde{\tau} \in S_{i_1 + \cdots + i_k}$ that permutes the blocks of sizes $i_1$, ..., $i_k$ (out of which the set of size $i_1+\cdots+i_k$ is made) according to $\tau$, and let it act on $\gamma_{k,i_1,\ldots,i_k}(F\otimes G_1\cdots\otimes G_k)$; the results must be the same, so that

$$\widetilde{\tau}(\gamma_{k,i_1,\ldots,i_k}(F\otimes G_1\otimes\cdots\otimes G_k))=\gamma_{k,i_1,\ldots,i_k}(\tau(F)\otimes G_1\otimes\cdots\otimes G_k).$$

By definition, a *symmetric operad* is a collection of right $S_n$-modules $\{\mathcal{P}(n)\}_{n\geq 0}$ equipped with structure maps

$$\gamma_{k,i_1,\ldots,i_k}\colon \mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k)\to\mathcal{P}(i_1+\cdots+i_k),$$

and the "unit" $\mathrm{id}\in\mathcal{P}(1)$ which satisfy the same "three-level tree associativity" between them (in this case, it is a requirement on the structure maps, not a property that is satisfied automatically as for a specific choice of those in the case of multilinear maps), the same unit axioms, and the same compatibility with the $S_n$-module structure.

Of course, by design the endomorphism operad is a symmetric operad. Another series of examples of algebraic nature is obtained as follows. For each algebraic structure $\mathcal{S}$ (associative algebras, Lie algebras, commutative algebras etc.), one defines an operad, putting $\mathcal{S}(n)$ to be the space of all multilinear operations one can define using just the structure operations guaranteed by $\mathcal{S}$. For example, a unit of an associative algebra would be modelled by an element $1\in\mathcal{S}(0)$, as it is a "constant", an operation with no arguments; similarly, a binary product would be modelled by an element $\mu\in\mathcal{S}(2)$. Symmetries of operations (e.g. skew-symmetry of a Lie bracket) is also incorporated trivially, since components of an operad are supposed to be modules over symmetric groups. Such an operad is called the operad controlling algebras of type $\mathcal{S}$.

## 3. Examples coming from topology

The axioms above can be written in any symmetric monoidal category, not necessarily the category of vector spaces. Two important (and related to each other) examples is the category of $\mathbb{Z}$-graded chain complexes (with the usual tensor product, and the symmetry isomorphism that incorporates the sign rule: swapping two elements of odd degree creates a factor $(-1)$) and the category of topological spaces (with the cartesian product). From a symmetric operad in the category of topological spaces we get a symmetric operad in the category of chain complexes by taking singular chains, and from the latter an operad in the category of graded vector spaces (chain complexes with zero differential) by taking homology. The Künneth formula guarantees that the latter is an operad.

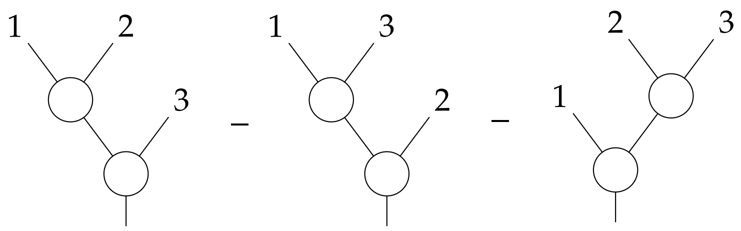
One of the most famous topological operads is the operad of little disks whose $n$th component is the space of configuration of $n$ non-overlapping disks inside the unit disk. Its homology is the operad of Gerstenhaber algebras; both of these operads are featured in many talks of this workshop.

## 4. Free operad, and operads presented by generators and relations

Similarly to how one constructs the free associative algebra generated by a vector space $V$ as the space spanned by words in the alphabet indexed by a basis of $V$, one can construct the *free operad* generated by the collection of right $S_n$-modules $\{\mathcal{V}(n)\}_{n\geq 0}$. Its spanning set consists of "tree-shaped tensors", that is rooted trees whose vertices are decorated by basis elements of the generating collection, so that each vertex with $k$ inputs is decorated by an element of $\mathcal{V}(k)$. The structure maps $\gamma_{k,i_1,\ldots,i_k}$ are given by grafting of trees onto one another.

An *two-sided ideal* of the free operad is a collection $\mathcal{I}$ of $S_n$-submodules which satisfies the following generalisation of the condition defining two-sided ideals of associative algebras: $\gamma_{k,i_1,\ldots,i_k}(F \otimes G_1 \cdots \otimes G_k)$ belongs to $\mathcal{I}$ whenever at least one of the elements $F, G_1, \ldots, G_k$ belongs to $\mathcal{I}$. A quotient of the free operad by a two-sided ideal has a natural operad structure. If we pick a subset $\mathcal{R}$ of components of the free operad, the quotient by the two-sided ideal generated by $\mathcal{R}$ (that is, by definition, the smallest ideal containing $\mathcal{R}$) is the operad generated by $V$ subject to relations $R$.

For example, if we take the free operad with one binary skew-symmetric generator, its quotient by the ideal generated by the element




is the operad controlling Lie algebras.

## 5. Further topics

In homological algebra, there are two important circles of questions. First, given a chain complex with some extra structure (multiplication, Lie bracket etc.), one wonders what structures are there to be found on its homology, provided that the given structure is compatible with the differential. Second, given some structure on the homology of a chain complex, one wonders what structures on the original complex may induce such a structure. A classical example of the first kind is given by higher Massey products on the singular homology of a topological space, an example of the second kind is what is often called the Deligne conjecture about Hochschild cohomology complex of an associative algebra. Both types of questions benefit a lot from operadic methods, including, but not limited to, bar-cobar duality, Koszul duality, homotopy transfer theorem, Quillen homology, Gröbner bases for operads etc. Interested readers are welcome to explore the introductory text [2] and the monograph [1].

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## Cyclic Homology

Ulrich Krähmer

### 1. Duchain complexes

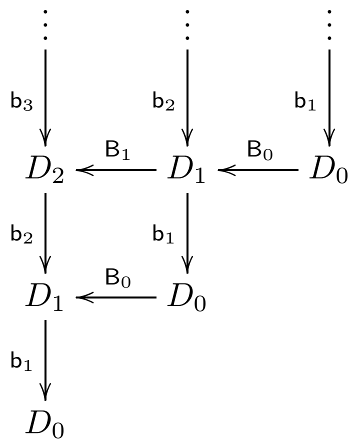
A *duchain complex* in an abelian category $\mathcal{C}$ is a sequence $\{D_n\}_{n\geq 0}$ of objects in $\mathcal{C}$ together with morphisms

$$\mathsf{b}_n : D_n \to D_{n-1}, \quad \mathsf{B}_n : D_n \to D_{n+1}, \quad \mathsf{b}_{n-1}\mathsf{b}_n = \mathsf{B}_{n+1}\mathsf{B}_n = 0.$$

For any duchain complex one defines $\mathsf{T}_n : D_n \to D_n$ by

$$\mathsf{b}_{n+1}\mathsf{B}_n + \mathsf{B}_{n-1}\mathsf{b}_n =: \mathrm{id} - \mathsf{T}_n.$$

If $\mathsf{T}_n = \mathrm{id}$ for all $n$, then $D$ is called a *mixed complex.* In this case,




is a bicomplex and the homology of its total complex is the *cyclic homology* $\mathrm{HC}_n(D)$ of $D$. For a general duchain complex one calls $\mathrm{M}(D) := \mathrm{coker}\,(\mathrm{id} - \mathsf{T})$ the *mixed complex associated to* $D$, and one defines $\mathrm{HC}_n(D) := \mathrm{HC}_n(\mathrm{M}(D))$.

### 2. Duplicial objects

In the main part of this introductory talk I recall the Dwyer-Kan correspondence which generalises the classical Dold-Kan correspondence: a *duplicial object* in a category $\mathcal{C}$ is a simplicial object $C_\bullet = \{C_n\}_{n\in\mathbb{N}}$ with face and degeneracy maps

$$\mathsf{d}_{n,i} \colon C_n \to C_{n-1}, \quad \mathsf{s}_{n,j} \colon C_n \to C_{n+1}, \quad 0 \leq i, j \leq n$$

together with additional morphisms $\mathsf{t}_n \colon C_n \to C_n$ satisfying

$$\mathsf{d}_{n,i}\mathsf{t}_n = \begin{cases} \mathsf{t}_{n-1}\mathsf{d}_{n,i-1}, & 1 \leq i \leq n \\ \mathsf{d}_{n,n}, & i = 0 \end{cases}$$

and

$$\mathsf{s}_{n,i}\mathsf{t}_n = \begin{cases} \mathsf{t}_{n+1}\mathsf{s}_{n,i-1}, & 1 \leq i \leq n \\ \mathsf{t}_{n+1}^2\mathsf{s}_{n,n}, & i = 0. \end{cases}$$

A duplicial object is *cyclic* if $\mathsf{T}_n := \mathsf{t}_n^{n+1} = \mathrm{id}$. One defines

$$\mathsf{s}_n := \mathsf{t}_{n+1}\mathsf{s}_{n,n}, \quad \tilde{\mathsf{t}_n} := (-1)^n \mathsf{t}_n, \quad \mathsf{N}_n := \sum_{i=0}^{n} \tilde{\mathsf{t}_n^i},$$

and

$$\mathsf{b}_n := \sum_{i=0}^{n} (-1)^i \mathsf{d}_{n,i}, \quad \mathsf{B}_n := (\mathrm{id} - \tilde{\mathsf{t}_{n+1}})\mathsf{s}_n \mathsf{N}_n.$$

If $\mathcal{C}$ is abelian, then for every duplicial object $C_\bullet$ one defines its *associated cyclic object* $C_\bullet^{\mathrm{cyc}} := \mathrm{coker}(\mathrm{id} - \mathsf{T}_\bullet)$.

The Dwyer-Kan correspondence now states that the normalised chain complex functor establishes an equivalence between duplicial objects and duchain complexes.

## 3. An example

Let $A$ be a unital associative algebra over a commutative ring $k$, and let $\sigma \in \mathrm{Aut}(A)$ be an algebra automorphism.

Then $C_n := A^{\otimes n+1}$ becomes a duplicial $k$-module with

$$\mathsf{d}_{n,i}(a_0 \otimes \cdots \otimes a_n) := a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n,$$

$$\mathsf{d}_{n,n}(a_0 \otimes \cdots \otimes a_n) := \sigma(a_n) a_0 \otimes \cdots \otimes a_{n+1},$$

$$\mathsf{s}_{n,i}(a_0 \otimes \cdots \otimes a_n) := a_0 \otimes \cdots a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n,$$

and

$$\mathsf{t}_n(a_0 \otimes \cdots \otimes a_n) = \sigma(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

In this case, the simplicial homology $\mathrm{H}(C, \mathsf{b})$ of $C$ is is the *Hochschild homology* of $A$ with coefficients in the $A$-bimodule ${}_\sigma A$ which is $A$ as right $A$-module with left action $a.b := \sigma(a)b$. For $\sigma = \mathrm{id}$ the duplicial object becomes cyclic and defines the cyclic homology $\mathrm{HC}(A)$ of the associative algebra $A$. This is for example used as the target space of the *Connes-Chern character*

$$\mathrm{ch} : K_i(A) \to \mathrm{HC}_i(A).$$

## 4. Calabi-Yau algebras

These are associative algebras which show a Poincaré-type duality

$$\mathrm{H}^i(A, A) \simeq H_{d-i}(A, {}_\sigma A)$$

on Hochschild (co)homology, where $\sigma$ is a distinguished automorphism called the *modular automorphism* of $A$. So for these algebras, the duplicial object defined above describes the Hochschild cohomology of $A$. In the last part of the talk I explain that in this case, the operator dual to $\mathsf{B}$ yields a Batalin-Vilkovisky algebra structure on Hochschild cohomology, provided $\sigma$ is semisimple.

## $E_2$-algebras

James T. Griffin

An $E_2$-*algebra* is an algebra for an $E_2$-operad, which is an operad homotopy equivalent to the little 2-discs operad, $\mathcal{LD}_2$ in the topological case, or to the singular chains of the little 2-discs operad $C_*(\mathcal{LD}_2)$ in the algebraic case. This means that $E_2$-structure on one example may appear in a different form to the $E_2$-structure on another example.

In the topological setting, the arity $k$ operations of little discs operad, $\mathcal{LD}_2(k)$ are parametrised by the space of $k$ non-overlapping 2-discs inside a larger unit disc. The multiplication in the operad, $\mu(C; C_1, \ldots, C_k)$ for $C$ a configuration of $k$ discs is given by scaling and translating the configuration $C_i$ down to the size of the $i$th disc of $C$, for each $i = 1, \ldots, k$. Taking all of the scaled discs, and removing the $k$ unscaled discs gives a new configuration of discs, defining the composition.

The primary examples of algebras for $\mathcal{LD}_2$ are provided by double loop spaces $[S^2, Y]_*$ for a pointed space $Y$. Each configuration of discs, $C$ should be viewed as a template, where $\mu(C; f_1, \ldots, f_k)$ is a new map of the sphere into $Y$ given by sending the $i$th little disc of $C$ into $Y$ using $f_i$, and sending all remaining points to the base point. The remarkable *Recognition Theorem* [1, 7] says that if a $\mathcal{LD}_2$-algebra is connected, or more generally 'group-like', then it is homotopy equivalent to a $\mathcal{LD}_2$-algebra of the form $[S^2, Y]$ for some pointed space $Y$.

The homology of the little discs operad forms an operad in the category of graded abelian groups, isomorphic to the operad for Gerstenhaber algebras. There is a variant of $\mathcal{LD}_2$, where a point on the boundary of each disc is chosen, this is called the *framed little discs operad*, $f\mathcal{LD}_2$. The homology of this operad is the operad for Batalin-Vilkovisky algebras, the BV operad.

Gerstenhaber showed that the Hochschild cohomology $HH^*(A, A)$ of an associative algebra carries both a cup product and a graded Lie bracket, which satisfy certain identities [4]. A graded abelian group with this structure is now called a Gerstenhaber algebra. Deligne conjectured that the Hochschild cochains are an algebra for an operad homotopy equivalent to the chains $C_*(\mathcal{LD}_2)$, lifting the Gerstenhaber algebra structure on cohomology to an $E_2$-algebra on chains. This has been proved many times, for one approach see [6].

The action of the operad of chains factors through operations called "braces". These are described by a dg-operad Br, whose space of operations $\mathrm{Br}(k)$ is spanned by the set of rooted bipartite planar trees; one set of vertices, the 'white' vertices are labelled 1 to $k$, the other 'black' set are unlabelled. The combinatorics of these trees can alternatively be used to describe a space of planar cacti. By building a chain of homotopy equivalences between the little discs operad, a planar cactus operad and the braces operad one may prove the Deligne conjecture.

We finish by reviewing conjectures from geometric group theory. Ruth Charney conjectured [3] that the Artin groups are CAT(0), that is they act properly and cocompactly by isometries on a CAT(0) space. In the type A case, Brady and McCammond constructed spaces with metrics on which the braid group $B_n$ acts [2]. These are referred to as the orthoscheme complexes of non-crossing partitions of the $n$-gon. They proved that these spaces are CAT(0) for $n = 4$ and 5, and conjectured this for all $n$. This has been verified for $n = 6$ in [5]. The connection with $E_2$-operads arises when one compares the orthoscheme complexes and the rooted planar bipartite trees. The talk ended with a conjecture that the normalised planar cacti and the orthoscheme complexes are isomorphic.

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## Schwede's Loop Bracket and Projective Classes

Paul Slevin

The concept of a Gerstenhaber algebra was invented by Murray Gerstenhaber in 1963, when he defined such a structure on $HH(A)$, the Hochschild cohomology of a $k$-algebra $A$. In 1998, Schwede came up with an interesting way to define the same bracket by constructing a loop in the category of Yoneda bimodule extensions of $A$ by $A$. In the first part of the talk, we recall the details of this construction.

In the second part of the talk, we will see how Schwede's method can be adapted for use in a different situation. Let $\mathcal{A}$ be an abelian category. A *projective class* in $\mathcal{A}$ is a pair $(\mathcal{P}, \mathcal{E})$, where $\mathcal{P}$ is a class of objects in $\mathcal{A}$ and $\mathcal{E}$ is a class of morphisms in $\mathcal{A}$, such that

- $\mathcal{P} = \{P \mid \mathcal{A}(P, f) \text{ is epic } \forall f \in \mathcal{E}\}$;
- $\mathcal{E} = \{f \mid \mathcal{A}(P, f) \text{ is epic } \forall P \in \mathcal{P}\}$;
- for all objects $A$ there exists a morphism $P \longrightarrow A$ in $\mathcal{E}$ with $P \in \mathcal{P}$.

Projective classes generalise many of the concepts in homological algebra. A complex $\mathbf{E}$ is a *$\mathcal{P}$-exact sequence* if $\mathcal{A}(P, \mathbf{E})$ is exact for all $P \in \mathcal{P}$. A chain morphism $f$ is a *$\mathcal{P}$-equivalence* if $\mathcal{A}(P, f)$ is a quasi-isomorphism in $\mathsf{Ab}$ for all $P \in \mathcal{P}$. A *$\mathcal{P}$-resolution* of an object $X$ is a $\mathcal{P}$-equivalence $P_\bullet \longrightarrow X$ where $P_\bullet$ is $\mathcal{P}$-exact and every object is in $\mathcal{P}$, and we can show that every object has a unique one up to homotopy. For any objects $M, N$ in $\mathcal{A}$ we define $\mathrm{Ext}_{\mathcal{P}}(M, N)$ to be the cohomology of the complex $\mathcal{A}(P_\bullet, N)$ where $P_\bullet$ is a $\mathcal{P}$-resolution of $M$.

Suppose further that $\mathcal{A}$ is abelian monoidal with monoidal unit $\mathbf{1}$, and in addition that the tensor product of $\mathcal{P}$-equivalences is a $\mathcal{P}$-equivalence. Then, using methods similar to Schwede's, we get a map

$$\Omega\colon\ \mathrm{Ext}^m_{\mathcal{P}}(\mathbf{1},\mathbf{1})\times\mathrm{Ext}^n_{\mathcal{P}}(\mathbf{1},\mathbf{1})\longrightarrow\mathrm{Ext}^{m+n-1}_{\mathcal{P}}(\mathbf{1},\mathbf{1}).$$

It is not known whether this is a Gerstenhaber bracket in general. As an example of this loop construction, we get a map $\Omega$ on $\mathrm{Ext}_{U/k}(k,k)$ when $U$ is a bialgebra over a commutative ring $k$.

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## Hopf algebroids

Gabriella Böhm

Our aim was to introduce the audience to bialgebroids and Hopf algebroids. These notions were motivated by the examples provided by bialgebras and Hopf algebras over a field. In the generalization to arbitrary base algebras, we focused on the subtleties coming from the various actions of the base algebra. The presentation was based on an essential use of monoidal categories.

Following an idea traced back to [2], two definitions of a *bialgebra* over a field were shown to be equivalent:

- an algebra $H$ together with a lifting of the monoidal structure of the category of vector spaces to the category of left (equivalently, right) $H$-modules;
- a comonoid $H$ in the monoidal category of algebras.

Dually, further two equivalent definitions of a bialgebra were given:

- a coalgebra $H$ together with a lifting of the monoidal structure of the category of vector spaces to the category of left (equivalently, right) $H$-comodules;
- a monoid $H$ in the monoidal category of coalgebras.

For any bialgebra $H$, the adjunctions provided by the free and forgetful functors give rise to a vector space isomorphism between the space of left $H$-module and right $H$-comodule homomorphisms $H\otimes H\to H\otimes H$, and the space of linear maps $H\to H$. This isomorphism can be used to transfer the algebra structure on the former space (given by the composition of maps) to the latter space, yielding the so-called convolution algebra of linear maps $H\to H$. These considerations immediately yield the equivalence of two definitions of *Hopf algebra*:

- A bialgebra $H$ such that the canonical map (that is, the image of the identity map $H\to H$ under the above isomorphism) is invertible;

- A bialgebra $H$ possessing an antipode; that is, a convolution inverse of the identity map $H \to H$.

Passing from a base field to an arbitrary (associative and unital but not necessarily commutative) base algebra $R$ essentially means to work in the monoidal category of $R$-bimodules instead of the category of vector spaces. However, in contrast to the category of vector spaces, the category of $R$-bimodules is not symmetric so the generalization is highly non-trivial.

As a generalization of (co)algebra, an *$R$-(co)ring* was defined as a (co)monoid in the category of $R$-bimodules. Based on [3], two equivalent definitions of a left (respectively, right) *bialgebroid* were given:

- as an $R^{\mathsf{e}} := R^{\mathsf{op}} \otimes R$ ring $H$ together with a lifting of the monoidal structure of the category of $R$-bimodules to the category of left (respectively, right) modules over the algebra $H$,
- by Takeuchi's earlier definition (who used the name *$\times_R$-bialgebra*) in terms of an $R$-coring structure on $H$ (with appropriately chosen $R$-actions).

The comultiplication $H \to H \otimes_R H$ of the $R$-coring $H$ in Takeuchi's definition factorizes through the so-called *Takeuchi product* which is a distinguished sub $R^{\mathsf{e}}$-bimodule of the $R$-module tensor product $H \otimes_R H$. While $H \otimes_R H$ is not an algebra, the Takeuchi product carries a natural $R^{\mathsf{e}}$-ring structure and the corestriction of the comultiplication to the Takeuchi product is a homomorphism of $R^{\mathsf{e}}$-rings. Also the counit $H \to R$ gives rise to a homomorphism of $R^{\mathsf{e}}$-rings from $H$ to the algebra of linear endomorphisms of $R$.

The definition of bialgebroid is not known to have an equivalent form based on the lifting of some monoidal structure to the category of comodules over a given coring. As a partial result, it is proven in [3] that the monoidal structure of the category of $R$-bimodules does lift to the category of (both left and right) comodules over any (left or right) $R$-bialgebroid. In particular, any comodule carries a natural $R$-bimodule structure. The coaction on the $R$-module tensor product of comodules is given by the so-called diagonal coaction and the coaction on $R$ is induced by the algebra homomorphism $R^{\mathsf{e}} \to H$.

For any right bialgebroid $H$ over an algebra $R$, the adjunctions by the forgetful and free functors yield an isomorphism between the space of left $H$-module and right $H$-comodule homomorphisms from an appropriate $R^{\mathsf{op}}$-module tensor product $H \otimes_{R^{\mathsf{op}}} H$ to the $R$-module tensor product $H \otimes_R H$, and the space of $R$-bimodule maps $H \to H$ in a suitable sense. The canonical map is defined as the image of the identity map $H \to H$ under this isomorphism. Requiring it to be an isomorphism, we obtain the definition of *right Hopf algebroid* (a.k.a. *$\times_R$-Hopf algebra*) in [4]. (A symmetric consideration for left bialgebroids yields the definition of *left Hopf algebroid*.) Since the space of left $H$-module and right $H$-comodule homomorphisms $H \otimes_{R^{\mathsf{op}}} H \to H \otimes_R H$ carries no algebra structure, it induces no convolution algebra structure on the isomorphic space of $R$-bimodule maps $H \to H$. Consequently in a left or right Hopf algebroid there may exist no antipode.

A more restrictive notion of so-called *full Hopf algebroid* was suggested in [1]. It consists of a left and a right bialgebroid structure on the same algebra $H$ and an antipode map which relates them. This antipode is not the inverse of the identity map $H \to H$ in some convolution algebra but in a suitable Morita context.

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## Hopf-cyclic cohomology

Tomasz Brzeziński

In an attempt to calculate the index of a transversally elliptic operator on a foliation [3], Alain Connes and Henri Moscovici described how an action of a particular Hopf algebra $\mathcal{H}$ on the convolution algebra $\mathcal{A}$ of smooth functions on the étale groupoid associated to the foliation can aid calculation of the index by the use of the cyclic cohomology of $\mathcal{H}$ [4]. $\mathcal{H}$ has a specific character, which allows one to construct a cocyclic module out of the tensor powers of $\mathcal{H}$. The cohomology of this module maps into the cyclic cohomology of $\mathcal{A}$, through the so-called *characteristic map*, so that the class needed for the index calculation is contained in the image of this map.

Motivated by this, Connes and Moscovici went on to associate a cocyclic module to any Hopf algebra $H$ over a field $k$, equipped with a character $\delta : H \to k$ and a group-like element $\sigma \in H$ such that $\delta(\sigma) = 1$ and the square of the $\delta$-twisted antipode $S_\delta = (\delta \otimes S) \circ \Delta$ acts on elements of $H$ by conjugation by $\sigma$ [5]. The couple $(\delta, \sigma)$ is called a *modular pair in involution* and the cyclic object is built by a modification of the standard Cartier complex for $H$.

Soon afterwards it has been realized that modular pairs in involution are special cases of vector spaces with compatible $H$-module and $H$-comodule structures [7], [8], [11], known as *stable anti-Yetter-Drinfeld modules* [6]. For a right $H$-module left $H$-comodule $M$, the anti-Yetter-Drinfeld compatibility condition is

$$\varrho(mh) = \sum S(h_{(3)})m_{(-1)}h_{(1)} \otimes m_{(0)}, \qquad \text{for all} \quad m \in M,\ h \in H,$$

where we use the standard Sweedler notation for the coproduct, while for left coaction $\varrho : M \to H \otimes M$ in the form $\varrho(m) = \sum m_{(-1)} \otimes m_{(0)}$. The stability condition reads $\sum m_{(0)}m_{(-1)} = m$.

Given a stable anti-Yetter-Drinfeld $H$-module $M$ one can associate a cocyclic object to any left $H$-module coalgebra $C$ by setting

$$C^n_H(C, M) := M \otimes_H C^{\otimes n+1},$$

where $H$ acts on $C^{\otimes n+1}$ diagonally, with the standard Cartier faces and degeneracies and the cyclic operator

$$\tau_n(m \otimes c_0 \otimes \cdots c_n) = \sum m_{(0)} \otimes \otimes c_1 \otimes \cdots c_n \otimes m_{(-1)} c_0.$$

The Connes-Moscovici object associated to a pair in involution $(\delta, \sigma)$ is obtained by specifying $C = H$ and $M = k$ with $H$-multiplication defined by $\delta$ and $H$-coaction by $\sigma$. The category of anti-Yetter-Drinfeld modules should be contrasted with that of Yetter-Drinfeld modules that plays natural role in the representation theory of (quasitriangular) Hopf algebras. The latter is a braided monoidal category, while the former is not. Nevertheless the latter acts on the former one. Both can be interpreted as subcategories of categories of differential graded modules over specific differential graded algebras associated to a Hopf algebra.

One of the main advantages of interpreting of modular pairs in involution in a more general framework of stable anti-Yetter-Drinfeld modules was opening the doors for several new classes of cocyclic objects associated to Hopf algebras. Rather than to an $H$-module coalgebra $C$ one can associate such an object also to an $H$-module algebra $A$ [7]. Rather than considering left $H$-comodule $M$ one can cosider an $H$-contramodule. Rather than considering Hopf algebras over a commutative base field $k$, one can consider Hopf algebroids over a non-commutative ring $R$ [1, 10], or, even more generally, distributive laws in category theory [1]. This last point of view reveals how cyclic duality connects various examples and constructions [2]. Another advantage is in interpreting of the Connes-Moscovici characteristic map as a form of pairing, which led to the definition of cup products [9].

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## Noncommutative differential calculi

Niels Kowalzig

A *precalculus* [GDTs, NTs] or *Gerstenhaber module* consists of a pair $(V^\bullet, \Omega_\bullet)$, where $(V^\bullet, \smile, \{.,.\})$ is a Gerstenhaber algebra and

(1) $(\Omega_{-\bullet}, \frown)$ is a graded module over $(V^\bullet, \smile)$ with action $\iota_\alpha := \alpha \frown -$, as well as

(2) $(\Omega_{-\bullet}, \mathcal{L})$ is a graded Lie algebra module over $(V^\bullet[1], \{.,.\})$ with action $\mathcal{L}$ such that for $\alpha, \beta \in V$ the mixed Leibniz rule

$$[\iota_\alpha, \mathcal{L}_\beta] = \iota_{\{\alpha,\beta\}}$$

holds true. A *calculus* or *Batalin-Vilkovisky module* is a Gerstenhaber module equipped with a differential $B$ of degree 1 such that the *Cartan-Rinehart homotopy formula*

$$\mathcal{L}_\alpha = [B, \iota_\alpha].$$

is fulfilled. One of the first (algebraic) examples unveiled in [GDTs, NTs] consists of the pair $\big(HH^\bullet(A,A), HH_\bullet(A,A)\big)$ of Hochschild cohomology and homology for an associative algebra $A$ over a commutative ring $k$. This example can be generalised [KoKr] to the realm of (left) Hopf algebroids $(U, A)$: if $U$ is right $A$-projective and $M$ a stable anti Yetter-Drinfel'd module, the pair $\big(\mathrm{Ext}^\bullet_U(A,A), \mathrm{Tor}^U_\bullet(M,A)\big)$ defines a calculus. Moreover, if $N$ is a braided commutative Yetter-Drinfel'd algebra such that $M \otimes_{A^{op}} N$ is a stable anti Yetter-Drinfel'd module, then even $\big(\mathrm{Ext}^\bullet_U(A,N), \mathrm{Tor}^U_\bullet(M,N)\big)$ carries the structure of a calculus [Ko].

### 1. The Kontsevich-Soibelman operad and an enhanced Deligne conjecture

Cohen's theorem [Co] states that (if $k$ is a field with characteristic zero) the singular homology operad $H_\bullet(dgD_2, k)$ of the little 2-discs operad $D_2$ is isomorphic to the Gerstenhaber operad $\mathsf{G}$. This, in particular, means that $H_\bullet(D_2, k)$ defines a left action on the Hochschild cohomology $HH^\bullet(A,A)$ for an associative $k$-algebra $A$. The Deligne conjecture (first proven by Tamarkin [Ta]) then states that the $H_\bullet(D_2,k)$-algebra structure on $HH^\bullet(A,A)$ is induced by a corresponding action on the (normalised singular) chain level, that is by an $\bar{S}_\bullet(D_2,k)$-algebra structure on the (normalised) Hochschild cochains.

These statements are, in some sense, only half of the story: Kontsevich and Soibelman [KS] introduced a coloured operad $\mathcal{KS}$ along with its topological partner, the coloured operad $\mathcal{C}yl$ of little discs on a cylinder (and two marked points on the top resp. bottom of the cylinder) such that (see [DoTaTs]) the singular homology operad $H_{-\bullet}(\mathcal{C}yl, k)$ with reversed grading gives the operad $\mathsf{calc}$, the algebras of which are precisely calculi in the sense mentioned above. While this obviously generalises Cohen's theorem, an enhanced Deligne conjecture is given by the result [KS] that the pair $\big((\bar{C}^\bullet(A,A), \bar{C}_\bullet(A,A)\big)$ of (normalised) Hochschild cochains and chains form an algebra over the singular chains operad of $\mathcal{C}yl$, which in turn is quasi-isomorphic to the operad $\mathcal{KS}$. The singular homology operad $H_{-\bullet}(\mathcal{KS}, k)$

is, on the other hand, generated by the operations $\smile, \frown, \{.,.\}, \mathcal{L}$, and $B$ that are defined for the pair $\big(\bar{C}^\bullet(A,A), \bar{C}_\bullet(A,A)\big)$ and that yield the well-known calculus structure on $\big(HH^\bullet(A,A), HH_\bullet(A,A)\big)$.

## 2. Calculi for cyclic modules over operads with multiplication

It is, at present, not obvious whether one can relate the (singular homology operad of the) Kontsevich-Soibelman operad or a suitable generalisation thereof to the pair $\big((C^\bullet(U,N), C_\bullet(U,M)\big)$ of cochains and chains for a (left) Hopf algebroid $(U,A)$ and with coefficients $M, N$ as stated above since the calculus operations $\smile, \frown, \{.,.\}, \mathcal{L}$, and $B$ are in this case of considerably higher complexity. On the other hand, it is known [GeSch, McCSm] that each operad $O^\bullet$ (in the category of $k$-modules) *with multiplication* $\mu$ defines a cosimplicial $k$-module, the corresponding cohomology $H^\bullet(O)$ of which carries the structure of a Gerstenhaber algebra. An extension of this result to a full calculus structure is developed in [Ko] by introducing a notion which is, in some sense, dual to that of a cyclic operad: if $-\bullet$ is a (unital) left module with reversed grading over the operad $(O^\bullet, \mu)$, one defines two additional structures on $\mathcal{M}_\bullet$: an *extra partial module map*

$$\bullet_0 : O^p \otimes_k \mathcal{M}_n \to \mathcal{M}_{n-p+1}, \quad 0 \le p \le n+1,$$

that is required to fulfil the same associativity relations as the "standard" partial module maps $\bullet_i$, $1 \le i \le n-p+1$, defining the left $O^\bullet$-action on $\mathcal{M}_{-\bullet}$, plus a morphism $t : \mathcal{M}_\bullet \to \mathcal{M}_\bullet$ that fulfils

$$(\varphi \bullet_i x) = \varphi \bullet_{i+1} t(x), \qquad i = 0, \ldots, n-p, \qquad \varphi \in O^p,\ x \in \mathcal{M}_n.$$

If $t^{n+1} = \mathrm{id}$, such an $\mathcal{M}_\bullet$ is called a *cyclic (unital) $O^\bullet$-modul* as it is straightforward to prove that $\mathcal{M}_\bullet$ gives rise to a cyclic $k$-module, with homology $H_\bullet(\mathcal{M})$ as the homology of the underlying simplicial $k$-module.

One can then show [Ko] that the pair $\big(H^\bullet(O), H_\bullet(\mathcal{M})\big)$ carries the structure of a noncommutative differential calculus. In particular, this contains the examples $\big(HH^\bullet(A,A), HH_\bullet(A,A)\big)$ and $\big(\mathrm{Ext}^\bullet_U(A,A), \mathrm{Tor}^U_\bullet(M,A)\big)$ mentioned above, but also directly applies to further pairs of cohomology and homology theories, as for example Poisson calculus.

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## Lie-Rinehart algebras, Hopf algebroids with and without an antipode

Ana Rovi

The enveloping algebra of a Lie algebra is a classical example of a Hopf algebra. Roughly speaking, the enveloping algebra of a Lie-Rinehart algebra [4] carries the structure of a Hopf algebroid [1]. More precisely, they are always *left bialgebroids* (introduced under the name $\times_R$-bialgebras by Takeuchi [7]), and in fact *left Hopf algebroids* (introduced under the name $\times_R$-Hopf algebras by Schauenburg [6]). However, the definitions of a Hopf algebroid due to Lu [3] and the one due to Böhm and Szlachányi [1], both assume the existence of an antipode satisfying certain axioms.

The aim of the talk was to communicate a concrete example [2, 5] of a Lie-Rinehart algebra whose universal enveloping algebra does not admit an antipode and hence is not a Hopf algebroid in the sense of [1, 3]. Hence we show that although every Hopf algebroid in the sense of [1] is left Hopf algebroid, see [1], there exist left Hopf algebroids without an antipode.

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## Homotopy Batalin-Vilkovisky algebras I

Brice Le Grignou

The first seminal example of what is now called a Batalin–Vilkovisky algebra first appeared in the paper [1] in the context of Mathematical Physics. The general notion was coined by Jean-Louis Koszul in his beautiful paper [3] a few years later. This kind of algebraic structure now plays an important role in many various fields of Mathematics.

**Definition.** A *Batalin–Vilkovisky algebra*, *BV algebra* for short, is a graded-commutative, or super-commutative in the physicists language, algebra $(A, ., \Delta)$ together with an order $\leq 2$, degree 1, square-zero operator.

Let $\langle -; - \rangle_2$ be the bracket which measures the default of $\Delta$ to be a derivation with respect to the product:

$$< a; b >_2 := \Delta(a.b) - \Delta(a).b - (-1)^{|a|} a.\Delta(b) \ .$$

The operator $\Delta$ is said to be of *order* $\leq 2$ if $\langle a; - \rangle_2$ is a derivation with respect to the product for any $a \in A$.

Here are two examples of BV-algebras, the first being the original one and the second one appears in Koszul's paper.

(1) Let $V$ a finite dimensional vector space and let $\mathcal{B} = \{x_1, ..., x_n\}$ be a basis. Let $\mathcal{S}(V^* \oplus s^{-1}V)$ be the space of polynomials on $x_1^*, \ldots, x_n^*$ and $s^{-1}x_1, \ldots, s^{-1}x_n$, where $\{x_1^*, \ldots, x_n^*\}$ is the dual base and where $s^{-1}x_i$ is the homological desuspension of $x_i$. The differential operator

$$\Delta := \sum_{i=1}^{n} \frac{\partial}{\partial x_i^*} \frac{\partial}{\partial s^{-1}x_i}$$

has order $\leq 2$, degree 1 and squares to zero.

(2) Let $\mathcal{M}$ be a manifold together with a Poisson structure $\omega \in \Gamma\big(\bigwedge^2 T\mathcal{M}\big)$. Consider the de Rham algebra of differential forms $(\Omega^\bullet \mathcal{M}, \wedge, d_{DR})$ with its usual product $\wedge$ and de Rham differential $d_{DR}$. The contraction $\iota_\omega$ is the endomorphism of $\Omega^\bullet \mathcal{M}$ which sends $f$ to $f(\omega, -, \ldots, -)$. The order $\leq 2$ operator

$$\Delta := [\iota_\omega, d_{DR}] = \iota_\omega \circ d_{DR} + d_{DR} \circ \iota_\omega$$

endows the de Rham algebra with a BV-algebra strcture.

For any BV-algebra $\mathcal{A} = (A, ., \Delta)$, the data $\big(A, ., \langle -; - \rangle_2\big)$ forms a *Gerstenhaber algebra*, i.e. a graded commutative algebra $(A, .)$ with a degree $+1$ skew-symmetric bracket $\langle -; - \rangle_2$ satisfying the Jacobi identity, with degree shift; it is called an odd Lie bracket. Moreover, the operator $\Delta$ is a derivation with respect to the bracket. So it is possible to define a BV-algebra as a Gerstenhaber algebra $\big(A, ., \langle -; - \rangle_2\big)$ with a degree $+1$ operator $\Delta$, which is a derivation with respect to $\langle -; - \rangle_1$ and such that $\Delta^2 = 0$ and $\langle a; b \rangle_2 = \Delta(a.b) - \Delta(a).b - (-1)^{|a|} a.\Delta(b)$.

Let $\mathcal{BV}$ denote the operad encoding BV-algebras. The two aforementioned definitions of BV-algebras induce two presentations of this operad. On the one hand, the operad $\mathcal{BV}$ is the free operad on a commutative product and an operator $\Delta$ modulo several the associativity, the order $\leq 2$ and the square-zero relations:

$$\mathcal{BV} \cong \mathcal{T}(\,.\,,\ \Delta)/(\text{associativity}, \Delta \text{ of order} \leq 2, \Delta^2 = 0)\ .$$

On the other hand, the operad $\mathcal{BV}$ is the free operad on a commutative product, a bracket $\langle -; - \rangle_2$ and an operator $\Delta$ modulo the other relations denoted $R$.

$$\mathcal{BV} \cong \mathcal{T}(\,.\,,\ \langle -; - \rangle_2,\ \Delta)/(R).$$

The second presentation is essential to study the notion of BV-algebra up to homotopy since it provides the Koszul model of the operad $\mathcal{BV}$.

**Theorem.**[2] Let $(A,\ d_A,\ .\,,\ \Delta)$ be a differential graded BV-algebra and let $(H, d_H)$ a deformation retract of $(A, d_A)$

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H), \qquad id_A - ip = hd_A + d_A h\ .$$

Then, any BV-algebra structure on $A$ induces a homotopy equivalent homotopy BV-algebra structure on $H$.

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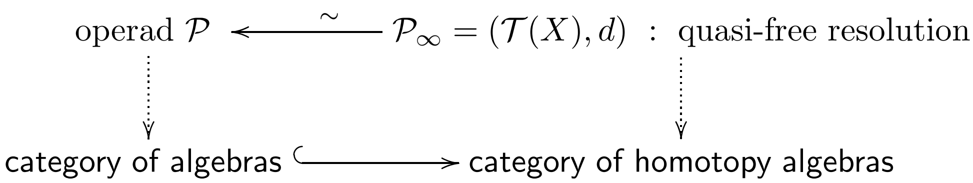
## Homotopy Batalin-Vilkovisky algebras II

Bruno Vallette

**Introduction.**Classical notions of algebras do not mix well a priori with homotopy theory. For instance, the transfer of an algebra of type $\mathcal{P}$ through a deformation retract does not produce in general a “strict” $\mathcal{P}$-algebra. Instead, the homotopy equivalent space carries a homotopy $\mathcal{P}$-algebra structure, which is an algebraic structure made up of infinitely many multilinear operations that are higher homotopies for the relations of type $\mathcal{P}$.

**Quasi-free resolutions.** To coin a good notion of homotopy $\mathcal{P}$-algebras, one proceeds in the following way. First, one encodes the category of algebras with an operad $\mathcal{P}$. Then, one tries to find a quasi-free operad $\mathcal{P}_\infty$, which resolves the operad $\mathcal{P}$, i.e. which is quasi-isomorphic to it. Finally, the category of homotopy $\mathcal{P}$-algebras is defined as the category of $\mathcal{P}_\infty$-algebras; it carries the required homotopy properties and the initial category of $\mathcal{P}$-algebras sits inside it.



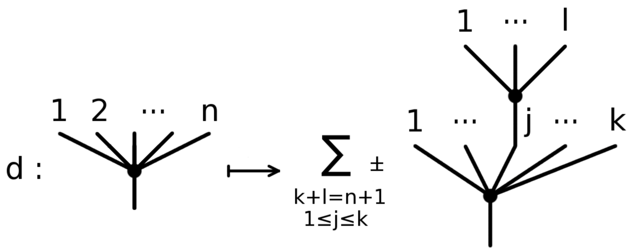

**Koszul–Tate resolutions.**The first method to find such resolutions is by hand, à la Koszul–Tate, that is one kills the homology groups step by step by inducing new syzygies at each step. For instance, in the case of the algebra of dual numbers $D := H_\bullet(S^1) = T(\Delta)/(\Delta^2)$ modeling mixed complexes, a resolution is given by the following quasi-free algebra

$$D_\infty := \big(T(\delta_1 \oplus \delta_2 \oplus \delta_3 \oplus \cdots), d\big) \xrightarrow{\sim} D \ ,$$

where the differential is equal to $d(\delta_n) = \sum_{i=1}^{n-1} \delta_i \otimes \delta_{n-1}$. In the example of the nonsymmetric operad $As$ modeling associative algebras, a resolution is given by the quasi-free operad

$$A_\infty := \left(\mathcal{T}\left(\text{Y}, \text{Y}, \text{Y}, \ldots\right), d\right) \xrightarrow{\sim} As \ ,$$

where the differential is equal to




**General methods.** The general complexity of the operads involved makes it impossible to always find resolutions by hand. The bar-cobar construction provides us with a functorial quasi-free resolution, but the price to pay is a huge underlying space (twice the tensor module construction). One can often simplify this resolution using the Koszul duality theory. At the very end, one can try to describe the minimal model of the operad $\mathcal{P}$, which is a quasi-free resolution with trivial internal part of the differential. While all the quasi-free resolutions (with a good filtration of the space of generators) are quasi-isomorphic and hence give rise to homotopy equivalent notions of homotopy $\mathcal{P}$-algebras, the minimal model is unique up to isomorphism, thereby inducing equivalent categories of algebras.

**Theorem.**[2, 3] In the case of the operad $\mathcal{BV}$ encoding Batalin–Vilkovisky algebras, the bar-cobar construction, the Koszul model and the minimal model provide us with three quasi-resolutions.

$$\begin{array}{ccc} \Omega\mathrm{B}\,\mathcal{BV} & & \\ \uparrow\sim & \searrow\sim & \\ \Omega\,\mathcal{BV}^{\text{¡}} & \xrightarrow{\sim} & \mathcal{BV} \\ \uparrow\sim & \nearrow\sim & \\ \Omega_\infty\big(\delta_1 \oplus \delta_2 \oplus \delta_3 \oplus \cdots \bigoplus H^\bullet(\mathcal{M}_{0,n+1})\big) & & \end{array} ,$$

where the generators of the minimal model are given by the resolution of the circle and the cohomology of the moduli space of genus 0 curves.

**Moduli spaces of curves.** Recall the homology of the moduli space of stable genus 0 curves $H_\bullet(\overline{\mathcal{M}}_{0,n+1})$ forms an algebraic operad, whose algebras are called *hypercommutative algebras*, or formal Frobenius manifolds or genus 0 cohomological field theories. Its Koszul dual cooperad is nothing but $H_\bullet(\overline{\mathcal{M}}_{0,n+1})^{\text{¡}} \cong H^\bullet(\mathcal{M}_{0,n+1})$, so an algebra over $\Omega\big(H^\bullet(\mathcal{M}_{0,n+1})$ is a homotopy hypercommutative algebra.

**Homotopy transfer theorem for BV-algebras.**One can prove a homotopy transfer theorem for the last notion of homotopy BV-algebras given by the minimal model of the operad $\mathcal{BV}$. The resulting algebraic structure is made up of a homotopy action of the circle and a certain action of the cohomology of the moduli space of genus 0 curves. In some cases, the transferred action of the circle is trivial, yielding a homotopy hypercommutative algebra as in the following example.

**Theorem.**[1] The de Rham cohomology of a Poisson manifold carries a homotopy hypercommutative algebra structure, which allows one to reconstruct the homotopy type of the BV-algebra of the de Rham forms.

**Definition.** A *Batalin–Vilkovisky algebra*, *BV algebra* for short, is a graded-commutative, or super-commutative in the physicists language, algebra $(A, ., \Delta)$ together with an order $\leq 2$, degree 1, square-zero operator.

Let $\langle -; - \rangle_2$ be the bracket which measures the default of $\Delta$ to be a derivation with respect to the product:

$$< a; b >_2 := \Delta(a.b) - \Delta(a).b - (-1)^{|a|} a.\Delta(b) .$$

The operator $\Delta$ is said to be of *order* $\leq 2$ if $\langle a; - \rangle_2$ is a derivation with respect to the product for any $a \in A$.

Here are two examples of BV-algebras, the first being the original one and the second one appears in Koszul's paper.

(1) Let $V$ a finite dimensional vector space and let $\mathcal{B} = \{x_1, ..., x_n\}$ be a basis. Let $\mathcal{S}(V^* \oplus s^{-1}V)$ be the space of polynomials on $x_1^*, \ldots, x_n^*$ and

$s^{-1}x_1, \dots, s^{-1}x_n$, where $\{x_1^*, \dots, x_n^*\}$ is the dual base and where $s^{-1}x_i$ is the homological desuspension of $x_i$. The differential operator

$$\Delta := \sum_{i=1}^{n} \frac{\partial}{\partial x_i^*} \frac{\partial}{\partial s^{-1}x_i}$$

has order $\leq 2$, degree 1 and squares to zero.

(2) Let $\mathcal{M}$ be a manifold together with a Poisson structure $\omega \in \Gamma\big(\bigwedge^2 T\mathcal{M}\big)$. Consider the de Rham algebra of differential forms $(\Omega^\bullet \mathcal{M}, \wedge, d_{DR})$ with its usual product $\wedge$ and de Rham differential $d_{DR}$. The contraction $\iota_\omega$ is the endomorphism of $\Omega^\bullet\mathcal{M}$ which sends $f$ to $f(\omega, -, \dots, -)$. The order $\leq 2$ operator

$$\Delta := [\iota_\omega, d_{DR}] = \iota_\omega \circ d_{DR} + d_{DR} \circ \iota_\omega$$

endows the de Rham algebra with a BV-algebra strcture.

For any BV-algebra $\mathcal{A} = (A, ., \Delta)$, the data $\big(A,\ .,\ \langle -; - \rangle_2\big)$ forms a *Gerstenhaber algebra*, i.e. a graded commutative algebra $(A, .)$ with a degree $+1$ skew-symmetric bracket $\langle -; - \rangle_2$ satisfying the Jacobi identity, with degree shift; it is called an odd Lie bracket. Moreover, the operator $\Delta$ is a derivation with respect to the bracket. So it is possible to define a BV-algebra as a Gerstenhaber algebra $\big(A,\ .,\ \langle -; - \rangle_2\big)$ with a degree $+1$ operator $\Delta$, which is a derivation with respect to $\langle -; - \rangle_1$ and such that $\Delta^2 = 0$ and $\langle a; b \rangle_2 = \Delta(a.b) - \Delta(a).b - (-1)^{|a|} a.\Delta(b)$.

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## BV algebras in quantum field theory (QFT)

Kasia Rejzner

In physics, QFT is a framework which combines special relativity with quantum mechanics.

In the algebraic approach (based on the idea of Haag and Kastler [3]), a QFT model is defined by a net (pre-cosheaf) $\{\mathfrak{A}\}_{\mathcal{O}\subset\mathbb{M}}$ of topological, unital *-algebras associated to bounded regions $\mathcal{O}\subset\mathbb{M}$ of Minkowski spacetime ($\mathbb{R}^4$ equipped with a bilinear form $\eta$ represented by the diagonal matrix $\mathrm{Diag}(1,-1,-1,-1)$). This idea can also be implemented in perturbation theory. In my talk I have shown how a BV algebra arises in this construction, even in case where no gauge symmetries are present[1].

I consider the example of the real scalar field. The construction starts by specifying the off-shell configuration space $\mathcal{E}$. For the real scalar field $\mathcal{E}=\mathcal{C}^\infty(\mathbb{R}^4,\mathbb{R})$. Classical observables of the theory are identified with functionals on $\mathcal{E}$. For simplicity, in this talk, I restricted myself to functionals which can be written as

$$F_f(\varphi)=\int_{\mathbb{R}^4} f(x_1,\dots,x_k)\varphi(x_1)\dots\varphi(x_k)d^4x_1\dots d^4x_k\,, \tag{1}$$

where $f\in\mathcal{C}^\infty_c(\mathbb{R}^{4k},\mathbb{R})$. Let $\mathcal{F}$ denote the algebra generated by such functionals with respect to the pointwise product $F\cdot G(\varphi):=F(\varphi)\cdot G(\varphi)$. Next, one introduces equations of motion (EOM's) by specifying $dS\in\Gamma(T^*\mathcal{E})$.[2] For the free scalar field, $dS(\varphi)=(\Box+m)\varphi$, where $\varphi\in\mathcal{E}$, $m\geq 0$ and $\Box=\sum_{\mu\nu}\eta^{\mu\nu}\partial_\mu\partial_n$ is the wave operator.

Next, we consider the subspace $\mathcal{V}$ of $\Gamma(T\mathcal{E})$ consisting of vector fields $X_f$ which are derivations of $\mathcal{F}$ acting by

$$\partial_{X_f}F:=\int_{\mathbb{M}} f(x_1,\dots,x_k,y)\varphi(x_1)\dots\varphi(x_k)\frac{\delta F}{\delta\varphi(y)}d^4x_1\dots d^4x_k,$$

where $F\in\mathcal{F}$, $\frac{\delta F}{\delta\varphi(y)}\equiv F^{(1)}(y)$ and $f\in\mathcal{C}^\infty_c(\mathbb{R}^{4(k+1)},\mathbb{R})$. We represent $X_f$ by $X_f=\int_{\mathbb{M}} f(x_1,\dots,x_k,y)\varphi(x_1)\dots\varphi(x_k)\frac{\delta}{\delta\varphi(y)}d^4x_1\dots d^4x_k$. In physics literature

[1] The original work of Batalin and Vilkovisky [1] was motivated by gauge theories, but it turns out (see [2]) that the BV algebra structure is more universal.

[2] With an appropriate choice of the topology on $\mathcal{E}$, $\Gamma(T^*\mathcal{E})$ is isomorphic to the space of smooth maps from $\mathcal{E}$ to $\mathcal{E}'_c$, where $\mathcal{E}_c:=\mathcal{C}^\infty_c(\mathbb{R}^4,\mathbb{R})$ are smooth functions with compact support and $\mathcal{E}'_c$ are distributions.

$\frac{\delta}{\delta\varphi(y)}$'s are called *antifields*. Let $\Lambda\mathcal{V}$ denote the space of poli-vector fields built from $\mathcal{V}$, i.e. the graded algebra generated by elements of the form

$$X_h = \int_{\mathbb{M}} h(x_1, \ldots, x_k, y_1, ..., y_m)\varphi(x_1)\ldots\varphi(x_k)\frac{\delta}{\delta\varphi(y_1)}\wedge\ldots\wedge\frac{\delta}{\delta\varphi(y_m)}d^4x_1\ldots d^4y_m\,,$$

where $h \in \mathcal{C}_c^\infty(\mathbb{R}^{4(k+m)}, \mathbb{R})$. On $\Lambda\mathcal{V}$ we introduce a differential $\delta$, which acts on $\mathcal{V}$ as contraction of a vector field with the distinguished 1-form $dS$, is 0 on $\mathcal{F}$ and gets extended to poli-vector fields by the graded Leibniz rule. We obtain a differential complex $(\Lambda\mathcal{V}, \delta)$. It can be shown that $H_0(\Lambda\mathcal{V}, \delta) = \mathcal{F}/\mathcal{F}_0$, where $\mathcal{F}_0$ consists of functionals $F \in \mathcal{F}$ such that $F(\varphi) = 0$ for all $\varphi$, which satisfy EOM's (i.e. $dS(\varphi) \equiv 0$). Note that on $\Lambda\mathcal{V}$ we can introduce the Schouten bracket $\{.,.\}$, which, for $X, Y \in \mathcal{V}$ is the commutator of vector fields $\{X, Y\} := [X, Y]$, for $F \in \mathcal{F}$ we set $\{X, F\} := \partial_X F$ and extend $\{.,.\}$ to $\Lambda\mathcal{V}$ by the graded Leibniz rule. $(\Lambda\mathcal{V}, \{.,.\})$ is a Gerstenhaber algebra.

At the end of the talk I discussed the quantization. An important result from PDE theory states that the operator $(\Box + m)$ possesses certain distinguished Green's functions and using these, one can construct the Feynman propagator $D_F$. The time-ordering operator $\mathcal{T}$ is defined on $\mathcal{F}[[\hbar]]$ as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \tfrac{\hbar^n}{n!}\langle F^{(2n)}(\varphi), (\tfrac{1}{2}D_F)^{\otimes n}\rangle,$$

Formally, it would correspond to the path integral

$$\mathcal{T}F(\varphi) \overset{\text{formal}}{=} \int F(\varphi - \phi)\, d\mu_{\hbar D_F}(\phi).$$

The time-ordered product $\cdot_{\mathcal{T}}$ is defined on $\mathcal{T}(\mathcal{F}_{\mathrm{reg}}(M)[[\hbar]])$ as

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G).$$

The BV algebra structure arises when we conjugate $\delta$ by the $\mathcal{T}$ map, i.e. we consider $\mathcal{T} \circ \delta \circ \mathcal{T}^{-1}$. It turns out (using the fact that $D_F$ is a distributional bisolution of EOM's) that

$$\mathcal{T} \circ \delta \circ \mathcal{T}^{-1} = \delta + \Delta\,,$$

where $\Delta$ acts on vector fields as a divergence operator, i.e.

$$\Delta X_f = \int_{\mathbb{M}} f(x_1, \ldots, x_k, y)\varphi(x_1)\ldots\varphi(x_k)d^4x_1\ldots d^4x_k d^4y\,.$$

It was shown in [2] that $(\Lambda\mathcal{V}, \{.,.\}, \Delta)$ is a BV algebra.

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## Multicomplexes and $S^1$-actions

Alexandru Oancea

I explained in this talk how the notion of multicomplex, or mixed complex up to homotopy, appears naturally in the context of $S^1$-equivariant homology. Denote the Borel construction of an $S^1$-space $X$ by $X_{S^1} := X \times_{S^1} ES^1$. If $X$ is a manifold and if one uses Morse homology as a model for singular homology, one can construct on the total space of $X_{S^1}$ Morse functions and pseudo-gradient vector fields which are adapted to the fibration $X \hookrightarrow X_{S^1} \to \mathbb{C}P^\infty$, in the sense that gradient lines on $X_{S^1}$ project onto gradient lines on $\mathbb{C}P^\infty$. This leads immediately to a description of the differential on $X_{S^1}$ in terms of multicomplex data on $X$. The Gysin exact triangle relating equivariant and non-equivariant homology is readily obtained, and so is the spectral sequence converging to the $S^1$-equivariant homology $H_\cdot^{S^1}(X) := H_\cdot(X_{S^1})$. If $X$ is a manifold and if one uses the de Rham model for singular cohomology with real coefficients, the Cartan model for $S^1$-equivariant cohomology is an example of multicomplex data in which all higher order terms vanish, i.e. a mixed complex. If $X$ is a topological $S^1$-space and one uses singular chains, a formula of Nancy Hingston based on the Eilenberg-MacLane shuffle map [3, p. 64] defines a mixed complex structure associated to the respective $S^1$-action. The verification of this formula was the topic of the after-hours session on Thursday.

To the best of the author's knowledge, the notion of multicomplex was first introduced by Wall [7] in a slightly different setting, then advertised by Meyer [5]. It was recently revisited by Lapin [4] from the point of view of homological perturbation and also by Dotsenko, Shadrin and Vallette [2]. We came upon it in the context of geometric $S^1$-actions together with Bourgeois [1], guided by an idea of Seidel [6].

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## BV and Feynman categories

Ralph Kaufmann

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BV equations arise when studying certain operadic structures. There are basically two instances, theories with self–gluing and theories with a cyclically invariant internal multiplication. The latter give rise to the solution of the cyclic Deligne conjecture [10] and the former appear in master equations [16]. There is a general framework for this called Feynman categories [15].

**Feynman categories**
Let $\mathcal{V}$ be a groupoid, $(\mathcal{F},\otimes)$ a monoidal category and $\imath\mathcal{V}\to F$ a functor. Denote by $\imath^{\otimes}:\mathcal{V}^{\otimes}\to\mathcal{F}$ the induced functor from the free monoidal category on $\mathcal{V}$. Such a triple $\mathfrak{F}=(\mathcal{V},\mathcal{F},\imath)$ is called a *Feynman category* if $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$, $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories $Iso(\mathcal{F}\downarrow\mathcal{V})^{\otimes}$ and $Iso(\mathcal{F}\downarrow\mathcal{F})$.[1]; for any $*\in\mathcal{V}$, $(\mathcal{F}\downarrow *)$ is essentially small.

A first example is $\mathcal{F}=(\mathcal{S}urj,\amalg)$, the category of finite sets and surjections and $\mathcal{V}=*$ the category with one object and its identity.

A second example: Let $\mathcal{V}=\mathcal{C}rl$ be the set of finite sets and isomorphisms considered as corollas and set $Iso(\mathcal{A}gg)=\mathcal{C}rl^{\otimes}$, i.e. aggregates of corollas and isomorphisms and let $\mathcal{A}gg\subset\mathcal{G}raphs$ be the full subcategory of objects from $\mathcal{A}gg$ in the category $\mathcal{G}raphs$ defined in [2].

Each morphism $\phi$ has an underlying graph $\mathbb{\Gamma}(\phi)$. By decorating or restricting $\mathbb{\Gamma}(\phi)$ for morphisms in $(\mathcal{F}\downarrow\mathcal{V})$, we get the Feynman categories whose $\mathcal{O}p$ are operads, PROPs, cyclic/modular operads etc.

There are also enriched versions having twisted (modular) operads and algebras over a given operad as $\mathcal{O}ps$. One general theorem is that the natural forgetful functor $G:\mathcal{V}-\mathcal{M}ods_{\mathcal{C}}\to\mathcal{F}-\mathcal{O}ps_{\mathcal{C}}$ has a monoidal left adjoint functor $F$. That is there is a free construction.

**Universal operations and BV**
Let $\hat{\mathcal{F}}$ be the cocompletion of $\mathcal{F}$ sitting inside $Fun(\mathcal{F}^{op},\mathcal{S}et)$ and $\jmath:\mathcal{F}\to\hat{\mathcal{F}}$ the inclusion. Set $\mathbb{1}=\mathrm{colim}_{\mathcal{V}}\jmath\circ\imath$ and consider $\mathcal{V}_{\mathcal{V}}$ with object $\mathbb{1}$ and the identity as well as $\mathcal{F}_{\mathcal{V}}$ the monoidal category generated by $\mathcal{V}$ inside $(Fun(\mathcal{F}^{op},\mathcal{S}et),\circledast)$ equipped with Day convolution and $\imath_{\mathcal{V}}$ be the inclusion of $\mathcal{V}$.

**Theorem 1.** $(\mathcal{V}_{\mathcal{V}},\mathcal{F}_{\mathcal{V}},\imath_{\mathcal{V}})$ is a Feynman category that we call the Feynman category of universal operations. For any $\mathcal{O}:\mathcal{F}\to\mathcal{C}$ it acts on $\hat{\mathcal{O}}(\mathbb{1})$, where $\mathcal{O}:\hat{\mathcal{F}}\to C$ is the universal extension.

For the Feynman category for operads $\mathcal{O}(\mathbb{1})=\bigoplus\mathcal{O}(n)_{\mathbb{S}_n}$ and we recover the result [11]. Considering non–Sigma operads, we see that the structure lifts to

[1] $Iso(\mathcal{C})$ is the restriction of a category $\mathcal{C}$ to its isomorphisms and $(\ \downarrow\ )$ denotes the comma category.

$\bigoplus \mathcal{O}(n)$ as in [5]. For odd or anti–cyclic operads, we obtain the cyclic bracket on $\bigoplus \mathcal{O}(n)_{\mathbb{S}_{n+1}}$ first defined in [16]. Considering the non–Sigma, the bracket lifts to $\bigoplus \mathcal{O}(n)_{\mathbb{Z}/n\mathbb{Z}}$. Special cases of this bracket are what underlies the three geometries of Kontsevich [12, 3].

**BV I** [16] Considering the Feynman category for nc–$\mathfrak{K}$–modular operads (nc for non–connected) the universal operations are generated by the cyclic bracket above, a differential $\Delta$ and a horizontal product, for which $\Delta$ is a BV operator.

The Feynman category of nc–$\mathfrak{K}$–modular operads has as objects corollas with a genus marking (as in modular operads) and as morphisms with appropriate genus markings together with an orientation of the edges, that is a mod 2 order.

**BV II** [6] In [6] we proved the cyclic Deligne conjecture: i.e. for a Frobenius algebra $A$ the Hochschild cochains $CH^*(A, A)$ carry an action of a chain model of the *framed* little discs, which induces a BV structure on $HH^*(A, A)$ whose bracket is the Gerstenhaber bracket. From the proof is it obvious that this can be generalized to an action on any cyclic operad with cyclic multiplication, viz. an element in $\mathcal{O}(2)$ which induces a multiplication that associative and cyclicly invariant. On $HH^*$ the BV structure was first given by [17], the $A_\infty$ version on the chain level is proved in [20].

Using Feynman categories we automatically get the operad of universal operations [2]. For the ordinary Deligne conjecture, we have the Feynman category for operads with multiplication. The universal operations are given by planted planar b/w bipartite trees with flags. The components are then just planted planar b/w bipartite trees as in [8]. For cyclic operads with cyclic multiplication, the universal operad is that of planted planted planar b/w trees with spines, which is isomorphic to the cell model of cacti given in [6, 8]. For the $A_\infty$–cases things are slightly more complicated, but essentially the same, see [14, 20].

**BV III Master equations** In [16], we also established that given any operadic type/viz. $\mathcal{F}$–$\mathcal{O}ps$ with odd self–gluing (say we are in the graph case), then there is a Feynman transform whose algebras are classified by a Master equation which includes a BV operator. To be careful a differential $\Delta$ that becomes BV if one considers the setup with non–connected graphs. In [15] Feynman transforms and master equations are treated in complete generality incluidng the necessary model category structures. [3].

It is important to note that BV structure I and III are strict, while BV structure II is up to homotopy.

**Outlook and questions** There is a graphical calculus for operations in Deligne's conjecture that even extends to String Topology and moduli space actions [7, 9]. What is the analogue for the Hopf case? Every Feynman category defines a Hopf algebra [15]. The ones that appear for the simplest examples all have

[2] One still has to prove that this gives a chain model for the little discs!

[3] Without this extension the Master equation for $\mathfrak{K}$–modular operads is contained in [1] and the one for wheeled properads can be deduced from [19, 18]. The corresponding non–connected versions can be found in [16]

number theoretic significance and are equal to Hopf algebras obtained from fiber functors [4]. Can one see the fiber functors and hence the associated Tannakian categories directly in the Feynman category picture? What is the role of Hopf cyclic cohomology in this story? Is there any categorification which related this story to Hopf algebroids?

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## Maurer-Cartan Elements and Cyclic Operads

Benjamin C. Ward

To an operad $\mathcal{O}$ in the category of differential graded (dg) vector spaces one can associate a dg Lie algebra, $\mathcal{O}^*$, the prototype being the construction of Gerstenhaber [2]. The choice of a Maurer-Cartan (MC) element $\eta$ in said Lie algebra allows the construction of a differential $\delta_\eta$ via the formula $\delta_\eta(-) = d_{\mathcal{O}}(-) + [\eta, -]$. To begin this talk we discuss a generalization of a conjecture of Deligne which states that the complex $(\mathcal{O}^*, \delta_\eta)$ is an $E_2$-algebra. In its original form the conjecture concerned the endomorphism operad of an associative algebra and this conjecture has been proven by several authors, see *eg* the MathSciNet review of [10] written by A.A. Voronov. As we discuss, the general setup in which we now consider the conjecture encompasses other operadic cohomology theories, including those considered in [11], as well as the singular cochain complex of a space, after [4], as well as the Hopf algebroid based constructions of [6]. The proof of the general version of the conjecture is sketched. In particular the proof relies on the construction of a chain model for the little disks constructed in [7], see also [8].

We then turn our attention to the relevant particular instance of the following general question: given a construction which produces a Gerstenhaber algebra, what additional requirements on the input of said construction produce a compatible BV operator? Answers to this question in related contexts considered during the conference have been a volume form, a flat connection, or a circle action, and the answer in this context is two-fold. First, the operad in question should be cyclic (see [3]) and second, the MC element should be cyclically symmetric, in a sense that we shall discuss. The proof of this result relies upon the construction of a chain model for the framed little disks operad given in [12] and generalizes the work of Kaufmann [5].

Time permitting we will then discuss the parallels between the above algebraic constructions and the geometric constructions in string topology [1]. In particular we can realize the analog of Chas and Sullivan's string bracket and infinite family of $L_\infty$ structures in the context of a cyclic operad and its associated complex of cyclic (co)invariants via a cyclic analog of the brace operations first defined in [13]. We may also discuss a direction for generalization of the above results from the perspective of Feynman categories [9]. The results discussed in this talk will appear in [14].

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## Gerstenhaber and BV-algebras, Hochschild and Hopf cyclic cohomology

Luc Menichi

In this talk, we explain some results of [7] and [8].

### 1. Gerstenhaber algebras and Operads

The first example of Gerstenhaber algebra, due to Gerstenhaber, is the Hochschild cohomology $HH^*(A, A)$ of an algebra $A$.

**Theorem 1.** [8] Let $A$ be a bialgebra. Then $\mathrm{Ext}^*_A(\Bbbk, \Bbbk)$ is a sub Gerstenhaber algebra of the Hochschild cohomology of the algebra $A$, $HH^*(A, A)$.

In this talk, every Gerstenhaber algebra comes from a (linear) operad with multiplication using the following general theorem:

**Theorem 2.** [4, 5, 6] a) Each operad with multiplication $O$ is a cosimplicial module.Denote by $\mathcal{C}^*(O)$ the associated cochain complex.
b) Its homology $H(\mathcal{C}^*(O))$ is a Gerstenhaber algebra.

### 2. BV-algebras

**Theorem 3.** [7] If $\mathcal{O}$ is a cyclic operad with a multiplication then

a) the structure of cosimplicial module on $\mathcal{O}$ extends to a structure of cocyclic module and

b) the Connes coboundary map $B$ on $\mathcal{C}^*(\mathcal{O})$ induces a natural structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra $H^*(\mathcal{C}^*(\mathcal{O}))$.

**Theorem 4.** [8] Let $\mathcal{O}$ be a linear cyclic operad with multiplication. Consider the associated cocyclic module. Then the cyclic cochains $\mathcal{C}^*_\lambda(\mathcal{O})$ forms a subcomplex of $\mathcal{C}^*(\mathcal{O})$, stable under the Lie bracket of degree $-1$. In particular, the cyclic cohomology $HC^*_\lambda(\mathcal{C}^*(\mathcal{O}))$ has naturally a graded Lie algebra structure of degree $-1$.

In representation theory [3], an algebra $A$ is *symmetric Frobenius* if $A$ is equipped with an isomorphism of $A$-bimodules $\Theta : A \overset{\cong}{\to} A^\vee$ between $A$ and its dual $\mathrm{Hom}(A, \Bbbk)$. As first application of Theorem 2, we show

**Corollary 5.** [7, 9] Let $A$ be a symmetric Frobenius algebra. Then the Connes coboundary map on $HH^*(A, A^\vee)$ defines via the isomorphism

$$HH^*(A, \Theta) : HH^*(A, A) \overset{\cong}{\to} HH^*(A, A^\vee)$$

a structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra $HH^*(A, A)$.

Corollary 5 was first proved by Tradler [9]. But we were unable to understand his proof. Our proof in [7] is the first published proof. As second application, we show

**Corollary 6.** [8] Let $K$ be a Hopf algebra equipped with a group-like element $\sigma$ such that for all $k \in K$, $S^2(k) = \sigma^{-1}k\sigma$. Let $t_n : K^{\otimes n} \to K^{\otimes n}$ be the linear map defined by

$$t_n(k_1 \otimes \cdots \otimes k_n) = \sigma S(k_1^{(1)} \dots k_{n-1}^{(1)} k_n) \otimes k_1^{(2)} \otimes \cdots \otimes k_{n-1}^{(2)}.$$

The dual of the Bar construction on $K$, $B(K)^\vee$ is a cyclic operad with multiplication. In particular, the Gerstenhaber algebra $\mathrm{Ext}^*_K(\Bbbk, \Bbbk)$, is in fact a Batalin-Vilkovisky algebra and the cyclic cohomology of $K$, $\widetilde{HC}^*_{(\varepsilon,\sigma)}(K)$ has a Lie bracket of degree $-1$. The dual of this last Corollary, which involves Connes-Moscovici cyclic cohomology of Hopf algebras [1, 2], first appeared in [7].

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## Brace Bar-Cobar Duality, $E_2$ cochains, and BV algebras

Justin Young

We consider the classical bar-cobar adjunction $\Omega : CoAlg \to Alg$ et $B : Alg \to CoAlg$, which gives a nice duality between the two categories. Gerstenhaber-Voronov [3] observed that if an algebra $A$ has the structure of an $\mathcal{S}_2$ algebra (other places called a homotopy G-algbra), where $\mathcal{S}_2$ is the $E_2$ suboperad of the sequence operad studied by McClure-Smith [7] or equivalently the operad of spineless cacti studied by Kaufmann [10], then the bar construction $BA$ has the structure of a bialgebra. Thus, they obtain a functor $B : \mathcal{S}_2 Alg \to BiAlg$ enhancing the classical bar construction. It was first observed explicitly by Menichi [11] that, by taking the tensor powers of a bialgebra, one obtains an operad with multiplication, and therefore by another result of [3] we have a functor $\Omega : BiAlg \to \mathcal{S}_2 Alg$.

In the classical case, the bar and cobar constructions give an adjunction such that the unit and counit maps are weak equivalences. With the new enhanced functors, the situation is a bit more messy. The old unit map $C \to B\Omega C$ descends to the category of bialgebras in the case when $C$ is a bialgebra. However, the counit map $\Omega BA \to A$ is almost never a map of $\mathbb{S}_2$ algebras. This situation can be remedied by studying the universal property of $B$, and we discover that $B$ has a left adjoint $\widetilde{\Omega} : BiAlg \to \mathcal{S}_{\in} Alg$, and that there is a natural transformation $\widetilde{\Omega} \to \Omega$ that is a strong deformation retract of algebras. Thus, $\Omega BA$ and $A$ are at least equivalent as $\mathcal{S}_2$ algebras. This is the main result of [12].

The motivation for this work was to study the cochains $S^*(X, R)$, with coefficients in a ring $R$ that has positive characteristic, as an $\mathcal{S}_2$ algebra. In my PhD thesis, I showed that if $R$ has characteristic $p$, and $X$ is a CW complex that is $r$ connected and $rp - p + 1$ dimensional, then $S^*(X, R)$ is equivalent as an $\mathcal{S}_2$ algebra to a commutative algebra. The method of the proof is to study $BS^*(X, R)$ as a Hopf algebra, and use techniques of Anick [8] to show that $BS^*(X, R)$ is the dual of a universal enveloping algebra $UL_X$. Then, we apply a classical Koszul duality result to see that $\Omega(UL_X)^\vee \simeq C^*(L_X)$ as $\mathcal{S}_2$ algebras where the latter is the Chevalley-Eilenberg cochains on the Lie algebra $L_X$, and in particular a strictly commutative algebra. Then, we use brace bar-cobar duality to recover $S^*(X, R) \simeq \Omega BS^*(X, R) \simeq \Omega(UL_X)^\vee \simeq C^*(L_X)$.

Finally, we connect to the BV part of the story, and mention some open problems. The paper of Kaufmann [10] together with work of Menichi [11] shows that when a bialgebra $H$ admits an involutive antipode, then the tensor powers become a cyclic operad with multiplication, and so $\Omega H$ admits an action of $f\mathcal{S}_2$ the operad of cacti studied by Kaufmann. We have an inclusion $\mathcal{S}_2 \subseteq f\mathcal{S}_2$ such that after taking homology we recover $G \subseteq BV$ where $G$ is the Gerstenhaber operad and $BV$ is the BV-operad. Thus, we have a functor $\Omega : HopfAlgInv \to f\mathcal{S}_2 Alg$. This leaves open the question of if we have a functor in the other direction, as well as the question of duality. In a different direction, Hess and collaborators [9] showed that $\Omega S_*(X)$ is equivalent to $S_*(\Omega X)$ as a bialgebra. By applying brace bar-cobar duality, we see that $\Omega^2 S_*(X) \simeq \Omega S_*(\Omega X)$ as $\mathcal{S}_2$ algebras. The latter is then equivalent to $S_*(\Omega^2 X)$ as a Hopf algebra. First of all, the algebra

$S_*(\Omega^2 X)$ is an $E_2$ algebra, so one question is can we find an $E_2$ operad such that $S_*(\Omega^2 X)$ is equivalent to $\Omega S_*(\Omega X)$ as $E_2$ algebras, and therefore to $\Omega^2 S_*(X)$ as well? Secondly, $S_*(\Omega X)$ is (if we choose our models correctly) a Hopf algebra with involutive antipode, and so $\Omega S_*(\Omega X)$ is an $f\mathcal{S}_2$ algebra. Thus, there should be some kind of $fE_2$ operad (equivalent to chains on the framed little disks) that acts also on $\Omega^2 S_*(X)$ and $S_*(\Omega^2 X)$ so that all three are equivalent. This question has been partially answered by Quesney [13] for the case $X = \Sigma^2 Y$.

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## Grothendieck-Teichmüller and Batalin-Vilkovisky

Sergey Merkulov

My talk is based on a joint work with Thomas Willwacher.

Let $M$ be a finite dimensional affine $\mathbb{Z}$-graded manifold $M$ over a field $\mathbb{K}$ equipped with a constant degree 1 symplectic structure $\omega$. In particular, the ring of functions $\mathcal{O}_M$ is a Batalin-Vilkovisky algebra, with Batalin-Vilkovisky operator $\Delta$ and bracket $\{\ ,\ \}$. A degree 2 function $S \in \mathcal{O}_M[[u]]$ is a solution the quantum master equation on $M$ if

$$u\Delta S + \frac{1}{2}\{S, S\} = 0,$$

where $u$ is a formal variable of degree 2. In other words $S$ is a Maurer-Cartan element in the differential graded (dg) Lie algebra $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$.

The Grothendieck-Teichmüller group $GRT$ is a pro-unipotent group introduced by Drinfeld; we denote its Lie algebra by $\mathfrak{grt}$. In this paper we show the following result.

**Main Theorem** *There is an $L_\infty$ action of the Lie algebra $\mathfrak{grt}$ on the differential graded Lie algebra $(\mathcal{O}_M[[u]][1], u\Delta, \{\ ,\ \})$ by $L_\infty$ derivations. In particular, it follows that there is an action of $GRT$ on the set of gauge equivalence classes of formal solutions of the quantum master equation, i. e., on gauge equivalence classes of Maurer-Cartan elements in the differential graded Lie algebra $(\hbar\mathcal{O}_M[[u]][[\hbar]][1], u\Delta, \{\ ,\ \})$, where $\hbar$ is a formal deformation parameter of degree 0.*

Our main technical tool is a version of the Kontsevich graph complex, $(\mathsf{GC}_2[[h]], d_h)$ which controls universal deformations of $(\mathcal{O}_M[[u]][1], h\Delta, \{\ ,\ \})$ in the category of $L_\infty$ algebras. We prove an isomorphism of Lie algebras

$$H^0(\mathsf{GC}_2[[h]], d_u) \simeq \mathfrak{grt}$$

from which we deduce our Main Theorem.

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