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Calculus of Variations

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Abstract. The Calculus of Variations is at the same time a classical subject, with long-standing open questions which have generated deep discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and by emergent applications. This workshop balanced the traditional variational problems with novel questions with origins in diverse areas, such as economic models of the academic labor market, or differential geometry in metric emasure spaces. In particular, the meeting featured presentations on regularity theory, existence, and classification questions related to minimal surfaces, surfaces of prescribed Gaussian curvature, and mean curvature flow; domain optimization problems; nonlinear elasticity; calibrated geometries; variational formulations of certain dynamical problems; and stochastic variational problems.

Mathematics Subject Classification (2010): Primary: 49-06; Secondary: 49Q20, 58E30.

Introduction by the Organisers

The workshop *Calculus of Variations* brought together 52 participants from 9 countries, including experts in minimal surfaces, optimal transportation, nonlinear elasticity, geometric flows, geometry in metric measure spaces, stochastic homogenization, geometric measure theory, elliptic partial differential equations, and general relativity. All of these topics were represented in the formal part of the scientific program, which consisted of 21 talks of 60 minutes each, including discussions. Eight of the 21 speakers were junior researchers – graduate students or postdoctoral fellows. The workshop was organized by Simon Brendle (Stanford), Camillo de Lellis (Zürich), and Robert Jerrard (Toronto).

Minimal surfaces were first considered by Lagrange over 250 years ago, and since then have been a continuing preoccupation in the Calculus of Variations. Together with related geometric variational problems, they have provided the impetus for the development of modern tools such as geometric measure theory, and for a deep regularity theory. Contributions in this direction included striking new regularity results for stationary varifolds, Brakke solutions of the mean curvature flow, and semi-calibrated integral currents. A high point of the conference was a breakthrough result that provides a lower bound on the filling area associated to an integer multiple of an integral cycle. New results relating the topology and index of minimal surfaces were presented, and shown to yield the proof of an old conjecture about nonexistence of a class of minimal surfaces. Novel directions relating to the classical theory were apparent in a talk on fractional minimal surfaces, which included some of the first nontrivial examples of such objects. A Frobenius property of integral currents was presented, and was used to complete a counterexample to a conjecture about the decomposability of normal currents.

Other topics connected to geometry included the shape on large surfaces of prescribed Gaussian curvature; a construction of mean curvature flow with surgery; and a sharp quantitative version of the classical Faber-Krahn inequality, describing domains that nearly optimize the principle eigenvalue of the Laplacian.

A number of talks explored different variational problems arising in materials science and nonlinear elasticity. These included a derivation of quasistatic evolution model for plasticity, obtained as a limit of parabolic problems; a fine analysis of the energy scaling associated to wrinkling in stressed thin elastic films; a derivation of Young's law, relying on deep new regularity results for free boundary problems for almost-minimizers of elliptic integrands; and new results on variational models for dislocations – an emerging area with numerous difficult and interesting open questions. Quite different connections to mathematical physics were exemplified in a talk that presented results on variational problems arising in General Relativity, illustrating the fact that General Relativity is a fertile source of compelling variational problems. Another talk connected to applications presented a variational model for the educational labor market, one that predicts certain singularities at the top of the wage scale.

Several talks highlighted the strong connections of the Calculus of Variations to other areas of mathematics. These included a novel approach to the foundations of differential geometry in very general metric measure spaces; a quantitative description of stochastic homogenization of convex integral functionals; and a provocative proposal for using variational methods to develop what might be called a "dissipative action principle" for some Hamiltonian systems.

Overall, the participants and lectures at the workshop represented well the diversity, vitality, and depth of the contemporary Calculus of Variations.

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Workshop: Calculus of Variations

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Abstracts

Frobenius property for integral currents and decomposition of normal currents

Annalisa Massaccesi
(joint work with Giovanni Alberti)

1. Frobenius property for integral currents

The Frobenius Theorem states the equivalence between the complete integrability of a k-dimensional simple vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k \in C^1(\mathbb{R}^d; \Lambda_k(\mathbb{R}^d))$ (that is, the existence of a local foliation of $\mathbb{R}^d$ by k-dimensional tangent submanifolds) and the involutivity condition for ξ, that is

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x) \ldots, \tau_k(x)\} \quad \forall\, x \in \mathbb{R}^d\,,\ \forall\, m, n = 1, \ldots, k.$$

It is natural to arise the following question: consider a k-vector field ξ, for which weak notion of k-dimensional surface the conclusions of the Frobenius Theorem still hold? There have been various attempts to answer this question, for instance when ξ is the horizontal distribution of the Heisenberg group $\mathbb{H}^d$ (see [3] and [4] for the cases of a 2-dimensional Lipschitz graph in $\mathbb{H}^1$ and the image of a function in $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^3)$ with maximal rank Jacobian matrix, respectively).

We prove that integral currents behave like submanifolds with respect to the integrability problem.

Theorem 1. *Let $\xi = \tau_1 \wedge \ldots \wedge \tau_k$ be a k-dimensional simple vector field in $\mathbb{R}^d$, with $\tau_1, \ldots, \tau_k \in C^1(\mathbb{R}^d)$, and let $R \in \mathcal{I}_k(\mathbb{R}^d)$ be a k-dimensional integral current with $R = [\![\Sigma, \xi, \theta]\!]$ (i.e., $R(\omega) = \int_\Sigma \langle \omega(x), \xi(x) \rangle \theta(x)\, d\mathcal{H}^1(x)$ for every $\omega \in C^\infty_c(\mathbb{R}^d; \Lambda^k(\mathbb{R}^d))$, where $\Sigma \subset \mathbb{R}^d$ is a rectifiable set), then*

$$[\tau_m, \tau_n](x) \in \operatorname{span}\{\tau_1(x), \ldots, \tau_k(x)\}$$

for every pair $m, n = 1, \ldots, k$ and for every x in the closure of the set of points of positive density of Σ.

The proof of the theorem is carried out by contradiction, exploiting the following results.

Lemma 1. *Given a non-involutive simple k-vector field $\xi = \tau_1 \wedge \ldots \wedge \tau_k$, with $\tau_1, \ldots, \tau_k \in C^1(\mathbb{R}^d)$, there exist an open subset $U \subset \mathbb{R}^d$ and a $(k-1)$-form α such that*

$$\langle \mathrm{d}\alpha(x), \xi(x) \rangle \neq 0 \quad \forall\, x \in U$$

and

$$\langle \alpha(x), \eta(x) \rangle = 0 \quad \forall\, x \in \mathbb{R}^d$$

whenever η is a simple $(k-1)$-vector field representing a linear subspace of ξ for every $x \in \mathbb{R}^d$.

Theorem 2. *Let ξ be a continuous k-dimensional vector field on $\mathbb{R}^d$ and let $R \in \mathcal{I}_k$ be a k-dimensional integral current with $R = [\![\Sigma, \xi, \theta]\!]$. If $\partial R = [\![\Sigma', \eta, \theta']\!]$, then*

$$\operatorname{span} \eta(x) \subset \operatorname{span} \xi(x) \quad \mathcal{H}^{k-1}\text{– } a.e.\ x \in \Sigma' .$$

The contradiction for the proof of Theorem 1 is obtained testing $\partial R = [\![\Sigma', \eta, \theta']\!]$ on the $(k-1)$-form α provided by Lemma 1. Indeed

$$0 = \int_{\Sigma'} \langle \alpha, \eta \rangle \theta' \, d\mathcal{H}^{k-1} = \partial R(\alpha) = R(\mathrm{d}\alpha) = \int_{\Sigma} \langle \mathrm{d}\alpha, \xi \rangle \theta \, d\mathcal{H}^k \neq 0 .$$

From the assumptions in Theorem 2 one sees why we can state Theorem 1 for integral currents, but not for rectifiable currents (see also [2]) nor for normal currents. Indeed, even if ξ is a non-involutive simple k-vector field of class C^1, the current $T := \xi \wedge \mathcal{L}^d$ is a locally normal current. An interesting open problem is to reduce all the normal k-currents which misbehave with respect to Theorem 1 to currents of a certain type. More specifically, let us consider the horizontal non-involutive simple 2-vector field $\xi = X \wedge Y$ in $\mathbb{H}^1 \cong \mathbb{R}^3$ (that is, $X(x, y, z) = (1, 0, -x/2)$ and $Y(x, y, z) = (0, 1, y/2)$) and a normal current $T = \xi \wedge \mu \in \mathbb{N}_2(\mathbb{R}^3)$: is it possible to prove that μ is necessarily absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^3$?

Concerning open problems, what happens if we drop the condition about the C^1-regularity of ξ and we use weaker definitions of involutivity, not involving the Lie bracket?

2. Decomposition of normal currents

In [1], F. Morgan formulated the following problem for the decomposition of normal currents: given a normal current $T \in \mathbb{N}_k(\mathbb{R}^d)$, we ask whether there exists a family of integral currents $(R_\lambda)_{\lambda \in L}$,where L is a suitable measure space, such that

(i) $T = \int_L R_\lambda \, d\lambda$;
(ii) $\mathbb{M}(T) = \int_L \mathbb{M}(R_\lambda) \, d\lambda$;
(iii) $\mathbb{M}(\partial T) = \int_L \mathbb{M}(\partial R_\lambda) \, d\lambda$.

Thanks to Theorem 1, we can complete the proof of a counterexample already proposed by M. Zworski in [6]: the aforementioned normal current $T := \xi \wedge \mathcal{L}^d \in \mathbb{N}_k(\mathbb{R}^d)$, with ξ non-involutive simple k-vector field of class C^1, does not even admit a decomposition in integral currents $(R_\lambda)_{\lambda \in L}$ satisfying the conditions (i) and (ii). In fact, conditions (i) and (ii) imply that $R_\lambda = [\![\Sigma_\lambda, \xi, \theta_\lambda]\!]$ for almost every $\lambda \in L$. Let us point out that the situation changes dramatically if we allow a decomposition in rectifiable currents, as pointed out by G. Alberti (see Section 4.5 of [5]).

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Transition between planar and wrinkled regions in uniaxially stretched thin elastic film

Peter Bella

When a thin elastic material is confined, it relaxes the compression by buckling/wrinkling out-of-plane. In [2], Kohn and the author studied a model problem for wrinkling of a thin elastic sheet caused by compression. We considered thin elastic annular sheet subject to radial dead loads T_{in} on the inner boundary and T_{out} on the outer boundary. If T_{in} is large enough compared to T_{out}, some concentric circles are pushed towards the center of the annulus and so the sheet is expected to wrinkle out-of-plane near the inner part of the annulus, and be planar near the outer boundary (see [3] for heuristic arguments). Rather than trying to explicitly find a minimizer of the elastic energy of the sheet (which in this case seems to be very hard, if not impossible, task), we first studied minimum value of the energy and its dependence on the thickness of the sheet. We showed that

$$\mathcal{E}_0 + C_0 h \leq \min E_h(u) \leq \mathcal{E}_0 + C_1 h, \tag{1}$$

where $0 < h < 1$ is thickness of the film, $\mathcal{E}_0$ and $0 < C_0 < C_1$ are constants (independent of h), E_h is energy of the system (normalized per unit thickness), and we minimized over deformations u. To obtain the upper bound, for each h one constructs a deformation (based on some ansatz) with small enough energy, whereas the lower bound needs to be ansatz-free.

In phycics literature one mostly finds upper bounds (i.e., constructions of deformations with small energy). For the above problem, deformations with energy $\mathcal{E}_0+Ch(|\log h|+1)$ were explicitly constructed in [3]. Those (suboptimal) deformations were obtained by superimposing wrinkles with radius-dependent amplitude onto some planar deformation. The number of wrinkles is obtain by optimization but does not vary with radius. To obtain the upper bound in (1) (i.e, to get rid of the logarithmic factor), we used construction where number of wrinkles changes (it increases near the free boundary between wrinkled and planar region). Deformations with such cascade of wrinkles achieves optimal energy scaling, but it is not clear whether minimizers (or low-energy configuration) have to exhibit such behavior. Moreover, there is no experimental evidence supporting such constructions (in the physical experiments the number of wrinkles did not vary with radius).

To understand whether such a cascade of wrinkles is necessary we considered a toy model, which should capture essential features of the original problem, but

should be easier to analyze [1]. We study minimization of Föppl-von Kármán energy with prescribed metric
(2)

$$E_h(w,u_3) = \int_\Omega |e(w) + \nabla u_3 \otimes \nabla u_3/2 - xe_2 \otimes e_2|^2 + h^2|\nabla\nabla u_3|^2 - 2\int_{x=\pm 1} w_1.x,$$

where $\Omega = [-1,1]\times[-1,1]$, $w:\Omega\to\mathbb{R}^2$ and $u_3:\Omega\to\mathbb{R}$ are respectively in-plane and out-of-plane displacement, both periodic in y, and $e(w) = (\nabla w + \nabla^T w)/2$ denotes the symmetric gradient. Using arguments behind (1) we immediately get that $C_0 \le \frac{1}{h}(\min E_h(w,u_3) - \mathcal{E}_0) \le C_1$.

For very small thickness h, wrinkling (more precisely, the out-of-plane displacement u_3) should play a dominant role in determining shape of low-energy deformations. In fact, we prove that

$$\lim_{h\to 0} \frac{\inf_{(w,u_3)} E_h(w,u_3) - \mathcal{E}_0}{h} = 2\inf_{L>0} \sigma_L, \tag{3}$$

where

$$\mathcal{S}_L(u) := \frac{1}{2L}\int_0^1\int_{-L}^{L} u_{,x}^2 + u_{,yy}^2, \quad \sigma_L := \inf_{u\in\mathcal{A}_L} \mathcal{S}_L(u)$$

with

$$\mathcal{A}_L := \left\{ \begin{matrix} u\in W^{1,2}((0,1)\times(-L,L)), u(0,\cdot)=0, \\ \text{for (a.e.) } x\in(0,1): u(x,\cdot) \text{ is } 2L\text{−periodic and } \frac{1}{2L}\int_{-L}^{L} u_{,y}^2(x,y)\mathrm{d}\,y = 2x \end{matrix} \right\}.$$

In order to show (3), one of the main steps was to show regularity of $u\in\mathcal{A}_L$, a ground state of $\mathcal{S}_L$. More precisely, for $L\ge 1$ we showed that there exists a global minimizer $u\in\mathcal{A}_L$ of the functional $\mathcal{S}_L$, u is odd in the y-variable, and for every $x\in(0,1)$ satisfies:

$$\frac{1}{2L}\int_{-L}^{L} u^2(x,y)\,\mathrm{d}y \le Cx^2(|\ln x|+1),$$
$$\frac{1}{2L}\int_{-L}^{L} u_{,x}^2(x,y)\,\mathrm{d}y \le C(|\ln x|+1),$$
$$\frac{1}{2L}\int_{-L}^{L} u_{,xx}^2(x,y)\,\mathrm{d}y \le Cx^{-2}(|\ln x|^7+1),$$
$$\frac{1}{2L}\int_{-L}^{L} u_{,xy}^2(x,y)\,\mathrm{d}y \le Cx^{-1}(|\ln x|^2+1),$$
$$\frac{1}{2L}\int_{-L}^{L} u_{,yy}^2(x,y)\,\mathrm{d}y \le C(|\ln x|+1),$$
$$\frac{1}{2L}\int_{-L}^{L} u_{,xyy}^2(x,y)\,\mathrm{d}y \le Cx^{-2}(|\ln x|^3+1),$$

where C does not depend on L.

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Regularity of semi-calibrated integral 2-cycles

Costante Bellettini

A calibration is a closed (smooth) differential form ϕ (say of degree k) on a Riemannian manifold $(\mathcal{M}^n, g)$ with the following key property that relates ϕ and the metric g. For any $x \in \mathcal{M}$ and for any oriented k-plane V in $T_x\mathcal{M}$ the action of ϕ on V is bounded from above by the action of the k-dimensional volume form (induced by g). In other words, for any unit simple k-vector ξ we have $\phi(\xi) \leq 1$. The unit simple k-vectors on which equality is realized are called calibrated planes, in other words they are the oriented k-planes on which the form ϕ acts just like the k-dimensional volume form. An integral k-current is said to be calibrated by ϕ when its approximate tangent planes (that exist $\mathcal{H}^k$-a.e.) are calibrated by ϕ. The notion of calibration appeared in full generality in the landmark paper [7] and using the closedness of the form it is straightforward to conclude that an integral current calibrated by ϕ is a mass-minimizer in its homology class. By dropping the closedness assumption on the differential form ϕ, and keeping the property relating it with the metric g, we obtain what is usually referred to as a semi-calibration. In the same way as before we can define semi-calibrated integral currents. From the variational point of view, due to the lack of closedness of ϕ, such currents are generally only almost-minimizers in their homology class. A very important example of semi-calibration arises when we look at tangent cones to a calibrated integral k-current, [7]. Each tangent cone here is an integral k-cycle invariant under dilations about the vertex and calibrated by a parallel form in $\mathbb{R}^n$. The radial invariance can simplify the study of the cone in that it is enough to look at the slice of the cone with a sphere centered at the vertex. We are then looking at a $(k-1)$-dimensional integral cycle which turns out to be semi-calibrated in the sphere S^{n-1}. The study of semi-calibrated integral cycles thus has a direct impact on the understanding of tangent cones to calibrated currents[1].

In the case $k = 2$ a classical example of calibration is the Kähler form on a Kähler manifold and calibrated currents are the holomorphic ones. In [5] it is proven that, given an arbitrary semi-calibration ω of degree 2, it is possible (locally) to treat any semi-calibrated integral cycle as a pseudo-holomorphic one,

[1]The interest in semi-calibrations goes however beyond this particular aspect, see e.g. the introduction of [5].

i.e. there exists[2] an almost complex structure J in the ambient manifold such that the approximate tangents to the cycle are invariant under the action of J and positively oriented by J. In other words, semi-calibrated integral 2-cycles can be studied by exploiting an extra structure in the ambient space. We present here the following regularity result:

Theorem 1. *Let ω be a semi-calibration of degree 2 on a Riemannian manifold $(\mathcal{M}^n, g)$ and let T be an integral 2-cycle semi-calibrated by ω. Then the singular set of T is made of (at most) isolated points.*

We recall here the notion of a smooth point. This means that in a neighbourhood of this point the current coincides with the current of integration on a smooth submanifold counted with an integer (constant) multiplicity. The complement of the set of smooth points is the so-called singular set and it is by definition always a closed set. In particular Theorem 1 states that there are at most finitely many singularities in every compact region. The counterpart of the previous theorem in terms of calibrated cones is the following

Theorem 2. *Let ϕ be a parallel calibration of degree 3 in $\mathbb{R}^n$ and let C be an integral 3-dimensional cone without boundary (in particular, C can be a tangent cone at any point of an arbitrary 3-dimensional calibrated integral cycle). Then the singular set of C is made of (at most) finitely many half-lines.*

Important examples of calibrations of degree 3 are the Special Lagrangian calibration on Calabi-Yau 3-folds and the associative calibration on $G2$-manifolds. Theorem 2 extends the result of [1] where the authors dealt with 3-dimensional Special Lagrangian cones.

The proof of Theorem 1 is based on a pseudo-holomorphic blow-up, a technique introduced in [3], [4]. This is an implementation, in an almost complex setting, of the classical blow-up of a point in algebraic geometry or symplectic geometry. In [4] the technique is employed to prove the uniqueness of tangent cones for pseudo-holomorphic integral cycles. The construction itself requires the presence of an almost-complex structure in the ambient manifold. In the proof of Theorem 1 such a structure can be assumed to exist by virtue of [5].

Semi-calibrated integral cycles have, by the monotonicity of the mass ratio, a density (or multiplicity) that is well-defined everywhere on their support (and not just almost everywhere) and in the case of cycles of dimension 2, this density is everywhere a positive integer. The proof of Theorem 1 proceeds by induction on the multiplicity[3]. A point of density 1 is a smooth point by Allard theorem and we must prove the inductive step: for any $Q \in \mathbb{N}$ we assume that the theorem is

[2] We are tacitly assuming that the ambient manifold is even dimensional. When this is not the case it is enough to embed it in its cartesian product with $\mathbb{R}$.

[3] The inductive scheme is used in [10], [8], [9], [1], [2] as well. All of these works deal with particular types of semi-calibrated integral 2-cycles with an almost complex structure that induces the semi-calibration: the almost complex structure is either given in the setting ([10], [8], [9]) or proven to exist ([1], [2]). The common conclusion is that the singular set is made of at most isolated points and our Theorem 1 contains these results as particular cases.

true under the assumption that the density is locally bounded by $Q-1$ and we want to show that it is also true when the density is locally bounded by Q. The proof requires repeated blow-ups of the current at a singular point. The intuitive idea is that this should mimic the resolution of singularities in the classical setting of algebraic curves.

This is quite evident in the case that we are dealing with a singular point (let us assume that it is the origin of coordinates) with tangent cone made of two distinct planes counted with multiplicity (e.g. a point with density 5 where the tangent cone is of the form $3[\![D_1]\!]+2[\![D_2]\!]$, where D_1 and D_2 are distinct 2-planes). As it is customary we localize our current and dilate it around the origin enough to obtain that in the ball of radius 1 around the origin the support of the current is contained in a small conic neighbourhood of D_1 and D_2. After the blow up, we find that the current actually becomes the union of two disjoint pseudo holomorphic integral 2-cycles with strictly lower multiplicities (in the example, a cycle with density bounded by 3 and a cycle with density bounded by 2). Here (after the blow up) we can use the inductive assumption, which tells us that all points except possibly finitely many are smooth. The blow up map is a diffeomorphism away from the origin and therefore the regularity properties are preserved (except possibly at the origin) when we perform it. In view of this also the original current must have at most finitely many singularities (we have added at most another singular point - the origin).

The situation is more complicated when the tangent is made of a single plane counted with multiplicity[4]. To give a concrete example, consider the curve $w = 3z^3 + z^5 + z^{20/3}$ in $\mathbb{C}^2$. The origin is a singular point of density 3 and the tangent cone there is the 2-plane $\{w = 0\}$ counted with multiplicity 3 (remark that the curve is expressed as a 3-valued graph on its tangent). Blowing up the origin corresponds essentially to the change of variable $\xi z = w$ (whilst the variable z stays the same) which leads to the curve $\xi = 3z^2 + z^4 + z^{17/3}$. With two more blow-ups (from now on we use variables ξ and z for all subsequent blow ups) we get the curve $\xi = 3 + z^2 + z^{11/3}$. At this stage we can translate the curve and get $\xi = z^2 + z^{11/3}$. The blow-up map is a diffeomorphism away from the origin and therefore the three blow-ups and the translation that we have performed have preserved the regularity properties of the original curve away from the origin[5]. If we perform two further blow-ups we obtain the curve $\xi = 1 + z^{5/3}$ and then with a translation we can still preserve the regularity and face the[6] curve $\xi = z^{5/3}$. Again, this curve has the same regularity as the original one away from the origin and its

[4]This always turns out to be the hard case, compare [10], [8], [9], [1], [2].

[5]In this specific example the regularity has been preserved at the origin as well, but, as it happens when resolving singularities, regularities could improve at the origin. Remark that at this stage we could blow-down three times and get the curve $w = z^5 + z^{20/3}$, in other words we have removed the first term in the Taylor expansion of the original 3-valued graph.

[6]Remark that at this stage we could blow-down two+three times and get the curve $w = z^{20/3}$, in other words we have removed the first two terms in the Taylor expansion for the original three-valued graph. This idea replaces the construction of the center manifold in the regularity results for mass-minimizers, see e.g. [6].

tangent cone at the origin is made of the 2-plane $\{w = 0\}$ counted with multiplicity 3. So what have we gained? The gain becomes evident in the next step. We can blow up again and we obtain the curve $\xi = z^{2/3}$. Two things have happened now: *the tangent cone has tilted*[7], as now the curve is tangent to $\{z = 0\}$, and, very importantly, *the density has strictly dropped*! The density at the origin is now 2. So, to sum up, we have passed to a curve with the same regularity as the original one (except possibly at the origin) and with stricly lower density. The proof of Theorem 1 requires to implement this pattern on semi-calibrated currents rather than curves: after a finite number of blow ups[8] we have a pseudo holomorphic integral 2-cycle with the same regularity, except possibly at the origin, but with strictly lower density at the origin itself. The inductive assumption then applies, proving the inductive step.

It is not obvious that the tilting of the tangent should happen (and that in that case density should drop) after a finite number of blow ups. We can prove that the only case when this does not happen is when the original current is made of a smooth submanifold counted an integer number of times. This can be understood in the sense of unique continuation. For example, can it happen that we keep blowing up and at all steps the current is tangent to $\{w = 0\}$? If this is the case, then we can observe the following. We restrict the current to the region $|z| \leq 1$ (this region does not change with the blowups) and the boundary of the restricted current lives in the region $\{|z| = 1, |w| \leq \delta\}$ for some $\delta > 0$ (recall that we are assuming that the tangent is $\{w = 0\}$ - by localizing and dilating around the origin we can assume that the support of the current in the unit ball lives a conic neighbourhood of $\{w = 0\}$). With the change of variable $\xi z = w$ that induces the blow up, we can see that the boundary will once again live in exactly the same region $\{|z| = 1, |\xi| \leq \delta\}$. By a *maximum principle* (to be shown) the current after the blow up must all live in the region $\{|z| \leq 1, |\xi| \leq \delta\}$. This means however that the original current lived in the region $\{|z| \leq 1, |\xi| \leq \delta|z|\}$. If we iterate n blow ups, we will have a current that lives in the same region after n blow ups, which would force the original current to live in the region $\{|z| \leq 1, |\xi| \leq \delta|z|^n\}$. Now the original current must have some strictly positive separation between the sheets (say at $|z| = \frac{1}{2}$), unless it is a single sheet counted an integer number of times. If n can be chosen as large as we want in the previous iteration, then this separation would become at some point too large to fit in the region $\{|z| \leq 1, |\xi| \leq \delta|z|^n\}$. A similar argument applies when, rather than having that the tangent is $\{w = 0\}$ at every iteration, we have more generally that the tangent is always of the form $\{w = 0\}$ or $\{w = \alpha z\}$, again yielding the conclusion that, locally around the chosen point, we are dealing with a smooth submanifold conted an integer number of times.

[7]When we speak of tilting we actually mean that the tangent goes from $\{w = 0\}$ to $\{z = 0\}$, we do not mean that it becomes of the form $\{w = \alpha z\}$ for some complex number α. In this latter case, indeed by performing one more blow up and a translation, we find again a current with the same density and tangent $\{w = 0\}$.

[8]What we show here is that if we repeatedly blow up around a singularity, after a finite number of step the tangent tilts as in the example of the curve and the density drops strictly.

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A dissipative least action principle for Hamiltonian ODEs and PDEs

Yann Brenier

At a formal level, the "inviscid Burgers" equation $\partial_t u + u\partial_x u = 0$ is a time-reversible, conservative equation, which preserves $E(t) = \int_{-\infty}^{+\infty} u^2(t,x)dx$ as time evolves. Furthermore, this equation has a Hamiltonian structure and can be derived from a suitable least action principle.

However, the so-called "entropy solutions" obtained (as in [7]) by letting ϵ go to zero in the Burgers equation $\partial_t u + u\partial_x u = \epsilon\partial_{xx}u$, behave in a very different way. For all smooth compactly supported initial conditions not identically equal to zero, they *always* produce discontinuities in finite time and, then, dissipate $E(t)$. These entropy solutions are definitely not time reversible. (One may conjecture -although this is highly controversial- a similar behaviour for the vanishing viscosity solutions of the Navier-Stokes equations, following Kolmogorov's theory of turbulence [6, 5].)

So a natural question arises: can we $modify$ the least action principe in order to get "dissipative" solutions for similar Hamiltonian ODEs or PDEs? This problem has been already addressed, in particular in [8]. Here, we refer to [2] and provide a simple class of Hamiltonian ODEs for which this seems possible. Given a closed bounded subset S of the Euclidean space (with inner product $((\cdot,\cdot))$ and norm $||\cdot||$), let us consider the Hamiltonian ODE

$$\frac{d^2X}{dt^2} = X - (\nabla K)[X], \quad K[X] = \sup\{((X,s)) - \frac{||s||^2}{2}\ ;\ s \in S\}. \tag{1}$$

Note that K is a Lipschitz convex function, everywhere differentiable except on a "meager" set $\mathcal{N}$, which is not empty, unless S is convex. [An example is

$$(2) \qquad S = \{(a_{\sigma_i}, \cdots, a_{\sigma_N}) \in \mathbf{R^N}, \quad \sigma \in \Sigma_N\}, \quad a_j = \frac{j}{N} - \frac{1}{2}, \quad j = 1, \cdots, N,$$

where Σ_N is the symmetric group. In that case, (1) reads

$$\frac{d^2 X_i}{dt^2} = X_i - \frac{1}{N}\sum_{j=1}^{N}(1\{X_i > X_j\} - 1/2), \quad i = 1, \cdots, N,$$

and describe the motion of a cloud of N self-gravitating particles moving along the real line (with a neutralizing background, as usual in Cosmology [3]).]
Because K is just a Lipschitz convex function, the Cauchy problem cannot be treated by the standard Cauchy-Lipschitz approach. Nevertheless, following Bouchut [1], equation (1) can be solved in the "almost everywhere" sense of DiPerna-Lions and Ambrosio: for almost every initial condition $(X^0, \frac{dX^0}{dt})$, equation (1) admits a unique global C^1 solution, which is time-reversible and conservative. [In case (2) this corresponds to gravitation with elastic collisions.] Equation (1) is Hamiltonian and can be derived by minimizing

$$(3) \qquad \int_{t_0}^{t_1} \{\frac{1}{2}||\frac{dX}{dt}||^2 + \Phi[X]\}dt, \quad \Phi[X] = \frac{1}{2}||X||^2 - K[X] = \frac{1}{2}\text{dist}^2(X, S),$$

as the end points $(t_0, X(t_0))$ and $(t_1, X(t_1))$ are fixed. Note that, except on $\mathcal{N}$, the squared distance function to S is everywhere differentiable and equal to its own squared gradient. Thus, the "action" can be written

$$(4) \qquad \int_{t_0}^{t_1} \frac{1}{2}\{||\frac{dX}{dt}||^2 + ||\nabla\Phi[X]||^2\}dt,$$

at least for those "good" curves X that touch the "bad" set $\mathcal{N}$ only for a negligible amount of time. For such good curves, the action principle is therefore equivalent to

$$(5) \qquad \inf \int_{t_0}^{t_1} \frac{1}{2}||\frac{dX}{dt} - \nabla\Phi[X]||^2\}dt$$

since the rectangle term in the square depends only on the end-points. So, obvious "action minimizers" are all solutions X of the "gradient flow" equation

$$(6) \qquad \frac{dX}{dt} = \nabla\Phi[X] = X - (\nabla K)[X],$$

as long as they almost never touch $\mathcal{N}$. Because K is a Lipschitz convex function, first order equation (6) can be easily solved in the sense of maximal monotone operator theory [4]: for every initial condition X_0, there is a unique global Lipschitz generalized solution, which is everywhere right-differentiable with

$$(7) \qquad \frac{dX(t+0)}{dt} = X(t) - d^0K[X(t)], \quad \forall t \geq 0,$$

where $d^0K[X]$ denotes the unique element in the subdifferential $\partial K[X]$ with minimal Euclidean norm. Typically, such a solution takes values in $\mathcal{N}$ for a non-negligible amount of time and is not C^1. In particular, it is not time-reversible and *cannot* be solution of (1) in the Bouchut-Ambrosio sense. We get a completely different type of solutions. [In case (2), this exactly corresponds to gravitation with sticky collisions.] So we suggest the use of a modified action

$$\int_{t_0}^{t_1} \frac{1}{2}||\frac{dX}{dt} - X + d^0K[X]||^2 \tag{8}$$

to get "dissipative solutions" of (1). The analysis of this minimization problem is still largely open. [In case (2), the analysis is greatly simplified by the use of rearrangement tools. In particular, a time-discrete scheme can be easily derived as follows. We first introduce an explicit time-discrete scheme for (7), namely

$$X^{n+1} = \mathcal{R}((1+h)X^n - hA), \tag{9}$$

where $\mathcal{R}$ denotes the sorting operator in increasing order, $A = (a_1, \cdots, a_N)$, $h > 0$ is the time step and $X^n = (X_1^n, \cdots, X_N^n)$ are the approximate positions of the gravitating sticky particles at time $t = nh$. (N.B. there is no need here for the "implicit" Euler scheme commonly used in maximal monotone operator theory [4] since K is not only convex but also Lipschitz continuous.) Then we look for minimizers of the time-discrete version of (8), namely

$$\sum_{n=n_0}^{n=n_1} ||X^{n+1} - \mathcal{R}((1+h)X^n + hA)||^2. \tag{10}$$

The resulting variational scheme simply reads

$$X^{n+1} = \mathcal{R}(\frac{X^n - Z^{n-1}}{1+h} + (1+h)X^n - hA), \quad Z^n = \mathcal{R}((1+h)X^n - hA) \tag{11}$$

and has provided very satisfactory numerical results.]

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Rigidity and flexibility phenomena in asymptotically flat spaces

Alessandro Carlotto

Over the last few decades, General Relativity has proven to be a wonderful source of deep variational problems and we will report here about our contributions to one of the most basic questions one might possibly ask:

Problem (A). *Are there complete, stable minimal hypersurfaces in asymptotically flat manifolds?*

Asymptotically flat manifolds should be thought of as natural models for *isolated gravitational systems*, that is the way our solar system was classically thought of. We specify here, once and for all, that the word *complete* is meant here to implicitly refer to *unbounded* minimal hypersurfaces. Apart from their intrinsic relevance, stable minimal hypersurfaces naturally arise as limits of two categories of variational objects:

(L1) sequences of minimizing currents solving the Plateau problem for diverging boundaries;

(L2) sequences of large isoperimetric boundaries or of large volume-preserving CMC hypersurfaces;

and thus their study plays a key role in the process of deeper understanding the large scale geometry of initial data sets for the Einstein field equations.

Even in the most basic of all cases, namely when (M, g) is $\mathbb{R}^n$ with its flat metric and Σ^{n-1} is assumed to be an entire minimal graph, the study of Problem (A) has played a crucial role in the development of Analysis along the whole course of the twentieth century:

Problem (B). *Are affine functions the only entire minimal graphs over $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$?*

Indeed, minimal graphs are automatically stable (in fact: area-minimizing) by virtue of a well-known calibration argument and thus (B) can be regarded as the most special subcase of (A). Such problem, which is typically named after S. N. Bernstein, was formulated around 1917 as an extension of the $n = 3$ case, which Bernstein himself had settled. In higher dimension, the answer is positive only up to ambient dimension 8 and is due to De Giorgi (for $n = 4$), Almgren (for $n = 5$) and Simons (for $6 \leq n \leq 8$) who also showed that the conjecture is false for $n \geq 9$ because of the existence of non-trivial area-minimizing cones in $\mathbb{R}^{n-1}$ (as later investigated in detail by Bombieri, De Giorgi and Giusti).

When the ambient manifold is Euclidean, but Σ is only known to be stable (and not necessarily graphical) a similar classification result is only known when $n = 3$ and it was obtained independently by do Carmo and Peng and Fischer-Colbrie and Schoen:

Theorem. [4, 5] *The only complete stable oriented minimal surface in $\mathbb{R}^3$ is the plane.*

However, the same statement is still not known to be true in $\mathbb{R}^n$ for $n \geq 4$ unless the minimal hypersurface Σ^{n-1} under consideration is assumed to have *polynomial volume growth* meaning that for some (hence for any) point p

$$\mathscr{H}^{n-1}(\Sigma \cap B_r(p)) \leq \theta^* r^{n-1}, \quad \text{for all } r > 0.$$

Before stating our main rigidity theorems concerning question (A), we need to recall an essential physical point. It is customary in General Relativity to assume that the mass density measured by any physical observer is *non-negative* at each point: as a result, time-symmetric data have non-negative scalar curvature and thus it is natural and standard to restrict our study to manifolds with this property.

Our first theorem states that there is a wide and phyisically relevant class of asymptotically flat manifolds for which the presence of a positive ADM mass is an obstruction to the existence of stable minimal surfaces. For the sake of this report, our readers may simply consider the ADM mass $\mathcal{M}$ a scalar quantity measuring the gravitational deformation of (M, g) from the trivial couple $(\mathbb{R}^n, \delta)$.

Theorem 1. *Let (M, g) be an asymptotically Schwarzschildean 3-manifold of non-negative scalar curvature. If it contains a complete, properly embedded stable minimal surface Σ, then (M, g) is isometric to the Euclidean space $\mathbb{R}^3$ and Σ is an affine plane.*

An analogous result is obtained for ambient dimension $4 \leq n \leq 7$ under an a-priori bound on the volume growth of Σ and provided the stability assumption is replaced by *strong* stability.

Theorem 2. *Let (M, g) be an asymptotically Schwarzschildean manifold of dimension $4 \leq n \leq 7$ and non-negative scalar curvature. If it contains a complete, properly embedded strongly stable minimal hypersurface Σ of polynomial volume growth, then (M, g) is isometric to the Euclidean space $\mathbb{R}^n$ and Σ is an affine hyperplane.*

These two theorems also apply to the physically relevant case when the ambient manifold M has a compact boundary (an *horizon*, for instance) once it is assumed that $\Sigma \subset M \setminus \partial M$. If instead Σ is allowed to intersect the boundary ∂M (and thus to have a boundary $\partial\Sigma$), then we need to add extra requirements. For instance, when $n = 3$ we need Σ to be a *free boundary* minimal surface (with respect to $\partial\Sigma \subset \partial M$) and $\int_{\partial\Sigma} \kappa \, d\mathscr{H}^1 \geq -2\pi$.

We do not expect the assumption of *properness* to be inessential to the above theorems, unless Σ is assumed to be (locally) area-minimizing in which case properness can be easily proved via a standard local replacement argument. Moreover, we expect Σ to be *automatically* proper whenever (M, g) has an outermost minimal horizon: however, we will not discuss these aspects here any further as we plan to analyse them carefully in a forthcoming paper with O. Chodosh and M. Eichmair.

Theorem 1 has several remarkable consequences and, among these, we would like to mention an application to the study of sequences of solutions to the Plateau problem for diverging boundaries belonging to a given hypersurface. As will be apparent from the statement, this can be interpreted as a result concerning the *failure of the convex hull property* in asymptotically flat spaces. Given an asymptotically flat manifold (M, g) *with one end* and, correspondingly, a system of asymptotically flat coordinates $\{x\}$ we call *hyperplane* a subset of the form $\Pi = \left\{x \in \mathbb{R}^n \setminus B \mid \sum_{i=1}^n a_i x^i = 0\right\}$ for some real numbers $a_1, a_2, \ldots, a_n$. Possibly by changing $\{x\}$ one can reduce to the case when $a_1 = \ldots = a_{n-1} = 0$ and $a_n = 1$. In this setting, we define *height* of a point in $M \setminus Z \simeq \mathbb{R}^n \setminus B$ the value of its x^n−coordinate.

Corollary 3. *Let (M, g) be an asymptotically Schwarzschildean 3-manifold of non-negative scalar curvature and positive ADM mass, let Π an hyperplane and let $(\Omega_i)_{i\in\mathbb{N}}$ any monotonically increasing sequence of regular, relatively compact domains such that $\cup_i \Omega_i = \Pi$. For any index i, define Γ_i to be a solution of the Plateau problem with boundary $\partial\Omega_i$. Then for each $x' \in \Pi$ the sequence $(\Gamma_i)_{i\in\mathbb{N}}$ cannot have uniformly bounded height at x', namely*

$$\liminf_{i\to\infty} \max_{(x',x^3)\in\Gamma_i} x^3 = +\infty.$$

As anticipated above, our proof of Theorem 1 is inspired by the proof given by Schoen-Yau of the *Positive Mass Theorem* in [6] where (arguing by contradiction) negativity of the ADM mass is exploited for constructing a (strongly) stable minimal surface of planar type, thereby violating the stability inequality by a preliminary reduction to a Riemannian metric of strictly positive scalar curvature (at least outside a compact set). In our case, we need to deal with two substantial differences:

(1) the hypersurface Σ is not constructed but is just assumed to exist and thus its structure and its behaviour at infinity are not known a priori;
(2) the metric g is only required to have non-negative scalar curvature, thereby admitting the (relevant) case when it is in fact scalar flat, as prescribed by the Einstein constraints in the *vacuum* case.

One crucial part of our study is indeed to characterize the structure at infinity of a complete minimal hypersurface having finite Morse index. Essentially, we extend to asymptotically flat manifolds the Euclidean structure theorem which states, roughly speaking, that any such minimal hypersurface has to be *regular at infinity* in the sense that it can be decomposed (outside a compact set) as a finite union of graphs with at most logarithmic growth when $n = 3$ or polynomial decay (like $|x'|^{3-n}$) when $n \geq 4$. For $n = 3$ Schoen proved this theorem making use of the Weierstrass representation, a tool which is peculiar of the Euclidean setting as its applicability relies on the fact that the coordinate function have harmonic restrictions to minimal submanifolds. In our case, a key point in the proof of Theorem 1 is showing (via a preliminary study of the limit laminations which may arise by blow-down of Σ) that any such Σ has finite total curvature and hence its Gauss map extends continuously at infinity. Instead, such an argument is not at disposal in the higher-dimensional case and hence a deep result of L. Simon [7, 8] concerning the analysis of isolated singularities of minimal subvarieties has to be used.

In fact, Theorem 1 follows from a more general rigidity result, given in [2], concerning marginally outer-trapped surfaces (or MOTS), a class of objects of fundamental importance in General Relativity and which coincide with minimal surfaces for time-symmetric data. This extension is not trivial since MOTS are not known to have a variational nature.

We would like to conclude this report by mentioning the paper [3] by the author and R. Schoen, where we show that the rigidity Theorem 1 (as well as Theorem 2) is essentially sharp by constructing asymptotically flat solutions of the Einstein constraint equations in $\mathbb{R}^n$ (for $n \geq 3$) that have positive ADM mass and are *exactly flat* outside of a solid cone (for any positive value of the corresponding opening angle) so that they contain plenty of complete, area-minimizing hypersurfaces. A posteriori, this strongly justifies our requirement that the metric g is asymptotically Schwarzschildean.

Theorem 4. *Let (M, g) be a scalar flat asymptotically flat manifold of dimension n and order of decay $p > (n-2)/2$. Given θ_0, θ_1, q with $0 < \theta_0 < \theta_1 < \pi$ and $(n-2)/2 < q < p \leq n-2$, there is an a_∞ sufficiently large so that for any $a \in \mathbb{R}^n$ with $|a| > a_\infty$ we can find a metric $\hat{g}$ on M with $\hat{g} = g$ in $C_{\theta_0}(a)$, $\hat{g} = \delta$ outside $C_{\theta_1}((|a|+1)a/|a|)$, and*

$$\hat{g}_{ij} = \delta_{ij} + O(|x|^{-q}), \ R(\hat{g}) \equiv 0.$$

(Here $C_\theta(a)$ is the cone of vertex a made of vectors making an angle less than θ with the vector $-a$).

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On the topology and index of minimal surfaces

Otis Chodosh
(joint work with Davi Maximo)

A well known result of Fischer–Colbrie [4] and Gulliver–Lawson [6, 7] states that for a minimal surface Σ^2 in $\mathbb{R}^3$, finite Morse index is equivalent to finite total curvature. To define the Morse index of Σ, we first define $\operatorname{index}(\Sigma \cap B_R^\Sigma(p))$ to be the number of negative Dirichlet eigenvalues of the second variation operator $L := -\Delta + 2\kappa$. Here, κ is the Gauss curvature of Σ and Δ is the intrinsic Laplacian. Then, Σ is said to have *finite Morse index* if

$$\lim_{R\to\infty} \operatorname{index}(\Sigma \cap B_R^\Sigma(p)) < \infty,$$

and in this case, *the Morse index* of Σ, denoted $\operatorname{index}(\Sigma)$ is defined to be the limit. The surface Σ is said to have *finite total curvature* if

$$\int_\Sigma |\kappa| < \infty.$$

Some examples of surfaces of finite Morse index include the plane (index 0), the catenoid (index 1), Enneper's surface (index 1), and the Costa–Hoffman–Meeks surfaces (the Costa surface with genus one, with two catenoidal ends and one flat end has index 5, as proven in [9]). Recently there have been a wide range of examples constructed by various authors.

A classical result of Osserman [10] says that a minimal surface Σ^2 in $\mathbb{R}^3$ of finite total curvature (equivalently, finite Morse index) is conformally equivalent to a punctured compact Riemann surface $\overline{\Sigma} \setminus \{p_1, \dots, p_r\}$, and the Gauss map extends meromorphically across the punctures. This places strong restrictions on the topology and geometry of such surfaces. As such, one might hope to classify such surfaces under a "small index" or "simple topology" assumption. Indeed, several such results have been obtained, including the following "small index" classification results:

- The plane is the unique two-sided stable (index 0) minimal surfaces, as proven independently by Fischer-Colbrie–Schoen [5], do Carmo–Peng [3], and Pogorelov [11].
- There are no one-sided stable minimal surfaces, as proven by Ros [12].
- The catenoid and Enneper's surface are the unique two-sided minimal surfaces of index 1, by work of López–Ros [8].

As such, it is natural to consider the case of classifying surfaces of index 2 and indeed, it was conjectured by Choe in [2] that there are no such surfaces. In [1] we confirmed this conjecture in the case of embedded surfaces.

Theorem 1. *There are no embedded minimal surfaces in $\mathbb{R}^3$ of index 2.*

The key ingredient in the proof of this is the following new estimate relating the index and topology of the surface:

Theorem 2. *Suppose that $\Sigma \to \mathbb{R}^3$ is an immersed, complete, two-sided, minimal surface of genus g and r ends. Then*

$$\operatorname{index}(\Sigma) \geq \frac{2}{3}(g+r) - 1.$$

To see how Theorem 2 allows one to prove Theorem 1, note that in the index 2 case, we obtain $g + r \leq 4$. This allows us to use classification results of "simple topology" minimal surfaces of finite total curvature to rule out such a surface, in the embedded case.

An additional consequence of Theorem 2 is the following (the upper bound in the following corollary is due to Tysk [13], it is the lower bound that follows from Theorem 2)

Corollary 1. *For Σ a two-sided minimal surface in $\mathbb{R}^3$ with embedded ends and finite total curvature, we have that*

$$-\frac{1}{3} + \frac{2}{3}\left(-\frac{1}{4\pi}\int_\Sigma \kappa\right) \leq \operatorname{index}(\Sigma) \leq (7.7)\left(-\frac{1}{4\pi}\int_\Sigma \kappa\right).$$

This shows that the index of such a surfaces is related to the total curvature in a linear sense (with quite reasonable bounds). This bound can be viewed as a partial answer to a remark of Fischer–Colbrie that there should be a relation between the total curvature and the index.

Finally, we briefly mention the key ingredients in the proof of Theorem 2. The basic tool in the argument is a link between harmonic 1-forms on the surface and the index; various authors have considered harmonic 1-forms as destabilizing directions, but our argument is inspired by the one of Ros [12], where he shows that the dimension of the space of harmonic 1-forms in L^2 provides a linear lower bound for the index. However, on minimal surface of finite total curvature having genus g and r ends, the dimension of the harmonic 1-forms in L^2 is $2g$ (this follows from the conformal invariance of the L^2-norm in this setting). So such an argument proves a weaker bound, which does not take into consideration the number of ends, only the genus.

On the other hand, it turns out that there are forms on $\overline{\Sigma} \setminus \{p_1, \dots, p_r\}$ which resemble $\pm\frac{dz}{z}$ near a pair of the p_i's. These forms are not in L^2, but by using the behavior of the ends, it is possible to show that they are in some slightly bigger weighted L^2 space $L^2_{-\delta}(\Sigma) \supset L^2(\Sigma)$. It does not seem possible to compare the forms in the weighed L^2 space with the L^2-index in the sense of Fischer–Colbrie, but a careful reworking of her work shows that it may be extended to the weighted setting. A key observation is that the "weighted" index and "standard" index are the same, because on a fixed compact set, the norms are equivalent, and hence the min-max definition of index via the Rayleigh quotient shows that the two notions of index agree at this scale; taking the limit as the compact set exhausts Σ proves the equality of both notions.

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Faber-Krahn inequalities in sharp quantitative form

Guido De Philippis
(joint work with Lorenzo Brasco and Bozhidar Velichkov)

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure, the *first eigenvalue of the Dirichlet-Laplacian* of Ω is defined by

$$\lambda(\Omega) = \min_{u \in W_0^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, dx \, : \, \|u\|_{L^2(\Omega)} = 1 \right\}.$$

If we denote by Δ the usual Laplace operator, $\lambda(\Omega)$ coincides with the smallest real number λ for which the Helmholtz equation

$$-\Delta u = \lambda \, u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

admits non-trivial solutions.

A classical optimization problem connected with λ is the following one: among sets with given volume, find the one which minimizes the principal frequency λ. Actually, balls are the (only) solutions to this problem. As λ has the dimensions of a length to the power -2, this "isoperimetric" property can be equivalently rewritten as

$$|\Omega|^{2/N}\,\lambda(\Omega) \geq |B|^{2/N}\,\lambda(B), \tag{1}$$

where B denotes a generic $N-$dimensional ball and $|\cdot|$ stands for the $N-$dimensional Lebesgue measure of a set. Moreover, equality holds in (1) if and only if Ω is a ball. The estimate (1) is the celebrated *Faber-Krahn inequality*.

The fact that balls can be characterized as the only sets for which equality holds in (1), naturally leads to consider the question of its *stability*. More precisely, one would like to improve (1), by adding in its right-hand side a reminder term measuring the deviation of a set Ω from spherical symmetry. A typical quantitative Faber-Krahn inequality then reads as follows

$$|\Omega|^{2/N}\,\lambda(\Omega) - |B|^{2/N}\,\lambda(B) \geq g(d(\Omega)), \tag{2}$$

where g is a modulus of continuity and $\Omega \mapsto d(\Omega)$ is some scaling invariant *asymmetry functional.* The quest for quantitative versions like (2) is not new and has attracted an increasing interest in the last years. To the best of our knowledge, the first ones to prove partial results in this direction have been Hansen and Nadirashvili in [10] and Melas in [12]. Both papers treat the case of simply connected sets in dimension $N = 2$ or the case of convex sets in general dimensions. These pioneering results prove inequalities like (2), with a modulus of continuity (typically a power function) depending on the dimension N and with the following asymmetry functionals

$$d_1(\Omega) = 1 - \frac{r_\Omega}{r_{B_\Omega}}, \qquad \text{where} \qquad \begin{array}{rl} r_\Omega = & \text{inradius of } \Omega, \\ r_{B_\Omega} = & \text{radius of the ball } B_\Omega, \end{array}$$

like in [10], or

$$d_2(\Omega) = \min\left\{ \frac{\max\{|\Omega \setminus B_1|, |B_2 \setminus \Omega|\}}{|\Omega|} \ :\ B_1 \subset \Omega \subset B_2 \text{ balls} \right\},$$

as in [12]. It is easy to see that for general sets an estimate like (2) with the previous asymmetry functionals *can not be true* (just think of a ball with a small hole at the center). In the general case, a better notion of asymmetry is the so called *Fraenkel asymmetry*, defined as

$$\mathcal{A}(\Omega) = \inf\left\{ \frac{|\Omega \Delta B|}{|B|} \ :\ B \text{ ball such that } |B| = |\Omega| \right\}, \tag{3}$$

where the symbol Δ now stands for the symmetric difference between sets. For such an asymmetry functional, Bhattacharya and Weitsman [4] and Nadirashvili [13] indipendently conjectured the following.

Conjecture There exists a dimensional constant $\sigma > 0$ such that

$$|\Omega|^{2/N}\,\lambda(\Omega) - |B|^{2/N}\,\lambda(B) \geq \sigma\,\mathcal{A}(\Omega)^2. \tag{4}$$

The aim of the talk is to provide a positive answer to the above conjecture, namely we have

Theorem([5]) There exists a constant $\sigma = \sigma(N)$, depending only on the dimension N, such that (4) is satisfied by every open set $\Omega \subset \mathbb{R}^N$ with finite measure.

Let us notice that the previous result is *sharp*, since the power 2 on the asymmetry can not be replaced by any smaller power. Indeed one can verify that the family of ellipsoids

$$\Omega_\varepsilon = \Big\{(x', x_N) \in \mathbb{R}^N \,:\, |x'|^2 + (1+\varepsilon)\, x_N^2 \le 1\Big\}, \qquad 0 < \varepsilon \ll 1,$$

are such that

$$\mathcal{A}(\Omega_\varepsilon) \simeq \varepsilon \qquad \text{and} \qquad |\Omega_\varepsilon|^{2/N}\, \lambda(\Omega_\varepsilon) - |B|^{2/N}\, \lambda(B) \simeq \varepsilon^2.$$

2. Strategy of the proof

Let us now explain the main ideas behind the proof of (4). First by an application of the *Kohler-Jobin inequality* ([11]) one can show that (4) is implied by the following inequality

$$E(\Omega) - E(B_1) \ge \sigma\, \mathcal{A}(\Omega)^2, \qquad \text{for every } \Omega \text{ such that } |\Omega| = |B_1|, \tag{5}$$

where σ is a dimensional constant and B_1 is the ball of radius 1 and centered at the origin. Here $E(\Omega)$ is the *energy functional* of Ω,

$$E(\Omega) = \min_{u \in W_0^{1,2}(\Omega)} \frac{1}{2} \int_\Omega |\nabla u|^2\, dx - \int_\Omega u\, dx = \frac{1}{2} \int_\Omega |\nabla u_\Omega|^2\, dx - \int_\Omega u_\Omega\, dx,$$

where $u_\Omega \in W_0^{1,2}(\Omega)$ is the (unique) function achieving the above minimum.

Suppose now by contradiction that (5) is false. Since it is pretty easy to see that (5) can only fail in the small asymmetry regime (i.e. on sets converging in L^1 to the ball), we find a sequence of sets Ω_j such that

$$|\Omega_j| = |B_1|, \qquad \varepsilon_j := \mathcal{A}(\Omega_j) \to 0 \qquad \text{and} \qquad E(\Omega_j) - E(B_1) \le \sigma \mathcal{A}(\Omega_j)^2, \tag{6}$$

with σ as small as we wish. We now look for an "improved" sequence of sets U_j, still contradicting (5) and enjoying some additional smoothness properties. In the spirit of Ekeland's variation principle, these sets will be selected through some minimization problem. Roughly speaking we look for sets U_j which solve the following

$$\min \Big\{ E(\Omega) + \sqrt{\varepsilon_j^2 + \sigma(\mathcal{A}(\Omega) - \varepsilon_j)^2} \,:\, |\Omega| = |B_1| \Big\}. \tag{7}$$

One can easily show that the sequence U_j still contradict (5) and that $\mathcal{A}(U_j) \to 0$. Relying on the minimality of U_j, one then would like to show that the L^1 convergence to B_1 can be improved to a smooth convergence. If this is the case, then the second order expansion of $E(\Omega)$ for smooth nearly spherical sets done in [9] shows that (6) cannot hold true if σ is sufficiently small.

The key point is thus to prove (uniform) regularity estimates for sets solving (7). For this, first one would like to get rid of volume constraints applying some sort of Lagrange multiplier principle to show that U_j minimizes

$$E(\Omega)+\sqrt{\varepsilon_j^2+\sigma(\mathcal{A}(\Omega)-\varepsilon_j)^2}+\Lambda\,|\Omega|. \tag{8}$$

Then, taking advantage of the fact that we are considering a "min–min" problem, the previous is equivalent to require that $u_j=u_{U_j}$ minimizes

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx-\int_{\mathbb{R}^N}v\,dx+\Lambda\,\big|\{v>0\}\big|+\sqrt{\varepsilon_j^2+\sigma(\mathcal{A}(\{v>0\})-\varepsilon_j)^2}, \tag{9}$$

among all functions with compact support. Since we are now facing a perturbed free boundary type problem, we aim to apply the techniques of Alt and Caffarelli [1] (see also [6, 7]) to show the regularity of $\partial U_j=\partial\{u_j>0\}$ and to obtain the smooth convergence of U_j to B_1.

Even if this will be the general strategy, several non-trivial modifications have to be done to the above sketched proof. In particular, although solutions to (9) enjoy some mild regularity property, we cannot expect $\partial\{u_j>0\}$ to be smooth. Indeed, by formally computing the first order optimality condition in (9) and assuming that B_1 is the unique optimal ball for $\{u_j>0\}$ in (3), one gets that u_j should satisfy

$$\left|\frac{\partial u_j}{\partial\nu}\right|^2=\Lambda+\frac{\sigma(\mathcal{A}(\{u_j>0\})-\varepsilon_j)}{\sqrt{\varepsilon_j^2+\sigma(\mathcal{A}(\{u_j>0\})-\varepsilon_j)^2}}\big(1_{\mathbb{R}^N\setminus\overline{B}_1}-1_{B_1}\big),\qquad\text{on }\partial\{u_j>0\},$$

where 1_A denotes the characteristic function of a set A and ν is the outer normal versor. This means that the normal derivative of u_j is discontinuous at points where $U_j=\{u_j>0\}$ crosses ∂B_1. Since classical elliptic regularity implies that if ∂U_j is $C^{1,\gamma}$ then $u_j\in C^{1,\gamma}(\overline{U_j})$, it is clear that the sets U_j can not enjoy too much smoothness properties.

To overcome this difficulty, inspired by [2], we replace the Fraenkel asymmetry with a new "distance" between a set Ω and the set of balls, which behaves like a squared L^2 distance between the boundaries, namely

$$\alpha(\Omega)=\int_{\Omega\Delta B_1(x_\Omega)}\big|1-|x-x_\Omega|\big|\,dx, \tag{10}$$

where x_Ω is the barycenter of Ω. One can then shows that

$$\alpha(\Omega)\geq c(N,\mathrm{diam}(\Omega))\mathcal{A}(\Omega)^2.$$

By exploiting the ideas described above one then obtain that

$$E(\Omega)-E(B_1)\geq\sigma_1(N,\mathrm{diam}\Omega)\alpha(\Omega)\geq\sigma_2(N,\mathrm{diam}(\Omega))\mathcal{A}(\Omega)^2$$

for every Ω such that $|\Omega|=|B_1|$, establishing the validity of (4) for bounded sets. In the last step of the proof one then shows how to pass from the case of general sets to the case of bounded ones.

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Quasistatic evolution as limit of dynamic evolutions: the case of perfect plasticity

Gianni Dal Maso

(joint work with Riccardo Scala)

The quasistatic evolution of rate independent systems has been often studied as the limit case of viscosity driven evolutions. In this talk we present a case study of approximation of a quasistatic evolution by dynamic evolutions, where all inertial effects are taken into account.

We consider the quasistatic evolution problem in linearly elastic perfect plasticity. In [6] it was obtained as a vanishing viscosity limit of Perzyna visco-plasticity, Therefore we consider a dynamic model which couples dynamic visco-elasticity with Perzyna visco-plasticity, and obtain the quasistatic case as limit when a parameter connected with the speed of the process tends to 0.

In our model the reference configuration is a bounded open set $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. The linearized strain Eu (the symmetric part of the gradient of the displacement u) is decomposed as $Eu = e + p$, where e is the elastic part and p is the plastic part. The stress $\sigma = A_0 e + A_1 \dot{e}$ is the sum of an elastic part $A_0 e$ and a viscous part $A_1 \dot{e}$, where A_0 is the elasticity tensor, A_1 is the viscosity tensor, and $\dot{e}$ is the derivative of e with respect to time. As usual we assume that A_0 is symmetric and positive definite, while we only assume that A_1 is symmetric and positive semidefinite, so that we are allowed to consider also $A_1 = 0$, which corresponds to a dynamic version of Perzyna visco-plasticity.

Assuming, for simplicity, that the mass density is identically equal to 1, the balance of momentum leads to the equation

$$\ddot{u} - \mathrm{div}\sigma = f,$$

where f is the volume force. As in Perzyna visco-plasticity, another important ingredient of the model is a convex set K in the space of deviatoric symmetric matrices. The evolution of the plastic part is determined by the flow rule

$$\dot{p} = \sigma_D - \pi_K \sigma_D,$$

where σ_D is the deviatoric part of σ and π_K is the projection onto K, which can be interpreted as the domain of visco-elasticity. Indeed, if σ_D belongs to K during the evolution, then there is no production of plastic strain, so that, if $p = 0$ at the initial time, then $p = 0$ for every time and the solution is purely visco-elastic.

The complete system of equations is then

$$Eu = e + p, \tag{1a}$$

$$\sigma = A_0 e + A_1 \dot{e}_{A_1}, \tag{1b}$$

$$\ddot{u} - \mathrm{div}\sigma = f, \tag{1c}$$

$$\dot{p} = \sigma_D - \pi_K \sigma_D, \tag{1d}$$

where e_{A_1} denotes the projection of e into the image of A_1. This system is supplemented by initial and boundary conditions.

Other dynamic models of elasto-plasticity with viscosity have been considered in [1] and [2]. The main difference with respect to our model is that they couple visco-elasticity with perfect plasticity, while we couple visco-elasticity with visco-plasticity.

Under natural assumptions on A_0, A_1, f, and K we prove (see [4])existence and uniqueness of a solution to (1) with initial and boundary conditions. In analogy with the energy method for rate independent processes developed by Mielke (see [5] and the references therein), we first prove that (1) has a weak formulation expressed in terms of a sort of stability condition together with an energy balance. The proof of the existence of a solution to this weak formulation is obtained by time discretization. In the discrete formulation we solve suitable incremental minimum problems and then we pass to the limit as the time step tends to 0.

Our main result concerns the behavior of the solution to system (1) as the data of the problem become slower and slower. After rescaling time we are led to the study of the system

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{2a}$$

$$\sigma^\epsilon = A_0 e^\epsilon + \epsilon A_1 \dot{e}^\epsilon_{A_1}, \tag{2b}$$

$$\epsilon^2 \ddot{u}^\epsilon - \mathrm{div}\sigma^\epsilon = f, \tag{2c}$$

$$\epsilon \dot{p}^\epsilon = \sigma^\epsilon_D - \pi_K \sigma^\epsilon_D, \tag{2d}$$

as ϵ tends to 0.

Under suitable assumptions we prove (see [4]) that these solutions converge, up to a subsequence, to a weak solution of the quasistatic evolution problem in perfect plasticity (see [6] and [3]), whose strong formulation is given by

$$Eu = e + p, \tag{3a}$$

$$\sigma = A_0 e, \tag{3b}$$

$$-\mathrm{div}\sigma = f, \tag{3c}$$

$$\sigma_D \in K \ \text{ and } \ \dot{p} \in N_K \sigma_D, \tag{3d}$$

where $N_K \sigma_D$ denotes the normal cone to K at σ_D.

The proof of this convergence result is obtained using the weak formulation of (1) mentioned above. We show that we can pass to the limit obtaining the energy formulation of (3) developed in [3]. A remarkable difficulty in this proof is due to the fact that problems (1) and (3) are formulated in completely different function spaces.

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Examples of fractional minimal surfaces

Manuel del Pino
(joint work with Juan Dávila, Juncheng Wei)

Let $0 < s < 1$. According to the notion introduced in [2] and [1], a s-minimal (stationary) surface $\Sigma = \partial E$ in $\Omega \subset \mathbb{R}^N$ is one that satisfies

$$H^s_\Sigma(p) := \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx = 0 \quad \text{for all } p \in \Sigma \cap \Omega. \tag{1}$$

Here χ denotes characteristic function, $E^c = \mathbb{R}^N \setminus E$. The integral is understood in a principal value sense

$$H^s_\Sigma(p) = \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B_\delta(p)} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx,$$

and it is well-defined provided that Σ is regular near p. It agrees with usual mean curvature in the limit $s \to 1$ by the relation

$$\lim_{s \to 1} (1-s)\, H^s_\Sigma(p) = c_N H_\Sigma(p). \tag{2}$$

Besides, (1) is the Euler-Lagrange equation for the fractional perimeter functional introduced in [1]

$$\mathcal{I}_s(E,\Omega) = \int_{E\cap\Omega} \int_{E^c} \frac{dx\, dy}{|x-y|^{N+s}} + \int_{E\cap\Omega^c} \int_{E^c\cap\Omega} \frac{dx\, dy}{|x-y|^{N+s}}.$$

Clearly a hyperplane in $\mathbb{R}^N$ is a s-minimal surface for any s. Next in complexity in $\mathbb{R}^3$ is the *axially symmetric case.* In the classical case, the minimal surface equation reduces to an ODE from which the catenoid C_1 is obtained:

$$C_1 = \{(x_1,x_2,x_3)\ /\ |x_3| = \log(r + \sqrt{r^2-1}), \quad r = \sqrt{x_1^2+x_2^2} > 1\}.$$

In [4] wa construct an axially symmetric s-minimal surface C_s for $s$close to 1 in such a way that $C_s \to C_1$ as $s \to 1$ on bounded sets. We call this surface the *fractional catenoid.* A striking feature of the surface of revolution C_s is that it becomes at main order as $r \to \infty$ a cone with small slope rather than having logarithmic growth. More precisely, in [4] we have established the following:

For all $0 < s < 1$ sufficiently close to 1 there exists a connected surface of revolution C_s such that if we set $\varepsilon = (1-s)$ then

$$\sup_{x \in C_s \cap B(0,2)} \operatorname{dist}(x, C_1) \le c \frac{\sqrt{\varepsilon}}{|\log \varepsilon|},$$

and, for $r = \sqrt{x_1^2+x_2^2} > 2$ the set C_s can be described as $|x_3| = f(r)$, where

$$f(r) = \begin{cases} \log(r+\sqrt{r^2-1}) + O\left(\frac{r\sqrt{\varepsilon}}{|\log\varepsilon|}\right) & \text{if} \quad r < \frac{1}{\sqrt{\varepsilon}} \\ r\sqrt{\varepsilon} + O(|\log\varepsilon|) + O\left(\frac{r\sqrt{\varepsilon}}{|\log\varepsilon|}\right) & \text{if} \quad r > \frac{1}{\sqrt{\varepsilon}}. \end{cases}$$

A plane is a s-minimal surface for any $0 < s < 1$. In the classical scenario, so is the union of two parallel planes, say $x_3 = 1$ and $x_3 = -1$. This is no longer the case when $0 < s < 1$ since the nonlocal interaction between the two components deforms them and in fact equilibria is reached when the two components diverge becoming cones. In [4] we have built a *two-sheet* nontrivial s-minimal surface D_s for s close to 1 where the components eventually become at main order the cone $x_3 = \pm r\sqrt{\varepsilon}$.

More precisely, for all $0 < s < 1$ sufficiently close to 1 there exists a two-component surface of revolution $D_s = D_s^+ \cup D_s^-$ such that if we set $\varepsilon = (1-s)$ then $D_s^\pm$ is the graph of the radial functions $x_3 = \pm f(r)$ where f is a positive function of class C^2 with $f(0) = 1$, $f'(0) = 0$, and

$$f(r) = \begin{cases} 1 + \frac{\varepsilon}{4} r^2 + O(\varepsilon r) & \text{if} \quad r < \frac{1}{\sqrt{\varepsilon}} \\ r\sqrt{\varepsilon} + O(1) + O(\varepsilon r) & \text{if} \quad r > \frac{1}{\sqrt{\varepsilon}}. \end{cases}$$

We consider, for given $n, m \geq 1$, and $0 < s < 1$ the problem of finding a value $\alpha > 0$ such that the *Lawson cone*

$$C_\alpha = \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n \ / \ |v| = \alpha |u|\} \tag{3}$$

is a s-minimal surface in $\mathbb{R}^{m+n} \setminus \{0\}$. For the classical case $s = 1$ we compute directly that mean curvature of this cone is zero if and only if $n = m = 1$ or

$$n \geq 2, \ m \geq 2, \quad \alpha = \sqrt{\frac{n-1}{m-1}}.$$

We have proven in [4] that for any given $m \geq 1$, $n \geq 1$, $0 < s < 1$, there is a unique $\alpha = \alpha(s, m, n) > 0$ such that the cone C_α given by (3) is an s-fractional minimal surface. We call this $C_m^n(s)$ the s-Lawson cone. For $n = 3$, $C_1^2(s)$ is precisely the s-minimal cone that represents at main order the asymptotic behavior of the revolution s-minimal surfaces described above.

In [6, 5] it is proven that smooth regularity of fractional s-perimeter minimizing surfaces except for a set of Hausdorff dimension at most $N-3$, improving previous results in [1]. In [3], regularity of non-local minimizers is found up to a $(N-8)$-dimensional set, whenever s is sufficiently close to 1. Thus, there remains a conspicuous gap between the best general regularity result found so far and the case s close to 1. We find in [4] the following result: there is a $s_0 > 0$ such that for each $s \in (0, s_0)$, all minimal cones $C_m^n(s)$ are unstable if $N = m+n \leq 6$ and stable if $N = 7$ (stability is understood in the sense of second variation of s-perimeter). This suggests that regularity only up to and $(N-7)$-dimensonal set may be the best possible for minimizers and general s.

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Nonsmooth differential geometry

Nicola Gigli

The first issue one encounters when doing analysis on metric measure spaces is the lack of all the classical tools of differential geometry used to perform computations.

In the talk I presented some recent results of mine [1], inspired by an earlier work of Weaver [2], which aim to build a working differential structure on metric measure spaces. In situations where a lower Ricci curvature bound is imposed on the space, the theory can be pushed up to the construction of Hessians, covariant and exterior derivative and of Ricci curvature, but the talk focussed on the first order structure which is present on general complete and separable metric spaces equipped with a nonnegative Radon measure $(X, \mathsf{d}, \mathfrak{m})$.

The crucial notion of the approach is that of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$ module, which is a structure $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}, \cdot, |\cdot|)$ where: $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ is a Banach space, $\cdot$ is a multiplication of elements of $\mathcal{M}$ with $L^\infty(\mathfrak{m})$ functions satisfying

$$f(gv) = (fg)v \qquad \text{and} \qquad \mathbf{1}v = v \qquad \text{for every} \qquad f, g \in L^\infty(\mathfrak{m}),\ v \in \mathcal{M},$$

with $\mathbf{1}$ being the function identically equal to 1, and $|\cdot| : \mathcal{M} \to L^2(\mathfrak{m})$ is the 'pointwise norm', i.e. a map assigning to every $v \in \mathcal{M}$ a non-negative function in $L^2(\mathfrak{m})$ such that

$$\begin{aligned} \|v\|_{\mathcal{M}} &= \||v|\|_{L^2(\mathfrak{m})}, \\ |fv| &= |f||v|, \quad \mathfrak{m}-\text{a.e.} \end{aligned} \qquad \text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathcal{M},$$

so that in particular we have

$$\|fv\|_{\mathcal{M}} \leq \|f\|_{L^\infty(\mathfrak{m})}\|v\|_{\mathcal{M}}, \qquad \text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathcal{M}.$$

The basic example of $L^2(\mathfrak{m})$-normed module is the space of L^2 (co)vector fields on a Riemannian/Finslerian manifold: here the norm $\|\cdot\|_{\mathcal{M}}$ is the L^2 norm and the multiplication with an L^∞ function and the pointwise norm are defined in the obvious way.

The job that the notion of $L^2(\mathfrak{m})$-normed module does is to revert this procedure and give the possibility of speaking about L^2 sections of a vector bundle without really having the bundle.

This fact is of help when trying to build a differential structure on metric measure spaces, because it relieves from the duty of defining a tangent space at every, or almost every, point, allowing one to concentrate on the definition of L^2 (co)vector field. Then one constructs the cotangent module starting from the notion of Sobolev function and of weak upper gradient. Thus let $S^2(X)$ be the class of real valued functions f on X having 2-weak upper gradient $|\mathrm{D}f|$ in $L^2(\mathfrak{m})$ and recall that the basic calculus rules for $|\mathrm{D}f|$ are:

$$\begin{aligned} |\mathrm{D}f| &= 0, \quad \mathfrak{m}-\text{a.e. on } \{f=0\}, && \forall f\in S^2(X),\\ |\mathrm{D}(\varphi\circ f)| &= |\varphi'|\circ f|\mathrm{D}f|, && \forall f\in S^2(X),\ \varphi\in C^1(\mathbb{R}),\\ |\mathrm{D}(fg)| &\le |f||\mathrm{D}g|+|g||\mathrm{D}f|, && \forall f,g\in S^2\cap L^\infty(X). \end{aligned}$$

The idea to define the cotangent module is then to 'pretend that it exists' and that for each $f\in S^2(X)$ and Borel set $E\subset X$ the abstract object $\chi_E\mathrm{d}f$, to be interpreted as the 1-form equal to the differential of f on E and 0 on $X\setminus E$, is an element of such module. The definition comes via explicit construction. We introduce the set 'Pre-cotangent module' Pcm as

$$\mathrm{Pcm} := \Big\{\{(f_i,A_i)\}_{i\in\mathbb{N}}\ :\ (A_i)_{i\in\mathbb{N}} \text{ is a Borel partition of } X,\\ f_i\in S^2(X)\ \forall i\in\mathbb{N},\ \text{and } \sum_{i\in\mathbb{N}}\int_{A_i}|\mathrm{D}f|^2\,\mathrm{d}\mathfrak{m}<\infty\Big\}$$

and an equivalence relation on it declaring $\{(f_i,A_i)\}_{i\in\mathbb{N}}\sim\{(g_j,B_j)\}_{j\in\mathbb{N}}$ provided

$$|\mathrm{D}(f_i-g_j)|=0,\quad \mathfrak{m}-\text{a.e. on } A_i\cap B_j\ \forall i,j\in\mathbb{N}.$$

Denoting by $[(f_i,A_i)]$ the equivalence class of $\{(f_i,A_i)\}_{i\in\mathbb{N}}$, the operations of addition, multiplication by a scalar and by a simple function (i.e. taking only a finite number of values) and the one of taking the pointwise norm can be introduced as

$$\begin{aligned} [(f_i,A_i)]+[(g_j,B_j)] &:= [(f_i+g_j,A_i\cap B_j)]\\ \lambda[(f_i,A_i)] &:= [(\lambda f_i,A_i)]\\ \Big(\sum_j\alpha_j\chi_{B_i}\Big)\cdot[(f_i,A_i)] &:= [(\alpha_j f_i,A_i\cap B_j)],\\ \big|[(f_i,A_i)]\big| &:= \sum_i\chi_{A_i}|\mathrm{D}f_i|, \end{aligned}$$

and it is not difficult to see that these are continuous on Pcm$/\sim$ w.r.t. the norm $\|[f_i,A_i]\|:=\sqrt{\int|[(f_i,A_i)]|^2\,\mathrm{d}\mathfrak{m}}$ and the $L^\infty(\mathfrak{m})$-norm on the space of simple functions. Thus they all can be continuously extended to the completion of (Pcm$/\sim,\|\cdot\|$): we shall call such completion together with these operation the cotangent module and denote it by $L^2(T^*X)$. When applied to a smooth Riemannian/Finslerian manifold, this abstract construction is canonically identifiable with the space of L^2 sections of the cotangent bundle T^*X, whence the notation chosen.

Given a Sobolev function $f\in S^2(X)$, its **differential** $\mathrm{d}f$ is a well defined element of $L^2(T^*X)$, its definition being

$$\mathrm{d}f := [(f,X)],$$

and from the properties of Sobolev functions one can verify that the differential is a closed operator. Directly from the definition we see that $|\mathrm{d}f| = |\mathrm{D}f|$ $\mathfrak{m}$-a.e., and with little work one can check that the calculus rules for $|\mathrm{D}f|$ can be improved to:

$$\begin{aligned} \mathrm{d}f &= 0, \qquad \mathfrak{m}-\text{a.e. on } \{f=0\} && \forall f \in S^2(X), \\ \mathrm{d}(\varphi\circ f) &= \varphi'\circ f \mathrm{d}f, && \forall f \in S^2(X),\ \varphi \in C^1(\mathbb{R}), \\ \mathrm{d}(fg) &= f\mathrm{d}g + g\mathrm{d}f, && \forall f,g \in S^2\cap L^\infty(X), \end{aligned}$$

where thanks to the L^∞-module structure the chain and Leibniz rules both make sense and the locality condition is interpreted as $\chi_{\{f=0\}}\mathrm{d}f = 0$.

Once the notion of cotangent module is given, the tangent module $L^2(TX)$ can be introduced by duality: it is the space of linear continuous maps $L : L^2(T^*X) \to L^1(\mathfrak{m})$ satisfying

$$L(f\omega) = fL(\omega), \qquad \forall \omega \in L^2(T^*X),\ f \in L^\infty(\mathfrak{m}),$$

and it is not hard to see that it carries a canonical structure of $L^2(\mathfrak{m})$-normed module as well, so that in particular for any vector field Z, i.e. every element of the tangent module $L^2(TX)$, the pointwise norm $|Z|$ is a well defined function in $L^2(\mathfrak{m})$.

Based on these grounds, a general first-order differential theory can be developed on arbitrary metric measure spaces. Properties worth of notice are:

- In the smooth setting, for every smooth curve γ the tangent vector γ'_t is well defined for any t and its norm coincides with the metric speed of the curve.

  A similar property holds on metric measure spaces provided one replaces 'smooth curve' be 'test plan', a rigorous meaning of this being also based on the notion of pullback of a module.
- (co)vector fields are transformed via 'regular' maps between metric measure spaces as in the smooth setting, i.e. we can speak of pullback of forms and these regular maps possess a differential acting on vector fields.

  Here the relevant notion of regularity for a map φ from $(X_2, \mathsf{d}_2, \mathfrak{m}_2)$ to $(X_1, \mathsf{d}_1, \mathfrak{m}_1)$ is to be Lipschitz and such that $\varphi_*\mathfrak{m}_2 \le C\mathfrak{m}_1$ for some $C > 0$.
- The gradient of a Sobolev function is in general not uniquely defined and even if so it might not linearly depend on the function, as it happens on smooth Finsler manifolds. Spaces where the gradient $\nabla f \in L^2(TX)$ of a Sobolev function $f \in S^2(X)$ is unique and linearly depends on f are those which, from the Sobolev calculus point of view, resemble Riemannian manifolds among the more general Finsler ones and can be characterized as those for which the energy $E : L^2(\mathfrak{m}) \to [0, +\infty]$ defined as

$$E(f) := \begin{cases} \dfrac{1}{2}\displaystyle\int |\mathrm{D}f|^2\,\mathrm{d}\mathfrak{m}, & \text{if } f \in S^2(X), \\ +\infty, & \text{otherwise.} \end{cases}$$

  is a Dirichlet form. On such spaces, the tangent module (and similarly the cotangent one) is, when seen as Banach space, an Hilbert space and its pointwise norm satisfies a pointwise parallelogram identity. Thus by polarization it induces a pointwise scalar product

$$L^2(TX) \ni Z, W \qquad \mapsto \qquad \langle Z, W \rangle \in L^1(\mathfrak{m}),$$

which we might think of as the 'metric tensor' on our space. It can then be verified that for $f, g \in L^2 \cap S^2(X)$ the scalar product $\langle \nabla f, \nabla g \rangle$ coincides with the Carré du champ $\Gamma(f, g)$ induced by the Dirichlet form E.

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Mean curvature flow with surgery

Gerhard Huisken

The lecture describes the construction and a priori estimates for mean curvature flow $F : M^n \times [0, T) \to (N^{n+1}, \bar{g})$

$$\frac{d}{dt} F = \bar{H} = -H\nu \qquad (MCF)$$

of smooth, closed mean-convex hypersurfaces in a smooth closed Riemannian manifold, interrupted by surgery at finitely many times $0 = t_0 < t_1 < \dots t_N < T$, where necks of type $S^{n-1} \times (a, b)$ are replaced by spherical caps of much lower curvature.

Main results are

Theorem A (joint with Carlo Sinestrari, 2009)[6])
If M_0^n is mean convex with $\lambda_i + \lambda_j > 0, i \neq j$ for all pairs of principal curvatures, $n \geq 3$, $(N^3, \bar{g}) = (R^3, \delta)$, then (MCF) with surgery always exists.

Theorem B (joint with Simon Brendle, 2013)[4])
If $n = 2$ and M_0^2 is mean convex and embedded, then (MCF) with at most finitely many surgeries always exists. If $T = \infty$, M_t^2 tends to a weakly stable minimal surface of smaller genus than the initial surface as $t \to \infty$.

The proof of Theorem B is based on a sharpening of the non-collapsing results of White[7] and Andrews [1] by Brendle [2], a pseudo locality estimate, an interior gradient estimate by Haslhofer-Kleiner [5] and new estimates on the improvement of collapsing properties in cylindrical necks. Convergence for $t \to \infty$ is shown with the help of regularity results due to White [7], an interior radius estimate in 3-manifolds of Brendle and a modified monotonicity formula of Brendle [3].

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Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law

Francesco Maggi
(joint work with G. De Philippis)

The classical description of capillarity phenomena involves the study of Gauss free energy for a liquid inside a container, which takes the form

$$\mathcal{H}^{n-1}(A\cap\partial E)+\int_{\partial A\cap\partial E}\sigma(x)\,d\mathcal{H}^{n-1}(x)+\int_E g(x)\,dx-l\,|E|\,.$$

Here A is an open set in $\mathbb{R}^n$ ($n\ge 2$), the container of the fluid; $E\subset A$ is the region occupied by the fluid, with volume $|E|$; $\mathcal{H}^{n-1}(A\cap\partial E)$ is the total surface tension energy of the interior interface $A\cap\partial E$; the surface tension between the liquid and the boundary walls of the container is obtained by integrating over the wetted surface $\partial A\cap\partial E$ the coefficient $\sigma(x)$; finally, $g(x)$ is the potential energy density (typically, when $n=3$ one considers $g(x)=\rho\, g_0\, x_3$ where ρ is the constant density of the fluid and g_0 the gravity of Earth), and l is a Lagrange multiplier. If $M=A\cap\partial E$ is smooth enough, then the equilibrium conditions are

$$H_E+g=l \qquad \text{on } A\cap\partial E\,, \tag{1}$$

$$\nu_E\cdot\nu_A=\sigma \qquad \text{on } \partial A\cap\partial E\,, \tag{2}$$

where ν_E is the outer unit normal to E and H_E is the mean curvature of $A\cap\partial E$. These conditions, first described by Young in [12], have then been expressed in analytic form by Laplace in 1805; see [5]. The second condition, commonly known as *Young's law*, enforces $|\sigma|\le 1$ and is independent from the potential energy g.

Volume constrained minimizers of the Gauss free energy are found in the class of sets of finite perimeter. One is thus lead to discuss a regularity problem in order to validate (1) and (2). Interior regularity has been addressed in the classical theory developed by De Giorgi, Federer, Almgren, and others in the Sixties: if ∂^*E denotes the reduced boundary of the minimizer E, and we set $M=\text{closure}(A\cap$

$\partial^* E$), then there exists a closed set $\Sigma \subset M$ such that $M \setminus \Sigma$ is "as smooth as g allows it to be" and Σ has Hausdorff dimension at most $n-8$. The situation concerning boundary regularity is less conclusive. Taylor [11] proved in dimension $n=3$ everywhere regularity of M at ∂A (in the more general context of $(\mathbf{M}, \xi, \delta)$-minimal sets). Cafferelli and Friedman [2] addressed the sessile droplet problem ($A = \{x_n > 0\}$ and $g(x) = g(x_n)$) in the case when $-1 < \sigma(x) < 0$ for every $x \in \{x_n = 0\}$ and $2 \le n \le 7$ by mixing symmetrization arguments, barrier techniques, interior regularity for perimeter minimizers, and the regularity theory of free boundary problems associated to quasilinear uniformly elliptic equations. Grüter [7, 8, 9] and Grüter and Jost [6] addressed the case when $\sigma \equiv 0$ by exploiting reflection techniques and interior regularity.

Motivated by applications to relative isoperimetric problems in Riemannian and Finsler geometry, one would also like to understand the regularity of minimizers of anisotropic surface energies of the form

$$I(E) = \int_{A \cap \partial E} \Phi(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\partial A \cap \partial E} \sigma \, d\mathcal{H}^{n-1}$$

where $\Phi : A \times \mathbb{R}^n \to [0, \infty)$ is such that $\Phi(x, \cdot)$ positively one-homogeneous and convex on $\mathbb{R}^n$ for every $x \in A$. The typical assumption to obtain regularity here is that $\Phi(x, \nu)$ is l-elliptic in ν: roughly speaking, one asks that for some $l \in (0,1]$, and for every $x \in A$ and $\nu \in S^{n-1}$,

$$l \le \Phi(x,\nu) \le \frac{1}{l}, \qquad \nabla^2 \Phi(x,\nu) \ge l \mathrm{Id} \qquad \text{on } \nu^\perp .$$

Under this assumption, interior regularity is known since the works of Almgren [1], and Schoen, Simon and Almgren [10]. In [3, 4] we address boundary regularity.

Theorem 1. *Let $\partial A \in C^{1,1}$, Φ be l-elliptic in ν and uniformly Lipschitz in x, let $\sigma \in \mathrm{Lip}(\mathbb{R}^n)$ be such that $-\Phi(x, -\nu_A) < \sigma(x) < \Phi(x, \nu_A)$ for $x \in \partial A$, and let $E \subset A$ be such that*

$$I(E) \le I(F) + \Lambda \, |E \Delta F| ,$$

whenever $F \subset A$, $E \Delta F \subset\subset B_{x,r}$ where $x \in A$ and $r < \delta$. Then E is equivalent to an open set, $\partial E \cap \partial A$ is a set of finite perimeter in ∂A, and there exists a closed set $\Sigma \subset \mathrm{closure}(A \cap \partial E) =: M$ such that $M \setminus \Sigma$ is a $C^{1,1/2}$-manifold with boundary, $\mathcal{H}^{n-3}(\Sigma) = 0$, and the (anisotropic) Young's law

$$\nabla \Phi(x, \nu_E(x)) \cdot \nu_A(x) = \sigma(x) ,$$

holds for every $x \in (M \setminus \Sigma) \cap \partial A$.

As said, the fact that $A \cap (M \setminus \Sigma)$ is a $C^{1,1/2}$-manifold for a closed set $\Sigma \subset M$ with $\mathcal{H}^{n-3}(A \cap \Sigma) = 0$ is proved in [1, 10]: our contribution here is addressing the situation at boundary points, namely, on $M \cap \partial A$. As explained above, this last problem was still partially open in the isotropic case, and, to the best of our knowledge, completely open in the genuinely anisotropic case. In [3] we have proved Theorem 1 in a weaker form, where one only concludes that $\mathcal{H}^{n-2}(\partial A \cap \Sigma) = 0$; starting from this dimensional estimate, in [4] we have further developed our analysis to conclude that $\mathcal{H}^{n-3}(\partial A \cap \Sigma) = 0$, thus matching the best known interior regularity results.

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The structure of minimum matchings

Mircea Petrache

(joint work with Roger Züst)

1. Calibrations without orientation

We recall here the setting of the theory of calibrations (see [5], [4]). The following is a simple proof that the shortest oriented curve connecting two points $a, b \in \mathbb{R}^n$ is the oriented segment $[a, b]$. Let α be the constant coefficient differential 1-form dual to the unit vector τ orienting $[a, b]$. Then for any other Lipschitz curve γ from a to b we have

$$\text{lenght}([a,b]) = \int_{[a,b]} \alpha = \int_{\gamma} \alpha \leq \text{lenght}(\gamma) \ , \tag{1}$$

where we used the fact that $d\alpha = 0$ for the middle equality and the fact that τ realizes the maximum of α and α measures the lenght along $[a, b]$

$$\langle \alpha, \tau \rangle = \max_{\tau' \in \mathbb{S}^{n-1}} \langle \alpha, \tau' \rangle = 1 \ , \tag{2}$$

for the remaining equality and inequality. More in general, we may apply the same method for minimizers of the following problem. Consider the 0-current $[\![X^\pm]\!] := \sum_{i=1}^n ([\![x_i^+]\!] - [\![x_i^-]\!])$. Consider then

$$\mathrm{Fill}([\![X^\pm]\!]) := \inf \left\{ \mathbf{M}(C) \ \middle| \ \begin{array}{c} C \text{ is an integer multiplicity 1-current} \\ \text{and } \partial C = [\![X^\pm]\!] \end{array} \right\} . \tag{3}$$

This can be generalized to prove the minimality of k-dimensional oriented submanifolds of $\mathbb{R}^n$ as well, using their duality with smooth k-forms. A *calibration* of dimension k is a comass-1 closed k-form. This is one of the most robust tools for testing the minimality of submanifolds. For more precise definitions and extensions see [5].

The above reasoning is strongly based on a linear structure, namely the duality between k-currents and k-forms. How much of the procedure can be retrieved if we discard part of this structure? The main goal of the work [7] which is described in this report is to give a complete answer to this in a simple case.

2. Results

We will consider the unoriented version of (3). Let $n \in \mathbb{N}$ and $X = \{x_1, \dots, x_{2n}\}$ a set with $2n$ points equipped with a pseudometric d. A *matching* on X is a partition π of X into n pairs of points, $\pi = \{\{x_1, x_1'\}, \dots, \{x_n, x_n'\}\}$. The set of all matchings on X is denoted by $\mathcal{M}(X)$. We consider the **minimum matching problem** for d:

$$m(X, d) := \min_{\pi \in \mathcal{M}(X)} \sum_{\{x, x'\} \in \pi} d(x, x') \ . \tag{4}$$

Note that this corresponds to the minimization of mass for flat **1-chains with coefficients in $\mathbb{Z}_2$**. The basic observation which we made is that the dual problem for (4) uncovers a wonderful geometric structure. Namely, dual objects are parameterized trees:

Theorem 1. *For any pseudometric d on X, there is a tree-like pseudometric D on X with $D \le d$ and $m(X, D) = m(X, d)$.*

A pseudometric space (X, d) is said to be *tree-like* if for any choice of points $x_1, x_2, x_3, x_4 \in X$,

$$d(x_1, x_3) + d(x_2, x_4) \le \max\{d(x_1, x_2) + d(x_3, x_4), d(x_1, x_4) + d(x_2, x_3)\} \ . \tag{5}$$

(X, d) is tree-like if and only if it can be realized as a subset of a metric tree, see [3]. *Metric trees* can be characterized as uniquely arcwise connected geodesic metric spaces. Throughout these notes we will also assume that metric trees are complete.

3. Unoriented Kantorovich duality

Based on Theorem 1, we obtain the following:

Theorem 2 (unoriented Kantorovich duality)**.** *Let (X,d) be a pseudometric space of cardinality $2n$. Then*

$$m(X,d)=\max\left\{m(X,f^*d_T)\,\middle|\,\begin{array}{l} f:X\to(T,d_T) \text{ is } 1\text{-Lipschitz}\\ \text{and } (T,d_T) \text{ is a metric tree}\end{array}\right\}\ . \tag{6}$$

This can be compared with the following classical result (see [6] for the originating idea, and see e.g. [8, Lemma 2.2] for a proof of this precise statement). Let (X,d) be a metric space of cardinality $2n$. Let $\Pi=\{\{x_1^+,\dots,x_n^+\},\{x_1^-,\dots,x_n^-\}\}$ be a partition of X into two n-ples of points. Consider $\mu^\pm:=\frac{1}{n}\sum_{i=1}^n\delta_{x_i^\pm}$ and define $M(\Pi,d):=W_1(\mu^+,\mu^-)$ where W_1 is the 1-Wasserstein distance defined on probability measures (cfr. [9], [1] and the references therein). By density considerations, if $\tilde{X}$ is Polish, then giving W_1 on averages of Dirac masses is the same as giving it on the whole set of probability measures on $\tilde{X}$.

Note that for any 1-Lipschitz function $f:X\to\mathbb{R}$ there holds

$$\sum_{i=1}^n f(x_i^+)-f(x_i^-)=\min_{\sigma\in S_n}\sum_{i=1}^n d_{\mathbb{R}}\Big(f\big(x_i^+\big),f\big(x_{\sigma(i)}^-\big)\Big)=M(\Pi,f^*d_{\mathbb{R}})\ . \tag{7}$$

Then one can compare Theorem 2 with the following:

Theorem 3 (Kantorovich duality, equivalent formulation)**.** *Let (X,d) be a metric space of cardinality $2n$. Let $\Pi=\{\{x_1^+,\dots,x_n^+\},\{x_1^-,\dots,x_n^-\}\}$ be a partition of X into two n-ples of points. Then the following holds,*

$$M(\Pi,d)=\max\left\{M(\Pi,f^*d_{\mathbb{R}})\ :\ f:X\to\mathbb{R}\text{ is }1\text{-Lipschitz}\right\}\ . \tag{8}$$

The important difference between this theorem and Theorem 2 is that here the minimization is done amongst a wider class of competitors. The set X has $\frac{(2n)!}{2^n n!}$ matchings and once we fix a partition Π only $n!$ of them are admissible connections for it. Therefore there holds

$$m(X,d)\le M(\Pi,d)\ , \tag{9}$$

with a strict inequality in general. It might then look slightly surprising that, while on the one hand in the unoriented version the minimum on the left decreased, on the other hand in order to achieve the same number by the maximum we have to enlarge the space of 1-Lipschitz maps competing for the dual problem on the right, passing form $\mathbb{R}$ to general metric trees.

4. Global calibrations modulo 2

We now describe a consequence of Theorem 1 in the spirit of unoriented mincut-maxflow theorems which allows to build a solid analogy with the result of the introduction.

A 1-Lipschitz function $\rho : T \to \mathbb{R}$ is an *orientation modulo 2 for* $A \subset T$ if for any arc $[a,b] \subset T$ we have $J(\rho|_{[a,b]})(t) = 1$ for $\mathcal{H}^1$-a.e. $t \in [a,b] \cap A$. Such orientations for T are given for example by the distance functions $t \mapsto d_T(p,t)$ for any $p \in T$. As defined in [2], the set $\mathcal{R}_1(\tilde{X}, \mathbb{Z}_2)$ of rectifiable 1-chains modulo 2 is composed of chains $[\![\Gamma]\!]$, where Γ is some $\mathcal{H}^1$-rectifiable set $\Gamma \subset X$. If $f : \tilde{X} \to \mathbb{R}$ is Lipschitz we can define its action on $[\![\Gamma]\!]$ as follows. Fix some countable parameterization $\gamma_i : K_i \subset \mathbb{R} \to \gamma_i(K_i) \subset \Gamma$, i.e. K_i is compact, the images $\gamma_i(K_i)$ are pairwise disjoint, all γ_i are bi-Lipschitz and $\mathcal{H}^1(\Gamma \setminus \cup_i \gamma_i(K_i)) = 0$. Then we define

$$[\![\Gamma]\!](df) := \sum_i \int_{K_i} |(f \circ \gamma_i)'(t)| \, d\mathcal{H}^1(t) \ .$$

It is not hard to check that this definition does not depend on the parameterization and on the choice of the set Γ representing $[\![\Gamma]\!]$ as above. Further, $[\![\Gamma]\!](df) \leq \mathrm{Lip}(f)\mathbf{M}([\![\Gamma]\!])$ and if $f \in C^1(\mathbb{R}^n)$ and $\gamma : [0,1] \to \mathbb{R}^n$ is Lipschitz and injective, then $[\![\mathrm{im}(\gamma)]\!](df) = \int_0^1 |df(\gamma'(t))| \, dt$, justifying the use of df in the definition of this action. In contrast to chains with coefficients in $\mathbb{Z}$, this action is not linear. For Lipschitz functions f, g and $C, C' \in \mathcal{R}_1(\tilde{X}, \mathbb{Z}_2)$ there holds, $C(d(f+g)) \leq C(df) + C(dg)$ and $(C + C')(df) \leq C(df) + C'(df)$, with strict inequalities in general.

Consider now the problem

$$\mathrm{Fill}_{\mathbb{Z}_2}([\![X]\!]) := \inf \left\{ \mathbf{M}(C) \,\middle|\, \begin{array}{c} C \text{ a 1-chain with coefficients} \\ \text{in } \mathbb{Z}_2 \text{ and } \partial C = [\![X]\!] \end{array} \right\} \tag{10}$$

We next describe the dual to it. Given a closed set $A \subset \tilde{X}$ and a set $X \subset \tilde{X}$ of even cardinality, we say that A is a $\mathbb{Z}_2$*-cut* of X if at least one of the connected components of $\tilde{X} \setminus A$ contains an odd number of points in X. Then denote

$$\mathrm{Cut}_{\mathbb{Z}_2}(A, X) := \# \left\{ \begin{array}{c} \text{connected components } A' \text{ of } A \\ \text{that are } \mathbb{Z}_2\text{-cuts} \end{array} \right\} . \tag{11}$$

For a Lipschitz function $\varphi : \tilde{X} \to \mathbb{R}$ we define

$$\mathrm{lev}_{\mathbb{Z}_2}(\varphi, X) := \int_{\mathbb{R}} \mathrm{Cut}_{\mathbb{Z}_2}(\varphi = t, X) \, dt \ . \tag{12}$$

We then consider the following real number:

$$\mathrm{Lev}_{\mathbb{Z}_2}(X) := \sup \left\{ \mathrm{lev}_{\mathbb{Z}_2}(\varphi) : \ \varphi : \tilde{X} \to \mathbb{R} \text{ is 1-Lipschitz} \right\} . \tag{13}$$

For a map $f : X \to T$ defined on an even cardinality metric space X into a tree, define

$$A_X := \bigcup \left\{ [f(x), f(y)] \,\middle|\, \begin{array}{c} \{x, y\} \text{ appears in some} \\ \text{minimal matching of } (X, d) \end{array} \right\} . \tag{14}$$

The analogue of $\mathrm{Cut}_{\mathbb{Z}_2}(A, X), \mathrm{lev}_{\mathbb{Z}_2}(\varphi, X)$ for the minimization on integral 1-currents like in the introduction is as follows. For a closed set $A \subset \tilde{X}$ and for

$\Pi = \{\{x_i^+\}, \{x_i^-\}\}$ a partition of X into two equal parts, define the quantity

$$\mathrm{Cut}_{\mathbb{Z}}(A, \Pi) := \left|\#A \cap \{x_i^+\} - \#A \cap \{x_i^-\}\right| \ .$$

Then for a 1-Lipschitz function $f : \tilde{X} \to \mathbb{R}$ define

$$\mathrm{lev}_{\mathbb{Z}}(f, \Pi) := \int_{\mathbb{R}} \mathrm{Cut}_{\mathbb{Z}}(\{f \leq t\}, \Pi)\, dt \leq \mathrm{Fill}_{\mathbb{Z}}([\![X^{\pm}]\!]) \ . \tag{15}$$

If $\mathrm{Lev}_{\mathbb{Z}}(\Pi)$ is defined to be the supremum of $\mathrm{lev}_{\mathbb{Z}}(f, \Pi)$ among all f as above, then we see immediately that Theorem 3 states exactly that $\mathrm{Fill}_{\mathbb{Z}}([\![X^{\pm}]\!]) = \mathrm{Lev}_{\mathbb{Z}}(\Pi)$. For usual calibrations we have the following characterizing properties.

Proposition 4. *Let $\tilde{X}$ be a connected Riemannian manifold with $H^1(\tilde{X}) = 0$ and Π be some partition $\{\{x_1^+, \ldots, x_n^+\}, \{x_1^-, \ldots, x_n^-\}\}$ of a finite subset X of $\tilde{X}$. Let C be an integer 1-chain with $\partial C = [\![X^{\pm}]\!]$ and $\mathbf{M}(C) = \mathrm{Fill}([\![X^{\pm}]\!])$. For a flat 1-form α on $\tilde{X}$ the following are equivalent:*

(1) α is a calibration for C.
(2) α is a calibration for any minimizer C as above.
(3) $\alpha = df$ for some 1-Lipschitz function $f : \tilde{X} \to \mathbb{R}$ for which

$$\min_{\sigma \in S_n} \sum_{i=1}^{n} d\big(x_i^+, x_{\sigma(i)}^-\big) = \sum_{i=1}^{n} f(x_i^+) - f(x_i^-) \ .$$

(4) $\alpha = df$ for some 1-Lipschitz function $f : \tilde{X} \to \mathbb{R}$ realizing the equality $\mathrm{lev}_{\mathbb{Z}}(f, \Pi) = \mathrm{Lev}_{\mathbb{Z}}(\Pi)$.

We have the following analogue of the above proposition.

Proposition 5. *Let $\tilde{X}$ be a connected Riemannian manifold with $H^1(\tilde{X}) = 0$ and let $X \subset \tilde{X}$ be an even cardinality set. Let C be a chain modulo 2 with $\partial C = [\![X]\!]$ and $\mathbf{M}(C) = \mathrm{Fill}_{\mathbb{Z}_2}([\![X]\!])$. For a closed flat 1-form α on $\tilde{X}$ consider the following statements:*

(1) α has comass 1 and for a fixed C as above, $C(\alpha) = \mathbf{M}(C)$.
(2) α has comass 1 and for any minimizer C as above, $C(\alpha) = \mathbf{M}(C)$.
(3) $\alpha = d(\rho \circ f)$, where $f : \tilde{X} \to T$ is a 1-Lipschitz map into a finite tree (T, d_T) such that $m(X, d) = m(X, f^* d_T)$ and ρ is an orientation for A_X defined in (14).
(4) $\alpha = d\varphi$ for some 1-Lipschitz function $\varphi : \tilde{X} \to \mathbb{R}$ realizing the equality $\mathrm{lev}_{\mathbb{Z}_2}(\varphi, X) = \mathrm{Lev}_{\mathbb{Z}_2}(X)$.

Then (4) $\Leftrightarrow$ (3)$\mathbb{R}ightarrow$(2)$\mathbb{R}ightarrow$(1) and the other implications are false in general.

Therefore we define:

Definition 6 (global calibrations modulo 2)**.** Let $(\tilde{X}, d)$ be a geodesic metric space and let $[\![X]\!]$ be a 0-boundary modulo 2 in $\tilde{X}$. The differential $d(\rho \circ f)$ for f, ρ as above is called a global calibration modulo 2 for $[\![X]\!]$.

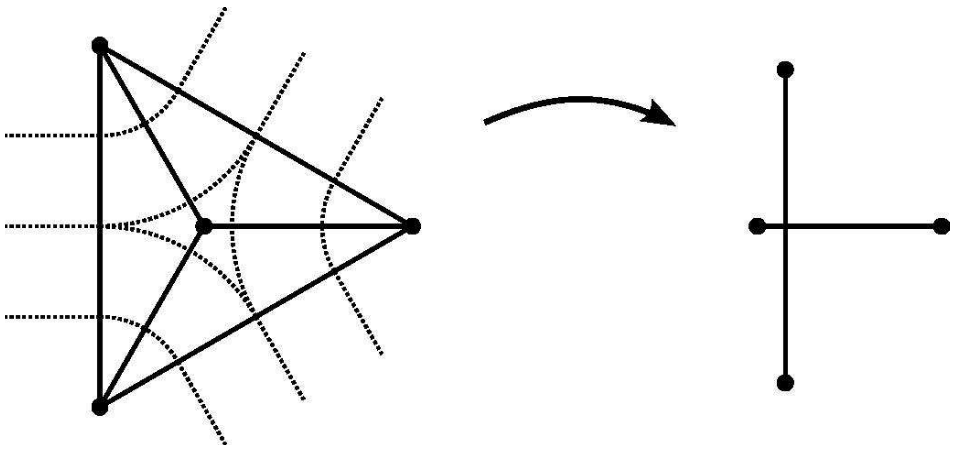

Figure 1. Illustrated is a 1-Lipschitz map $f : \mathbb{R}^2 \to T$ corresponding to a global calibration modulo 2 for the set X displayed by the four dots and mutual geodesics by thick lines on the left. The dotted lines indicate some possible level sets.

5. Matching dimension

As a concrete application of our new global duality result for matchings, we prove an incompressibility property for minimum matchings. For a metric space (X, d), an even number $k \in \mathbb{N}$ and $\epsilon > 0$ define the *matching numbers*

$$m_k(X, d) := \sup\{m(X', d) : X' \subset X, |X'| = k\} \ ,$$
$$m'_\epsilon(X, d) := \sup\{m(X', d) : X' \text{ is } \epsilon\text{-separated in } X\} \ .$$

Depending on some geometric conditions on a metric space we give some bounds to these matching numbers.

Proposition 7. *Let (X, d) be a compact metric space and $n \geq 1$. Assume that there are constants $0 < c_1 < C_1$ such that for every $0 < \epsilon < \operatorname{diam}(X)$,*

$$c_1\epsilon^{-n} < \sup\{|X'| : X' \subset X \textit{ has even cardinality and is } \epsilon - \textit{separated}\} \leq C_1\epsilon^{-n} \ .$$

Then, there is a constant $c > 0$ such that for all $0 < \epsilon < \operatorname{diam}(X)$ and all even numbers k,

$$m_k(X, d) \geq ck^{\frac{n-1}{n}} \ , \quad \textit{and } m'_\epsilon(X, d) \geq \quad c_1\epsilon^{1-n} \ . \tag{16}$$

Let $Y \subset X$. Assume that $\mathcal{H}^n(X) < \infty$ and that there are constants $C_2 > 0$ and $0 < \lambda_2 < \frac{1}{2}$ such that for all points $x, x' \in Y$ and all open sets $U \subset X$ with $\mathbf{B}(x, \lambda_2 d) \subset U$ and $\mathbf{B}(x, \lambda_2 d) \subset X \setminus \bar{U}$ there holds

$$\mathcal{H}^{n-1}(\partial U) \geq C_2 d^{n-1} \ .$$

Then, there is a constant $C > 0$ such that for all $0 < \epsilon < \operatorname{diam}(X)$ and all even numbers k,

$$m_k(Y, d) \leq C\mathcal{H}^n(X)^{\frac{1}{n}} k^{\frac{n-1}{n}} \ , \quad \textit{and} \quad m'_\epsilon(Y, d) \leq C\mathcal{H}^n(X)\epsilon^{1-n} \ . \tag{17}$$

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The shape of “large” conformal metrics with prescribed Gauss curvature

Michael Struwe

1. Main result

Let (M, g_0) be a closed Riemann surface of genus at least 2, endowed with a smooth background metric g_0. Following [8], [9], for a given function $f \in C^\infty(M)$ we consider the equation

$$-\Delta_{g_0} u + K_{g_0} = f e^{2u} \quad \text{on } M. \tag{1}$$

whose solutions induce metrics $g = e^{2u} g_0$ of Gauss curvature $K_g = f$ on M. By the uniformization theorem we may assume that g_0 has constant Gauss curvature $K_{g_0} \equiv k_0$. Moreover, we normalize the volume of (M, g_0) to unity.

Solutions u of (1) can be characterized as critical points of the functional

$$E_f(u) = \frac{1}{2} \int_M \left(|\nabla u|^2_{g_0} + 2k_0 u - f e^{2u} \right) d\mu_{g_0},\ u \in H^1(M, g_0)\ .$$

Note that E_f is strictly convex and coercive on $H^1(M, g_0)$ when $f \le 0$ does not vanish identically. Hence for such f the functional E_f admits a strict absolute minimizer $u \in H^1(M, g_0)$ which is the unique solution of (1).

Let $f_0 \le 0$ be a smooth, non-constant function, all of whose maximum points p_0 are non-degenerate with $f_0(p_0) = 0$, and for $\lambda \in \mathbb{R}$ let $f_\lambda = f_0 + \lambda$, $E_\lambda(u) = E_{f_\lambda}$. Then for $\lambda \le 0$ by the preceding observation E_λ admits a strict absolute minimizer $u_\lambda \in H^1(M, g_0)$ which is the unique solution of (1) for f_λ. Moreover, by the implicit function theorem also for suitably small $\lambda > 0$ the functional E_λ admits a strict relative minimizer $u_\lambda \in H^1(M, g_0)$, smoothly depending on λ. However, for $\lambda > 0$ the functional E_λ is no longer bounded from below, as can be seen by choosing a comparison function $v \geq 0$ supported in the set where $f_\lambda > \lambda/2$ and looking at $E_\lambda(sv)$ for large $s > 0$. Thus, for small $\lambda > 0$ the functional E_λ exhibits a *"mountain pass"* geometry and one may expect the existence of a further critical point of saddle-type.

In fact, Ding-Liu [6] show the following result.

Theorem 1 (Ding-Liu [6]). *For any smooth, non-constant function $f_0 \leq 0 = \max_{p \in M} f_0(p)$ consider the family of functions $f_\lambda = f_0 + \lambda$, $\lambda \in \mathbb{R}$, and the associated family of functionals $E_\lambda(u) = E_{f_\lambda}(u)$ on $H^1(M, g_0)$. There exists a number $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ the functional E_λ admits a local minimizer u_λ and a further critical point $u^\lambda \neq u_\lambda$ not of minimum type.*

Unaware of the Ding-Liu [6] result, in 2011 together with Franziska Borer and Luca Galimberti we gave a different proof of Theorem 1, reported on in my Oberwolfach report [14]. This new approach crucially relies on the "entropy bound"

$$\liminf_{\lambda \downarrow 0} \big(\lambda \int_M e^{2u^\lambda}\, d\mu_{g_0}\big) \leq 8\pi. \tag{2}$$

for the "large" solutions u^λ that we obtained using the *"monotonicity trick"* from [11], [12], [13] in a way similar to [15]. Note that the bound (2) by the Gauss-Bonnet identity

$$2\pi\chi(M) = \int_M f_\lambda e^{2u^\lambda} d\mu_{g_0} = \lambda \int_M e^{2u^\lambda} d\mu_{g_0} - \int_M |f_0| e^{2u^\lambda} d\mu_{g_0} \tag{3}$$

gives a bound on the total curvature of the metrics $g^\lambda = e^{2u^\lambda} g_0$ as $\lambda \downarrow 0$ suitably, which allows to invoke results of Brezis-Merle [3] and others to study their blow-up behavior. In are thus able to establish the following result.

Theorem 2 (Borer-Galimberti-Struwe [2]). *Let $f_0 \leq 0$ be a smooth, non-constant function, all of whose maximum points p_0 are non-degenerate with $f_0(p_0) = 0$, and for $\lambda \in \mathbb{R}$ also let $f_\lambda = f_0 + \lambda$, $E_\lambda(u) = E_{f_\lambda}$ as in Theorem 1 above. There exist $I \in \mathbb{N}$, a sequence $\lambda_n \downarrow 0$ and a sequence of non-minimizing critical points $u_n = u^{\lambda_n}$ of E_{λ_n} such that for suitable $r_n^{(i)} \downarrow 0$, $p_n^{(i)} \to p_\infty^{(i)} \in M$ with $f(p_\infty^{(i)}) = 0$, $1 \leq i \leq I$, the following holds.*

i) We have smooth convergence $u_n \to u_\infty$ locally on $M_\infty = M \setminus \{p_\infty^{(i)};\ 1 \leq i \leq I\}$, and u_∞ induces a complete metric $g_\infty = e^{2u_\infty} g_0$ on M_∞ of finite total curvature $K_{g_\infty} = f_0$.

ii) For each $1 \leq i \leq I$, either a) there holds $r_n^{(i)}/\sqrt{\lambda_n} \to 0$ and in local conformal coordinates around $p_n^{(i)}$ we have

$$w_n(x) := u_n(r_n^{(i)} x) - u_n(0) + \log 2 \to w_\infty(x) = \log\big(\frac{2}{1 + |x|^2}\big)$$

smoothly locally in $\mathbb{R}^2$, where w_∞ induces a spherical metric $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ of curvature $K_{g_\infty} = 1$ on $\mathbb{R}^2$, or b) we have $r_n^{(i)} = \sqrt{\lambda_n}$, and in local conformal *coordinates around* $p_\infty^{(i)}$ *with a constant* $c_\infty^{(i)}$ *there holds*

$$w_n(x) = u_n(r_n^{(i)}x) + log(\lambda_n) + c_\infty^{(i)} \to w_\infty(x)$$

smoothly locally in $\mathbb{R}^2$*, where the metric* $g_\infty = e^{2w_\infty} g_{\mathbb{R}^2}$ *on* $\mathbb{R}^2$ *has finite volume and finite total curvature with* $K_{g_\infty}(x) = 1 + (Ax, x)$*, where* $A = \frac{1}{2}Hess_f(p_\infty^{(i)})$*.*

Remark 1. *i) Without the bound* (2) *neither the results of Brezis-Merle* [3] *nor those of Martinazzi* [10] *can be applied to* (1)*.*

ii) Comparing the bound (2) *with the threshold value* 2π *for blow-up at a point* p_0 *resulting from work of Brezis-Merle* [3] *we see that our sequence* (u_n) *can blow up in at most* $I = 4$ *points, regardless of how many maximum points the function* f_0 *possesses. Thus if there are* $m > 4$ *distinct maximum points* p_i *where* $f(p_i) = 0$*, we may conjecture that* E_λ *for sufficiently small* $\lambda > 0$ *admits multiple non-minimizing critical points. In fact, in a recent preprint Manuel Del Pino and Carlos Román* [5] *via matched asymptotic expansions achieve a construction of precisely* $2^m - 1$ *different families of solutions* u_k^λ*,* $1 \le k < 2^m$*, blowing up as* $\lambda \downarrow 0$*.*

iii) We do not know if solutions of the type arising in case ii.b) exist; see Cheng-Lin [4] *for related results.*

iv) In a forthcoming paper, Luca Galimberti [7] *proves the analogue of Theorem 2 for a closed surface of genus* 1*.*

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A regularity theorem for curvature flow of networks

Yoshihiro Tonegawa
(joint work with Neshan Wickramasekera)

We report some regularity results of general 1-dimensional Brakke curvature flow of networks on $\mathbb{R}^2$ which exhibits various singularities such as triple junctions and their collisions. Due to the singular nature of the flow, the problem is set most naturally in the framework of varifold as was originally formulated by Brakke [1]. We first state the definition of general Brakke mean curvature flow for any dimension and co-dimension. Let $1 \le k < n$ be integers. A one parameter family of k-dimensional varifolds $\{V_t\}_{t\ge 0}$ in $U \subset \mathbb{R}^n$ is called Brakke mean curvature flow if

(i) V_t is integral for a.e. $t \ge 0$,
(ii) generalized mean curvature $h(V_t,\cdot)$ exists for a.e. $t \ge 0$ and $h(V_t,\cdot) \in L^2_{loc}(\|V_t\| \times dt)$,
(iii) for all $0 \le t_1 < t_2 < \infty$ and $\phi \in C^1_c(U \times [0,\infty);\mathbb{R}^+)$,

$$(1)\qquad \int_U \phi(\cdot,t)\, d\|V_t\|\Big|_{t=t_1}^{t_2} \le \int_{t_1}^{t_2} dt \int_U (\nabla\phi - \phi h(V_t,\cdot))\cdot h(V_t,\cdot) + \frac{\partial\phi}{\partial t}\, d\|V_t\|.$$

If spt $\|V_t\|$ is compact in U and if we take ϕ such that $\phi = 1$ on spt $\|V_t\|$, we have

$$\|V_{t_2}\|(U) - \|V_{t_1}\|(U) \le -\int_{t_1}^{t_2} dt \int_U |h(V_t,\cdot)|^2\, d\|V_t\| \le 0$$

which shows that the total mass of varifold is monotone decreasing. The integrability condition in (ii) may be considered as a natural and necessary condition to make sense of (1). Otherwise, it is reasonable to assume that V_t vanishes when $h(V_t,\cdot)$ stops being L^2 integrable with respect to space-time. Though it may not be so obvious, the condition (iii) is equivalent to the definition of classical mean curvature flow when the varifolds are smoothly moving k-dimensional surfaces. The general existence of such family of varifolds for any dimension and co-dimension was studied by Brakke [1], where he proved that there exists such a solution given any initial integral varifold with some mild finiteness assumption. For the 1-dimensional case on the plane, there have been a number of papers on the existence, stability and asymptotic behavior of regular network solutions. On the other hand, much is unknown about the fine regularity property of general Brakke curvature flow. Here we present a general regularity statement for any 1-dimensional Brakke curvature flow which may be the starting point for further studies on similar higher dimensional singularities. First we note that the following partial regularity result has been known [1, 3, 5] for any dimension and co-dimension:

Theorem 1. *In addition to* $\{V_t\}_{t\geq 0}$ *being k-dimensional Brakke mean curvature flow, for a.e.* $t \geq 0$*, assume that* V_t *is a unit density varifold (i.e., the multiplicity function is equals to 1,* $\|V_t\|$ *a.e.). Then for a.e.* $t \geq 0$*, there exists a closed set* $C_t \subset U$ *such that* $\mathcal{H}^k(C_t) = 0$*. For any* $x \in U \setminus C_t$*, there exists a space-time neighborhood* $O \subset U \times \mathbb{R}$ *containing* (x,t) *such that* $\cup_{s>0} \operatorname{spt} \|V_s\| \times \{s\}$ *is a smooth* $k+1$*-dimensional surface in* O *moving by mean curvature.*

The flow considered in [3] is more general and their local regularity theorem may be also seen as a strict parabolic generalization of the Allard regularity theorem.

For one dimension, one wonders naturally if the size of the singular set C_t may be much smaller than $\mathcal{H}^1(C_t) = 0$ in general. It is expected that C_t may constitute a discrete set for most of the time and furthermore, they are regular triple junctions meeting at 120 degrees and moving smoothly in space-time. The result pointing toward such speculation is what we report in the following. We report two main results, and the first one, stated rather imprecisely, is as follows.

Theorem 2. *Any Brakke curvature flow sufficiently close to a static triple junction in space-time in measure must be regular triple junctions moving by curvature.*

Note that, even if a Brakke flow is close to a static triple junction in measure, it is not apparent whether or not there may be a fine complexity persisting around the junction point. The above theorem basically says that such phenomena cannot occur and that the regular triple junction is in some sense a dynamical attractor for the curvature flow. For stating above claim precisely, define one standard triple junction

$$J := \{(s,0) : s \geq 0\} \cup \{(-s/2, \sqrt{3}s/2) : s \geq 0\} \cup \{(-s/2, -\sqrt{3}s/2) : s \geq 0\}$$

and let ϕ_1, ϕ_2, ϕ_3 be non-negative smooth approximations of characteristic functions

$$\chi_{B_{1/2}(1,0)},\ \chi_{B_{1/2}(-1/2,\sqrt{3}/2)},\ \chi_{B_{1/2}(-1/2,-\sqrt{3}/2)},$$

respectively, so that they are identical to each other under 120 degree rotations centered at the origin. Define

$$\lambda := \int_J \phi_1 \, d\mathcal{H}^1 \Big(= \int_J \phi_2 \, d\mathcal{H}^1 = \int_J \phi_3 \, d\mathcal{H}^1 \Big)$$

which is close to 1 due to the definition.

Theorem 3. *Given* $\nu \in (0,1)$ *and* $E_1 \in (1,\infty)$*, there exists* $\varepsilon = \varepsilon(\nu, E_1) \in (0,1)$ *with the following. Suppose that* $\{V_t\}_{t\in[-2,2]}$ *is 1-dimensional Brakke curvature flow in* B_2 *(i.e., it satisfies (i)-(iii) with* $U = B_2$*) and assume in addition that*

(a) $\sup_{B_r(x)\subset B_2,\, t\in[-2,2]} \frac{\|V_t\|(B_r(x))}{2r} \leq E_1$,
(b) $\exists j_1 \in \{1,2,3\} \,:\, \int_{B_2} \phi_{j_1} \, d\|V_{-2}\| \leq \lambda(2-\nu)$,
(c) $\exists j_2 \in \{1,2,3\} \,:\, \int_{B_2} \phi_{j_2} \, d\|V_2\| \geq \lambda\nu$,

(d) $\mu := \left(\int_{-2}^{2} dt \int_{B_2} (\mathrm{dist}\,(x, J))^2 \, d\|V_t\|(x)\right)^{\frac{1}{2}} \leq \varepsilon.$

Then there exists a family of smooth curves $\{l_1(t), l_2(t), l_3(t)\}_{t\in[-1,1]}$ *in* B_1 *meeting at* $a(t) \in B_1$ *with 120 degree angles such that* $\mathrm{spt}\,\|V_t\| \cap B_1 = \cup_{j=1}^{3} l_j(t)$*. Moreover, the deviation of* $\cup_{j=1}^{3} l_j(t)$ *from* J *in* $C^{1,\alpha}$ *norm as well as* $\|a\|_{C^{\frac{1+\alpha}{2}}}$ *are bounded by a constant multiple (depending only on* ν, E_1, α*) of* μ*.*

We remark that the condition (a) is always satisfied for some finite E_1 (assuming that V_t is defined on a larger space-time domain) due to Huisken's monotonicity formula [2], thus it is included merely as a quantifier. The condition (b) excludes the possibility of having two static triple junctions which are both close to J. Obviously, they cannot be represented as such three curves stated in the conclusion. The condition (c) excludes $V_t \equiv 0$, which happens to satisfy all the other conditions trivially. The condition (d) requires V_t is close to J in L^2 sense in space-time. Such distance is an appropriate one under the weak topology of measure. The conclusion is that the support of $\|V_t\|$ is a regular triple junction in B_1 for $t \in [-1, 1]$, with the stated estimates in terms of μ.

The proof of Theorem 3 borrows ideas from [4] on the analysis of singular set of minimal submanifolds, with a number of new and extra estimates which are not needed for Simon's case. The key estimate is the so called 'no L^2 concentration estimate', which shows that the portion of L^2 distance norm near the triple junction is small with respect to μ. In the talk, an outline of proof is given as a sequence of propositions.

By applying White's stratification theorem [7] of mean curvature flow, we obtain the second main result:

Theorem 4. *Suppose that we have a Brakke curvature flow* $\{V_t\}_{t\geq 0}$ *in* U *with an additional property that there is no static tangent flow of density* ≥ 2 *at each space-time point* (x, t)*. Then there exists a closed set* $C \subset U \times [0, \infty)$ *with parabolic Hausdorff dimension at most 1 such that* $\cup_{s>0} \mathrm{spt}\,\|V_s\| \times \{s\}$ *is a regular 2-dimensional triple junction in* $U \times (0, \infty) \setminus C$*.*

If we have the stated assumption on the static tangent flow, we either have a unit density static line or a triple junction as a tangent flow outside of a set C of parabolic Hausdorff dimension at most 1. In the former case, we may apply [3, 5] to show the regularity, and in the latter case of triple junction, we apply Theorem 3. These regularity theorems show that such regular (including regular triple junction) points constitute an open set in space-time, thus the set C is closed. We note that C can be further stratified into two disjoint sets, one being a discrete set of 'shrinking tangent flow' and the other being the complement within C, which is a set of 'quasi-static tangent flow'. The detail is described in [6].

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Filling multiples of embedded cycles

Robert Young

Given a smooth curve T in $\mathbb{R}^N$, there is a minimal surface U with boundary T. If we trace T twice to get a curve $2T$, there is a minimal surface U' with boundary $2T$. One might guess that $U' = 2U$, and, by a theorem of Federer [1], this holds when $N \leq 3$, but a remarkable example of L. C. Young shows that U' and U may be very different. Young [5] constructed a smooth curve T drawn on a nonorientable surface in $\mathbb{R}^4$ such that $\operatorname{area} U' \approx (1 + 1/\pi) \operatorname{area} U$. Morgan [3] and White [4] later found other examples of this phenomenon with different multipliers. A version of Young's example is shown in Figure 1.

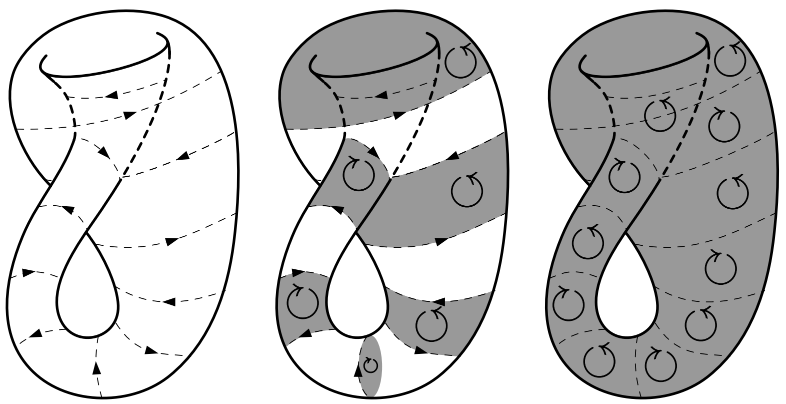

Figure 1. Fillings of a 1-cycle on a Klein bottle. The 1-cycle T consists of $2k+1$ loops in alternating directions. In the middle, we fill T with k cylindrical bands and a disc, and on the right, we fill $2T$ with $2k+1$ cylindrical bands with alternating orientations.

One can generalize this to arbitrary dimensions. If T is a d-cycle in $\mathbb{R}^N$, we define $\operatorname{FV}(T)$ to be the minimum mass of an integral $d+1$-chain with boundary T. Young's example can then be generalized to an example of a d-cycle in $\mathbb{R}^{d+3}$ such that $\operatorname{FV}(T) > \operatorname{FV}(2T)/2$.

One might ask whether the ratio $\operatorname{FV}(2T)/\operatorname{FV}(T)$ can be made arbitrarily small. In fact, the following holds:

Theorem 1. *Let $0 < d < N$ be natural numbers. There is a $C > 0$ depending on d and N such that if $T \in C_{d-1}(\mathbb{R}^N; \mathbb{Z})$ is a boundary, then*

$$\mathrm{FV}(T) \leq C\, \mathrm{FV}(2T).$$

This theorem can be reduced to the problem of proving that any mod-2 cellular cycle U in $\mathbb{R}^N$ (for instance, a minimal filling of $2T$) is congruent mod-2 to an integral cycle of comparable mass. That is,

Proposition 1. *There is a $c > 0$, depending on d and N such that for every mod-2 cellular d-cycle U in the unit grid in $\mathbb{R}^N$, there is an integral d-cycle R such that $U \equiv R \pmod 2$ and $\operatorname{mass} R \leq c \operatorname{mass} U$.*

A weaker version of the proposition, showing that there is an R such that $U \equiv R \pmod 2$ and $\operatorname{mass} R \leq c \operatorname{mass} U(\log \operatorname{mass} U)$, can be proved by using the Federer-Fleming Deformation Theorem to construct a sequence of approximations of U, a method similar to those used in [6] and [2].

To remove this factor of $\log \operatorname{mass} U$, we use uniform rectifiability. Uniformly rectifiable sets were developed by David and Semmes as a quantitative version of the notion of rectifiable sets. Recall that a set $E \subset \mathbb{R}^n$ is d-rectifiable if it can be covered by countably many Lipschitz images of $\mathbb{R}^d$. Uniform rectifiability quantifies this by bounding the Lipschitz constants and the number of images necessary to cover E. We introduce a decomposition of cellular cycles in $\mathbb{R}^N$ into sums of cellular cycles supported on uniformly rectifiable sets.

Specifically, we prove:

Theorem 2. *If $A \in C_d(\tau; \mathbb{Z}/2)$ is a d-cycle in the unit grid in $\mathbb{R}^N$, then there are cycles $M_1, \dots, M_k \in C_d(\tau; \mathbb{Z}/2)$ and uniformly rectifiable sets $E_1, \dots, E_k \subset \mathbb{R}^N$ such that*

(1) $A = \sum_i M_i$,
(2) $\operatorname{supp} M_i \subset E_i$,
(3) $\operatorname{mass} M_i \sim |E_i|$, *and*
(4) $\sum_i |E_i| \lesssim \operatorname{mass} A$.

Here, $|\cdot|$ represents d-dimensional Hausdorff measure.

This reduces the proof of Proposition 1 to the case where T is supported on a uniformly rectifiable set. We then prove Proposition 1 by using a corona decomposition to break T into pieces that are close to d-planes in $\mathbb{R}^N$.

Open questions

We can ask a similar question about the relationship between real filling volume and integral filling volume. That is, if $\mathrm{FV}_{\mathbb{R}}(T)$ is the minimum mass of a $d{+}1$-chain with boundary T and real coefficients, then is $\mathrm{FV}_{\mathbb{R}}(T)/\mathrm{FV}(T)$ bounded below?

There are several related questions in geometric measure theory about the relationship between real chains, integral chains, and mod-2 chains, several of which were studied by Almgren. For instance, is the integral flat norm of a chain bounded in terms of its real flat norm? Is every normal mod-2 current equivalent to a normal integral current? Generalizing Theorem 2 to the context of currents rather than cellular cycles might help answer some of these questions.

References

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Boundary Behavior in Mean Curvature Flow

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Consider a compact variety $M \subset \mathbb{R}^3$ that has a connected boundary, near which M is a smooth manifold-with-boundary. Thus if ϵ is sufficiently small, then

$$\{x \in M : \text{dist}\,(x, \partial M) \le \epsilon\}$$

is an annulus. We let $L(M)$ denote the mod 2 linking number of the two boundary components of that annulus. For example, if M is an embedded disk, then $L(M) = 0$, and if M is an embedded Möbius Strip, then $L(M) = 1$.

Theorem 1. *There exists a smooth, simple closed curve* $\Gamma \subset \mathbb{R}^3$ *with total curvature* $< 4\pi$ *and a mean curvature flow:*

$$t \in \mathbb{R} \mapsto M(t) \subset \mathbb{R}^3$$

with the following properties:

(1) At all times, $\partial M(t) = \Gamma$.
(2) At almost all times, $M(t)$ *is a compact, smoothly embedded surface.*
(3) As $t \to -\infty$, $M(t)$ *converges smoothly to a compact, smoothly embedded minimal surface* $M_{-\infty}$ *with* $L(M_{-\infty}) = 1$.
(4) As $t \to \infty$, $M(t)$ *converges smoothly to a compact, smoothly embedded minimal surface* M_{∞} *with* $L(M_{\infty}) = 0$.

Theorem 2. *The flow in Theorem 1 must have one or more boundary singularities. At such a singularity, the tangent flow is given by a smoothly embedded, nonorientable, self-similarly shrinking surface* $\Sigma \subset \mathbb{R}^3$ *whose boundary is a straight line through the origin.*

Theorem 2 makes no assertion as to whether the flow also has interior singularities.

Sketch of proof of Theorem 2. By Theorem 1, there are times t_0 and t_1 such that for all $t < t_0$, the surface $M(t)$ is isotopic to $M_{-\infty}$, and for all $t > t_1$, the surface $M(t)$ is isotopic to M_∞. Since $M_{-\infty}$ and M_∞ are topologically different, one or more singularities must occur in the time interval $[t_0, t_1]$. Moreover, because $L(M_{-\infty}) \neq L(M_\infty)$, there must be at least one boundary singularity.

(Note that no surgeries away from the boundary on a surface M can change the linking number $L(M)$.)

The tangent flow at a boundary singularity is given by a self-similar shrinker Σ with straight line boundary through the origin.

One can show that monotonicity and the total curvature bound on Γ imply that the Gaussian density of the flow is < 2 at each interior point, and that the Gaussian density is $< \frac{3}{2}$ at each boundary point. That in turn implies (with a little work) that Σ is smoothly embedded.

Because the flow has a singularity at the space-time point in question, Σ cannot be a half-plane. Non-orientability of Σ then follows from Theorem 3 below. □

Remark. *I believe the method of proof used to show Theorem 1 actually produces an example in which $M_{-\infty}$ is a Möbius Strip and M_∞ is a disk. I expect such a flow to have a boundary singularity at which the tangent flow Σ is a self-similarly shrinking Möbius Strip (with straight line boundary).*

Theorems 1 and 2 are in sharp contrast to the following theorems:

Theorem 3. *Let Σ be a smoothly embedded, orientable, m-dimensional manifold-with-boundary in $\mathbb{R}^{m+1}$ such that $\partial\Sigma$ is an $(m-1)$-dimensional linear subspace and such that Σ is a self-similar shrinker under mean curvature flow. Then Σ is a half-plane.*

Theorem 4. *Let N be a mean convex, smooth, compact $(m+1)$-dimensional Riemannian-with-boundary (such as a ball in $\mathbb{R}^{m+1}$). Let $M \subset N$ be a smoothly embedded, compact m-dimensional manifold-with-boundary such that $\Gamma := \partial M \subset \partial N$. Let $t \in [0,\infty) \mapsto M(t)$ be a mean curvature flow, as constructed by elliptic regularization, with $M(0) = M$ and with $\partial M(t) \equiv \Gamma$ for all t.*

Then $M(t)$ is smooth near Γ at all times.

In other words, although the flow may have interior singularities, it can never have boundary singularities.

Remark. *Though mean curvature flow is unique up until the first singularity, it might not be afterwards. In case of non-uniqueness, Theorem 4 does not rule out some of the flows having boundary singularities. Rather, it asserts that there is a natural class of flows (including those produced by elliptic regularization) for which boundary singularities do not occur.*

These theorems are from a work in preparation. Theorems 3 and 4 are inspired by the work of Hardt and Simon on boundary regularity for area-minimizing integral currents in codimension 1 (Annals of Math. **110** (1979), 439–486).

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