Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2014

DOI: 10.4171/OWR/2014/34

Real Analysis, Harmonic Analysis and Applications

Organised by
Michael Christ, Berkeley
Detlef Müller, Kiel
Christoph Thiele, Bonn

20 July – 26 July 2014

Abstract. The workshop has focused on important developments within the last few years in the point of view and methods of real and harmonic Analysis as well as significant concurrent progress in the application of these to various other fields.

Mathematics Subject Classification (2010): 42B20.

Introduction by the Organisers

This workshop, which continued the triennial series at Oberwolfach on Real and Harmonic Analysis that started in 1986, has brought together experts and young scientists working in harmonic analysis and its applications (such as nonlinear dispersive and elliptic PDE, number theory, geometric measure theory) with the objective of furthering the important interactions between these fields.

Major areas and results represented at the workshop are:

- Fourier restriction theorems and Strichartz estimates.
- The study of sharp constant estimates for classical inequalities such as Hausdorff Young inequalities or restriction inequalities has lead to approaches to these inequalities quite different from those merely aiming at existence proofs for constants. These methods include group theoretic methods and special functions, variational methods, and very fine geometric arguments.
- Uniformity questions for oscillatory integrals play a role in various contexts in mathematics, the workshop has seen a discussion of progress on uniform lower bounds on Bergman kernels, a coordinate free approach to uniform

oscillatory estimates, and uniform bounds for Fourier restriction operators on polynomial curves. Uniformity in the dimension of maximal operator bounds is a related area of recent interest.
- Discrete analogues of results in harmonic analysis and connections to number theory.
- Multilinear singular integral theory in several dimensions studies operators formed by integrating a multidimensional singular kernel against a product of functions, each of these functions factoring through its own low dimensional projection of the integration domain. The behavior of the multilinear singular integral depends critically on the dimensionality and relative position of these projections, leading to a variety of interesting phenomena. These phenomena are only very partially understood, recent progress has been on various generalizations of commutator estimates and on very singular operators of entangled type.
- Analysis on spaces of Carnot-Caratheodory type, estimates on the Heisenberg group.
- A number of presentations of the workshop discussed various applications of harmonic analysis to PDE and further ares of mathematics. Applications included the magnitude of balls arising in category theory, calculated by solving a PDE, weighted integrability of polyharmonic functions, and a uniqueness theorem of Holmgren, and a Paley Wiener type theorem for Schrödinger evolutions, and inverse spectral theory for unbounded domains, and a discussion of minimal surfaces and sets.
- The interplay between martingale methods and harmonic analysis, for example to obtain sharp weighted estimates on singular integrals, including the recent progress on characterizing the two weight bounds for the Hilbert transform by testing conditions. The methods are also applicable in the study of questions in geometric measure theory, which require understanding of singular integral theory in very hostile environments such as spaces not of homogeneous type.

The meeting took place in a lively and active atmosphere, and greatly benefited from the ideal environment at Oberwolfach. It was attended by 53 participants. The program consisted of 28 lectures of 40 minutes. The organizers made an effort to include young mathematicians, and greatly appreciate the support through the Oberwolfach Leibniz Graduate Students Program, which allowed to invite several outstanding young scientists.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: Real Analysis, Harmonic Analysis and Applications

Table of Contents

Abstracts

Maximal functions: Boundedness and dimensions

Jesús Munárriz Aldaz

The centered Hardy-Littlewood maximal operator M has turned out to be a tool of considerable interest in the fields of Real and Harmonic Analysis. This is mainly due to the fact that $|f| \le Mf$ a.e. (so Mf is larger than $|f|$) but Mf is not much larger than $|f|$, as M satisfies the strong type (p,p) inequality $\|Mf\|_p \le C_p \|f\|_p$ for $1 < p \le \infty$. For $p = 1$, M satisfies instead the weak type $(1,1)$ inequality $\sup_{\alpha>0} \alpha\mu(\{M_\mu f \ge \alpha\}) \le c_1 \|f\|_1$.

Several authors have studied the behavior of the constants C_p, as different ingredients of the operator are changed (for instance, the measure, the dimension d of the underlying euclidean space $\mathbb{R}^d$, the sets over which averages are taken, etc.).

Trivially, $\|Mf\|_\infty \le \|f\|_\infty$ since averages never exceed a supremum, so $C_\infty = 1$. But when $1 < p < \infty$, how does C_p depend, for instance, on the dimension of $\mathbb{R}^d$? E. M. Stein showed that for the centered Hardy-Littlewood maximal operator associated with euclidean balls and Lebesgue measure in $\mathbb{R}^d$, C_p could be taken to be independent of d (cf. for instance, [19], [20]). A motivation for the study of L^p bounds that are uniform in d, comes from the interest in extending harmonic analysis from $\mathbb{R}^d$ to the infinite dimensional setting. Also, since the maximal operator appears often in chains of inequalities, better bounds for the latter operator lead to improvements in several other inequalities. Finally, it is interesting to have a better understanding of the operator. For instance, the doubling constant of Lebesgue measure in dimension d, plays no role in the optimal C_p bounds. In fact, for $p \ge 2$, P. Auscher and M. J. Carro gave the explicit (and surprisingly small) bound $C_p \le (2 + \sqrt{2})^{2/p}$ ([5]).

Stein's L^p result was generalized to the centered maximal function defined using arbitrary (norm) balls by J. Bourgain [6], [7], and A. Carbery [9], when $p > 3/2$, with bounds that were not only independent of the dimension, but also of the balls being used. For ℓ_q balls, $1 \le q < \infty$, D. Müller [17] has shown that uniform bounds hold for every $p > 1$ (given $1 \le q < \infty$, the ℓ_q balls are defined using the norm $\|x\|_q := (x_1^q + x_2^q + \cdots + x_d^q)^{1/q}$). However, these bounds did depend on q and diverged to ∞ as $q \to \infty$. In particular, the question whether the maximal operator associated to cubes (ℓ_∞ balls) is bounded uniformly in d for $1 < p \le 3/2$, remained open until 2012, when Bourgain provided a positive answer ([8]). Thus, while the two most important cases, cubes and euclidean balls, are both settled, it would still be interesting to prove the general case (constants with bounds independent of both the balls and the dimensions, for all $p > 1$). From the point of view of a better understanding of the operator, and of possible extensions to the metric space setting, it would be worthwhile to obtain proofs that do not depend on the particular shape of the balls.

Denote by $c_{1,d}$ the optimal (that is, lowest) weak type $(1,1)$ bound in dimension d, and by $C_{p,d}$ the optimal strong type (p,p) bound, $p > 1$. In [20], a paper that has had considerable influence in latter developments, E. M. Stein and J.-O. Strömberg proved that the best constants $c_{1,d}$ grow at most like $O(d)$ for euclidean balls (by semigroup methods, utilizing the maximal ergodic theorem) and like $O(d \log d)$ for general balls (by a very astute covering theorem). Stein and Strömberg also asked wheter the constants $c_{1,d}$ for euclidean balls, were bounded independently of d. While this question remains open, in the case of cubes it was shown by me that the answer is negative ([1]). Explicit lower bounds were later given by G. Aubrun in [4], and more recenty, it has been shown by A.S. Iakovlev and J.-O. Strömberg that $c_{1,d} \geq \Theta(d^{1/4})$ (cf. [12]). Of course, the higher these bounds are, the more unlikely it seems that uniform bounds exist for euclidean balls.

These questions have also been explored for measures different from Lebesgue measure in $\mathbb{R}^d$, and for measures in more general spaces. We sumarize a few recent developments, skipping over some intermediate results and without stating the most general versions.

For euclidean balls, $p > 1$, and certain doubling measures that include $d\mu_t := \|x\|_2^{-t} dx$ on $\mathbb{R}^d$ (fixed $t > 0$, $d > t$), $\sup_d C_{p,d} < \infty$ ([11]). However, for the doubling measures $d\mu_{t,d} := \|x\|_2^{-td} dx$ on $\mathbb{R}^d$, $t \in (1/2, 1)$ and all $p < \infty$, the weak type (p,p) constants $c_{p,d}$ grow exponentially in d ([3]). If we consider measures "like the gaussian", that is, defined by finite, rotationally invariant, radially decreasing densities, then the weak type (p,p) constants $c_{p,d}$ grow exponentially with d, whenever $1 \leq p < 1.037$ ([2]). And in the specific case of the gaussian, $c_{p,d}$ increases exponentially with d, for all $p < \infty$ ([10]).

Finally, in the more general settings of manifolds, and of metric measure spaces, we mention the following results:

Regarding volume in d-dimensional hyperbolic space, with geodesic balls, $c_{1,d} \leq O(d \log d)$ ([16]), so the Stein-Strömberg bound reappears once more, and for $p > 1$, $\sup_d C_{p,d} < \infty$ ([15]). These results are remarkable, as the doubling property is completely missing in this context.

With respect to volume in the $d-1$-dimensional euclidean unit sphere, and geodesic balls, $c_{1,d} \leq O(d)$ ([13], [14]). For $p > 1$, $\sup_d C_{p,d} < \infty$ is to be expected, but nobody has proven it yet.

As for metric measure spaces, Naor and Tao ([18]) show that the Stein-Strömberg bound $c_{1,d} \leq O(d \log d)$ holds for metric measure spaces that satisfy the "Strong Microdoubling Condition" (ex., Ahlfors-David regular spaces). Such notion abstracts a key part of the proof of the Stein-Strömberg covering theorem mentioned above. Furthermore, in this context, the $\Theta(d \log d)$ bounds are optimal.

References

[1] J.M. Aldaz, *The weak type* $(1,1)$ *bounds for the maximal function associated to cubes grow to infinity with the dimension*, Ann. of Math. (2) **173** (2011), no. 2, 1013–1023.

[2] J.M. Aldaz and J. Pérez Lázaro, *Behavior of weak type bounds for high dimensional maximal operators defined by certain radial measures*, Positivity **15** (2011), 199–213.

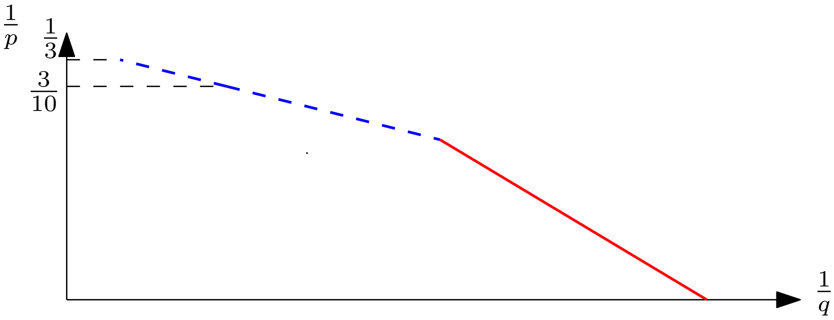
[3] J.M. Aldaz and J. Pérez Lázaro, *On high dimensional maximal operators*, Banach J. Math. Anal. 7 (2013) **7** (2013), 225–243.
[4] G. Aubrun, *Maximal inequality for high-dimensional cubes*, Confluentes Math. **1** (2009), no. 2, 169–179.
[5] P. Auscher and M.J. Carro, *Transference for radial multipliers and dimension free estimates*, Trans. Amer. Math. Soc. **342** (1994), no. 5, 575–593.
[6] J. Bourgain, *On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math. **108** (1986), no. 6, 1467–1476.
[7] J. Bourgain, *On the L^p-bounds for maximal functions associated to convex bodies in R^n*. Israel J. Math. **54** (1986), no. 3, 257–265.
[8] J. Bourgain, *On the Hardy-Littlewood maximal function for the cube*, arXiv:1212.2661.
[9] A. Carbery, *An almost-orthogonality principle with applications to maximal functions associated to convex bodies*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 269–273.
[10] A. Criado and P. Sjögren, *Bounds for maximal functions associated with rotational invariant measures in high dimensions*, J. Geom. Anal. **24** (2014), 595–612.
[11] A. Criado and F. Soria, *Localization and dimension free estimates for maximal functions*, J. Funct. Anal. 265 (2013), no. 10, 2553–2583. **265** (2013), 2553–2583.
[12] A.S. Iakovlev and J.-O. Strömberg, *Lower bounds for the weak type* (1, 1) *estimate for the maximal function associated to cubes in high dimensions*, Math. Res. Letters, to appear.
[13] P.M. Knopf, *Maximal functions on the unit n-sphere*, Pacific J. Math. **129** (1987), 77–84.
[14] H.-Q. Li, *Remark on "Maximal functions on the unit n-sphere" by Peter M. Knopf (1987)*, Pacific J. Math. **263** (2013), 253–256.
[15] H.-Q. Li, *Centered Hardy-Littlewood maximal function on hyperbolic spaces, $p > 1$*, (2013), arXiv:1304.3261.
[16] H.-Q. Li and N. Lohoué, *Fonction maximale centrées de Hardy-Littlewood sur les espaces hyperboliques*, Ark. för Mat. 50 (2012), no. 2, 359–378.
[17] D. Müller, *A geometric bound for maximal functions associated to convex bodies*, Pacific J. Math. **142** (1990), no. 2, 297–312.
[18] A. Naor and T. Tao, *Random martingales and localization of maximal inequalities*, J. Funct. Anal. **259** (2010), no. 3, 731–779.
[19] E.M. Stein, *The development of square functions in the work of A. Zygmund. Bull. Amer. Math. Soc*, (N.S.) **7** (1982), no. 2, 359–376.
[20] E.M. Stein and J.O. Strömberg, *Behavior of maximal functions in R^n for large n*, Ark. Mat. **21** (1983), no. 2, 259–269.

A Fourier restriction estimate for a surface of finite type

Stefan Buschenhenke
(joint work with Detlef Müller & Ana Vargas)

During the last decades, restrictions of the Fourier transform were studied mainly for surfaces with non-vanishing curvatures, or a certain number of non-vanishing curvatures. We present a new result for a class of surfaces S where the principle curvatures are allowed to vanish, but which still fulfill a so-called finite type condition. A model case is the surface $S = \{(x_1, x_2, x_1^{m_1} + x_2^{m_2}) | 0 \leq x_1, x_2 \leq 1\}$, $m_1 \geq m_2 \geq 2$. Our result for the extension operator $R^*f = \widehat{f d\sigma}$, the dual of the restriction operator, is as follows:

Theorem. *The extension operator $R^* : L^{q,p}(S, \sigma) \to L^p(\mathbb{R}^3)$ is bounded if $\frac{1}{q'} \geq \frac{h+1}{p}$, $p > \max\{\frac{10}{3}, h+1\}$ and $\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2}$.*



The Lorentz space $L^{q,p}(S,\sigma)$ can be improved to $L^q(S,\sigma)$ except the part of the critical line $\frac{1}{q'} = \frac{h+1}{p}$ where $q > p$. Here, the strong type estimate fails.
The condition

$$\frac{1}{q'} \geq \frac{h+1}{p} \tag{SB 1}$$

is necessary, as can be seen by a classical Knapp box example. $p > h+1$ is necessary as well, but

$$p > \frac{10}{3} \tag{SB 2}$$

is not sharp. The corresponding necessary condition is $p > 3$, but notice that the best known result on the sharp line for the fully curved case, the paraboloid (recovered by taking $m_1 = 2 = m_2$), is for $p > \frac{10}{3}$ due to Tao's bilinear approach [SB1]. Since we use bilinear techniques as well, it is clear that we cannot exceed this range.
A surprising new feature is the condition

$$\frac{1}{q} + \frac{2m_1+1}{p} < \frac{m_1+2}{2} \tag{SB 3}$$

which is neither a threshold depending only on p, nor a Knapp Box example, but which is sharp as well. Such a condition, corresponding to the dashed line with the slope in the picture, was never encountered before in Fourier restriction theory. This is related to the restriction of the surfaces $S(K) = \{(x,|x|^2) \in \mathbb{R}^3 : |x_1| \leq 1, |x_2| \leq K\}$. Although the paraboloid has been extensively studied, we are not aware of any quantitative results with respect to the dependence on the parameter $K \gg 1$.
Observe that the situation may differ from that one shown in the picture, depending on the choice of powers m, l. Our new condition (SB 3) will only show up in cases of anisotropic surfaces, i.e. $m_1 \gg m_2$. For instance, if $m_1 = 2 = m_2$, the conditions (SB 1) and (SB 2) are stronger than (SB 3). On the other hand, if $m_1 \gg m_2 = 2$ the condition is non-trivial. Heuristically speaking, there has to be some condition: If $m_2 = 2$, as $m_1 \to \infty$, the surface approximates a cylinder, whereas (SB 3) becomes $p > 4$ at $m_1 = \infty$, which is a well known necessary condition for the parabola/cylinder.
In some cases (SB 2) and (SB 3) are stronger than (SB 1), namely if the exponents are "large". Observe that in this cases, our result is sharp.
Methodically, we follow the bilinear approach developed by Bourgain, Wolff, Tao

and others. Since the surface has non-constant curvatures, we have to develop modified, more quantitative bilinear techniques, taking into account the size of the local curvature.

References

[SB1] T. Tao, *A Sharp bilinear restriction estimate for paraboloids*, Geometric and Functional Analysis **13** (2003), 1359-1384.

Magnitudes of Balls – an Application of Analysis to the Theory of Enriched Categories

Anthony Carbery
(joint work with Juan Antonio Barceló)

This is part of a much bigger project – of Tom Leinster and Mark Meckes and also Simon Willerton – to understand the notion of magnitude or Euler characteristic in a category-theoretic setting. To every finite category one can associate an invariant called its Euler characteristic which encapsulates its important geometric information. The class of finite categories is contained in the class of finite **enriched categories** and to each of these we can also associate an Euler characteristic. The class of **metric spaces** is also contained in a natural way in the class of enriched categories and so each finite metric space has a numerical invariant, its Euler characteristic, associated to it. Since metric spaces will generally also have a *topological* Euler characteristic, the term **magnitude** is used to refer to the category-theoretic Euler characteristic.

Let (X, d) be a finite metric space. A **weighting** for X is a function $w : X \to \mathbb{R}$ such that $\sum_{y \in X} \exp(-d(x, y))w(y) = 1$ for all $x \in X$. If a finite metric space X has a weighting w, the **magnitude** of X, denoted $|X|$, is given by $|X| := \sum_{x \in X} w(x)$. In view of a lack of homogeneity when we replace the metric d by td for $t > 0$, it is beneficial to study the *family* of metric spaces $tX := (X, td)$ and their corresponding magnitudes $|tX|$. There exist finite metric spaces whose "magnitude spectrum" is quite wild. For example, if $K_{3,2}$ is the bipartite graph on $3 + 2$ vertices, then $|tK_{3,2}|$ is 1 at $t = 0$, approaches 5 from below as $t \to \infty$ but has a vertical asymptote at $t = \log \sqrt{2}$. However, if X is a finite subset of a euclidean space magnitude is better-behaved; for example it is nonnegative and monotonic.

Let (X, d) be a compact subset of a euclidean space. We define $|X| = \sup\{|A| : A \subseteq X,\ A \text{ finite}\}$.

Proposition[M. Meckes] If there exists a finite signed Borel measure μ on X such that for all $x \in X$

$$\int \exp(-d(x, y))\mathrm{d}\mu(y) = 1$$

then $|X| = \mu(X)$.
Such a measure is called a **weight measure** for X.

Let $X = [-R, R] \subseteq \mathbb{R}$ with the usual metric. One simply checks (using high-school integration by parts) that $\frac{1}{2}(\delta_{-R} + \delta_R + t\lambda|_{[-R,R]})$ is a weight measure for tX where λ is Lebesgue measure. Thus $|tX| = tR + 1$. This is the only example of a compact convex set in euclidean space whose magnitude was (hitherto) known. Motivated by a similar result in the cateogory-theoretic world, the case $n = 1$, and numerical evidence, Leinster and Willerton conjectured that if $X \subseteq \mathbb{R}^n$ is compact and convex, then $t \mapsto |tX|$ is a polynomial of degree n and moreover

$$|tX| = \frac{\mathrm{Vol}(X)}{n!\omega_n}t^n + \frac{\mathrm{Surf}(X)}{2(n-1)!\omega_{n-1}}t^{n-1} + \cdots + 1 = \sum_{i=0}^{n} \frac{1}{i!\omega_i} V_i(X)t^i$$

where ω_i is the volume of the unit ball in $\mathbb{R}^i$ and $V_i(X)$ is the i'th intrinsic volume of X. It is easy to predict the top and bottom terms $\lim_{t\to 0} |tX| = 1$ and $\lim_{t\to\infty} t^{-n}|tX| = \frac{\mathrm{Vol}(X)}{\int_{\mathbb{R}^n} e^{-|x|}\mathrm{d}x} = \frac{\mathrm{Vol}(X)}{n!\omega_n}$. The others are more mysterious.

To fix ideas, the Leinster–Willerton conjecture predicts that for the ball B_R of radius R in $\mathbb{R}^n$ we will have $|B_R| =$

$$\begin{aligned}
n = 1 &: R + 1\\
n = 2 &: \frac{R^2}{2} + \frac{\pi R}{2} + 1\\
n = 3 &: \frac{R^3}{6} + R^2 + 2R + 1\\
n = 4 &: \frac{R^4}{24} + \frac{\pi R^3}{8} + \frac{3R^2}{2} + \frac{3\pi R}{4} + 1\\
n = 5 &: \frac{R^5}{120} + \frac{R^4}{9} + \frac{2R^3}{3} + 2R^2 + \frac{8R}{3} + 1\\
n = 6 &: \frac{R^6}{720} + \frac{\pi R^5}{128} + \frac{5R^4}{24} + \frac{5\pi R^3}{16} + \frac{5R^2}{2} + \frac{15\pi R}{16} + 1\\
n = 7 &: \frac{R^7}{5040} + \frac{R^6}{225} + \frac{R^5}{20} + \frac{R^4}{3} + \frac{4R^3}{3} + 3R^2 + \frac{16R}{5} + 1
\end{aligned}$$

etc.

There is a strong analogy between what we are addressing and classical potential theory. Indeed, a variant of our problem is to calculate $\sup\{\mu(X) : \int_X e^{-|x-y|}\mathrm{d}\mu(y) \leq 1 \text{ on } X\}$ where the sup is taken over all finite Borel measures supported in X. Compare this with the classical Newtonian capacity ($n \geq 3$ case) $\mathrm{Cap}(X) = \sup\{\mu(X) : \int_X \frac{\mathrm{d}\mu(y)}{|x-y|^{n-2}} \leq 1 \text{ on } X\}$ – which can also be calculated as the energy integral $C_n \inf\left\{\int_{\mathbb{R}^n} |\nabla h|^2 : h \in \dot{H}^1(\mathbb{R}^n),\ h \equiv 1 \text{ on } X\right\}$. Similarly:

Theorem[M. Meckes] If $X \subseteq \mathbb{R}^n$ is compact, then

$$|X| = \frac{1}{n!\omega_n} \inf\left\{\|h\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} : h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X\right\},$$

and moreover there exists a unique extremiser.

The Euler–Lagrange equation for the extremal problem is, unsurprisingly,

$$(I-\Delta)^{(n+1)/2}h = 0 \text{ on } X^c$$

(in the weak sense, testing against functions in $\mathcal{C}_c^\infty(X^c)$). So a related PDE problem is

$$(I-\Delta)^{(n+1)/2}h = 0 \text{ weakly on } X^c$$
$$h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X.$$

Note that this is really a PDE problem only when n is odd. It is not a standard BVP – more a mixed BVP/extension problem of higher-order in an exterior domain. Results in the literature tend to deal with classical BVP for $(-\Delta)^m$ for $m = 1, 2, \ldots$; there is no "off the shelf" theory available to handle this equation.

Consider the problem

$$(I-\Delta)^{m}h = 0 \text{ weakly on } X^c$$
$$h \in H^{m}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X.$$

Theorem[JAB and AC] If $m \in \mathbb{N}$ and $X \subseteq \mathbb{R}^n$ is convex and compact, then there is a unique solution h to this problem which moreover satisfies

$$\|h\|^2_{H^m(\mathbb{R}^n)} = \mathrm{Vol}(X) - \sum_{\frac{m}{2}<j\le m} (-1)^j \binom{m}{j} \int_{\partial X_+} \frac{\partial}{\partial \nu}\Delta^{j-1} h \mathrm{d}S.$$

Here, ν is the unit normal pointing into X and $\int_{\partial X_+}$ is a limit of integrals taken over $\partial(rX)$ as $r \downarrow 1$. The (standard) techniques in the proof include integration by parts, Hilbert space methods, elliptic regularity, weak*-compactness and expoliting lots of cancellation.

So the game in odd dimensions is now to find the unique solution to

$$(I-\Delta)^{(n+1)/2}h = 0 \text{ weakly on } X^c$$
$$h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X$$

and then calculate the quantities

$$\mathcal{A}_j(X) = \int_{\partial X_+} \frac{\partial}{\partial \nu}\Delta^{j-1} h \mathrm{d}S.$$

The magnitude of the compact convex X for n odd will then be given by

$$|X| = \frac{1}{n!\omega_n}\left(\mathrm{Vol}(X) - \sum_{\frac{n+1}{4}<j\le\frac{n+1}{2}} (-1)^j \binom{\frac{n+1}{2}}{j} \mathcal{A}_j(X)\right).$$

When X is a ball we can work in polar coordinates to reduce matters to ODEs, and make explicit calculations. The upshot is:

Theorem[JAB and AC] The magnitude of the ball of radius R in $\mathbb{R}^n$ is:

$$n = 1 : R + 1$$
$$n = 3 : \frac{R^3}{6} + R^2 + 2R + 1$$
$$n = 5 : \frac{R^5}{120} + \frac{R^4}{9} + \frac{2R^3}{3} + 2R^2 + \frac{8R}{3} + 1$$
$$n = 7 : \frac{R^7}{5040} + \frac{R^6}{225} + \frac{R^5}{20} + \frac{R^4}{3} + \frac{4R^3}{3} + 3R^2 + \frac{16R}{5} + 1$$

etc., thus verifying the Leinster–Willerton conjecture in odd dimensions.

Subharmonicity and the regularity of maximal operators

Emanuel Carneiro
(joint work with Benar F. Svaiter)

Background. Let $\varphi \in L^1(\mathbb{R}^d)$ be a nonnegative function such that

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = 1.$$

We let $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$ be the usual approximations of the identity and consider the associated maximal operator M_φ given by

$$M_\varphi f(x) = \sup_{t>0} \big(|f| * \varphi_t\big)(x).$$

The centered Hardy-Littlewood maximal function, henceforth denoted by M, occurs when we consider $\varphi(x) = (1/m(B_1))\chi_{B_1}(x)$, where B_1 is the d-dimensional ball centered at the origin with radius 1 and $m(B_1)$ is its Lebesgue measure. If our φ admits a radial non-increasing majorant in $L^1(\mathbb{R}^d)$ with integral A, then a classical result of Stein gives

$$M_\varphi f(x) \leq A\, Mf(x)$$

for all $x \in \mathbb{R}^d$, and thus we obtain the boundedness of M_φ from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ if $p > 1$, and from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ if $p = 1$.

In this talk we are interested in the following question: *does a maximal operator of convolution type M_φ increase the variation of a function?* We use the word *variation* here to mean either the classical total variation for one-dimensional functions or, more generally, the L^p-norm of the gradient for some $p \geq 1$.

The first result in this direction was obtained by J. Kinnunen [4], who proved that M maps $W^{1,p}(\mathbb{R}^d)$ into $W^{1,p}(\mathbb{R}^d)$ boundedly, for $p > 1$. In fact, Kinnunen showed that if $f \in W^{1,p}(\mathbb{R}^d)$, then Mf has a weak derivative and the pointwise estimate

$$|\nabla Mf(x)| \leq M|\nabla f|(x)$$

holds almost everywhere. In particular, this implies that

$$\|\nabla Mf\|_p \leq C\|\nabla f\|_p \tag{1}$$

for $p > 1$ and a certain constant $C > 1$. His proof relies only on the convolution nature of the operator and on functional analysis tools for the reflexive space $W^{1,p}(\mathbb{R}^d)$, $p > 1$, and can easily be extended to M_φ. In particular, one obtains

$$\|\nabla M_\varphi f\|_p \leq C\|\nabla f\|_p$$

for $p > 1$ and a certain constant $C > 1$. When $p = 1$ the situation is more delicate, since the maximal function does not map into $L^1(\mathbb{R}^d)$. In [3], Hajłasz and Onninen asked the following question: is the operator $f \mapsto \nabla Mf$ bounded from $W^{1,1}(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$? By dilation considerations, this comes down to prove that Mf has a weak derivative and that

$$\|\nabla Mf\|_1 \leq C\|\nabla f\|_1. \tag{2}$$

There has been some partial progress on the question (2) posed above, but restricted only to dimension $d = 1$. For the non-centered Hardy-Littlewood maximal operator $\widetilde{M}$, Tanaka [6] verified (2) with constant $C = 2$. This was later improved by Aldaz and Pérez-Lázaro [1], who obtained (2) with constant $C = 1$. They showed that if a function $f : \mathbb{R} \to \mathbb{R}$ has bounded variation, then

$$V(\widetilde{M}f) \leq V(f), \tag{3}$$

where $V(f)$ denotes the total variation of f. Inequality (3) is easily seen to be sharp. For the centered maximal function M, Kurka [5] recently showed that (2) holds with constant $C > 1$ (in fact his proof gives roughly $C \sim 240,000$), and also showed that

$$V(Mf) \leq C\,V(f), \tag{4}$$

for the same $C > 1$.

Main results. We prove estimates like (1), (2) and (4), with constant $C = 1$, for maximal operators of convolution type related to two kernels of interest: the Gauss (heat) kernel and the Poisson kernel. For this, let

$$\varphi_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} \tag{5}$$

be the heat kernel (note the harmless change of variables $t \mapsto \sqrt{t}$ with respect the previous notation) or

$$\varphi_t(x) = c_d\, \frac{t}{(|x|^2 + t^2)^{(d+1)/2}}, \tag{6}$$

be the Poisson kernel for the upper half-space, where $c_d = \Gamma\left(\frac{d+1}{2}\right)\pi^{-(d+1)/2}$. Given $u_0 \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, we define $u : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ by

$$u(x,t) = (|u_0| * \varphi_t)(x).$$

Note that $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and solves either: (i) the heat equation with initial datum $|u_0|$, when φ_t is given by (5); or (ii) Laplace's equation with initial datum $|u_0|$, when φ_t is given by (6). To simplify notation we write

$$u^*(x) = M_\varphi u_0(x) = \sup_{t>0} (|u_0| * \varphi_t)(x) \tag{7}$$

for the maximal function. The following result was obtained in [2, Theorems 1 and 2].

Theorem 1. Let u^* be the heat flow maximal function or the Poisson maximal function defined in (7). The following propositions hold.

(i) Let $1 < p \leq \infty$ and $u_0 \in W^{1,p}(\mathbb{R})$. Then $u^* \in W^{1,p}(\mathbb{R})$ and

$$\|(u^*)'\|_p \leq \|u_0'\|_p.$$

(ii) Let $u_0 \in W^{1,1}(\mathbb{R})$. Then $u^* \in L^\infty(\mathbb{R})$ and has a weak derivative $(u^*)'$ that satisfies

$$\|(u^*)'\|_1 \leq \|u_0'\|_1.$$

(iii) Let u_0 be of bounded variation on $\mathbb{R}$. Then u^* is of bounded variation on $\mathbb{R}$ and

$$V(u^*) \leq V(u_0).$$

(iv) Let $d > 1$ and $u_0 \in W^{1,p}(\mathbb{R}^d)$, for $p = 2$ or $p = \infty$. Then $u^* \in W^{1,p}(\mathbb{R}^d)$ and

$$\|\nabla u^*\|_p \leq \|\nabla u_0\|_p.$$

Discrete analogues of this result were obtained in [2, Theorems 3 and 4]. The proof relies on a nice interplay between the analysis of maximal functions and qualitative properties of the underlying partial differential equations (heat equation and Laplace's equation). In particular, we use the maximum principles associated the these equations to establish the fundamental geometric lemma.

Lemma 2. Let $u_0 \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$ or u_0 be bounded and Lipschitz continuous. Then u^* is subharmonic in the open set $A = \{x \in \mathbb{R}^d;\, u^*(x) > u_0(x)\}$.

References

[1] J. M. Aldaz and J. Pérez-Lázaro, *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2443–2461.

[2] E. Carneiro and B. F. Svaiter, *On the regularity of maximal operators of convolution type*, J. Funct. Anal. **265** (2013), no.5, 837–865.

[3] P. Hajłasz and J. Onninen, *On boundedness of maximal functions in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 167–176.

[4] J. Kinnunen, *The Hardy-Littlewood maximal function of a Sobolev function*, Israel J. Math. **100** (1997), 117–124.

[5] O. Kurka, *On the variation of the Hardy-Littlewood maximal function*, preprint at http://arxiv.org/abs/1210.0496.

[6] H. Tanaka, *A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function*, Bull. Austral. Math. Soc. **65** (2002), no. 2, 253–258.

Best constants for the local Hausdorff–Young inequality on compact Lie groups

Michael G. Cowling
(joint work with Alessio Martini, Detlef Müller and Javier Parcet)

This is an account of work in progress.

Given a function f on $\mathbb{R}^n$, we define its Fourier transform $\hat{f}$ on $\mathbb{R}^n$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi\cdot x}\, \mathrm{d}x.$$

Babenko (1961, for the case where $q \in 2\mathbb{Z}^+$) and Beckner (1975, for general q) proved the following result.

Theorem 1. *If $f \in L^p(\mathbb{R}^n)$ where $1 \le p \le 2$, then $\hat{f} \in L^q(\mathbb{R}^n)$ and*

$$\|\hat{f}\|_q \le B_p^n \|f\|_p\,,$$

where $1/q = 1 - 1/p$ and $B_p^2 = p^{1/p}/q^{1/q}$.

The extremal functions are Gaussians; once one knows this, computing the constants is easy.

We identify the n-torus $\mathbb{T}^n$ with $(-\frac{1}{2}, \frac{1}{2}]^n$, and define the Fourier transform of f on $\mathbb{T}^n$ by

$$\hat{f}(\xi) = \int_{(-\frac{1}{2},\frac{1}{2}]^n} f(x)\, e^{-2\pi i \xi\cdot x}\, \mathrm{d}x.$$

For the n-torus $\mathbb{T}^n$, consideration of the function 1 shows that the best that we can do is the classical inequality

$$\|\hat{f}\|_q \le \|f\|_p\,,$$

However, take $\delta < \frac{1}{2}$, and let $C_p(\mathbb{T}^n, \delta)$ be the best constant in the inequality

$$\|\hat{f}\|_q \le C\|f\|_p \qquad \operatorname{supp}(f) \subseteq (-\delta, \delta)^n.$$

Clearly $C_p(\mathbb{T}^n, \delta) \le 1$, and it is certainly possibly that it is strictly less than 1. Between 1993 and 1996, Andersson [1], Sjölin [4], and finally Kamaly [3] managed to prove the following theorem.

Theorem 2. *With the above notation,*

$$\lim_{\delta\to 0+} C_p(\mathbb{T}^n, \delta) = B_p(\mathbb{R}^n).$$

In 2003, García-Cuerva, Marco, and Parcet [2] proved a central local Hausdorff–Young type theorem for a compact Lie group G, and applied this to functional analytic questions about Banach spaces. In this context, the classical Hausdorff–Young theorem may be stated as follows.

$$\Big(\sum_{\lambda \in \Lambda^+} d_\lambda \left\| \pi_\lambda(f) \right\|_{C_q}^q \Big)^{1/q} \le C \left\| f \right\|_p . \tag{1}$$

Here the set Λ^+ (of dominant integral weights) parametrises the collection of irreducible unitary representations π_λ of G;

$$\pi_\lambda(f) = \int_G f(x)\, \pi_\lambda(x) \, \mathrm{d}x,$$

a linear operator on the finite-dimensional vector space of π_λ; and C_q denotes the Schatten space. Again, unless some support restriction is imposed, the best value for C is 1. García-Cuerva, Marco, and Parcet considered $C_p^c(U)$, the best constant in the inequality above, with two restrictions on f, namely, $\operatorname{supp}(f) \subset U$ and f is central, that is, $f(xy) = f(yx)$. They showed that $C_p^c(U) < 1$ for small U and $p \in (1, 2)$, but they did not find exact values (which is probably impossible in general) or the limit as U shrinks (which is possible).

Our theorem may be stated as follows.

Theorem 3. *With the above notation,*

$$\lim_{U \to \{e\}} C_p^c(U) = B_p(\mathbb{R}^{\dim(G)}).$$

We do not give the details of our proof here, but remark that there are at least two different proofs for $\mathbb{T}^n$, with different strengths and weakness, and only one of these generalises nicely to compact Lie groups. Given f in $\mathbb{T}^n$, we define F on $\mathbb{R}^n$ by

$$F(x) = \begin{cases} f(x) & \text{when } x \in (-\frac{1}{2}, \frac{1}{2}]^n \\ 0 & \text{otherwise.} \end{cases}$$

The proof uses the Hausdorff–Young theorem for F and transfers it to f. One of the proofs for $\mathbb{T}^n$ uses a formula related to the Shannon sampling theorem, while the other uses the Poisson summation formula to show that, with some appropriate restrictions on the support of F,

$$\sum_{m \in \mathbb{Z}^n} \hat{F} = \int_{\mathbb{R}^n} \hat{F}(\xi) \, \mathrm{d}\xi.$$

This result is what is needed to prove the theorem in the case where $q \in 2\mathbb{Z}^+$; the general case requires an additional interpolation argument—one that does not lose the sharp constant! Kirillov's orbit method, and in particular his character formula, may be applied to prove a version of the Poisson summation formula for a compact Lie group. In the case of SU(2), the formula shows that for certain functions with small support on the Lie algebra $\mathfrak{su}(2)$, the integral of the Fourier transform may also be expressed as a sum of integrals over spheres. With this modification, the second proof for $\mathbb{T}^n$ goes through.

References

[1] M. E. Andersson, *The Hausdorff–Young inequality and Fourier type*, Ph.D. Thesis, Uppsala (1993).

[2] José García-Cuerva, José Manuel Marco and Javier Parcet, *Sharp Fourier type and cotype with respect to compact semisimple Lie groups*, Trans. Amer. Math. Soc. **355** (2003), 3591–3609.

[3] A. Kamaly, *A new local variant of the Hausdorff-Young inequality.* International Symposium on Complex Analysis and Related Topics (1996), 18-22.

[4] P. Sjölin, A remark on the Hausdorff–Young inequality, Proc. Amer. Math. Soc. **123** (1995), 3085–3088.

A new monotonicity formula for minimal sets with a sliding boundary condition

Guy David

Let us define almost minimal sets of dimension d in R^n. Given a closed set E with locally compact Hausdorff measure H^d, we say that the functions $\varphi_t : E \to \mathbb{R}^n$, $0 \le t \le 1$, define a deformation of E in a ball $B(X, R)$ if the following properties are satisfied:

$$(t, x) \to \varphi_t(x) \text{ is a continuous mapping from } E \times [0, 1] \text{ to } R^n;$$
$$\varphi_t(x) = x \text{ for } t = 0 \text{ and for } x \in E \setminus B(X, R);$$
$$\varphi_t(E \cap B(X, R)) \subset B(X, R) \text{ for } 0 \le t \le 1;$$
$$\varphi_1 \text{ is Lipschitz.}$$

Essentially following Almgren [1] we say that E is almost minimal, with the gauge function h, when

$$H^d(E \cap B(X, R)) \le H^d(\varphi_1(E) \cap B(X, R)) + R^d h(R)$$

for every deformation $\{\varphi_t\}$ as above. Here h is a nondecreasing function of R that tends to 0 when R tends to 0; for instance, $h(R) = CR^\alpha$ for some choice of $C > 0$ and $\alpha > 0$ would do.

For the present lecture, in addition we are given a nice boundary set L, and we say that the deformation preserves the boundary when, in addition to the properties above,

$$\varphi_t(x) \in L \text{ when } x \in E \cap L \text{ and } 0 \le t \le 1.$$

The set E is said to be almost minimal minimal with a boundary condition given by L if the defining inequality above holds for all the deformations that satisfy the additional constraint.

This seems to be a nice (if seldom used) way to encode boundary constraints, as in the Plateau problem. Lots of variations on this definition are possible; for instance we could localize the notion to an open set, or use more than one boundary constraint, or account for almost minimality in slightly different ways. We refer to [2] for this, and for most of the regularity and limiting properties that the author knows. In particular, it is important to know that a (local Hausdorff) limit of almost minimal sets with a given boundary constraint coming from L is also (for reasonable sets L) almost minimal, in the same class. Also, monotonicity results are often useful.

The goal of the lecture was to present a new monotonicity formula for minimal sets, say, associated to a boundary set L which is a $(d-1)$-dimensional vector subspace.

Standard techniques (essentially, comparing E with the cone over $E \cap \partial B(0,R)$) show that the quantity

$$\theta(R) = R^{-d} H^d(E \cap B(0,R))$$

is nondecreasing. This fails when we take $x \in R^n \setminus L$ and consider

$$\theta_x(R) = R^{-d} H^d(E \cap B(x,R)),$$

for instance when $n = 3$, $d = 2$, L is a line, and E is a half plane bounded by L. What we show instead is that the quantity

$$\theta_x(R) + R^{-d} H^d(S \cap B(x,R))$$

is nondecreasing, where S denotes the shade of L seen from x, i.e., the set of points y such that $[x,y]$ meets L.

This allows us to give a rough description of E when it is very close to the simplest minimal cones (again, with a boundary condition coming from a $(d-1)$-plane), but for the moment we only get a very limited number of examples.

Hopefully the author will put out a preprint with much more detail within two or three months.

References

[1] F. J. Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Memoirs of the Amer. Math. Soc. 165, volume 4 (1976), i-199.

[2] G. David, *Local regularity properties of almost- and quasiminimal sets with a sliding boundary condition*, preprint, arXiv:1401.1179.

On an endpoint Strichartz estimate

Damiano Foschi

(joint work with Abdelhakim Mouhamed Ahmed Abdelsattar)

We consider the solutions $u(t,x)$ of the inhomogeneous Schrödinger equation with a forcing term which concentrates at one point in $\mathbb{R}^4$:

$$i\partial u + \Delta u = f(t)\delta(x), \quad u(0,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^4,$$

with the function $f(t)$ supported on the unit time interval $[0,1]$. We show that the estimate $\|u\|_{L^4([2,3]\times\mathbb{R}^4)} \leq C \|f\|_{L^4([0,1])}$ is "almost" true. More precisely [1]:

(1) We have the estimate $\|u\|_{L^4([2,3]\times\mathbb{R}^4)} \leq C(\log N)^{3/4} \|f\|_{L^4([0,1])}$, for all functions f which are piecewise constant on a partition of $[0,1]$ into N intervals of equal size $1/N$.

(2) For every $\varepsilon > 0$ we have $\left\|u \exp(-\varepsilon|x|^2)\right\|_{L^4([2,3]\times\mathbb{R}^4)} \leq C|\log\varepsilon|^{1/2} \|f\|_{L^4([0,1])}$.

This result is a first step to understand if the following endpoint estimate holds true for generic forcing terms:

$$\|u\|_{L^{2n/(n-2)}([2,3];L^n(\mathbb{R}^n))} \leq C\,\|F\|_{L^n([0,1];L^1(\mathbb{R}^n))}\,,$$

when $u(t,x)$ is the solution of

$$i\partial u + \Delta u = F(t,x), \quad u(0,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}^n,$$

and $F(t,x)$ is supported on $[0,1] \times \mathbb{R}^n$. This is still an open problem [2].

References

[1] Abdelhakim Mouhamed Ahmed Abdelsattar, *On space-time integrability properties of solutions of the inhomogeneous Schrödinger equation*, PhD Thesis, Università di Ferrara (2013).

[2] Damiano Foschi, *Inhomogeneous Strichartz estimates*, Journal of Hyperbolic Differential Equations **2**, no 1 (2005), 1–24.

Coordinate-independent approaches to uniform oscillatory integral estimates

Philip T. Gressman

A common and effective strategy in the study of degenerate oscillatory integrals is to identify and employ highly-tailored coordinate systems (see, for example, [5, 6, 3, 2]). One drawback of an approach like this, though, is that such coordinates are typically unstable with respect to small perturbations of the phase, which makes their use especially challenging when one specifically needs uniform estimates. Here we discuss some recent work on uniform oscillatory integral estimates which takes a complementary approach, namely, it attempts to proceed in an essentially coordinate-independent way.

The first such approach to uniform estimation of the integral

$$\int_\Omega e^{if}\psi d\mu \tag{1}$$

is inspired by Bruna, Nagel, and Wainger [1] and Street [9], and only assumes an underlying smooth structure of homogeneous type as can be built, for example, in any Carnot-Carathéodory geometry. The full hypotheses may be found in [4], but roughly correspond to a family of nice balls $B_j(x) \subset \Omega$ (imagined to be dyadic in the integer index j) imbued with smoothness structure: for each ball $B_j(x)$, we assume that there is a homeomorphism $\Phi_{j,x} : \mathbb{B}^d \to B_j(x)$ which maps 0 to x, where $\mathbb{B}^d$ is the open Euclidean unit ball in dimension d. In a nutshell, we will assume that these homeomorphisms are smooth with respect to each other when compared on two comparable balls, that the balls foliate the space into leaves, and that $d\mu$ is appropriately smooth. Under these assumptions, it is possible to quantify the smoothness of a function f at any particular point x and any given scale j. We specifically define

$$|d_x^k f|_j := \sup_{1\leq|\alpha|\leq k} \left| \partial_t^\alpha \left[f \circ \Phi_{j,x}(t) \right] \right|_{t=0} \Big| \tag{2}$$

for any $k = 1, \ldots, m$. Under the assumptions of [4], the quantity $|d_x^k f|_j$ satisfies a sort of weak invariance: compositions of f with "tame" diffeomorphisms (measured by composing with $\Phi_{j,x}$ for each j and x) preserve the magnitude of $d_x^k f$ at scale j up to a bounded factor. The main theorem in this direction is the following:

Theorem 1 ([4])**.** *Assuming the suitable structure can be built on $\Omega_0 \subset \Omega$, fix any $\epsilon \in (0,1)$, and suppose $m \geq 2$. Let $E \subset \Omega_0$ consist of those points where $d_x^1 f \neq 0$, and suppose $R : E \to \mathbb{Z}$ some Borel measurable function such that $B_{R(x)+1}(x)$ is well-defined for each $x \in E$ and each of the following holds:*

$$B_{R(x)}(x) \cap B_{R(y)}(y) \neq \emptyset \Rightarrow |R(x) - R(y)| \lesssim 1, \tag{3}$$

$$|d_x^m f|_{R(x)} \lesssim \sum_{k=1}^{m-1} \epsilon^{k-m} |d_x^k f|_{R(x)}, \tag{4}$$

$$\sup_{y \in B_{R(x)}(x)} \epsilon |d_y^1 f|_{R(y)} \lesssim 1 + \inf_{y \in B_{R(x)}(x)} \epsilon |d_y^1 f|_{R(y)} \tag{5}$$

with implicit constants uniform with respect to $x, y \in E$ and ϵ. Then for any smooth, bounded ψ whose support has finite measure in Ω_0, there is another function ψ_m such that

$$\int_{\Omega_0} e^{if} \psi d\mu = \int_{\Omega_0} e^{if} \psi_m d\mu$$

and

$$|\psi_m(x)| \lesssim \frac{\sum_{k=0}^{m-1} \epsilon^k |d_x^k \psi(x)|_{R(x)}}{(1 + \epsilon |d_x^1 f(x)|_{R(x)})^{m-1}}. \tag{6}$$

This theorem may be used to establish a uniform version of the Bruna, Nagel, Wainger result which does not require strict convexity.

More recent work, relating to the program of Phong, Stein, and Sturm [8], uses a coordinate-free approach to consider the question of when it is possible to establish an estimate of the form

$$\sup_{\xi \in \mathbb{R}^2} \left| \int_{[-1,1]^2} e^{i(\lambda \Phi(x) + \xi \cdot x)} \chi(\epsilon^{-1} \det \text{Hess } \Phi(x)) \psi(x) dx \right| \leq C \lambda^{-1} \epsilon^{-\frac{1}{2}}, \tag{7}$$

for all positive λ and ϵ. (Note that summing over ϵ and using a sublevel set estimate would give a different approach to some of the results of [7].) Unlike the robust analogue appearing in [8], the inequality (7) can fail under fairly ordinary circumstances: e.g., (7) cannot hold when $\Phi(x_1, x_2) := (x_2 - x_1^2)^2$. It is nevertheless possible to identify a broad class of phases for which (7) holds by requiring that the compact faces of the Newton polygon satisfy appropriate constraints on the singularities their gradients may possess. The interesting feature of (7) is that, while it only involves critical points which are qualitatively nondegenerate, there are only very weak quantitative bounds on the nondegeneracy (which of course fails as $\epsilon \to 0$).

In this problem, an important part of the analysis that, if the geometry includes an intrinsically-defined, torsion-free connection, then it is possible to identify an intrinsic second-order elliptic operator $\Box$ (of possibly mixed signature) which governs the uniform behavior of scalar oscillatory integrals via a connection to the (pseudo-)Schrödinger equation

$$\frac{\partial \Psi}{\partial t} - \frac{i}{2} \Box \Psi = 0.$$

This relationship allows us to make uniform, coordinate independent estimates for all terms of the asymptotic expansion of the corresponding oscillatory integral as well as uniform estimates for the corresponding remainder terms. The result is summarized in the following lemma:

Lemma 1. *Suppose Φ has an isolated critical point at p and consider the function*

$$I_\psi(t) := \int_U t^{-\frac{n}{2}} e^{it^{-1}\Phi} \psi \, d\mu$$

for ψ smooth and compactly supported near p and $d\mu$ some measure generated by a smooth, nonvanishing density μ on $\mathcal{M}$. Let ω equal the number of positive eigenvalues of $\nabla^2 \Phi(p)$ minus the number of negative eigenvalues. Then as $t \to 0^+$, the difference

$$I_\psi(t) - \pi^{\frac{n}{2}} e^{i\frac{\pi}{4}\omega} e^{it^{-1}\Phi(p)} \left(\frac{\sqrt{|\det \nabla^2 \Phi|}}{\mu}(p) \right)^{-1} \sum_{\ell=0}^{N} \frac{(i\Box^*)^\ell \psi(p)}{2^\ell \ell!} t^\ell \tag{8}$$

is $O(t^{N+1})$ as $t \to 0^+$ for each N. The operator $\Box^*$ is the adjoint of $\Box$ with respect to $d\mu$. If the magnitude of the difference (8) *is denoted $E_\psi^N(t)$ and if k is any integer strictly greater than $\frac{n}{2}$, then*

$$E_\psi^N(t) \lesssim t^{N+1} \left(\int |(\Box^*)^{N+1} \psi| d\mu \right)^{1-\frac{n}{2k}} \left(\int |(\Box^*)^{N+k+1} \psi| d\mu \right)^{\frac{n}{2k}}, \tag{9}$$

where the implicit constant depends only on N, n, and k.

References

[1] J. Bruna, A. Nagel, and S. Wainger. Convex hypersurfaces and Fourier transforms. *Ann. of Math. (2)*, **127**(2):333–365, 1988.

[2] T. C. Collins, A. Greenleaf, and M. Pramanik. A multi-dimensional resolution of singularities with applications to analysis. *Amer. J. Math.*, **135**(5):1179–1252, 2013.

[3] M. Greenblatt. Resolution of singularities in two dimensions and the stability of integrals. *Adv. Math.*, **226**(2):1772–1802, 2011.

[4] P. T. Gressman. Scalar oscillatory integrals in smooth spaces of homogeneous type. arXiv:1205.5774.

[5] I. A. Ikromov, M. Kempe, and D. Müller. Estimates for maximal functions associated with hypersurfaces in $\mathbb{R}^3$ and related problems of harmonic analysis. *Acta Math.*, **204**(2):151–271, 2010.

[6] I. A. Ikromov and D. Müller. On adapted coordinate systems. *Trans. Amer. Math. Soc.*, **363**(6):2821–2848, 2011.

[7] I. A. Ikromov and D. Müller. Uniform estimates for the Fourier transform of surface carried measures in $\mathbb{R}^3$ and an application to Fourier restriction. *J. Fourier Anal. Appl.*, **17**(6):1292–1332, 2011.

[8] D. H. Phong, E. M. Stein, and J. Sturm. Multilinear level set operators, oscillatory integral operators, and Newton polyhedra. *Math. Ann.*, **319**(3):573–596, 2001.
[9] B. Street. Multi-parameter Carnot-Carathéodory balls and the theorem of Frobenius. *Rev. Mat. Iberoam.*, **27**(2):645–732, 2011.

Weighted integrability of polyharmonic functions and the uniqueness theorem of Holmgren

Haakan Hedenmalm

We investigate the boundary properties of polyharmonic functions in a planar domain. More precisely, given that the function divided by a power of the distance to the boundary is L^p-integrable, we investigate when this implies that the function vanishes identically. In joint work with Alexander Borichev, this question is fully resolved for the open unit disk [1]. For other domains, the matter is not fully resolved, but turns out to be related with the theory of quadrature domains, see [2].

References

[1] A. Borichev, H. Hedenmalm, *Weighted integrability of polyharmonic functions.* Adv. Math. **264** (2014], 464-505.
[2] H. Hedenmalm, *On the uniqueness theorem of Holmgren.* Submitted.

The two-weight inequality for the Hilbert transform

Tuomas Hytönen

The (dual-weight form of the) two-weight problem for the Hilbert transform

$$H(f\,d\sigma)(x) = \int_{\mathbb{R}} \frac{1}{x-y} f(y)\,d\sigma(y), \qquad x \notin \operatorname{spt} f, \quad \operatorname{spt} f \text{ compact},$$

is to characterize the pairs of non-negative weight functions $\sigma, w \in L^1_{\mathrm{loc}}(\mathbb{R})$ for which the weighted measures $d\sigma := \sigma\,dx$ and $dw := w\,dx$ satisfy

$$\|H(f\,d\sigma)\|_{L^2(w)} \leq C\|f\|_{L^2(\sigma)} \qquad \forall\; f \in L^2(\sigma). \tag{1}$$

A variant of this problem was solved in a complex-variable framework by Cotlar and Sadosky [1] as early as 1979, but obtaining a real-variable characterization turned out to be a much more difficult problem.

In the formulation (1), the classical one-weight theory correspond to $\sigma = 1/w$. In this case, Muckenhoupt's elegant A_2 condition

$$[w,\sigma]_{A_2} := \sup_I \frac{w(I)}{|I|}\frac{\sigma(I)}{|I|} < \infty \tag{2}$$

characterizes (1), as well as several related inequalities, and it was natural to look for a two-weight theory along the same lines. It is easy to see that (2) is still *necessary* for (1) for arbitrary pairs of weights, and so is a stronger variant with Poisson averages:

$$(3)\qquad [w,\sigma]^{**}_{A_2} := \sup_I P(w,I)P(\sigma,I) < \infty, \quad P(w,I) = \int_{\mathbb{R}} \frac{|I|\,dw(x)}{|I|^2 + (x-c_I)^2}.$$

While it is relatively easy to see that (2) is insufficient for (1) in general, it was believed for some time that the stronger (3) might yield a characterization. This was disproven by a counterexample of Nazarov [6] from 1997.

Soon after this, the attention shifted towards characterizations in terms of so-called *testing conditions*: An obvious necessary condition for (1) to hold for all $f \in L^2(\sigma)$ is that it holds for all functions from a more restricted class, for example all indicators of intervals $I \subset \mathbb{R}$. As H is a linear operator, by duality the same should also hold for the adjoint $H^* : L^2(w) \to L^2(\sigma)$, which is also a Hilbert transform, except for a minus sign. So a necessary condition for (1) is given by the *global testing conditions*

$$(4)\qquad \|H(1_I\,d\sigma)\|_{L^2(w)} \leq C\sigma(I)^{1/2}, \quad \|H^*(1_I\,dw)\|_{L^2(\sigma)} \leq Cw(I)^{1/2} \qquad \forall I,$$

and *a fortiori* by the *local testing conditions*

$$(5)\quad \|1_I H(1_I\,d\sigma)\|_{L^2(w)} \leq C\sigma(I)^{1/2}, \quad \|1_I H^*(1_I\,dw)\|_{L^2(\sigma)} \leq Cw(I)^{1/2} \qquad \forall I,$$

where we have inserted the indicator also on the other side of the Hilbert transform.

Characterizations of norm inequalities by such conditions have two independent historical roots: several two-weight inequalities for *positive* operators were characterized in such terms by Sawyer [8, 9] during the 1980s, and the $T(1)$ theorem of David and Journé, in its local formulation, takes exactly the same form for $d\sigma = dw = dx$ and a general Calderón–Zygmund operator in place of H. The following conjecture was formulated in Volberg's 2003 monograph [10]:

Conjecture 1 ([10]). *For two Radon measures σ, w on $\mathbb{R}$, with no common atoms (i.e., $\sigma\{a\}w\{a\} = 0$ for all $a \in \mathbb{R}$), the bound* (1) *holds if and only if both the local testing conditions* (5) *and the Poisson A_2-condition* (3) *are satisfied.*

This goes somewhat beyond the classical two-weight problem, in that possibly singular Radon measures, rather than just weights, are allowed; this generality becomes natural as soon as the dual-weight formulation is adopted.

Over the following ten years, the conjecture was confirmed under increasingly general side conditions on the measures σ and w: the doubling condition ($\sigma(2I) \leq C\sigma(I)$ and $w(2I) \leq Cw(I)$ for all I) by Volberg [10], and more general but also more technical "pivotal" and "energy" conditions by Nazarov–Treil–Volberg [7] and Lacey–Sawyer–Uriarte-Tuero [5], respectively. Finally, the conjecture, and thus the classical two-weight problem, was fully solved in a sequence of two papers in 2012–13: a reduction to a local problem by Lacey–Sawyer–Shen–Uriarte-Tuero [4]), and the solution of the local problem by Lacey [3].

What about the condition of "no common atoms"? On the one hand, this is necessary for the A_2 condition (2), thus also for (3). Namely,

$$\sigma\{a\}w\{a\} = \lim_{I\downarrow a} \sigma(I)w(I) \leq \lim_{I\downarrow a} [\sigma, w]_{A_2}|I|^2 = 0,$$

provided that $[\sigma, w]_{A_2} < \infty$. On the other hand, it is not necessary for the boundedness (1), as shown by the very classical example of the discrete Hilbert transform with $\sigma = w = \sum_{k\in\mathbb{Z}} \delta_k$, which has *only* common atoms.

Thus the mentioned solution of the conjecture can be restated as:

Theorem 2 (Lacey–Sawyer–Shen–Uriarte-Tuero [4, 3])**.** *For two Radon measures σ, w on $\mathbb{R}$, the following are equivalent:*

- *The inequality* (1) *holds, and σ and w have no common atoms.*
- *The local testing conditions* (5) *and the Poisson A_2 condition* (3) *hold.*

Moreover, the first condition is strictly stronger than the mere estimate (1); thus a characterization of (1) alone should be strictly weaker than the second condition of the theorem. As the testing conditions (5) are clearly necessary, the only possibility is to weaken the A_2 condition, whose necessity breaks down for general measures. This was achieved in the following:

Theorem 3 ([2])**.** *For two Radon measures σ, w on $\mathbb{R}$, the estimate* (1) *holds, if and only if we have the local testing conditions* (5) *and the following pair of one-sided A_2 conditions, involving a local average of one measure, times the Poisson tail of the other one:*

$$[\sigma, w]^*_{A_2} := \sup_I \sigma(I) \int_{I^c} \frac{dw(x)}{(x - c_I)^2} < \infty, \quad [w, \sigma]^*_{A_2} := \sup_I w(I) \int_{I^c} \frac{d\sigma(x)}{(x - c_I)^2} < \infty.$$

Moreover, (1) *is also equivalent to the global testing conditions* (4) *alone.*

Interestingly, the last mentioned equivalence seems to have been previously unnoticed even in the case of no common atoms.

The main additional difficulty in Theorem 3, compared to Theorem 2 (which is already hard!), is in the estimation of certain tails of the Hilbert transform. The argument of Lacey et al. [4] proceeds as

$$\begin{aligned}\text{Hilbert tail} &\approx \text{Poisson tail} \leq \text{full Poisson integral} \\ &\overset{(*)}{\lesssim} \text{Poisson testing} \lesssim \text{Hilbert testing} + \text{Poisson } A_2,\end{aligned}$$

where $(*)$ is an application of Sawyer's characterization [9] for the two-weight inequality of the Poisson integral. The last step uses the full power of (3), which is not available in Theorem 3. Instead, we argue that

$$\text{Hilbert tail} \approx \text{Poisson tail} \overset{(**)}{\lesssim} \text{Poisson tail testing} \lesssim \text{Hilbert testing} + \text{new } A_2,$$

where $(**)$ is a new two-weight inequality for the tail of the Poisson integral. Since this is smaller than the full Poisson integral, it also admits an estimate in terms of the smaller A_2 constant available in Theorem 3.

The first and the last steps in both chains above use a two-sided "monotonicity estimate" of [5], in both directions. Finding a substitute for this lemma appears to be the key obstacle to extending this argument to any other operators than the Hilbert transform.

References

[1] M. Cotlar and C. Sadosky. On the Helson-Szegő theorem and a related class of modified Toeplitz kernels. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 383–407. Amer. Math. Soc., Providence, R.I., 1979.
[2] T. P. Hytönen. The two-weight inequality for the Hilbert transform with general measures. Preprint, arXiv:1312.0843, 2013.
[3] M. T. Lacey. Two weight inequality for the Hilbert transform: A real variable characterization, II. Preprint, arXiv:1301.4663, 2013.
[4] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, and I. Uriarte-Tuero. Two weight inequality for the Hilbert transform: A real variable characterization, I. Preprint, arXiv:1201.4319, 2012.
[5] M. T. Lacey, E. T. Sawyer, and I. Uriarte-Tuero. A two weight inequality for the Hilbert transform assuming an energy hypothesis. *J. Funct. Anal.*, 263(2):305–363, 2012.
[6] F. Nazarov. A counterexample to Sarason's conjecture. Unpublished manuscript, http://www.math.msu.edu/~fedja/prepr.html, 1997.
[7] F. Nazarov, S. Treil, and A. Volberg. Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures. Unpublished manuscript, arXiv:1003.1596, 2005/2010.
[8] E. T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.
[9] E. T. Sawyer. A characterization of two weight norm inequalities for fractional and Poisson integrals. *Trans. Amer. Math. Soc.*, 308(2):533–545, 1988.
[10] A. Volberg. *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, volume 100 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2003.

Multilinear singular integrals with entangled structure

Vjekoslav Kovač
(joint work with Frédéric Bernicot, Kristina Ana Škreb, and Christoph Thiele)

Multilinear singular integral operators in higher dimensions can have more complicated structure than their one-dimensional analogues. An interesting example is a two-dimensional variant of the bilinear Hilbert transform introduced by Demeter and Thiele in [5]. It is defined as the bilinear operator

$$T(f,g)(x,y) = \text{p.v.} \int_{\mathbb{R}^2} f\big((x,y)+A(s,t)\big)\, g\big((x,y)+B(s,t)\big)\, K(s,t)\, ds dt, \tag{1}$$

where K is a Calderón-Zygmund kernel, while A and B are 2×2 real matrices. The same authors observed that L^p estimates for (1) imply boundedness of the Carleson operator in a certain range of exponents. This fact guarantees that the proof of any bounds for (1) has to use some techniques from time-frequency analysis.

It is natural to pair a multilinear operator with an extra function and reduce its boundedness to proving estimates for the corresponding multilinear form. We focus on a class of multilinear singular integral forms acting on two-dimensional functions that "partially share variables". Schematically and somewhat informally we want to study forms such as

$$(2)\qquad \Lambda(f_1, f_2, \ldots) = \int_{\mathbb{R}^n} f_1(x_1, x_2)\, f_2(x_1, x_3)\, K(x_1, \ldots, x_n)\, dx_1 dx_2 dx_3 \ldots dx_n,$$

in which a variable x_1 appears in the arguments of functions f_1 and f_2. It turns out that wave packet decompositions are no longer efficient for bounding such forms, because of the appearance of the pointwise product $x_1 \mapsto f_1(x_1, x_2) f_2(x_1, x_3)$. The structure of any such form is determined by the kernel K and a simple undirected graph G on the set of variables $x_1, x_2, \ldots$, where two vertices x_i and x_j are joined with an edge if and only if there exists a function depending on the pair (x_i, x_j) in the definition of Λ.

General results in the dyadic setting. A rather complete theory of entangled forms is possible in the case when the graph G is bipartite and K is a "perfect" dyadic model of a multilinear Calderón-Zygmund kernel. More precisely, we take positive integers $m, n \geq 2$ and consider the "diagonal" in $\mathbb{R}^{m+n}$,

$$D = \big\{ (\underbrace{x, \ldots, x}_{m}, \underbrace{y, \ldots, y}_{n}) : x, y \in \mathbb{R} \big\}.$$

We require that the kernel K satisfies the usual "size condition",

$$|K(x_1, \ldots, x_m, y_1, \ldots, y_n)| \leq C \Big(\sum_{i_1 < i_2} |x_{i_1} - x_{i_2}| + \sum_{j_1 < j_2} |y_{j_1} - y_{j_2}| \Big)^{2-m-n}$$

and also that K is constant on all $(m+n)$-dimensional dyadic cubes disjoint from D. The last condition is in analogy with the setting from [1]. For technical reasons we also assume that K is bounded and compactly supported. Take $E \subseteq \{1, \ldots, m\} \times \{1, \ldots, n\}$ and interpret it as the set of edges of a simple bipartite undirected graph on $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$. The corresponding $|E|$-linear singular form (2) becomes

$$\Lambda\big((F_{i,j})_{(i,j)\in E}\big) = \int_{\mathbb{R}^{m+n}} K(x_1, \ldots, y_n) \prod_{(i,j)\in E} F_{i,j}(x_i, y_j)\ dx_1 \ldots dy_n.$$

In order to avoid degeneracy of Λ we assume that there are no isolated vertices in G. Recall that there are $|E|$ mutually adjoint $(|E|-1)$-linear operators $T_{u,v}$, $(u, v) \in E$ corresponding to Λ.

The main result from the pair of papers by Thiele and the author [6], [11] is a $T(1)$-type characterization of L^p boundedness.

Theorem. *There exist positive integers $d_{i,j}$ such that the following holds. If*

$$|\Lambda(\mathbf{1}_Q, \ldots, \mathbf{1}_Q)| \leq C_1 |Q| \quad \textit{for every dyadic square } Q$$

and

$$\|T_{u,v}(\mathbf{1}_{\mathbb{R}^2}, \ldots, \mathbf{1}_{\mathbb{R}^2})\|_{\mathrm{BMO}(\mathbb{R}^2)} \leq C_2 \quad \textit{for each } (u, v) \in E,$$

then

$$\big|\Lambda\big((F_{i,j})_{(i,j)\in E}\big)\big| \leq C_3 \prod_{(i,j)\in E} \|F_{i,j}\|_{\mathrm{L}^{p_{i,j}}(\mathbb{R}^2)}$$

for exponents $p_{i,j}$ satisfying $\sum_{(i,j)\in E} 1/p_{i,j} = 1$ and $d_{i,j} < p_{i,j} \leq \infty$.

Let us remark that the numbers $d_{i,j}$ depend on the graph G in a rather complicated way, but they always determine a non-empty range of exponents in which L^p estimates hold.

The first step in the proof of the above theorem is a decomposition into two types of paraproduct-type operators. "Cancellative entangled paraproducts" satisfy L^p estimates provided their coefficients are bounded, while the coefficients of "non-cancellative entangled paraproducts" need to satisfy a Carleson-type condition. The second step consists of a certain multilinear variant of the Bellman function technique developed in [6] and [7]. This method primarily applies to multilinear forms described above, but it can occasionally also provide results on some forms possessing explicit modulation invariance, for which more involved wave packet analysis would normally be required, see [8]. However, the technique usually needs to be combined with other ideas in order to transfer the results from the perfect dyadic or other "algebraic models" to the actual singular integral operators.

Applications of multilinear forms with entangled structure. Most of the previously mentioned techniques and results were motivated by the only case of (1) that was left out from [5] as an open problem. After a change of variables it turns into

$$\Lambda(f,g,h) = \int_{\mathbb{R}^4} f(u,y)\, g(x,v)\, h(x,y)\, K(u,v,x,y)\, du dv dx dy. \tag{3}$$

Since the graph u—y—x—v is bipartite, at least the dyadic model of (3) falls into the realm of the general theory from the previous section. The first bounds for (3) were established in the paper by the author [7], while the result of Bernicot [3] significantly expanded the range of L^p estimates. Later Bernicot and the author [4] studied some Sobolev norm inequalities for the same type of bilinear multipliers. Škreb and the author [10] generalized some of the L^p estimates to the setting of general dilations found in [12].

Yet another source of motivation are quantitative convergence results on ergodic averages. Such results in ergodic theory are often established by proving relevant estimates for integral operators on the real line. Particularly interesting are double ergodic averages along orbits of two commuting invertible measure-preserving transformations, motivated by multidimensional Szemerédi's theorem. Their convergence in L^p, $p < \infty$ is a classical result, but recently Avigad and Rute [2] raised a question of showing any norm-variation estimates, which would quantify this convergence. The author was able to prove a finite group model for such an estimate [9], where the $\mathbb{Z}$-actions are replaced by two commuting measure-preserving actions of the direct sum of countably many copies of a finite abelian group. The techniques used in [9] very closely resemble the methods in the proof of the above theorem.

References

[1] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, C. Thiele, *Carleson measures, trees, extrapolation, and T(b) theorems*, Publ. Mat. 46 (2002), no. 2, 257–325.

[2] J. Avigad, J. Rute, *Oscillation and the mean ergodic theorem for uniformly convex Banach spaces*, to appear in Ergodic Theory Dynam. Systems, available at arXiv:1203.4124 [math.DS].
[3] F. Bernicot, *Fiber-wise Calderón-Zygmund decomposition and application to a bi-dimensional paraproduct*, Illinois J. Math. 56 (2012), 415–422.
[4] F. Bernicot, V. Kovač, *Sobolev norm estimates for a class of bilinear multipliers*, Commun. Pure Appl. Anal. 13 (2014), no. 3, 1305–1315.
[5] C. Demeter, C. Thiele, *On the two-dimensional bilinear Hilbert transform*, Amer. J. Math. 132 (2010), no. 1, 201–256.
[6] V. Kovač, *Bellman function technique for multilinear estimates and an application to generalized paraproducts*, Indiana Univ. Math. J. 60 (2011), no. 3, 813–846.
[7] V. Kovač, *Boundedness of the twisted paraproduct*, Rev. Mat. Iberoam. 28 (2012), no. 4, 1143–1164.
[8] V. Kovač, *On a trilinear form related to the Carleson theorem*, J. Math. Anal. Appl. 405 (2013), no. 1, 220–226.
[9] V. Kovač, *Quantitative norm convergence of double ergodic averages associated with two commuting group actions*, to appear in Ergodic Theory Dynam. Systems, available at arXiv:1405.3499 [math.DS].
[10] V. Kovač, K. A. Škreb, *One modification of the martingale transform and its applications to paraproducts and stochastic integrals*, preprint, available at arXiv:1307.6403 [math.PR].
[11] V. Kovač, C. Thiele, *A T(1) theorem for entangled multilinear dyadic Calderón-Zygmund operators*, to appear in Illinois J. Math., available at arXiv:1305.4752 [math.CA].
[12] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. 84 (1978), no. 6, 1239–1295.

Some results about centered Hardy-Littlewood maximal function on manifolds

Hong-Quan Li

A very interesting result due to Stein and Strömberg ([17]) says that the standard centered Hardy-Littlewood maximal function on $\mathbb{R}^n$ ($n \in \mathbb{N}^*$), $M_{\mathbb{R}^n}$, has the following properties:
1. the weak $(1,1)$ norm of $M_{\mathbb{R}^n}$ grows at most linearly in the dimension n;
2. the strong (p,p) $(p > 1)$ norms of $M_{\mathbb{R}^n}$ have a bound independent of n.

Further, when the usual metric of $\mathbb{R}^n$ is replaced by the one induced by the Minkowski functional defined by a symmetric convex bounded open set U, Stein and Strömberg obtained an uniform weak $(1,1)$ bound of $O(n \ln n)$. Also, some (partial) uniform dimension-free L^p $(p > 1)$ estimates were shown by Bourgain, Carbery and Müller, c.f. [3]-[7], [15].

Recently, Aldaz [2] proved that the weak type $(1,1)$ bounds for the maximal function associated to cubes grow to infinity with the dimension. Naor and Tao [16] extended the $O(n \ln n)$ result of Stein-Strömberg to the context of "strong n-microdoubling" metric measure spaces and demonstrated the sharpness of this result in this general setting.

On the other hand, such problems have received much attention in the context of Riemannian manifolds or manifolds equipped with a measure and an (essentially) self-adjoint second order differential operator. In contrast with the above results, the weak $(1,1)$ and strong (p,p) norms of maximal function can grow exponentially fast with the dimension. For example, see the results of Aldaz [1], Criado and Sjögren [8] in the setting of Ornstein-Uhlenbeck operator, $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2x_i \frac{\partial}{\partial x_i}$. Notice that in the case of the Laplacian with drift on real hyperbolic space of dimension $n \geq 2$,

$$y^2 \frac{\partial^2}{\partial y^2} - (n-2) y \frac{\partial}{\partial y} + y^2 \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \alpha y \frac{\partial}{\partial y},$$

a sufficient and necessary condition such that the maximal function is strong (p,p) has been obtained, and it only depends on n and the choice of α, c.f. [9], [10].

Our aim in this talk is to provide some positive results in the setting of some typical manifolds.

Theorem 1 (Li, Lohoué [13]). *The weak* $(1,1)$ *bound of* $O(n \ln n)$ *is valid for maximal functions on real hyperbolic spaces of dimension* $n \geq 2$.

Theorem 2 (Li [12]). L^p $(p > 1)$ *dimension-free estimates of maximal functions hold on real hyperbolic spaces of dimension* $n \geq 2$.

Theorem 3 (Li [11], Li, Qian [14]). *Let* $\mathrm{H}(2n, m)$ *denote the Heisenberg type group with dimension of center* m *and* M_K *the centered Hardy-Littlewood maximal function defined by the Korányi norm. Then the uniform weak* $(1,1)$ *bound of* $O(n)$ *is valid for* $m^2 \ll \ln n$.

Remark that in the setting of Heisenberg groups, $\mathrm{H}(2n, 1)$, L^p dimension-free estimates were obtained by Zienkiewicz [18] for centered Hardy-Littlewood maximal function defined either by Carnot-Carathéodory distance, or by Korányi norm.

Moreover, the weak $(1,1)$ bound of $O(n)$ and L^p dimension-free estimates are valid in the setting of Grushin operator $\Delta_G = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + (\sum_{i=1}^{n} x_i^2) \frac{\partial^2}{\partial u^2}$ and Laplacian with drift in $\mathbb{R}^n$.

References

[1] J. M. Aldaz, *Dimension dependency of the weak type (1,1) bounds for maximal functions associated to finite radial measures*, Bull. Lond. Math. Soc. **39** (2007), 203–C208.

[2] J. M. Aldaz, *The weak type* $(1,1)$ *bounds for the maximal function associated to cubes grow to infinity with the dimension*, Ann. of Math. **173** (2011), 1013-1023.

[3] J. Bourgain, *On the* L^p*-bounds for maximal functions associated to convex bodies in* $\mathbb{R}^n$, Israel J. Math. **54** (1986), 257–265.

[4] J. Bourgain, *On high-dimensional maximal functions associated to convex bodies*, Amer. J. Math. **108** (1986), 1467–1476.

[5] J. Bourgain, *On dimension free maximal inequalities for convex symmetric bodies in* $\mathbb{R}^n$, Lecture Notes in Math. **1267** (1987), 168–176.

[6] J. Bourgain, *On the Hardy-Littlewood maximal function for the cube*, Israel Journal of Mathematics 2014.

[7] A. Carbery, *An almost-orthogonality principle with applications to maximal functions associated to convex bodies*, Bull. Amer. Math. Soc. **14** (1986), 269–273.

[8] A. Criado, P. Sjögren, *Bounds for Maximal Functions Associated with Rotational Invariant Measures in High Dimensions*, J. Geometric Anal. **24** (2014), 595–612.

[9] H.-Q. Li, *La fonction maximale de Hardy-Littlewood sur une classe d'espaces métriques mesurables*, C. R. Acad. Sci. Paris **338** (2004), 31–34.

[10] H.-Q. Li, *Les fonctions maximales de Hardy-Littlewood pour des measures sur les variétés cuspidales*, J. Math. Pures Appl. **88** (2007), 261–275.
[11] H.-Q. Li, *Fonctions maximales centrées de Hardy-Littlewood sur les groupes de Heisenberg*, Studia Math. **191** (2009), 89–100.
[12] H.-Q. Li, Centered Hardy-Littlewood maximal function on hyperbolic spaces, $p > 1$. Preprint.
[13] H.-Q. Li, N. Lohoué, *Fonction maximale centrée de Hardy-Littlewood sur les espaces hyperboliques*, Ark. Mat. **50** (2012), 359–378.
[14] H.-Q. Li, B. Qian, *Centered Hardy-Littlewood maximal functions on Heisenberg type groups*, Trans. Amer. Math. Soc. **366** (2014), 1497–1524.
[15] D. Müller, *A geometric bound for maximal functions associated to convex bodies*, Pacific J. Math. **142** (1990), 297–312.
[16] A. Naor, T. Tao, *Random martingales and localization of maximal inequalities*, J. Funct. Anal. **259** (2010), 731–779.
[17] E. M. Stein, J.-O. Strömberg, *Behavior of maximal functions in $\mathbb{R}^n$ for large n*, Ark. Mat. **21** (1983), 259–269.
[18] J. Zienkiewicz, *Estimates for the Hardy-Littlewood maximal function on the Heisenberg group*, Colloq. Math. **103** (2005), 199–205.

Hypoelliptic operators and sharp multiplier theorems

Alessio Martini
(joint work with Detlef Müller)

Let $L = -\Delta$ be the Laplace operator on $\mathbb{R}^d$. A corollary of the Mihlin-Hörmander multiplier theorem states that an operator of the form $F(L)$ is of weak type $(1,1)$ and bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ whenever the "spectral multiplier" $F : \mathbb{R} \to \mathbb{C}$ satisfies a local, scale-invariant Sobolev condition

$$\sup_{t>0} \|F(t\cdot)\,\chi\|_{W_2^s} < \infty \tag{1}$$

of order $s > d/2$. Here W_2^s denotes the L^2 Sobolev space on $\mathbb{R}$ of order s and $\chi \in C_c^\infty((0,\infty))$ is a nonzero cutoff. It is well known that this result is sharp, that is, the threshold $d/2$ in the smoothness condition cannot be lowered. Further this sharp result extends — at least locally — to the case where L is a more general second-order elliptic differential operator on a manifold of dimension d [11].

The situation is much less clear when L is not supposed to be elliptic, but just hypoelliptic. Consider the prototypical example of such an operator, that is, the case of a homogeneous sublaplacian L on a stratified group G. A theorem due to Christ [1] and to Mauceri and Meda [9] states that $F(L)$ is of weak type $(1,1)$ and bounded on $L^p(G)$ for $1 < p < \infty$, provided the multiplier F satisfies the condition (1) for some $s > Q/2$, where Q is the homogeneous dimension of G. Although for many purposes Q is the natural replacement for the dimension d of $\mathbb{R}^d$, it turns out that in several cases the threshold $Q/2$ in the multiplier theorem is not sharp.

For instance, in the case where G is a Heisenberg group, Müller and Stein [10] improved the above theorem, by proving that the smoothness condition can be pushed down to $s > d/2$, where d is the topological dimension of G; this result is sharp. An analogous result with condition $s > d/2$ was proved by Hebisch [2] for a larger class of 2-step stratified groups, namely, when G is a direct product of Métivier and abelian groups. It is still an open problem, however, if the same improvement of the Christ–Mauceri–Meda theorem holds true for an arbitrary stratified group. In fact in the last twenty years there has been apparently no substantial progress in the investigation of this question.

Recently we managed to prove a multiplier theorem with condition $s > d/2$ for several 2-step stratified groups G that do not fall into the class of groups considered by Hebisch and Müller and Stein. The groups that we consider include:

- the free 2-step group $N_{3,2}$ on 3-generators [5];
- the Heisenberg–Reiter groups $\mathbb{R}^{d_1 \times d_2} \ltimes (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ [4];
- the 2-step groups G with $d \leq 7$ or with 2-dimensional second layer [6].

The technique that we developed for this purpose can be adapted to deal with hypoelliptic operators L in settings other than 2-step groups. In particular we could obtain a sharp multiplier theorem for the Grushin operators $-\Delta_x - |x|^2 \Delta_y$ on $\mathbb{R}^{d_1}_x \times \mathbb{R}^{d_2}_y$ [7]; this improves the result of [8], where the sharp theorem is obtained only under the assumption that $d_1 \geq d_2$.

The proof of our result for homogeneous sublaplacians L on 2-step groups G is reduced, by a standard argument based on the Calderón-Zygmund theory of singular integral operators, to the following L^1-estimate for the convolution kernel $\mathcal{K}_{F(L)}$ of the operator $F(L)$ corresponding to a compactly supported multiplier F: for all compact sets $K \subseteq \mathbb{R}$, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq K$, and for all $s > d/2$,

$$\|\mathcal{K}_{F(L)}\|_1 \leq C_{K,s} \|F\|_{W_2^s}. \tag{2}$$

In [9] this L^1-estimate is proved for $s > Q/2$, as a consequence of a weighted L^2-estimate: if $|\cdot|_G : G \to \mathbb{R}$ is a homogeneous norm on G, then, for all $\beta > \alpha \geq 0$ and all multipliers $F : \mathbb{R} \to \mathbb{C}$ supported in a compact set $K \subseteq \mathbb{R}$,

$$\|(1 + |\cdot|_G)^\alpha \mathcal{K}_{F(L)}\|_2 \leq C_{K,\alpha,\beta} \|F\|_{W_2^\beta}. \tag{3}$$

The known improvements of the Christ–Mauceri–Meda theorem are all based on an improved version of (3) entailing an "extra weight" $w : G \to [1,\infty)$, i.e.,

$$\|(1 + |\cdot|_G)^\alpha \, w \, \mathcal{K}_{F(L)}\|_2 \leq C_{K,\alpha,\beta} \|F\|_{W_2^\beta}. \tag{4}$$

Different types of weights w are used in the aforementioned works; in particular, [2] uses an extra weight depending (in exponential coordinates) only on the variables on the first layer $\mathfrak{g}_1$, whereas [5, 4] exploit a weight depending only on the variables on the second layer $\mathfrak{g}_2$. In any case, the presence of the extra weight is sufficient to compensate the difference between the homogeneous dimension and the topological dimension. In [6], however, no "global" L^2-estimate of the form (4) is obtained. More precisely, if $\mathbf{U}$ denotes the vector of the "central derivatives" on G, then, by the use of a suitable partition of unity $\{\zeta_\iota\}_\iota$ we decompose the operator $F(L)$ along the spectrum of $\mathbf{U}$, thus

$$\mathcal{K}_{F(L)} = \sum_\iota \mathcal{K}_{F(L)\,\zeta_\iota(\mathbf{U})}. \tag{5}$$

For each piece $\mathcal{K}_{F(L)\,\zeta_\iota(\mathbf{U})}$ we prove a weighted L^2-estimate of the type (4), where the extra weight w may depend on the piece, hence these estimates cannot be directly summed; however they can be summed at the level of L^1, after the application of Hölder's inequality, thus yielding the L^1-estimate (2) for all $s > d/2$.

The decomposition (5) is related to the possible "singularities" of the algebraic structure of G. Namely, let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{g}_1$ determined by the sublaplacian, and define for all $\eta \in \mathfrak{g}_2^*$ the skewsymmetric endomorphism J_η of $\mathfrak{g}_1$ by $\langle J_\eta x, x' \rangle = \eta([x, x'])$ for all $x, x' \in \mathfrak{g}_1$. Then, by the spectral theorem,

$$-J_\eta^2 = \sum_{j=1}^{M} (b_j^\eta)^2 P_j^\eta$$

for some distinct $b_1^\eta, \dots, b_M^\eta \in (0, \infty)$ and some projections $P_1^\eta, \dots, P_M^\eta$ on mutually orthogonal subspaces of $\mathfrak{g}_1$ of even ranks. By the use of the representation theory of the nilpotent group G, a formula for the Fourier transform $\widehat{\mathcal{K}}_{H(L,\mathbf{U})}$ of the convolution kernel of an operator $H(L, \mathbf{U})$ can be written, involving the quantities $b_1^\eta, \dots, b_M^\eta$, $P_1^\eta, \dots, P_M^\eta$. Weighted L^2-estimates of $\mathcal{K}_{H(L,\mathbf{U})}$ correspond, roughly speaking, to L^2-estimates of derivatives of $\widehat{\mathcal{K}}_{H(L,\mathbf{U})}$; therefore we are interested in controlling the derivatives of the algebraic functions $\eta \mapsto b_j^\eta$ and $\eta \mapsto P_j^\eta$.

The singularities of these functions lie on a homogeneous Zariski-closed subset of $\mathfrak{g}_2^*$. For the groups considered in [10, 5, 4], the only relevant singularity is at the origin of $\mathfrak{g}_2^*$, hence the derivatives of the b_j^η and P_j^η can be simply controlled by homogeneity and no decomposition of the form (5) is needed. This is not the case for more general 2-step groups. Nevertheless, if $\dim \mathfrak{g}_2 = 2$, then the singular set is a finite union of rays emanating from the origin; by the use of a finite decomposition (5) we can consider each of these rays separately, and classical results for the resolution of singularities of algebraic curves allow us to obtain the desired estimate.

For the case $d \leq 7$, it remains to consider some examples where $\dim \mathfrak{g}_2 = 3$. It turns out that in most examples the singular set is a finite union of lines and planes, and an adaptation of the technique used when $\dim \mathfrak{g}_2 = 2$ works here too. However there is an example where the singular set has a nonflat component, namely a conic surface. In this case, in the neighborhood of the cone we exploit an infinite decomposition (5) analogous to the "second dyadic decomposition" used in [12] to prove sharp L^p estimates for Fourier integral operators. Due to the "too large amount" of pieces, this technique alone would give only a partial improvement of the Christ–Mauceri–Meda theorem; however a further extra weight can be gained in this case by a variation of the technique of [2, 3], and the combination of the two techniques yields eventually the wanted result.

References

[1] M. Christ, *L^p bounds for spectral multipliers on nilpotent groups*, Trans. Amer. Math. Soc. **328** (1991), 73–81.

[2] W. Hebisch, *Multiplier theorem on generalized Heisenberg groups*, Colloq. Math. **65** (1993), 231–239.

[3] W. Hebisch and J. Zienkiewicz, *Multiplier theorem on generalized Heisenberg groups. II*, Colloq. Math. **69** (1995), 29–36.
[4] A. Martini, *Spectral multipliers on Heisenberg–Reiter and related groups*, Ann. Mat. Pura Appl. (to appear), `doi:10.1007/s10231-014-0414-6`.
[5] A. Martini and D. Müller, *L^p spectral multipliers on the free group $N_{3,2}$.*, Studia Math. **217** (2013), 41–55.
[6] A. Martini and D. Müller, *Spectral multiplier theorems of Euclidean type on new classes of 2-step stratified groups*, Proc. Lond. Math. Soc. (to appear), `doi:10.1112/plms/pdu033`.
[7] A. Martini and D. Müller, *A sharp multiplier theorem for Grushin operators in arbitrary dimensions*, Rev. Mat. Iberoam. (to appear), `arXiv:1210.3564`.
[8] A. Martini and A. Sikora, *Weighted Plancherel estimates and sharp spectral multipliers for the Grushin operators*, Math. Res. Lett. **19** (2012), 1075–1088.
[9] G. Mauceri and S. Meda, *Vector-valued multipliers on stratified groups*, Rev. Mat. Iberoam. **6** (1990), 141–154.
[10] D. Müller and E. M. Stein, *On spectral multipliers for Heisenberg and related groups*, J. Math. Pures Appl. (9) **73** (1994), 413–440.
[11] A. Seeger and C. D. Sogge, *On the boundedness of functions of (pseudo-) differential operators on compact manifolds*, Duke Math. J. **59** (1989), 709–736.
[12] A. Seeger, C. D. Sogge, and E. M. Stein, *Regularity properties of Fourier integral operators*, Ann. Math. **134** (1991), 231–251.

Discrete analogues in harmonic analysis: maximal functions and singular integral operators

Mariusz Mirek
(joint work with Jim Wright, Jacek Zienkiewicz)

In the continuous setting, the Hardy–Littlewood maximal function was initially used to establish the Lebesgue Differentiation Theorem; its discrete analogue is a notable object in pointwise ergodic theory. One can note a resemblance between discrete Hardy–Littlewood maximal function

$$\mathcal{M}f(n) = \sup_{N\in\mathbb{N}} \frac{1}{N}\Big|\sum_{m=1}^{N} f(n-m)\Big|.$$

and the maximal function $\mathcal{A}f = \sup_{N\in\mathbb{N}} |A_N f|$ associated with Birkhoff's averaging operators

$$A_N f(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x),$$

defined on a σ-finite measure space $(X, \mathcal{B}, \mu)$ where T is an invertible and measure preserving transformation on X. In view of Calderón's transference principle, for any dynamical system $(X, \mathcal{B}, \mu, T)$, $L^p(X, \mu)$ bounds with $p > 1$ or weak type $(1, 1)$ bounds for $\mathcal{A}$, can be deduced from the corresponding results for $\mathcal{M}$ on the set of integers. In light of this transference, one sees that the set of functions for which pointwise convergence of the A_N means holds is *closed* in $L^p(X, \mu)$ for $p \geq 1$. Consequently, to prove pointwise convergence for the A_N means one needs only to find a dense class of functions, say, in $L^2(X,\mu)$ for which we have the pointwise convergence. A good candidate is the set

$$\mathcal{I} \oplus \mathcal{J}_\infty \subseteq L^2(X,\mu),$$

where

$$\mathcal{I} = \{f \in L^2(X,\mu) : f \circ T = f\},$$

and

$$\mathcal{J}_\infty = \{g \circ T - g : g \in L^2(X,\mu) \cap L^\infty(X,\mu)\}.$$

Indeed, it is easy to observe that $A_N f = f$ if $f \in \mathcal{I}$ and $\lim_{N\to\infty} A_N h = 0$ for $h \in \mathcal{J}_\infty$ due to the telescoping nature of A_N on $\mathcal{J}_\infty$.

The situation changes dramatically when one wants to consider more general ergodic averages, for instance along integer-valued polynomials $\mathcal{P}$, i.e.

$$A_N^{\mathcal{P}} f(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^{\mathcal{P}(n)}x).$$

Even for $\mathcal{P}(n) = n^2$ it is difficult, a priori, to find a class of dense functions on which we have pointwise convergence. The class $\mathcal{I} \oplus \mathcal{J}_\infty$ is inadequate for the averages $A_N^{\mathcal{P}}$ because the sequence $(n^2)_{n\in\mathbb{N}}$ has unbounded gaps $(n+1)^2 - n^2 = 2n+1$ and we do not have the telescoping property on $\mathcal{J}_\infty$ any more.

In spite of these difficulties, in the mid 1980's, Bourgain in [1], [2] and [3], generalized Birkhoff's pointwise ergodic theorem, showing that almost everywhere convergence still holds when averages are taken only along the squares or an integer-valued polynomial $\mathcal{P}$.

Bourgain introduced new ideas, arising from analytic number theory (the *circle method* of Hardy and Littlewood), and applied them in an ergodic context, shedding new light on various discrete analogues in harmonic analysis.

In proving his pointwise convergence result, Bourgain showed that for every $p > 1$, there[1] is a constant $C_p > 0$ such that

$$\big\|\mathcal{M}^{\mathcal{P}} f\big\|_{\ell^p(\mathbb{Z})} \le C_p \|f\|_{\ell^p(\mathbb{Z})},$$

for every $f \in \ell^p(\mathbb{Z})$, where

$$\mathcal{M}^{\mathcal{P}} f(n) = \sup_{N\in\mathbb{N}} \frac{1}{N}\Big|\sum_{m=1}^{N} f\big(n - \mathcal{P}(m)\big)\Big|.$$

Bourgain's papers [1, 2, 3] initiated extensive study both in pointwise ergodic theory along various arithmetic subsets of the integers and investigations of discrete analogues of classical operators with arithmetic features, see for instance [6, 7, 9, 10, 12, 14]. We refer the reader to [12] and [15] for more detailed explanations and the references given there.

[1]Recently, it was shown that the restriction to the range $p > 1$ is essential as LaVictoire [8], extending work of Buczolich and Mauldin [4], showed the pointwise convergence result for $A_N^{\mathcal{P}} f$ fails on $L^1(X,\mu)$ when $\mathcal{P}(n) = n^k$ with $k \ge 2$.

Our motivations to study discrete maximal functions and operators of singular integrals (modeled on subsets of integers) are: firstly: they are considered in pointwise ergodic theory, secondly: the phenomena occurring in the discrete "world" may completely differ from what happens when the continuous analogues are considered, see [4], [8] and [9]. Although several aspects of maximal functions and singular integral operators along various subsets of integers have been developed by many authors, we have recently touched upon a path which is much less developed: ℓ^p–boundedness of *multi-parameter* discrete maximal functions and singular integral operators along polynomial surfaces. Our investigations show that these problems, as before, have a strong 'number theoretic' flavour, and there are also interesting connections with their continuous counterparts [5, 11, 13].

One of our recent results in discrete multi-parameter theory is the following.

Theorem 4 (Mirek, Wright and Zienkiewicz). *Let $\mathcal{P} : \mathbb{Z}^2 \mapsto \mathbb{Z}$ be an integer-valued polynomial of two variables and define*

$$\mathcal{M}_2 f(n) = \sup_{M,N\in\mathbb{N}} \frac{1}{MN}\Big|\sum_{m=1}^{M}\sum_{n=1}^{N} f\big(n-\mathcal{P}(m,n)\big)\Big|,$$

Then for every $1 < p \leq \infty$, there exists a constant $C_p > 0$ such that for all $f \in \ell^p(\mathbb{Z})$ we have that

$$\|\mathcal{M}_2 f\|_{\ell^p(\mathbb{Z})} \leq C_p \|f\|_{\ell^p(\mathbb{Z})}.$$

The proof of our theorem strongly uses the geometry of the 'backward' Newton diagram $\Pi_{\mathcal{P}}$ corresponding with the polynomial

$$\mathcal{P}(m,n) = \sum_{(k,l)\in\Lambda} c_{k,l} m^k n^l,$$

which is the closed convex hull of the set

$$\bigcup_{(k,l)\in\Lambda} \{(x+k, y+l) \in \mathbb{R}^2 : x \leq 0, y \leq 0\},$$

where $\Lambda = \{(k,l) \in (\mathbb{N}\cup\{0\})^2 : c_{k,l} \neq 0\}$. The backward Newton diagram allows us to split the set $(\mathbb{N}\cup\{0\})^2$ into finitely many cones, each with a dominating monomial. This reduction is critical, since we can restrict our attention to the maximal function with the supremum taken over pairs from a fixed cone and the polynomial $\mathcal{P}$ replaced by a dominating monomial associated with the cone.

Another important ingredient in the proof is the circle method of Hardy and Littlewood which gives good $\ell^2(\mathbb{Z})$ theory. However, we had to adjust its aspects to the multi-parameter settings and this might be interesting in its own right.

Our methods are robust enough and extend to multi-parameter cases in higher dimensions as well or singular integrals of the form

$$\mathcal{H} f(x,y,z) = \sum_{m\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{f\big(x-m, y-n, z-\mathcal{P}(m,n)\big)}{mn}, \quad (x,y,z) \in \mathbb{Z}^3.$$

We hope to extend these techniques to cover discrete *bilinear* averages in the near future.

References

[1] J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers.* Israel J. Math. **61** (1988), 39–72.

[2] J. Bourgain, *On the pointwise ergodic theorem on L^p for arithmetic sets.* Israel J. Math. **61** (1988), 73–84.

[3] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets, with an appendix by the author, H. Furstenberg, Y. Katznelson, and D. S. Ornstein.* Inst. Hautes Etudes Sci. Publ. Math. **69** (1989), 5–45.

[4] Z. Buczolich, R. D. Mauldin, *Divergent square averages.* Ann. Math. **171** (2010), no. 3, 1479–1530.

[5] A. Carbery, S. Wainger, J. Wright, *Double Hilbert transform along polynomial surfaces in* $\mathbb{R}^3$. Duke Math. J. **101**, (2000), no. 3, 499–513.

[6] A. D. Ionescu, A. Magyar, E. M. Stein, and S. Wainger, *Discrete Radon transforms and applications to ergodic theory.* Acta Math. **198** (2007), 231–298.

[7] A. D. Ionescu, S. Wainger, *L^p boundedness of discrete singular Radon transforms.* J. Amer. Math. Soc. **19** (2005), no. 2, 357–383.

[8] P. LaVictoire, *Universally L^1–Bad Arithmetic Sequences.* J. Anal. Math., **113** (2011), no. 1, 241–263.

[9] A. Magyar, E. M. Stein, and S. Wainger, *Discrete analogues in harmonic analysis: spherical averages.* Ann. Math. **155** (2002), 189–208.

[10] M. Mirek and B. Trojan, *Discrete maximal functions in higher dimensions and applications to ergodic theory.* Preprint (2014). Available at `http://arxiv.org/abs/1405.5566`

[11] S. Patel, *Weak-type estimate for a maximal function.* Acta Sci. Math. (Szeged) **75** (2009) 241–264.

[12] L. B. Pierce, *Discrete analogues in harmonic analysis.* PhD Thesis, Princeton University (2009).

[13] F. Ricci and E. M. Stein, *Multiparameter singular integrals and maximal functions.* Ann. Inst. Fourier (Grenoble) **42** (1992), 637–670.

[14] E. M. Stein, S. Wainger, *Discrete analogues in harmonic analysis I: ℓ^2 estimates for singular Radon transforms.* Amer. J. Math. **121** (1999), 1291–1336.

[15] J. Wright, *Discrete analogues in harmonic analysis.* Poland Lectures–November 2013.

Nonembeddability of the Heisenberg group via Littlewood-Paley theory

Assaf Naor

(joint work with Vincent Lafforgue)

This talk is devoted to the following inequality, which holds true for every smooth and compactly supported $f:\mathbb{R}^3\to\mathbb{R}$ and every $p\in(1,2]$.

$$(1)\quad \left(\int_0^\infty\left(\int_{\mathbb{R}^3}\left(\frac{|f(x,y,z+t)-f(x,y,z)|}{\sqrt{t}}\right)^p dxdydz\right)^{\frac{2}{p}}\frac{dt}{t}\right)^{\frac12} \lesssim_p \left(\int_{\mathbb{R}^3}\left(\left|\frac{\partial f}{\partial x}(x,y,z)\right|^p+\left|\frac{\partial f}{\partial y}(x,y,z)+x\frac{\partial f}{\partial z}(x,y,z)\right|^p\right)dxdydz\right)^{\frac1p}.$$

We also present a discrete version of this inequality (which follows from (1) via a partition of unity argument), and show how it implies sharp distortion bounds for bi-Lipschitz embeddings of balls in the discrete Heisenberg group into ℓ_p. This is the only known approach that yields such sharp results. The proof of (1) relies on the Littlewood-Paley inequality, and as such it departs from methods that were previously used to prove nonembeddability of the Heisenebrg group, which were based on various notions of metric differentiation. The endpoint case $p = 1$ of (1) is conjectured to hold true as well, i.e., the implicit constant that is indicated by the sign $\lesssim_p$ is conjectured to remain bounded as $p \to 1$. We present an important application of this conjecture to theoretical computer science, namely it yields a sharp integrality gap lower bound for the Goemans-Linial semidefinite program for the Sparsest Cut problem.

Mixed commutators and little product BMO

Stefanie Petermichl
(joint work with Yumeng Ou, Elizabeth Strouse)

A classical result of Nehari [8] shows that a Hankel operator with anti-analytic symbol b mapping analytic functions into the space of anti-analytic functions by $f \mapsto P_-bf$ is bounded if and only if the symbol belongs to BMO. This theorem has an equivalent formulation in terms of the boundedness of the commutator of the multiplication operator with symbol function b and the Hilbert transform $[H, b] = Hb - bH$. To see this correspondence one rewrites the commutator as a sum of Hankel operators with orthogonal ranges.

Let $H^2(\mathbb{T}^2)$ denote the Banach space of analytic functions in $L^2(\mathbb{T}^2)$. In [5], Ferguson and Sadosky study the symbols of bounded 'big' and 'little' Hankel operators on the bidisk. Big Hankel operators are those which project on to a 'big' subspace of $L^2(\mathbb{T}^2)$ - the orthogonal complement of $H^2(\mathbb{T}^2)$; while little Hankel operators project onto the much smaller subspace of complex conjugates of functions in $H^2(\mathbb{T}^2)$ - or anti-analytic functions. The corresponding commutators are

$$[H_1H_2, b], \text{ and } [H_1, [H_2, b]]$$

where $b = b(x_1, x_2)$ and H_i are the Hilbert transforms acting in the i^{th} variable. Ferguson and Sadosky show that the first commutator is bounded if and only if the symbol b belongs to the so called little BMO class, consisting of those functions that are uniformly in BMO in each variable separately. They also show that if b belongs to the product BMO space, as identified by Chang and Fefferman then the second commutator is bounded. The fact that boundedness of the commutator implies that b is in product BMO was shown in the groundbreaking paper of Ferguson and Lacey [4]. The techniques to tackle this question in several parameters are very different and have brought valuable new insight and use to existing theories, for example in the interpretation of Journé's lemma in combination with Carleson's example. Lacey and Terwilliger extended this result to an arbitrary number of iterates, requiring thus, among others, a refinement of Pipher's iterated multi-parameter version of Journé's lemma.

The first part of the lecture is concerned with mixed Hankel operators or commutators such as

$$[H_1, [H_2H_3, b]].$$

In [9], we classify boundedness of these commutators by a mixed BMO class (little product BMO): those functions $b = b(x_1, x_2, x_3)$ so that $b(\cdot, x_2, \cdot)$ and $b(\cdot, \cdot, x_3)$ are uniformly in product BMO. Similar arguments can be made for any finite iteration of commutators of the same type.

In this part of the argument, Toeplitz operators with operator symbol arise naturally - they can be used to facilitate the use of lower estimates for iterated commutators that do not contain any tensor products.

When leaving the notion of Hankel operators behind, their interpretation as commutators allow for natural generalizations. The classical text of Coifman, Rochberg and Weiss [1] extended the one-parameter theory to real analysis in the sense that the Hilbert transforms were replaced by Riesz transforms. In their text, they obtained sufficiency, i.e. that a BMO symbol b yields an L^2 bounded commutator for certain more general, convolution type singular integral operators. For necessity, they showed that the collection of Riesz transforms was representative enough:

$$\|b\|_{\mathrm{BMO}} \leq c \sup_j \|[R_j, b]\|_{2\to 2}.$$

Due to a line of investigation of Uchiyama and Li [7], other representative classes than those of the Riesz transforms are known, amongst them, Calderón-Zygmund operators adapted to cones.

The result of Coifman, Rochberg and Weiss [1] using Riesz transforms instead of Hilbert transforms, was extended to the multi-parameter setting in [6]. In [3] this classification result was extended to more general classes than those of the Riesz transforms. In a recent paper [2] it is shown that iterated commutators formed with any arbitrary Calderón-Zygmund operators are bounded if the symbol belongs to product BMO.

The second part of this lecture is concerned with an explanation of the universal class of operators defined in [3] as well as a real variable analog or commutators of the form

$$[T_1, [T_2T_3, b]],$$

where T_i are Calderón-Zygmund operators. We first extend the main result of [2] and demonstrate an upper bound for any choice of T_i by means of the appropriate little product BMO norm. We then show necessity of the little product BMO condition when the T_i are allowed to run through a representative class. This result can be found in [9].

It is well known that two-sided commutator estimates have an equivalent formulation in terms of weak factorization. We find the pre-duals of our little product BMO spaces and prove a classification by means of factorization.

References

[1] Coifman, R. R., Rochberg, R., Weiss, G., *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103**, (1976), no. 3, 611–635.
[2] Dalenc, L., Ou, Y., *Upper bound for multi-parameter iterated commutators*, Preprint, (2014).
[3] Dalenc, L., Petermichl, S., *A lower bound criterion for iterated commutators*, J. Funct. Anal., **266**, (2014) no. 8, 5300–5320.
[4] Ferguson, Sarah H., Lacey, Michael T., *A characterization of product BMO by commutators*, Acta Math., **189**, (2002), no. 2, 143–160.
[5] Ferguson, Sarah H., Sadosky, Cora, *Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures*, J. Anal. Math., **81**, (2000), 239–267.
[6] Lacey, Michael T., Petermichl, Stefanie, Pipher, Jill C., Wick, Brett D., *Multiparameter Riesz commutators*, Amer. J. Math., **131**, (2009), no. 3, 731–769.
[7] Li, Song-Ying, *Characterization of the boundedness for a family of commutators on L^p*, Colloq. Math., **70**, (1996), no. 1, 59–71.
[8] Nehari, Zeev, *On bounded bilinear forms*, Ann. of Math., **65**, (1957), 153–162.
[9] Ou, Yumeng, Petermichl, Stefanie, Strouse, Elizabeth, *Mixed commutators and little product BMO*, Preprint, (2014).

On uniform lower bounds for the Bergman kernel

Duong H. Phong
(joint work with Jian Song, Jacob Sturm)

An important problem in analysis is to determine the decay rates of oscillatory integrals, and the families of phases with respect to which these rates can be uniform. This problem has been studied extensively (see e.g. [9, 10, 11, 1, 13, 4, 6, 7] and references therein), but it is still largely open. It is likely that further progress will require significant new ideas. In this talk, we describe some important new progress, due to Donaldson and Sun [5], on a related problem from complex geometry, which is that of uniform lower bounds for the Bergman kernel. We shall also take the opportunity to describe an extension of the Donaldson-Sun techniques to Kähler-Ricci solitons.

Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X. Assume that L is positive, in the sense that L admits a metric h whose curvature $\omega = -\frac{i}{2}\partial\bar\partial \, log\, h$ is a strictly positive $(1,1)$-form. This form ω defines then a Kähler metric on X. Let $H^0(X,L)$ be the space of sections of L which are holomorphic. It can be viewed as a Hilbert space, with the L^2 inner product

$$\|\varphi\|^2 = \int_X |\varphi(z)|_h^2 \frac{\omega^n}{n!} \tag{1}$$

where $|\varphi(z)|_h$ is the norm of the section $\varphi(z)$ with respect to the metric h. The Bergman kernel $\rho(z)$ of the bundle L equipped with the metric h is defined by

$$\rho(z) = \sum_{\alpha=0}^{N} |s_\alpha(z)|_h^2 \tag{2}$$

where $\{s_\alpha(z)\}_{\alpha=0}^N$ is an orthonormal basis for $H^0(X,L)$.

It is in general an important issue whether the Bergman kernel $\rho(z)$ has a zero, or equivalently, whether all holomorphic sections of L vanish at some point. We shall be interested in examining this issue in the high energy limit, which is constructed as follows. Let k be a positive integer, and replace in the above set-up L by L^k, h by h^k, ω by $k\omega$. The Bergman kernel for the bundle L^k is then constructed out of an orthonormal basis for $H^0(X, L^k)$ and will be denoted by $\rho_k(z)$. It is a classic theorem of Kodaira that $\rho_k(z)$ will not have a zero if k is large enough. The issue is whether $\rho_k(z)$ admits a strictly positive bound which is uniform with respect to the underlying geometry.

We can now describe the results of Donaldson and Sun [5]. Fix a dimension n, and positive constants c and V. Let $\mathbf{K}(n, V, c)$ be the space of all positive bundles $L \to X$ with metric h so that $\dim X = n$, $-\frac{1}{2}\omega \leq Ric(\omega) \leq \omega$, $Vol(X) \leq V$, and $Vol\, B_r(z) \geq c\, r^{2n}$. Here Vol denotes the volume with respect to the metric ω, and $Ric(\omega) = -\frac{i}{2}\partial\bar\partial\, \omega^n$ is the Ricci curvature of ω. Then we have

Theorem 5. *(Donaldson-Sun) There exists k a positive integer and $a > 0$ depending only on n, V and c so that*

$$\inf_{\mathbf{K}(n,V,c)} \rho_k(z) \geq a \qquad \text{for all } z \in X. \tag{3}$$

The proof of Theorem 5 is built on several key ingredients:

(a) The first is a compactification of the space of parameters $\mathbf{K}(n, V, c)$. This is achieved by taking the Gromov-Hausdorff limit points of the metric spaces $(X, k\omega)$. The theory of Cheeger-Colding [2, 3] guarantees that, in presence of two-sided uniform Ricci bounds and Kähler structure, the limit points will be regular outside of a singular variety of codimension at least 4.

(b) The second is an estimate of the form

$$\|Ds\| \leq K\|s\| \tag{4}$$

for all $s \in H^0(X, L^k)$, where K is a constant depending only on n, V and c. This estimate is a consequence of the boundedness of the Ricci curvature, and of Moser iteration, the constants in which depend only on n, V and c.

(c) The third is the existence of cut-off functions supported arbitrarily near the singular set of limit manifolds and with arbitrary small L^2 norm for their gradients. This is a consequence of the fact that the singular set has codimension strictly greater than 2.

(d) The fourth is Hörmander's L^2 method for constructing holomorphic sections from a smooth section s with $\|\bar\partial s\|/\|s\|$ sufficiently small.

In essence, (a) reduces the problem to proving uniform estimate near each limit point of the space $\mathbf{K}(n, V, c)$. One can then construct holomorphic sections peaking at a distance $\sim K^{-1}$ away from the singular set in view of (b). This can be done using (d) and a smooth section supported near a given point at a distance $\sim K^{-1}$ from the singular set. The ratio $\|\bar\partial s\|/\|s\|$ can be chosen small using (c).

By combining the methods of Donaldson-Sun with those of Perelman [8], we can extend the above uniform bounds to the setting of Kähler-Ricci solitons. More precisely, we say that a compact n-dimensional Kähler manifold is a Kähler-Ricci soliton if the metric ω satisfies the equation

$$Ric(\omega) = \omega + \frac{i}{2}\partial\bar{\partial}u \tag{5}$$

where u is a smooth function with the property that $V^j = g^{j\bar{k}}\partial_{\bar{k}}u$ is a holomorphic vector field. We define the Futaki invariant of (X, ω) by $Fut(X, \omega) = \int_X |V|^2 \frac{\omega^n}{n1}$. Let V and F be positive numbers. We define $\mathbf{KR}(n, F)$ to be the space of all Kähler-Ricci solitons of dimension n and with $Fut(X, \omega) \leq F$. Then the following theorem was proved in [12]:

Theorem 6. *(P., J. Song, J. Sturm) There exists k a positive integer and $a > 0$ depending only on n and F so that*

$$\inf_{\mathbf{KR}(n,F)} \rho_k(z) \geq a \qquad \text{for all } z \in X. \tag{6}$$

The proof of Theorem 6 makes use of the idea of Z. Zhang [15] to transform by a conformal transformation the metrics ω to metrics of bounded Ricci curvature. A key step is then to prove an extension of the estimate (4) to the case of Kähler-Ricci solitons.

The Bergman kernels $\rho_k(z)$ of Theorem 5 can be realized as oscillatory integrals. Thus Theorem 5 provides a striking example where uniform bounds can hold. It is a very interesting question whether the above methods can be adapted in any way to the theory of oscillatory integrals.

References

[1] Carbery, A., M. Christ, and J. Wright, "Multidimensional van der Corput and sublevel set estimates", J. Amer. Math. Soc. 12 (1999), no. 4, 981–1015

[2] Cheeger, J. and T. Colding, "On the structure of spaces with Ricci curvature bounded below. III", J. Diff. Geom., 54 (2000), 37-74

[3] Cheeger, J., Colding, T.H. and Tian, G., "On the singularities of spaces with bounded Ricci curvature", Geom. Funct. Anal., 12 (2002), 873-914

[4] Collins, T., A. Greenleaf, and M. Pramanik, "A multi-dimensional resolution of singularities with applications to analysis", arXiv: 1007.0519, Amer. J. Math. 135 (2013), no. 5.

[5] Donaldson, S. and S. Sun, "Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry", arXiv: 1206. 2609.

[6] Gressman, P., "Uniform geometric estimates of sublevel sets", arXiv: 0909.0875, to appear in J. d'Analyse Math.

[7] Müller, D., "Problems of Harmonic Analysis related to finite type hypersurfaces in $\mathbf{R}^3$, and Newton polyhedra", arXiv:1208.6411.

[8] Perelman, G., "The entropy formula for the Ricci flow and its geometric applications", arXiv:math.DG/0211159

[9] Phong, D.H. and E.M. Stein, "The Newton polyhedron and oscillatory integral operators", Acta Math. 179 (1997), 107-152

[10] Phong, D.H., E. M. Stein and J. Sturm, "On the growth and stability of real-analytic functions", Amer. J. Math 121 (1999) no. 3, 519-554.

[11] Phong, D.H., E.M. Stein, and J. Sturm, "Multilinear level set operators, oscillatory integral operators, and Newton polyhedra", Math. Ann. 319 (2001), no. 3, 573–596

[12] Phong, D.H., J. Song, and J. Sturm, "Degeneration of Kähler-Ricci solitons on Fano manifolds", arXiv: 1211.5849

[13] Seeger, A., "Radon transforms and finite type conditions", Journal of the Amer. Math. Soc. 11 (1998), no. 4, 869-897
[14] Tian, G. and Z. Zhang, "Degeneration of Kähler-Ricci solitons", Int. Math. Res. Not. IMRN 2012, no. 5, 957-985
[15] Zhang, Z., "Degeneration of shrinking Ricci solitons", Int. Math. Res. Not. IMRN 2010, no. 21, 4137-4158

Novel bounds for exponential sums arising in number theory

L. B. Pierce
(joint work with D. R. Heath-Brown)

Bounds for oscillatory integrals play a critical role in harmonic analysis; analogous bounds for exponential and character sums are powerful tools in analytic number theory. This talk reviews familiar bounds for oscillatory integrals and then introduces the concept of short character sums, leading to a discussion of their many applications as well as recent work on proving new upper bounds.

Let $\chi(n)$ be a non-principal multiplicative Dirichlet character to a modulus q, and consider the character sum

$$S(N,H) = \sum_{N<n\leq N+H} \chi(n).$$

The Pólya-Vinogradov inequality states that

$$|S(N,H)| \ll q^{1/2}\log q,$$

which is nontrivial only if the length H of the character sum is longer than $q^{1/2}\log q$. Burgess improved on this in a series of papers in the 1950's and 60's, proving (among more general results) that for χ a non-principal multiplicative character to a prime modulus q,

$$|S(N,H)| \ll H^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} \log q,$$

for any integer $r \geq 1$, uniformly in N. This provides a nontrivial estimate for $S(N,H)$ as soon as $H > q^{1/4+\epsilon}$. The Burgess bound found immediate applications in an upper bound for the least quadratic non-residue modulo a prime and a celebrated sub-convexity estimate for Dirichlet L-functions, and has since been used in a wide range of problems in analytic number theory. The original method of proof has also been refined and simplified, but its main utility currently remains limited to a few types of short character sums. It would be highly desirable to generalize the Burgess method further to a wide range of character sums involving additive and multiplicative characters, polynomial arguments, and multiple dimensions.

Recent work of the author with D. R. Heath-Brown [1] has applied the Burgess method to short mixed character sums of the form

$$S(f;N,H) = \sum_{N<n\leq N+H} e(f(n))\chi(n),$$

where f is a real-valued polynomial of degree $d \geq 1$ and χ is a non-principal character to a prime modulus q. For integers $r, d \geq 1$, let $J_{r,d}(X)$ denote the number of solutions to the system of Diophantine equations given by

$$x_1^m + \cdots + x_r^m = x_{r+1}^m + \cdots + x_{2r}^m, \qquad \text{for all } 1 \leq m \leq d,$$

with $1 \leq x_1, \ldots, x_{2r} \leq X$. The main conjecture for Vinogradov's Mean Value Theorem states that for all $r \geq 1, d \geq 1$,

$$J_{r,d}(X) \ll_{r,d,\epsilon} X^{\epsilon}(X^r + X^{2r-D}), \tag{1}$$

with

$$D = \frac{1}{2}d(d+1).$$

Under the assumption that (1) holds, for any integer $r > D$ and $H < q^{\frac{1}{2}+\frac{1}{4(r-D)}}$, we prove that

$$\sum_{N<n\leq N+H} e(f(n))\chi(n) \ll_{r,d,\epsilon} H^{1-\frac{1}{r}} q^{\frac{r+1-D}{4r(r-D)}+\epsilon},$$

uniformly in N, for any $\epsilon > 0$. This is as strong as the original Burgess bound, and in many cases this result holds true unconditionally, due to significant recent work of Wooley on the main conjecture (1).

We also discuss more recent work of the author on Burgess bounds for multi-dimensional short mixed character sums of the following form. For each $i = 1, \ldots, k$, let χ_i be a non-principal multiplicative character modulo a prime q_i. Let f be a real-valued polynomial of total degree d in k variables and set

$$S(f; \mathbf{N}, \mathbf{H}) = \sum_{\mathbf{x} \in \mathbb{Z}^k \cap (\mathbf{N}, \mathbf{N}+\mathbf{H}]} e(f(\mathbf{x}))\chi_1(x_1) \cdots \chi_k(x_k)$$

for any k-tuple $\mathbf{N} = (N_1, \ldots, N_k)$ of real numbers and k-tuple $\mathbf{H} = (H_1, \ldots, H_k)$ of positive real numbers. Here

$$(\mathbf{N}, \mathbf{N}+\mathbf{H}] = (N_1, N_1 + H_1] \times (N_2, N_2 + H_2] \times \cdots \times (N_k, N_k + H_k]$$

denotes the corresponding box in $\mathbb{R}^k$. Note that we do not assume the primes q_i are distinct, and in particular an interesting special case arises when all the q_i are equal to a fixed prime q.

We prove bounds for $S(f; \mathbf{N}, \mathbf{H})$ by applying recent work of Parsell, Prendiville and Wooley on generalizations of the Vinogradov Mean Value theorem in the setting of translation-dilation invariant systems. Instead of defining a translation-dilation invariant system here, we merely mention that given a real-valued polynomial f, the set of all monomials of positive degree that appear in any partial derivative of f comprises a translation-dilation invariant system. Given such a translation-dilation invariant system $\mathbf{F} = \{F_1, \ldots, F_R\}$ with monomials $F_j \in \mathbb{Z}[X_1, \ldots, X_k]$, consider for any integer $r \geq 1$ the system of R Diophantine equations given by

$$\sum_{j=1}^{r} (\mathbf{F}(\mathbf{x}_j) - \mathbf{F}(\mathbf{y}_j)) = \mathbf{0}, \tag{2}$$

where $\mathbf{x}_j, \mathbf{y}_j \in \mathbb{Z}^k$ for $j = 1, \ldots, r$. Define $J_r(\mathbf{F}; X)$ to be the number of integral solutions of the system (2) with $1 \leq x_{j,i}, y_{j,i} \leq X$ for all $1 \leq j \leq r$, $1 \leq i \leq k$. Recently Parsell, Prendiville and Wooley have proved strong upper bounds for $J_r(\mathbf{F}; X)$ when $\mathbf{F}$ is a translation-dilation invariant system. Indeed, if $\mathbf{F}$ is a reduced translation-dilation invariant system having dimension k, degree d, rank R and weight M, they have proved that for any $r \geq R(d+1)$,

$$J_r(\mathbf{F}; X) \ll X^{2rk - M + \epsilon}.$$

We may apply this result to prove bounds of Burgess type for multi-dimensional short mixed character sums $S(f; \mathbf{N}, \mathbf{H})$ that are uniform as f varies over the linear span of any translation-dilation invariant system of monomials. In particular, by constructing the smallest translation-dilation invariant monomial system in which a given polynomial f lies, we may prove results of the following type: Suppose that $\mathbf{H} = (H_1, \ldots, H_k)$ satisfies $q_i^{\frac{1}{2(r-M(f))}} < H_i < q_i^{\frac{1}{2} + \frac{1}{4(r-M(f))}}$ for each $i = 1, \ldots, k$, and set $\|\mathbf{H}\| = H_1 \cdots H_k$. Given a real-valued polynomial f of total degree $d \geq 1$ in $k \geq 1$ variables, there is an associated reduced monomial translation-dilation invariant system $\mathbf{F}(f)$ with degree d, dimension k, rank $R(f)$, weight $M(f)$, and density $\gamma(f)$ such that for any $r \geq R(f) \cdot (d+1)$,

$$S(f; \mathbf{N}, \mathbf{H}) \ll \|\mathbf{H}\|^{1 - \frac{1}{r}} \|\mathbf{q}\|^{\frac{-r - M(f) + 1}{4r(r - M(f))} + \epsilon} (\mathbf{q}^{\gamma(f)})^{\frac{1}{4r(r-M(f))}} q_{\max}^{\frac{2rk}{4r(r-M(f))}} q_{\min}^{-\frac{M(f)}{4r(r-M(f))}},$$

uniformly in $\mathbf{N}$, with implied constant independent of the coefficients of f.

The construction of a tailor-made translation-dilation invariant system (and in particular one with rank, weight, and density as small as possible) is particularly advantageous if the polynomial f is such that it has higher degree in some variables than in others. Note that this is a genuinely multi-dimensional phenomenon. In the case of dimension $k = 1$, given a seed polynomial $f(x) = x^d$, the corresponding reduced monomial translation-dilation invariant system $\mathbf{F}(f)$ is $\{x, x^2, \ldots, x^d\}$, so that filling out f to $\mathbf{F}(f)$ naturally leads to a system that spans all real-valued single-variable polynomials of degree d. The strength of the multi-dimensional theorem stems from the fact that filling out a seed polynomial $f(\mathbf{x})$ of total degree d in $k \geq 2$ variables to a translation-dilation invariant system can, for particular polynomials, lead to a system that spans a smaller family of polynomials than the full family of all polynomials of total degree d in k variables.

References

[1] D. R. Heath-Brown and L. B. Pierce, *Burgess bounds for short mixed character sums*, arXiv:1404.1677 (2014)

[2] L. B. Pierce, *Burgess bounds for multi-dimensional short mixed character sums*, arXiv:1404.6994 (2014)

Unbounded potential recovery in the plane

Keith M. Rogers
(joint work with Kari Astala, Daniel Faraco)

We consider the inverse scattering problem at a fixed energy in the plane. That is to recover a potential from the scattered plane waves with a fixed frequency. More precisely, let $V : \mathbb{R}^2 \to \mathbb{R}$ be compactly supported and fix $k > 0$. For each $\theta \in \mathbb{S}^1$, let u be the outgoing scattered solution to the Schrödinger equation

$$(-\Delta + V)u = k^2 u \tag{1}$$

with direction θ. In other words, u solves the Lippmann–Schwinger equation

$$u(x,\theta) = e^{ikx\cdot\theta} - \int_{\mathbb{R}^2} G_0(x,y)V(y)u(y,\theta)\,dy,$$

where the outgoing Green's function satisfies

$$G_0(x,y) = e^{-ik\frac{x}{|x|}\cdot y}\,\frac{e^{i\frac{\pi}{4}}e^{ik|x|}}{\sqrt{8\pi k|x|}} + o\Big(\frac{1}{\sqrt{|x|}}\Big).$$

Plugging the asymptotics into the Lippmann–Schwinger equation yields

$$u(x,\theta) = e^{ik\theta\cdot x} - A[V]\Big(\frac{x}{|x|},\theta\Big)\frac{e^{i\frac{\pi}{4}}e^{ik|x|}}{\sqrt{8\pi k|x|}} + o\Big(\frac{1}{\sqrt{|x|}}\Big),$$

where $A[V] : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}$ satisfies

$$A[V](\vartheta,\theta) = \int_{\mathbb{R}^2} e^{-ik\vartheta\cdot y}V(y)u(y,\theta)\,dy. \tag{2}$$

This function, the scattering amplitude, can be measured from the scattered waves, and the challenge is then to recover the potential V from this information alone. Taking $u(y,\theta) = e^{ik\theta\cdot y}$ in (2), naively we might expect

$$A[V](\vartheta,\theta) \approx \widehat{V}\big(k(\vartheta-\theta)\big)$$

and so if we knew $A[V]$ for k infinitely large, we would reduce the problem to that of inverting the Fourier transform. For practical applications however it is desirable to keep the energy finite.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain that contains the support of V. Under mild assumptions, there is a unique solution to (1) inside Ω with Dirichlet data $u|_{\partial\Omega} = f$ allowing us to define the Dirichlet-to-Neumann map Λ_{V-k^2} formally given by

$$\Lambda_{V-k^2} \,:\, f \mapsto \nabla u \cdot n|_{\partial\Omega}.$$

The Dirichlet-to-Neumann map is uniquely determined by the scattering amplitude [1]. On the other hand, recovery of the potential from the Dirichlet-to-Neumann map has been considered in connection with Calderón's inverse problem [3].

Here we report on recent work in which we first adapt constructive arguments due to Nachman [6, 7] and Stefanov [9] in order to provide explicit formulae that recover Λ_{V-k^2} from $A[V]$ for compactly supported potentials with some Sobolev regularity. We then give explicit formulae with which one can recover V from Λ_{V-k^2} as long as the potential has half a derivative in L^2. For this we make a connection between the recent work of Bukhgeim [2] and Carleson's pointwise convergence question for time-dependent Schrödinger equations [4].

After recovering the Dirichlet-to-Neumann map, Alessandrini's identity gives us that

$$\int_{\partial\Omega} (\Lambda_{V-k^2} - \Lambda_0)[u]v = \int_{\Omega} (V - k^2)uv$$

as long as u solves (1) and $\Delta v = 0$, and so, after recovering Λ_{V-k^2} from $A[V]$, we are required to come up with solutions from which we can recover the potential from the right-hand side. Following Bukhgeim, we use exponentially increasing solutions with quadratic phases rather than the more traditional linear phases. After some further manipulations we can interpret the right-hand side as a perturbation of a solution to the nonelliptic time-dependent Schrödinger equation

$$\begin{cases} i\partial_t w + \Box w = 0 \\ w(\cdot, 0) = V, \end{cases}$$

where $\Box = \partial_{x_1x_1} - \partial_{x_2x_2}$. Certain choices of solutions u and v correspond to the same solution w at different times, and it turns out that as long as the solution w converges almost everywhere to V as time tends to zero, we can recover the potential. To prove the convergence, we factorise the problem into two one-dimensional problems as in [5] and then using the Kolmogorov–Seliverstov–Plessner method as in [4].

By adapting work from [8], we also provide examples of compactly supported potentials with $s < 1/2$ derivatives in L^2 for which the recovery process fails completely. Interpreting the problem acoustically however, it is unsurprising that we are unable to recover all the potentials with this level of regularity. Taking

$$V(x) = k^2(1 - c^{-2}(x)),$$

where $c(x)$ denotes the speed of sound at x, the scattered solutions u also satisfy $c^2\Delta u + k^2 u = 0$. Now there are potentials with $s < 1/2$ derivatives in L^2, which are singular on closed curves. Thus the speed of sound is zero on the curve and so a continuous solution u would be zero there. Interpreting physically, incident waves cannot pass through vacuum and so it would be unreasonable to expect to be able to detect what is inside.

References

[1] Ju. M. Berezanskiĭ, The uniqueness theorem in the inverse problem of spectral analysis for the Schrödinger equation, Trudy Moskov. Mat. Obšč. **7** (1958), 1–62.

[2] A.L. Bukhgeim, Recovering a potential from Cauchy data in the two-dimensional case, J. Inverse Ill-Posed Probl. **16** (2008), no. 1, 19–33.

[3] A.P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matematica, Rio de Janeiro, (1980), 65-73.

[4] L. Carleson, Some analytic problems related to statistical mechanics, Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), Lecture Notes in Math. **779**, 5–45, Springer, Berlin, 1980.

[5] B.E.J. Dahlberg and C.E. Kenig, A note on the almost everywhere behavior of solutions to the Schrödinger equation, Harmonic analysis (Minneapolis, Minn., 1981), Lecture Notes in Math. **908**, 205–209, Springer, Berlin, 1982.

[6] A.I. Nachman, Reconstructions from boundary measurements. Ann. of Math. **128** (1988), no. 3, 531–576.

[7] A.I. Nachman, Inverse scattering at fixed energy, in *Mathematical physics, X (Leipzig, 1991)*, 434–441, Springer, Berlin.

[8] K.M. Rogers, A. Vargas and L. Vega, Pointwise convergence of solutions to the nonelliptic Schrödinger equation. Indiana Univ. Math. J. **55** (2006), no. 6, 1893–1906.

[9] P. Stefanov, Stability of the inverse problem in potential scattering at fixed energy, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 4, 867–884 (1991).

L^p estimates for the wave equation on Heisenberg groups

Andreas Seeger

(joint work with Detlef Müller)

Let G be a Heisenberg group of topological dimension d and let L be the usual sums of squares sub-Laplacian L on G. We consider the problem of L^p regularity for solutions for the wave equation associated to L,

$$(\partial_\tau^2 + L)u = 0, \quad u\big|_{\tau=0} = f, \quad \partial_\tau u\big|_{\tau=0} = g. \tag{1}$$

Various qualitative statements for the kernel of the wave operator were obtained by Nachman [3]. Müller and Stein [2] proved almost sharp L^p estimates, namely

$$\|u(\cdot,\tau)\|_p \le C\big[\|(I+\tau^2 L)^{\frac{\gamma}{2}} f\|_p + \|\tau(I+\tau^2 L)^{\frac{\gamma}{2}-1} g\|_p\big], \tag{2}$$

for $\gamma > (d-1)|1/p-1/2|$.

In the recent work [1] we provide a parametrix in the form of a sum of oscillatory integral operators with singular phase functions which enables us to prove the sharp endpoint results. In the construction of our parametrix we use a subordination formula which expresses spectrally-localized versions of the wave operators in terms of Schrödinger operators. Let $\chi_1 \in C^\infty$ so that $\chi_1(s) = 1$ for $s \in [1/4, 4]$. Let g be a C^∞ function supported in $(1/2, 2)$ and let $\lambda \gg 1$. Then there are C^∞ functions a_λ depending linearly on g, with a_λ supported in $[1/16, 4]$, so that

$$g(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}} = A_\lambda + E_\lambda,$$

the negligible operators E_λ are bounded on $L^p(G)$ with norm $O(\lambda^{-N})$ and the main term A_λ is given by

$$A_\lambda = \chi_1(\lambda^{-2})L)\sqrt{\lambda}\int e^{i\frac{\lambda}{4s}} a_\lambda(s)\, e^{isL/\lambda}\, ds.$$

For this term one can use known explicit formulas for the Schrödinger operators to get an oscillatory integral representation for the kernel K_λ of A_λ, as follows:

$$K_\lambda(x,u) \;=\; \lambda^{1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\frac{\lambda}{4s}} a_\lambda(s) \eta_0(2\pi|\mu|\tfrac{s}{\lambda}) \times \\ \Big(\frac{|\mu|}{2\sin(2\pi|\mu|s/\lambda)}\Big)^{(d-1)/2} e^{-i|x|^2 \frac{\pi}{2}|\mu|\cot(2\pi|\mu|s/\lambda)} ds\; e^{2\pi i u\mu} d\mu\,.$$

One can understand the singularities of the kernels by making an equally spaced decomposition in the variable $\tilde{\mu} = \mu s/\lambda$ and relate the pieces to Nachman's description of the singular support. A careful analysis of these oscillatory integrals leads to the estimate $\|K_\lambda\|_{L^1} = O(\lambda^{(d-1)/2})$, and then to

Theorem. *For $\lambda \gg 1$,*

$$\|g(\lambda^{-1}\sqrt{L})e^{i\sqrt{L}}\|_{L^1\to L^1} \;=\; O(\lambda^{(d-1)/2})\,.$$

We can also combine estimates for different dyadic scales $\lambda = 2^j$ to prove bounds for functions in a suitable local Hardy space on the group. An interpolation argument then yields sharp L^p bounds:

Theorem. *For $1 < p < \infty$*

$$\big\|(I+t^2L)^{-\alpha/2}e^{it\sqrt{L}}\big\|_{L^p(G)\to L^p(G)} < \infty, \quad \alpha = (d-1)|1/p-1/2|\,.$$

Consequently estimate (2) holds for $\gamma = (d-1)|1/p-1/2|$, $1 < p < \infty$. The L^p regularity estimates for the wave equation can be used to sharpen known results for general multiplier transformations $m(L)$. All boundedness results can be extended to the more general setting of groups of Heisenberg type.

References

[1] D. Müller, A. Seeger, *Sharp L^p bounds for solutions of the wave equation on groups of Heisenberg type*, preprint, 2014.

[2] D. Müller, E.M. Stein, *L^p-estimates for the wave equation on the Heisenberg group,* Rev. Mat. Iberoamericana 15 (1999), no. 2, 297–334.

[3] A.I. Nachman, *The wave equation on the Heisenberg group,* Comm. in PDE, 7 (1982), 675–714.

Uniform bounds for Fourier restriction to polynomial curves

Betsy Stovall

Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a d-times continuously differentiable curve. Then the affine arclength measure along γ is given (in parametrized form) by

$$\lambda_\gamma(t)\, dt = |\det(\gamma'(t), \ldots, \gamma^{(d)}(t))|\, dt.$$

It is easy to verify that this defines a parametrization-invariant measure along γ that transforms nicely under affine transformations of $\mathbb{R}^d$. In our talk, we outlined a proof of the following theorem.

Theorem 7. *For each dimension $d \geq 2$, positive integer N, and exponent pairs (p, q) satisfying*

$$p' = \tfrac{d(d+1)}{2} q, \qquad q > \tfrac{d^2+d+2}{d^2+d},$$

there exists a constant $C_{N,d,p}$ such that for all polynomials $\gamma : \mathbb{R} \to \mathbb{R}^d$ of degree less than or equal to N, the Fourier restriction estimate

$$\|\hat{f}(\gamma(t))\|_{L^q(\lambda_\gamma\, dt)} \leq C_{N,d,p} \|f\|_{L^q(dx)},$$

holds for all Schwartz functions $f : \mathbb{R}^d \to \mathbb{C}$.

This theorem is sharp in the sense that this the largest possible range of exponents for which such a result is possible and in the sense that this is essentially the largest measure for which such a result can hold.

Prior work had established the theorem in dimension $d = 2$ [10], for nondegenerate curves such as $\gamma(t) = (t, t^2, \ldots, t^d)$ [9, 3, 6], for monomial curves [7, 8, 1], for general polynomial curves in a restricted range [5, 2], for monomial-like curves in the full range [4], and for "simple" curves in the full range [2]. In the talk we sketched a new proof from [11], which resolves the remaining cases. We outline the argument below.

Let $\mathcal{E}_\gamma$ denote the dual/extension operator

$$\mathcal{E}_\gamma f(x) = \int e^{ix\gamma(t)} f(t) \lambda_\gamma(t)\, dt.$$

By duality, it suffices to prove estimates of the form

$$\|\mathcal{E}_\gamma f\|_{L^p(\mathbb{R}^d)} \leq C_{N,d,p} \tag{1}$$

in the dual range, $q = \frac{d(d+1)}{2} p'$, $q > \frac{d^2+d+2}{2}$.

The first step is to prove a uniform local estimate, i.e. to prove that (1) holds for the full (p, q) range when f is supported on an interval I on which λ_γ is proportional to some fixed constant, $\lambda_\gamma \sim C$. This is done by a rescaling, followed by a compactness argument, and then an adaptation of the interpolation/bootstrap approach of Drury from [6]. Restricting to regions where λ_γ is essentially constant avoids some of the geometric and analytic difficulties faced using earlier methods.

The next step is to prove a uniform square function estimate of the form

$$\|\mathcal{E}_\gamma f\|_{L^q(\mathbb{R}^d)}\|_{L^q} \leq C_{q,d,N} \|(\sum_k |\mathcal{E}_\gamma \chi_{I_k} f|^2)^{\frac{1}{2}}\|_{L^q(\mathbb{R}^d)} \tag{2}$$

to separate out the different torsion scales $I_k = \{t \in \mathbb{R} : \lambda_\gamma(t) \sim 2^k\}$. Heuristically (after a reparametrization), polynomials are locally well-approximated by monomials, and if $\lambda_\gamma(t) \sim t^{k_1}$ and $\gamma_1(t) \sim t^{k_2}$, then we can apply the Littlewood–Paley square function estimate in the x_1 variable (and then Fubini) to obtain (2). Some care must be taken, however, to ensure that the 'local' and 'well-approximated' aspects are sufficiently uniform.

The last step is to put the pieces together. In proving (1), we may assume (by interpolation) that $q \leq d(d+1)$. Thus by (2) and Minkowski's inequality,

$$\|\mathcal{E}_\gamma f\|_{L^q}^q \lesssim \sum_{n_1 \leq \cdots \leq n_D} \int \prod_{j=1}^{D} |\mathcal{E}_\gamma \chi_{I_{k_j}} f|^{\frac{q}{D}},$$

where $D = \frac{d(d+1)}{2}$. The main tool in controlling the right hand side is a multilinear estimate with decay, namely that if $n_1 \leq \cdots \leq n_D$, then

$$\|\prod_{j=1}^{D} \mathcal{E}_\gamma \chi_{I_{k_j}} f\|_{q/D} \lesssim 2^{-\varepsilon_{q,d}(n_D - n_1)} \prod_{j=1}^{D} \|\chi_{I_{k_j}} f\|_{L^p}.$$

This is proved using an interpolation argument, a lemma of Christ [3] that arose in the nondegenerate case, and a geometric inequality of Dendrinos–Wright [5].

References

[1] J.-G. Bak, D. M. Oberlin, A. Seeger, *Restriction of Fourier transforms to curves. II. Some classes with vanishing torsion.* J. Aust. Math. Soc. 85 (2008), no. 1, 1–28.

[2] J.-G. Bak, D. M. Oberlin, A. Seeger, *Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure.* J. Reine Angew. Math. 682 (2013), 167–205.

[3] M. Christ, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case.* Trans. Amer. Math. Soc. 287 (1985), no. 1, 223–238.

[4] S. Dendrinos, D. Müller, *Uniform estimates for the local restriction of the Fourier transform to curves.* Trans. Amer. Soc. 365 (2013), 3477–3492.

[5] S. Dendrinos, J. Wright, *Fourier restriction to polynomial curves I: a geometric inequality.* Amer. J. Math. 132 (2010), no. 4, 1031–1076.

[6] S. W. Drury, *Restrictions of Fourier transforms to curves.* Ann. Inst. Fourier (Grenoble) 35 (1985), no. 1, 117–123.

[7] S. W. Drury, B. P. Marshall, *Fourier restriction theorems for curves with affine and Euclidean arclengths.* Math. Proc. Cambridge Philos. Soc. 97 (1985), no. 1, 111–125.

[8] S. W. Drury, B. P. Marshall, *Fourier restriction theorems for degenerate curves.* Math. Proc. Cambridge Philos. Soc. 101 (1987), no. 3, 541–553.

[9] E. Prestini, *Restriction theorems for the Fourier transform to some manifolds in $\mathbb{R}^n$.* Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, pp. 101–109, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979.

[10] P. Sjölin, *Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\mathbb{R}^2$.* Studia Math. 51 (1974), 169–182.

[11] B. Stovall, *Uniform estimates for Fourier restriction to polynomial curves in $\mathbb{R}^d$*, preprint.

Multilinear singular integrals of Christ-Journé type

Brian Street
(joint work with Andreas Seeger, Charlie Smart)

In [1], Christ and Journé studied multilinear operators of the form

$$\begin{aligned} &T(a_1,\ldots,a_{n+2}) \\ &= \iiint K(x-y)\chi_{[0,1]^n}(\alpha)a_1(\alpha_1 x+(1-\alpha_1)y)\cdots a_n(\alpha_n x+(1-\alpha_n)y) \\ &\quad \times a_{n+1}(x)a_{n+2}(y)\,dx\,dy\,d\alpha. \end{aligned}$$

Here, $K(x)$ is a Calderón-Zygmund kernel on $\mathbb{R}^d$. They proved, for $a_1,\ldots,a_n \in L^\infty(\mathbb{R}^d)$, $a_{n+1}\in L^p(\mathbb{R}^d)$, $a_{n+2}\in L^{p'}(\mathbb{R}^d)$ (with $1<p<\infty$) that for any $\epsilon>0$,

$$|T(a_1,\ldots,a_{n+2})| \le C_{d,p,\epsilon} n^{n+\epsilon}\left[\prod_{l=1}^n \|a_l\|_{L^\infty}\right]\|a_{n+1}\|_{L^p}\|a_{n+2}\|_{L^{p'}}. \tag{1}$$

Careful inspection of the proof in [1] shows that the conclusion (1) is invariant under permuting the roles of $a_1,\ldots,a_{n+2}$, but the class of operators is not. I.e., if σ is a permutation of $\{1,\ldots,n+2\}$, then $T(a_{\sigma(1)},\ldots,a_{\sigma(n+2)})$ is not of the same form as $T(a_1,\ldots,a_{n+2})$.

The goal of this talk is to present a generalization of Christ and Journé's result which is symmetric under adjoints. The main result is the following. We consider kernels $K(\alpha,v)$ ($\alpha\in\mathbb{R}^n$, $v\in\mathbb{R}^d$ which have the following norm finite, for some $\epsilon>0$:

$$\begin{aligned} \|K\|_{\mathcal{K}_\epsilon} := &\sup_{\substack{1\le i\le n\\ R>0}} \iint_{R\le|x|\le 2R} (1+|\alpha_i|)^\epsilon |K(\alpha,x)|\,dx\,d\alpha \\ &\sup_{\substack{1\le i\le n\\ R>0\\ 0<h\le 1}} h^{-\epsilon}\iint_{R\le|x|\le 2R} |K(\alpha+he_i,x)-K(\alpha,x)|\,dx\,d\alpha \\ &\sup_{\substack{R>0\\ y\in\mathbb{R}^d}} R^\epsilon \iint_{|x|\ge R|y|} |K(\alpha,x-y)-K(\alpha,x)|\,dx\,d\alpha \\ &+ \text{(cancellation condition)}, \end{aligned}$$

where the "cancellation condition" is of the normal sort of cancellation condition one assumes for Calderón-Zygmund kernels. We also define another norm $\|K\|_{\mathcal{K}_0}$ which satisfies $\|K\|_{\mathcal{K}_0}\le\|K\|_{\mathcal{K}_\epsilon}$. The main result comes as two theorems.

Theorem: Suppose $\|K\|_{\mathcal{K}_\epsilon}<\infty$. Then $\forall\delta>0$, $\exists\epsilon'=\epsilon'(\epsilon,d)>0$ such that $\forall$ permutations σ of $\{1,\ldots,n+2\}$,

$$T[K](a_{\sigma(1)},\ldots,a_{\sigma(n+2)}) = T[K_\sigma](a_1,\ldots,a_{n+2}),$$

with $\|K_\sigma\|_{\mathcal{K}_{\epsilon'}}\le Cn^\delta\|K\|_{\mathcal{K}_\epsilon}$, and $\|K_\sigma\|_{\mathcal{K}_0}=\|K\|_{\mathcal{K}_0}$.

Theorem: Suppose $\|K\|_{\mathcal{K}_\epsilon} < \infty$ and $p_1, \dots, p_{n+2} \in (1, \infty]$ with $\sum p_j^{-1} = 1$. Then,

$$|T[K](a_1, \dots, a_{n+2})| \leq C\|K\|_{\mathcal{K}_0} n^2 \log^3 \left(2 + n\frac{\|K\|_{\mathcal{K}_\epsilon}}{\|K\|_{\mathcal{K}_0}}\right) \prod_{l=1}^{n+2} \|a_l\|_{L^{p_l}}.$$

In this talk we present the main motivation for the above definitions. The ideas take their roots in understanding the connections between these operators and $\mathbb{R}P^n$.

References

[1] M. Christ, J-L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. **159** (1987), no 1-2, 51-80.

Square functions, doubling conditions, densities, and rectifiability

Xavier Tolsa

A set $E \subset \mathbb{R}^d$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i = 1, 2, \dots$, such that

$$\mathcal{H}^n\left(\mathbb{R}^d \setminus \bigcup_i f_i(\mathbb{R}^n)\right) = 0, \tag{1}$$

where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. Also, one says that a a Radon measure μ on $\mathbb{R}^d$ is n-rectifiable if μ vanishes out of an n-rectifiable set $E \subset \mathbb{R}^d$ and moreover μ is absolutely continuous with respect to $\mathcal{H}^n|_E$. In the case $n = 1$, instead of saying that a set or a measure is 1-rectifiable, one just says that it is rectifiable.

Given a Radon measure μ and $x \in \mathbb{R}^d$ we denote

$$\Theta^{n,*}(x, \mu) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}, \qquad \Theta^n_*(x, \mu) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}.$$

These are the upper and lower n-dimensional densities of μ at x. If they coincide, they are denoted by $\Theta^n(x, \mu)$. In the case when $\mu = \mathcal{H}^n|_E$ for some set $E \subset \mathbb{R}^d$, we also write $\Theta^{n,*}(x, E)$, $\Theta^n_*(x, E)$, $\Theta^n(x, E)$ instead of $\Theta^{n,*}(x, \mathcal{H}^n|_E)$, $\Theta^n_*(x, \mathcal{H}^n|_E)$, $\Theta^n(x, \mathcal{H}^n|_E)$, respectively.

A fundamental result concerning the relationship between rectifiability and densities is given by the following celebrated theorem of Preiss [Pr].

Theorem. *A Radon measure μ in $\mathbb{R}^d$ is n-rectifiable if and only if the density*

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n} \tag{2}$$

exists and is non-zero for μ-a.e. $x \in \mathbb{R}^d$.

In particular, for $\mu = \mathcal{H}^n|E$ with $\mathcal{H}^n(E) < \infty$, the preceding theorem ensures the n-rectifiability of E just assuming that the density $\Theta^n(x,E)$ exists and is non-zero for $\mathcal{H}^n$-a.e. $x \in E$.

Quite recently, in the works [CGLT] and [TT], the authors have obtained some results which can be considered as square function versions of Preiss theorem. In particular, in [TT] the following is proved:

Theorem 8. *Let μ be a Radon measure in $\mathbb{R}^d$ such that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Then μ is n-rectifiable if and only if*

$$\int_0^1 \left| \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{3}$$

This theorem was preceded by the proof of a related result in [CGLT] which characterizes the so called uniform n-rectifiability in terms of a square function similar to the one in (3).

A natural question is if the condition (3) above implies the n-rectifiability of E just under the assumption that $0 < \Theta^{n,*}(x,\mu) < \infty$ μ-a.e. If this were true, then we would deduce that a set $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$ is n-rectifiable if and only if

$$\int_0^1 \left| \frac{\mathcal{H}^n(E\cap B(x,r))}{r^n} - \frac{\mathcal{H}^n(E\cap B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E.$$

The arguments used in [TT] make an essential use of the assumption that the lower density $\Theta_*(x,\mu)$ is positive. So different techniques are required if one wants to extend Theorem 8 to the case of vanishing lower density. Quite recently this problem has been solved by the author in the case $n = 1$:

Theorem 9. *Let μ be a Radon measure in $\mathbb{R}^d$ such that $\Theta^{1,*}(x,\mu) > 0$ for μ-a.e. $x \in \mathbb{R}^d$. Then μ is rectifiable if and only if*

$$\int_0^1 \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{4}$$

Corollary 10. *Let $E \subset \mathbb{R}^d$ be a Borel set with $\mathcal{H}^1(E) < \infty$. The set E is rectifiable if and only if*

$$\int_0^1 \left| \frac{\mathcal{H}^1(E\cap B(x,r))}{r} - \frac{\mathcal{H}^1(E\cap B(x,2r))}{2r} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in E.$$

I do not know if the analogous result in the case $n > 1$ holds.

Let us remark that the "only if" part of Theorem 9 is an immediate consequence of Theorem 8 above. Indeed, if μ is rectifiable, then it follows easily that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. So the assumptions of Theorem 8 are fulfilled and thus (3) holds.

The proof of the "if" implication of Theorem 9 combines a compactness argument which originates from [CGLT] and constructive techniques involving stopping time conditions. One of the main difficulties, which is absent in [TT], consists in controlling the oscillations of the densities $\frac{\mu(B(x,r))}{r}$ as $r \to 0$. If the power in the integrand of (4) were 1 instead of 2, then this task would be significantly easier, and we could argue as in [TT].

In our arguments, a basic tool for the control of such oscillations of the density is the construction of suitable measures σ^k supported on some approximating curves Γ^k so that, for each k, σ^k has linear growth with some absolute constant and such that the $L^2(\sigma^k)$ norm of a smooth version of the square function in (4), with μ replaced by σ^k, is very small. The main obstacle to extend Theorem 9 to higher dimensions lies in the difficulty to extend this construction to the case $n > 1$.

Another recent result obtained by the author deals with the connection between the boundedness in $L^2(\mu)$ of the square function

$$T\mu(x) = \left(\int_0^\infty \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r} \right)^{1/2}$$

and the $L^2(\mu)$ boundedness of the Cauchy transform. Recall that given a complex Radon measure ν on $\mathbb{C}$, its Cauchy transform is defined by

$$\mathcal{C}\nu(z) = \int \frac{1}{z-\xi}\, d\nu(\xi),$$

whenever the integral makes sense.

In the particular case when $\mu = \mathcal{H}^1|E$ with $\mathcal{H}^1(E) < \infty$, by the theorem of David-Léger [Lé], the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ implies the rectifiability of E. So it is natural to expect some relationship between the behaviors of the Cauchy transform of μ and of the square function $T\mu$. The next theorem shows that indeed there is a very strong and precise connection between the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ and the $L^2(\mu)$ behavior of $T\mu$ for arbitrary measures μ with linear growth.

Theorem 11. *Let μ be a finite Radon measure in $\mathbb{C}$ satisfying the linear growth condition*

$$\mu(B(x,r)) \leq c\, r \qquad \text{for all } x \in \mathbb{C} \text{ and all } r > 0.$$

The Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ if and only if

$$\int_{x\in Q} \int_0^\infty \left| \frac{\mu(Q\cap B(x,r))}{r} - \frac{\mu(Q\cap B(x,2r))}{2r} \right|^2 \frac{dr}{r}\, d\mu(x) \leq c\,\mu(Q) \tag{5}$$

for every square $Q \subset \mathbb{C}$.

The behavior of the square integral $T\mu$ is related to the cancellation properties of the densities $\frac{\mu(B(x,r))}{r}$, $x \in \mathbb{C}$, $r > 0$. On the other hand, heuristically the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ seems to be more connected to the behavior of the approximate tangents to μ. So it is quite remarkable (to the author's point of view) that the behavior of $T\mu$ is so strongly connected to the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$, as shown in the preceding theorem.

The proof of Theorem 11 uses a corona decomposition analogous to the one of [To]. Loosely speaking, the condition (5) is equivalent to the existence of a corona decomposition such as the one mentioned above, which in turn is equivalent to the $L^2(\mu)$ boundedness of the Cauchy transform because of the results of [To].

References

[CGLT] V. Chousionis, J. Garnett, T. Le and X. Tolsa, *Square functions and uniform rectifiability.* Preprint (2014).

[Lé] J.C. Léger. *Menger curvature and rectifiability.* Ann. of Math. 149 (1999), 831–869.

[Pr] D. Preiss, *Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities,* Ann. of Math. 125 (1987), 537–643.

[To] X. Tolsa, *Bilipschitz maps, analytic capacity, and the Cauchy integral*, Ann. of Math. 162:3 (2005), 1241–1302.

[TT] X. Tolsa and T. Toro, *Rectifiability via a square function and Preiss' theorem.* To appear in Internat. Math. Res. Not.

Hardy type spaces on noncompact manifolds with exponential volume growth

Maria Vallarino

(joint work with Giancarlo Mauceri, Stefano Meda)

We shall discuss the results contained in a series of papers [3, 4, 5, 6] with G. Mauceri and S. Meda where we develop a theory of Hardy type spaces on a class of noncompact Riemannian manifolds with bounded geometry and spectral gap and we apply this theory to prove endpoint results for certain spectral multipliers of the Laplacian and for Riesz transforms.

Let (M, g) be a complete connected noncompact Riemannian manifold of dimension n and denote by $\mathcal{L}$ the Laplace–Beltrami operator on M. We assume that M has positive injectivity radius, Ricci curvature bounded from below and that the bottom of the spectrum of $\mathcal{L}$ on $L^2(M)$ is strictly positive, i.e. $b = \inf \sigma_2(\mathcal{L}) > 0$.

Under these assumptions the Riemannian measure μ is locally doubling but not globally doubling. More precisely, if we denote by $B(p, r)$ the geodesic ball with centre p and radius r, there exist a positive constant C such that

$$\frac{1}{C}\, r^n \leq \mu\big(B(p,r)\big) \leq C\, r^n \qquad \forall r \in (0,1),\, p \in M\,,$$

and there exist a nonnegative constant α and positive constants C, β such that

$$\mu\big(B(p,r)\big) \leq C\, r^\alpha\, e^{\beta r} \qquad \forall r \in [1,\infty),\, p \in M\,.$$

Examples of manifolds satisfying our geometric assumptions are noncompact symmetric spaces with the Killing metric, Damek–Ricci spaces, nonamenable connected unimodular Lie groups with a left invariant Riemannian distance.

There are interesting operators which are bounded on $L^p(M)$ for every $p \in (1, \infty)$, but which are neither bounded on $L^1(M)$, nor of weak type $(1, 1)$. Examples include Riesz potentials and Riesz transforms of high order on noncompact symmetric spaces. One motivation of our work is to find an endpoint result for $p = 1$ for such operators. More generally, our aim is to find a suitable Hardy type space X strictly included in $L^1(M)$ such that

- X is "sufficiently large" to have nice interpolation properties, namely

$$\big(X, L^2(M)\big)_\theta = L^p(M) \qquad \theta \in (0,1),\ \frac{1}{p} = 1 - \frac{\theta}{2}\,;$$

- X is "small enough" to prove endpoint results for interesting singular integral operators, like the imaginary powers of the Laplacian $\mathcal{L}^{iu}, u \in \mathbb{R}$, and the Riesz transform $\nabla \mathcal{L}^{-1/2}$.

In the geometric setting described above A. Carbonaro, G. Mauceri and S. Meda [1] (inspired by a previous work by A. Ionescu in the setting of rank one symmetric spaces [2]) introduced an atomic Hardy space $H^1(M)$, where atoms are supported only on "small balls". In [1] the authors showed that $L^p(M)$ spaces, $p \in (1,2)$, are intermediate spaces between $H^1(M)$ and $L^2(M)$ and proved that certain functions of the Laplacian whose kernel is integrable at infinity are bounded from $H^1(M)$ to $L^1(M)$. However, the Hardy space $H^1(M)$ turns out to be too large to obtain endpoint results for singular integrals which are "singular at infinity". For example, if M is a noncompact symmetric space, then the Riesz transform $\nabla \mathcal{L}^{-1/2}$, the Riesz potentials $\mathcal{L}^{-\sigma}$, $\sigma > 0$, and the imaginary powers $\mathcal{L}^{iu}$, $u \in \mathbb{R}$, do not map $H^1(M)$ in $L^1(M)$.

Thus with Mauceri and Meda [4] we introduce an atomic Hardy type space defined in terms of atoms supported in "small balls" which satisfy an infinite dimensional cancellation condition. More precisely, fix $s_0 = \frac{1}{2} Inj_M$. An *admissible* X^1 *atom* is function A supported in a geodesic ball B of radius $r_B \leq s_0$, which satisfies the following conditions:

(a) $\|A\|_{L^2} \leq \mu(B)^{-1/2}$ (size condition)
(b) $\int A\, v\, d\mu = 0$ for all $v \in q(B) = \{u \in L^2(B) : \mathcal{L}u = cost \ on \ B\}$ (cancellation condition).

It is worth noticing that the cancellation condition (ii) is stronger than the usual cancellation condition for atoms (which simply requires the vanishing average of an atom). The *atomic Hardy space* $X^1(M)$ is the space of all functions in $L^1(M)$ which admit an atomic decomposition in terms of admissible X^1 atoms and we define

$$\|F\|_{X^1} = \inf \left\{ \sum_j |c_j| \, : \, F = \sum_j c_j\, A_j,\ A_j\, admissible\, X^1\, atom \right\} \qquad \forall F \in X^1(M)\,.$$

The Hardy space $X^1(M)$ is strictly included in $H^1(M)$ and can be described by the following equivalent characterization.

Theorem 1. The operator $\mathcal{U} = \mathcal{L}(\mathcal{I} + \mathcal{L})^{-1}$ is an isomorphism between $H^1(M)$ and $X^1(M)$. Moreover there exist a positive constant C such that

$$\frac{1}{C}\|F\|_{X^1} \leq \|\mathcal{U}^{-1}F\|_{H^1} \leq C\|F\|_{X^1} \qquad \forall F \in X^1(M)\,.$$

By using Theorem 1 and the fact that $H^1(M)$ has nice interpolation properties, we deduce that

$$\big(X^1(M), L^2(M)\big)_\theta = L^p(M) \qquad \theta \in (0,1),\, \frac{1}{p} = 1 - \frac{\theta}{2}\,.$$

The space $X^1(M)$ turns out to be useful to study the boundedness of singular integrals on the manifold. Indeed, we prove the following boundedness result.

Theorem 2. Let $\mathcal{T}$ be a singular integral operator bounded on $L^2(M)$ such that

$$\sup_{y\in B} \int_{(4B)^c} |K_{\mathcal{T}\circ\mathcal{L}}(x,y)|d\mu(x) \leq C\, r_B^{-2} \qquad \forall B = B(c_B, r_B),\, r_B \leq s_0\,,$$

for some positive constant C, where $K_{\mathcal{T}\circ\mathcal{L}}$ denotes the integral kernel of $\mathcal{T}\circ\mathcal{L}$. Then $\mathcal{T}$ extends to a bounded operator from $X^1(M)$ to $L^1(M)$.

In [6] we apply Theorem 2 to obtain the following sharp endpoint result for the imaginary powers of the Laplacian and the first order Riesz transform.

Theorem 3. The operators $\mathcal{L}^{iu}, u \in \mathbb{R}$, and $\nabla\mathcal{L}^{-1/2}$ are bounded from $X^1(M)$ to $L^1(M)$.

Actually in [3, 4] we construct a strictly decreasing sequence of Hardy type spaces $\{X^k(M)\}_{k\geq 1}$, defined by taking admissible atoms orthogonal to k-quasi harmonic functions instead of quasi-harmonic functions in the cancellation condition (b) above. The characterization and the interpolation results described above for the space $X^1(M)$ have a counterpart for the spaces $X^k(M)$, $k > 1$, provided that the covariant derivatives of the Ricci tensor $\nabla^j Ric$, $j = 0, \ldots, 2k-2$, are uniformly bounded on the manifold. The Hardy type spaces $X^k(M)$ with $k > 1$ are useful to obtain endpoint results for singular integral operators on the manifold which are "more singular at infinity". We prove for example the following result for a class of functions of the Laplacian which are more general than that considered by M. Taylor [7, 8].

Theorem 4. Let M be a noncompact Riemannian manifold satisfying our geometric assumptions and let $b > 0$ be the bottom of the spectrum of the Laplacian $\mathcal{L}$ on $L^2(M)$. Let m be an even holomorphic function in the strip $\Sigma_{\sqrt{b}} = \{z \in \mathbb{C} : |\Im z| < \sqrt{b}\}$ for which there exist a positive constant C such that

$$|D^j m(\zeta)| \leq C \max\big(|\zeta^2 + b|^{-\tau-j}, |\zeta|^{-j}\big) \qquad \forall \zeta \in \Sigma_{\sqrt{b}}, j = 0, \ldots, J\,,$$

for some $\tau \geq 0$ and for some $J > \frac{n}{2} + 2$. If $k > \tau + J$, then $m(\sqrt{\mathcal{L} - b})$ is bounded from $X^k(M)$ to $H^1(M)$.

References

[1] A. Carbonaro, G. Mauceri, S. Meda, *H^1 and BMO for certain locally doubling metric measure spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **8** (2009), no. 3, 543–582.

[2] A. Ionescu, *Fourier integral operators on noncompact symmetric spaces of real rank one*, J. Funct. Anal. **174** (2000), no. 2, 274–300.

[3] G. Mauceri, S. Meda, M. Vallarino, *Hardy type spaces on certain noncompact manifolds and applications*, J. Lond. Math. Soc. (2) **84** (2011), no. 1, 243–268.

[4] G. Mauceri, S. Meda, M. Vallarino, *Atomic decomposition of Hardy type spaces on certain noncompact manifolds*, J. Geom. Anal. **22** (2012), no. 3, 864–891.
[5] G. Mauceri, S. Meda, M. Vallarino, *Harmonic Bergman spaces, the Poisson equation and the dual of Hardy-type spaces on certain noncompact manifolds*, arXiv:1305.1127, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.
[6] G. Mauceri, S. Meda, M. Vallarino, *Sharp endpoint results for imaginary powers and Riesz transforms on certain noncompact manifolds*, arXiv:1305.7109.
[7] M. Taylor, *L^p-estimates on functions of the Laplace operator*, Duke Math. J. **58** (1989), no. 3, 773–793.
[8] M. Taylor, *Hardy spaces and BMO on manifolds with bounded geometry*, J. Geom. Anal. **19** (2009), no. 1, 137–190.

A Theorem of Paley-Wiener Type for Schrödinger Evolutions

Luis Vega
(joint work with C.E. Kenig and G. Ponce)

Define the Fourier transform of a function f as

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

Then, for solutions $e^{it\Delta}u_0(x)$ of the free Schrödinger equation

$$\partial_t u = i\Delta u, \qquad u(x,0) = u_0(x), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

one has

$$u(x,t) = e^{it\Delta}u_0(x) = \int_{\mathbb{R}^n} \frac{e^{|x-y|^2/4t}}{(4\pi it)^{n/2}} u_0(y)\, dy$$

$$(1) \qquad = \frac{e^{|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-2ix\cdot y/4t} e^{i|y|^2/4t} u_0(y)\, dy$$

$$= \frac{e^{|x|^2/4t}}{(4\pi it)^{n/2}} \left(\widehat{e^{i|\cdot|^2/4t} u_0}\right)\left(\frac{x}{2t}\right).$$

From identity (1) it is easy to establish a relation between properties of the free solution of the Schrödinger equation with some classical uncertainty principles for the Fourier transform as the well known result of G. H. Hardy [3]:

If $f(x) = O\big(E^{-x^2/\beta^2}\big)$, $\quad \widehat{f}(\xi) = O\big(e^{-4\xi^2/\alpha^2}\big)$ and $\alpha\beta < 4$, then $f \equiv 0$,
and if $\alpha\beta = 4$, then $f(x) = ce^{-x^2/\beta^2}$.

The following generalization in terms of the L^2–norm was established in [1]

$$\text{If } e^{\frac{|x|^2}{\beta^2}} f(x), \quad e^{\frac{4|\xi|^2}{\alpha^2}} \widehat{f}(\xi) \in L^2(\mathbb{R}^n), \quad \text{and } \alpha\beta < 4, \quad \text{then } f \equiv 0.$$

For solutions of the free Schrödinger equation the L^2–version of Hardy Uncertainty Principle reads

$$\text{(2)} \quad \text{If } e^{\frac{|x|^2}{\beta^2}} u_0(x), \quad e^{\frac{|x|^2}{\alpha^2}} e^{it\Delta} u_0(x) \in L^2(\mathbb{R}^n), \quad \text{and } \alpha\beta < 4t, \quad \text{then } u_0 \equiv 0.$$

In a series of papers in collaboration with L. Escauriaza, C.E. Kenig, and G. Ponce we gave proofs of the above results in L^2 that avoid the use of complex analysis and rely just on methods of PDE's. This allowed us to extend (2) to solutions of Schrödinger equations with potentials that can be non-regular and time dependent. A survey of our results can be found in [2] together with a non-sharp version of Paley-Wiener theorem. More recently in [5] we obtain the following:

Theorem 0.1. *Let $u \in C\big([0,1] : L^2(\mathbb{R}^n)\big)$ be a strong solution of the equation*

$$\text{(3)} \qquad \partial_t u = i\big(\Delta u + V(x,t)u\big), \qquad (x,t) \in \mathbb{R}^n \times [0,1].$$

Assume that

$$\text{(4)} \qquad \sup_{0\le t\le 1} \int_{\mathbb{R}^n} |u(x,t)|^2 dx \le A_1,$$

$$\text{(5)} \qquad \int_{\mathbb{R}^n} e^{2\alpha_1 |x_1|} |u(x,0)|^2 dx \le A_2, \qquad \textit{for some } a_1 > 0,$$

$$\text{(6)} \qquad \operatorname{supp} u(\cdot,1) \subset \{x \in \mathbb{R}^n : x_1 \le a_2\}, \qquad \textit{for some } a_2 < \infty,$$

with

$$\text{(7)} \qquad V \in L^\infty\big(\mathbb{R}^n \times [0,1]\big), \qquad \|V\|_{L^\infty(\mathbb{R}^n\times[0,1])} = M_0,$$

and

$$\text{(8)} \qquad \lim_{\rho\to+\infty} \|V\|_{L^1([0,1]:L^\infty(\mathbb{R}^n\setminus B_\rho))} = 0.$$

then $u \equiv 0$.

Remarks:

(a) By rescaling it is clear that the result in Theorem 0.1 applies to any time interval $[0,T]$.

(b) We recall that in Theorem 0.1 there are no hypotheses on the size of the potential V in the given class or on its regularity.

As an almost straightforward consequence we get the following result.

Theorem 0.2. *Given*

$$u_1,\, u_2 \in C\big([0,T] : H^k(\mathbb{R}^n)\big), \qquad 0 < T \le \infty,$$

strong solutions of

$$\partial_t u = i\big(\Delta u + F(u,\overline{u})\big),$$

with $k \in \mathbb{Z}$, $k > n/2$, $F : \mathbb{C}^2 \to \mathbb{C}$, $F \in C^k$ and $F(0) = \partial_u F(0) = \partial_{\overline{u}} F(0) = 0$ such that

$$\text{(9)} \qquad \operatorname{supp}\big(u_1(\cdot,0) - u_2(\cdot,0)\big) \subset \{x \in \mathbb{R}^n : x_1 \le a_2\}, \quad a_2 < \infty.$$

If for some $t \in (0,T)$ *and for some* $\epsilon > 0$

$$u_1(\cdot,t) - u_2(\cdot,t) \in L^2\big(e^{\epsilon|x_1|}dx\big), \tag{10}$$

then $u_1 \equiv u_2$.

Remark:
In the case $F(u,\overline{u}) = |u|^{\alpha-1}u$, with $\alpha > n/2$ if α is not an odd integer, we have that if Q is the unique non–negative, radially symmetric solution of

$$-\Delta Q + \omega Q = Q^{\alpha},$$

then

$$u_1(x,t) = e^{i\omega t}Q(x) \tag{11}$$

is a solution ("standing wave") of

$$\partial_t u = i\big(\Delta u + |u|^{\alpha-1}u\big). \tag{12}$$

We can use the above theorem to conclude that if at time $t = 0$ we consider an initial datum that is a perturbation of Q with compact support, then the corresponding solution cannot have an exponential decay at $t \neq 0$.

The proof of 0.1 follows by contradiction. The following lemma plays a crucial role.

Lemma 1. *There exists* $\epsilon_n > 0$ *such that if*

$$\mathbb{V} : \mathbb{R}^n \times [0,1] \to \mathbb{C}, \quad \textit{with} \quad \|\mathbb{V}\|_{L^1_t L^\infty_x} \le \epsilon_n, \tag{13}$$

and $u \in C\big([0,1] : L^2(\mathbb{R}^n)\big)$ *is a strong solution of the IVP*

$$\begin{cases} \partial_t u = i\big(\Delta + \mathbb{V}(x,t)\big)u + \mathbb{G}(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{14}$$

with

$$u_0, u_1 \equiv u(\,\cdot\,,1) \in L^2(e^{2\lambda\cdot x}dx), \qquad \mathbb{G} \in L^1\big([0,1] : L^2(e^{2\lambda\cdot x}dx)\big), \tag{15}$$

for some $\lambda \in \mathbb{R}^n$, *then there exists* c_n *independent of* λ *such that*

$$\begin{aligned} &\sup_{0\le t\le 1} \big\|e^{\lambda\cdot x}u(\,\cdot\,,t)\big\|_{L^2(\mathbb{R}^n)} \\ &\le c_n \left(\big\|e^{\lambda\cdot x}u_0\big\|_{L^2(\mathbb{R}^n)} + \big\|e^{\lambda\cdot x}u_1\big\|_{L^2(\mathbb{R}^n)} + \int_0^1 \big\|e^{\lambda\cdot x}\mathbb{G}(\,\cdot\,,t)\big\|_{L^2(\mathbb{R}^n)}dt\right). \end{aligned} \tag{16}$$

With the above lemma an upper bound of exponential type for the solutions of (3) that satisfy (5) and (6) is obtained. This contradicts a lower bound that can be proved using the following Carleman estimate -see [4].

Lemma 2. *Assume* $R > 0$ *large enough and such that* $\varphi : [0,1] \to \mathbb{R}$ *is a smooth function. Then, there exists* $c = c\big(n; \|\varphi'\|_\infty + \|\varphi''\|_\infty\big) > 0$ *such that the inequality*

$$\frac{\sigma^{3/2}}{R^2}\left\|e^{\sigma|\frac{x_1-x_{0,1}}{R}+\varphi(t)|^2}g\right\|_{L^2(dxdt)} \le c\left\|e^{\sigma|\frac{x_1-x_{0,1}}{R}+\varphi(t)|^2}(i\partial_t+\Delta)g\right\|_{L^2(dxdt)} \tag{17}$$

holds when $\sigma \geq cR^2$ *and* $g \in C_0^\infty(\mathbb{R}^{n+1})$ *is supported on the set*

$$\left\{(x,t) = (x_1,\ldots,x_n,t) \in \mathbb{R}^{n+1} \,:\, \left|\frac{x_1 - x_{0,1}}{R} + \varphi(t)\right| \geq 1\right\}.$$

References

[1] M. Cowling, J.F. Price, *Generalizations of Heisenberg's inequality.* Harmonic Analysis (Cortona, 1982) Lecture Notes in Math., **992** (1983), 443–449.

[2] L. Escauriaza, C.E. Kenig, G. Ponce, L. Vega, *Uniqueness properties of solutions to Schrödinger equations.* Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 3, 415–442.

[3] G.H. Hardy. *A theorem concerning Fourier transform.* Journal London Math. Soc. **8**, (1933), 227–231.

[4] V. Izakov, *Carleman type estimates in an anisotropic case and applications.* J. Diff . Eqs. **105**, (1993), 217–238.

[5] C.E. Kenig, G. Ponce, L. Vega. *A theorem of Paley-Wiener type for Schrödinger evolutions.* Ann. Sci. Ec. Norm. Super. (4) **47**, (2014), no. 3, 539–557.

Variational methods in estimating singular integrals

Alexander Volberg

We will explain the Bellman function approach to some singular integral estimates. There is a dictionary that translates the language of singular integrals to the language of stochastic optimization. The main tool in stochastic optimization is a Hamilton–Jacobi–Bellman PDE. We show how this technique (the reduction to a Hamilton–Jacobi–Bellman PDE) allows us to get many recent results in estimating (often sharply) singular integrals of classical type. For example, the solution of A_2 conjecture will be given by the manipulations with convex functions of special type, which are the solutions of the corresponding HJB equation.

As an illustration we also compute the numerical value of the norm in L^p of the real and imaginary parts of the Ahlfors–Beurling transform. The upper estimate is again based on a solution of corresponding HJB, which one composes with the heat flow. The estimate from below (we compute the norm, so upper and lower estimates coincide) is obtained by the method of laminates, which misrelated to a problem of C. B. Morrey from the calculus of variations. We also give a certain (not sharp) estimate of the Ahlfors–Beurling operator itself and explain the connection with Morrey's problem.

Abundant examples of using this technique can be found in [1]–[5].

If time permits we also show how HJB approach allows us to build counterexamples to weak Muckenhoupt conjecture and some of its relatives.

References

[1] S. Petermichl, A. Volberg, "Heating the Ahlfors–Beurling operator: weakly quasiregular maps on the plane are quasiregular", Duke Math. J., 112 (2002), no.2, pp. 281–305.

[2] Dragicevic, Oliver; Volberg, Alexander, "Linear dimension-free estimates in the embedding theorem for Schrödinger operators". J. Lond. Math. Soc. (2) 85 (2012), no. 1, 191–222.

[3] Nazarov, Fedor; Reznikov, Alexander; Volberg, Alexander, "The proof of A_2 conjecture in a geometrically doubling metric space". Indiana Univ. Math. J. 62 (2013), no. 5, 1503–1533.

[4] Borichev, Alexander; Janakiraman, Prabhu; Volberg, Alexander, "Subordination by conformal martingales in L^p and zeros of Laguerre polynomials". Duke Math. J. 162 (2013), no. 5, 889–924.
[5] Andrea Carbonaro, Oliver Dragicevic, "Functional calculus for generators of symmetric contraction semigroups", arXiv:1308.1338, pp. 1–26.

On a class of pseudodifferential operators with mixed homogeneities

Po-Lam Yung
(joint work with Elias M. Stein)

In this talk, we discuss the phenomena that arise when we combine the standard pseudodifferential operators with those operators that appear in the study of some sub-elliptic estimates, and on strongly pseudoconvex domains. The algebra of operators we introduce is geometrically invariant, and is adapted to a smooth distribution of tangent subspaces of constant rank. Certain ideals in the algebra can be isolated, whose analysis is of particular interest.

We point out that related questions have been considered, for instance, by Phong and Stein [11], and have a number of applications (e.g. in Greiner and Stein [3], Müller, Peloso and Ricci [4]). We also remark that our work can be seen as a pseudodifferential realization of the two-flag kernels of Nagel, Ricci, Stein and Wainger [9], in the step-2 case, but we consider it when only a distribution of tangent subspaces are given (without requiring the underlying space to be a Lie group), and we consider operators of all orders (not just those of order 0). Furthermore, we refer the reader to the series of work by Müller, Nagel, Ricci, Stein and Wainger [5], [6], [7], [8] on one-flag kernels, and Nagel and Stein [10], Beals and Greiner [1] for earlier work on a family of non-isotropic pseudodifferential operators on the boundary of strongly pseudoconvex domains.

To set things up, suppose we are given a smooth distribution $\mathcal{D}$ of tangent subspaces in $\mathbb{R}^N$. To fix ideas, for simplicity, assume $\mathcal{D}$ is of codimension 1, and is given by the nullspace of a 1-form whose coefficients have uniformly bounded derivatives. Then there exists a global frame $X_1, \dots, X_N$ of tangent vectors on $\mathbb{R}^N$, such that the first $N-1$ vectors form a basis of $\mathcal{D}$ at every point. Furthermore, one can pick

$$X_i = \sum_{j=1}^N A_i^j(x) \frac{\partial}{\partial x^j}, \quad i = 1, \dots, N,$$

such that all coefficients $A_i^j(x)$ are C^∞ functions, with $\|\partial_x^I A_i^j\|_{L^\infty} \leq C_I$ for all multiindices I, and $|\det(A_i^j(x))| \geq c > 0$ uniformly as x varies over $\mathbb{R}^N$. Below we will fix such a frame $X_1, \dots, X_N$, and use that to construct our class of symbols. It is to be pointed out, however, that our symbol class will ultimately depend only on the distribution $\mathcal{D}$, and not on the particular choice of $X_1, \dots, X_N$. We also remark that no curvature assumption on $\mathcal{D}$ is necessary (or relevant).

Let $\theta^1, \dots, \theta^N$ be the dual frame of $X_1, \dots, X_N$. We define a variable coefficient semi-norm on the cotangent bundle of $\mathbb{R}^N$ as follows. Given a cotangent vector

$\xi = \sum_{i=1}^N \xi_i dx^i$ at a point $x \in \mathbb{R}^N$, we express ξ in terms of a linear combination of $\theta^1, \dots, \theta^N$, say

$$\xi = \sum_{i=1}^N (M_x\xi)_i \theta^i.$$

($(M_x\xi)_i = \sum_{j=1}^N A_i^j(x)\xi_j$ will do.) Then we define

$$\rho_x(\xi) = \sum_{i=1}^{N-1} |(M_x\xi)_i|.$$

Also define $|\xi| = \sum_{i=1}^N |\xi_i|$, the Euclidean norm of ξ.

The variable semi-norm $\rho_x(\xi)$ induces a quasi-metric $d(x,y)$ on $\mathbb{R}^N$, by

$$d(x,y) = \sup \left\{ (\rho_x(\xi) + |\xi|^{1/2})^{-1} \colon (x-y)\cdot\xi = 1 \right\}.$$

The Euclidean distance between x and y is just $|x-y|$.

To describe our class of symbols, we will need some special derivatives on the cotangent bundle of $\mathbb{R}^N$. First, let (B_j^k) be the inverse of the matrix (A_i^j). In other words,

$$\sum_{j=1}^N A_i^j(x) B_j^k(x) = \delta_i^k$$

for every $x \in \mathbb{R}^N$. Then we define a 'good' derivative on the cotangent bundle, by

$$D_\xi = \sum_{i=1}^N B_i^N(x) \frac{\partial}{\partial \xi_i}.$$

Next, we need some 'good' modification of the coordinate derivatives $\frac{\partial}{\partial x_j}$: for $1 \le j \le N$, define

$$D_j = \frac{\partial}{\partial x^j} + \sum_{k=1}^N \sum_{p=1}^N \sum_{l=1}^N \frac{\partial B_k^p}{\partial x^j}(x) A_p^l(x) \xi_l \frac{\partial}{\partial \xi_k}.$$

We write $D_J = D_{j_1} D_{j_2} \dots D_{j_k}$, if $J = (j_1, \dots, j_k)$, where each $j_r \in \{1, \dots, N\}$.

Our class of symbols will come with two different 'orders', one which is isotropic (which we denote by m), and another which is non-isotropic (which we denote by n). Given any $m, n \in \mathbb{R}$, if $a(x,\xi) \in C^\infty(T^*\mathbb{R}^N)$ is such that

$$|\partial_\xi^\alpha D_\xi^\beta D^J a(x,\xi)| \le C_{\alpha,\beta,J}(1+|\xi|)^{m-\beta}(1+\rho_x(\xi)+|\xi|^{1/2})^{n-|\alpha|},$$

then we say a is a symbol of orders (m,n) adapted to the distribution $\mathcal{D}$, and write $a \in S^{m,n}(\mathcal{D})$. This class of symbols is fairly big: for instance, $S^{m,0}(\mathcal{D})$ contains the standard (isotropic) symbols of order m, namely those that satisfy

$$|\partial_\xi^\alpha \partial_x^J a(x,\xi)| \le C_{\alpha,J}(1+|\xi|)^{m-|\alpha|},$$

Also, $S^{0,n}(\mathcal{D})$ contains the following class of symbols, which we think of as non-isotropic symbols of order n adapted to $\mathcal{D}$:

$$|\partial_\xi^\alpha D_\xi^\beta D^J a(x,\xi)| \le C_{\alpha,\beta,J}(1+\rho_x(\xi)+|\xi|^{1/2})^{n-|\alpha|-2\beta}.$$

Many of the results below have an easier variant for such non-isotropic symbols.

To each symbol $a \in S^{m,n}(\mathcal{D})$, we associate a pseudodifferential operator

$$T_a f(x) = \int_{\mathbb{R}^N} a(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

We denote the set of all such operators by $\Psi^{m,n}(\mathcal{D})$.

One can show that the class of operators we get exhibits geometric invariance, in the sense that $\Psi^{m,n}(\mathcal{D})$ depends only on the distribution $\mathcal{D}$, and not on the choice of the vector fields $X_1, \dots, X_N$ (nor on the choice of a coordinate system on $\mathbb{R}^N$).

Furthermore, $\Psi^{m,n}(\mathcal{D})$ are psuedodifferential analogs of the 'two-flag' kernels of Nagel, Ricci, Stein and Wainger [9], as can be seen from the following:

Theorem 4. *If $T \in \Psi^{m,n}(\mathcal{D})$ with $m > -1$ and $n > -(N-1)$, then one can write*

$$Tf(x) = \int_{\mathbb{R}^N} f(y)K(x,y)dy,$$

whenever $f \in C_c^\infty(\mathbb{R}^N)$, and x is not in the support of f, where the kernel $K(x,y)$ satisfies

$$|(X')^\gamma_{x,y}\partial^\delta_{x,y}K(x,y)| \leq C_{\gamma,\delta}|x-y|^{-(N-1+n+|\gamma|)}d(x,y)^{-2(1+m+|\delta|)}.$$

Here X' refers to any of the 'good' vector fields $X_1, \dots, X_{N-1}$ that are tangent to $\mathcal{D}$, and the subscripts x, y indicates that the derivatives can act on either the x or y variables.

We also have the following composition law, and L^p mapping properties:

Theorem 5. *If $T_1 \in \Psi^{m,n}(\mathcal{D})$ and $T_2 \in \Psi^{m',n'}(\mathcal{D})$, then*

$$T_1 \circ T_2 \in \Psi^{m+m',n+n'}(\mathcal{D}).$$

Furthermore, if T_1^* is the adjoint of T_1 with respect to the standard L^2 inner product on $\mathbb{R}^N$, then we also have

$$T_1^* \in \Psi^{m,n}(\mathcal{D}).$$

In particular, the class of operators $\Psi^{0,0}$ form an algebra under composition, and is closed under taking adjoints.

Theorem 6. *If $T \in \Psi^{0,0}(\mathcal{D})$, then T maps $L^p(\mathbb{R}^N)$ into itself for all $1 < p < \infty$.*

Note that operators in $\Psi^{0,0}(\mathcal{D})$ are standard pseudodifferential operators of type $(1/2, 1/2)$. As such they are bounded on L^2. But operators in $\Psi^{0,0}(\mathcal{D})$ may not be of weak-type (1,1); this forbids one to run the Calderon-Zygmund paradigm in proving L^p boundedness.

Nonetheless, there are two very special ideals of operators inside $\Psi^{0,0}(\mathcal{D})$, namely $\Psi^{\varepsilon,-2\varepsilon}(\mathcal{D})$ and $\Psi^{-\varepsilon,\varepsilon}(\mathcal{D})$ for $\varepsilon > 0$. (The fact that these are ideals of the algebra $\Psi^{0,0}(\mathcal{D})$ follows from Theorem 5). They satisfy:

Theorem 7. *If $T \in \Psi^{\varepsilon,-2\varepsilon}(\mathcal{D})$ or $\Psi^{-\varepsilon,\varepsilon}(\mathcal{D})$ for some $\varepsilon > 0$, then*

(a) T is of weak-type (1,1), and

(b) T maps the Hölder space $\Lambda^\alpha(\mathbb{R}^N)$ into itself for all $\alpha > 0$; similarly for a suitable non-isotropic Hölder space $\Gamma^\alpha(\mathbb{R}^N)$.

The operators in $\Psi^{m,n}(\mathcal{D})$, where say m, n are slightly negative, possess some surprising smoothing properties:

Theorem 8. *Suppose $T \in \Psi^{m,n}(\mathcal{D})$ with*

$$m > -1, \quad n > -(N-1), \quad m+n \leq 0, \quad \text{and} \quad 2m+n \leq 0.$$

For $p \geq 1$, define an exponent p^* by

$$\frac{1}{p^*} := \frac{1}{p} - \gamma, \qquad \gamma := \min\left\{\frac{|m+n|}{N}, \frac{|2m+n|}{N+1}\right\}$$

if $1/p > \gamma$. Then:

(i) $T\colon L^p \to L^{p^*}$ for $1 < p \leq p^* < \infty$; and
(ii) if $\frac{m+n}{N} \neq \frac{2m+n}{N+1}$, then T is weak-type $(1, 1^*)$.

These estimates for $\Psi^{m,n}(\mathcal{D})$ are better than those obtained by composing between the optimal results for $\Psi^{0,n}(\mathcal{D})$ and $\Psi^{m,0}(\mathcal{D})$. For example, take the example of the 1-dimensional Heisenberg group $\mathbb{H}^1$ (so $N = 3$), and let $\mathcal{D}$ be the contact distribution on $\mathcal{H}^1$. If T_1 is a standard (or isotropic) pseudodifferential operator of order $-1/2$ (so $T_1 \in \Psi^{-1/2,0}(\mathcal{D})$), and T_2 is a non-isotropic pseudodifferential operator of order -1 (so $T_2 \in \Psi^{0,-1}(\mathcal{D})$), then the best we could say about the operators individually are just

$$T_2\colon L^{4/3} \to L^2, \quad T_1\colon L^2 \to L^3.$$

But according to the previous theorem,

$$T_1 \circ T_2\colon L^{4/3} \to L^4,$$

which is better.

We close by mentioning some applications. First, one can localize the above theory of operators of class $\Psi^{m,n}(\mathcal{D})$ to compact manifolds without boundary.

Let $M = \partial\Omega$ be the boundary of a strongly pseudoconvex domain Ω in $\mathbb{C}^{n+1}$, $n \geq 1$, and $\mathcal{D}$ be the distribution spanned by the $2n$ 'good' tangent vectors. Then the Szegö projection S is an operator in $\Psi^{\varepsilon,-2\varepsilon}(\mathcal{D})$ for all $\varepsilon > 0$; c.f. Phong and Stein [11]. Furthermore, the Dirichlet-to-$\overline{\partial}$-Neumann operator $\Box_+$, which one needs to invert in solving the $\overline{\partial}$-Neumann problem on Ω, has a parametrix in $\Psi^{1,-2}(\mathcal{D})$; c.f. Greiner and Stein [3], Chang, Nagel and Stein [2].

References

[1] Richard Beals, Peter Greiner, *Calculus on Heisenberg manifolds*, Annals of Mathematics Studies, volume 119, Princeton University Press, Princeton, N.J., 1988.

[2] Der-Chen Chang, Alexander Nagel, Elias M. Stein, *Estimates for the $\overline{\partial}$-Neumann problem in pseudoconvex domains of finite type in* $\mathbf{C}^2$, Acta Math. **169** (1992), no. 3-4, 153–228.

[3] Peter Greiner, Elias M. Stein, *Estimates for the $\overline{\partial}$-Neumann problem*, Mathematical Notes, No. 19, Princeton University Press, Princeton, N.J., 1977.

[4] Detlef Müller, Marco Peloso, Fulvio Ricci, *Analysis of the Hodge Laplacian on the Heisenberg group*, preprint.

[5] Detlef Müller, Fulvio Ricci, Elias M. Stein, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. I*, Invent. Math. **119** (1995), no. 2, 199–233.
[6] Detlef Müller, Fulvio Ricci, Elias M. Stein, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. II*, Math. Z. **221** (1996), no. 2, 267–291.
[7] Alexander Nagel, Fulvio Ricci, Elias M. Stein, *Singular integrals with flag kernels and analysis on quadratic CR manifolds*, J. Funct. Anal. **181** (2001), no. 1, 29–118.
[8] Alexander Nagel, Fulvio Ricci, Elias M. Stein, Stephen Wainger, *Singular integrals with flag kernels on homogeneous groups, I*, Rev. Mat. Iberoam. **28** (2012), no. 3, 631–722.
[9] Alexander Nagel, Fulvio Ricci, Elias M. Stein, Stephen Wainger, *Singular integrals with flag kernels on homogeneous groups, II*, Manuscript in preparation.
[10] Alexander Nagel, Elias M. Stein, *Lectures on pseudodifferential operators: regularity theorems and applications to nonelliptic problems*, Mathematical Notes, No. 24, Princeton University Press, Princeton N.J., 1979.
[11] D. H. Phong, Elias M. Stein, *Some further classes of pseudodifferential and singular-integral operators arising in boundary value problems. I. Composition of operators*, Amer. J. Math. **104** (1982), no. 1, 141–172.

Reporter: Diogo Oliveira e Silva

Participants

Prof. Dr. Pascal Auscher
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Dr. Michael Bateman
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Stefan Buschenhenke
ICMAT
Instituto de Sciencas Matemáticas
C/Nicolas Cabrera, No. 13-15
28049 Madrid
SPAIN

Prof. Dr. Anthony Carbery
School of Mathematics
University of Edinburgh
King's Buildings
Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Prof. Dr. Emanuel Carneiro
Instituto de Matematica Pura e
Aplicada - IMPA
Jardim Botanico
Estrada Dona Castorina, 110
Rio de Janeiro, RJ 22460-320
BRAZIL

Marcos Charalambides
82 Kritis St.
Lemesos 3087
CYPRUS

Prof. Dr. Michael Christ
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Michael G. Cowling
School of Mathematics and Statistics
University of New South Wales
Sydney NSW 2052
AUSTRALIA

Prof. Dr. Guy David
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Dr. Spyridon Dendrinos
School of Mathematical Sciences
University College Cork
Cork
IRELAND

Prof. Dr. Francesco Di Plinio
Dipartimento di Matematica
Universita degli Studi di Roma II
Tor Vergata
Via della Ricerca Scientifica
00133 Roma
ITALY

Prof. Dr. Jacek Dziubanski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. M. Burak Erdogan
Dept. of Mathematics, University of
Illinois at Urbana-Champaign
273 Altgeld Hall; MC-382
1409 West Green Street
Urbana, IL 61801-2975
UNITED STATES

Prof. Dr. Damiano Foschi
Dipartimento di Matematica
Universita di Ferrara
Via Machiavelli 35
44121 Ferrara
ITALY

Prof. Dr. Philip Gressman
Department of Mathematics
David Rittenhouse Laboratory
University of Pennsylvania
209 South 33rd Street
Philadelphia, PA 19104-6395
UNITED STATES

Shaoming Guo
Mathematisches Institut
Universität Bonn
Endenicher Allee 62
53115 Bonn
GERMANY

Prof. Dr. Hakan Hedenmalm
Department of Mathematics
Royal Institute of Technology
10044 Stockholm
SWEDEN

Prof. Dr. Dirk Hundertmark
Fakultät für Mathematik
Institut für Analysis
Karlsruher Institut f. Technologie (KIT)
76128 Karlsruhe
GERMANY

Dr. Tuomas Hytönen
Dept. of Mathematics & Statistics
University of Helsinki
P.O.Box 68
00014 University of Helsinki
FINLAND

Prof. Dr. Alexander Iosevich
Department of Mathematics
University of Rochester
Rochester, NY 14627
UNITED STATES

Prof. Dr. Herbert Koch
Mathematisches Institut
Universität Bonn
53115 Bonn
GERMANY

Prof. Dr. Vjekoslav Kovac
Department of Mathematics
University of Zagreb
Bijenicka cesta 30
10000 Zagreb
CROATIA

Prof. Dr. Hong-Quan Li
School of Mathematical Sciences
Fudan University
220 Handan Road
Shanghai 200 433
CHINA

Dr. Alessio Martini
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
24098 Kiel
GERMANY

Dr. Mariusz Mirek
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Detlef Müller
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
24098 Kiel
GERMANY

Prof. Dr. Jesus Munarriz Aldaz
Departamento de Matematicas
Universidad Autonoma de Madrid
Ciudad Universitaria de Cantoblanco
28049 Madrid
SPAIN

Prof. Dr. Camil Muscalu
Department of Mathematics
Cornell University
White Hall
Ithaca, NY 14853-7901
UNITED STATES

Prof. Dr. Assaf Naor
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES

Prof. Dr. Richard Oberlin
Department of Mathematics
Florida State University
LOV 116
Tallahassee, FL 32306-4510
UNITED STATES

Dr. Diogo Oliveira e Silva
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Stefanie Petermichl
Institut de Mathématiques de Toulouse
Université Paul Sabatier
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Duong H. Phong
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Lillian Beatrix Pierce
Hausdorff Center for Mathematics
Endenicher Allee 62
53115 Bonn
GERMANY

Prof. Dr. Malabika Pramanik
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Prof. Dr. Maria C. Reguera
Department of Mathematics
The University of Birmingham
P.O.Box 363
Birmingham B15 2TT
UNITED KINGDOM

Prof. Dr. Keith M. Rogers
Instituto de Ciencias Matematicas
Consejo Superior de Investigaciones
Cientificas (CSIC)
28049 Madrid
SPAIN

Prof. Dr. Andreas Seeger
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Prof. Dr. Adam S. Sikora
Mathematics Department
Macquarie University
NSW 2109
AUSTRALIA

Prof. Dr. Elias M. Stein
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Stefan Steinerberger
Department of Mathematics
Yale University, Office: 456 L
P.O. Box 208283
New Haven CT 06520
UNITED STATES

Prof. Dr. Betsy Stovall
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison WI 53706
UNITED STATES

Prof. Dr. Brian T. Street
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Prof. Dr. Christoph Thiele
Hausdorff Center for Mathematics
Universität Bonn
Villa Maria
Endenicher Allee 62
53115 Bonn
GERMANY

Prof. Dr. Xavier Tolsa
Departament de Matematiques
ICREA - Universitat Autonoma de
Barcelona
Campus Universitari
08193 Bellaterra
SPAIN

Prof. Dr. Maria Vallarino
Dipartimento di Science Matematiche
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino
ITALY

Prof. Dr. Luis Vega
Departamento de Matematicas
Facultad de Ciencias
Universidad del Pais Vasco
Apartado 644
48080 Bilbao
SPAIN

Prof. Dr. Alex Volberg
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
UNITED STATES

Prof. Dr. Jim Wright
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Prof. Dr. Po-Lam Yung
Mathematical Institute
Oxford Center for Nonlinear PDE
University of Oxford
Andrew Wiles Bldg.
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Jacek Zienkiewicz
Institute of Mathematics
Wroclaw University
50-384 Wroclaw
POLAND

Eugen Zimmermann
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
Ludewig-Meyn-Str. 4
24118 Kiel
GERMANY

Dr. Pavel Zorin-Kranich
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY