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Topologie

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Abstract. The Oberwolfach conference "Topologie" is one of only a few opportunities for researchers from many different areas in algebraic and geometric topology to meet and exchange ideas. The program covered new developments in fields such as automorphisms of manifolds, applications of algebraic topology to differential geometry, quantum field theories, combinatorial methods in low-dimensional topology, abstract and applied homotopy theory and applications of L^2-cohomology. We heard about new results describing the cohomology of the automorphism spaces of some smooth manifolds, progress on spaces of positive scalar curvature metrics, a variant of the Segal conjecture without completion, advances in classifying topological quantum field theories, and a new undecidability result in combinatorial group theory, to mention some examples. As a special attraction, the conference featured a series of three talks by Dani Wise on the combinatorics of CAT(0)-cube complexes and applications to 3-manifold topology.

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Introduction by the Organisers

This conference was the third topology conference in Oberwolfach organized by Thomas Schick, Peter Teichner, Nathalie Wahl and Michael Weiss. About 50 mathematicians participated, working in many different areas of algebraic and geometric topology.

The talks were of three types. There were 14 regular one-hour talks, 3 one-hour talks by keynote speaker Dani Wise and a "gong show" where 12 young speakers had the opportunity to present their research in 10 minutes each, including question time.

The 15 regular talks of the conference covered a wide range of topics such as spaces of automorphisms of highly connected manifolds, spaces of Riemannian metrics with positive scalar curvature, new developments in abstract homotopy theory, techniques for solving equations in groups, undecidability results in combinatorial group theory, algorithms in 3-manifold topology, a variant of the (affirmed) Segal conjecture which does away with the need for finite completion, and new developments in topological and other quantum field theories. Speakers were instructed to give talks that could be appreciated by an audience of topologists of many different kinds, and they were generally very successful in doing so.

Keynote speaker Dani Wise spoke on CAT(0)-cube complexes, his work in the theory and how it became an essential ingredient in the recent spectacular proof of the virtual Haken conjecture by Ian Agol (Berkeley). He concentrated on the combinatorial aspects, giving a very patient introduction to the geometric properties of CAT(0)-cube complexes in the first two talks and sketching applications to 3-manifold topology in the last one. His vigorous delivery made these talks as riveting as we could have wished.

The gong show with 12 speakers took place on Wednesday morning. In the opinion of this writer, it is a hard training for the young, but there is no doubt at all that the speakers rose to the occasion. Dieter Degrijse and Irakli Patchkoria both talked on their joint work relating the virtual cohomological dimension of groups G to the homotopy theory of what they call proper G-spectra. Lukasz Grabowski spoke on his work in the theory of L^2 invariants to disprove the conjecture of Lott and Lück that the Novikov-Shubin invariants are always positive. Holger Kammeyer reported on a proof of the Farrell-Jones conjecture in algebraic K- and L-theory for arbitrary lattices in connected Lie groups. Christina Pagliantini presented a new result on Gromov's simplicial volume for hyperbolic 3-manifolds. Daniel Kasprowski spoke on the Farrell-Jones conjecture in algebraic K-theory for groups with finite decomposition complexity. Daniela Egas Santander offered a comparison of various combinatorial models of moduli spaces of two-dimensional cobordisms and some compactifications, relating for example the graph models of Godin and Costello to Bödigheimer's model designed along more classical lines. Pedro Boavida de Brito talked about his work in functor calculus, spaces of smooth embeddings and operad theory with applications to spaces of higher-dimensional long knots. Both Daniel Tubbenhauer and Lukas Lewark spoke on developments in the Khovanov homology of knots, Tubbenhauer more on relations with representation theory and Lewark more on applications to genus-type invariants of slice knots. Markus Upmeier spoke on a theorem of his establishing the existence of a moment map for certain actions of symplectomorphism groups. Finally Nat Stapleton reported on a new proof and generalization of a result of Strickland's regarding some generalized cohomology of symmetric groups.

We now describe the themes of the regular 1-hour talks.

Oscar Randal-Williams talked about his joint work with Galatius (Stanford) on the cohomology and related invariants of spaces of automorphisms of smooth manifolds. One point of departure for this, years ago, was the affirmed Mumford conjecture on the cohomology H^* of spaces of automorphisms of surfaces of genus g where $g \gg *$. But the current level of generality allows for manifolds of dimension $2n$, where $n \neq 2$, having the form of a connected sum of a fixed and rather arbitrary smooth manifold with g copies of $S^n \times S^n$.

Category theory teaches us, as soon as we have learned to reason with sets and elements, that we should not reason quite so much with elements. In Emily Riehl's talk about formal category theory we learned the next lesson: do not reason quite so much with objects and morphisms. To begin with she described axioms/conditions isolating key features of the category of categories as a 2-category. More desirable features were added to the framework as the talk went on. In a similar vein, Ieke Moerdijk talked about categories for homotopy theorists, specifically about categories of functors from a small category $\mathcal{A}$ to spaces, and a comparison of that, for homotopy theorists, with the category of spaces over the classifying space $B\mathcal{A}$.

Wolfgang Lück spoke on L^2-torsion invariants and relations between these and the Thurston norm on the first cohomology of irreducible 3-manifolds.

Johannes Ebert spoke about new results on spaces of positive scalar curvature metrics. The proofs rely on the results of Randal-Williams and Galatius on automorphisms of some smooth manifolds, but also on steady progress in the theory of surgery on manifolds with a positive scalar curvature metric.

Both Owen Gwilliam and Chris Schommer-Priess talked on aspects of quantum field theories. Schommer-Priess talked about progress related to the Stolz-Teichner program, a conjectural parameterization of some elementary quantum field theories. Gwilliam's talk was on the deformation quantization of a type of classical field theory, emphasizing the cohomological meaning of existence and (non-)uniqueness of such deformation quantizations and describing some associated computations. The talk by Zsuzsanna Dancso appeared to have some intriguing connections with the mathematics of quantum field theory though she did not say so. Her theme, abstractly stated, was homomorphic expansions of planar algebras. She explained how in the case of a particular planar algebra, obtained from a topological setup, these homomorphic expansions are equivalent to solutions of an important equation in Lie theory, and how they can be constructed using the topological setup.

Andrew Putman spoke on homological stability phenomena with dimension shift in the cohomology of $\mathrm{SL}_n\mathbb{Z}$. His talk had strong connections to number theory.

Gerd Laures' talk was a survey of recent computations of the generalized cohomology (closely related to TMF) of important classifying spaces such as $B\mathrm{Spin}$ and $B\mathrm{String}$.

The Segal conjecture, affirmed in the mid 1980s, states that (a form of) the 0-th stable cohomotopy group of the classifying space of a finite group G is isomorphic to the Burnside ring of G completed at the augmentation ideal. It is one of the great triumphs of stable homotopy theory. In Jesper Grodal's talk we heard about a variant of the Segal conjecture which involves the uncompleted Burnside ring.

Quite a few talks were related to algorithms or solvability statements of use in algebraic topology. Martin Bridson spoke on the non-decidability of the existence of finite-index subgroups in finitely presented groups. Andreas Thom's talk was on methods for solving certain equations in countable groups. Saul Schleimer gave a survey of algorithms and algorithmic problems in 3-manifold topology.

Once again the Oberwolfach staff, not least the kitchen staff, helped to make this meeting pleasant and memorable. Our thanks go to the institute for creating this atmosphere and making the conference possible.

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Workshop: Topologie

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Equations over groups

Andreas Thom
(joint work with Anton Klyachko)

Let Γ be a countable group and let $n \in \mathbb{N}$ be a positive integer. Consider an equation $w \in \Gamma * \mathbb{F}_n$ and the associated word map $w\colon \Gamma^n \to \Gamma$ that is obtained by evaluation. We say that w has a solution in Γ, if there exists $g_1, \dots, g_n \in \Gamma$ such that $w(g_1, \dots, g_n) = e$ in Γ. Similarly, we say that w has a solution over Γ if there exists some overgroup $\Lambda \supset \Gamma$, such that w has a solution in Λ. It is well-known that not every equation with coefficients in Γ has a solution over Γ, e.g., $\Gamma = \langle a, b \mid a^2, b^3 \rangle$ and $w(t) = tat^{-1}b$. The study of equations goes back to work of Bernhard Neumann [6] and attracted much attention because of various applications in low-dimensional topology.

We also consider the natural augmentation $\varepsilon\colon \Gamma * \mathbb{F}_n \to \mathbb{F}_n$ and call $w \in \Gamma * \mathbb{F}_n$ non-singular, if $\varepsilon(w) \neq e$ in $\mathbb{F}_n$. There is no example of a singular equation that cannot be solved in some overgroup.

Conjecture A: Every non-singular equation with coefficients in a group Γ can be solved over Γ.

The case $n = 1$ is the classical Kervaire-Laudenbach Conjecture. Based on results [3], it has been observed by Pestov [7] that the Conjecture A holds provided that the group Γ is hyperlinear and $n = 1$ – see [7] for more information on the class of hyperlinear groups. All sofic groups are hyperlinear (in particular all amenable groups and all residually finite groups) and there is no group known to be non-sofic.

Our main result is the following theorem.

Theorem: Conjecture A holds for $w \in \Gamma * \mathbb{F}_2$ provided that Γ is hyperlinear and $\varepsilon(w) \notin [\mathbb{F}_2, [\mathbb{F}_2, \mathbb{F}_2]]$.

The proof is based on a detailed study of the effect of word maps on the cohomology of $\mathrm{PU}(p)$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$. It uses computations of Baum-Browder [1] that established that the co-product on cohomology is not co-commutative in this case, and various other explicit computations of Hamanaka-Kishimoto-Kono [4] and Kishimoto-Kono [5].

The strategy has potential to provide information about the following conjecture by Larsen.

Conjecture B: For non-trivial $w \in \mathbb{F}_2$, there exists n_0, such that for all $n \geq n_0$ the associated word map $w\colon \mathrm{PU}(n) \times \mathrm{PU}(n) \to \mathrm{PU}(n)$ is surjective.

In [2], it was proved that Conjecture B holds for all $w \notin [[\mathbb{F}_2, \mathbb{F}_2], [\mathbb{F}_2, \mathbb{F}_2]]$ and infinitely many n. On the contrary, in [8] it was shown that for fixed n and any neighborhood V of $1_n \in \mathrm{U}(n)$, there exist a non-trivial word $w \in \mathbb{F}_2$, such that the image of the associated word map $w\colon \mathrm{U}(n) \times \mathrm{U}(n) \to \mathrm{U}(n)$ lies in V.

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Stable moduli spaces of high dimensional manifolds

Oscar Randal-Williams

(joint work with Søren Galatius)

For a closed smooth manifold W, we consider the *moduli space of manifolds of type* W to be the classifying space $B\mathrm{Diff}(W)$ of the group of diffeomorphisms of W. One justification for this name is that $B\mathrm{Diff}(W)$ carries a smooth fibre bundle with fibre W, and this is the universal example of such a bundle, i.e. any such bundle $\pi : E \to B$ is obtained up to isomorphism by pulling back this universal bundle along a unique homotopy class of map $f : B \to B\mathrm{Diff}(W)$. Hence the cohomology ring $H^*(B\mathrm{Diff}(W))$ is precisely the ring of characteristic classes of smooth fibre bundles with fibre W.

Suppose that W has dimension $2n$, and define the *genus* of W by

$$g(W) := \max\{g \in \mathbb{N} \,|\, \#^g S^n \times S^n \text{ is a connect-summand of } W\}.$$

When $2n = 2$ and W is orientable, this coincides with the usual genus of a surface. The tangent bundle of W is classified by a Gauss map $\tau_W : W \to BO(2n)$, and we may form the Moore–Postnikov n-stage of this map,

$$\tau_W : W \xrightarrow{\ell} B \xrightarrow{\theta} BO(2n),$$

where θ is a fibration. Recall that this is a factorisation of τ_W having the property that ℓ is n-connected and θ is n-co-connected. It is characterised up to homotopy equivalence by these properties.

We may form the following two objects associated to the fibration θ. Firstly, let $\mathbf{MT}\theta$ be the Thom spectrum associated to the virtual vector bundle $-\theta^*\gamma_{2n}$ over B. Secondly, let $\mathrm{hAut}(\theta)$ denote the grouplike topological monoid of self-homotopy equivalences of B over $BO(2n)$, i.e. homotopy equivalences $f : B \xrightarrow{\sim} B$ such that $\theta \circ f = \theta$. As the Thom spectrum construction is functorial, we obtain an action of $\mathrm{hAut}(\theta)$ on the spectrum $\mathbf{MT}\theta$, and hence on its associated infinite loop space $\Omega^\infty\mathbf{MT}\theta$. The set of path components $\pi_0(\Omega^\infty\mathbf{MT}\theta)$ has a cobordism-theoretic description—via the Pontrjagin–Thom construction—in terms of $2n$-dimensional manifolds equipped with a lift of their Gauss map along the fibration θ, and we let $\Omega^\infty_{[W]}\mathbf{MT}\theta$ be the union of those path components given by the $\mathrm{hAut}(\theta)$-orbit of $[W,\ell] \in \pi_0(\Omega^\infty\mathbf{MT}\theta)$.

With this preparation, our main theorem is as follows, which extends the Madsen–Weiss theorem and related homological stability results when $2n = 2$.

Theorem A. *There is a map*

$$\alpha_W : B\mathrm{Diff}(W) \longrightarrow \big(\Omega^\infty_{[W]}\mathbf{MT}\theta\big)/\!\!/\mathrm{hAut}(\theta),$$

which, if W is simply-connected and $2n \geq 6$, induces an isomorphism in integral (co)homology in degrees $* \leq \frac{g(W)-3}{2}$.

Many variations of this theorem also hold: there is a version for orientation-preserving diffeomorphisms, where one replaces θ by a lift $\theta^+ : B \to BSO(2n)$ to the classifying space for oriented $2n$-dimensional vector bundles; there is a version for manifolds W with non-empty boundary, where the target of α_W is again modified slightly.

As the target of the map α_W is constructed in purely homotopy-theoretic terms, it is amenable to calculation using the traditional tools of algebraic topology. In particular, for various interesting manifolds (such as $\#^g S^n \times S^n$, or smooth hypersurfaces in $\mathbb{CP}^4$) one can now compute the ring $H^*(B\mathrm{Diff}(W);\mathbb{Q})$ in this stable range of degrees.

Theorem A is a consequence of a collection of more technical results, spread throughout [1, 2, 3]. However, these more technical results are of independent interest, and for theoretical rather than computational applications may be more useful than Theorem A itself. (See J. Ebert's report in this volume for an example.)

Moduli spaces of θ-manifolds. Rather than constructing the fibration θ from the manifold W, we may take a different point of view: fix an n-co-connected fibration $\theta : B \to BO(2n)$, and consider the space of all n-connected maps $W \to B$ which are lifts of τ_W along the fibration θ (and are specified on ∂W). Alternatively, we may fix a bundle map $\ell_\partial : TW|_{\partial W} \to \theta^*\gamma_{2n}$ and consider the homotopy equivalent space

$$\mathrm{Bun}^\theta_n(W) := \{\ell : TW \to \theta^*\gamma_{2n} \text{ an } n\text{-connected bundle map extending } \ell_\partial\}.$$

From this we may form the Borel construction

$$B\mathrm{Diff}^\theta_\partial(W) := \mathrm{Bun}^\theta_n(W)/\!\!/\mathrm{Diff}_\partial(W),$$

a moduli space of θ-manifolds of type W. When $\partial W = \emptyset$ the monoid $\mathrm{hAut}(\theta)$ can be made to act on this space, and there is a homotopy equivalence

$$B\mathrm{Diff}^\theta(W)/\!\!/\mathrm{hAut}(\theta) \simeq B\mathrm{Diff}(W).$$

Hence Theorem A is a consequence of the following theorem, which we now formulate for manifolds with boundary.

Theorem B. *There is a map*

$$\alpha^\theta_W : B\mathrm{Diff}^\theta_\partial(W) \longrightarrow \Omega^\infty_{[W]}\mathbf{MT}\theta,$$

which, if W is simply-connected and $2n \geq 6$, induces an isomorphism in integral (co)homology in degrees $* \leq \frac{g(W)-3}{2}$.

Theorem A is obtained by taking Borel constructions of both sides by $\mathrm{hAut}(\theta)$, using the fact that α^θ_W may be chosen to be $\mathrm{hAut}(\theta)$-equivariant.

Homology stability with respect to $S^n \times S^n$. An immediate consequence of Theorem B is the fact that $B\mathrm{Diff}^\theta(W)$ and $B\mathrm{Diff}^\theta(W \# S^n \times S^n)$ have the same homology in the stable range of degrees, as they both have the homology of a collection of path components of $\Omega^\infty\mathbf{MT}\theta$. When W has non-empty boundary there is a *stabilisation map*

$$B\mathrm{Diff}^\theta_\partial(W) \longrightarrow \mathrm{Diff}^\theta_\partial(W \# S^n \times S^n)$$

inducing this homology isomorphism, given by gluing on $([0,1] \times \partial W)\#(S^n \times S^n)$ with some θ-structure. This is an independent ingredient of these theorems, and holds in greater generality. The following theorem is proved in [2].

Theorem C. *Let $\theta : B \to BO(2n)$ be spherical (every θ-structure on D^{2n} extends to S^{2n}), but not necessarily n-co-connected. Let $B\mathrm{Diff}^\theta_\partial(W)$ be defined as above, but using all bundle maps, not just the n-connected ones. Then the stabilisation map induces an isomorphism in integral (co)homology in degrees $* \leq \frac{g(W)-3}{2}$ as long as W is simply-connected and $2n \geq 6$.*

Homology stability with respect to higher handles. A further immediate consequence of Theorem B is an analogous homological stability theorem for gluing on to W a θ-cobordism $K : \partial W \rightsquigarrow P$ such that $(K, \partial W)$ is $(n-1)$-connected, i.e. attaching to W handles of index n or higher. This is again an independent ingredient; given Theorem C can be phrased as follows, which will appear in [3].

Theorem D. *Let $\theta : B \to BO(2n)$ be n-co-connected, partition $\partial W = Q \cup D^{2n-1}$, and let $K : Q \rightsquigarrow Q'$ be a θ-cobordism which is trivial on the boundary and such that (K, Q) is $(n-1)$-connected. Let $S : D^{2n-1} \rightsquigarrow D^{2n-1}$ be $([0,1] \times D^{2n-1})\#(S^n \times S^n)$. Then the map*

$$- \cup K : \underset{k\to\infty}{\mathrm{hocolim}} B\mathrm{Diff}^\theta_\partial(kS \cup W) \longrightarrow \underset{k\to\infty}{\mathrm{hocolim}} B\mathrm{Diff}^\theta_\partial(kS \cup W \cup K)$$

induces an isomorphism on homology as long as $2n \geq 4$.

Stable homology. A further immediate consequence of Theorem B is that the induced map after stabilising by $S := ([0,1] \times \partial W)\#(S^n \times S^n)$,

$$\underset{k\to\infty}{\mathrm{hocolim}} B\mathrm{Diff}^\theta_\partial(kS \cup W) \longrightarrow \underset{k\to\infty}{\mathrm{hocolim}} \Omega^\infty_{[kS \cup W]}\mathbf{MT}\theta,$$

is a homology equivalence (as $g(kS \cup W) \geq k + g(W)$ diverges). This again holds in much greater generality: θ can be just spherical rather than n-co-connected; W need not be simply connected; we can take $B\mathrm{Diff}^\theta$ in the sense given in Theorem

C (though in this generality we may need to stabilise by more than just S). A statement of the general result is complicated, and we refer to [1] for a detailed statement, and the proof.

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CAT(0) cube complexes

Dani Wise

In my talks I will attempt to give a flavor for the central role that CAT(0) cube complexes are now playing in geometric group theory. A survey of this topic together with references to the literature can be found in [1].

1. "Scheme" for understanding a group G. A strategy for understanding a group G follows the following:

(1) Find sufficiently many codimension-1 $\{H_i\}$ subgroups of G
(2) Cubulate to obtain action of G on $\widetilde{X}$
(3) If codimension-1 subgroups are nice enough then G acts nicely
(4) Find finite index subgroup G' such that $G'\backslash\widetilde{X}$ is very organized it is a "special cube complex"
(5) Obtain an embedding G' into a Right Angled Artin Group. RAAGs are easy groups and we can conclude G has nice properties

There are discrete groups for which the above scheme cannot apply. Most notably, infinite nonabelian amenable groups and infinite groups with property-(T) cannot act properly on a CAT(0) cube complex.

Some prominent examples where this scheme has been successful are:

(1) Coxeter Groups (Niblo-Reeves + Haglund-W)
(2) One-relator Groups with Torsion (Magnus-Moldavananskii + W)
(3) Hyperbolic Free-by-Cyclic Groups (Hagen+W)
(4) Mixed 3-manifolds (Liu + Przytycki-W)
(5) Simple-Type Hyperbolic Arithmetic Lattices (Bergeron-Haglund-W)
(6) Hyperbolic 3-manifolds with boundary (W)
(7) Closed Hyperbolic 3-manifolds (Kahn-Markovic+Agol)
(8) C'(1/6) small-cancellation groups (W+Agol)

2. Cube complexes. An *n-cube* is a copy of $[-1,1]^n$, its subcubes correspond to the subspaces obtained by restricting one or more coordinate to ± 1. A *cube complex* X is a obtained from a collection of cubes of various dimensions by gluing them along subcubes. A *flag complex* is a simplicial complex where $n+1$ vertices span an n-simplex if and only if they are pairwise adjacent. The cube complex X is *nonpositively curved* if the link of each 0-cube is a flag complex. The *link* of 0-cube v is the complex built with an n-simplex for each corner of each $(n+1)$-cube at v.

Some examples of nonpositively curved cube complexes: Any graph is nonpositively curved, that is a 1-dimensional cube complex. If $\dim(X) = 2$ then X is nonpositively curved if and only if $\text{link}(v)$ has girth ≥ 4 for each $v \in X^0$. Any closed surface except S^2 or P^2 is homeomorphic to a nonpositively curved cube complex. If A and B are nonpositively curved cube complexes then so is $A \times B$.

3. Right-Angled Artin Group (RAAG). Let Γ be a simplicial graph. The *Right-Angled Artin Group* $A(\Gamma)$ is presented by:

$$A(\Gamma) = \langle v \in \text{Vertices}(\Gamma) \mid uv = vu : (u,v)\text{Edges}(\Gamma)\rangle$$

Note: $A(\Gamma) = \pi_1 R(\Gamma)$ with $R(\Gamma)$ a nonpositively curved cube complex called a *Salvetti complex*. For each n-clique of Γ, the cube complex $R(\Gamma)$ has an n-cube attached as an n-torus in the usual way.

A RAAG A is *residually finite* which means that for each $a \neq 1_A$ there is a finite quotient $A \to \bar{A}$ such that $\bar{a} \neq 1_{\bar{A}}$.

Every finitely generated RAAG A is linear. Moreover, for some $n = n(A)$ we have an embedding $A \subset SL_n(Z)$.

4. CAT(0) Cube Complexes. A CAT(0) cube complex $\widetilde{X}$ is a simply-connected nonpositively curved cube complex. These are "generalized trees" $\widetilde{X}$ has a geodesic metric with n-cubes isometric to $[-1,1]^n \subset E^n$. This geodesic metric satisfies the CAT(0) inequality: Geodesic triangles in $\widetilde{X}$ are at least as thin as their comparison triangles in E^2. Specifically, let $\Delta(a,b,c)$ be a geodesic triangle in $\widetilde{X}$, and let $\Delta'(a',b',c')$ be a triangle in E^2 with the exact same side lengths. For points p,q in Δ, we let p',q' denote the points in the same relative positions in Δ'. Then the CAT(0) inequality requires that:

$$d_{\widetilde{X}}(p,q) \leq d_{E^2}$$

5. Hyperplanes. A midcube in $[-1,1]^n$ is a subspace restricting one coordinate to 0. A Hyperplane is a connected subspace of $\widetilde{X}$ intersecting each cube in either $\emptyset$ or in a single midcube.

- Every midcube in $\widetilde{X}$ lies in a unique hyperplane.
- Each hyperplane separates $\widetilde{X}$ into two parts.
- Each hyperplane is itself a CAT(0) cube complex.

Hyperplanes give a CAT(0) cube complex $\widetilde{X}$ its personality, and generalize role of edges in a tree.

6. Special Cube Complexes. The *Immersed hyperplanes* in a nonpositively curved cube complex X look locally like hyperplanes in $\widetilde{X}$.

Definition 1. A nonpositively curved cube complex X is special if its immersed hyperplanes satisfy:

(1) No hyperplane V self-crosses: (It does not pass through two midcubes of the same cube)
(2) No hyperplane V is 1-sided: (The cubical neighborhood $N(V)$ is isomorphic to $V \times [-1, 1]$.)
(3) No hyperplane self-osculates (V does not pass through two different 1-cubes with the same initial vertex)
(4) No pair of hyperplanes U, V inter-osculate (If U, V pass through different midcubes of the same cube, then they cannot pass through different 1-cubes that do not form the sides of a 2-cube)

Ex: any subcomplex of the product of two graphs is special.

Theorem 2 (Haglund-W). *X is special $\Leftrightarrow$ There is a local-isometry $X \to R(\Gamma)$ to the Salvetti complex of a RAAG.*

G is special if $G = \pi_1 X$ where X is special. Equivalently: G is special if G is a subgroup of a RAAG. A special cube complex is a "high-dimensional graph". A special group is a relaxed version of a free group.

The following generalizes Marshall Hall's theorem for graphs as reexpressed by Stallings. It plays a fundamental role in the theory of special cube complexes.

Theorem 3 (Canonical Completion and Retraction). *Let X be a special cube complex. $f : Y \to X$ be a local isometry with Y a compact cube complex. There exists a finite cover $\rho : \widehat{X} \to X$ and a lift $\widehat{f} : Y \to X$ such that $\widehat{f}$ is an embedding, and $\widehat{X}$ retracts to $\widehat{f}(Y)$.*

7. Cayley Graphs and Hyperbolicity. Let G be a group with generators $\{s_1, s_2, \ldots, s_r$. Its *Cayley graph* has a vertex for each $g \in G$ and has an edge joining g, g' whenever $gs_i = g'$ for some generator s_i.

The Cayley graph has a *path metric* induced by regarding each edge as a unit interval. G is hyperbolic if there exists $\delta \geq 0$ such that all geodesic triangles in the Cayley graph are *δ-thin* in the sense that each side lies in the δ-neighborhood of the union of the other two sides.

A subgroup $H \subset G$ is *quasiconvex* if there exists $\kappa > 0$ such that for any geodesic γ in Cayley(G), if the endpoints of γ lie in H then γ lies in $\mathcal{N}_\kappa(H)$.

8. Codimension-1 subgroups and Dual Cube Complexes. A *codimension-1* subgroup H of G is a subgroup that "cuts G in half". More precisely, there exists $r > 0$ such that Cayley$(G) - \mathcal{N}_r(H)$ has at least two H-orbits of components K that are *deep* in the sense that $K \not\subset \mathcal{N}_s(H)$ for any $s > 0$.

Examples include $Z^n \subset Z^{n+1}$ or more generally, $\pi_1 M^n \subset \pi_1 M^{n+1}$ where $M^n \to M^{n+1}$ is a π_1-injective map between closed aspherical manifolds of those dimensions.

A *wallspace* is a space S together with a locally finite system of walls $\{W_i\}$. Each *wall* W_j decompose S into two halfspaces intersecting at W_j.

Construction 4 (Sageev). Inputs codim-1 subgroup $H \subset G$, and outputs action of G on a CAT(0) cube complex $\widetilde{X}$.

- G acts cocompactly on $\widetilde{X}$ when G is hyperbolic and H is quasiconvex.
- G acts freely on $\widetilde{X}$ when the G-translates of H cut G sufficiently well.

A codimension-1 subgroup H yields a wall in the space $S = \text{Cayley}(G)$. The G-translates of W provides a wallspace with a G-action.

A notable application of Sageev's construction is the following:

Theorem 5 (Wise)**.** *Let G be a $C'(\frac{1}{6})$ group. Then G acts properly and cocompactly on a CAT(0) cube complex.*

A $C'(\frac{1}{6})$ group has a presentation $\langle x_1, \ldots, x_r \mid R_1, \ldots R_s \rangle$ with the property that for each piece P in R_i, we have $|P| < \frac{1}{6}|R_i|$. A *piece* of a relator R_i is a subword that occurs in some other way as a subword of one of the relators.

9. Surfaces in Closed Hyperbolic 3-manifolds and Cubulating from the Boundary.

Theorem 6 (Kahn-Markovic)**.** *Let M be a closed hyperbolic 3-manifold. Then there is a π_1-injective quasifuchsian surface $K \to M$. Moreover, for each circle $C \subset \partial\widetilde{M}$, there exists K with $\partial\widetilde{K} \sim C$.*

Theorem 7. *(Bergeron-W, Dufour) Let G be hyperbolic. Suppose each $p, q \in \partial G$ are cut by a quasiconvex subgroup H in sense that ∂H separates p, q. Then G acts properly and cocompactly on a CAT(0) cube complex.*

Corollary 8. *Let M be closed hyperbolic 3-manifold. $\pi_1 M$ acts freely and cocompactly on CAT(0) cube complex.*

10. Groups with a quasiconvex hierarchy. A group G has a *hierarchy* if G can be built from trivial groups using finitely many amalgamated free products $A *_C B$ and HNN extensions $A*_{C^t=C'}$

It is a *quasiconvex hierarchy* if amalgamated subgroups C are f.g. and $C \subset A *_C B$ and $C \subset A*_{C^t=C'}$ are quasi-isometric embeddings at all stages.

Theorem 9. *If G is a hyperbolic group with a quasiconvex hierarchy then G is virtually special. That is, G has a finite index subgroup G' such that G' is a subgroup of a RAAG.*

A relatively hyperbolic generalization of the above theorem yields:

Theorem 10. *Let M be a hyperbolic 3-manifold with cusps. Then $\pi_1 M$ is is virtually special.*

Another consequence resolves an old problem of Baumslag's on one-relator groups:

Theorem 11. *Every one-relator group with torsion $\langle a, b | W^n \rangle$ is residually finite when $n \geq 2$.*

11. Some Ingredients. A subgroup $M \subset G$ is malnormal if $M^g \cap M = 1$ whenever $g \notin M$.

Theorem 12 (Specializing Amalgams (Haglund-W))**.** *Let X be a compact nonpositively curved cube complex with $\pi_1 X$ hyperbolic, and containing an embedded hyperplane D with $\pi_1 D$ malnormal. Suppose each component of $X - N(D)$ is virtually special. Then X is virtually special.*

We use the notation $N(D)$ for the union of all open cubes intersecting D.

Theorem 13 (Cubulating Amalgams: (Hsu-W))**.** *Let $G = A *_C B$ or $G = A*_{C^t=C'}$ be a hyperbolic group that splits over malnormal quasiconvex C. If A, B are virtually compact special then G is π_1 of a compact nonpositively curved cube complex.*

Ex: $\langle a, b, c, d \mid ababb = cdc^{-1}d^{-1} \rangle$ is π_1 of a compact nonpositively curved cube complex.

Theorem 14 (Special Quotient Theorem)**.** *Let G be hyperbolic and virtually compact special. Let $H \subset G$ be quasiconvex. Then H has finite index H' such that $G/ << H' >>$ is virtually compact special.*

Ex: $\langle a, b \mid (ababb)^8 \rangle$ is virtually compact special. Here $G = Free(a, b)$ and $H = \langle ababb \rangle$ and $H' = \langle (ababb)^8 \rangle$.

A final ingredient is: *Cubical Small-Cancellation Theory* which is a high-dimensional generalization of classical small cancellation theory - joining theories of: Gromov, Osin, Groves-Manning, Dahmani-Guirardel, and others.

An ordinary presentation $\langle a, b, c \mid R_1, R_2, \ldots, R_n \rangle$ can be represented as: $\langle X \mid Y_1, Y_2, \ldots, Y_n \rangle$ where X is a bouquet of circles and each $Y_i \to X$ is an immersion of a circle corresponding to the word R_i. The group G of the presentation corresponds to $\pi_1 X/ << \pi_1 Y_1 \ldots \pi_1 Y_n >>$.

Likewise, we can let X be compact nonpositively curved cube and let each $Y_i \to X$ be a local isometry of cube complexes, and we obtain a "cubical presentation". The group this yields follows the same formula as above. The standard definitions of small-cancellation theory have generalization to this framework, and when $\langle X \mid Y_1, Y_2, \ldots, Y_n \rangle$ is $C'(\frac{1}{12})$ one obtains very strong control of its properties as in the classical small-cancellation theory. In particular, one is able to study walls in the generalized Cayley graph of this group, and this theory becomes a very convenient organizing tool, that played an important role in proofs of some of the above theorems.

12. All closed hyperbolic 3-manifolds are virtually special. By applying the special quotient theorem in an ingenious fashion, Ian Agol proved the following result that completes the main goals of this research program:

Theorem 15 (Agol)**.** *Let X be a compact nonpositively curved cube complex. Suppose $\pi_1 X$ is hyperbolic. Then X is virtually special.*

Given the cubulation from Kahn-Markovic we have the following:

Corollary 16. *Every closed hyperbolic 3-manifold M is virtually Haken, virtually fibered, has positive virtual first Betti number, lies in $SL(m, Z)$, ...*

Given the cubulation of C'(1/6) small-cancellation groups we have:

Corollary 17. *Let G be a C'(1/6) small-cancellation group. Then G is virtually special.*

We conclude that for fixed r, s, a random finitely presented group with r-generators and s-relators almost certainly residually finite. This is a remarkable consequence for combinatorial group theory.

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Toward the formal theory of (∞, n)-categories

Emily Riehl
(joint work with Dominic Verity)

"Formal category theory" refers to a commonly applicable framework (i) for defining standard categorical structures – monads, adjunctions, limits, the Yoneda embedding, Kan extensions – and (ii) in which the classical proofs can be used to establish the expected relationships between these notions: e.g. that right adjoints preserve limits. One such framework is a 2-category equipped with a bicategory of "modules." (A *module* or *profunctor* from a category A to a category B is a functor $A^{\text{op}} \times B \to \mathbf{Set}$, for instance hom: $A^{\text{op}} \times A \to \mathbf{Set}$.)

In previous work, we show the basic category theory of quasi-categories can be developed formally in a strict 2-category, the "homotopy 2-category" of quasi-categories. A main point is that certain weak 2-limits present in this 2-category, particularly *comma objects*, encode universal properties up to the appropriate notion of equivalence for quasi-categories. An important feature of these "formal" definitions and proofs is that they apply representably in other higher homotopical contexts, including Rezk objects (e.g., complete Segal spaces). In the quasi-categorical context, we are reprising the foundational work pioneered by Joyal, Lurie, and others. Our aim is to develop new tools to prove further theorems, but an important side benefit is that this work applies equally to other models.

The aforementioned comma objects are precisely those modules that are represented by ordinary functors. In work in progress, we have developed a general theory of modules between quasi-categories, which is robust enough to support a complete formal category theory. (Modules appear under the guise of *correspondences* in Lurie's work, but our presentation, as *two-sided discrete fibrations*, is different.) This allows us to prove, for instance, the familiar (co)limit formula for pointwise Kan extensions.

At present, these new results do not immediately translate to other flavors of $(\infty, 1)$-categories, or to (∞, n)-categories, because there is one key technical property (a "homotopy exponentiability" criterion for maps) that we prove in specific reference to the quasi-categorical model. In what follows, we explain some of the basic ideas behind formal category theory and explore future vistas.

Basic formal category theory

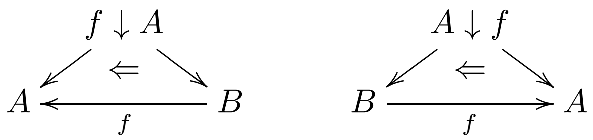
The simplest framework for formal category theory is a strict 2-category. The prototypical example might be the 2-category of categories, functors, and natural transformations. Our particular interest will be in a 2-category whose objects are (∞, n)-categories, whose morphisms are functors of such, and whose 2-cells are homotopy classes of 1-simplices in appropriate hom-spaces.

There is a 2-categorical definition of an adjunction: an *adjunction* consists of objects A, B; 1-cells $u\colon A \to B$, $f\colon B \to A$; and 2-cells $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ satisfying the triangle identities. The standard proofs demonstrate that (i) any two left adjoints to a common 1-cell are isomorphic and (ii) adjunctions compose.

Now suppose the 2-category has some notion of "exponentiation," indexed by objects X in some other category. The object A^X is thought of as the object of X-shaped diagrams in A. A morphism $X \to Y$ should induce a map $A^Y \to A^X$. In particular, assuming that exponentiation by the terminal object is the identity, this gives rise to a "constant diagram map" $\mathrm{const}\colon A \to A^X$.

Declare that an object A in the 2-category *has X-shaped limits* if the 1-cell $\mathrm{const}\colon A \to A^X$ has a right adjoint $\lim\colon A^X \to A$. As an immediate consequence of propositions (i) and (ii) above, if A and B have X-shaped limits, any right adjoint $u\colon A \to B$ preserves them.

Further results are possible if the 2-category has comma objects. Given $f\colon B \to A$, we may define a pair of (weak) *comma objects* consisting of the data



and satisfying a weak universal property. A *generalized element* of $f \downarrow A$, meaning a morphism $X \to f \downarrow A$, corresponds to a generalized element $a\colon X \to A$ of A and a generalized element $b\colon X \to B$ of B, together with a 2-cell $fb \Rightarrow a$.

There is a formula for the adjunct morphism $b \Rightarrow ua$ to $fb \Rightarrow a$ in terms of the unit and counit of the adjunction. By a 2-categorical encoding of precisely this argument, the comma objects $f \downarrow A$ and $B \downarrow u$ are equivalent over $A \times B$.

The homotopy 2-category

The strict 2-categories of interest arise as the *homotopy 2-category* of a *quasi-categorical context*: a simplicially enriched category whose hom-spaces are quasi-categories, satisfying the properties enjoyed by the fibrant objects in a model category that is enriched over simplicial sets (with the Joyal model structure) and in which all fibrant objects are cofibrant. A main point of this talk is that for much of the formalism, the precise definition of quasi-category does not matter.

The prototypical example is given by the category of quasi-categories. Other examples include complete Segal spaces or more general categories of *Rezk objects*:

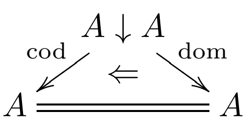
simplicial objects in a model category that are Reedy fibrant and satisfy the Segal and completeness conditions. For instance, Barwick's n-fold complete Segal space model of (∞, n)-categories has this form. If $\mathcal{K}$ is a quasi-categorical context, so is the slice category $\mathcal{K}/A$ over any object A.

The *homotopy 2-category* $\mathcal{K}_2$ of a quasi-categorical context $\mathcal{K}$ is the strict 2-category defined by applying the homotopy category functor to each of the hom-spaces. Its objects and 1-cells are the same as in $\mathcal{K}$, and its 2-cells are homotopy classes of 1-simplices in the hom-spaces. A quasi-categorical context admits exponentials by arbitrary simplicial sets and comma objects constructed as homotopy limits. This structure descends to the aforementioned structures on the homotopy 2-category. The content of the papers [1, 2, 3] is stated in the language of quasi-categories but all of the results appearing there apply, essentially without change, in any quasi-categorical context. This means that the definitions of the basic categorical concepts can be interpreted there and the proofs, largely taking place in the homotopy 2-category, are also unchanged.

Two-sided discrete fibrations

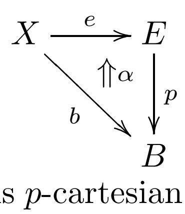
Missing from the basic framework of a 2-category with comma objects is the Yoneda embedding (classically, the "hom" bifunctor $A^{\mathrm{op}} \times A \to \mathbf{Set}$) and its generalizations (arbitrary functors $B^{\mathrm{op}} \times A \to \mathbf{Set}$). These go by a variety of names: *modules*, *profunctors*, *distributors*, or *correspondences*. There are several possible ways to encode modules in a 2-category. Given the structures that are present in a homotopy 2-category, our preference will be to use comma objects.

For example, the Yoneda embedding for A is encoded by the comma object:



The generalized elements of $A \downarrow A$ encode 2-cells with codomain A. But $A \downarrow A$ has additional universal properties relating to the pre- and post-composition actions by arrows in A, which are expressed by saying that $A \xleftarrow{\mathrm{cod}} A \downarrow A \xrightarrow{\mathrm{dom}} A$ is a *two-sided discrete fibration* in the homotopy 2-category.

To state this definition, we first need a notion of cartesian fibration. For quasi-categories, this coincides exactly with the notion introduced by Lurie, but our 2-categorical definition can be interpreted in any homotopy 2-category. An isofibration $p\colon E \to B$ is a *cartesian fibration* if

(i) Every 

admits a p-*cartesian lift* $\chi\colon \bar{e} \Rightarrow e$ along p. Here a 2-cell χ is p-cartesian if it satisfies a weak form of the expected factorization axiom and also has a 2-cell conservativity property: any endomorphism of χ sitting over the identity on b is an isomorphism.

(ii) The p-cartesian 2-cells are stable under restriction along any functor.

The domain projection dom: $E \downarrow E \to E$ is a cartesian fibration. Reversing the 2-cells but not the 1-cells, we obtain the notion of a *cocartesian fibration.*

A cartesian fibration $p\colon E \to B$ is *discrete* if any 2-cell over p is an isomorphism. If $b\colon 1 \to B$ is a point, then dom: $B \downarrow b \to B$ is a discrete cartesian fibration. A span $A \xleftarrow{q} E \xrightarrow{p} B$ is a *two-sided discrete fibration* if

(i) $E \to A \times B$ is a discrete cartesian fibration in $\mathcal{K}_2/A$.
(ii) $E \to A \times B$ is a discrete cocartesian fibration in $\mathcal{K}_2/B$.

The comma objects $f \downarrow A$ and $A \downarrow f$ are two-sided discrete fibrations.

The equipment for quasi-categories

With the notion of a two-sided discrete fibration to encode modules, we can establish a complete framework for formal category theory.

Theorem. *There is a bicategory* $\mathbf{qMod}_2$ *of quasi-categories; modules, i.e., two-sided discrete fibrations* $A \xleftarrow{q} E \xrightarrow{p} B$*, written* $E\colon A \nrightarrow B$*; and isomorphism classes of maps of spans. Moreover:*

(i) $\mathbf{qMod}_2$ *is biclosed: the functors* $E \otimes_B -$ *and* $- \otimes_A E$ *admit right biadjoints.*
(ii) The identity-on-objects homomorphism $\mathbf{qCat}_2 \hookrightarrow \mathbf{qMod}_2$ *that carries* $f\colon A \to B$ *to* $B \downarrow f\colon A \nrightarrow B$ *is locally fully faithful.*
(iii) The covariant represented module $B \downarrow f\colon A \nrightarrow B$ *is left adjoint to the contravariant represented module* $f \downarrow B\colon B \nrightarrow A$.

In summary, $\mathbf{qCat}_2 \hookrightarrow \mathbf{qMod}_2$ *is an equipment in the sense of Wood.*

The proof of this result uses the Yoneda lemma for maps between modules. Note that the proposition proven above asserts that if $f \dashv u$, then the modules $f \downarrow A$ and $B \downarrow u$ are isomorphic as 1-cells $A \nrightarrow B$.

At present, we must specialize to quasi-categories because we have yet to explore how our *conduché condition* for homotopy exponentiability may be generalized to other contexts. For the time being, we might note that the structures on $\mathbf{qMod}_2$ requiring this condition are convenient, but not strictly necessary.

With this theorem, we can now commence with the formal category theory. For example, there is a standard definition of a right (Kan) extension diagram in any 2-category. In $\mathbf{qCat}_2$ this is too weak (failing, in general, to be "pointwise"), but in $\mathbf{qMod}_2$ it gives the correct notion.

Definition. Consider a pair of functors $f\colon A \to B$ and $g\colon A \to C$. The right extension E

$$\begin{array}{ccc} A & \xrightarrow{\;C\downarrow g\;} & C \\ & \Leftarrow & \\ {\scriptstyle B\downarrow f} \searrow & & \swarrow {\scriptstyle E} \\ & B & \end{array}$$

exists because $\mathbf{qMod}_2$ is closed. If the module E is covariantly representable, i.e., if $E \cong B \downarrow r$ for some $r\colon C \to B$, then r is the *right extension of* f *along* g.

Theorem. *A module* $C \xleftarrow{q} E \xrightarrow{p} B$ *is covariantly representable if and only if the following equivalent conditions hold:*

(i) q has a right adjoint right inverse
(ii) each fiber of q has a terminal object

Condition (ii) can be used to establish the expected result: if B has limits indexed by certain comma objects, then the right extension of f along g exists.

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The twisted L^2-torsion function and its application to 3-manifolds

Wolfgang Lück

The talk is about an ongoing project joint with Stefan Friedl.

Let G be a group, $G \to \overline{X} \to X$ be a G-covering over a finite CW-complex X and $\phi\colon G \to \mathbb{Z}$ be a group homomorphism. If G is residually finite and $\overline{X}$ is L^2-acyclic, i.e., all L^2-Betti numbers $b_n^{(2)}(\overline{X}, \mathcal{N}(G))$ vanish, we can assign to it a function

$$\rho^{(2)}(\overline{X}, \mathcal{N}(G); \phi)\colon (0,\infty) \to \mathbb{R}$$

which is essentially the L^2-torsion of $\overline{X}$ twisted with the 1-dimensional real representation $\mathbb{R}$ on which $g \in G$ acts by multiplication with $t^{\phi(g)}$. (Actually this function is only well-defined up to adding $k \cdot \ln(t)$ for some $k \in \mathbb{Z}$). If $G = \pi_1(X)$ and $\overline{X}$ is the universal covering $\widetilde{X}$, then we abbreviate $\rho^{(2)}(\widetilde{X};\phi) := \rho^{(2)}(\widetilde{X}, \mathcal{N}(\pi_1(X)); \phi)$. See [5, 4, 3]. For basics about L^2-invariants we refer to [7].

We present some basic properties such as homotopy invariance, sum formula, product formula or more generally a formula for fibrations with L^2-acyclic fiber, passage to finite covering, scaling ϕ, Poincaré duality, and compute it for S^1-spaces with appropriate S^1-action and mapping tori T_f for ϕ the canonicial homomorphism $\pi_1(T_f) \to \pi_1(S^1) = \mathbb{Z}$.

Then we pass to 3-manifolds and compute it for graph manifolds and 3-manifolds which fiber over S^1. We show that for a knot $K \subseteq S^3$ with knot complement $X(K)$ and $\phi \in H^1(X(K);\mathbb{Z}) \cong \mathbb{Z}$ a generator that $\rho^{(2)}(\widetilde{X(K)}, \phi)$ detects the trivial knot, see [1, 8].

A function ρ is asymptotically monomial if for some constants C_0 and C_∞ the limits $\lim_{t\to 0}\bigl(\rho(t) - C_0 \cdot \ln(t)\bigr)$ and $\lim_{t\to\infty}\bigl(\rho(t) - C_\infty \cdot \ln(t)\bigr)$ exists. In this case we define the degree $\deg(\rho)$ to be $C_\infty - C_0$. Denote by $x_M(\phi)$ the Thurston norm of $\phi \in H^1(X;\mathbb{Z})$.

Our main the theorem is

Theorem 1. *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group π and empty or incompressible torus boundary. Consider $\phi \in H^1(X;\mathbb{Z})$. Then*

$$\deg\big(\rho^{(2)}(\widetilde{M};\phi)\big) = -x_M(\phi).$$

We can actually generalize it to other coverings than the universal covering.

Theorem 2. *Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group π and empty or incompressible torus boundary which is not a closed graph manifold.*

Then there is a virtually finitely generated free abelian group Γ, and a factorization $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f := H_1(M)/\operatorname{tors}(H_1(M))$ of the canonical projection into epimorphisms, an element $m \in H_1(M)_f$, an integer $k \geq 1$ such that the following holds:

For any group homomorphism $\phi\colon H_1(\pi)_f := H_1(\pi)/\operatorname{tors}(H_1(\pi)) \to \mathbb{Z}$ and any factorization of $\alpha\colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ for a residually finite group G, there exists real numbers constants $D_2 \geq 0$ and $D_4 \geq 0$ such that for the G-covering $\overline{M} \to M$ associated to μ we get

$$\frac{\phi(m)}{k} \cdot \ln(t) - D_2 \leq \rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \beta \circ \nu)(t) \leq \frac{\phi(m)}{k} \cdot \ln(t) \quad \textit{for } t \leq 1;$$

and

$$\left(-x_M(\phi) + \frac{\phi(m)}{k}\right) \cdot \ln(t) - D_4 \leq \rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \nu)(t) \leq \left(-x_M(\phi) + \frac{\phi(m)}{k}\right) \cdot \ln(t) \quad \textit{for } t \geq 1.$$

In particular $\rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \nu)$ is asymptotically monomial and satisfies

$$\deg\big(\rho^{(2)}(\overline{M}, \mathcal{N}(G); \phi \circ \nu)\big) = -x_M(\phi).$$

We use this to show for the higher order Alexander polynomial of Cochrane and Harvey, see [2, 6], that their degree coincides with the Thurston norm in the situation of the last theorem provided that G is torsionfree elementary amenable and residually finite. Previously only an inequality was known.

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Infinite loop spaces and positive scalar curvature

Johannes Ebert

(joint work with Boris Botvinnik, Oscar Randal-Williams)

Definition 1. Let W be a compact d-dimensional manifold. By $\mathcal{R}^+(W)$, we denote the space of all Riemann metrics on W that have positive scalar curvature. If W has boundary M, we fix a collar $M \times [0,1] \subset W$ and a point $g \in \mathcal{R}^+(M)$. By $\mathcal{R}^+(W)_g$, we denote the space of all $h \in \mathcal{R}^+(W)$, such that on the collar, h is of the form $g + dt^2$.

If W^d has a spin structure, then index theory provides a powerful tool for the study of positive scalar curvature, namely the Atiyah-Singer Dirac operator $\not{\mathfrak{D}}$ on the spinor bundle. The index of the operator $\not{\mathfrak{D}}$ is an element $\operatorname{ind}(\not{\mathfrak{D}}) \in KO^{-d}(*)$. The relevance to positive scalar curvature is given by the Schrödinger-Lichnerowicz formula

$$\not{\mathfrak{D}}^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}.$$

As a consequence, $\operatorname{ind}(\not{\mathfrak{D}})$ is trivial if $\mathrm{scal} > 0$. For two psc metrics h_0, h_1, one obtains a secondary invariant, the index difference $\operatorname{inddiff}(h_0, h_1) \in KO^{-d-1}(*)$ (note the dimension shift). The construction of the index difference generalizes well to the family situation and to compact manifolds with boundary, and this yields a map (depending on a basepoint h)

$$\operatorname{inddiff}_h : \mathcal{R}^+(W)_g \to \Omega^{\infty+d+1}\mathbf{ko}.$$

At the time of writing, this map seems to be the only available tool to detect homotopy information on $\mathcal{R}^+(W)_g$.

To find nontrivial homotopy classes in $\mathcal{R}^+(W)_g$, one needs to map well-understood spaces to $\mathcal{R}^+(W)_g$. Our construction uses Galatius' and Randal-Williams' theory of moduli spaces of high-dimensional manifolds [5]. Let $\Theta : BO(2n)\langle n\rangle \to BO(2n)$ be the n-connected cover and let $\mathrm{MT}\Theta(2n)$ be the Thom spectrum of the additive inverse to the vector bundle classified by Θ (this spectrum is one of the Madsen-Tillmann-Weiss spectra). If $n \geq 2$, the map Θ factors through $B\mathrm{Spin}(2n)$ and so the spectrum $\mathrm{MT}\Theta(2n)$ has an Atiyah-Bott-Shapiro Thom class $\hat{\mathfrak{a}} : \mathrm{MT}\Theta(2n) \to \Sigma^{-2n}\mathbf{ko}$.

Theorem 1. *([1]) Let W be a compact spin manifold with boundary M, let $g \in \mathcal{R}^+(M)$ and assume that $h \in \mathcal{R}^+(W)_g \neq \emptyset$.*

(1) If $\dim(W) = 2n \geq 6$*, there exists a map*

$$\Phi_h : \Omega^{\infty+1}\mathrm{MT}\Theta(2n) \to \mathcal{R}^+(W)_g$$

such that the composition $\mathrm{inddiff}_h \circ \Phi_h$ *is homotopic to the infinite loop map induced by* $\hat{\mathfrak{a}}$*.*

(2) If $\dim(W) = 2n+1 \geq 7$*, there exists a map*

$$\Phi_h : \Omega^{\infty+2}\mathrm{MT}\Theta(2n) \to \mathcal{R}^+(W)_g$$

such that the composition $\mathrm{inddiff}_h \circ \Phi_h$ *is homotopic to the infinite loop map induced by* $\hat{\mathfrak{a}}$*.*

This result has various computational consequences because the map $\hat{\mathfrak{a}}$ can be studied by the traditional tools of algebraic topology. For example, the index difference map is surjective on rational homotopy groups (on π_0, this was proven by Gromov and Lawson [7], and if the dimension is large compared to the homotopical degree, by Hanke-Steimle-Schick [8]). Moreover, all the $\mathbb{Z}/2$-groups in the homotopy of **ko** are hit (in low degrees, this was shown by Hitchin [9] and for one half of these groups in higher degrees by Crowley-Schick [3]). Using more advanced computations in homotopy theory, one can refine these results considerably.

Theorem 1 is proven first for even-dimensional manifolds, and then the result in odd dimensions is derived from that. For the even-dimensional case, we use three main ingredients. The first is a result of Chernysh [2] and Walsh [10]: if the d-manifold W' is obtained from W by a surgery of index $3 \leq k \leq d-2$, then the spaces $\mathcal{R}^+(W)_g$ and $\mathcal{R}^+(W')_g$ are homotopy equivalent. It is a classical result by Gromov and Lawson [6] that one of these spaces is nonempty iff the other is, and the proof by Chernysh and Walsh is indeed an elaboration of Gromov-Lawson's proof. The second ingredient is the due to Galatius and Randal-Williams [5]. Namely, let $W_k := \sharp^k(S^n \times S^n) \setminus D^{2n}$. They showed that the classifying spaces $B\mathrm{Diff}_\partial(W_k)$ homologically approximate $\Omega^\infty\mathrm{MT}\Theta(2n)$ (here the assumption $n \geq 3$ comes into play). We use the Borel construction of the action of $\mathrm{Diff}_\partial(W_k)$ on $\mathcal{R}^+(W_k)_{g_{\mathrm{round}}}$ and the Chernysh-Walsh theorem to construct a fibration with fibre $\mathcal{R}^+(D^{2n})_{g_{\mathrm{round}}}$ over $\Omega_0^\infty\mathrm{MT}\Theta(2n)$, which is the Quillen Plus construction of $\mathrm{hocolim}_k B\mathrm{Diff}_\partial(W_k)$. The map Φ is the fibre transport of this fibration. Using the Atiyah-Singer family index theorem, we then prove that the composition of the fibre transport with the index difference is $\Omega^{\infty+1}\hat{\mathfrak{a}}$. This establishes Theorem 1 for $W = D^{2n}$. The general even-dimensional case is proven by a simple cut-and paste technique. To pass to odd-dimensional manifolds, we use a geometrically constructed map $\Omega_g\mathcal{R}^+(M) \to \mathcal{R}^+(W)_g$ (as before, $\partial W = M$ and $\mathcal{R}^+(W)_g \neq \emptyset$). On the index theoretic side, we use a different construction of inddiff and the equality of both constructions proven by [4], which is a generalization of the classical spectral-flow-index theorem.

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Tits buildings, class numbers, and the high-dimensional cohomology of $\mathbf{SL}_n(\mathcal{O})$

Andrew Putman
(joint work with Thomas Church, Benson Farb)

Let $\mathcal{O}$ be the ring of integers in an algebraic number field K. In this talk, we discuss a new structural result about the spherical Tits building attached to $\mathrm{SL}_n(\mathcal{O})$. Applications include vanishing and nonvanishing theorems for the top degree rational cohomology groups of $\mathrm{SL}_n(\mathcal{O})$.

The Tits building and the Steinberg module One of the most fundamental geometric objects attached to $\mathrm{SL}_n(K)$ is its *Tits building*, denoted $\mathcal{T}_n(K)$. The space $\mathcal{T}_n(K)$ is the simplicial complex whose $(p-1)$-simplices are flags

$$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq K^n.$$

The group $\mathrm{SL}_n(K)$ acts on $\mathcal{T}_n(K)$ by simplicial automorphisms. As we will explain in more detail below, $\mathcal{T}_n(K)$ plays an important role in the study of arithmetic groups and their cohomology.

The Solomon–Tits Theorem says that $\mathcal{T}_n(K)$ is $(n-3)$-connected, and thus is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres. The *Steinberg module* for $\mathrm{SL}_n(K)$, denoted $\mathrm{St}_n(K)$, is $\widetilde{\mathrm{H}}_{n-2}(\mathcal{T}_n(K);\mathbb{Z})$. This is an important representation of $\mathrm{SL}_n(K)$. The Solomon-Tits Theorem also gives a generating set for $\mathrm{St}_n(K)$ in terms of apartments, which we now define. A *frame* for K^n is an ordered set $\mathbf{L} = \{L_1, \ldots, L_n\}$ of lines in K^n such that $K^n = L_1 \oplus \cdots \oplus L_n$. The *apartment* corresponding to $\mathbf{L}$, denoted $A_{\mathbf{L}}$, is the $(n-2)$-sphere in $\mathcal{T}_n(K)$ obtained as follows. Let σ be the barycentric subdivision of an $(n-1)$-simplex. The vertices of σ can be identified with nonempty subsets of $\{1, \ldots, n\}$, and the vertices in the boundary $\partial\sigma \cong S^{n-2}$ are identified with proper subsets. There is then a map $\partial\sigma \to \mathcal{T}_n(K)$ taking the vertex corresponding to $\emptyset \subsetneq I \subsetneq \{1, \ldots, n\}$

to $\langle L_i \mid i \in I \rangle$; the image is the apartment $A_{\mathbf{L}}$. Each apartment determines an *apartment class*

$$[A_{\mathbf{L}}] \in \widetilde{\mathrm{H}}_{n-2}(\mathcal{T}_n(K);\mathbb{Z}) = \mathrm{St}_n(K).$$

The Solomon–Tits theorem says that $\mathrm{St}_n(K)$ is generated by the set of apartment classes. Since $\mathrm{SL}_n(K)$ acts transitively on apartments, this implies that $\mathrm{St}_n(K)$ is a cyclic $\mathrm{SL}_n(K)$-module.

Integrality and non-integrality. While the action of $\mathrm{SL}_n(K)$ on $\mathcal{T}_n(K)$ is transitive, the action of $\mathrm{SL}_n(\mathcal{O})$ is not always so, and indeed this action encodes arithmetic information about $\mathcal{O}$. For example, the number of orbits of the $\mathrm{SL}_2(\mathcal{O})$-action on $\mathcal{T}_2(\mathcal{O})$ is the class number of $\mathcal{O}$. In this context one has the following natural notion. A frame $\mathbf{L} = \{L_1, \ldots, L_n\}$ of K^n is *integral* if

$$\mathcal{O}^n = (L_1 \cap \mathcal{O}^n) \oplus \cdots \oplus (L_n \cap \mathcal{O}^n).$$

In this case we call $A_{\mathbf{L}}$ an *integral apartment.* Whenever $\mathcal{O} = \mathbb{Z}$, and more generally whenever $\mathcal{O}$ is Euclidean, Ash and Rudolph [1] proved that $\mathrm{St}_n(K)$ is generated by integral apartments. To do this they give a beautiful generalization of the method of continued fractions to higher dimensions. We prove the following generalization of Ash–Rudolph's theorem. Let $\mathrm{cl}(\mathcal{O})$ denote the class number of $\mathcal{O}$.

Theorem 1 (Church–Farb–Putman [3]). *Let $\mathcal{O}$ be the ring of integers in an algebraic number field K with $\mathrm{cl}(\mathcal{O}) = 1$. Assume either that $\mathcal{O}$ has a real embedding or that $\mathcal{O}$ is Euclidean. Then $\mathrm{St}_n(K)$ is spanned by integral apartment classes.*

Under these conditions $\mathrm{SL}_n(\mathcal{O})$ acts transitively on integral apartments, so Theorem 1 implies that $\mathrm{St}_n(K)$ is a cyclic $\mathrm{SL}_n(\mathcal{O})$-module.

Remark 2. Ash–Rudolph's proof of Theorem 1 when $\mathcal{O}$ is Euclidean is based on an algorithm to write a non-integral apartment class as a sum of integral apartment classes. They use the Euclidean function on $\mathcal{O}$ to measure the "complexity" of the non-integral apartment classes. Our proof is quite different: non-integral apartments never actually show up, and our proof does not even make use of the fact that $\mathrm{St}_n(K)$ is generated by apartment classes.

The assumption in Theorem 1 that $\mathrm{cl}(\mathcal{O}_S) = 1$ obviously excludes many examples; however, this assumption is necessary in a very strong sense.

Theorem 3 (Church–Farb–Putman [3]). *Let $\mathcal{O}$ be the ring of integers in an algebraic number field K. If $\mathrm{cl}(\mathcal{O}) > 1$ and $n \geq 2$, then $\mathrm{St}_n(K)$ is not generated by integral apartment classes.*

Bieri-Eckmann duality and the Steinberg module The above results are closely connected to the high-dimensional cohomology groups of $\mathrm{SL}_n(\mathcal{O})$. Recall that for any virtually torsion-free group Γ, the virtual cohomological dimension is

$$\mathrm{vcd}(\Gamma) := \max\{k \mid \mathrm{H}^k(\Gamma; M \otimes \mathbb{Q}) \neq 0 \text{ for some } \Gamma\text{-module } M\}.$$

Define $\nu_n = \mathrm{vcd}(\mathrm{SL}_n(\mathcal{O}))$. While the group $\mathrm{SL}_n(\mathcal{O})$ does not satisfy Poincaré duality, Borel-Serre [2] proved that it does satisfy *Bieri-Eckmann duality* with *rational dualizing module* $\mathrm{St}_n(K)$. This means that for any $\mathrm{SL}_n(\mathcal{O})$-module V and all $i \geq 0$ we have

$$\mathrm{H}^{\nu_n - i}(\mathrm{SL}_n(\mathcal{O}); V \otimes \mathbb{Q}) \cong \mathrm{H}_i(\mathrm{SL}_n(\mathcal{O}); V \otimes_{\mathbb{Q}} \mathrm{St}_n(K)).$$

This implies that $\mathrm{H}^k(\mathrm{SL}_n(\mathcal{O}); \mathbb{Q}) = 0$ for $k > \nu_n$, and

$$\mathrm{H}^{\nu_n}(\mathrm{SL}_n(\mathcal{O}); \mathbb{Q}) \cong \mathrm{H}_0(\mathrm{SL}_n(\mathcal{O}); \mathrm{St}_n(K) \otimes \mathbb{Q}) \cong (\mathrm{St}_n(K) \otimes \mathbb{Q})_{\mathrm{SL}_n(\mathcal{O})}.$$

Combining this with Theorem 1, we are able to prove the following theorem.

Theorem 4 (Church–Farb–Putman [3])**.** *Let $\mathcal{O}$ be the ring of integers in an algebraic number field K such that $\mathrm{cl}(\mathcal{O}) = 1$. Assume either that $\mathcal{O}$ has a real embedding or that $\mathcal{O}$ is Euclidean. Then $\mathrm{H}^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q}) = 0$ for all $n \geq 2$.*

For $\mathcal{O}$ Euclidean, Theorem 4 was originally proved by Lee–Szczarba [4].

Remark 5. Some condition on K beyond the class number assumption $\mathrm{cl}(\mathcal{O}) = 1$ is necessary in Theorem 4. Indeed, for $d < 0$ squarefree let $\mathcal{O}_d$ denote the ring of integers in the quadratic imaginary field $K_d = \mathbb{Q}(\sqrt{d})$. Those $d < 0$ such that $\mathcal{O}_d$ is non-Euclidean but $\mathrm{cl}(\mathcal{O}_d) = 1$ are exactly $d \in \{-19, -43, -67, -163\}$. Although $\mathrm{H}^2(\mathrm{SL}_2(\mathcal{O}_{-19}); \mathbb{Q}) = 0$, we have

$$\mathrm{H}^2(\mathrm{SL}_2(\mathcal{O}_{-42}); \mathbb{Q}) = \mathbb{Q}, \quad \mathrm{H}^2(\mathrm{SL}_2(\mathcal{O}_{-67}); \mathbb{Q}) = \mathbb{Q}^2, \quad \mathrm{H}^2(\mathrm{SL}_2(\mathcal{O}_{-163}); \mathbb{Q}) = \mathbb{Q}^6.$$

For these calculations, see [5, 6]. Presumably similar things occur for $\mathrm{SL}_n(\mathcal{O}_{-d})$ for $n \geq 3$, but we could not find such calculations in the literature. It is likely that imaginary quadratic fields provide the only such counterexamples to Theorem 4; indeed, Weinberger [7] proved that the generalized Riemann hypothesis implies that if K has class number 1 and $\mathcal{O}_K$ has infinitely many units, then $\mathcal{O}_K$ is Euclidean.

Nonvanishing. In Theorem 4 we assumed the class number was 1. The following theorem shows this assumption is necessary.

Theorem 6 (Church–Farb–Putman [3])**.** *Let $\mathcal{O}$ be the ring of integers in an algebraic number field K. Then for all $n \geq 2$,*

$$\dim \mathrm{H}^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q}) \geq (\mathrm{cl}(\mathcal{O}) - 1)^{n-1}.$$

The cohomology classes we construct in Theorem 6 were known classically when $n = 2$. To illustrate this, consider a quadratic imaginary field K_d as in Remark 5. The Bianchi group $\mathrm{SL}_2(\mathcal{O}_d)$ is a lattice in $\mathrm{SL}_2 \mathbb{C} = \mathrm{Isom}(\mathbb{H}^3)$. The associated locally symmetric space $X_d = \mathrm{SL}_2(\mathcal{O}_d) \backslash \mathbb{H}^3$ is a noncompact arithmetic 3-dimensional hyperbolic orbifold of cohomological dimension 2. The cusps of X_d are in bijection with the $\mathrm{SL}_2(\mathcal{O}_d)$-conjugacy classes of parabolic subgroups in $\mathrm{SL}_2(\mathcal{O}_d)$; one can show that there are $\mathrm{cl}(\mathcal{O}_d)$ such conjugacy classes. An embedded path in X_d connecting one cusp to another defines an element of the locally finite homology group $\mathrm{H}_1^{\mathrm{lf}}(X_d; \mathbb{Q})$, which is dual to $\mathrm{H}^2(X; \mathbb{Q}) \cong \mathrm{H}^2(\mathrm{SL}_2(\mathcal{O}_d); \mathbb{Q})$. Since there are $\mathrm{cl}(\mathcal{O}_d)$ cusps, intersecting with such paths gives a $(\mathrm{cl}(\mathcal{O}_d)-1)$-dimensional projection of $\mathrm{H}^2(X; \mathbb{Q}) \cong \mathrm{H}^2(\mathrm{SL}_2(\mathcal{O}_d); \mathbb{Q})$. A similar procedure works for $\mathrm{SL}_2(\mathcal{O}_K)$

for any number field K. However, the case $n \geq 3$ is more complicated: the cusps overlap in complicated ways, so this simple argument does not work.

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Session of 10 minute talks

Dieter Degrijse and Irakli Patchkoria: What is the geometric meaning of the virtual cohomological dimension of a group?

In this talk we present a triangulated category of proper G-spectra where G is an infinite discrete group with bounded torsion. The triangulated category is generated by the orbits with finite isotropy and admits restriction functors to genuine H-spectra for any finite subgroup H of G. We also indicate how a proper G-spectrum gives rise to a Mackey functor for G. This is joint work with Lück and Schwede. We will also discuss a second project, joint with Bárcenas, where we apply this setup to define a notion of stable geometric dimension for proper actions of G. We are trying to show that this notion of dimension coincides with the Mackey cohomological dimension of G. If G is virtually torsion free, the Mackey cohomological dimension is known to equal the virtual cohomological dimension of G. Hence, we would obtain a geometric interpretation of the virtual cohomological dimension of G.

Lukasz Grabowski: Group ring elements with large spectrum near zero

Motivated by the theory of l^2-invariants of CW-complexes, Lott and Lück conjectured that for every element T of the integral group ring of a group G there exists $c > 0$ such that for sufficiently small x the proportion of the eigenvalues of T which are in the interval $[0, x]$ is at most x^c. Subsequently Lück proved a much weaker bound $1/|\log(x)|$ for a large class of groups G. I'll present a result which shows that the Lück's bound is not far away from optimal, in the sense that for every

$d > 1$ there exists a group ring element with the proportion of the eigenvalues in the interval $[0, x]$ at least $1/|\log(x)|^d$.

Holger Kammeyer: The Farrell–Jones conjecture for lattices in virtually connected Lie groups

We reported on recent progress on the Farrell-Jones conjecture in algebraic K- and L-theory which includes a proof for *cocompact* lattices in virtually connected Lie groups by Bartels–Farrell–Lück. We then gave a rough outline of how we exploit Weil's local rigidity theorem in joint work with Lück and Rüping to extend this result to *all* lattices in virtually connected Lie groups.

Cristina Pagliantini: Simplicial volume versus integral foliated simplicial volume

The *simplicial volume* is a homotopy invariant of compact manifolds introduced by Gromov. For a compact connected oriented manifold the simplicial volume is the ℓ^1-seminorm of the real fundamental class of the manifold itself. Even if the simplicial volume depends only on the homotopy type of a manifold, it is deeply related to the geometric structures that a manifold can carry.

Gromov conjectured that an aspherical oriented closed connected manifold with vanishing simplicial volume has zero Euler characteristic. Gromov himself suggested to use the *integral foliated simplicial volume* for which the corresponding statement is true. In a joint work with C. Löh we proved that the simplicial volume and the integral foliated simplicial volume are equal for hyperbolic 3-manifolds.

Daniel Kasprowski: On the K-theory of groups with finite decomposition complexity

Decomposition complexity is a property of metric spaces generalizing the concept of asymptotic dimension. It was first introduced by Guentner, Tessera and Yu. By a result of Ramras, Tessera and Yu the K-theoretic assembly map

$$H_n^G(\underline{E}G; \mathbb{K}_R) \to K_n(R[G])$$

is split injective for every group G with finite decomposition complexity that admits a compact model for BG (and therefore is torsion-free) and for every ring R. We give a generalization of this result, which in particular allows for groups with torsion and show that the above assembly map is split injective for every subgroup of a virtually connected Lie group, that admits a finite dimensional model for $\underline{E}G$.

Daniela Egas Santander: Combinatorial models of Moduli space of Riemann surfaces

We compare several combinatorial models of the Moduli space of two dimensional cobordisms and their compactifications. More precisely, we construct direct connections between the space of metric admissible fat graphs due to Godin, the chain complex of black and white graphs due to Costello, and the space of radial slit configurations due to Bödigheimer. In particular, these constructions show that the space of Sullivan diagrams, which is a quotient of the space of metric admissible fat graphs, is homotopy equivalent to Bödigheimer's Harmonic compactification of Moduli space. Furthermore, we construct a PROP structure on admissible fat graphs, which models the PROP of Moduli spaces of two dimensional cobordisms. We use the connections above to give black and white graphs a PROP structure with the same property.

Pedro Boavida: Spaces of smooth embeddings and the little discs operad

We describe the homotopy theoretical obstructions to lifting a smooth immersion into a smooth embedding in operadic terms and extend earlier work of Arone-Turchin and Dwyer-Hess. This is joint work with Michael Weiss.

Daniel Tubbenhauer: Categorification and topology

The Jones polynomial is a celebrated invariant of links with connections reaching from combinatorics over low-dimensional topology to mathematical physics.

Shortly after its discovery by Vaughan Jones in the mid 80ties a whole family of "Jones like polynomials" was discovered. This is the "Jones revolution": Before his discovery there was a lack of link polynomials and after him there where too many. A new question was to order them.

A representation theoretical "explanation" was found by Reshetikhin and Turaev around 1990. They constructed the "Jones like polynomials" as intertwiners of $\mathbf{U}_q(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$. The Jones polynomial is the special case $\mathfrak{g} = \mathfrak{sl}_2$. Hence the name $\mathfrak{sl}_2$-link polynomial.

The Jones polynomial also has a categorification called Khovanov homology, i.e. a chain complex of graded vector spaces whose graded Euler characteristic gives the $\mathfrak{sl}_2$-link polynomials.

It was introduced by Mikhail Khovanov around 1999. History repeats itself: Before Khovanov there was a lack of link homologies and after him there where too many. Namely, all the $\mathfrak{sl}_n$-link polynomials have categorifications called the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies.

In this talk I will sketch how one can obtain all of them using Khovanov-Lauda's categorification $\mathcal{U}(\mathfrak{sl}_m)$ of the quantum group $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ using "categorified" representation theory of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$.

As an outcome, in addition to the fact that this "explains" their appearance, we shortly sketch how rigidity of "categorified" representation theory of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ says that, morally, "Khovanov like homologies" are "unique" homologies theories one can associate to links.

Lukas Lewark: Khovanov-Rozansky homologies induce non-additive four-ball genus bounds (joint with A. Lobb)

The classical knot signature is a lower bound to the four-ball genus of a knot, i.e. the minimal genus of a surface in the four-ball bounding the knot. It is a concordance homomorphism, in particular additive with respect to the connected sum; so is the Rasmussen invariant, a four-ball genus bound coming from a perturbed version of Khovanov homology, the categorification of the Jones polynomial. We will see a large family of similar four-ball genus bounds that emerge from the Khovanov-Rozansky homologies (categorifications of Reshetikhin and Turaev's sl(n)-polynomials). However, not all of these are concordance homomorphisms. A possible application is to bound the four-ball genus of knots whose stable four-ball genus vanishes, such as amphichiral knots.

Markus Upmeier: Extremal Metrics on Transverse Symplectic Foliations (joint with M. Lejmi)

A transversely symplectic foliation of codimension $2q$ is the kernel of a closed 2-form ω with ω^q never zero and ω^{q+1} identically zero. If non-empty, the corresponding space of basic ω-compatible transverse almost complex structures $\mathcal{AC}(\omega)$ is contractible. Many of the familiar aspects of almost Kähler geometry, such as the transverse Kähler identities, continue to hold for $J \in \mathcal{AC}(\omega)$.

We show that a result of Fujiki [2], generalized by Donaldson [1] to the non-integrable case, and proven by He [3] for Sasakian manifolds, holds in greater generality: the action of the basic symplectomorphism group on the (infinite-dimensional) Kähler manifold $\mathcal{AC}(\omega)$ admits a moment map which is given by the Hermitian transverse scalar curvature.

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Nat Stapleton: A transchromatic proof of Strickland's theorem

Strickland showed that the Morava E-theory of the symmetric group (modulo a transfer ideal) is the ring of functions on the scheme that classifies subgroup schemes in the formal group associated to E. In joint work with Schlank we have given a generalization of this result to p-divisible groups as well as a new proof of Strickland's result. The main technical tool is a character map from E-theory to p-adic K-theory developed in joint work with Barthel.

Characteristic classes and families of Batalin-Vilkovisky field theories

Owen Gwilliam

Although the notion of a quantum field theory plays a central role in modern physics, I think it is safe to say that this notion has not yet found a mathematical formalization that captures the breadth of examples and the various flavors of reasoning that appear in physics. There are, in fact, many different mathematical objects inspired by quantum field theory that *a priori* look quite different. The focus of the talk is on an approach to quantum field theory that arose from the work of Batalin and Vilkovisky, which is homological in nature. As a quick gloss, one might say that the Batalin-Vilkovisky (BV) formalism is a version of deformation quantization for field theory (by contrast to the usual meaning of "deformation quantization," which is aimed at mechanics).

In particular, there are two aspects to the BV formalism: first, one constructs a classical BV theory, and second, one attempts to quantize, which is a deformation procedure. For a given classical BV theory T, there may not exist a BV quantization or there may exist many. It should come as no surprise that one can formulate a cochain complex (Def_T, ∂) in which

(1) the obstruction to the existence of a BV quantization is a cocycle ob, and
(2) if the obstruction is cohomologically trivial, $[ob] = 0$, then the set of allowable quantizations is given by trivializations def such that $\partial(def) = ob$.[1]

The goal of the talk is to sketch what Def_T is and describe how one can compute it. Before we can do that, however, we need to give a precise definition of a classical and quantum BV theory! Throughout the talk, we work with the mathematical machine developed by Kevin Costello in [2], who found a beautiful way to formalize mathematically the BV approach (while also adding a lot too!).

The culmination of the talk is a description of recent works of Kevin Costello [1], Si Li and Qin Li [4], and Ryan Grady and myself [3], in which the obstruction-deformation complexes of several theories are computed. These theories are all nonlinear σ-models, by which I will mean a theory whose fields consist of maps from one manifold Σ to another manifold X. For the σ-models in the works mentioned above, the obstructions to quantization are actually characteristic classes of Σ and X, just as asserted in the physics literature.

Theorem 1 (Li-Li)**.** *Let X be a complex manifold. There is a classical BV theory encoding the B-twisted topological σ-model of maps from Σ to X.*

(1) The obstruction to the existence of a BV quantization is given by $c_1(\Sigma) \otimes c_1(X)$. If either characteristic class vanishes, a quantization on Σ exists.
(2) If $c_1(X) = 0$, each choice of holomorphic volume form on X determines a unique quantization for every Riemann surface.

In short, the target X needs to be Calabi-Yau, just as one learns from physicists, and the choice of a trivialization of the canonical bundle on X fixes the quantization. (This amounts to a choice of string coupling constant.) See [6] for a discussion of this system.

Similarly, Costello showed that for the curved $\beta\gamma$ system on an elliptic curve, the obstruction depends only on the target. It is the first Pontryagin class $p_1(X)$ in $H^4(X, \mathbb{C})$. It must vanish for a quantization to exist, and each choice of trivializing 3-form leads to a distinct quantization. For comparison with the physics literature, see [7] or [5] for the theory constructed by Costello.

[1] To anticipate the more precise description given in the talk, let me add that Def_T describes deformations to first order in the deformation parameter, which is usually called $\hbar$. That is, Def_T will describe ways of quantizing T modulo $\hbar^2$.

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Flying rings and the Kashiwara–Vergne problem

Zsuzsanna Dancso

(joint work with Dror Bar-Natan)

This talk gave an outline of a topological context for the Kashiwara–Vergne problem in Lie theory [WKO2], and some hints to a topological proof [WKO3]. To do this we use a "machine" whose input is a topological structure (usually some space of knotted objects), and whose output is a set of equations in a graded space. In this abstract we first introduce the Kashiwara–Vergne problem, then describe the topological input to the aforementioned machine, followed by a general description of the machine itself. Finally, we'll explain why the machine outputs the Kashiwara–Vergne equations in this specific case, and how we can to use this to obtain a topological proof that solutions exist.

1. The Kashiwara–Vergne problem

The Kashiwara–Vergne (abbreviated KV from now on) problem, put forth in 1978 [KV], is a set of equations involving the Baker–Campbell–Hausdorff (BCH) series which has strong consequences in Lie theory and harmonic analysis. Solutions were first proven to exist in 2006 by Alekseev and Meinrenken [AM]. In 2008 Alekseev and Torossian [AT] re-phreased and proved the conjecture in a more algebraic setting and related it to Drinfel'd associators, and shortly thereafter Alekseev, Enriquez and Torossian [AET] provided a formula for solutions in terms of Drinfel'd associators. We are going to partially state the Alekseev–Torossian formulation here.

To do this we need to set up some notation. Recall that the BCH series is the infinite Lie series given by $\log(e^x e^y)$, where x and y are non-commuting variables. Let $\mathfrak{lie}_2$ denote the degree-completed free Lie algebra on two generators x and y, and let $\mathfrak{der}_2$ denote the set of derivations of $\mathfrak{lie}_2$ (that is, maps $D : \mathfrak{lie}_2 \to \mathfrak{lie}_2$ satisfying the Leibnitz rule). Let $\mathfrak{tder}_2$ stand for the set of "tangential derivations", i.e. derivations D with the additional property that $D(x) = [x, a_x]$ and $D(y) = [y, a_y]$ for some $a_x, a_y \in \mathfrak{lie}_2$. Finally, let $\mathrm{TAut}_2 := \exp(\mathfrak{tder}_2)$ denote the group of tangential automorphisms. The KV conjecture states that there exists $F \in \mathrm{TAut}_2$ for which the main KV equation

$$F(x+y) = \log(e^x e^y) \tag{1}$$

holds, as well as another equation which we omit here for brevity (but which admits a similar topological interpretation).

2. Topology

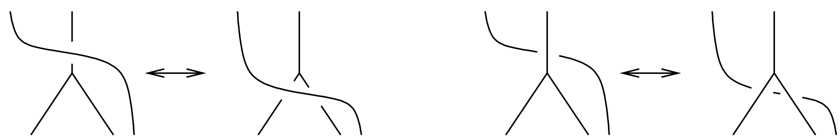
Recall that B_n, the ordinary braid group on n strands, can be thought of as the "group of crawling ants": the fundamental group of the configuration space of n distinct points in $\mathbb{R}^2$. However, the braid group also has a "Reidemeister presentation" in terms of generators $\{\sigma_i \mid i = 1, \ldots, n-1\}$, where σ_i represents a crossing between strands i and $i+1$, and modulo relations R3 and LS (Reidemeister 3 and Locality in Space).

We define the group of w-braids wB_n (here w stands for "welded" or "weakly virtual") analogously as the "group of flying rings" in $\mathbb{R}^3$; that is, the fundamental group of the configuration space of n disjoint geometric circles in $\mathbb{R}^3$, all of which are parallel to the xy-plane. Alternatively, wB_n has a Reidemeister presentation, where "virtual crossings" s_i are added as generators: the topological meaning of σ_i now is that ring number i switches places with ring $(i+1)$ *by flying through it*, while s_i represents the two rings switching places with no interaction, as shown below:

The set of relations is correspondingly larger, in particular, notice that $s_i^2 = 0$, and that the "Overcrossings Commute", or OC relation $\sigma_i^{-1} s_{i+1} \sigma_i = \sigma_{i+1} s_i \sigma_{i+1}^{-1}$ holds.

The topological structure required to gain insight into the KV equations is slightly richer than wB_n, called w-Tangled Foams, or *wTF*. Instead of a group, it is a finitely presented *planar algebra*: an algebraic structure whose elements are pictures and whose operations are given by planar connections of said pictures. In addition to crossings and virtual crossings, the generators include *singular vertices*: these can be thought of as a flying ring doubling itself to produce an inner and an outer ring, followed by the inner ring flying out and starting its own life. The relations include all of the wB_n relations as well as several additional ones, most importantly the Reidemeister 4 relation which describes the interaction of crossings and vertices:

R4 :



For a detailed definition of *wTF* see [WKO2], the space there is called *wTF*^{o}.

3. The Machine

The "machine" as described here can be applied to planar algebras or groups, and in fact makes sense in a much more general setting (described in [WKO2]). We start with a planar algebra (or group) $\mathcal{K}$ given by a finite presentation in terms of generators and relations. We allow formal $\mathbb{Q}$-linear combinations (only of elements with the same "skeleton"; this can be ignored at the present level of detail). The *augmentation ideal* $\mathcal{I}$ consists of elements of $\mathbb{Q}\mathcal{K}$ whose coefficients sum to zero. Powers of the augmentation ideal provide a decreasing filtration of $\mathbb{Q}\mathcal{K}$. Let $\mathcal{A} := \oplus_{n=0}^{\infty}\mathcal{I}^n/\mathcal{I}^{n+1}$ be the associated graded space, usually a space with a diagrammatic description: "chord diagrams" for classical braids, "arrow diagrams" in the case of *wTF*. An *expansion* (also called universal finite type invariant) is a map $Z : \mathbb{Q}\mathcal{K} \to \mathcal{A}$, with the non-degeneracy property that the associated graded map gr $Z : \mathcal{A} \to \mathcal{A}$ is the identity. Z is *homomorphic* if it is well-behaved with respect to all operations (e.g. planar algebra composition, as well as other operations defined on $\mathcal{K}$ but not mentioned here).

Does a (homomorphic) expansion exist? To find one, it is enough to determine the Z-images of the generators in $\mathcal{A}$. These values are subject to a number of equations arising from the relations of $\mathcal{K}$ (and homomorphicity), hence the problem of finding Z amounts to solving a system of equations in $\mathcal{A}$. Often this set of equations turns out to be interesting and hard, as in the case of *wTF* discussed below.

4. *wTF* and the KV Equations

When looking for a (homomorphic) expansion Z for *wTF*, the main difficulty lies in finding the Z-value of the vertex; let us denote it by V. One finds that V lies in a subalgebra of $\mathcal{A}$ which can be identified as the universal enveloping algebra of "$\mathfrak{tder}_2 \rtimes$(something easier)". Hence, finding V can largely be translated to a problem in $\mathfrak{tder}_2$, and the main equation induced by the $R4$ relation is the KV equation (1). This leads to the following theorem (rough formulation, see [WKO2] for details):

Theorem. [WKO2] *There is a bijection between homomorphic expansions for wTF and solutions of the KV problem.*

Finally, in order to use this insight to provide a topological proof of the KV conjecture, we need to construct such a homomorphic expansion by topological means. This is possible using the classical analogue of the space *wTF*: knotted trivalent graphs (or KTGs). It has long been known (works of Murakami, Ohtsuki, Cheptea, Le, Bar-Natan, and the speaker) that homomorphic expansions of KTGs are determined by Drinfel'd associators, and there is a map from KTGs to *wTF*, known as Satoh's "tubing map". This relationship can be exploited to construct a homomorphic expansion for *wTF* out of one for KTGs, which provides a formula for solutions of the KV equations in terms of associators, recovering that of [AET].

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Characteristic Classes in TMF with level structures

Gerd Laures

(joint work with Martin Olbermann)

Characteristic numbers play an important role in the determination of the structure of cobordism rings. For unoriented, oriented and *Spin* manifolds the cobordism rings were calculated in the 50s and 60s with the help of Stiefel-Whitney, $H\mathbb{Z}$- and KO-Pontryagin classes. However, it is known that for manifolds with lifts of the tangential structure to the 7-connective cover *String* of BO these numbers do not determine the bordism classes.

Locally at the prime 2, the Thom spectrum $MSpin$ splits into summands of connective covers of KO and an Eilenberg-MacLane part. A similar splitting is conjectured for $MString$ where KO is replaced by suitable versions of the spectrum TMF: the Witten orientation provides a surjection of the string bordism ring to the ring of topological modular forms and there is evidence that another summand of $MString$ is provided by the 16 connective cover of $TMF_0(3)$. In order to provide maps to this possible summand one has to study $TMF_0(3)$-characteristic classes for string manifolds.

There is a much easier theory $TMF_1(3)$ which is complex orientable. Its formal group is the completion of the universal elliptic curve with $\Gamma_1(3)$ structures. Its relation to $TMF_0(3)$ is analogous to the relation between complex and Real K-theory: a $\Gamma_1(3)$-structure is a choice of point of exact order 3 on an elliptic curve. A $\Gamma_0(3)$-structure is the choice of subgroup scheme of the form $\mathbb{Z}/3$ of the points of order 3. Given such a subgroup scheme there are exactly two choices of points of exact order 3 and they differ by a sign. Hence the corresponding cohomology theory $TMF_0(3)$ is the 'Real' version of the complex theory $TMF_1(3)$. It can be obtained by taking homotopy fixed points under the action which changes the sign of the 3 division point.

In [Lau] the $TMF_1(3)$ cohomology rings of $BSpin$ and $BString$ were computed. It was shown that the diagram

$$\begin{array}{ccc} TMF_1(3)^*BSpin & \xrightarrow{\lambda} & K^*_{Tate}BSpin \\ \downarrow & & \downarrow ch \\ H^*(BSpin, TMF_1(3)^*_{\mathbb{Q}}) & \longrightarrow & H^*(BSpin, K_{Tate}{}^*_{\mathbb{Q}}) \end{array}$$

is a pullback. The horizontal map is Miller's elliptic character which corresponds to the evaluation at the Tate curve on the moduli stack of elliptic curves. On coefficients this map is just the traditional q-expansion map for modular forms. The theory K_{Tate} is the power series ring $K[1/3]((q))$ of K-theories. The right vertical map is the Chern character and the left vertical map is the Dold character, that is, the map to rational cohomology induced by the exponential of the formal group law. The theorem determines the ring of $TMF_1(3)$-characteristic classes for spin bundles as follows. An element of $K^*_{Tate}BSpin$ is a K_{Tate}-characteristic class for spin bundles, that is, a formal series of virtual vector bundles which is naturally defined for spin bundles. If its Chern character is invariant under the appropriate Möbius transformations then it gives rise to a unique $TMF_1(3)$-characteristic class. This property allows the construction of many natural classes such as Pontryagin classes.

Theorem 1. *(1) There are unique classes $p_i \in TMF_1(3)^{4i}BSpin$ with the following property: the formal series $p(t) = 1 + p_1 t + p_2 t^2 \dots$ is given by*

$$\prod_{i=1}^{m}(1 + t\rho^*(x_i\overline{x}_i))$$

when restricted to the classifying space of each maximal torus of $Spin(2m)$. Here, ρ is the map to the maximal torus of $SO(2m)$ and the x_i (and $\overline{x}_i$) are the first $TMF_1(3)$-Chern classes of the canonical line bundles L_i (resp. $\overline{L}_i$) over the classifying spaces of the tori.

(2) The classes p_i freely generate the $TMF_1(3)$-cohomology of $BSpin$, that is,

$$TMF_1(3)^*BSpin \cong TMF_1(3)^*[\![p_1, p_2, \dots]\!].$$

(3) Let $\widehat{TMF}_1(3)$ be the $K(2)$-localization of $TMF_1(3)$ at the prime 2. Then there is an isomorphism of algebras

$$\widehat{TMF}_1(3)^*BString \cong \widehat{TMF}_1(3)^*[\![r, p_1, p_2, \dots]\!]$$

where $p_1, p_2, \dots$ are the Pontryagin classes coming from $BSpin$ and r restricts to a topological generator of degree 6 in the $K(2)$-cohomology of $K(\mathbb{Z}, 3)$.

It is useful to consider $TMF_1(3)$ as a Real theory in the sense of Atiyah, which means that there is a $\mathbb{Z}/2$-equivariant spectrum ("the Real theory") whose non-equivariant restriction ("the complex theory") is $TMF_1(3)$ and whose fixed point spectrum ("the Real theory") is $TMF_0(3)$. This allows a lift of the Pontryagin classes to $TMF_0(3)$ for spin bundles. For string bundles one has to provide a more geometric construction of the class r described above. Here, the theory of cubical structures on elliptic curves comes in which also played a role in the construction of the Witten orientation. It turns out that a convenient choice of a generator r is the defect class which compares the Witten orientation with the complex orientation.

Theorem 2. *(1) There are classes $\pi_i \in TMF_0(3)^{-32i}BSpin$ which lift the products $v_2^{6i}p_i$ for the $TMF_1(3)$ Pontryagin classes p_i. Moreover, we have*

$$TMF_0(3)^*BSpin \cong TMF_0(3)^*[\![\pi_1, \pi_2, \ldots]\!]$$

(2) For String bundles ξ over X there is a natural stable class

$$r(\xi) \in TMF_0(3)^0X$$

and an isomorphism

$$\widehat{TMF_0(3)}^*[\![r, \pi_1, \pi_2, \ldots]\!] \longrightarrow \widehat{TMF_0(3)}^*BString.$$

In terms of the Chern character of its elliptic character it is given by the formula

$$ch(\lambda(r(\xi))) = \prod_i \frac{\Phi(\tau, x_i - \omega)}{\Phi(\tau, -\omega)}$$

where the x_i are the formal Chern roots of $\xi \otimes \mathbb{C}$, $\omega = 2\pi i/3$ and Φ is the theta function

$$\begin{aligned} \Phi(\tau, x) &= (e^{x/2} - e^{-x/2}) \prod_{n=1}^{\infty} \frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2} \\ &= x \exp\Big(-\sum_{k=1}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau) x^{2k}\Big). \end{aligned}$$

As a consequence we give a proof of the stable version of a conjecture by Brylinski. For sufficiently large n divisible by 3, there is a group homomorphism

$$\varphi : (P_m)_{\Gamma(n)} \longrightarrow TMF(n)^*BString$$

from a stable group of positive energy representations V of the free loop group $LSpin$ to the TMF-cohomology with level n-structure. Here, the congruence group is larger than the one considered before since the character of the representation is only known to be invariant under the action of $\Gamma(n)$ by a theorem of Kac, Peterson and Wakimoto.

We will describe the map φ in terms of its elliptic character. Suppose P is a $String$-principal bundle over X. Let $\tilde{L}Spin$ be the universal central extension of $LSpin$. Then LX carries a $\tilde{L}Spin$-principal bundle $\tilde{L}P$ whose associated $LSpin$-bundle is LP. In particular, this holds for the universal $String$-bundle $EString$ over $BString$. The elliptic character of φ gives the bundle

$$\lambda\varphi(V) = (\tilde{L}ESpin \times_{\tilde{L}Spin} V)_{|BString}$$

(when V is suitably normalized with a character of the rotation circle). In this formula, the right hand side is considered as a formal power series of virtual bundles by decomposing the bundle as a representation of the circle group which reparameterizes the loops. The evaluation of this class on the fundamental class of a string manifold is the formal index of the Dirac operator on LM with coefficient in the bundle associated to the representation V.

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Finite-sheeted coverings and undecidability

Martin R. Bridson

(joint work with Henry Wilton)

In the middle of the twentieth century, most of the basic decision problems for finitely presented groups were proved to be undecidable and attention shifted towards more refined questions concerning the existence of algorithms within specific classes of groups, and to connections with geometry and topology. However, certain basic decision problems about general finitely presented groups were not covered by the techniques developed at that time and did not succumb to the geometric techniques developed in the 1990s. The most obvious of these is settled by the following theorem from [3], which is at the heart of the project reported on here.

Theorem 1. *There is no algorithm that can determine if a finitely presented group has a non-trivial finite quotient.*

This talk focusses mainly on applications of this theorem, emphasising how different refinements (of a geometric and topological nature) are needed for each. The first refinement shows that the existence of finite-index subgroups remains unsolvable among the fundamental groups of compact, non-positively curved cube complexes, even in dimension 2. This should be contrasted with Agol's Theorem [1] (explained by Wise at this meeting), from which it follows that if the fundamental group of the complex is hyperbolic, then it is residually finite.

Theorem 2. *There is no algorithm that can determine if a compact square complex of non-positive curvature has a non-trivial, connected, finite-sheeted covering.*

The technical meaning of this theorem is that there is a recursive sequence of finite squared 2-complexes K_n such that the set of n where there is a proper subgroup of finite index $H_n < \pi_1 K_n$ is recursively enumerable but not recursive.

To pass from Theorem 1 to Theorem 2 we use an idea that arises in the Kan-Thurston construction. Roughly speaking, the idea is to replace the discs in a cell complex by spaces which are "trivial" in a sense that one wants to control but are complicated enough to allow one to stay in a desirable category of objects. In the Kan-Thurston construction, one replaces discs by more complicated acyclic spaces; in our setting, we replace 2-cells in presentation 2-complexes by other non-positively curved spaces with fundamental groups that have no finite quotients; the desirable category of objects that we strive to stay in is that of compact non-positively curved squared complexes.

Theorem 2 is more striking than Theorem 1 because, whereas one is used to the idea that arbitrary finite group presentations encode unlimited pathology, one expects much more controlled behaviour in the presence of non-positive curvature.

1. Permutoids and rigid pseudogroups. Across many contexts in mathematics one encounters extension problems of the following sort: given a set S of partially-defined automorphisms of an object X, one seeks an object $Y \supset X$ and a set of automorphisms $\tilde{S}$ of Y such that each $s \in S$ has an extension $\tilde{s} \in \tilde{S}$. In the category of finite sets, this problem is trivial because any partial permutation of a set can be extended to a permutation of that set. But if one requires extensions to respect (partially defined) compositions in S, such existence problems become more subtle. In 2004 Peter Cameron conjectured that there does not exist an algorithm that can solve the following extension problem.

Problem 1. Given *partial permutations* $p_1, ..., p_m$ *of a finite set* X *(that is, bijections between subsets of* X*) such that*

(1) $p_1 = \mathrm{id}_X$*, and*
(2) for all i, j *with* $\mathrm{dom}(p_i) \cap \mathrm{ran}(P_j) \neq \emptyset$*, there is at most one* k *such that* p_k *extends* $p_i \cdot p_j$

decide *whether or not there exists a finite set* Y *containing* X*, and permutations* f_i *of* Y *extending* p_i *for* $i = 1, ..., m$*, such that if* p_k *extends* $p_i \cdot p_j$ *then* $f_i \circ f_j = f_k$*.*

In [5] we prove that this problem is indeed algorithmically unsolvable by considering the partial permutations defined by left-multiplication on balls of finite radius in Cayley graphs; we use the Cayley graphs of the groups constructed in the proof of Theorem 2. The existence of a uniform solution to the word problem in this class of groups plays a crucial role, as does the formalism of *permutoids* that we develop. A similar result is proved for rigid pseudogroups.

2. Profinite Isomorphism. The profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ; it is denoted $\widehat{\Gamma}$. The natural map $\Gamma \to \widehat{\Gamma}$ is injective if and only if Γ is residually finite. A *Grothendieck pair* is a monomorphism $u : P \hookrightarrow \Gamma$ of finitely presented, residually finite groups such that $\hat{u} : \widehat{P} \to \widehat{\Gamma}$ is an isomorphism but P is not isomorphic to Γ. In 1970 Grothendieck asked if such pairs exist; in 2004 Bridson and Grunewald [2] proved that they do, raising the problem: *is there an algorithm that, given a monomorphism of finitely presented, residually finite groups* $u : P \hookrightarrow \Gamma$*, can determine whether or not* $\hat{u}$ *is an isomorphism?* In [4] we prove that no such algorithm exists.

Theorem 3. *There are recursive sequences of finite presentations for residually finite groups* $P_n = \langle A_n \mid R_n \rangle$ *and* $\Gamma_n = \langle B_n \mid S_n \rangle$ *together with explicit monomorphisms* $u_n : P_n \hookrightarrow \Gamma_n$*, such that:*

(1) $\widehat{P}_n \cong \widehat{\Gamma}_n$ *if and only if the induced map of profinite completions* $\hat{u}_n$ *is an isomorphism;*
(2) $\hat{u}_n$ *is an isomorphism if and only if* $\hat{u}_n$ *is surjective; and*
(3) *the set* $\{n \in \mathbb{N} \mid \widehat{P}_n \not\cong \widehat{\Gamma}_n\}$ *is recursively enumerable but not recursive.*

The groups Γ_n that we construct are of the form $H_n \times H_n$ where H_n is a residually finite hyperbolic group with a 2-dimensional classifying space. This construction is based on the original template for Grothendieck pairs from [2], but this requires as input super-perfect groups whose classifying spaces have finite 3-skeletons. To obtain such input, we need the following extension of Theorem 1.

Theorem 4. *There is a recursive sequence of finite combinatorial CW-complexes* K_n *so that*

(1) *each* K_n *is aspherical;*
(2) $H_1(K_n, \mathbb{Z}) \cong H_2(K_n, \mathbb{Z}) \cong 0$ *for all* $n \in \mathbb{N}$*; and*
(3) *the set* $n \in \mathbb{N} \mid \widehat{\pi_1 K_n} \not\cong 1\}$ *is not recursive.*

3. Homology Spheres. A further application of Theorem 4 is the following.

Theorem 5. *If* $d \geq 5$*, then there is no algorithm that will determine, given a closed homology* d*-sphere* Σ*, whether or not* Σ *has a finite-sheeted covering.*

The precise meaning of this theorem is that there is a recursive sequence (Σ_n) of finite simplicial complexes, the geometric realisation of each being homeomorphic to a closed orientable d-manifold with the integral homology of a sphere, such that the set of n such that Σ_n has no finite-sheeted cover is not recursive.

Kervaire [6] proved that for $d \geq 5$, a finitely presented group G is the fundamental group of a closed homology d-sphere if $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$. One can use this to deduce Theorem 5 from Theorem 4, arranging that $\pi_1 \Sigma_n = \pi_1 K_n$.

4. Hyperbolic groups. Most decision problems for hyperbolic groups can be solved efficiently. Even the isomorphism problem is solvable in this class of groups. But it is unknown whether or not there exists a non-trivial hyperbolic group with no finite quotients. If there were such a group, then a variation on our arguments would prove that there is no algorithm that can determine whether or not a given hyperbolic group has a non-trivial finite quotient. Without knowing this, we can already prove, for example, that there is no algorithm that given a hyperbolic group can determine whether or not it has a finite-dimensional linear representation with infinite image. Also, in contrast to Agol's virtual fibering theorem, we prove:

Theorem 6. *There does not exist an algorithm that, given a finite presentation of a hyperbolic group* Γ*, can determine whether or not* Γ *has a subgroup of finite index* $\Gamma_0 < \Gamma$ *with* $\dim H_1(\Gamma_0, \mathbb{Q}) \geq 1$.

5. On the proof of Theorem 1. The (non)existence of a finite quotient for a finitely presented group can be expressed as a sentence in the first order theory of finite groups. For example, in all finite groups, $\forall a, b, c, d :$

$$(bab^{-1} \neq a^2) \vee (cbc^{-1} \neq b^2) \vee (dcd^{-1} \neq c^2) \vee (ada^{-1} \neq d^2) \vee (a = b = c = d = 1)$$

because $\langle a, b, c, d \mid bab^{-1}a^{-2}, cbc^{-1}b^{-2}, dcd^{-1}c^{-2}, ada^{-1}d^{-2} \rangle$ has no non-trivial finite quotients.

The initial seed of undecidability for Theorem 1 comes from work of Solbodskoi [7], which provides a finitely presented group G such that there is no algorithm that, given a word w in the generators of G can determine whether or not $w = 1$ in $\hat{G}$. The meat of the proof is topological in nature, in the spirit of John Stallings's work on the topology of graphs (as developed by Scott, Wise and others), wherein the profinite topology of a group is studied via the geometry and topology of finite covers.

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Rational Homotopy Theory via Quantum Field Theory

Chris Schommer-Pries

(joint work with Nathaniel Stapleton)

Cohomology theories such as real cohomology, K-theory, and cobordism theories have the distinct advantage of a geometric description. They are built out of geometric cochains such as differential forms, vector bundles, or cobordism classes of manifolds. This significantly aids our ability to compute with these theories while also allowing methods from algebraic topology to be used to solve geometric problems.

Chromatic homotopy theory organizes cohomology theories according to their height, which is a measure of the complexity of the theory. Real cohomology and K-theory are at heights 0 and 1, respectively. The theory of *topological modular forms TMF* introduced by Hopkins and Miller is of height 2, while there are numerous theories, such as Morava E_n-theory and $K(n)$-theory, which exist for arbitrary heights n.

In contrast to real cohomology and K-theory, there are no known geometric descriptions of these latter theories. In fact, aside from bordism theories (which are manifestly geometric), to our knowledge the only known geometric construction of a cohomology theory of complexity greater than K-theory is via the Baas-Dundas-Richter-Rognes theory of '2-vector bundles' [1, 2]; it produces $K(ku)$, the algebraic K-theory of topological K-theory, a theory of telescopic complexity two.

Nevertheless, several years ago the enticing idea was put forward that quantum field theories could provide some of the best candidates for geometric cochains for higher height cohomology theories. This idea was pioneered by Graeme Segal [9] who proposed to use 2-dimensional conformal field theories to give geometric cocycles for elliptic cohomology.

This idea has been further developed in the work of Stolz-Teichner [11, 12], and has been quite successful in low heights. Namely they have shown that supersymmetric Euclidean field theories of dimensions $1|1$ and $0|1$ correspond, respectively, to K-theory and periodic de Rham cohomology; see [5] for the latter case.

Quantum field theories of different dimensions can be related by the process of *dimensional reduction*. For example a 1-dimensional topological field theory gives rise to a 0-dimensional theory by precomposing with the "cross with a circle" map between bordism categories. In the presence of geometric structures and supersymmetry the situation is more complicated, nevertheless it has been shown that dimensional reduction does give a quantum field theoretic interpretation of the (Bismut) Chern character map [4].

$$bCh : K^*(X) \to HP^*_{S^1}(LX)^\wedge_{(u^{-1})}$$

This map goes from the K-theory of a manifold X to the negatively completed periodic S^1-equivariant cohomology of the free loop space LX [3].

Higher analogs of the Chern character have also been an important tool for studying cohomology theories of higher height [6, 10]. This theory provides a generalized character map that approximates high height cohomology theories by a form of rational cohomology. The form of rational cohomology has coefficients that are a ring extension of the rationalization of the coefficients of the high height cohomology theory. These rings are often algebras over the p-adic rationals.

Many features of these character maps are reminiscent of the dimensional reduction maps between field theories. For example the target of such character maps is (a form of) the p-adic rational cohomology of a p-adic analog of the iterated free loop space. Periodic de Rham cohomology cannot be a suitable target for the higher height character maps that take place at a prime p. This is essentially because there is no (interesting) map from the real numbers $\mathbb{R}$ to the p-adic rationals $\mathbb{Q}_p$. For example the p-adic Chern character may be obtained as the completion of the ordinary Chern character, but only once it is factored through periodic *rational* cohomology.

In this talk, based on [7], we introduce a new notion of space which mixes ideas from supermanifolds, schemes from algberaic geometry, and simplicial sets. We call this notion *superalgebraic Cartesian sets*. This theory of spaces is flexible and is defined, functorially, over any base ring R. Moreover it retains enough of the structure to allow us to define and compute the collection of $0|1$-dimensional supersymmetric topological field theories over a simplicial set X. Following [8] we introduce twisted field theories. We show that when R is a $\mathbb{Q}$-algebra (for example $R = \mathbb{Q}_p$), then concordance classes of topological field theories, twisted by a natural choice of *degree n twist*, are in bijection with $H^n(X;R)$. Moreover we provide a family of more exotic twists which allow one to recover not just the cohomology of X, but the entire rational homotopy type.

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Homotopy colimits and left fibrations

Ieke Moerdijk

(joint work with Gijs Heuts)

For a small category $\mathbf{A}$, I consider the category $\mathbf{sSets}^{\mathbf{A}}$ of diagrams of simplicial sets ('spaces') parametrized by $\mathbf{A}$. The usual homotopy colimit functor construction can be consider as a functor

$$h_! : \mathbf{sSets}^{\mathbf{A}} \longrightarrow \mathbf{sSets}/N\mathbf{A},$$

where $N\mathbf{A}$ is the nerve of $\mathbf{A}$. It is well known that this functor gives an equivalence of homotopy categories when $\mathbf{A}$ is a group (viewed as a category with one object). I will show that $h_!$ *always* gives an equivalence of homotopy categories, in the following precise way: one equips $\mathbf{sSets}^{\mathbf{A}}$ with the projective model structure and

sSets/N**A** with the covariant model structure. The fibrant objects $X \to N\mathbf{A}$ in the latter model category are characterized by the lifting property

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & N\mathbf{A}, \end{array}$$

for all $n > 0$ and $0 \leq k < n$. This makes the functor $h_!$ into (the left part of) a Quillen equivalence. The proof makes use of another left Quillen equivalence $r_! : \mathbf{sSets}/N\mathbf{A} \to \mathbf{sSets}^{\mathbf{A}}$ in the opposite direction and is completely elementary. The talk is based on joint work with Gijs Heuts [HM] and simplifies the treatment in Lurie's Higher Topos Theory.

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Low-dimensional algorithmic topology

Saul Schleimer

(joint work with Ben Burton, Marc Lackenby)

Our goal in this note is to show that many three-manifold recognition problems lie in the complexity class **NP**.

1. Decision problems. Recall that a *decision problem* Q is a collection of instances $\{\omega\}$ together with a uniform question. The answer $Q(\omega)$ is required to be YES or NO. Some care is needed to give a uniform *encoding* of ω, in binary. We define $|\omega|_2$ to be the length of this binary encoding.

As a simple example: given a two-dimensional triangulation $\mathcal{T}$ the problem CONNECTED SURFACE asks if the underlying topological space $S = |\mathcal{T}|$ is a connected surface.

A decision problem Q lies in **P**, that is, is solvable in polynomial time, if there is a Turing machine A and a polynomial q so that $A(\omega)$ computes $Q(\omega)$ and halts, all in time at most $q(|\omega|_2)$. The above example, CONNECTED SURFACE, lies in **P**.

A decision problem Q lies in **NP** if there is an algorithm A and a polynomial q with the following property. For every instance ω we have $Q(\omega) =$ YES if and only if there is a binary string α so that $A(\omega, \alpha)$ prints YES and halts in time at most $q(|\omega|_2)$.

In what follows we concentrate on decision problems in three-dimensional topology, stated in terms of triangulated manifolds. There is a similarly rich, and older, family of problems concerning knot diagrams. However, in the interest of space we will content ourselves with a single reference [6].

2. Results past. We begin with a previous result from [10]. See also [5]. Given a three-dimensional triangulation $\mathcal{T}$, the problem THREE-SPHERE RECOGNITION asks if the underlying topological space $M = |\mathcal{T}|$ is homeomorphic to S^3.

Theorem 1. [Schleimer, Ivanov] THREE-SPHERE RECOGNITION *lies in* **NP**.

Here is a sketch of the proof. We begin with a bit of useful preprocessing; in polynomial time we can decide if $M = |\mathcal{T}|$ is a three-manifold and if M has the same homology as the three-sphere. This done, the instance $\omega = \mathcal{T}$ has, as its certificate, a sequence $\alpha = \{(\mathcal{T}_i, S_i)\}_{i=0}^N$ of triangulations and (almost) normal spheres as follows:

- $\mathcal{T}_0 = \mathcal{T}$,
- $\mathcal{T}_{i+1}$ is obtained by *crushing* $\mathcal{T}_i$ along S_i, and
- $\mathcal{T}_N = \emptyset$.

That the certificate α exists, and that it can be verified in polynomial time, relies crucially on the ideas of [8, 12, 2].

Theorem 1 is foundational; most three-manifold recognition problem (in **NP**) require it as a subroutine. Here are several applications.

Corollary 2. *The recognition problems for the following manifolds lie in* **NP**: B^3, P^3, $S^2 \times S^1$, $D^2 \times S^1$, $P^3 \# P^3$, $T^2 \times I$, *and* $K^2 \tilde{\times} I$.

We provide very brief sketches of the proofs to give a sense of the necessary techniques. Let $M = |\mathcal{T}|$ be the underlying space. In all cases we begin with the usual preprocessing of $\mathcal{T}$. We then check that ∂M is the correct surface.

To recognize B^3, we apply Alexander's theorem [1]; thus M is homeomorphic to B^3 if and only if its double $D(M)$ is homeomorphic to the three-sphere. To recognize P^3 we apply Livesay's theorem [7]; thus M is homeomorphic to P^3 if and only if it has a double cover M' which is homeomorphic to the three-sphere.

The certificate for $S^2 \times S^1$ begins with a non-separating normal two-sphere $S \subset M$.

After crushing $\mathcal{T}$ along S we certify that the resulting manifolds are three-spheres. To recognize $D^2 \times S^1$ it suffices to double M and recognize that $D(M) \cong S^2 \times S^1$. To recognize $P^3 \# P^3$, the certificate provides a normal separating sphere which we crush. The resulting manifolds are all copies of S^3 and P^3, which are certified as discussed above.

To recognize $T^2 \times I$ we apply the cyclic surgery theorem. Suppse that T is a torus. Four slopes $\{\alpha_i\}_{i=0}^3$ in T form a *Farey square* if

$$\Delta(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = 0 \text{ and } j = 2 \\ 1 & \text{otherwise.} \end{cases}$$

We check that M has $\partial M = S \sqcup T$, both tori. If $M(\alpha_i)$ is a solid torus for all four slopes in a Farey square then $M \cong T^2 \times I$, essentially by [3, Theorem 2.4.4]

To recognize $K^2 \tilde{\times} I$ we first check that M has ∂M homeomorphic to a single torus. The certificate for M provides a double cover $M' \to M$ together with a certificate that $M' \cong T^2 \times I$. An elementary version of topological rigidity (using vertical annuli) now implies M is homeomorphic to $K^2 \tilde{\times} I$.

3. Results present. We may now discuss our current work. A three-manifold M is called *elliptic* if M is a quotient of S^3 by a freely acting subgroup $\Gamma < \mathrm{SO}(4)$.

Theorem 3. [Lackenby-Schleimer] *The problem of recognizing elliptic manifolds lies in* **NP***.*

In fact more is true – the name of the elliptic manifold can also be certified. The proof here uses standard notions from the theory of Seifert fibered spaces, the covering space ideas mentioned above, and our recognition theorem for lens spaces [11].

Suppose that $K \subset S^3$ is a knot and $n(K)$ is a small regular neighborhood. Define $X_K = S^3 - n(K)$ to be the associated knot complement.

Theorem 4. [Burton-Schleimer] *The problem of recognizing knot complements lies in* **NP***.*

This is proven by extending the crushing operation from spheres to planar surfaces; the result of crushing a planar surface is that of first filling the boundary slope with a layered solid torus and then crushing along the resulting normal sphere.

Theorem 5. *The problem of recognizing torus knot complements lies in* **NP***.*

In fact more is true – the name of the torus knot can also be certified. This is proved by crushing the essential annulus and obtaining a pair of lens spaces. That this is a certificate relies on a partial solution to the cabling conjecture due to Greene [4].

Given a triangulated three-manifold $(M, \mathcal{T})$ and a one-cocycle $\alpha \in Z^1(M)$, the problem FIBERED CLASS asks if there is a surface bundle structure $F \to M \to S^1$ so that $[F] \in H_2(M, \partial M)$ is Poincaré dual to $[\alpha] \in H^1(M)$.

Theorem 6. [Burton-Schleimer] *The problem of recognizing fibered classes lies in* **NP***.*

The heart of the proof relies is a polynomial bound on the bit-complexity of normal representatives of the fibers and on the technique of *dicing* surface bundles discussed in [9].

Theorem 7. [Burton-Schleimer] *The problem of recognizing* E^3, Nil*, or* Sol *manifolds lies in* **NP***.*

This is essentially a special case of Theorem 6; detecting the Thurston geometries of a torus bundle relies on the fact that homology of cyclic covers can be computed in polynomial time. In similar fashion we may deduce the following.

Theorem 8. *The problem of recognizing X_K, for K the figure-eight knot, lies in* **NP***.*

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The Segal conjecture, uncompleted

Jesper Grodal

In my talk I identified the Grothendieck group of maps between p–completed classifying spaces $Gr([BG\hat{_p}, \coprod_n B\Sigma_n\hat{_p}])$, for G an arbitrary finite group, as the Burnside ring $A(\mathcal{F}_p(G))$ of the p–fusion system $\mathcal{F}_p(G)$ of G. In fact I gave a calculation of the whole mapping space $\Omega B\,\mathrm{map}(BG, \coprod_n (B\Sigma_n)\hat{_p})$ as described below.

Let S be a Sylow p–subgroup of G. A finite S–set X is said to be G–stable if for all subgroups $Q \leq S$ and all $g \in G$ such that $gQg^{-1} \leq S$, the Q–set obtained by restricting the S–action on X to Q is isomorphic, as a Q–set, to the Q–set obtained by restricting the S-action to gQg^{-1}, and then viewing X as a Q–set via the conjugation map c_g. The Burnside ring $A(\mathcal{F}_p(G))$ is defined as the Grothendieck group of the monoid of G–stable finite S–sets, under disjoint union — this is easily seen only to depend on the p–fusion system $\mathcal{F}_p(G)$ of G, explaining the notation. Our first main theorem can now be stated as follows:

Theorem 1. *Let G be a finite group with Sylow p-subgroup S, and let $A(\mathcal{F}_p(G))$ be the Burnside ring of G–stable finite S–sets, as defined above. Then*

$$Gr([BG\hat{_p}, \coprod_n (B\Sigma_n)\hat{_p}]) \xrightarrow{\cong} A(\mathcal{F}_p(G))$$

It is easy to see that $A(\mathcal{F}_p(G))$ is a free abelian group of rank the number of conjugacy classes of p-subgroups in G, and it is naturally as a subring of the ordinary Burnside ring $A(S)$ of the Sylow p–subgroup S. The algebraic properties of $A(\mathcal{F}_p(G))$ has been studied by several authors, see e.g., [3, 8]. It was proven in [2] that the map in Theorem 1 had finite kernel and cokernel, by observing that the relevant obstructions took values in finite groups; we here show that the obstruction groups in fact vanish.

Applying Theorem 1 for the different primes p dividing the order of G, one obtains an 'integral' version, where the p–completion is replaced by Bousfield–Kan $\mathbb{Z}$–completion $(-)\hat{_\mathbb{Z}}$. (By general localization theoretic facts, $\mathbb{Z}$–completion can here also be replaced by Quillen's plus construction or $\mathbb{Z}$–homology localization.)

Corollary 1. *The following diagram is a pull-back of rings*

$$\begin{array}{ccc} Gr([BG, \coprod_n (B\Sigma_n)\hat{_\mathbb{Z}}]) & \longrightarrow & \prod_{p||G|} A(\mathcal{F}_p(G)) \\ \downarrow{\scriptstyle aug} & & \downarrow{\scriptstyle aug} \\ \mathbb{Z} & \xrightarrow{diag} & \prod_{p||G|} \mathbb{Z} \end{array}$$

Said differently, $Gr([BG, \coprod_n (B\Sigma_n)\hat{_\mathbb{Z}}])$ is the product of the $A(\mathcal{F}_p(G))$, for $p \mid |G|$, in the category of $\mathbb{Z}$–augmented rings.

Theorem 1 can be viewed as an 'uncompleted' version of the Segal conjecture as shown by the following commutative diagram

$$\begin{array}{ccc} Gr([BG\hat{_p}, \coprod_n (B\Sigma_n)\hat{_p}]) & \xrightarrow{\cong} & A(\mathcal{F}_p(G)) \\ \downarrow & & \downarrow{\scriptstyle (-)\hat{_I}} \\ [BG\hat{_p}, \Omega^\infty S^\infty\hat{_p}] & \xrightarrow{\cong} & A(\mathcal{F}_p(G))\hat{_I} \end{array} \tag{1}$$

where I is the augmentation ideal. Here the top isomorphism is Theorem 1 and bottom isomorphism is the Segal conjecture, in Ragnarsson's version [6, 3].

We now give our result for the whole mapping space.

Theorem 2 (The Burnside space of $BG\hat{_p}$)**.** *Let p be a prime and G a finite group. Then*

$$\Omega B \coprod_n \mathrm{map}(BG, (B\Sigma_n)\hat{_p}) \simeq \prod_{[Q]} (\Omega^\infty \Sigma^\infty (BW_G(Q)_+))\hat{_p}$$

where the product runs over G–conjugacy classes $[Q]$ of p–subgroups Q in G, and the plus denotes disjoint basepoint.

In particular

$$\pi_i(\Omega B \coprod_n \mathrm{map}(BG, (B\Sigma_n)\hat{_p})) \cong \begin{cases} A(\mathcal{F}_p(G)) & \text{for } i = 0 \\ \lim_{G/Q \in \mathbf{O}_p(G)} \pi_i^Q (\Omega^\infty S^\infty)_{(p)} & \text{for } i > 0 \end{cases}$$

where $\mathbf{O}_p(G)$ is the p-orbit category of G and π^Q denotes Q-equivariant homotopy groups.

Assembling the information at the different primes, we also get an integral version of Theorem 2.

Corollary 2 (The Burnside space of $BG\hat{_\mathbb{Z}}$)**.** *Let G be a finite group then*

$$\Omega B(\coprod_n \mathrm{map}(BG, (B\Sigma_n)\hat{_{\mathbb{Z}}})) \xrightarrow{\sim} \Omega^\infty\Sigma^\infty(BG_+) \times \prod_{p||G|,1<[Q]} (\Omega^\infty\Sigma^\infty(BW_G(Q)_+))\hat{_p}$$

where the product is taken over G-conjugacy classes $[Q]$ of non-trivial p-subgroups Q of G for all primes p dividing the order of G.

This result fits as the middle line in the following commutative diagram:

(2)

$$\begin{array}{ccc}
\Omega B(\coprod_n \mathrm{map}(BG, B\Sigma_n)) & \xrightarrow{\sim} & \Omega^\infty\Sigma^\infty(BG_+) \times \prod_{1<[H]} \Omega^\infty\Sigma^\infty(BW_G(H)_+) \\
\downarrow & & \downarrow \\
\Omega B(\coprod_n \mathrm{map}(BG, (B\Sigma_n)\hat{_{\mathbb{Z}}})) & \xrightarrow{\sim} & \Omega^\infty\Sigma^\infty(BG_+) \times \prod_{p||G|,1<[Q]} (\Omega^\infty\Sigma^\infty(BW_G(Q)_+))\hat{_p} \\
\downarrow & & \downarrow \\
\mathrm{map}(BG, \mathbb{Z} \times (B\Sigma_\infty)\hat{_{\mathbb{Z}}}) & \xrightarrow{\sim} & \Omega^\infty\Sigma^\infty(BG_+) \times \prod_{p||G|,1<[Q]} \Omega^\infty((\Sigma^\infty(BW_G(Q)_+))\hat{_p})
\end{array}$$

The top product is taken over G–conjugacy classes $[H]$ of non-trivial subgroups H of G, whereas the two latter sums are taken over G–conjugacy classes $[Q]$ of non-trivial p-subgroups Q of G for all primes p dividing $|G|$. The top horizontal homotopy equivalences is the equivariant Barrett–Priddy–Quillen theorem and the Segal–tom Dieck splitting; this essentially goes back to Segal's 1970 ICM address [9, Prop. 7]. The middle homotopy equivalence is Theorem 2. The bottom homotopy equivalences is Carlsson's strong form of the Segal conjecture [1], in the p–primary version obtained by Ragnarsson [7, Thm. D].

Notice that on π_0 Diagram 2 from top to bottom gives the factorization

$$A(G) \longrightarrow \prod_p^{aug} A(\mathcal{F}_p(G)) \longrightarrow \prod_p^{aug} A(\mathcal{F}_p(G))\hat{_I} \cong A(G)\hat{_I}$$

from Diagram 1, where superscript *aug* means that the product is taken in the category of $\mathbb{Z}$–augmented rings. It is the placement of the p–completion in the right-hand column of Diagram 2 which accounts for the p–completion on components or lack thereof.

The proofs of the above results evolve around showing that certain obstruction groups vanish. Our approach mirrors that used in an celebrated result of Jackowski–Oliver on vector bundles over classifying spaces [5]. To make this approach work we have to replace their use of equivariant K–theory by a link to the equivariant sphere spectrum. This in turn requires us to revisit various constructions in equivariant stable homotopy theory, such as group completions and homological stability, and modify and assemble them for our purpose. The results described here will appear in [4].

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