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Arbeitsgemeinschaft: Totally Disconnected Groups

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Abstract. Locally compact groups are ubiquitous in the study of many continuous or discrete structures across geometry, analysis and algebra. Every locally compact group is an extension of a connected group by a totally disconnected group. The connected case has been studied in depth, notably using Lie theory, a culminating point being reached in the 1950s with the solution to Hilbert's 5th problem. The totally disconnected case, by contrast, remains full of challenging questions. A series of new results has been obtained in the last twenty years, and today the activity in this area is witnessing a sharp increase. These texts report on the recent Arbeitsgemeinschaft on this topic.

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Introduction by the Organisers

Locally compact groups arise as the symmetry groups of all sorts of structures across many areas of mathematics. This includes Lie groups, p-adic and adélic groups, isometry groups of general proper metric spaces. Even discrete structures such as locally finite graphs give rise to very interesting locally compact automorphism groups. Besides the groups themselves, one of the most important motivations to study locally compact groups is that they frequently appear as the "envelope" in which abstract groups of interest appear as lattices. This is notably the case for arithmetic groups and Kac–Moody groups. It has often happened that the most interesting theorems about those abstract groups are proved by transfering the problem to the ambient locally compact group and solving it there.

In the study of locally compact groups, it is usually understood that the focus is on *non-discrete* groups since otherwise it remains within "abstract" group theory. The case of Lie groups has been extensively studied for well over a century and largely classified in the early twentieth century. The next significant period of research culminated in the 1950s with the solution to Hilbert's Fifth Problem, giving a satisfactory picture of the conected case.

Therefore, the main locus of modern research on locally compact groups is the study of non-discrete *totally disconnected* locally compact groups, since a general locally compact group decomposes as an extension of a connected group by a totally disconnected group.

The revival of this topic can arguably be dated to the work of G. Willis starting two decades ago. This gave a new impetus to the study of the *local structure* of totally disconnected groups. More recently, there has been progress both on the global and local structure. In addition, the compact case (i.e. profinite groups) has also witnessed important recent progress on the algebraic side.

The goals of the Arbeitsgemeinschaft are: to learn the necessary prerequisites, to study substantial parts of the recent developments and to reach the point where open problems can be discussed.

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Arbeitsgemeinschaft: Totally Disconnected Groups

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Abstracts

Locally compact groups: around Van Dantzig's theorem

Marc Burger

Given a locally compact group G and its connected component G^0 of e, we obtain an exact sequence

$$(e) \to G^0 \to G \to G/G^0 \to (e),$$

where G/G^0 is a totally disconnected locally compact group; recall that a topological space is said totally disconnected if the connected subsets are reduced to points.

In the first part of the lecture we establish a fundamental theorem (Van Dantzig) which says that if G is locally compact and totally disconnected, every neighborhood of $e \in G$ contains a compact open subgroup. Since such groups are profinite, it implies that the multiplication in a neighborhood of e can be approximated with arbitrary accuracy by the multiplication in finite groups.

In the second part of the lecture we address the problem to which extent a general locally compact group is a product of a Lie group and a totally disconnected group, using the solution to Hilbert's 5th problem.

We first show the existence of the amenable radical $A(G)$ of a locally compact group G: it is the unique largest closed amenable normal subgroup of G; furthermore it is a radical in that $A\big(G/A(G)\big) = (e)$.

We deduce that since $A(G^0)$ is a topologically characteristic subgroup of G^0, $A(G^0)$ is normal in G. Let $L := G/A(G^0)$ denote the quotient. Using the Gleason–Yamabe structure theorem we show (compare with [1], Thm. 3.3.3):

Theorem: *The group L^0 is a direct product of adjoint connected, simple, non-compact Lie groups. Its centraliser $\mathcal{Z}_L(L^0)$ is totally disconnected and the product*

$$L^0 \cdot \mathcal{Z}_L(L^0)$$

is a direct product which is an open subgroup of finite index in L.

Observe that $\mathcal{Z}_L(L^0)$ is locally isomorphic to G/G^0; we can thus lift locally G/G^0 to $G/A(G^0)$.

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Transformation groups and permutation actions

Andreas Thom

I talked about the general theory of automorphism groups of connected locally finite graphs and explained why these groups are totally disconnected and locally compact. Important examples are given by automorphism groups of so-called Cayley–Abels graphs; references: e.g. §11 in [2] and §2 in [1]. Moreover, I explained the link between totally disconnected and locally compact groups and abstract groups with commensurated subgroups and (relative) Schlichting completions. The relevant referene here was Section 3 in [3].

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Closed groups of tree automorphisms and Tits simplicity theorem.

Yair Glasner

Let T be a locally finite tree, $\mathrm{Aut}(T)$ its automorphism group endowed with the topology of pointwise convergence. $\mathrm{Aut}(T)$ is a compactly generated t.d.l.c. group. The maximal compact (profinite) subgroups are the vertex and edge stabilizers. We have a standard classification of automorphisms into inversions, elliptic and hyperbolic elements.

$$\mathrm{Aut}(T) = \mathrm{Inv} \sqcup \mathrm{Ell} \sqcup \mathrm{Hyp}\,.$$

All three sets are open. For example an element $g \in \mathrm{Aut}(T)$ being hyperbolic, is characterized by the open condition that there exits an edge $e = (x, y)$ that is co-oriented with its image $ge = (gx, gy)$. Two directed edges are co-oriented if they both point in the same direction along the geodesic that connects them.

I discussed totally elliptic groups (consisting only of elliptic elements) and totally hyperbolic groups. Tits' lemma says that if g, h, gh are all elliptic then g, h must have a common fixed point. Combining this with a Helly's type argument shows that every totally elliptic subgroup has a common fixed point either in the tree or in its boundary. When the totally elliptic subgroup is finitely generated then there must be a fixed vertex inside the tree. As for totally hyperbolic groups, I showed that these are free groups acting freely on the tree. I showed how such groups can be realized as fundamental groups of graphs, whose universal cover is the given tree. I also discussed how such groups can be realized as Schotkey groups satisfying the conditions of the ping-pong Lemma.

Definition 1. Let $C = x_i, x_{i+1}, x_{i+2}, \dots, x_j, \quad i, j \in [-\infty, \infty]$ be a finite or infinite geodesic and $\pi : T \to C$ the closest point projection. We denote by $T_i = \pi^{-1}(x_i)$. If $G < \mathrm{Aut}(T)$ is a subgroup let G_C be the pointwise stabilizer of C, $\phi_i : G_C \to \mathrm{Aut}(T_i)$ the natural restriction map and $G_i = \phi_i(G)$ the image of this map.

We say that G satisfies *Tits' property (P)* if for every such path C we have an isomorphism

$$\prod_{n=i}^{j} \phi_n : G \xrightarrow{\sim} \prod_{n=i}^{j} G_i.$$

Definition 2. Say that a group $G < \mathrm{Aut}(T)$ is *geometrically dense* if it is minimal in the sense that it does not stabilize any proper subtree and in addition it does not fix any point at infinity.

Theorem 3. *(Tits' simplicity theorem) Let $G < \mathrm{Aut}(T)$ be a geometrically dense subgroup satisfying Tits' property (P). Let G^+ be the subgroup generated by all edge stabilizers. Then every nontrivial subgroup of G normalized by G^+ contains G^+.*

Proof. (sketch) Let N be such a group. Let $(x, y) = e \in ET$ be an edge and T_x, T_y the corresponding two half trees. By property (P) we may identify G_x with the pointwise stabilizer of T_y and vice versa and with this identification $G = G_x \times G_y$. Since e, x, y are arbitrary it is enough to show that $N > G_x$.

Geometric density is essentially inherited by non-trivial normal subgroups. So, after some work, it follows from geometric density that N contains a hyperbolic element n. Furthermore, after replacing n by an appropriate conjugate one can arrange for $C = \mathrm{Axis}(g) \subset T_y$. Thus $G_C > G_x$. So it would be enough to show that $N > G_C$. We show this by showing that the following map, whose image is clearly contained in N, is surjective.

$$\begin{array}{rcl} G_C & \to & G_C \\ h & \mapsto & [h, n] = hnh^{-1}n^{-1} \end{array}$$

Indeed let us denote by $m|_i$ the restriction of an element $m \in \mathrm{Aut}(T)$ to the subtree T_i. With this notation we have $(hnh^{-1}n^{-1})|_i = h|_i \circ n|_{i-\ell} \circ h^{-1}|_{i-\ell} \circ n^{-1}|_i$. Solving for $[h, n] = f$ where $f \in G_C$ is any given element we obtain

$$h|_i := f|_i \circ n|_{i-\ell} \circ h|_{i-\ell} \circ n^{-1}|_i.$$

Assuming we have arbitrarily fixed the values of $h|_0, h|_1, \dots, h|_{\ell-1}$ we can now solve recursively for $h|_\ell, h|_{\ell+1}, h|_{\ell+2}, \cdot$ by substituting into the above formula. Solving for $h|_{i-\ell}$ and applying a similar argument will yield also all the values $h|_{-1}, h|_{-2}, \dots$. Thus we end up with a complete set of values

$$h = (\dots, h|_{-1}, h|_0, h|_1, h|_2, \dots)$$

solving the desired equation $[h, n] = f$. □

Simplicity of the Neretin group

Łukasz Garncarek and Nir Lazarovich

The boundary ∂T of an infinite regular tree T can be identified with the set of infinite rays in T with fixed initial vertex o. This allows to define a metric d_o on ∂T, called the *visual metric* with respect to o, in which two paths are close if they have large overlap; more precisely, we define

$$d_o(\xi, \eta) = e^{-(\xi,\eta)_o},$$

where $(\xi, \eta)_o$ is the length of the common initial segment of ξ and η. A *spheromorphism* of ∂T can then be defined as a local similarity, i.e. a map $\phi\colon \partial T \to \partial T$ such that there is a finite partition of $\partial T = B_1 \cup \cdots \cup B_k$ into disjoint balls, such that the restrictions $\phi|_{B_i}$ are similarities with respect to the visual metric.

The group of all spheromorphisms of a $(n+1)$-regular tree is denoted by N_n and called the *Neretin group*. The automorphism group $\mathrm{Aut}(T)$ of T embeds into the Neretin group, which can be given a unique totally disconnected locally compact group topology in which this embedding is open. This is not the compact-open topology arising from the action of N_n on ∂T!

Another class of groups related to the story are the Higman–Thompson group $G_{n,r}$. This time we consider a forest F of r rooted $(n+1)$-ary trees, endowed with an additional structure comprised of a linear orders on the sets of children of each vertex. Such orders induce linear orders on the boundaries of the trees in F. An element of the group $G_{n,r}$ is a local order-preserving similarity of the boundary of F, which is just a disjoint union of the boundaries of trees in F.

Returning to the original tree T and the Neretin group, if we remove an edge from T, we get a forest of two rooted $(n+1)$-ary trees. If we put linear orders on sets of children, we get an embedding of $G_{n,2}$ into N_n. Any two such embeddings have conjugate images, and the conjugating element can be chosen from the image of $\mathrm{Aut}^+(T)$, the group of *type-preserving* automorphisms of T, generated by the stabilizers of edges in $\mathrm{Aut}(T)$. An important observation is that N_n is generated by the image of $\mathrm{Aut}^+(T)$, and any of the embedded copies of $G_{n,2}$.

In [2], Higman proved that the commutator group $G'_{n,r} = [G_{n,r}, G_{n,r}]$ of the Higman–Thompson group is simple. Moreover, it is of index 1 or 2 in $G_{n,r}$ depending respectively on whether n is even or odd.

In [4], Tits described sufficient criteria for a group G of automorphisms of a tree to have a simple or trivial type-preserving subgroup G^+. As a corollary he deduced that the group of all type-preserving automorphisms $\mathrm{Aut}^+(T)$ is simple.

Following [3] we combine the two theorems with some added tricks from [1] to obtain that the Neretin group N_n is simple.

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Locally compact groups as metric spaces

Romain Tessera

The aim of this talk is to give a short introduction to the large-scale geometry of locally compact groups. The general idea is that one can attach to every σ-compact locally compact group an essentially unique coarse metric structure. There is a large literature on this topic, and we refer to the book [1] for references. Before introducing more precise notions, let us briefly recall a few classical instances of left-invariant metrics on groups. The first class of examples is that of connected Lie groups, which can be endowed with a left-invariant Riemannian metric. In the case of semi-simple Lie groups though, it is sometimes more fruitful to consider their actions on their associated symmetric spaces, which in some sense reflect better their large-scale geometry: for instance these spaces are simply connected, and admit non-positively curved Riemannian metrics. On the other hand, Gromov has popularized the study of the large-scale geometry of Cayley graphs of finitely generated groups. One of the aims of this lecture is to introduce the notion of Cayley–Abels graph, which is a natural generalization of Cayley graphs for totally discontinuous, compactly generated, locally compact groups. Roughly speaking a Cayley–Abels graph is a G-invariant locally finite graph structure on $X = G/K$, where K is a compact open subgroup of G.

An admissible metric on a σ-compact locally compact group is a left-invariant pseudo-distance $d : G \times G \to [0, \infty)$, which takes bounded values on compact subsets of $G \times G$ and such that balls are relatively compact. When the group G is generated by a compact subset S, i.e. is *compactly generated*, then one checks that the word metric d_S is admissible. It is not hard to see every σ-compact locally compact groups admit admissible metrics. Recall that a coarse equivalence between two (pseudo)-metric spaces is a map $F : X \to Y$ such that there exists proper functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ and a constant C, satisfying for all $x, x' \in X$

$$\rho_1(d(x, x')) \leq d(F(x), F(x')) \leq \rho_2(d(x, x'),$$

and for all $y \in Y$, there exists $x \in X$ such that $d(y, f(x)) \leq C$. One checks that being coarse equivalent defines an equivalence relation between (pseudo)-metric spaces, and that two admissible metrics on a locally compact group are coarse equivalent.

The main goal of this lecture is to characterize algebraically the property of a group G to act properly cocompactly by isometries on a connected (resp. simply connected) metric space. The strategy is first to show that this can be encoded into coarse-equivalent properties: respectively, coarse connectedness, and coarse simple-connectedness. For illustration, let us define the first of these two notions:

a metric space X is coarsely connected if there exists $C > 0$ such that every pair of points can be connected by a "discrete path" $x = x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq C$. The interest of this notion is illustrated by the following easy fact.

Proposition. *A σ-compact locally compact group is compactly generated if and only if it admits an admissible coarse connected (pseudo)-metric, if and only if it acts properly cocompactly by isometries on a geodesic metric space.*

A similar characterization holds for coarse simple connectedness, which involves the notion of compact presentability, a direct generalization of finite presentability for discrete groups. A group is compactly presentable if it admits a presentation $\langle S; R\rangle$, where S is a compact generating subset of G, and R is a set of word in S of length bounded by some constant k. One has the following important fact.

Theorem. *A σ-compact locally compact group is compactly presented if and only if it admits an admissible coarse simply connected (pseudo)-metric, if and only if it acts properly cocompactly by isometries on a geodesic simply connected metric space.*

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Introduction to p-adic Lie groups

Yves Cornulier

This is an extended abstract for an introduction to p-adic Lie groups. The main reference is Bourbaki [B2-3].

Let $\mathbf{K}$ be a non-discrete complete normed field, with norm written as $|\cdot|$.

Power series and analytic manifolds. Write $\mathbf{N} = \{0, 1, 2, \ldots\}$. For $n = (n_1, \ldots, n_d) \in \mathbf{N}^d$, write $|n| = \sum n_i$.

If $(a_n)_{n\in\mathbf{N}^d}$ is a sequence in $\mathbf{K}$, and if $\rho = \underline{\lim}\, |a_n|^{-1/|n|} \in [0, \infty]$, then the series $\sum a_n x^n$ is locally uniformly convergent on an open ball of radius $< \rho$ centered at 0. The function $x \mapsto \sum a_n x^n$ is said to have a power series expansion at zero. A function f from an open subset Ω of $\mathbf{K}^d$ and valued into a subset of $\mathbf{K}$ is called *analytic at 0* if $0 \in \Omega$ and f coincides, in a neighborhood of 0, with a series as above; f is called *analytic* if for every $x_0 \in \Omega$, the function $x \mapsto f(x + x_0)$ defined on $\Omega - x_0$ is analytic at zero. A function from Ω to a subset of $\mathbf{K}^\ell$ is called analytic if all its ℓ coordinates are analytic, and a function between two open subsets of $\mathbf{K}^d$ is called bianalytic if it is analytic, bijective, and its inverse is also analytic. When necessary, we say $\mathbf{K}$-analytic instead of analytic.

An *analytic* $\mathbf{K}$*-manifold* is a Hausdorff topological space with an atlas into open subsets of $\mathbf{K}^d$ (for various d), such that change of charts are bianalytic. If d is fixed beforehand, this is called an analytic $\mathbf{K}$-manifold of pure dimension d. This allows to define the notion of analytic function on an analytic manifold.

Exercise. Let p be prime and fix $d \in \mathbf{N} \smallsetminus \{0\}$. Show that there are exactly $p-1$ nonempty compact analytic $\mathbf{Q}_p$-manifolds of pure dimension d up to bianalytic transformation, namely $\mathbf{Z}_p^d \times \{1, \dots, k\}$ for $1 \leq k \leq p-1$.

An *analytic* $\mathbf{K}$*-Lie group* is an analytic $\mathbf{K}$-manifold endowed with a group law such that both the law and the inversion map are analytic. For $\mathbf{K} = \mathbf{Q}_p$, we say "$p$-adic Lie group" rather than "$\mathbf{Q}_p$-Lie group".

Instances of p-adic Lie groups are

- the additive group $\mathbf{Q}_p$ itself, its open subgroup $\mathbf{Z}_p$, and their direct products such as $\mathbf{Q}_p^k \times \mathbf{Z}_p^\ell$.
- the general linear group $\mathrm{GL}_d(\mathbf{Q}_p)$, and the special linear group $\mathrm{SL}_d(\mathbf{Q}_p)$.
- all discrete groups. Although these are trivial examples, it is good to keep in mind, since any theorem about general p-adic Lie groups should include all discrete groups.

Any $\mathbf{K}$-analytic Lie group $(G, *)$ has its law, given in a chart centered at the unit element, given as $x * y = L(x) + L'(y) + B(x, y) + O(\|x\|^3 + \|y\|^3)$, with L, L' linear and B a quadratic polynomial in (x, y). Using only that 0 is the neutral element, we obtain that L, L' are identity and that B is actually bilinear in the variables x and y.

A first consequence is that $x^p = px + O(\|x\|^2)$. This implies, in the case $\mathbf{K} = \mathbf{Q}_p$, that the p-power map is conjugate around 0 to the contracting map $x \mapsto px$. In particular, we deduce that every p-adic Lie group G admits an open torsion-free pro-p-subgroup.

For $\mathbf{K}$ arbitrary again, using associativity and a Taylor expansion of order 3, we see that the alternating bilinear map $(x, y) \mapsto B(x, y) - B(y, x)$ satisfies the Jacobi identity, and that the resulting Lie algebra $\mathfrak{g}$ depends functorially on G (in the category of $\mathbf{K}$-adic Lie groups with analytic homomorphisms).

An *exponential map* is defined as an analytic map f from an open neighborhood Ω of 0 in $\mathfrak{g}$, stable under multiplication by elements of the closed 1-ball $\mathbf{K}_{\leq 1}$ of $\mathbf{K}$, such that

- $f(0) = 1$ and the differential of f at 0 is $\mathrm{Id}_{\mathfrak{g}}$
- $f(tx)f(t'x) = f((t + t')x)$ for all $x \in \Omega$ and all $t, t' \in \mathbf{K}$ such that $|t|$, $|t|$, and $|t + t'|$ all belong to $\mathbf{K}_{\leq 1}$.

In the case when $\mathbf{K}$ is a p-adic field ($\mathbf{Q}_p$ and its finite extensions), we have a good correspondence: let $\mathcal{S}_{\mathbf{K}}^{\omega}(G)$ be the set of closed $\mathbf{K}$-analytic subgroups of G, and $\mathrm{Sub}_{\mathbf{K}}(\mathfrak{g})$ the set of Lie $\mathbf{K}$-subalgebras of $\mathfrak{g}$. Let $\simeq$ be the equivalence relation on the set of subgroups of G defined by: $H \simeq H'$ if H and H' locally coincide: there exists a compact open subgroup L of G such that $H \cap G = H' \cap G$.

Given a subgroup H of G, its Lie algebra is defined as the set of $x \in \mathfrak{g}$ that occur as $\mathfrak{g}amma'(0)$, where $\mathfrak{g}amma$ ranges over germs of analytic functions from $\mathbf{K}^{\dim(G)}$ to G valued in H; this is indeed a Lie $\mathbf{K}$-subalgebra. Of course, it only depends on H modulo $\simeq$. If H is a closed analytic subgroup, then this is just the tangent space of H at the neutral element.

Given a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $\mathfrak{h}'$ be a small enough open compact $\mathbf{K}_{\leq 1}$-submodule of $\mathfrak{g}$, and define H as its exponential: this is a compact analytic subgroup, uniquely defined in $\mathcal{S}^{\omega}_{\mathbf{K}}(G)/\simeq$.

Then these two maps are inverse maps between $\mathcal{S}^{\omega}_{\mathbf{K}}(G)/\simeq$ and $\mathrm{Sub}_{\mathbf{K}}(\mathfrak{g})$.

When $\mathbf{K} = \mathbf{Q}_p$, we have two improvements, showing that the analytical structure is governed by the structure of topological group.

- Every continuous homomorphism between p-adic Lie groups is analytic; in particular the analytical structure is uniquely determined by the underlying topological group;
- Every closed subgroup is analytic.

Thus in a p-adic Lie group, there is a natural bijection between the poset of closed subgroups modulo $\simeq$ and the poset of Lie subalgebras of the Lie algebra. Another consequence is that the quotient of a p-adic Lie group by any closed subgroup is naturally a p-adic manifold with an analytic action, and the quotient by a closed normal subgroup is naturally a p-adic Lie group.

This shows that there is a bound on the length of chains of non-open inclusions of closed subgroups in p-adic Lie group. In particular, if G is a p-adic Lie group, then the set of n such that $\mathbf{Z}_p^n$ is a subquotient of G is bounded. Interestingly, the converse holds, in the following form: if G is a locally compact group with an open pro-p-subgroup P, and if there is a bound on the set of n such that $\mathbf{Z}_p^n$ is a subquotient of P (this clearly does not depend on the choice of P), then G is a p-adic Lie group [DDMS]. This is a characterization of p-adic Lie groups among topological groups, not referring to p-adic manifolds.

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The scale function

Albrecht Brehm and Rafaela Rollin

The aim of our talk was to give a motivation around the notions arising in Willis' Theory of the scale function and to prove some elemental properties of the scale. All theorems and definitions are drawn from the papers which appear in the references. In the sequel let G denote a totally disconnected locally compact group, $\mathcal{B}$ the collection of compact open subgroups and α a continuous automorphism of G. Furthermore let U be a compact open subgroup of G.

At first we observe that the index $|\alpha(U) : \alpha(U) \cap U|$ is finite, because the cosets of $\alpha(U) \cap U$ form an open covering of $\alpha(U)$. Thus the scale function can be defined as

$$s\colon \text{Aut } G \to \mathbb{N},\ \alpha \mapsto \min_{U \in \mathcal{B}} |\alpha(U) : \alpha(U) \cap U|.$$

Let us call a compact open subgroup U minimizing for α iff

$$s(\alpha) = |\alpha(U) : \alpha(U) \cap U|,$$

i.e. iff the minimum is attained at U.

This leads to the question which properties minimizing subgroups necessarily have? For a given arbitrary compact open U there are two possibilites to get $|\alpha(U) : \alpha(U) \cap U|$ smaller without destroying the compact open property.

(1) "Cutting U down", an operation which decreases the enumerator faster than the denominator and
(2) "adding an α-invariant compact space to U", an operation which increases the denominator faster than the enumerator.

Therefore we conclude that a necessary condition for a compact open subgroup to be minimizing is to be constant under both of the operations described above. This motivates the following definitions:

Definition 1. Let

$$U_+ := \bigcap_{n \geq 0} \alpha^n(U), \qquad U_{++} := \bigcup_{n \geq 0} \alpha^n(U_+)$$

and

$$U_- := \bigcap_{n \leq 0} \alpha^n(U), \qquad U_{--} := \bigcup_{n \leq 0} \alpha^n(U_-).$$

A compact open subgroup U is said to be semi-tidy for α iff

$$|\alpha(U_+) : U_+| = |\alpha(U) : \alpha(U) \cap U|.$$

A compact open subgroup U is said to be tidy for α iff it is semi-tidy and $\mathcal{L} \subseteq U$, where $\mathcal{L} := \{x \in G : \text{for all but finitely many } n \in \mathbb{Z} :\ \alpha^n(x) \in U\}$.

Remark 2.

(1) There is a nice trick of Rosendal which you can find in [2] to prove that $\mathcal{L}$ is relatively compact.
(2) If you wonder why $\mathcal{L}$ is the right choice for the second procedure look at the example given by Willis in [3] p. 344 (e).
(3) The ideas mentioned above give a guide how to prove the existence of semi-tidy or tidy groups.

We give the most important characterisations of these notions:

Proposition 3. *The following assertions are equivalent for a compact open subgroup U:*

(1) $U = U_+U_-$,
(2) U is semi-tidy,

(3) $U = U_+\big(U \cap \alpha^{-1}(U)\big)$.
Furthermore U *is tidy if and only if* U *is semi-tidy and* U_{++} *is closed.*

All the notions we defined for automorphisms of G can also be used for elements $g \in G$ by replacing each α in the definitions above by the inner automorphism induced by g. This way the scale function for $g \in G$ is given by $s\colon G \to \mathbb{N},\ g \mapsto \min_{U\in\mathcal{B}} |gUg^{-1} : gUg^{-1} \cap U|$ and analogously one can define (semi-) tidy subgroups for g. We use this interpretation in the following part.
As a first application of semi-tidy subgroups there is an elementary proof of the next proposition and of the following theorem one can find in [2].

Proposition 4. *If* U *is semi-tidy for* $g \in G$*, then* $(UgU)^n = Ug^nU$ *and* $(Ug^{-1}U)^n = Ug^{-n}U$ *for every* $n \geq 1$.

Theorem 5. *The set* $P_1(G) := \{g \in G : \overline{\langle g\rangle} \text{ is compact}\}$ *of periodic elements is closed.*

The next theorem connects the scale function to the modular function Δ for the left invariant Haar measure on G. Proofs for this theorem and the following basic properties of the scale function can be found in [1].

Theorem 6. *The equality* $\dfrac{s(g^{-1})}{s(g)} = \Delta(g)$ *holds for every* $g \in G$.

Corollary 7. *For* $g \in G$ *we have* $s(g) = |gUg^{-1} : gUg^{-1} \cap U|$ *if and only if* $s(g^{-1}) = |g^{-1}Ug : g^{-1}Ug \cap U|$.

Corollary 8. *An element* $g \in G$ *normalises some* $U \in \mathcal{B}$ *if and only if* $s(g) = 1 = s(g^{-1})$.

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Tidy subgroups

Maxime Gheysens and Adrien Le Boudec

The aim of the talk is to give an account of Willis' theory of tidy subgroups and scale function of a totally disconnected locally compact group G.

The notion of compact open subgroup *tidy* for a given $\alpha \in \mathrm{Aut}(G)$ was introduced in [3]. One of its major interests comes from a result from [4], that characterizes tidy subgroups as those compact open subgroups at which the value of the scale function of α is attained, revealing that the tidiness criteria can be seen as structure results for minimizing subgroups for α.

In the first part of the talk we give a detailed exposition of the construction of tidy subgroups, following [4] and [2]. The idea is to start with an arbitrary compact open subgroup, and to follow a procedure that leads to a new compact open subgroup satisfying the two tidiness criteria. The main ingredient is the introduction of a compact subgroup that is normalized by the automorphism under consideration.

This construction is then illustrated with two examples, namely the general linear group $\mathrm{GL}(2, \mathbb{Q}_p)$ and the automorphism group of a regular locally finite tree $\mathrm{Aut}(T)$. In the first example, we explain why the compact open subgroup $\mathrm{SL}(2, \mathbb{Z}_p)$ fails to be tidy for the element $g = \mathrm{diag}(p, 1)$, and how to obtain a tidy subgroup and compute the scale function of g by following the aforementioned procedure. In the case of the group $\mathrm{Aut}(T)$, elliptic isometries have scale one, and we explain how tidy subgroups for a hyperbolic isometry in some sense lie along its axis.

Finally the end of the talk is devoted to show how a result of Willis on tidy subgroups [3] yields a compelling proof of the following result: if a locally compact totally disconnected group G admits an automorphism that is ergodic with respect to a Haar measure on G, then G must be compact [1].

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Margulis' normal subgroup theorem

Alex Furman

The goal of the talk was to discuss the following theorem of Margulis:

Theorem 1 (Margulis)**.** *Let G be a connected, semi-simple, real Lie group, with finite center and of higher rank: $\mathrm{rk}(G) \geq 2$. Let $\Gamma < G$ be an irreducible lattice. Then any normal subgroup $N \triangleleft \Gamma$ is either of finite index in Γ, or N is finite and central: $N \subset \Gamma \cap \mathcal{Z}(G)$.*

This theorem applies to any lattice in a simple Lie group of higher rank, such as $G = \mathrm{SL}_3(\mathbb{R})$, and to any irreducible lattice in semi-simple groups like $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. It applies more generally to (irreducible) lattices in (products) of groups of k-points for algebraic groups over local fields. The theorem does not apply to groups of rank one: uniform lattices in such groups are Gromov-hyperbolic, and as such have (each) uncountably many normal subgroups. So this is pure higher-rank phenomenon.

The proof can be split into two parts:

(i) Proving that if Γ/N is amenable, then Γ/N is finite.
(ii) Proving that if Γ/N is not amenable, then $N \subset \Gamma \cap \mathcal{Z}(G)$.

If G has Kazhdan's property (T) then Γ and Γ/N also have (T), and (i) follows. This argument applies to many examples, but for irreducible lattices in (some) products, such as $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, a more complicated argument is needed.

Remarkably, the proof of (ii) consists to the following purely measure-theoretic theorem

Theorem 2 (Margulis)**.** *Let G be a connected, semi-simple, real Lie group, with finite center and of higher rank:* $\mathrm{rk}(G) \geq 2$*. Let $P < G$ be a minimal parabolic subgroup, and $\Gamma < G$ an irreducible lattice. Then the only Γ-invariant, complete, sub-σ-algebras of the Lebesgue σ-algebra on G/P, are G-invariant ones, they are pull-backs of the Lebesgue σ-algebras from $G/P \to G/Q$ where $P < Q < G$ are various parabolic subgroups (there are $2^{\mathrm{rk}(G)}$-many such parabolics).*

The proof of this result relies (in a very clever way) on a version of Lebesgue differentiation theorem. Specifically Margulis shows that

Proposition 3. *Let V be a locally compact second countable group, and $\phi \in$* $\mathrm{Aut}(V)$ *such that for any open neighborhood of identity $1 \in U \subset V$ and any compact subset $K \subset V$, one has $\phi^n(K) \subset U$ for all $n \geq n(U, K)$. Then for any Borel subset $E \subset V$ for a.e. $v \in V$ one has*

$$\lim_{n\to\infty} \phi^{-n}(vE) = \Psi(v) \subset E$$

where $\Psi(v) = V$ if $v^{-1} \in E$ and $\Psi(v) = \emptyset$ if $v^{-1} \neq E$, where the convergence is "in measure on finite measure sets".

In the proof one identifies (measure-theoretically) G/P with U — the unipotent radical of the opposite parabolic, and the role of V is played by certain subgroups of this nilpotent groups; contracting automorphisms $\phi \in \mathrm{Aut}(V)$ are conjugation by some singular elements $a \in \mathfrak{a}^+$ in the positive Weyl chamber that contract V as above, while acting trivially on U/V. Howe–Moore's theorem (with some small additional argument) allows to mimic the action of any such a on G/P by an appropriate sequence in Γ.

Bader–Shalom Normal Subgroup Theorem

Światosław Gal

The theme of the talk is the following result of Uri Bader and Yehuda Shalom [1].

Theorem 1. *Let G_1 and G_2 be locally compact, nondiscrete, compactly generated groups, not both isomorphic to the group of real numbers. Let $\Gamma < G_1 \times G_2$ be a discrete cocompact irreducible subgroup. If both G_i's are just noncompact, then Γ is just infinite.*

The proof consist of showing that if N is a proper normal subgroup of Γ then Γ/N is both amenable and Kazhdan, thus finite. We discuss in further details the amenable part. To do this we equip G_i's with auxiliary finite measures and study the dynamics of the actions on Poisson boundaries. The main ingredients are Kaimanovich-Vershik-Rosenblatt characterisation of amenable groups, Zimmer result of the amenability of an action of a group on its Poisson boundary, and the Margulis Factor Theorem stating that every Γ-factor of the Poisson boundary of $G_1 \times G_2$ is isomorphic to the product of G_i-factors of Poisson boundaries of G_i's.

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L^2-Betti numbers of locally compact groups

Roman Sauer

L^2-Betti numbers of Riemannian manifolds were introduced by Atiyah in 1976. Since then their range of definition has been extended several times: By Dodziuk to finite simplicial complexes, by Cheeger-Gromov to arbitrary discrete groups, by Lück [3] to arbitrary spaces with group actions, and, more recently, by Petersen [4] to locally compact groups. In this talk we gave an overview of L^2-Betti numbers of (discrete and locally compact) groups.

The theory of L^2-Betti numbers of measured equivalence relations, which started with Gaboriau's fundamental paper [1], implies for the L^2-Betti numbers of lattices Λ, Γ in the same locally compact group G that $\mathrm{covol}(\Lambda)^{-1}\beta^n(\Lambda) = \mathrm{covol}(\Gamma)^{-1}\beta^n(\Gamma)$. This equality was the main motivation to introduce a notion of L^2-Betti number of G, which is equal to $\mathrm{covol}(\Gamma)^{-1}\beta^n(\Gamma)$. In full generality, this was achieved in [2].

The definitions for discrete and locally compact groups by Lück and Petersen, respectively, are similar on a formal level but important differences are the lack of a finite trace on the von Neumann algebra of a non-discrete group and the technical difficulties involved in dealing with continuous cohomology.

In the talk we explained computations of L^2-Betti numbers for $SL_3(\mathbb{Q}_p)$ and free groups.

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Commensurated subgroups, after Shalom and Willis

László Márton Tóth and Samuel Mellick

The aim of this talk is to present a result of Shalom and Willis answering a question of Margulis and Zimmer about the structure of commensurated subgroups in arithmetic groups. To sidestep the need for technical background, we discuss the specific case of SL_n for $n \geq 3$ – this particular example has the trappings of the general proof. Enthused readers are invited to look at the original paper [1].

The Margulis–Zimmer problem. Let $S = \{p_1, p_2, \ldots, p_s\}$ be a set of primes. Consider the group $SL_n(\mathbb{Z}[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_s}])$, denote it by $SL_n(\mathbb{Z}[\frac{1}{S}])$ for short. The subgroups $SL_n(\mathbb{Z})$, $SL_n(\mathbb{Z}[\frac{1}{p_i}])$, $SL_n(\mathbb{Z}[\frac{1}{p_i}, \frac{1}{p_j}])$, or generally $SL_n(\mathbb{Z}[\frac{1}{S'}])$ for any $S' \subseteq S$ are all commensurated. These subgroups, and any subgroup commensurable with a subgroup of this form, are deemed **standard commensurated subgroups**.

Question 1 (Margulis, Zimmer)**.** Are all commensurated subgroups of $SL_n(\mathbb{Z}[\frac{1}{S}])$ necessarily standard or finite?

Work of Venkataramana in [3] shows that the answer is *yes* for $SL_n(\mathbb{Z})$, and more generally for the integral points of higher-rank $\mathbb{Q}$-groups. Shalom and Willis have provided a conceptually different proof with wider scope, and have explored the connections of this problem with other deep properties of arithmetic groups.

The key new property elucidated by Shalom and Willis lets us answer the Margulis–Zimmer question, and can be viewed as a kind of fixed point statement:

Definition 2. A group Γ has the **outer commensurator-normalizer property** if for any group Δ and any homomorphism $\varphi : \Gamma \to \Delta$, if a subgroup $\Lambda \leq \Delta$ is commensurated by $\text{im}(\varphi)$ then there is some Λ' commensurable to Λ which is normalized by $\text{im}(\varphi)$.

To understand this an iota better, recall that for a subgroup H of G, commensuration means the indices $[H : H \cap gHg^{-1}]$ are all finite, and commensuration by another subgroup $K \leq G$ means g varies only over K (the indices $[H : H \cap kHk^{-1}]$ are all finite). One can retranslate this in terms of the action of H on the cosets G/H. It just means the H-orbits of points kH are all finite.

Theorem 3 (Bergman, Lenstra). *Let G be a group, $H \leq G$ a subgroup. Then H is uniformly commensurated in the sense that there is a bound C such that $[H : H \cap gHg^{-1}] \leq C$ for all $g \in G$ if and only if H is commensurable to a normal subgroup of G. More generally, if $K \leq G$ is another subgroup then H is uniformly commensurated by K in the sense that there is a bound C such that $[H : H \cap kHk^{-1}] \leq C$ for all $k \in K$ if and only if H is commensurable to a subgroup normalized by K.*

Through the lens of this theorem, one sees the outer commensurator-normalizer property as a statement that commensuration can only happen uniformly.

An important point missed in the particular case we're talking about is that the outer commensurator-normalizer property is a *commensurability invariant*. This is what is needed to make sense of the analogous question of our "every commensurated subgroup looks like one on this list" for more general algebraic groups. More concretely, think about $SL_3(\mathbb{Z})$ – the theorem says that every commensurated group is finite or finite-index. Is the same true for every finite-index subgroup of $SL_3(\mathbb{Z})$? This is a priori unclear, but turns out to be true.

Theorem 4 (Shalom, Willis). *$SL_n(\mathbb{Z})$ has the outer commensurator-normalizer property. Consequently, every commensurated subgroup of $SL_n(\mathbb{Z}[\frac{1}{S}])$ is standard or finite.*

One solves the Margulis–Zimmer question in terms of the outer commensurator-normalizer property by invoking strong approximation results and the Margulis normal subgroup theorem. Establishing the outer commensurator-normalizer property itself involves looking at the "relative profinite completion" (also known as a Schlichting completion) associated to a commensurated subgroup inside a group. This is a totally disconnected locally compact group on which $SL_n(\mathbb{Z})$ acts by automorphisms. From here one combines structural results about arithmetic groups, bounded generation theorems, Willis' theory of flatness and scale (as expounded in [4] and [5]), and the Bergman and Lenstra theorem.

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Minimal closed normal subgroups in certain compactly generated locally compact groups

Thibaut Dumont and Dennis Gulko

In [2], P.-E. Caprace and N. Monod proved the existence of minimal non-trivial closed normal subgroups in certain compactly generated locally compact groups and discussed how many could there be. We present the last version of these results to appear in the note [3] and we prove fractions of them. The totally disconnected case is discussed in details and a similar result for non-necessarily totally disconnected groups is stated.

Throughout this abstract the abbreviation *t.d.l.c.* stands for *totally disconnected locally compact*. A subgroup H of a topological group G is called *locally elliptic* (or topologically locally finite) if every finitely generated subgroup of H has compact closure. For example, any compact subgroup of G is locally elliptic. Van Dantzig's theorem guaranties non-discrete t.d.l.c. groups to have numerous locally elliptic subgroups. Any topological group G possesses a unique maximal normal locally elliptic subgroup denoted by $\mathrm{Rad}_{\mathcal{LE}}(G)$. It is closed and topologically characteristic; moreover $\mathrm{Rad}_{\mathcal{LE}}(G/\mathrm{Rad}_{\mathcal{LE}}(G)) = 1$.

The main result is the following.

Proposition (Proposition 2.6, [3]). *Let G be a compactly generated t.d.l.c. group without non-trivial compact or discrete normal subgroup. Then,*

(i) Every non-trivial closed normal subgroup contains a minimal one.
(ii) Let $\mathcal{M}$ be the set of minimal closed normal subgroups and $\mathcal{M}_{na}$ be the subset of non-abelian ones. Then $\mathcal{M}$ might be infinite but $\mathcal{M}_{na}$ is finite.
(iii) Each abelian $M \in \mathcal{M}$ is locally elliptic, hence contained in $\mathrm{Rad}_{\mathcal{LE}}(G)$. *In particular, if* $\mathrm{Rad}_{\mathcal{LE}}(G) = 1$*, then $\mathcal{M} = \mathcal{M}_{na}$ is finite.*
(iv) For any proper $\mathcal{E} \subset \mathcal{M}$, the subgroup $N_{\mathcal{E}} = \overline{\langle M \mid M \in \mathcal{E}\rangle}$ is properly contained in G.

Using the solution to Hilbert's fifth problem and some more machinery, they proved the following result for arbitrary compactly generated locally compact groups.

Theorem (Theorem B, [3]). *Let G be a compactly generated locally compact group. Then one of the following holds.*

(i) G has an infinite discrete normal subgroup.
(ii) G as a non-trivial closed normal subgroup which is {compact}-by-{soluble connected}.
(iii) There exist non-trivial closed normal subgroups, of which only finitely many are non-abelian.

The fact that a Hausdorff quotient of a compactly generated group is again compactly generated seems to allow iterations of Theorem B. If G is *Noetherian*, such a procedure will stop after finitely many steps. We refer to [2, Theorem C] for more details. Another approach would be to consider the quotient by the locally elliptic radical. Then $G/\operatorname{Rad}_{\mathcal{LE}}(G)$ could potentially satisfy the hypothesis of Proposition 2.6, in which case it would have only finitely many minimal closed normal subgroup by (i), all non-abelian. This motivates the next talk on the class of elementary t.d.l.c. groups, see [4].

The following proposition was the starting point leading to Proposition 2.6 and it is inspired by the work [1] of M. Burger and S. Mozes.

Proposition (Proposition 2.5, [2]). *Let G be a compactly generated t.d.l.c. group and V be an identity neighbourhood. Then there exists a compact normal subgroup $Q_V \subset V$ such that any filtering family of non-discrete closed normal subgroups of G/Q_V has non-trivial intersection.*

The idea behind Proposition 2.5 is that a compactly generated t.d.l.c. group acting faithfully on one of its Cayley-Abels graphs has the non-trivial intersection property for any filtering family of non-discrete closed normal subgroups. On the one hand if G has no compact normal subgroup then the action is indeed faithful since $Q_V = \{1\}$. On the other hand if every closed normal subgroup of G is non-discrete then the filtering property holds for any filtering family of closed normal subgroups. Those observations together prove (i) of Proposition 2.6.

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Elementary totally disconnected locally compact groups

Morgan Cesa and François Le Maître

We presented the class of elementary tdlc (totally disconnected locally compact) second-countable groups, which was introduced by Wesolek in [Wes14]. This class $\mathcal{E}$ is defined as the smallest class of tdlc second-countable groups such that

- $\mathcal{E}$ contains all profinite second-countable groups and all countable discrete groups.
- Whenever $N \leq G$ is a normal subgroup, if $N \in \mathcal{E}$ and G/N is profinite metrizable or countable discrete, then $G \in \mathcal{E}$.
- Whenever G may be written as a countable increasing union of open subgroups belonging to $\mathcal{E}$, then $G \in \mathcal{E}$.

Much like in the case of elementary amenable groups, the class of elementary tdlc groups enjoys stronger closure properties: for instance, it is closed under group extensions, taking closed subgroups, Hausdorff quotients, and inverse limits (for more details, see [Wes14, Thms. 1.3 and 1.4]).

Examples of elementary tdlc second-countable groups include solvable groups and small invariant neighbourhood groups. A wealth of non-elementary tdlc groups is provided by compactly generated topologically simple non-discrete non-profinite groups [Wes14, Prop. 6.2]. In particular, for all $n \geq 3$, neither the group of automorphisms of the n-regular tree nor the projective linear group of dimension n over $\mathbb{Q}_p$ are elementary. It is not known whether every tdlc second-countable amenable group is an elementary tdlc group.

We presented a very interesting feature of the class of elementary tdlc groups: the existence of a maximal normal elementary closed subgroup inside every tdlc second-countable group. Such a subgroup is unique, and is called the elementary radical. It can be used to show that elementary tdlc groups and topologically characteristically simple non-elementary tdlc groups may be seen as building blocks for general tdlc second-countable groups. More precisely, we proved the following theorem of Wesolek, using a result of Caprace and Monod [CM11].

Theorem 1 ([Wes14, Thm. 1.6])**.** *Let G be a compactly generated tdlc second-countable group. Then there exists a finite increasing sequence*

$$H_0 = \{e\} \leq \cdots \leq H_n$$

of closed characteristic subgroups of G such that

(1) G/H_n is an elementary tdlc group and
(2) for all $i = 0, ..., n-1$, the group $(H_{i+1}/H_i)/\mathrm{Rad}_{\mathcal{E}}(H_{i+1}/H_i)$ is a finite quasi-product of topologically characteristically simple non-elementary subgroups, where $\mathrm{Rad}_{\mathcal{E}}(H)$ denotes the elementary radical of H.

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Invariant Random Subgroups in rank one and higher rank Lie groups

Tsachik Gelander

An Invariant Random Subgroups (IRS) in a locally compact group G is a conjugacy invariant probability measure on the space $\mathrm{Sub}(G)$ of closed subgroups of G. Special examples are normal subgroups (Dirac mass) and finite volume homogeneous spaces (corresponding to lattices). The space $\mathrm{IRS}(G)$, of all IRS's, equipped with the weak-* topology is a compact space which one wishes to analyse, in particular, in order to understand better some of its special points (e.g. the lattices). In recent years there has been a largely growing interest in studying IRS in various groups. I will concentrate in the case that G is a simple Lie group. The first part of the talk will be dedicated to basic definitions and properties, including the analogy to Benjamini–Schramm topology. The second part will be devoted to the higher rank case (following my joint work with Abert, Bergeron, Biringer, Nikolov, Raimbault and Samet). In particular I will explain Stuck–Zimmer rigidity theorem, its application to the asymptotic shape of locally symmetric spaces, and some further application concerning L_2 invariants. The third part will be devoted to rank one groups, where the lack of rigidity plays an important role and could be used for instance to show that almost all hyperbolic manifolds are non-arithmetic (in an appropriate sense).

Automorphism Groups of Trees: Prescribed Local Actions

Alejandra Garrido and Stephan Tornier

This talk split into two parts. First, the second author introduced basic properties of the universal group construction by Burger and Mozes, see Section 3.2 of [2]. Second, the first author described some variations of this construction by Banks, Elder and Willis (see [1]) and explained how this construction can be used to find infinitely many locally compact compactly generated non-discrete simple subgroups of tree automorphisms.

Universal Groups. Let $T_d = (X, Y)$ denote the d-regular tree ($d \geq 3$) and let $l : Y \to \{1, \ldots, d\}$ be a legal labelling of T_d. We adopt Serre's conventions for graph theory, see [4]. Given a vertex $x \in X$, every automorphism $g \in \mathrm{Aut}(T_d)$ induces a permutation at x given by $c(g, x) := l|_{E(gx)} \circ g|_{E(x)} \circ l|_{E(x)}^{-1} \in S_d$, where $E(x) := \{y \in Y \mid o(y) = x\}$.

Definition 1. Let $F \leq S_d$. Define $\mathrm{U}(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in X : \ c(g, x) \in F\}$.

The following proposition collects several basic properties of $\mathrm{U}(F)$. Also, it exemplifies the principle that properties of $\mathrm{U}(F)$ should correspond to properties of the finite permutation group F, which is part of the beauty of the construction.

Proposition 2. *Let $F \leq S_d$. Then the following statements hold.*

(i) $\mathrm{U}(F)$ *is closed in* $\mathrm{Aut}(T_d)$.
(ii) $\mathrm{U}(F)$ *is locally permutation isomorphic to* F.
(iii) $\mathrm{U}(F)$ *is vertex-transitive.*
(iv) $\mathrm{U}(F)$ *is edge-transitive if and only if* F *is transitive.*
(v) *Given legal labellings* l *and* l' *of* T_d*, the groups* $\mathrm{U}_{(l)}(F)$ *and* $\mathrm{U}_{(l')}(F)$ *are conjugate in* $\mathrm{Aut}(T_d)$.

Furthermore, it is immediate from Definition 1 that $\mathrm{U}(F)$ satisfies Tits' Independence Property. More precisely, we have the following.

Proposition 3. *Let $F \leq S_d$. Then $\mathrm{U}(F)^+$ is either trivial or simple. If F is transitive and generated by its point stabilizers, then $\mathrm{U}(F)^+ = \mathrm{U}(F) \cap \mathrm{Aut}(T_d)^+$ and hence $\mathrm{U}(F)^+ \leq \mathrm{U}(F)$ is of index two.*

Here, $\mathrm{U}(F)^+ := \langle \{g \in \mathrm{U}(F) \mid \exists y \in Y : \ gy = y\} \rangle$ is the subgroup of $\mathrm{U}(F)$ generated by edge-stabilizers. It is edge-transitive if and only if F is transitive and generated by its point stabilizers.

Finally, the term "universal" is justified by the following result.

Proposition 4. *Let $G \leq \mathrm{Aut}(T_d)$ be vertex-transitive and locally permutation isomorphic to a transitive permutation group $F \leq S_d$. Then there is a legal labelling l of T_d such that $G \leq \mathrm{U}_{(l)}(F)$.*

Universal groups have come up in the theory of lattices in products of two trees, see [3], but constitute interesting objects of study in themselves, too.

k-closures and Property P_k. Let T denote an infinite and locally finite tree (not necessarily regular) and $B(x,n)$ the ball of radius n centred at vertex x of T.

Definition 5. Let $G \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$. The *k-closure* of G is

$$G^{(k)} := \{h \in \mathrm{Aut}(T) \mid \forall x \in X : \exists g \in G : h|_{B(x,k)} = g|_{B(x,k)}\}.$$

That is, the automorphisms of T that agree on each ball of radius k with some element of G.

In this setting, G is the analogue of F in the definition of $\mathrm{U}(F)$, providing a list of "allowed" actions. Notice also that $G^{(k)}$ is in some sense a "thicker" version of $\mathrm{U}(F)$ in that it has a prescribed local action on bigger balls (when $k > 1$).

Proposition 6. *The k-closure of G has the following basic properties.*

(i) *$G^{(k)}$ is a closed subgroup of $\mathrm{Aut}(T)$.*
(ii) *For every $k, l \in \mathbb{N}$ with $l > k$ we have $G \leq G^{(l)} \leq G^{(k)}$.*
(iii) *$\bigcap_{k\in\mathbb{N}} G^{(k)} = \overline{G}$ (the topological closure of G in $\mathrm{Aut}(T)$).*

Just as $\mathrm{U}(F)$ satisfies Tits' Independence Property (or Property P), the k-closure of G satisfies a "thicker" version of this property.

Definition 7. For any finite or (bi-)infinite path C in T and any $n \in \mathbb{N}$ let C^n be the subtree of T spanned by all vertices at distance at most n from C.

Let $G \leq \mathrm{Aut}(T)$, $k \in \mathbb{N}$ and C be a finite or infinite path in T. Then, for each vertex x of C, the point-wise stabilizer $\mathrm{Fix}_G(C^{k-1})$ of C^{k-1} in G acts on the "subtree rooted at x" (the subtree of T whose vertices are closer to x than to any other vertex of C) and we denote by F_x the permutation group induced by this action. We therefore have a map $\Phi : \mathrm{Fix}_G(C^{k-1}) \to \prod_{x\in C} F_x$ which is clearly an injective homomorphism.

We say that G satisfies *Property P_k* if for every finite or infinite path C the map Φ is an isomorphism.

Notice that when $k = 1$ we recover the original Property P defined by Tits ([5]).

Proposition 8. *Let $G \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$, then $G^{(k)}$ satisfies Property P_k.*

It is almost immediate that this holds when C is an edge, whence it can easily be extended to finite paths. That it holds for (bi-)infinite paths follows from a limiting argument and the fact that $G^{(k)}$ is a closed subgroup of $\mathrm{Aut}(T)$.

Satisfying Property P_k characterizes when the process of taking k-closures stabilizes.

Theorem 9. *The group $G \leq \mathrm{Aut}(T)$ satisfies Property P_k for some k if and only if $G^{(k)} = \overline{G}$.*

More importantly, we deduce the following which will be used when finding infinitely many distinct simple subgroups.

Corollary 10. *There are infinitely many distinct k-closures of G if and only if $\overline{G}$ does not satisfy Property P_k for any k.*

To find simple subgroups we will use an analogous result to Tits' theorem ([5, Théorème 4.5]), with a similar proof. Let $G^{+_k} := \langle \mathrm{Fix}_G(e^{k-1}) \mid e \in Y \rangle$ denote the subgroup of G generated by pointwise stabilizers of "$(k-1)$-thick" edges.

Theorem 11. *Suppose $G \leq \mathrm{Aut}(T)$ does not stabilize a proper non-empty subtree or an end of T, and satisfies Property P_k. Then G^{+_k} is simple (or trivial).*

We have the following recipe to find simple subgroups of Aut(T): start off with some $G \leq \mathrm{Aut}(T)$ which does not stabilize a proper subtree of T, form its k-closures (they all satisfy Property P_k), use Theorem 11 to obtain the simple subgroups $(G^{(k)})^{+_k}$. We still need to ensure that the latter subgroups are non-discrete and different from each other, which will follow from the results below.

Lemma 12. *If $G \leq \mathrm{Aut}(T)$ does not stabilize a proper subtree of T we have*
- (i) *$(G^{(k)})^{+_k}$ is an open subgroup of $G^{(k)}$.*
- (ii) *$(G^{(k)})^{+_k}$ is non-discrete if and only if $G^{(k)}$ is non-discrete.*
- (iii) *$(G^{(k)})^{+_k}$ satisfies Property P_k.*

Theorem 13. *Suppose that $G \leq \mathrm{Aut}(T)$ does not stabilize a proper subtree of T. Then $(G^{(r)})^{+_r} \leq (G^{(k)})^{+_k}$ for every $r \geq k$, with equality if and only if $G^{(r)} = G^{(k)}$.*

Thus, in order to construct infinitely many distinct t.d.l.c. simple non-discrete subgroups of $\mathrm{Aut}(T)$ it suffices to find examples with infinitely many distinct k-closures. By Corollary 10, this amounts to finding examples which do not satisfy Property P_k for any k.

Example 14. The following groups do not satisfy Property P_k for any k.
- (i) $\mathrm{PSL}(2, \mathbb{Q}_p)$ acting on its Bruhat–Tits tree (which is isomorphic to T_{p+1}).
- (ii) $\mathrm{BS}(m,n) := \langle a, t \mid t^{-1}a^m t = a^n \rangle$ (Baumslag–Solitar group) for coprime m, n acting on its Bass–Serre tree (which is isomorphic to T_{m+n}).

We note that this method finds infinitely many t.d.l.c. simple non-discrete groups which are pairwise distinct as subgroups of $\mathrm{Aut}(T)$. It would be desirable to know whether these subgroups are pairwise non-isomorphic. This is stated as work in progress in [1]. Using different methods, Simon Smith has found uncountably many t.d.l.c. simple non-discrete groups which are pairwise non-isomorphic. This was discussed in the talk by C. Reid and G. Willis.

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Simple totally disconnected groups, after Smith

Colin Reid and George Willis

We presented a recent construction ([6]) by Simon M. Smith of a product of permutation groups, which is a modification of the "universal group" construction of Marc Burger and Shahar Mozes ([1]). As with the Burger–Mozes construction, the product is defined as a group of automorphisms of a tree, and under some mild assumptions, the product (or a large subgroup of it) is a simple group, as it satisfies a criterion of Tits ([7]) for simplicity of groups acting on trees. However, the Smith construction is more flexible in that it takes as input two (possibly infinite) permutation groups instead of one, and the resulting product may be locally compact even if one of the input groups was an infinite discrete group. This can be used to produce a large class of examples of simple groups that are totally disconnected, locally compact and non-discrete. In particular, Smith obtains a continuum of non-isomorphic simple groups that are totally disconnected, locally compact, compactly generated and non-discrete. Prior to this work, only countably many examples of groups of this kind were known (see for instance [2]).

By Ol'shanskii ([3],[4]), for each prime $p > 10^{75}$, there are a continuum of non-isomorphic infinite simple groups T (known as *Tarski–Ol'shanskii monsters*) such that every proper non-trivial subgroup of T has order p. One combines T (in its action on the left cosets of one of its non-trivial finite subgroups) with a transitive finite permutation group S generated by point stabilisers (for example, the symmetric group of degree 3), to produce a topological group $T \boxtimes S$ that is totally disconnected, locally compact, compactly generated and non-discrete. Using Serre's property (FA) (see [5]), it is shown that if T and T' are non-isomorphic Tarski–Ol'shanskii monsters, then $T \boxtimes S$ and $T' \boxtimes S$ are non-isomorphic as abstract groups. Interestingly, $T \boxtimes S$ has an open compact subgroup that depends only on p and S, not on the choice of T, so the given examples fall into only countably many local isomorphism classes. In particular, we see that in contrast to the situation in Lie groups, the structure of a totally disconnected, locally compact simple group in a neighbourhood of the identity is very far from determining the global structure.

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Burger-Mozes' simple lattices

Laurent Bartholdi

The purpose of my talk is to present a complete proof (modulo some computer calculations) of the following

Theorem (Burger-Mozes, [BM]). *There exist groups* Γ *that are*

(1) finitely presented;
(2) simple;
(3) torsion-free;
(4) biautomatic;
(5) fundamental groups of non-positively curved 2*-dimensional complexes; hence of cohomological dimension* 2*;*
(6) presentable as amalgams $F *_E F$ *for finitely generated free groups* E, F *and finite-index inclusions of* E *in* F*.*

These groups Γ will appear as lattices in a product of "universal groups" $U(F_v)\times U(F_h)$, for finite permutation groups F_v, F_h; see the talk by Stefan Tornier and Alejandra Garrido, page 2641, in the Arbeitsgemeinschaft.

The main ingredient is a variant of Margulis' "normal subgroup theorem", proven by Burger-Mozes, and which is a special case of the Bader-Shalom "normal subgroup theorem" explained in the talk by Światosław Gal, page 2635:

Theorem 1 (Burger-Mozes, [BM, Corollary 5.1]). *Let* $d_v, d_h \geq 3$ *be integers, let* F_v, F_h *respectively be* 2*-transitive subgroups of* $\mathrm{Sym}(d_v), \mathrm{Sym}(d_h)$*, and let* $\Gamma \leq U(F_v)\times U(F_h)$ *be a cocompact lattice with dense projections:* $\overline{\mathrm{pr}_v(\Gamma)} \supseteq U(F_v)^+$ *and* $\overline{\mathrm{pr}_h(\Gamma)} \subseteq U(F_h)^+$*.*

Then Γ *is just infinite; i.e. all non-trivial normal subgroups of* Γ *have finite index.*

It follows that, if Γ is not residually finite, then $\bigcap_{1\neq N\lhd\Gamma} N$ is simple: it is again a lattice to which Theorem 1 applies, and has no non-trivial normal subgroup.

The second ingredient is a step-by-step construction of a non residually finite lattice. This can be done by pure thought, as in the original article; but I have rather explained tricks from combinatorial group theory when they were simple, and relied on Rattaggi's method of computer calculation, when they were not.

1. Lattices in products of trees

Consider two regular trees T_v, T_h of degree d_v, d_h respectively, and a group Γ acting freely and transitively on $T_v \times T_h$.

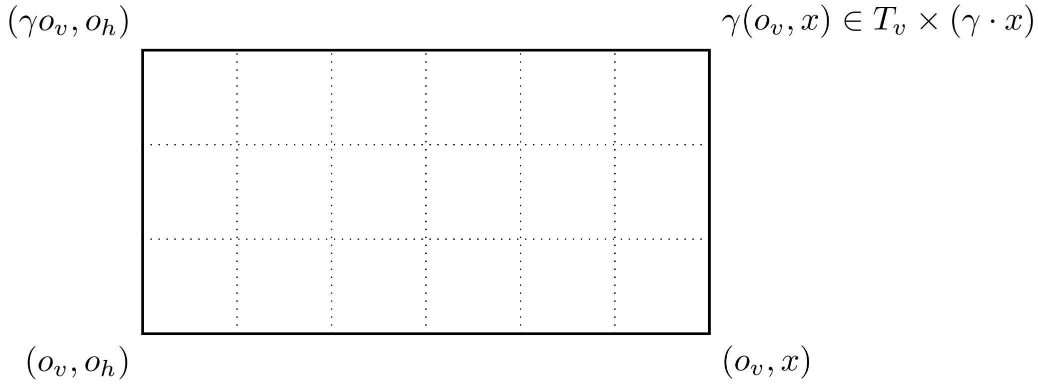
On the one hand, Γ may then be described as the fundamental group of the geometric object $\Gamma\backslash(T_v \times T_h)$. This is a 2-complex consisting of a single vertex; a collection of "vertical" and "horizontal" oriented loops at that vertex, respectively written S_v, S_h, and in bijection with the Γ-orbits of edges in T_v, T_h; and a collection of "squares" whose perimeter reads a vertical, horizontal, vertical, horizontal edge in sequence. The squares' labels are such that, for every $(s_v, s_h) \in S_v^{\pm 1} \times S_h^{\pm 1}$, there exists a single corner of a square two-cell at which the two incident edges carry the labels s_v, s_h with correct orientation. In particular, there is a single zero-cell, $(d_v + d_h)/2$ geometric one-cells, and $d_v d_h/4$ two-cells. Algebraically, this amounts to a presentation

$$\Gamma = \langle S_v \sqcup S_h \mid \text{relations } s_v s_h s'_v s'_h \rangle. \tag{1}$$

Fix a basepoint $(o_v, o_h) \in T_v \times T_h$, and consider

$$\Gamma_v = \big\{\gamma \in \Gamma : \gamma(T_v \times \{o_h\}) = T_v \times \{o_h\}\big\},$$

the stabilizer of o_h in the action of Γ on T_h. On the one hand, Γ_v is a group acting freely on T_v, and therefore is a free group. In fact, it is the subgroup of Γ generated by S_v, of rank $d_v/2$, and T_v may be identified with the Cayley graph of Γ_v. On the other hand, Γ_v acts on T_h fixing the basepoint o_h. This action may be described quite concretely, in terms of the presentation of Γ, as follows. Given $\gamma \in \Gamma_v$ and $x \in T_h$, consider the paths from o_v to γo_v in T_v and from o_h to x in T_h; these are the left and bottom edges of a unique rectangle in $T_v \times T_h$:



By the "corner" condition on the complex $\Gamma\backslash(T_v \times T_h)$, there is a single way of filling in this rectangle with labeled squares, and the top label gives the image of x under γ. Naturally the same considerations lead to a subgroup $\Gamma_h = \langle S_h \rangle$ acting on the rooted tree (T_v, o_v).

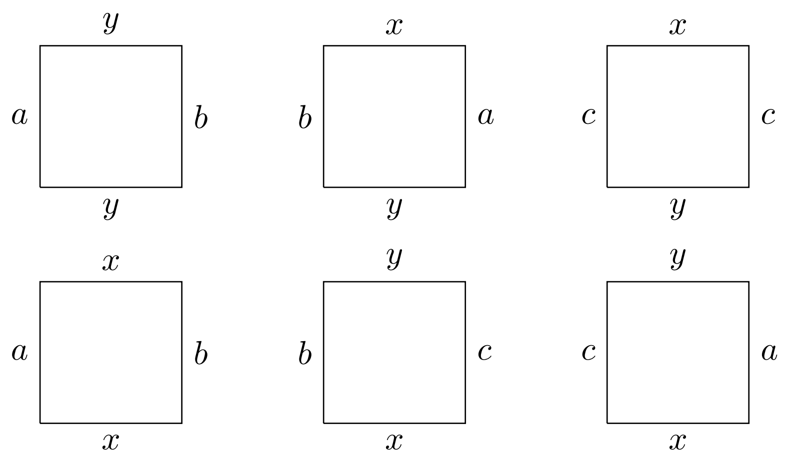
It is in general difficult to check whether a lattice, given e.g. by its presentation (1), has dense projections. However, the actions on rooted trees of Γ_v, Γ_h are computable, and this leads to a computable, sufficient condition:

Lemma 2 ([BM, Proposition 5.2]). *Assume in the notation of Theorem 1 that the groups F_v, F_h are primitive, with simple, non-abelian point stabilizers $\cong L_v, L_h$ respectively. Consider the action of Γ_v on the ball of radius 2 around o_h; choose an edge e at o_h and let K_h denote the fixator of the ball of radius 1 around e. Define similarly K_v. Then either $(K_v, K_h) = (1, 1)$, or $(K_v, K_h) \cong (L_v^{d_v-1}, L_h^{d_h-1})$ and the projections of Γ to $U(F_v)^+, U(F_h)^+$ are both dense.*

The proof of this lemma is difficult (for the speaker), and is essentially a variant of the Thompson-Wielandt theorem (asserting that, if a group acts transitively on a graph with finite vertex stabilizers, then the fixator of a ball of radius 1 must be a p-group).

2. A nonseparable lattice

We now construct explicitly some lattices. The first step is to produce a lattice Γ' that is not "subgroup separable", namely there exist a subgroup Δ and $g \in \Gamma' \setminus \Delta$ such that, in every finite quotient of Γ', the image of g belongs to the image of Δ. The lattice is given by $S'_v = \{a, b, c\}^{\pm 1}$, $S'_h = \{x, y\}^{\pm 1}$, and squares

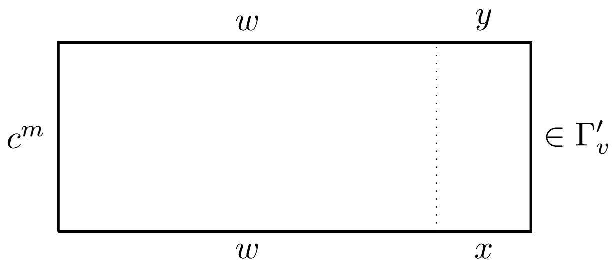


We first claim that the element $c \in \Gamma'_v$ acts transitively on $\{x, y\}^n \subset T_h$ for all $n \in \mathbb{N}$. This is proven by induction, using the following criterion: the action of c on $\{x, y\}^n$ must have odd signature for all $n \geq 1$. Indeed then, since $\{x, y\}^{n-1}$ is a single $\langle c \rangle$-orbit, there exists $w \in \{x, y\}^{n-1}$ such that $c \cdot wx = wy$ so that $\{x, y\}^n$ is a single orbit. For $\gamma \in \Gamma'_v$, let $\sigma_n(\gamma)$ denote the signature of the action of γ on $\{x, y\}^n$. Now the squares above give the relations

$$\sigma_n(a) = \sigma_{n-1}(b)^2 = 1, \quad \sigma_n(b) = \sigma_n(c) = \sigma_{n-1}(c)\sigma_{n-1}(a), \quad \sigma_1(c) = -1$$

from which $\sigma_n(c) = -1$ for all $n \geq 1$.

We then claim that, for all $m \geq 1$, the elememt $x^{-1}y$ belongs to $\Gamma_v \langle c^m \rangle^{\Gamma'}$. Indeed, since c acts transitively on $\{x, y\}^m$ for all m, it has infinite order; so there exists $w \in \{x, y\}^*$ such that $c^m \cdot wx = wy$. Consider the corresponding rectangle in $T_v \times T_h$:



It gives in Γ' the relation $(c^m)^{wx} x^{-1} y \in \Gamma'_v$, as desired.

Lemma 3 (Wise, [Wis, Corollary 6.4])**.** *For every homomorphism $\pi \colon \Gamma' \to Q$ to a finite group, $\pi(x^{-1}y) \in \pi(\Gamma'_v)$.*

Proof. Since Q is finite, there exists $m \geq 1$ such that $\pi(c^m) = 1$; then $\pi(x^{-1}y) \in \pi(\Gamma_v \langle c^m \rangle^{\Gamma'}) = \pi(\Gamma_v)$. □

3. A non residually finite lattice

We next construct a lattice that is not residually finite. It has presentation

$$\Gamma'' = \langle S'_v \sqcup S'_h \sqcup \overline{S'_h} \mid \text{two copies of the squares from } \Gamma' \rangle.$$

Note the following automorphism of Γ'': it fixes S'_v, and exchanges the two copies $S'_h, \overline{S'_h}$ by $x \leftrightarrow \overline{x}, y \leftrightarrow \overline{y}$. Its fixed point set is precisely Γ''_v. The claim follows from

Lemma 4 (Long-Niblo [LN])**.** *Let G be a residually finite group, and let θ be an automorphism of G. Then $\mathrm{Fix}(\theta)$ is separable in G.*

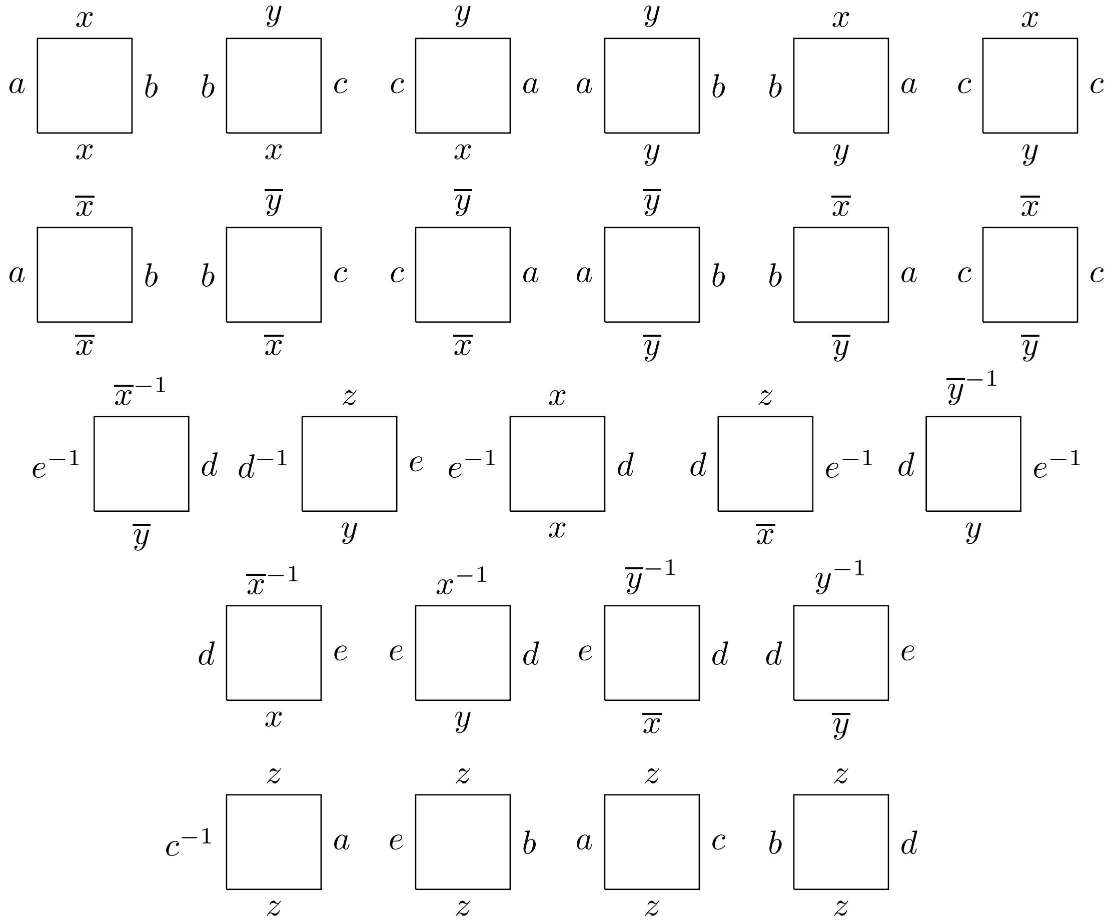
Proof. Choose $g \in G \setminus \mathrm{Fix}(\theta)$; so $g^{-1}\theta(g) \neq 1$. Thus there exists $\pi \colon G \to Q$ with Q finite and $\pi(g^{-1}\theta(g)) \neq 1$. Define $\phi \colon G \to Q \times Q$ by $\phi(g) = (\pi(g), \pi(\theta g))$. Note then that $\mathrm{Fix}(\theta)$ maps to the diagonal of $Q \times Q$, while g does not. □

In fact, we obtain a bit more than non-residual finiteness: there is a specific element $y^{-1} x \overline{x}^{-1} y \in \Gamma''_h$ that belongs to every finite-index subgroup of Γ''.

4. A simple lattice

We finally imbed Γ'' in a larger complex Γ of degrees $(d_v, d_h) = (10, 10)$, imbedding in $U(A_{10}) \times U(A_{10})$. Many examples are possible, but we content ourselves with a single one, constructed by Rattaggi [Rat]. Set $S_v = S'_v \sqcup \{d, e\}$ and $S_h =$

$S'_h \sqcup \overline{S'_h} \sqcup \{z\}$; the first two rows of squares are those of Γ'':



It is easy for a computer to check that Γ_v, Γ_h act by A_{10} on 1-balls, and that the finite groups K_v, K_h defined in Lemma 2 are non-trivial; so that both projections of Γ in $U(A_{10})$ have dense image in $U(A_{10})^+$, so that Γ is just infinite by Theorem 1.

The group Γ cannot be simple: there is always a homomorphism

$$\sigma\colon \Gamma \to \{\pm 1\} \times \{\pm 1\}, \quad S_v \mapsto (-1,1), \quad S_h \mapsto (1,-1).$$

Set $\Gamma_0 = \ker(\sigma)$.

Proof of the main theorem. Set $\Gamma_1 = \bigcap_{1 \neq N \triangleleft \Gamma} N$. Then Γ_1 is non-trivial, because it contains $w = y^{-1} x \overline{x}^{-1} y$, so it is simple and has finite index in Γ. A computer algebra program such as GAP can compute the normal closure of w in Γ, and check that it has index 4 in Γ, whence coincides with Γ_0. Therefore $\Gamma_0 = \Gamma_1 = \langle w \rangle^\Gamma$ so Γ_0 is simple.

To see that Γ_0 is torsion-free, it suffices to note that it is the fundamental group of a complex with contractible universal cover $T_v \times T_h$.

The decomposition of Γ_0 as amalgam comes from its action on T_h with fundamental domain an edge. The group F is in fact Γ_v, and E is the stabilizer of an edge touching o_h in T_h. $\square$

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The structure lattice, part I

John Wilson

The structure lattice $\mathcal{LN}(G)$ of a totally disconnected locally compact group G is the quotient of the family $\mathrm{LN}(G)$ of compact subgroups with open normalizer modulo the equivalence relation of commensurability. It is an analogue of the structure lattice of a just infinite group; however the latter lattice is automatically Boolean, because centralizers are plentiful and supply complements in the lattice. Basic properties of $\mathcal{LN}(G)$ were discussed, together with conditions ensuring that a certain subset $\mathcal{LC}(G)$ of $\mathcal{LN}(G)$ is a Boolean lattice; the elements of $\mathcal{LC}(G)$ are those containing subgroups $\mathrm{C}_U(H)$ with $H \in \mathcal{LN}(G)$ and with U compact and open. The proof was sketched that if G is compactly generated and topologically simple then $\mathcal{LN}(G)$ contains no non-trivial abelian subgroups and G has no non-trivial elements with open subgroups. It was also explained why that the only compactly generated and topologically characteristically simple groups for which these two properties do not hold are the obvious ones.

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Abstract quotients of profinite groups, after Nikolov and Segal, part I

Benjamin Klopsch

In my talk I presented and discussed results of Nikolay Nikolov and Dan Segal on abstract quotients of compact Hausdorff topological groups, paying special attention to the class of finitely generated profinite groups.

In [5], Nikolov and Segal streamline and generalise their earlier results [3, 4] which led to the solution of a problem raised by Jean-Pierre Serre [7, I.§4.2].

Theorem 1 (Nikolov, Segal [3])**.** *Let G be a finitely generated profinite group. Then every abstract finite-index subgroup H of G is necessarily open in G.*

Serre proved this assertion in the special case, where G is a finitely generated pro-p group for some prime p, by a neat and essentially self-contained argument. The proof of the general theorem is considerably more involved and makes use, for instance, of the Classification of Finite Simple Groups; the same is true for several of the results stated below. We refer to the survey article [8] for a discussion of the background to Serre's problem and further information on finite-index subgroups and verbal subgroups in profinite groups.

The key theorem in [5] concerns normal subgroups in finite groups. For a finite group Γ, let $d(\Gamma)$ denote the minimal number of generators of Γ, write Γ' for the derived subgroup of Γ and set

$$\begin{aligned}\Gamma_0 &= \bigcap\{T \trianglelefteq \Gamma \mid \Gamma/T \text{ almost-simple}\}\\ &= \bigcap\{\mathrm{C}_G(M) \mid M \text{ a non-abelian simple chief factor}\},\end{aligned}$$

where H is almost-simple if $S \trianglelefteq H \leq \mathrm{Aut}(S)$ for some non-abelian finite simple group S. For $X \subseteq \Gamma$ and $f \in \mathbb{N}$ we write $X^{*f} = \{x_1 \cdots x_f \mid x_1, \ldots, x_f \in X\}$.

Theorem 2 (Nikolov, Segal [5])**.** *Let Γ be a finite group and $\{y_1, \ldots, y_r\} \subseteq \Gamma$ a symmetric subset, i.e. a subset that is closed under taking inverses. Let $H \trianglelefteq \Gamma$.*

(1) *If $H \subseteq \Gamma_0$ and $H\langle y_1, \ldots, y_r\rangle = \Gamma'\langle y_1, \ldots, y_r\rangle = \Gamma$ then*

$$\langle [h, g] \mid h \in H, g \in \Gamma\rangle = \{[h_1, y_1] \cdots [h_r, y_r] \mid h_1, \ldots, h_r \in H\}^{*f},$$

where $f = f(r, d(\Gamma)) = O(r^6 d(\Gamma)^6)$.

(2) *If $\Gamma = \langle y_1, \ldots, y_r\rangle$ then the conclusion in* (1) *holds without assuming $H \subseteq \Gamma_0$ and with better bounds on f.*

While the proof of the key theorem is rather involved, the basic underlying idea is simple to sketch. Suppose that $\Gamma = \langle g_1, \ldots, g_r\rangle$ is a finite group and M a non-central chief factor. Then the set $[M, g_i] = \{[m, g_i] \mid m \in M\}$ must be 'relatively large' for at least one generator g_i. Hence $\prod_{i=1}^r [M, g_i]$ is 'relatively large'. In order to transform this observation into a rigorous proof of Theorem 2 one employs a combinatorial result, discovered by Timothy Gowers, which – informally speaking – comes down to the following: to show that a finite group is equal to a product of some of its subsets, it suffices to know that the cardinalities of these subsets are 'sufficiently large'. For a precise statement see [1, Corollary 2.6].

By standard compactness arguments, Theorem 2 yields a corresponding result for normal subgroups of finitely generated profinite groups. For a profinite group G, let $d(G)$ denote the minimal number of topological generators of G, write G' for the abstract derived subgroup of G and set

$$G_0 = \bigcap\{T \trianglelefteq G \mid T \text{ open in } G \text{ and } G/T \text{ almost-simple}\}.$$

For $X \subseteq G$ and $f \in \mathbb{N}$ we write $X^{*f} = \{x_1 \cdots x_f \mid x_1, \ldots, x_f \in X\}$ as before. The topological closure of X in G is denoted by $\overline{X}$.

Theorem 3 (Nikolov, Segal [5])**.** *Let G be a profinite group and $\{y_1, \ldots, y_r\} \subseteq G$ a symmetric subset. Let $H \trianglelefteq G$ be a closed normal subgroup.*

(1) *If* $H \subseteq G_0$ *and* $H\overline{\langle y_1, \ldots, y_r \rangle} = \overline{G'\langle y_1, \ldots, y_r \rangle} = G$ *then*

$$\langle [h, g] \mid h \in H, g \in G \rangle = \{[h_1, y_1] \cdots [h_r, y_r] \mid h_1, \ldots, h_r \in H\}^{*f},$$

where $f = f(r, d(G)) = O(r^6 d(G)^6)$.

(2) *If* $y_1, \ldots, y_r$ *topologically generate* G *then the conclusion in* (1) *holds without assuming* $H \subseteq G_0$ *and better bounds on* f.

In particular, the theorem shows that, if G is a finitely generated profinite group and $H \trianglelefteq G$ a closed normal subgroup, then the group $[H, G] = \langle [h, g] \mid h \in H, g \in G \rangle$ is closed. Thus G' and more generally all terms $\gamma_i(G)$ of the abstract lower central series of G are closed; these consequences were already established in [3].

Furthermore, one obtains the following tool for studying abstract normal subgroups of a finitely generated profinite group G, reducing certain problems more or less to the abelian profinite group G/G' or the profinite group G/G_0 which is semisimple-by-(soluble of bounded derived length).

Corollary 4 (Nikolov, Segal [5]). *Let* G *be a finitely generated profinite group and* $N \trianglelefteq G$ *an abstract normal subgroup. If* $NG' = NG_0 = G$ *then* $N = G$.

Using Corollary 4 and features of products of powers in non-abelian finite simple groups (cf. [2, 6]), it is not difficult to derive Theorem 1. Moreover, the methods developed in [5] lead to new consequences for abstract quotients of finitely generated profinite groups and, more generally, compact Hausdorff topological groups.

Every compact Hausdorff topological group G is an extension of a compact connected group G°, its identity component, by a profinite group G/G°. By the Levi–Mal'cev Theorem, the connected component G° is essentially a product of compact Lie groups and thus relatively tractable.

We conclude by stating two results whose proof requires the new machinery developed in [5], as they were not covered by the methods used in [3, 4].

Theorem 5 (Nikolov, Segal [5]). *Let* G *be a compact Hausdorff topological group. Then every finitely generated abstract quotient of* G *is finite.*

Theorem 6 (Nikolov, Segal [5]). *Let* G *be a compact Hausdorff topological group such that* G/G° *is topologically finitely generated. Then* G *has a countably infinite abstract quotient if and only if* G *has an infinite virtually-abelian continuous quotient.*

A basic example, illustrating these theorems is the additive group $(\mathbb{Z}_p, +)$ of p-adic integers: as an abstract group it does not map onto $(\mathbb{Z}, +)$ but onto $(\mathbb{Q}, +)$.

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Contraction groups and the scale

Phillip Wesolek

The primary resource for this talk is the work of U. Baumgartner and G. Willis [1]. Our discussion closely follows their paper. We cover the basic properties of contraction groups, the relation with the scale function, the closure of the contraction group, and an application. In the following discussion G always denotes a totally disconnected locally compact (t.d.l.c.) group and $\mathrm{Aut}(G)$ the collection of topological group isomorphisms. (We always assume our groups are Hausdorff.)

We begin by defining the contraction subgroup.

Definition 1. Let $\alpha \in \mathrm{Aut}(G)$. The **contraction group** associated to α is defined to be

$$\mathrm{con}(\alpha) := \{x \in G \mid \alpha^n(x) \to e \text{ as } n \to \infty\}.$$

It is easy to see $\mathrm{con}(\alpha)$ is a subgroup; it is, however, not in general closed.

Taking $U \leq G$ a compact open subgroup and $\alpha \in \mathrm{Aut}(G)$, we put

$$U_s := \bigcap_{i \in \mathbb{Z}} \alpha^i(U).$$

We then obtain the following:

Theorem 2. *If U is a compact open subgroup of G tidy for α, then $U_{--} = \mathrm{con}(\alpha)U_s$.*

From Theorem 2, we derive the following interesting relationship between the contraction group and the scale function.

Theorem 3. *Let $\alpha \in \mathrm{Aut}(G)$. Then $\mathrm{con}(\alpha)$ is relatively compact if and only if $s_G(\alpha^{-1}) = 1$.*

It is enlightening to note a consequence of the above: Suppose $\alpha \in \mathrm{Aut}(G)$ is non-unimodular. Say $\Delta(\alpha) > 1$. It follows $s(\alpha^{-1}) \neq 1$, hence $\mathrm{con}(\alpha)$ is not relatively compact.

We next show the closure of the contraction group again has a nice structure.

Definition 4. Let G be a t.d.l.c. group and $\alpha \in \mathrm{Aut}(G)$. The **nub** group is defined to be

$$\mathrm{nub}(\alpha) := \bigcap \{V \mid V \text{ is tidy for } \alpha\}.$$

Theorem 5. $\overline{\mathrm{con}(\alpha)} = \mathrm{con}(\alpha)\,\mathrm{nub}(\alpha)$.

We conclude by presenting an application of contraction groups. This application is due to P-E. Caprace, C. Reid, and Willis [2].

Definition 6. Let G be a t.d.l.c. group. The **Tits core** of G is defined to be

$$G^\dagger := \langle \overline{\mathrm{con}(g)} \mid g \in G \rangle.$$

It turns out $\mathrm{Aut}^+(\mathcal{T}_n)$ for $\mathcal{T}_n$ the n-regular tree and $G^+(k)$ for G a simple isotropic k-algebraic group with k a non-archimedean field are exactly the respective Tits cores.

We then note two interesting properties of the Tits core.

Theorem 7. *Let G be a t.d.l.c. group and D a dense subgroup. If $G^\dagger$ normalizes D, then $G^\dagger \leq D$.*

Corollary 8. *Let G be a t.d.l.c. group. If G is topologically simple, then $G^\dagger$ is abstractly simple.*

We lastly note two open questions.

Question 9. Does every compactly generated t.d.l.c. group that is topologically simple and non-discrete have at least one non-trivial contraction group?

A positive answer to the following question would imply a negative answer to the above. Recall $g \in G$ is **periodic** if $\langle g \rangle$ is relatively compact; this is the topological analogue of torsion.

Question 10. Is there a compactly generated t.d.l.c. group that is topologically simple, non-discrete, and periodic? That is to ask, do topological Tarski monster groups exist?

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The centraliser lattice

David Hume and Thierry Stulemeijer

1. Introduction

In this talk, we review the definition of the centraliser lattice $\mathcal{LC}(G)$ of a totally disconnected locally compact (t.d.l.c.) group, and explain why it is a Boolean algebra. We then study the action of G on Ω, the associated Stone space of $\mathcal{LC}(G)$, and prove that under certain hypotheses on G, it is continuous, minimal, strongly proximal and weakly branch. Moreover, Ω satisfies a universal property in the category of profinite G-spaces that are weakly branch.

2. The centraliser lattice as a Boolean algebra

The **centraliser lattice** is a subset of the structure lattice $\mathcal{LN}(G)$ defined in a previous talk. We first recall the definition of $\mathcal{LN}(G)$.

Definition 1.

(1) A subgroup $K \leq G$ is called **locally normal** if it is compact and normalised by an open subgroup of G.
(2) Two subgroups H, K of G are **locally equivalent** if there exists a compact open subgroup U of G such that $H \cap U = K \cap U$, or equivalently if $H \cap K$ has finite index in both H and K.
(3) The set of all local equivalence classes having a locally normal representative is called the **structure lattice** of G, and is denoted by $\mathcal{LN}(G)$.

So, informally, $\mathcal{LN}(G)$ is the lattice of subgroups that are compact and normal in an open subgroup, everything happening 'up to finite index'. As explained in a previous talk, this is a lattice in a natural way. Before defining the centraliser lattice, it will be more intuitive to first define local decomposition lattices.

Definition 2.

(1) Given a topological group H and a subgroup K, say K is an **almost direct factor** of H if there is a closed subgroup L of H of finite index such that K is a direct factor of L.
(2) Let $\alpha \in \mathcal{LN}(G)$, say $\alpha = [H]$, where H is locally normal. We define the **local decomposition lattice** $\mathcal{LD}(G; H)$ of G at H to be the subset of $\mathcal{LN}(G)$ consisting of elements $[K]$ where K is locally normal in G and K is an almost direct factor of H.

Note that a natural candidate for the complement of a locally normal subgroup is to take its centraliser. Lemma 4 shows that this map is well defined 'up to finite index', whenever H is **C-stable**.

Definition 3. Let G be a topological group, and $H \leq G$ be a subgroup.

(1) We define the **quasi-centraliser** of H in G, denoted by $QC_G(H)$, to be the subgroup of G consisting of those elements that centralise an open subgroup of H.
(2) H is called **C-stable** if $QC_G(H) \cap QC_G(C_G(H)) \cap U$ is trivial, for all open compact subgroup U of G.

Lemma 4. *Let G be a t.d.l.c. group and let K be a locally normal subgroup which is C-stable. Then for all open compact subgroups U of G, we have*

$$[C_U(K \cap U)] = [QC_G(K)]$$

Note that the C-stability condition can now be rephrased in the following more intuitive way :

$$H^{\perp} \wedge (H^{\perp})^{\perp} = 0$$

Using this complementation, we obtain the following result.

Lemma 5. *Let G be a t.d.l.c. group and let $\alpha \in \mathcal{LN}(G)$ have a C-stable representative. Then $\mathcal{LD}(G;\alpha)$ is a sublattice of $\mathcal{LN}(G)$ and every $\beta \in \mathcal{LD}(G;\alpha)$ has a locally normal C-stable representative. Furthermore, $\mathcal{LD}(G;\alpha)$ is internally a Boolean algebra (relative to the maximum α), with complementation map $\perp: [K] \to [QC_G(K)] \wedge \alpha$.*

We finally arrive at the definition of the **centraliser lattice**, which is a Boolean algebra whenever G is **locally C-stable**.

Definition 6. A t.d.l.c. group G is called **locally C-stable** if $QC_G(G) = \{1\}$ and if all locally normal subgroup are C-stable. Or equivalently, if $QC_G(G) = \{1\}$ and the only abelian locally normal subgroup is the trivial one.

Definition 7. Let G be a locally C-stable t.d.l.c. group. As above, define the map $\perp: \mathcal{LN}(G) \to \mathcal{LN}(G) : \alpha \to [QC_G(\alpha)]$. This is well-defined by Lemma 4. The centraliser lattice $\mathcal{LC}(G)$ is defined to be the set $\{\alpha^\perp | \alpha \in \mathcal{LN}(G)\}$ together with the map $\perp$ restricted to $\mathcal{LC}(G)$, the partial order inherited from $\mathcal{LN}(G)$ and the binary operations $\wedge_c$ and $\vee_c$ given by:

$$\alpha \wedge_c \beta = \alpha \wedge \beta$$
$$\alpha \vee_c \beta = (\alpha^\perp \wedge \beta^\perp)^\perp$$

Theorem 8. *Let G be a locally C-stable t.d.l.c. group. The poset $\mathcal{LC}(G)$ is a Boolean algebra and $\perp^2: \mathcal{LN}(G) \to \mathcal{LC}(G)$ is a surjective lattice homomorphism.*

3. Topological dynamics

In this section we concentrate on the dynamics of the action of a group G in the class $\mathcal{S}$ — compactly generated, topologically simple, non-discrete, totally disconnected locally compact groups — on the Stone space Ω of its centraliser lattice $\mathcal{LC}(G)$.

We recall that the Stone representation theorem assigns a profinite space $\Omega(\mathcal{B})$ to any Boolean algebra $\mathcal{B}$ where elements of $\mathcal{B}$ are in $1-1$ correspondence with clopen subsets of $\Omega(\mathcal{B})$.

As soon as the centraliser lattice is non-trivial this action is faithful. Moreover, the action of G on Ω shares many dynamical characteristics with the action of $\mathrm{Aut}(T)^+$ on the boundary of the regular tree T. In particular, the boundary of T is naturally homeomorphic to the Stone space of the centraliser lattice of $\mathrm{Aut}(T)^+$. The purpose of the talk was to highlight those similarities in order to illuminate the following theorem.

Theorem 9. [2, Theorem F]

(1) The G-action on Ω is continuous, minimal, strongly proximal and weakly branch; moreover, Ω contains a compressible clopen subset.

(2) Given a profinite space X with a continuous G-action, the G-action on X is weakly branch if and only if there is a continuous G-equivariant surjective map $\Omega \to X$. In particular, every weakly branch G-action is minimal and strongly proximal.

Let v be a vertex in T and let T' be a component of $T \setminus v$. The boundary of T' is a clopen subset $V \subset \partial_\infty T$ and the subgroup H of $\mathrm{Aut}(T)^+$ which fixes $\partial_\infty T \setminus V$ pointwise represents a class in $\mathcal{LD}(\mathrm{Aut}(T)^+)$. Let $g \in \mathrm{Aut}(T)^+$ be a hyperbolic isometry such that the repelling fixed point η_- of g in $\partial_\infty T$ lies in V. Notice that for each n, $[g^{-n}Hg^n]$ defines a strictly smaller element of the centraliser lattice and in the Stone space these clopen subsets converge in the weak-$*$ topology to the singleton $\{\eta_-\}$ (V is *compressible* and the action is *strongly proximal*).

Using the above we can also deduce that the action is *minimal* — all orbits are dense — and *weakly branch* — the pointwise stabiliser of any proper clopen subset is non-trivial.

The first part of the above theorem states that these phenomena hold whenever $G \in \mathcal{S}$, while the second part states that any weakly branch action is a 'quotient' of the action of G on Ω.

One important distinction between what is currently known about the dynamics of $\mathrm{Aut}(T)^+$ and a general $G \in \mathcal{S}$ is highlighted by the following corollary.

Corollary 10. [2, Corollary H] *Let $G \in \mathcal{S}$ and suppose $\mathcal{LC}(G) \neq \{0, \infty\}$. Then G contains a non-Abelian discrete free subsemigroup.*

While $\mathrm{Aut}(T)^+$ has North-South dynamics — hyperbolic isometries have attracting and repelling fixed points — so a standard ping-pong argument finds non-Abelian free subgroups, for a general $G \in \mathcal{S}$ it is currently only possible to find isometries which are attracting, hence we obtain free subsemigroups. It is therefore appropriate — as mentioned after the talk by Rémi Coulon — to say that the topological dynamics of such groups may resemble the action of a Baumslag–Solitar group more closely than the action of $\mathrm{Aut}(T)^+$. However, by [2, Corollary G] a group $G \in \mathcal{S}$ with non-trivial centraliser lattice cannot be amenable.

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Abstract quotients of profinite groups and applications, after Nikolov and Segal, part II

Jakub Gismatullin

My talk was devoted to explaining recent results of Nikolay Nikolov and Dan Segal. I also described how their results might be applied to some problems in group theory.

Let us start with some definitions. An arbitrary group G (not necessarily profinite) can be regarded as a topological group with the *profinite topology*, that is, a topology having as a basis of open subsets *all* cosets of normal subgroups of finite index. G with the profinite topology may not be compact. This topology is

Hausdorff if and only if G is residually finite. The *profinite completion* $\widehat{G}$ *of* G is the completion of G with respect to this topology. That is, $\widehat{G}$ is the inverse limit

$$\widehat{G} = \varprojlim (G/P)$$

of the system $\{(G/P, G/P \to G/Q) : P, Q \lhd G, P < Q, [G:P], [G:Q] < \infty\}$. Another way of constructing $\widehat{G}$ is to consider the closure $\widehat{i(G)}$ of $i(G)$ in $\prod_{P \lhd G, [G:P] < \infty} G/P$, where i is the diagonal map.

Let G be a profinite group. What happens if we construct the profinite completion $\widehat{G}$ of G itself? When the inclusion map from G to $\widehat{G}$ is bijective? Could $\widehat{G}$ be strictly bigger that G? According to Serre [8], this question has the following reformulation: when all subgroups of finite index in G are open? There are examples of profinite groups, with non-open finite index subgroups (e.g. $\mathbb{Z}/2\mathbb{Z}^{\aleph_0}$ has $\aleph_0$ open subgroups, but $2^{2^{\aleph_0}}$ subgroups of index 2). Such examples are not topologically finitely generated. Therefore, we assume that G is topologically finitely generated. Serre himself proved [8] that the answer is positive when G is a topologically finitely generated pro-p-finite group. In 2003 N. Nikolov and D. Segal answered [5] Serre's question in the positive: In every finitely generated profinite group G, every finite index subgroup is open.

Here is another translation of their result: A finitely generated profinite group G has no *strange* finite quotients (by a *strange* quotient of G we mean G/P, for non-closed $P \lhd G$). As a consequence one can prove that: A residually finite image of a finitely generated profinite group G is either finite or uncountable. Notice that a finitely generated profinite group can have a countable infinite image (for example $\mathbb{Q}$ is the image of $\mathbb{Z}_2$). Therefore, it is natural to consider the following question (from Blaubeuren Conference in 2007):

(A) Is it possible for a topologically finitely generated profinite group to have a finitely generated infinite homomorphic image?

In addition to the question (A), I would like to propose the following questions (B) and (C) (see below for the relationship of (C) with the class of (weak) sofic groups).

(B) What are possible homomorphic images of (topologically finitely generated) profinite groups?

(C) Is it possible to embed every finitely generated group as a subgroup of a quotient of a topologically finitely generated profinite group?

Let us stress the following fact. If a finitely generated group H is a subgroup of a quotient of some profinite group, then one can prove [2] that H is also a subgroup of a quotient of topologically *finitely generated* profinite group. Hence, without loss of generality the assumption that a profinite group is topologically finitely generated can be removed from (C).

The following results from [6, 7] give the answer to (A).

(1) Let G be a compact Hausdorff group, $N \lhd G$ and suppose that G/N is finitely generated. Then G/N is finite. [6, 1.13]

(2) Let G be a topologically finitely generated profinite group.

(a) If $N \lhd G$ and G/N is countably infinite, then G/N has an infinite virtually abelian quotient.
(b) G has countably infinite quotient iff some open subgroup of G has infinite abelianization.

The key to proving the above results are some theorems on finite groups. Then, using some standard transfer arguments, one can obtain versions for profinite groups. As an example we state the following achievement from [6].

We need some notation. For a group G, elements $a, b \in G$ and $A, B \subseteq G$ we use the following notation: $A \cdot B = \{ab : a \in A, b \in B\}$, $a^b = b^{-1}ab$, $A^B = \bigcup_{a \in A, b \in B} a^b$, $A^n = \underbrace{A \cdot \ldots \cdot A}_{n \text{ times}}$ and $[a, b] = a^{-1}b^{-1}ab$. By $[H, a]$ we denote ${a^{-1}}^H \cdot a = \{[h, a] : h \in H\}$. Recall that $[H, G]$ is the subgroup of G generated by $\{[h, g] : h \in H, g \in G\}$.

Theorem 1. [6, Theorems 1.2 and 1.6] *Let G be a finite (profinite) group and $\{a_1, \ldots a_n\}$ be a symmetric generating set (topological generating set resp.) for G. Then for some constant $M = M(n) \in \mathbb{N}$, for every normal subgroup (closed normal subgroup resp.) H of G,*

$$[H, G] = \left(\prod_{r=1}^{n} [H, a_r] \right)^M,$$

where $M = M(n)$ is $O(n^3)$.

Let us sketch some applications of Theorem 1. This theorem can be generalized to arbitrary group G with the profinite topology (assuming topological finite generation) in the following way. For $S \subseteq G$ we denote by $\overline{S}$ the closure of S in G in the profinite topology.

Theorem 2. [2] *Suppose G is an arbitrary group and $\{a_1, \ldots a_n\} \subseteq G$ is a symmetric subset which generates a dense subgroup of G with respect to the profinite topology. Then for every normal subgroup $H \lhd G$*

$$\overline{[H, G]} = \overline{\left(\prod_{r=1}^{n} [H, a_r] \right)^M} = \overline{\left(\prod_{r=1}^{n} {a_r^{-1}}^H \cdot {a_r}^H \right)^M},$$

where $M = M(n)$ is the constant from Theorem 1 (the second equality is because of $[H, a][H, a^{-1}] = {a^{-1}}^G a a^G a^{-1} = {a^{-1}}^G \cdot a^G$).

As a corollary of 2 one can prove the following fact related to Stallings Conjecture from [9]. It is well known that every conjugacy class in a finitely generated free group is closed in the profinite topologies. Stalings, Glebsky and Rivera [4] asked whether in a free group $\mathbb{F}_n$, any product of finitely many conjugacy classes is closed in the profinite topology. Using machinery from [1] we obtain the following negative answer not only for free groups.

Corollary 3. [3] *Suppose G is a group from the following list of groups: free groups, torsion free hyperbolic groups, right angled Artin groups, pure braid groups,* *commutator subgroups in a right angled Coxeter groups. Let* $\overline{b} = \{b_1, \ldots, b_m\} \subseteq G$ *and* $G_1 = \langle b_1, \ldots, b_m \rangle$. *Take* $H \lhd G$ *such that* $[H \cap G_1, G_1]$ *is nontrivial. Then* $\left(\prod_{r=1}^{m} {b_r^{-1}}^H \cdot {b_r}^H\right)^{M(m)}$ *is not closed the profinite topology on* G *(where* $M(m)$ *is from 1).*

Another application comes from the theory of (weak) sofic groups. There is notion of a *sofic group*. Several important conjectures in group theory are true for sofic groups. It is an open problem if every group is sofic. In [4, 4.1] the notion of a *w-sofic* group was introduced, and proved that every sofic group is w-sofic. Hence every non-w-sofic group is not sofic. One can prove [2] that G is w-sofic if and only if G can be embedded into a quotient of a profinite group. Thus the problem whether all groups are weak-sofic is just Question (C). Using Baire category theorem we have the following criterion for w-soficity of simple finitely generated groups, which is closely related to Theorems 1, 2 for free groups.

Theorem 4. *Suppose* $G = \mathbb{F}_m/H$ *is a finitely generated non-abelian simple group (where* $H \lhd \mathbb{F}_m$*). Then* G *is not w-sofic if and only if there are* $h_1, \ldots, h_n \in H$ *such that* $[\mathbb{F}_m, \mathbb{F}_m] = \overline{\prod_{r=1}^{n} [\mathbb{F}_m, h_r]}$.

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