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Mini-Workshop: Deformation quantization: between formal to strict

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Abstract. The philosophy of deformation was proposed by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in the seventies and since then, many developments occurred. Deformation quantization is based on such a philosophy in order to provide a mathematical procedure to pass from classical mechanics to quantum mechanics. Basically, it consists in deforming the pointwise product of functions to get a non-commutative one, which encodes the quantum mechanics behaviour. In formal deformation quantization, the non-commutative product (also said, star product) is given by a formal deformation of the pointwise product, i.e. by a formal power series in the deformation parameter which physically play the role of Planck's constant $\hbar$. From a physical point of view this is clearly not sufficient to provide a reasonable quantum mechanical description and hence one needs to overcome the formal power series aspects in some way. One option is strict deformation quantization, which produces quantum algebras not just in the space of formal power series but in terms of C^*-algebras, as suggested by Rieffel, with e.g. a continuous dependence on $\hbar$. There are several other options in between continuous and formal dependence on $\hbar$ like analytic or smooth deformations.

The Oberwolfach workshop *Deformation quantization: between formal to strict* consolidated, continued, and extended these research activities with a focus on the study of the connection between formal and strict deformation quantization in their various flavours and their applications in particular those in quantum physics and non-commutative geometry. It brought together specialists in differential geometry, operator algebras, non-commutative geometry, and quantum field theory with research interests in the mentioned quantization procedures. The aim of the workshop was to develop a coherent viewpoint of the many recent diverse developments in the field and to initiate new lines of research.

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Introduction by the Organisers

Formal deformation quantization as introduced by Bayen et al. has reached by now a very satisfying state: with the highly non-trivial formality theorem of Kontsevich the questions on existence as well as on classification of formal star products on general Poisson manifolds have been settled and answered in the positive in 1997. Alternative approaches to the globalization of Kontsevich's result were also obtained by Cattaneo, Felder, and Tomassini in 2002 as well as by Dolgushev 2005. Before, the symplectic case was investigated by various groups. Here the existence of star products was shown by Lecomte and DeWilde already 1983, later independently by Fedosov in 1986 and by Omori, Maeda, and Yoshioka in 1991. The classification of star products in the symplectic case was obtained by Nest and Tsygan in 1995 and independently by Deligne in 1995 and Bertelson, Cahen, and Gutt in 1997. The representation theory of the deformed algebras, which is crucial for a physical application, has been investigated in detail by many people: among other things, the full classification of the star product algebras up to Morita equivalence was obtained by Waldmann, Bursztyn and Dolgushev in 2012.

For a physical interpretation of the star product algebras as observable algebras of a quantized physical system, the formal parameter has to be identified with Planck's constant $\hbar$. Hence a convergence of the formal series in $\hbar$ is crucial. In the early era of deformation quantization the formal star products have been constructed by means of asymptotic expansions of other quantizations like Berezin-Toeplitz quantizations on quantizable Kahler manifolds or symbol calculus quantizations on cotangent bundles. Beside producing rather explicit examples like the constructions of Cahen, Gutt and Rawnsley case as well as Karabegov in the Kahler case or Bordemann, Neumaier, Pflaum and Waldmann in the cotangent bundle case, the good understanding of the formal star products also led to interesting results on the convergent origins: here the computations of characteristic classes by Karabegov and Schlichenmaier or the index theorems of Fedosov as well as Nest and Tsygan should be mentioned. For the whole world beyond smooth Poisson manifolds the works of Pflaum, Posthuma, and Tang show first deep results on deformation quantization also in this case.

On a more analytic oriented approach based on a C^*-algebraic formulation using continuous fields of C^*-algebras, Rieffel showed how an action of $\mathbb{R}^d$ on a C^*-algebra can be used to deform this C^*-algebra in a continuous way. Applied to the bounded continuous functions on a manifold, this ultimately leads again to a formal star product by an asymptotic expansion of the continuous deformation for $\hbar \longrightarrow 0$, at least on sufficiently smooth vectors of the action. Ever since, Rieffel's paradigma of deformation by group actions was studied in many contexts and substantially extended recently to other (non-abelian) Lie groups than $\mathbb{R}^d$ by Bieliavsky and Gayral and coworkers. On a more abstract level, Natsume, Nest, and Peter considered symplectic manifolds with a topological condition (trivial second fundamental group) and showed that a strict quantization always exists, based on the usage of Darboux charts and a Čech cohomological argument.

The relation between formal and strict deformation quantization has been subject of several studies, but there still remain deep open questions. Since the approach of formal deformation quantization is universal, as proved by Kontsevich, it is natural to try to find the way back: from the easy formal situation to the more complicated convergent one. Since the above mentioned quantization schemes all use particular geometric features, one can hope to recover not only a convergent quantization as required by physics, but also interesting information about the underlying geometry. There are only few examples where this way backwards was investigated: in the flat case, Beiser, Rmer and Waldmann considered the convergence of the Wick star product on $\mathbb{C}^n$ and recovered the full symmetry, coherent states, and the Bargmann-Fock representation from the convergence conditions. While this example is still geometrically rather trivial, it already shows a rich structure beyond the locally multiplicatively convex theory. It can be extended to infinite dimensions in a rather conceptual way as recently shown by Waldmann. The relations to the approches of Dito's star products on Hilbert spaces still remain to be investigated. Later, Beiser and Waldmann considered a Wick-type star product on the Poincaré disk. Here the underlying geometry is topologically still trivial but enjoys a curved Kahler structure. Again, in this example the full symmetry of the problem is recovered and the foundations of a representation theory to establish the relations with the Berezin-Toeplitz quantizations are formed. Bieliavsky, Detournay, and Spindel gave a deformation of the Poincaré disk in a C^*-algebraic approach thus complementing the picture from the other side. However, the precise relations between the different versions of convergence remain unclear. Even though these examples seem to be isolated at the moment,they can be seen as a proof of concept that investigating the convergence of formal star products gives both physically relevant and manageable observable algebras and interesting information about the underlying geometry.

Understanding the analytic aspects of deformation quantization has led to many non-trivial and surprising applications beyond the field of deformation quantization itself . Here we only want to mention a few: the works of Anderson and coworkers on the mapping class group where the results of Bordemann, Meinrenken, and Schlichenmaier on the asymptotic properties of Berezin-Toeplitz quantization enter in a crucial way. The works of Lechner show how one can use Rieffel's deformations to construction new examples of quantum field theories as deformations of free theories. In some sense they can be seen as quantum field theories on a non-commutative Minkowski spacetime. Quantum deformations of classical geometries lead to interesting spaces in non-commutative geometry, here the quantum spheres of Connes and Landi provide a non-trivial and rich class where concepts of non-commutative geometry can be tested explicitly.

Still many questions remain open: first, the above mentioned examples have to be investigated further to understand their relations and connections. Moreover, the quest for convergence of star products in order to produce (ultimately) a continuous field of C^*-algebras has to be extended beyond the above examples. Here one can think of other types of algebras between the formal power series on the one hand and the C^*-algebras on the other hand: in particular locally convex algebras and also bornological algebras may provide a good bridge. Here the techniques developed by Meyer on bornological algebras will play a crucial role.

The overall goal of the workshop was to develop a coherent viewpoint of the many recent developments on the analytic aspects of deformation quantization as described above with particular emphasis on the connection between formal and strict and their potential applications in physics.

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Mini-Workshop: Deformation quantization: between formal to strict

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Abstracts

Leibniz algebras: deformation quantization and integration

Martin Bordemann
(joint work with C.Alexandre, S.Benayadi, S.Rivière, F.Wagemann)

1. Let K be a commutative associative unital ring containing the rational numbers (e.g. a field of characteristic 0), let $\mathfrak{h}$ be a K-module, and let $[\ ,\ ] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ be a bilinear map. Recall that the pair $\big(\mathfrak{h}, [\ ,\ ]\big)$ is called a *(left) Leibniz algebra (over K)* (see [5],[9]) iff the *left Leibniz identity* holds

$$[x,[y,z]] - [[x,y],z] - [y,[x,z]] = 0. \tag{1}$$

for all $x, y, z \in \mathfrak{h}$. In case $\big(\mathfrak{h}, [\ ,\ ]\big)$ satisfies the left Leibniz identity for the opposite bracket $[\ ,\ ]^{\mathrm{opp}}$ defined by $[x,y]^{\mathrm{opp}} = [y,x]$ for all $x, y \in \mathfrak{h}$ it is called a right Leibniz algebra: this case is frequently dealt with in the literature, but we shall stick to left Leibniz algebras and refer to them as Leibniz algebras. They form an obvious category whose morphisms consist of bracket preserving linear maps.
Every Lie algebra is a Leibniz algebra leading to the obvious inclusion functor.

2. For $K = \mathbb{R}$ and $\big(\mathfrak{g}, [\ ,\ ]\big)$ a finite-dimensional real Lie algebra it is well-known that is dual space $\mathfrak{g}^*$ becomes a Poisson manifold by means of the so-called linear Poisson structure $\pi_\alpha := \alpha\big([\ ,\ ]\big)$ for all $\alpha \in \mathfrak{g}^*$. Hence the space of all real-valued smooth functions $A = \mathcal{C}^\infty(\mathfrak{g}^*, \mathbb{R})$ beomes a Poisson algebra whose commutative multiplication is the usual pointwise multiplication of functions and the Poisson bracket $\{\ ,\ \}_\pi$ is given by $\{f,g\}_\pi(\alpha) = \alpha\big([df(\alpha), dg(\alpha)]\big)$ for all $f, g \in A$. In 1983, Simone Gutt [7] has given a simple explicit formula for a formal bidifferential associative deformation $*_G$ of A, a so-called *star-product*, in terms of the Baker-Campbell-Hausdorff (BCH) series of the Lie algebra $\mathfrak{g}$: for any $x \in \mathfrak{g}$ let $e_x \in A$ be the exponential function $e_x(\alpha) = e^{\alpha(x)}$ for all $\alpha \in \mathfrak{g}$. Then for all $x, y \in \mathfrak{g}$

$$e_x *_G e_y = e_{BCH(x,y)} \tag{2}$$

where $BCH(x,y) = x + y + \frac{\lambda}{2}[x,y] + \ldots$ is the usual BCH-series.
In 2013, B.Dherin and F.Wagemann succeeded in giving an analogue of Gutt's formula on the dual space of a finite-dimensional real Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$, [6]: writing $\mathrm{ad}_x : \mathfrak{h} \to \mathfrak{h}$ for the linear map $y \mapsto \mathrm{ad}_x(y) = [x,y]$ for any $x \in \mathfrak{h}$, they get –by analytic techniques– the following nonassociative star-product formula $*_{DW}$

$$e_x *_{DW} e_y = e_{e^{\lambda \mathrm{ad}_x}(y)}. \tag{3}$$

In the particular case of a Lie algebra $\mathfrak{h} = \mathfrak{g}$ it is not hard to see that $e_x *_{DW} e_y = e_x *_G e_y *_G e_{-x}$ –I owe this important remark to Pierre Bieliavsky–which makes it more plausible to understand the 'awkward' classical limit

$$(f *_{DW} g)(\alpha) = f(0)g(\alpha) + \lambda\alpha\big([df(0), dg(\alpha)]\big) + O(\lambda^2).$$

3. In [8] and [6], the importance of the structure of a *pointed Lie rack* $(M, e, \mathbf{m})$ has been emphasized: (M, e) is a pointed manifold, and $\mathbf{m} : (M, e) \times (M, e) \to$

(M,e) is a smooth map written $\mathbf{m}(x,y) = x \triangleright y$ and satisfying for all $x,y,z \in M$ the equations $e \triangleright y = y$, $x \triangleright e = e$, $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$. The standard example is a Lie group G with e its unit element and $g \triangleright g' = gg'g^{-1}$ for all $g, g' \in G$, but for any real finite-dimensional Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$ the pointed manifold $\big(\mathfrak{h}, 0\big)$ is well-known to become a Lie rack with respect to $x \triangleright y = e^{\mathrm{ad}_x}(y)$ so that the exponential functions e_x constitute a 'Lie' rack with respect to $*_{DW}$.
In [1] we defined a class of nonassociative bialgebras $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ which we call *rack bialgebras*: here $(B, \Delta, \epsilon, \mathbf{1})$ is a coassociative counital coaugmented coalgebra over K, and $\mu : B \otimes B \to B$ (written $\mu(a \otimes b) = a \triangleright b$ is a linear map of counital coaugmented coalgebras satisfying

$$\mathbf{1} \triangleright b = b,\ a \triangleright \mathbf{1} = \epsilon(a)\mathbf{1},\ a \triangleright (b \triangleright c) = \sum_{(a)} (a_{(1)} \triangleright b) \triangleright (a_{(2)} \triangleright c). \tag{4}$$

This identity can be obtained from a Lie rack by using the *Serre functor* which associates to any pointed manifold (M,e) the real vector space of all distributions supported in e: being an obvious functor in the category of real vector spaces it is not hard to see upon using the diagonal map that the functor map into the category of all coassociative cocommutative counital coaugmented connected coalgebras over $\mathbb{R}$.
We show furthermore that for any Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$ over K its symmetric algebra $\mathsf{S}(\mathfrak{h})$ is equipped with such a structure: it already comes with a cocommutative coassociative comultiplication Δ, a counit ϵ and a unit $\mathbf{1}$. Since $\mathfrak{g} = \overline{h}$ acts on $\mathfrak{h}$ via ad, it acts on $\mathsf{S}(\mathfrak{h})$ as derivations of the standard commutative multiplication. Hence its universal envelopping algebra $\mathsf{U}(\mathfrak{g})$ acts on $\mathsf{S}(\mathfrak{h})$, written $u.b$ for all $u \in \mathsf{U}(\mathfrak{g})$ and $b \in \mathsf{S}(\mathfrak{h})$. The canonical surjection $p : \mathfrak{h} \to \mathfrak{g}$ induces a morphism of coalgebras $\mathsf{S}(p) : \mathsf{S}(\mathfrak{h}) \to \mathsf{S}(\mathfrak{g})$ which can be followed by the canonical symmetrization map $\omega : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ which is an isomorphism of coalgebras by the Poincaré-Birkhoff-Witt Theorem. Defining $a \triangleright b := \Big(\omega\big(\mathsf{S}(p)(a)\big)\Big).b$ for all $a, b \in \mathsf{S}(\mathfrak{h})$ we get a rack-bialgebra. It is then not hard to show that in the case of a real finite-dimensional Leibniz algebra the restriction of the star-product $*_{DW}$ to the subspace of all polynomial functions on $\mathfrak{h}^*$ (which is isomorphic to $\mathsf{S}(\mathfrak{h})$) coincides with $\triangleright$ for $\lambda = 1$.

4. Another feature of finite-dimensional real Leibniz algebras is the notorious *integration problem*: Lie's Third Theorem states that there is a functor from the category of all finite-dimensional real Lie algebras to the category of all simply connnected connected Lie groups (in fact an equivalence of categories). In general, for any category $\mathcal{C}$ of 'structured' finite-dimensional real vector spaces an integration functor would be a functor $\mathbf{J}$ from $\mathcal{C}$ in the category of all pointed manifolds such that the composition $T_e\mathcal{M} \circ \mathbf{J}$ (with $T_e\mathcal{M}(M,e) = T_eM$) is naturally isomorphic to the forgetful functor of $\mathcal{C}$ to the category of all underlying vector spaces. In the case of the category of all finite-dimensional real Leibniz algebras one could demand an additional compatibility condition, namely that the restriction of $\mathbf{J}$ to the subcategory of all Lie algebras be naturally isomorphic to the above-mentioned 'Lie Three'-integration functor. The question whether such a compatible integration functor **J** exists for the category of all finite-dimensional real Leibniz algebras is –surprisingly– still open (the *coquecigrue* according to J. L. Loday). Wagemann and I could recently show (still unpublished work) that the pointed manifolds integrating Leibniz algebras in a functorial way have to be equipped with the structure of a pointed Lie rack.

M.Kinyon [8] found an integration functor for the important subcategory of the so-called *hemi-semi direct product* Leibniz algebras: let $\big(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}\big)$ be a Lie algebra, and V be a $\mathfrak{g}$-module. On the direct sum $V \times \mathfrak{g}$ the following 'half' of the semi-direct product of Lie algebras is a Leibniz bracket: for all $v, w \in V$ and $\xi, \eta \in \mathfrak{g}$ define $\big[(v,\xi),(w,\eta)\big] = \big(\xi.w, [\xi,\eta]_{\mathfrak{g}}\big)$ Morphisms in that subcategory are pairs of Lie algebra morphisms and module morphisms. In the finite-dimensional real case it is not hard to see that the association $V \times \mathfrak{g} \to V \times G$ (where G is a connected simply connected Lie group having Lie algebra $\mathfrak{g}$) is a compatible integration functor where the pointed manifold $\big(V \times G, (0,e)\big)$ carries the structure of a Lie rack via the map $(v,g) \rhd (v',g') = \big(g.v', gg'g^{-1}\big)$.

Another 'integable' subcategory is the category of all *symmetric Leibniz algebras* $\big(\mathfrak{h}, [\ ,\ ]\big)$: these are Leibniz algebras which are at the same time right Leibniz algebras. Upon passing to antisymmetric and symmetric part of the Leibniz bracket, it was shown (see e.g. [4], [2]) that this category is isomorphic to the category of all Lie algebras $\big(\mathfrak{h}^-, [\ ,\ ]_-\big)$ carrying a symmetric bilinear map $[\ ,\ ]_+ : \mathfrak{h}^- \times \mathfrak{h}^- \to \mathfrak{h}^-$ whose image lies in the centre of $\mathfrak{h}^-$ and whose kernel includes its image and the derived ideal of $\mathfrak{h}^-$. In [3] we have shown that an integration functor exists by associating to a symmetric Leibniz algebra $\mathfrak{h}$ the connected simply connected Lie group H^- having Lie algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$. The Lie rack structure on H^- is given by $h \rhd h' = hh'h^{-1}\chi(h,h')$ where $\chi(h,h') = \exp\big([\kappa(h),\kappa(h')]'_+\big)$, and $\kappa : H^- \to \mathfrak{a}(\mathfrak{h}) = \mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$ is the canonical morphism of Lie groups, and $[\ ,\ ]'_+ : \mathfrak{a}(\mathfrak{h}) \times \mathfrak{a}(\mathfrak{h}) \to \mathfrak{h}^-$ is the symmetric bilinear map induced by $[\ ,\ ]_+$.

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Symmetries of non-formal deformation quantizations

Axel de Goursac

In this Oberwolfach contribution, we are interested in non-formal deformation quantizations of the Weyl-type and their symmetries. This kind of deformation quantization considers noncommutative products $\star_\theta$ on a "regular" subspace of the smooth functions $\mathcal{C}^\infty(M)$ of a Poisson manifold M, and these star-products $\star_\theta$ depend on a real deformation parameter θ such that we recover the usual pointwise product for $\theta = 0$. For the existing examples, i.e. the Moyal-Weyl product for $\mathbb{R}^{2n}$ [6], the Bieliavsky-Gayral product for the Kählerian Lie groups with negative curvature [3], the product on the hyperbolic plane [1], they are explicitly given by an integral kernel.

We first mention that these deformation quantization possess a Hilbert algebra structure, which just correspond to the Hilbert-Schmidt operators $\mathcal{L}^2(\mathcal{H})$ via the Weyl-type quantization map. We call them Hilbert deformation quantization [5]. Let us analyze this observation in an heuristic way and go further in this direction. What are ingredients of a non-formal deformation quantization?

- Basically speaking, this Hilbert algebra structure $(\mathbf{A}_\theta, \star_\theta)$, or more precisely its associated von Neumann algebra (isomorphic to $\mathcal{L}(\mathcal{H})$) corresponds to the noncommutative measured space underlying the deformation quantization $\star_\theta$. In the case of the Moyal-Weyl product, we have $\mathbf{A}_\theta = L^2(\mathbb{R}^{2n})$.
- Then come other ingredients of a non-formal deformation quantization, such as its symmetries: for example, the Moyal-Weyl product is invariant and covariant under the translation group $\mathbb{R}^{2n}$,
- the data of a topological *-subalgebra $\mathbf{B}_\theta$ of $\mathbf{A}_\theta$ that we can consider as the data of a "smooth structure" (and a topology) on the noncommutative space underlying the deformation quantization $\star_\theta$: for example, the *-subalgebra $\mathbf{B}_\theta = \mathcal{S}(\mathbb{R}^{2n})$ of Schwartz functions for the Moyal-Weyl product,
- the continuity of the family of pre-C*-algebras $(\mathbf{B}_\theta, \star_\theta)$,
- and the commutative limit $\theta \to 0$ of the $\star_\theta$-commutator going to the Poisson bracket.

In general, it is not easy to find an adapted topological *-subalgebra of $(\mathbf{A}_\theta, \star_\theta)$ with interesting properties. In the work [5], for an arbitrary Hilbert deformation quantization, we found a way to construct such topological *-subalgebras, as generalized Schwartz or Sobolev spaces, from its covariant symmetries. So we claim here that ingredient 3 of the above list can be seen as a consequence of the ingredient 2: the symmetries determine the topology and the smoothness of the noncommutative space. This reflects the well-known fact in the commutative setting that for a homogeneous space M acted by a Lie group G, the smooth structure of M is totally determined by the one of G.

Applications of this result are the concrete determination of various topological *-subalgebras of the deformation quantizations [6, 3, 1], but also the explicit computation of non-formal star-exponentials [2, 4] and their relations with these topological *-subalgebras.

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Completions of group algebras, growth and nuclearity

Simone Gutt
(joint work with Michel Cahen and Stefan Waldmann)

The results presented here the object of the joint work [2]. We study several completions of the group algebra $\mathbb{C}[G]$ of a finitely generated group G. The norms are inspired by those constructed by Beiser and Waldmann in [1] to get convergence of deformation quantization. We relate nuclearity of such a completion to a growth property of the group.

1. Growth and submultiplicative functions

Let G be a finitely generated infinite group with a finite set S of generators. The choice of S defines a *length* $L\colon G \longrightarrow \mathbb{N}_0$ by counting the minimal number $L(g)$ of generators needed to write $g \in G$ as a product of generators. By convention, $L(e) = 0$ for the group unit $e \in G$ and we assume that $S = S^{-1}$ so that $L(g) = L(g^{-1})$ and $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$. If L' is another word length corresponding to a different set of generators S' then we have constants $c, c' \in \mathbb{N}$ with $L(g) \leq cL'(g)$ and $L'(g) \leq c'L(g)$ for all $g \in G$. Using the word length one defines the *surface growth* and the *volume growth* of the group G by

$$\sigma_G(n) = \#\{g \in G \mid L(g) = n\} \quad \text{and} \quad \beta_G(n) = \#\{g \in G \mid L(g) \leq n\},$$

where we omit the dependence on L in the notation. Clearly $\sigma_G(n) \leq \beta_G(n)$ and $\beta_G(n)$ grows at most exponentially in n. Since we assume G to be infinite, we have $\sigma_G(n) \geq 1$ for all n and β_G is strictly increasing. In particular $\beta_G(n) > n$. Moreover, we have

$$\sigma_G(n+m) \leq \sigma_G(n)\sigma_G(m) \quad \text{and} \quad \beta_G(n+m) \leq \beta_G(n)\beta_G(m)$$

for all $n, m \in \mathbb{N}_0$. We define a class of functions to which we can compare the functions σ_G and β_G. We will not use the standard way of comparing growth functions, as usually done in geometric group theory. Instead, we use a slightly coarser notion. A map $\sigma : \mathbb{N}_0 \longrightarrow [0, \infty)$ is *submultiplicative* if

$$\sigma(n+m) \leq \sigma(n)\sigma(m)$$

for all $n, m \in \mathbb{N}_0$ and *almost submultiplicative* if for every $\epsilon > 0$ there is a constant $c > 0$ such that

$$\sigma(n+m) \leq c\sigma(n)^{1+\epsilon}\sigma(m)^{1+\epsilon}$$

for all $n, m \geq \mathbb{N}_0$. The almost submultiplicative functions will allow for a slightly greater flexibility. A map $\sigma : \mathbb{N}_0 \longrightarrow [1, \infty)$ is called a *growth function* if it is monotonically increasing and unbounded; monotonic meaning $\sigma(n) \leq \sigma(n+1)$. Let $\sigma, \sigma' : \mathbb{N}_0 \longrightarrow [1, \infty)$ be two maps. We define a relation $<$ by $\sigma < \sigma'$ if there are constants $c, k \geq 1$ with $\sigma(n) \leq c\sigma'(cn)^k$ for all $n \in \mathbb{N}_0$.

2. Completions of the group algebra

The *group algebra* $\mathbb{C}[G]$ of the group G is the complex vector space spanned by the set G; we denote a basis of $\mathbb{C}[G]$ by $\{ e_g \,|\, g \in G \}$; it is a *cocommutative Hopf* **-algebra* , with the associative algebra multiplication inherited from G :

$$ab = (\sum_{g \in G} a_g e_g)(\sum_{h \in G} b_h e_h) = \sum_{g,h \in G} a_g b_h \; e_{gh} = \sum_{h \in G}(\sum_{g \in G} a_g b_{g^{-1}h})e_h,$$

where only finitely many of the coefficients a_g and b_h are different from zero,. the unit element given by e_e, the *-involution is given by $a^* = \sum_{g \in G} \overline{a_g} e_{g^{-1}}$, the antipode $S(a) = \sum_{g \in G} a_g e_{g^{-1}}$, the counit $\epsilon(a) = \sum_{g \in G} a_g$, and the coproduct $\Delta(a) = \sum_{g \in G} a_g \, e_g \otimes e_g$. Note that the tensor product $\mathbb{C}[G] \otimes \mathbb{C}[G]$ is canonically isomorphic as algebra to $\mathbb{C}[G \times G]$ where $G \times G$ has the product group structure.

We consider completions of this albebra in the space of *formal power series* in G which we denote by $\mathbb{C}[[G]] = \{ \sum_{g \in G} a_g e_g$ where now a_g is unrestricted$\}$. Given a submultiplicative or an almost submultiplicative growth function σ, given a set of generators with corresponding word length L on G, and given $R \geq 0$, we define

$$\|a\|_{L,\sigma,R} = \sum_{g \in G} |a_g| \sigma(L(g))^R$$

for $a \in \mathbb{C}[[G]]$, allowing for the value $+\infty$ and we set

$$\ell^1_{L,\sigma,R}(G) = \left\{ a \in \mathbb{C}[[G]] \;\middle|\; \|a\|_{L,\sigma,R} < \infty \right\},$$

$$\mathcal{A}_\sigma(G) = \left\{ a \in \mathbb{C}[[G]] \;\middle|\; \|a\|_{L,\sigma,R} < \infty \text{ for all } R \geq 0 \right\} = \bigcap_{R \geq 0} \ell^1_{L,\sigma,R}(G),$$

equipped with the projective locally convex topology of all the seminorms $\| \; \|_{L,\sigma,R}$. We prove:

(1) The projective limit $\mathcal{A}_\sigma(G)$ of the Banach spaces $\ell^1_{L,\sigma,R}$ is independent of the chosen word length L.
(2) The group algebra $\mathbb{C}[G]$ is dense in $\mathcal{A}_\sigma(G)$ and all algebraic structures are continuous yielding a Fréchet-Hopf *-algebra structure on $\mathcal{A}_\sigma(G)$.

(3) If σ is submultiplicative then $\mathcal{A}_\sigma(G)$ is a locally multiplicatively convex algebra: the norms $\| \ \|_{L,\sigma,R}$ are submultiplicative.

Our main result is the following crucial relation with the growth of the group : Let $(1+n) < \sigma$ be an almost submultiplicative growth function. Then $\mathcal{A}_\sigma(G)$ is nuclear iff $\beta_G < \sigma$.

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Deformation of fibred manifolds and related Hochschild cohomologies

Benedikt Hurle

We consider the situation of a fibred manifold, i.e. a surjective submersion $p : P \to M$ between two manifolds M, P. We assume that M is equipped with a (differential) $\star$-product on $C^\infty(M)[\![\lambda]\!]$. Now we want to deform the classical module structure given by $a \cdot f = p^* a f$. Especially the case of fibre bundles is interesting in theoretical physics, where one tries to deform a classical gauge theory into a noncommutative (quantum) field theory.

Definition. In this situation we call a $(C^\infty(M)[\![\lambda]\!], \star)$-left module structure $\bullet$ on $C^\infty(P)[\![\lambda]\!]$, such that $a \bullet f = (p^*a)f + \sum_{k=1}^\infty \lambda^k L_k(a,f)$ where the $L_k \in \mathrm{DiffOp}^\bullet(C^\infty(M), C^\infty(P); C^\infty(P))$ are bidifferential operators, a (left) module deformation of $P \xrightarrow{p} M$. It is called fibre preserving, if $a \bullet p^* b = p^*(a \star b)$

A bimodule deformation of a fibred manifold is a left and right module deformation, both denoted $\bullet$, such that

$$(a \bullet f) \bullet b = a \bullet (f \bullet b) \tag{1}$$

for all $a, b \in C^\infty(M)$ and $f \in C^\infty(P)$, i.e. $C^\infty(P)[\![\lambda]\!]$ becomes a $(C^\infty(M)[\![\lambda]\!], \star)$-bimodule. It is called fibre preserving if both module structures are fibre preserving.

For the module case the problem of existence and uniqueness up to equivalence was solved in [1].

Given a bimodule deformation we define the semi-Poisson bracket [3, 4] by $\{a, f\} = \frac{i}{2\lambda}(a \bullet f - f \bullet a)|_{\lambda=0}$ for $a \in C^\infty(M)$ and $f \in C^\infty(P)$, so it is the semiclassical limit of the bimodule. This bracket has the following properties:

(1) $\{ab, f\} = p^*a\{b, f\} + p^*b\{a, f\}$
(2) $\{a, p^*bf\} = p^*\{a, b\}f + p^*b\{a, f\}$
(3) $\{a, \{b, f\}\} - \{b, \{a, f\}\} - \{\{a, b\}, f\} = 0,$

for all $a, b \in C^\infty(M)$ and $f \in C^\infty(P)$. It is called natural if it is a derivation in the second argument. In particular it is a Poisson module of $C^\infty(M)$.

If M is symplectic and the semi-Poisson bracket is natural, one can define a horizontal lift by $X_a^h(f) = \{a, f\rbrace\!\!\rbrace$, where $X_a(b) = \{a, b\}$ is the symplectic vectorfield of $a \in C^\infty(M)$. This can be shown to be well defined. It turns out that the corresponding connection is flat.

So we get that in this case we can only get a bimodule deformation, if the fibre bundle admits a flat connection, which is a quite strong obstruction.

Given a flat lift it is even possible to get a $\star$-product on $C^\infty(P)[\![\lambda]\!]$, s.t. $(C^\infty(M)[\![\lambda]\!], \star)$ is a subalgebra, by lifting the differential operators in $\star$. Note that in this case the Poisson bracket on P in fibre direction is trivial.

In the general Poisson case it would be interesting to know if the existence of a semi-Poisson bracket is enough to get a bimodule deformation. Another problem is their classification up to equivalence.

The obstruction for an order by order construction of a module structure is in the Hochschild cohomology $\mathbf{HH}^2(C^\infty(M), \mathrm{DiffOp}(P))$ for the bimodule in $\mathbf{HH}^2(C^\infty(M \times M), \mathrm{DiffOp}(P))$.

So we consider the differential Hochschild cohomology $\mathbf{HH}^\bullet_{\mathrm{diff}}(C^\infty(M), C^\infty(N))$ and $\mathbf{HH}^\bullet_{\mathrm{diff}}(C^\infty(M), \mathrm{DiffOp}(N))$, where we have a map $p : N \to M$, s.t. $p(N)$ is a submanifold of M and the bimodule structure on $C^\infty(N)$ is given by the pullback along p.

This cohomology can be computed using an explicit homotopy to the Koszul complex [2, 1], for the local situation $p : \mathbb{R}^n \to \mathbb{R}^m$, and then gluing things together using a partition of unity.

We get the following result:

Theorem. [3, 4] In the above situation

$$\mathbf{HH}^\bullet_{\mathrm{diff}}(C^\infty(M), C^\infty(N)) \cong \mathfrak{X}^\bullet(M)|_{p(N)} \otimes_{C^\infty(M)} C^\infty(N) \tag{2}$$

and

$$\mathbf{HH}^\bullet_{\mathrm{diff}}(C^\infty(M), \mathrm{DiffOp}(N)) \cong \Lambda^\bullet(TM/Tp(N)) \otimes_{C^\infty(M)} \mathrm{DiffOp}_{\mathrm{ver}}(N) \tag{3}$$

as $C^\infty(M)$-bimodules. One can also show that these are isomorphic to vector bundles over N.

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Deformation quantization with separation of variables of an endomorphism bundle

Alexander Karabegov

During the mini conference "Deformation quantization: from formal to strict" held at MFO in February 2015 I gave a talk "Deformation quantization with separation of variables of an endomorphism bundle" on a graph-theoretic formula for a star product with separation of variables on endomorphism-valued symbols. Since the introduction of deformation quantization in [1] in 1978 until the work of Kontsevich [3] in 1997 explicit formulas were known for a very limited number of invariant star products on homogeneous symplectic manifolds. For non-invariant star products there are explicit formulas expressed in terms of directed graphs. The first such formula is the celebrated Kontsevichs formula for a star product on Rn equipped with an arbitrary Poisson structure (see [3]). There are several explicit graph-theoretic formulas for star products with separation of variables on K?ahler manifolds by Reshetikhin and Takhtajan [4], Gammelgaard [2], and Hao Xu [5]. In my talk I gave a generalization of Gammelgaards graph-theoretic formula to the star products with separation of variables on the sections of the endomorphism bundle of a holomorphic Hermitian vector bundle on a Kahler manifold.

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Representations of Rieffel-deformed C^*-algebras and deformations of nets

Gandalf Lechner

In deformation quantization, one is usually interested in deformations of some "global" algebra $\mathbb{A}$ – such as the algebra of *all* observables of a system in classical mechanics – by some formal or strict procedure, typically involving a change in the (associative) product of $\mathbb{A}$. The material presented in this talk is motivated by a different situation, inspired by QFT, where one is rather interested in deformations of "local" subalgebras of $\mathbb{A}$. In the first part, I reviewed this motivation and stated the aims. In the second part, I described representations of Rieffel-deformed C^*-algebras and discussed some additional results that are available in concrete representations. In the last part, I applied this technique to certain situations of the type introduced in the first part. This talk touches upon various joint projects, with H. Grosse [4, 5], D. Buchholz and S. J. Summers [3, 2], and S. Waldmann [9].

1) Motivation/Introduction. We consider an algebra $\mathbb{A}$ with subalgebras: There is a partially ordered index set $\mathbb{I}$, and for each $i \in \mathbb{I}$, a subalgebra $\mathbb{A}_i \subset \mathbb{A}$ such that $i \mapsto \mathbb{A}_i$ is inclusion preserving (net). The set $\mathbb{I}$ also carries a complementation $\perp$, and for $i \perp j$ we require that $\mathbb{A}_i$ and $\mathbb{A}_j$ commute. Finally, we typically have a group G acting on $\mathbb{I}$ and $\mathbb{A}$ such that $\alpha_g(\mathbb{A}_i) = \mathbb{A}_{gi}$ in obvious notation.

Examples of this structures are given by taking a Lorentzian manifold M, the index set as (some subset of) the open subsets of M, partially ordered by inclusion, with $i \perp j$ denoting the spacelike complement, and G the isometries of M. Under further conditions, the algebras $\mathbb{A}_i$ can then be interpreted as the observables of a local QFT on M [6].

The aim is to find some strict deformation procedure which takes the data $\mathbb{A}_i \subset \mathbb{A}, \mathbb{I}, G, \alpha$ and produces "deformed" data which still satisfy the same properties as before. As shown in examples, this typically involves deformations $\mathbb{A}_i \to \mathbb{A}_i^{(a_i)}$ of the "local" algebras $\mathbb{A}_i$ with deformation parameters a_i depending on i, i.e., several different deformations appear at the same time, and their interplay has to be studied. Furthermore, the assumptions listed above are in many cases so strong that one expects the internal algebraic structure of each $\mathbb{A}_i$ to be (essentially) fixed. That is, in the situation considered here each "local" deformation $\mathbb{A}_i \to \mathbb{A}_i^{(a_i)}$ is expected to be trivial, whereas the net structure changes in a non-trivial manner.

2) Representations of Rieffel-deformed C^*-algebras and warped convolutions. This part reviewed the notion of "warped convolution", which amounts to covariant representations of Rieffel-deformed C^*-algebras in concrete Hilbert space situations. The precise relation to Rieffel's approach was explained together with some further results, involving in particular a "spectral commutator theorem" which states commutativity of two operators, deformed with opposite deformation parameters, under certain assumptions of spectral nature.

During this workshop it also become apparent that recent work of S. Neshveyev on a crossed product formulation of Rieffel's procedure is closely related to the warped convolution [11], see also [8].

A generalization in the direction of considering suitable locally convex algebras or modules with polynomially bounded $\mathbb{R}^n$-actions instead of C^*-algebras with isometric actions was also briefly mentioned [9].

3) Applications to deformations of nets. In the last part the deformation was applied to a particular situation of the type outlined in 1). Namely, M is taken to be $\mathbb{R}^n$, $n \geq 2$, and $\mathbb{I}$ the family of all wedges, i.e. all Poincaré transforms of $W := \{x \in \mathbb{R}^n \; : \; x_1 > |x_0|\}$. Since $\mathbb{I}$ is just a single orbit, it is sufficient to consider as local algebra the one corresponding to W. This algebra, denoted $\mathbb{M}$ and taken here to be a von Neumann algebra, is assumed to satisfy three natural conditions w.r.t. a unitary positive energy representation of the Poincaré group U on the space $\mathbb{M}$ acts on: 1) $\mathrm{Ad}U_g$ acts by endomorphisms on $\mathbb{M}$ for all g preserving

W, 2) AdU_g maps $\mathbb{M}$ into its commutant $\mathbb{M}'$ for all g mapping W into its causal complement, and 3) there exists a vector which is invariant under U and cyclic separating for $\mathbb{M}$.

In this setting, it was explained how $\mathbb{M}$ can be deformed by warped convolution to another von Neumann algebra $\mathbb{M}_\theta$ so that all three conditions (with unchanged U and cyclic separating vector) are still satisfied for $\mathbb{M}_\tau$ instead of $\mathbb{M}$. Since the stated conditions imply that $\mathbb{M}$ (and its deformed version) are type III_1 factors [10], this demonstrates the effect that the internal structure of the local algebras is typically (essentially) fixed. The net structure can in this example be shown to depend on the deformation parameters by scattering theory.

The participants then discussed in particular the questions if/how to conclude injectivity of $\mathbb{M}_\tau$ from injectivity of $\mathbb{M}$ (since the injective type III_1 factor is unique [7]), and how the setting could be generalized to non-abelian group actions such as the ones studied by Bieliavsky and Gayral [1].

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Cocycle deformation of operator algebras

Sergey Neshveyev

Let A be a C*-algebra and α be a continuous action of a vector group $V \cong \mathbb{R}^d$ on A. Denote by $\mathcal{A} \subset A$ the algebra of smooth vectors for this action. Fix a scalar product $\langle \cdot, \cdot \rangle$ on V. Consider the space $S(V; \mathcal{A})$ of $\mathcal{A}$-valued Schwartz functions on V. It can be made into a Fréchet algebra by defining the convolution product by

$$(f * g)(x) = \int_V f(y)\alpha_y(f(x-y))dy.$$

The space $S(V;\mathcal{A})$ with this product is denoted by $V \ltimes_\alpha \mathcal{A}$ and called the smooth crossed product of $\mathcal{A}$ by the action of V.

Let J be a skew-symmetric operator on V. Then Rieffel's deformation $\mathcal{A}_J$ of $\mathcal{A}$ is the Fréchet space $\mathcal{A}$ equipped with the new product

$$a \times_J b = \int_{V\times V} \alpha_{Jx}(a)\alpha_y(b)e(x\cdot y)dx\,dy,$$

where $e(x\cdot y)$ stands for $e^{2\pi i\langle x,y\rangle}$ and the integral is understood in the oscillatory sense [5]. The automorphisms α_x of $\mathcal{A}$ remain automorphisms of $\mathcal{A}_J$ and define an action of V on $\mathcal{A}_J$, which we denote by α^J.

The algebra $\mathcal{A}_J$ completes to a C*-algebra A_J as follows. We have a representation π_J of $\mathcal{A}_J$ on $S(V;\mathcal{A})$ defined by $\pi_J(a)\xi = \alpha(a)\times_J \xi$, where $\alpha(a)$ is the $\mathcal{A}$-valued function on V given by $\alpha(a)(x) = \alpha_{-x}(a)$, and the deformed product $\times_J$ for $\mathcal{A}$-valued functions is defined using the action of V on itself by left translations. The space $S(V;\mathcal{A})$ is a dense subspace of the Hilbert A-module $L^2(V)\otimes A$. The operators $\pi_J(a)$ extend by continuity to bounded operators on this Hilbert module, and we let A_J to be the norm-closure of $\pi_J(\mathcal{A}_J)$ in $\mathrm{End}_A(L^2(V)\otimes A) = M(K(L^2(V))\otimes A)$.

Although Rieffel's deformation procedure is highly nontrivial, there is a simple relation between the crossed products of A and A_J. In order to describe it, consider the Fourier transform on $S(V;\mathcal{A})$,

$$\hat{\xi}(x) = \int_V \xi(y)e(-x\cdot y)dy,$$

and define an operator Θ_J on $S(V;\mathcal{A})$ by

$$\Theta_J(f)(x) = \int_V \alpha_{Jy}(\hat{f}(y))e(x\cdot y)dy.$$

Note that this operator is invertible, with inverse equal to Θ_{-J}.

Theorem 1. *The operator Θ_J, viewed as a map $S(V;\mathcal{A}_J)\to S(V;\mathcal{A})$, is an algebra isomorphism of the smooth crossed products $V\ltimes_{\alpha^J}\mathcal{A}_J \cong V\ltimes_\alpha \mathcal{A}$. It extends by continuity to an isomorphism of the C*-algebra crossed products $V\ltimes_{\alpha^J} A_J \cong V\ltimes_\alpha A$.*

The existence of an isomorphism $V\ltimes_{\alpha^J} A_J \cong V\ltimes_\alpha A$ was proposed by Kasprzak [2], who checked it in some cases. A complete proof was first given in [1], while the above simple form of this isomorphism was found in [3].

This isomorphism led Kasprzak to develop a new approach to Rieffel's deformation. His idea was that instead of trying to define a new product on $\mathcal{A}\subset A$, we can just consider $V\ltimes_\alpha A$ and then try to recover A_J as a subalgebra of the multiplier algebra $M(V\ltimes_\alpha A)$. This is possible to do using a twisted dual action on $V\ltimes_\alpha A$ and the theory of Landstad algebras [2]. A different way has been proposed in [1, 4]. It relies on certain 'quantization maps' $A\to A_J$.

In order to define these maps, consider first the action of V by translations on $A = C_0(V)$. It is known that $C_0(V)_J$ is isomorphic to the twisted group C*-algebra $C^*(V;\Omega_J)$, where the 2-cocycle Ω_J is given by $\Omega_J(x,y) = e(x\cdot Jy)$. This C*-algebra can be defined using the projective representation $\gamma_J: V \to B(L^2(V))$, $(\gamma_J(x)f)(y) = e(x\cdot y)f(y+Jx)$. The genuine representation $\gamma_J \otimes \gamma_{-J}$ is quasi-equivalent to the regular representation, hence any normal state ν on $W^*(V;\Omega_{-J})$ defines a normal ucp map $(\iota\otimes\nu)(\gamma_J\otimes\gamma_{-J}): W^*(V) \to W^*(V;\Omega_J)$. Identifying $W^*(V)$ with $L^\infty(V)$ we get a 'quantization map' $T_\nu: C_0(V) \to C_0(V)_J$, which extends to a normal ucp map $L^\infty(V) \to W^*(V;\Omega_J)$ of the corresponding von Neumann algebras. Explicitly,

$$T_\nu(f) = \int_V \nu(\gamma_{-J}(x))\hat{f}(x)\gamma_J(x)dx, \quad \text{if } f \in L^\infty(V)\cap L^1(V),\ \hat{f}\in L^1(V).$$

Turning to the case of a general C*-algebra A equipped with an action α of V, we can now consider the maps

$$(T_\nu\otimes\iota)\alpha: A \to M(C_0(V)_J\otimes A) \subset M(K(L^2(V))\otimes A).$$

We then have the following description of A_J.

Theorem 2. *The Rieffel deformation A_J of A coincides with the norm closure of the subspace of $M(K(L^2(V))\otimes A)$ spanned by the elements of the form $(T_\nu\otimes\iota)\alpha(a)$ for all $a\in A$ and $\nu\in W^*(V;\Omega_{-J})_*$. Explicitly, for any $\nu\in W^*(V;\Omega_{-J})_*$ such that the function $x\mapsto\nu(\gamma_{-J}(x))$ lies in the Fourier algebra of V, so it is the Fourier transform of a function $g_\nu\in L^1(V)$, we have*

$$(T_\nu\otimes\iota)\alpha(a) = \int_V g_\nu(x)\pi_J(\alpha_x(a))dx.$$

One advantage of this picture of Rieffel's deformation is that its ingredients, such as the twisted group algebras and the quantization maps, make sense in a much greater generality. Namely, it is possible to develop a deformation procedure for any action of a locally compact group G on a C*-algebra A and a unitary 2-cocycle Ω on the dual quantum group $\hat{G}$ [4].

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A convergent star-product on the symmetric tensor algebra over a Hilbert space

Matthias Schötz

In a recent work [1] it was shown how to construct a topology on the symmetric tensor algebra $\mathcal{S}^\bullet(V)$ over an arbitrary locally convex space V such that the usual star-products of exponential type are continuous. This can be achieved by extending all the continuous semi-norms on V in a suitable manner to $\mathcal{S}^\bullet(V)$ by means of projective tensor products. In my talk I have shown that for the special case of a Hilbert space $\mathcal{H}$, a similar construction yields slightly better results: By using not projective tensor products but Hilbert tensor products, the resulting topology is in general coarser and can be shown to be the coarsest possible under some additional assumptions. This is a results from [2] and is to be published in [3].

The topology on $\mathcal{S}^\bullet(\mathcal{H})$ can be constructed as the topology that is defined by a suitable extensions of all the equivalent inner products (in the topological sense) on $\mathcal{H}$: For a bounded, positive-definite and invertible linear operator A on $\mathcal{H}$ we define an inner product $\langle \cdot \,|\, \cdot \rangle_A^\bullet$ on the tensor algebra over $\mathcal{H}$ by demanding that the subspaces of k- and ℓ-fold tensors are orthogonal if $k \neq \ell$, that $\langle \xi \,|\, \eta \rangle_A^\bullet = \bar{\xi}\eta$ for $\xi, \eta \in \mathbb{C}$ and that

$$\left\langle x_1 \otimes \ldots \otimes x_k \,\middle|\, y_1 \otimes \ldots \otimes y_k \right\rangle_A^\bullet = k! \, \Pi_{i=1}^k \langle x_i \,|\, A y_i \rangle$$

holds for all $k \in \mathbb{N}$ and $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathcal{H}$, where $\langle \cdot \,|\, \cdot \rangle$ denotes the inner product of $\mathcal{H}$.

Let $b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a continuous bilinear form, then one can define a star-product $\star_b$ on $\mathcal{S}^\bullet(\mathcal{H})$ – interpreted as a subspace of the tensor algebra – that satisfies $x \star_b y = x \vee y + b(x, y)$ for all $x, y \in \mathcal{H}$. Here $\vee$ denotes the symmetric tensor product, see [1] for details. It can now be shown that $\star_b$ is continuous under the locally convex topology defined by the Hilbert norms induced by all the above inner products with A running over all bounded, invertible and self-adjoint linear operators on $\mathcal{H}$ (or equivalently, with A running over all multiplications with natural numbers).

Conversely, it is possible to reconstruct a sufficiently many inner products $\langle \cdot \,|\, \cdot \rangle_A^\bullet$ out of suitable star-products: Therefore, assume that $\mathcal{H}$ is additionally equipped with an antilinear involution $\overline{\cdot}$ that fulfills $\langle \overline{x} \,|\, \overline{y} \rangle = \overline{\langle x \,|\, y \rangle}$. We can extend $\overline{\cdot}$ to a *-involution of the symmetric tensor algebra by demanding that $x^* = \overline{x}$ holds for all $x \in \mathcal{H}$. This is also a *-involution with respect to the product $\star_b$ if b fulfills $b(\overline{x}, \overline{y}) = \overline{b(y, x)}$. For a bounded, invertible and self-adjoint linear operator A on $\mathcal{H}$ that fulfills $A\overline{x} = \overline{Ax}$ for all $x \in \mathcal{H}$, define the continuous bilinear form $a(x, y) := \langle \overline{x} \,|\, A y \rangle$. Then $\cdot^*$ is a *-involution with respect to $\star_a$ and

$$\langle X \,|\, Y \rangle_A^\bullet = \pi_0\big(X^* \star_a Y \big)$$

holds for all $X, Y \in \mathcal{S}^\bullet(\mathcal{H})$ where $\pi_0 : \mathcal{S}^\bullet(\mathcal{H}) \to \mathbb{C}$ denotes the projection on the 0-component of a tensor.

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From Formal to Strict Deformation Quantization

Stefan Waldmann

In this first talk of the workshop, the main aim was to find a common language and a suitable platform to launch the various other talks: thus I presented some historical overview on deformation quantization focusing on the questions of the workshop.

In a first step, I recalled the motivations for deformation quantization arising from the quest for a construction of a quantum mechanical description of a given mechanical system. The starting point is the Hamiltonian point of view where the phase space of the mechanical system is modeled as a symplectic or Poisson manifold M while the dynamics is implemented as a Hamiltonian time evolution for a specific Hamiltonian. The basic idea of deformation quantization then consists in finding a star product for this situation: a formal power series of bidifferential operators C_r such that

$$f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f,g) \tag{1}$$

turns to be a $\mathbb{C}[[\hbar]]$-bilinear associative deformation of $C^\infty(M,\mathbb{C})[[\hbar]]$ into the direction of the Poisson bracket, i.e. one wants $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = i\{f,g\}$ for all $f,g \in C^\infty(M,\mathbb{C})$. In the formal power series setting, the existence and classification of such star products is by now well-understood. I recalled the basic results like the existence proofs of DeWilde and Lecomte in the early eighties, Fedosov's beautiful construction, as well as Kontsevich's general solution to existence and classification by means of his formality theorem. In the symplectic case the classification was obtained more easily by Nest and Tsygan as well as by Bertelson, Cahen, Gutt and many others. Furthermore, I commented on several particular constructions where additional properties of the star products like e.g. symmetries can be implemented.

From a physical point of view such a formal star product is of course far from being the end of the story: the deformation parameter $\hbar$ has to be Planck's constant, not a formal parameter, in order to have a physically reasonable quantum theory. Moreover, one needs to implement enough analytic structure to guarantee that the resulting quantum observable algebra has a reasonable representation theory by operators on a Hilbert space. This is required for physical reasons as one needs to implement the super-position principle for states. The states themselves can be implemented as positive functionals on the algebra itself.

This physical demand then provides the core of the workshop, the understanding of the analytic properties of the formal deformations and an implementation of convergence results in one or the other way.

Thus, in a second step I recalled several ideas to approach this convergence issue. Here the situation is less clear concerning the existence and classification. In fact, several competing definitions of what a strict deformation quantization should be are on the market. I tried to give an overview on the various possibilities: most notably, Rieffel suggested to replace the formal deformation by a C^*-algebraic deformation using the language of continuous fields of C^*-algebras. This was further developed by Landsman and exemplified in several works by Bieliavsky. General results on the existence of such strict deformation quantizations were obtained by Natsume and Nest, both construction were discussed in later talks in detail.

In a last part, I reported briefly on some recent work on the convergence of the prototype of every star product: the Weyl star product on a vector space V. Here the new aspect was to investigate the convergence of the formal series directly from the point of view of locally convex algebras. To this end, I established a locally convex topology on the symmetric algebra over V, which was assumed to be locally convex and equipped with a continuous constant Poisson bracket. Such a Poisson bracket is encoded in a continuous bilinear form on V. Then for this locally convex topology on the symmetric algebra, the Weyl star product becomes continuous. It therefore extends to the completion of this polynomial algebra. The completion contains several interesting entire functions like exponentials but not all of them.

Deformation of algebras by group 2-cocycles

Makoto Yamashita

We consider the deformation problem for algebras graded by a discrete group Γ. More precisely, let A be an associative algebra admitting a compatible grading by Γ, so that $A = \oplus_{g \in \Gamma} A_g$ and $A_g A_h \subset A_{gh}$. If $\omega(g, h)$ is a 2-cocycle on Γ into the multiplicative scalar group, one may define a new associative product structure $*_\omega$ on by setting $x *_\omega y = \omega(g, h) xy$ if $x \in A_g$ and $y \in A_h$. In the framework of $*$-algebras, the exponentiation $\omega^t(g, h) = \exp(\sqrt{-1}\omega_0(g, h))$ of $\mathbb{R}$-valued 2-cocycle ω_0 on Γ leads to the smooth deformation of Γ-graded algebras $(A, *_{\omega^t})_{t \in \mathbb{R}}$.

Carrying out this deformation scheme in the framework of C*-algebras, we obtain a generalization the Rieffel deformation of torus actions to reduced Fell bundles over discrete groups. Varying the parameter t in ω^t on an interval I, we obtain a $C(I)$-algebra with fibers (A, ω^t). We show that this $C(I)$-algebra has a structure of KK-fibration if Γ satisfies the strong Baum–Connes conjecture. The proof is based on the idea of Echterhoff–Lück–Phillips–Walters: namely, interpreting the Γ-grading as a $C^*_r(\Gamma)$-coaction, we can replace the algebras (A, ω^t) by the double crossed products $(A, \omega^t) \rtimes C_0(\Gamma) \rtimes \Gamma$ by Takesaki–Takai duality. The key observation is that $(A, \omega^t) \rtimes C_0(\Gamma)$ does not depend on t, so the problem is reduced to the continuous deformation of the dual Γ-actions. In order to apply the Baum–Connes conjecture one must show that crossed products by finite subgroups of Γ give KK-fibration, but that follows from the triviality of the $\mathbb{R}$-valued 2-cohomology of finite groups.

In the purely algebraic framework, we consider the analogous problem for cyclic cohomology groups. Here, the fibration structure is described by Getzler's Maurer–Cartan connection on periodic cyclic theory. Using the coaction map corresponding to the Γ-grading, we have the action of group cohomology to the periodic cyclic cohomology of algebras, denoted by $\xi \triangleright \phi$ for $\xi \in H^*(\Gamma; \mathbb{C})$ and $\phi \in \mathrm{HP}^*(A)$. The action of the Maurer–Cartan form on cyclic cohomology is shown to be cohomologous to the cup product action of the group cocycle. This allows us to compute the monodromy of the Gauss–Manin connection in the strict deformation setting. Moreover, the cyclic cocycles which are invariant under the coaction naturally induce cyclic cocycles on the deformed algebras. Denoting this induction $\phi \mapsto \phi^{(t)}$, the action of $\phi^{(t)}$ on the K-group of $(A, *_{\omega^t})$ is shown to be equal to the action of $e^{\sqrt{-1}[\omega_0]} \triangleright \phi$ on the K-group of A.

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