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New Perspectives on the Interplay between Discrete Groups in Low-Dimensional Topology and Arithmetic Lattices

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Abstract. This workshop brought together specialists in areas ranging from arithmetic groups to topological quantum field theory, with common interest in arithmetic aspects of discrete groups arising from topology. The meeting showed significant progress in the field and enhanced the many connections between its subbranches.

Mathematics Subject Classification (2010): 11-XX, 20-XX, 22-XX, 53-XX.

Introduction by the Organisers

The workshop *New Perspectives on the Interplay between Discrete Groups in Low-Dimensional Topology and Arithmetic Lattices* was held June 22 – June 26, 2015. The participants were specialists in areas ranging from arithmetic groups to topological quantum field theory, with common interest in arithmetic aspects of discrete groups arising from topology.

The mornings of the first two days of the meeting were devoted to longer lectures by senior participants aimed at surveying the new developments in their respective areas of research. These lectures were given by Alexander Lubotzky (on arithmetic representations of mapping class groups), Julien Marché (on topological quantum field theory), Eduard Looijenga (on algebraic-geometric aspects of mapping class groups) and Joachim Schwermer (on automorphic structures). All the remaining talks were reports on recent progress on one of the themes of the meeting.

The meeting showed significant progress in the field and enhanced the many connections between its subbranches. The workshop was attended by researchers from around the world, mainly ranging from postdocs to senior scientific leaders in their areas.

Thursday afternoon was reserved to short lectures by very recent graduates. On Monday evening, there was an introductory gathering giving each participant the opportunity of a short presentation of some of the research interests.

The meeting took place in a lively and active atmosphere, and greatly benefited from the ideal environment at Oberwolfach.

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Workshop: New Perspectives on the Interplay between Discrete Groups in Low-Dimensional Topology and Arithmetic Lattices

Table of Contents

Tara E. Brendle (joint with Dan Margalit)
Combinatorial models for mapping class groups 1705

Francesco Costantino
Non semi-simple TQFTs 1706

Renaud Detcherry
Asymptotics of matrix coefficients of quantum representations 1707

Elena Fuchs
Generic properties of finitely generated subgroups of $SL_n(\mathbb{Z})$ 1708

Tsachik Gelander (joint with Arie Levit)
Most hyperbolic manifolds are non-arithmetic 1712

Patrick M. Gilmer (joint with Gregor Masbaum)
Integral TQFT and modular representations 1713

Robert A. Kucharczyk (joint with John Voight)
Modular embeddings for triangle and Veech groups 1713

Christopher J. Leininger (joint with Ian Agol and Dan Margalit)
Pseudo-Anosov stretch factors and homology 1715

Eduard Looijenga
Some algebraic geometry related to the mapping class group 1718

Alexander Lubotzky, Justin Malestein (joint with Fritz Grunewald, Michael Larsen)
Arithmetic quotients of the mapping class group 1723

Julien Marché
Introduction to quantum representations of mapping class groups 1725

Bruno Martelli (joint with Alexander Kolpakov)
Constructing hyperbolic four-manifolds 1726

Maria Beatrice Pozzetti
Maximal representations of complex hyperbolic lattices 1728

Andrew Putman (joint with Thomas Church, Benson Farb)
The high dimensional cohomology of $\mathrm{SL}_n\mathcal{O}$ 1730

Nick Salter
4-manifolds can be surface bundles over surfaces in many ways 1730

Joachim Schwermer
On arithmetically defined groups and their cohomology – some examples, and some results 1732

Juan Souto (joint with Javier Aramayona, Dieter Degreijse, Conchita Martinez)
Geometric dimension of mapping class groups and lattices in classical simple groups 1735

Anna Wienhard (joint with François Guéritaud, Olivier Guichard, Fanny Kassel)
Geometric compactifications of locally symmetric spaces of infinite co-volume 1738

Helen Wong (joint with Francis Bonahon)
Character varieties and representations of the Kauffman bracket skein algebra 1741

Abstracts

Combinatorial models for mapping class groups

Tara E. Brendle
(joint work with Dan Margalit)

In 1997 Ivanov proved that the automorphism group of the complex of curves associated to a surface S is isomorphic to the extended mapping class group $\mathrm{Mod}(S)$, for most surfaces [9]. He applied this work to show that the abstract commensurator of $\mathrm{Mod}(S)$ is also isomorphic to $\mathrm{Mod}(S)$, and also gave a new proof of Royden's theorem that the isometry group of Teichmüller space is isomorphic to $\mathrm{Mod}(S)$.

Ivanov's work generated a flurry of activity, with similar results obtained by several different authors for various other simplicial complexes associated to surfaces. By the work of Farb, Irmak, Ivanov, Kida, Korkmaz, McCarthy, Papadopoulos, Schmutz Schaller, Yamagata, and Brendle–Margalit, the automorphism groups of the following curve complexes are all known to be isomorphic to the extended mapping class group (with finitely many exceptions in each case): the systolic complex of curves [16], the complex of nonseparating curves [5], the pants complex [14], the cut system complex [7], the Torelli complex [4], the complex of separating curves [1], the truncated complex of domains [15], the arc complex [8], the arc and curve complex [12], the ideal triangulation graph [13], and the complex of hole-bounding curves and hole-bounding pairs [6, 11].

Ivanov then posed a metaconjecture stating that every "sufficiently rich" complex associated to a surface S has $\mathrm{Mod}(S)$ as its group of automorphisms [10]. In recent joint work with Dan Margalit, we resolve Ivanov's metaconjecture for a wide class of complexes. Roughly speaking, these simplicial complexes have vertices corresponding to isotopy classes of a specified collection of connected compact subsurfaces of S, with edges corresponding to disjointness up to homotopy. The curve complex is an example of such a complex; one simply takes all annuli as the collection of subsurfaces.

We give two characterizations of sufficient richness for such complexes. The first is topological: it gives a short list of easy-to-check combinatorial criteria for the objects that generate the complex. The second is algebraic: sufficient richness is equivalent to the complex admitting no *exchange automorphisms*, that is, automorphisms interchanging two vertices of the complex while fixing all others. Exchange automorphisms were first observed by McCarthy–Papadopoulos as obstructions to Ivanov's metaconjecture for the so-called "complex of domains" of a surface with boundary [15]; the result obtained here shows that exchange automorphisms are the only obstruction for this class of complexes.

As one application of this work, we recover a theorem of Bridson–Pettet–Souto [3] stating that the abstract commensurator of each term in the Johnson filtration of the mapping class group is isomorphic to $\mathrm{Mod}(S)$ (the case of the Torelli group is originally due to Farb–Ivanov [4]; the case of the Johnson group is originally due to Brendle–Margalit [1, 2]). We further show that if N is any normal subgroup of $\mathrm{Mod}(S)$ containing an element with "small" support, then $\mathrm{Aut}(N) \cong \mathrm{Mod}(S)$.

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Non semi-simple TQFTs

Francesco Costantino

We will start by recalling the definition of Topological Quantum Field Theory (TQFT) and in particular of its byproducts: quantum invariants of 3-manifolds and quantum representations of mapping class groups. After this general introduction we will recall the main properties of the famous "quantum representations of mapping class groups" associated to the SU(2)-TQFTs constructed by Reshetikhin and Turaev ([3]) and also by Blanchet, Habegger, Masbaum and Vogel ([1]) using skein theory. We will compare these properties with those of the new family of representations associated to the new "non semi-simple TQFTs" we constructed recently in collaboration with Christian Blanchet, Nathan Geer and Bertrand Patureau ([2]). In particular, we will point out one crucial property of these new representations: namely that the order of the action of Dehn twists is infinite; this contrasts sharply with the behavior of the former representations.

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Asymptotics of matrix coefficients of quantum representations

Renaud Detcherry

The classical constructions of TQFTs from the Reshetikhin-Turaev invariants associates to any closed oriented surface Σ a sequence of finite dimensional Hermitian vector spaces $V_r(\Sigma)$ on which the mapping class group $MCG(\Sigma)$ acts projectively. These projective representations ρ_r of the mapping class groups are called the quantum representations. Also the alternative definition of Blanchet, Habegger, Masbaum and Vogel constructed natural basis $(\varphi_\alpha)_{\alpha\in I_r}$ associated to any pair of pants deomposition of the surface Σ. Finally, simple closed curves γ on Σ induce curve operators $T_r^\gamma \in \mathrm{End}(V_r(\Sigma))$.
Although the definitions of these TQFTs by Reshetikhin and Turaev is based on combinatorial calculations from surgeries presentations of 3-manifolds, they were inspired by the work of Witten who gave a formal definition of TQFTs from quantum Chern-Simons theory with gauge group SU_2. The approach of Witten, using path integral also yielded an asymptotic expansion for quantum invariants of closed 3-manifolds $Z_r(M)$ in terms of Reidemeister torsion and Chern-Simons invariants.
This talk attempts to give an asymptotic formula for matrix coefficients of the form $\langle \rho_r(\phi)\varphi_\alpha, \varphi_\beta\rangle$ where $\phi \in MCG(\Sigma)$ and ρ_r are the quantum representations. We use the approach of geometric quantification to turn the combinatorial model of TQFT of Reshetikhin and Turaev into a geometric model, where the vector spaces $V_r(\Sigma)$ are represented as spaces $H_r \subset H^0(M, L^r)$ of holomorphic sections of prequantizing complex line bundle L over an open dense subset M of the SU_2 character variety $\mathscr{M}(\Sigma) = \mathrm{Hom}(\pi_1\Sigma, \mathrm{SU}_2)/\mathrm{SU}_2$. Then we show that curve operators T_r^γ in this framework are Toeplitz operators of principal symbols the trace function $f_\gamma(\rho) = -\mathrm{Tr}(\rho(\gamma))$. We can then use the analytic properties of Toeplitz operators to give an estimation of basis vectors φ_α, and of some pairings $\langle \varphi_\alpha, \rho_r(\phi)\varphi_\beta\rangle$. The formula obtained is reminiscent of Witten's asymptotic expansion conjecture.

Generic properties of finitely generated subgroups of $\mathbf{SL}_n(\mathbb{Z})$

Elena Fuchs

Given a subgroup Γ of $\mathrm{GL}_n(\mathbb{Z})$ with Zariski closure $\overline{\Gamma}$ in $\mathrm{GL}_n(\mathbb{C})$, Γ is called a *thin group* if it is of infinite index in $\overline{\Gamma} \cap \mathrm{GL}_n(\mathbb{Z})$.

Such groups have become of great interest in number theory in the last decade, particularly after the development of the Affine Sieve by Bourgain–Gamburd–Sarnak in [BGS]. The main purpose of the sieve in their work is to answer the following kind of question.

Suppose one is given a finitely generated subgroup $\Gamma \leq \mathrm{GL}_n(\mathbb{Z})$, a vector $\mathbf{v} \in \mathrm{GL}_n(\mathbb{Z})$, and a nonzero polynomial $f(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_n]$. When does there exist an $r \in \mathbb{N}$ such that the set

$$\mathscr{O}_{\mathbf{v}}(f, r) := \{\mathbf{w} \in \Gamma\mathbf{v} \mid f(\mathbf{w}) \text{ has at most } r \text{ prime factors}\}$$

is Zariski-dense in $\mathrm{Zcl}(\Gamma\mathbf{v})$? The answer is, as shown in [BGS], essentially when Γ satisfies a rich combinatorial property having to do with expander graphs. This property depends only on the Zariski closure of Γ, and hence has nothing to do with whether or not the group is thin. To answer the question above, Bourgain–Gamburd–Sarnak count points in balls in $\mathscr{O}_{\mathbf{v}}(f, r)$, which in the context of thin groups was very much a new tool: previously such counting was only considered in the case where the group Γ is arithmetic.

With these new methods which apply to thin groups has come a surge of interest in studying such groups in their own right. One important question that has been considered is how to tell whether a given finitely generated group (given in terms of its generators) is thin. While this problem is probably too hard to answer in general, [FMS], [SVe], and [BT] have answered this question in the context of certain hypergeometric monodromy groups.

In this report, we describe a related question of a slightly different flavor: is a generic finitely generated subgroup of $\mathrm{SL}_n(\mathbb{Z})$ thin? Specifically, we discuss joint work with Rivin [FR] which shows that the generic 2-generator subgroup of $\mathrm{SL}_n(\mathbb{Z})$ is not only thin, but free when generic is defined appropriately.

Similar questions regarding subgroups of $\mathrm{SL}_n(\mathbb{Z})$ generated by random elements of *combinatorial* height at most X – i.e. elements obtained by a random walk of length X on the Cayley graph $\mathrm{Cay}(\mathrm{SL}_n(\mathbb{Z}), S)$ where S is a fixed finite set of generators of $\mathrm{SL}_n(\mathbb{Z})$ – have been previously addressed by Rivin in [R1, R2] and by Aoun in [A]. In fact, Rivin considers a much broader family of lattices in semisimple Lie groups beyond $\mathrm{SL}_n(\mathbb{Z})$, and Aoun's results apply also to finitely generated non-virtually solvable subgroups of $\mathrm{GL}(V)$ where V is a finite dimensional vector space over an arbitrary local field K.

Aoun's result in the context of $\mathrm{SL}_n(\mathbb{Z})$ that any two independent random walks on $\mathrm{SL}_n(\mathbb{Z})$ generate a free group implies that with this combinatorial definition of genericity, a finitely generated subgroup of $\mathrm{SL}_n(\mathbb{Z})$ is generically of infinite index in $\mathrm{SL}_n(\mathbb{Z})$ if $n \geq 3$. Combining this with Rivin's result in [R1] that in the combinatorial model a generic finitely generated subgroup of $\mathrm{SL}_n(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_n(\mathbb{C})$, we have that thinness is generic in the combinatorial setup.

One might expect then, that generic thinness will also be true if one starts with the following Euclidean model. Let $G = \mathrm{SL}_n(\mathbb{Z})$, and let B_X denote the set of all elements in G of norm at most X, where norm is defined as

$$\|\gamma\|^2 := \lambda_{\max}(\gamma^t \gamma) \tag{1}$$

where $\lambda_{\max}$ denotes the largest eigenvalue. Choose two elements g_1, g_2 uniformly at random from B_X and consider

$$\lim_{X\to\infty} \mu_X(\{g = (g_1, g_2) \in G^2 \mid \Gamma(g) \text{ is of infinite index in } G\})$$

where $\Gamma(g) = \langle g_1, g_1^{-1}, g_2, g_2^{-1}\rangle$, and μ_X is the measure on $G \times G$ induced by the normalized counting measure on B_X^2. If the above limit is 1, we say that the generic subgroup of G generated by two elements is infinite index in G. In this model, two randomly chosen elements *do* generate a Zariski-dense subgroup, and one might guess that, just as in the combinatorial setting, two randomly chosen elements of G in the Euclidean model will form a ping-pong pair with probability tending to 1 (i.e. that the generic 2-generator subgroup of G is free in particular). However, we use Breuillard-Gelander's [BG] characterization of ping-pong for $\mathrm{SL}_n(\mathbb{R})$ over projective space to show that while this is the case for $n = 2$, it is not true if $n > 2$ (see [FR]), and so whether or not thinness is generic in this model is still an open question.

However, if one "symmetrizes" the ball of radius X in a natural way, by imposing a norm bound on both the matrix and its inverse, we show in [FR] that two elements chosen at random in such a modified model *will,* in fact, be a ping-pong pair over a suitable space with probability tending to 1, and hence, in this modified setup, the generic subgroup of $\mathrm{SL}_n(\mathbb{Z})$ generated by two elements is thin. This modified Euclidean model is identical to the one described above, but B_X will be replaced by

$$B'_X(G) := \{g \in G \mid g, g^{-1} \in B_X\}, \tag{2}$$

and the measure μ_X is replaced by μ'_X, the normalized counting measure on $(B'_X)^2$. With this notation, we show the following.

Theorem 1 ([FR])**.** *Let $G = \mathrm{SL}_n(\mathbb{Z})$ where $n \geq 2$, and let $B'_X(G)$ and μ'_X be as above. Then we have*

$$\lim_{X\to\infty} \mu'_X(\{(g_1, g_2) \in (B'_X(G))^2 \mid \langle g_1, g_2\rangle \text{ is thin}\}) = 1$$

We remark that it is very natural to consider the region B'_X, rather than the usual ball B_X: it is in fact a more suitable analog of Aoun's combinatorial setup, in which an element of combinatorial height X has inverse whose combinatorial height is also X (unlike the Euclidean ball model, in which an element of norm X can have inverse of much larger norm).

We now briefly describe the methods used to prove Theorem 1, which are also the methods used to show that generically two elements chosen at random in the original Euclidean model do not form a ping-pong pair.

Consider the Cartan decomposition of $\mathrm{SL}_n(\mathbb{R})$, namely that it decomposes as

$$\mathrm{SL}_n(\mathbb{R}) = K\,A\,K$$

where $K = \mathrm{SO}_n(\mathbb{R})$ is the maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ and

$$A = \{\mathrm{diag}(e^{j_1}, \dots, e^{j_n}) \mid j_i \in \mathbb{R}, j_i \leq j_{i+1}, \sum_i j_i = 0\}.$$

Specifically, any element $g \in \mathrm{SL}_n(\mathbb{Z})$ can be written as

$$g = k_g a_g k'_g \tag{3}$$

where $k_g, k'_g \in K$ and $a_g \in A$. The matrix a_g is uniquely determined by g: it is the diagonal matrix with the eigenvalues of $g^t g$ on the diagonal, ordered from highest to lowest.

We recall the definition of a ping-pong pair from [BG] below (see [BG] for a description of the metric used on projective space, as well as certain properties of the action of $\mathrm{PSL}_n(\mathbb{R})$ on $\mathbb{P}^{n-1}(\mathbb{R})$).

Definition 2. *Two elements $g_1, g_2 \in \mathrm{SL}_n(\mathbb{R})$ are called a ping-pong pair if both g_1 and g_2 are (r, ϵ)-very proximal with respect to some $r > 2\epsilon > 0$, and if the attracting points of g_i and g_i^{-1} are at least distance r apart from the repulsive hyperplanes of g_j and g_j^{-1} in $\mathbb{P}^{n-1}(\mathbb{R})$, where $i \neq j$.*

In the above definition, an element $\gamma \in \mathrm{SL}_n(\mathbb{R})$ is said to be (r, ϵ)-very proximal if both γ and γ^{-1} are (r, ϵ)-proximal. Namely, both γ and γ^{-1} are ϵ-contracting with respect to some attracting point $v_\gamma \in \mathbb{P}^{n-1}(\mathbb{R})$ and some repulsive hyperplane H_γ, such that $d(v_\gamma, H_\gamma) \geq r$. Finally, γ is called ϵ-contracting if if there exists a point $v_\gamma \in \mathbb{P}^{n-1}(\mathbb{R})$ and a projective hyperplane H_γ, such that γ maps the complement of the ϵ-neighborhood of H_γ into the ϵ-ball around v_γ.

In fact, Proposition 3.1 in [BG] gives a necessary and sufficient condition for γ to be ϵ-contracting can be stated simply in terms of the top two singular values of γ:

Theorem 3 (Proposition 3.1 [BG])**.** *Let $\epsilon < 1/4$ and let $\gamma \in \mathrm{SL}_n(\mathbb{R})$. Let $a_1(\gamma)$ and $a_2(\gamma)$ be the largest and second-largest singular values of γ, respectively (i.e. largest and second-largest eigenvalues of $\gamma^t\gamma$). If $\frac{a_2(\gamma)}{a_1(\gamma)} \leq \epsilon^2$, then γ is ϵ-contracting. More precisely, writing $\gamma = k_\gamma a_\gamma k'_\gamma$, one can take H_γ to be the projective hyperplane spanned by $\{k'^{-1}_\gamma(e_i)\}_{i=2}^n$, and $v_\gamma = k_\gamma(e_1)$, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$.*

Conversely, suppose γ is ϵ-contracting. Then $\frac{a_2(\gamma)}{a_1(\gamma)} \leq 4\epsilon^2$.

The strategy in [FR] is then to consider the probability that this condition on the top two singular values will be satisfied both by a random element as defined above and by its inverse. The idea is, rather than to compare the number of elements in $\mathrm{SL}_n(\mathbb{Z})$ in a ball which are ϵ-contracting to the total number of elements in the ball, to compare the measures of the analogous sets in $\mathrm{SL}_n(\mathbb{R})$, using Haar measure on SL_n. This becomes a problem in analysis of bounding certain integrals from above and below. Since this is essentially identical to the comparison over $\mathbb{Z}$ by Theorem 1.4 of [EM] once one proves that the sets

(4)
$$C_{X,\eta} := \{\mathrm{diag}(\alpha_1,\dots,\alpha_n) \mid \alpha_i \in \mathbb{R}, X \geq \alpha_1 \geq \eta\alpha_2;\ \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_n, \prod_i \alpha_i = 1\}$$

where $\eta > 16$ is fixed make up a well-rounded sequence of sets (for two equivalent definitions of well-roundedness, see [EM]). In this way, one finds that in the usual Euclidean model the probability of a large gap between the kth and $k+1$st singular value is positive but small whenever $k < n-1$ (it goes to 0 if one lets n tend to infinity), but that the probability tends to 1 for $k = n-1$. Note that the gaps between singular values aside from the top two are relevant for us because if one considers the action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathbb{P}(\bigwedge^k(\mathbb{R}^n))$ for a suitable $k \geq 1$, any of these gaps can be shifted up to a gap between the top two singular values. However, for our purposes we need both the randomly chosen element *and* its inverse to have a large gap between the top two singular values, and the best we can do, even after considering actions on these larger spaces, is to prove that there is a positive but small probability that an element chosen at random out of a usual Euclidean ball will be ϵ-contracting.

We hence turn to the symmetrized ball model as defined in (2). Here one has that the probability of a large gap between the middle two singular values of a randomly chosen element tends to 1, meaning that both an element chosen uniformly at random in the symmetrized model, and its inverse, will be ϵ-contracting with probability tending to 1 when one considers the action of $\mathrm{SL}_n(\mathbb{R})$ on $\mathbb{P}(\bigwedge^k(\mathbb{R}^n))$, where $k = n/2$ if n is even, and $k = (n+1)/2$ if n is odd. Again, this is computed using Theorem 1.4 in [EM] and after proving that the relevant sequences of sets are all well-rounded.

One then has to show that the other conditions for ping-pong hold with probability tending to 1. Namely, we need that the fixed points and repelling hyperplanes of the two generators that were chosen uniformly at random are spaced far apart with probability tending to 1. Since these points and hyperplanes associated to an ϵ-contracting element γ are determined just by k_γ and k'_γ where $\gamma = k_\gamma a k'_\gamma$, this can be proven using the following equidistribution theorem of Gorodnick–Oh:

Theorem 4 ([GO])**.** *Let $G = SL_n(\mathbb{R})$ and $\Gamma = SL_n(\mathbb{Z})$. Let Ω_1, Ω_2 be Borel subsets of K with boundaries of measure zero. Let $A'_X \subset A$ be the elements of A belonging to B'_X. Then*

$$\#(\Gamma \cap \Sigma_1 A'_X \Sigma_2) \sim_{X\to\infty} \frac{Vol(\Sigma_1 A'_X \Sigma_2)}{Vol(G/\Gamma)} = \nu(\Sigma_1)\nu(\Sigma_2)\cdot \frac{Vol(G'_X)}{Vol(G/\Gamma)},$$

where ν is the probability Haar measure on K.

This is essentially Theorem 1.6 in [GO] stated in the context relevant to us, and upon proving that the sequence of regions B'_X is well rounded. Combining this strategy with the one above of showing that ϵ-very-contraction is generic, we are able to prove Theorem 1.

We end by noting that, while there are many reasons why the symmetrized ball model is a natural one, one still wants to show generic thinness in the usual ball model. This is widely expected to be true. However, whether or not a generic finitely generated group should be generically free in this model is unclear and, if it is not, one would need an entirely new method of approaching thinness.

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Most hyperbolic manifolds are non-arithmetic

Tsachik Gelander

(joint work with Arie Levit)

A celebrated result of Margulis asserts that finite volume locally symmetric manifolds of rank > 1 are arithmetic. Corlette and Gromov–Schoen extended this result to rank one spaces, with the two exceptions of real and complex hyperbolic.

A remarkable paper of Gromov and Piatetski-Shapiro establishes the existence of a non-arithmetic (real) hyperbolic manifold of finite volume, in any given dimension.

We prove that in fact almost all hyperbolic manifolds are non-arithmetic, with respect to a certain way of counting. Recall that two manifolds are commensurable if they share a common finite cover. Fixing the dimension $d > 3$, and counting up to commensurability, we show that the number of non-arithmetic hyperbolic d-manifolds of volume bounded by V is superexponential in V, while the number of arithmetic ones tends to be polynomial.

Integral TQFT and modular representations

Patrick M. Gilmer
(joint work with Gregor Masbaum)

Let $p = 2d + 1$ be a prime greater than three. If $p \equiv 1 \pmod 4$, this story is a little more complicated than described below. Let Σ be a surface of genus g with at most one colored point (colored $2c$ from the set of colors $\{0, 2, \cdots, p-3\}$). We recall the definition of the integral TQFT module $S(\Sigma)$ inside the SO(3) TQFT vector space $V(\Sigma)$. One has that $V(\Sigma)$ is a $\mathbb{Q}[\zeta_p]$-vector space and $S(\Sigma)$ is the $\mathbb{Z}[\zeta_p]$-submodule of $V(\Sigma)$ generated by the collection of all vacuum states. This is a free $\mathbb{Z}[\zeta_p]$-module according to [1, 2]. Then $F(\Sigma) = S(\Sigma)/(1-\zeta_p)S(\Sigma)$ is a vector space over $\mathbb{F}$ (the field with p elements) equipped with a representation of the mapping class group of Σ. There is an irreducible subrepresentation denoted $F^{\text{ odd}}(\Sigma)$ with an irreducible quotient $F(\Sigma)/F^{\text{ odd}}(\Sigma)$ [3]. These two irreps factor through $\mathrm{Sp}(2g, \mathbb{F})$ [4]. Let K be the algebraic closure of $\mathbb{F}$. Our main new result identifies these irreps by the highest weights of the unique $\mathrm{Sp}(2g, K)$ K-representation with p-restricted highest weight which restricts to $F^{\text{ odd}}(\Sigma) \otimes K$ and $(F(\Sigma)/F^{\text{ odd}}(\Sigma)) \otimes K$. We refer to these weights as the highest weights of $F^{\text{ odd}}(\Sigma)$ and $F(\Sigma)/F^{\text{ odd}}(\Sigma)$. The highest weight of $F^{\text{ odd}}(\Sigma)$ is $(d-2)\,\omega_g + \omega_{g-3}$ if $c = 0$, and is $(d-c-1)\,\omega_g + (c-1)\,\omega_{g-1} + \omega_{g-2}$ otherwise. The highest weight of $F(\Sigma)/F^{\text{ odd}}(\Sigma)$ is $(d-c-1)\,\omega_g + c\,\omega_{g-1}$. If $g \leq 2$, interpret an appearance of ω_i with $i < 0$ to mean that the representation is zero.

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Modular embeddings for triangle and Veech groups

Robert A. Kucharczyk
(joint work with John Voight)

For this extended abstract a *Fuchsian group* will always be a discrete subgroup of finite covolume in $\mathrm{SL}_2(\mathbb{R})$. If Δ is a Fuchsian group, the quotient $\Delta\backslash\mathbb{H}$ (here $\mathbb{H}$ is the upper half plane) is a Riemann surface of finite type and hence possesses a unique structure as a smooth complex algebraic curve C. To relate geometric and arithmetic properties of Δ to those of C is a difficult task in general. Much more is known when Δ is arithmetic. For instance, for any abstract field automorphism τ of $\mathbb{C}$ the curve $\tau(C)$ is given by ${}^{\tau}\Delta\backslash\mathbb{H}$ for another arithmetic Fuchsian group ${}^{\tau}\Delta$, see [3, 7]. Furthermore, ${}^{\tau}\Delta$ is a congruence subgroup if and only if Δ is, and in this case there is a very explicit description of ${}^{\tau}\Delta$.

In this talk we presented a class of Fuchsian groups slightly enlarging the class of arithmetic groups and containing many interesting examples which still enjoys similar properties: the *semi-arithmetic Fuchsian groups with modular embeddings.* A Fuchsian group $\Delta \subset \mathrm{SL}_2(\mathbb{R})$ is semi-arithmetic if (after possibly passing to a finite index subgroup) all traces $\operatorname{tr}\delta$ for $\delta \in \Delta$ lie in the ring of integers $\mathfrak{o}_K$ of some totally real algebraic number field $K \subset \mathbb{R}$. If this is the case then the $\mathfrak{o}_K$-subalgebra $\mathscr{O} = \mathfrak{o}_K\langle\Delta\rangle \subset \mathrm{SL}_2(\mathbb{R})$ generated by Δ is an order in a quaternion algebra $B = K\mathscr{O}$ over K.

Let $\varrho_1, \dots, \varrho_g \colon K \to \mathbb{R}$ be the distinct field embeddings, and assume they are ordered in such a way that ϱ_1 is the identity and that B splits at ϱ_j, i.e. $B \otimes_{K,\varrho_j} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$, if and only if $j \leq r$. For each $j \leq r$ fix some $\mathbb{R}$-algebra isomorphism $B \otimes_{K,\varrho_j} \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$. Then $(\varrho_1, \dots, \varrho_r)$ embeds the group $\mathscr{O}^1$ of units in $\mathscr{O}$ with reduced norm one into $\mathrm{SL}_2(\mathbb{R})^r$, and the image is a lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^r$ operating on $\mathbb{H}^r$.

Definition 1. *With the preceding notation, a* modular embedding *for Δ is a holomorphic map $f \colon \mathbb{H} \to \mathbb{H}^r$ which is equivariant for the natural inclusion $\varphi \colon \Delta \hookrightarrow \Gamma$.*

A modular embedding $F \colon \mathbb{H} \to \mathbb{H}^r$ descends to a regular map $f \colon \Delta\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}^r$ of algebraic varieties: $\Delta\backslash\mathbb{H}$ underlies an algebraic curve and $\Gamma\backslash\mathbb{H}^r$ underlies a *quaternionic Shimura variety*, i.e. a (possibly twisted) Hilbert–Blumenthal variety. Today only three classes of examples are known.

(i) If Δ is *arithmetic*, then Γ is commensurable to Δ and after passing to finite index subgroups we may choose $\Gamma = \Delta$ and $F = \mathrm{id}$. More generally this setup yields (twisted) diagonal embeddings $\mathbb{H} \to \mathbb{H}^r$ and Hirzebruch–Zagier cycles, see [4].

(ii) If Δ is a *triangle group* then there always exists a modular embedding for Δ. This embedding was constructed in [2] by three different methods: via Schwarz triangle maps, via hypergeometric differential equations and via period maps for certain special families of abelian varieties. The arithmetic aspects of these modular embeddings will be studied in [5], compare also [1].

(iii) If Δ is the *Veech group* uniformising a Teichmüller curve (see [6]) then there exists a modular embedding by [6] (genus two) and [9] (arbitrary genus). In this case Γ is always an arithmetic subgroup of $\mathrm{SL}_2(K)$. These modular embeddings are studied in [10].

In [4] a geometric criterion is proved for a map $f \colon C \to S$ from a smooth complex curve C to a quaternionic Shimura variety S to arise from a modular embedding. For sake of simplicity we only state the cocompact case. Let $S(\mathbb{C}) = \Gamma\backslash\mathbb{H}$ with Γ a torsion-free congruence subgroup of $\mathscr{O}^1$. Then the cotangent bundle of $\mathbb{H}^r$ splits naturally as a direct sum of line bundles coming from the r factors $\mathbb{H}$, and since $\mathscr{O}^1$ preserves this decomposition, it descends to a decomposition

$$\Omega^1_{S/\mathbb{C}} = \bigoplus_{j=1}^{r} \mathscr{M}_j.$$

Theorem 2 (K. [4]). *Let C be a smooth projective curve, let S be a quaternionic Shimura variety and let $f\colon C \to S$ be a regular map. Then f is covered by a modular embedding $F\colon \mathbb{H} \to \mathbb{H}^r$ if and only if the line bundle $f^*\mathscr{M}_1$ is isomorphic (as an algebraic line bundle on C) to the canonical bundle ω_C.*

This theorem is proved using Simpson's correspondence between variations of Hodge structure and Higgs bundles developed in [12]. As a corollary, if τ is an abstract field automorphism of $\mathbb{C}$ and if $f\colon C \to S$ is covered by a modular embedding, then so is $\tau(f)\colon \tau(C) \to \tau(S)$. In [4] an adelic formalism for modular embeddings, compatible with that for Shimura varieties and automorphic bundles as in [8], is developed, and this corollary is made more explicit.

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Pseudo-Anosov stretch factors and homology

Christopher J. Leininger

(joint work with Ian Agol and Dan Margalit)

A pseudo-Anosov homeomorphism of a surface $f\colon S \to S$ is a homeomorphism which in certain preferred local coordinates (defined off of a finite, f-invariant set) is given by $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$ for some $\lambda > 1$; see [9]. The number $\lambda = \lambda(f)$ is an invariant of f called the *stretch factor*.

The *mapping class group of* S, denoted $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation preserving homeomorphisms of S. For the closed orientable surface of genus g, S_g, the group $\mathrm{Mod}(S_g)$ is the (orbifold) fundamental group of the moduli space $\mathscr{M}_g$. Thurston's classification theorem [9, 24] implies that the pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ (that is, the elements with pseudo-Anosov representative) are precisely those infinite order elements with geodesic representatives in $\mathscr{M}_g$, with respect to the Teichmüller metric. Moreover, the length of the geodesic corresponding to $[f] \in \mathrm{Mod}(S_g)$ is exactly $\log(\lambda(f))$. In particular, the *length spectrum* of $\mathscr{M}_g$ is

$$\mathscr{L}(S_g) = \{\log(\lambda(f)) \mid f\colon S_g \to S_g \text{ pseudo-Anosov}\}.$$

According to work of Ivanov [15] and Arnoux–Yoccoz [4], this is a closed discrete subset of $\mathbb{R}$. In particular there is a smallest element denoted ℓ_g. While the exact value of ℓ_g is only known for $g = 1$ and $g = 2$ (see [25, 6]), in general it is unknown. On the other hand, Penner [19] described the behavior of ℓ_g as $g \to \infty$:

Theorem (Penner). *For all $g \geq 1$ we have $\ell_g \asymp \frac{1}{g}$.*

This means that there are constants $0 < C_1 < C_2 < \infty$ such that $\frac{C_1}{g} \leq \ell_g \leq \frac{C_2}{g}$. Penner's constants are explicit, and since publication of his paper [19], the value of C_2 has been improved, with the best known constants coming from the work of Hironaka [14], Aaber–Dunfield [1], and Kin–Takasawa [16].

We are interested in qualitative information about the elements realizing ℓ_g. More broadly, we want to understand the qualitative information about the pseudo-Anosov homeomorphisms $f\colon S_g \to S_g$ with $\log(f) \asymp \frac{1}{g}$. To this end, we investigate how the stretch factor $\lambda(f)$ is related to other invariants of f. Perhaps the most studied, classical invariant is the action on the first homology $f_*\colon \mathrm{H}_1(S_g) \to \mathrm{H}_1(S_g)$ (here we take $\mathrm{H}_1(S_g) = \mathrm{H}_1(S_g; \mathbb{R}) \cong \mathbb{R}^{2g}$, but our results remain valid with other coefficients; see [3]).

The stretch factor is related to the action on homology by the following; see [9].

Theorem (Manning). *For any pseudo-Anosov $f\colon S \to S$, we have $\lambda(f) \geq \lambda(f_*)$, where $\lambda(f_*)$ is the spectral radius of the linear transformation f_*.*

This is very useful as it provides a simple method for showing that $\lambda(f)$ is large. There are, however, pseudo-Anosov homeomorphisms $f\colon S \to S$ with $f_* = \mathrm{Id}$ (see [24]). In this case, the inequality may provides no information. In fact, the *Torelli group*, $\mathscr{I}(S) < \mathrm{Mod}(S)$, consisting of those elements represented by homeomorphisms f with $f_* = \mathrm{Id}$, is quite large and contains many pseudo-Anosov elements. In [8], the stretch factors of the pseudo-Anosov homeomorphisms in the Torelli group were analyzed. Writing

$$\ell_g^{\mathscr{I}} = \min\{\log(\lambda(f)) \mid f\colon S_g \to S_g \text{ pseudo-Anosov, and } f_* = \mathrm{Id}\}$$

we may state the main result of [8], which provides a sharp contrast to Manning's and Penner's results.

Theorem (Farb–Leininger–Margalit). *For all $g \geq 2$, $\ell_g^{\mathscr{I}} \asymp 1$.*

Ellenberg [7] asked whether there was a "linear interpolation" between Penner's result and this one, in terms of the dimension of the fixed subspace of homology.

More precisely, let $\kappa(f) = \dim(\mathrm{Fix}(f_*))$, where $\mathrm{Fix}(f_*) < H_1(S)$ is the subspace of elements fixed by f_*. Then we set

$$\ell_g(k) = \min\{\log(\lambda(f)) \mid f\colon S_g \to S_g \text{ pseudo-Anosov, and } \kappa(f) \geq k\}.$$

Note that $\ell_g(0) = \ell_g$ and $\ell_g(2g) = \ell_g^{\mathcal{I}}$. The main result of [3] answers Ellenberg's question with the following.

Theorem (Agol–Leininger–Margalit)**.** *For all $g \geq 2$ and $0 \leq k \leq 2g$, we have $\ell_g(k) \asymp \frac{k+1}{g}$. More precisely,*

$$.00031\left(\frac{k+1}{2g-2}\right) \leq \ell_g(k) \leq 12\log(2)\left(\frac{k+1}{2g-2}\right).$$

The proofs of both inequalities appeal to the rich geometric structure of the mapping torus of a pseudo-Anosov homeomorphism, drawing on the results from [2, 4, 5, 10, 11, 12, 13, 17, 18, 20, 21, 22, 23]. For complete details, see [3].

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Some algebraic geometry related to the mapping class group

Eduard Looijenga

This is a summary of the talk I gave in response to the request of the organizers of this workshop to survey the algebro-geometric aspects of the mapping class group. I am grateful to them for asking me to do so, as this led me to revisit (and to rethink) a few these connections. But the reader be warned that although I have tried to tailor this survey to an audience of geometers of a non-algebraic denomination, it remained one *au goût du jour* and consisted in the end more of asking questions than of stating results.

A moduli space of curves. Let S be an oriented surface of *hyperbolic type*, in the sense that it has finite Betti numbers and each connected component has negative Euler characteristic. We consider complex structures on S compatible with the given orientation that make it in fact a nonsingular complex algebraic curve (in other words, a compact Riemann surface minus a finite subset). The group $\mathrm{Diff}^+(S)$ of orientation preserving diffeomorphisms of S acts in an evident manner on this space of complex structures. The *Teichmüller space* $\mathcal{T}(S)$ of S is the orbit space of the identity component $\mathrm{Diff}^\circ(S)$ of this group, in other words, the space of isotopy classes of such structures. It is a basic fact that $\mathcal{T}(S)$ is contractible and comes with a natural structure of a complex manifold. The action of $\mathrm{Diff}^+(S)$ on $\mathcal{T}(S)$ clearly descends to one of the *mapping class group* $\Gamma(S) := \mathrm{Diff}^+(S)/\mathrm{Diff}^\circ(S)$ of S. This action is proper and virtually free and hence the orbit space is $\mathcal{M}(S) = \Gamma(S)\backslash\mathcal{T}(S)$ is an orbifold that is a virtual Eilenberg-Mac Lane space for $\Gamma(S)$. In particular, $H^\bullet(\Gamma(S)) \cong H^\bullet(\mathcal{M}(S))$ (in this note we shall always take (co)homology with $\mathbb{Q}$-coefficients).

Deligne-Mumford compactification. A compact 1-dimensional submanifold $A \subset S$ is necessarily a disjoint union of a finitely many embedded circles. Say that A is *admissible* if $S - A$ is of hyperbolic type (this includes the case $A = \emptyset$). Then $\Gamma(S - A)$ and $\mathcal{T}(S - A)$ are defined and only depend on the isotopy class (i.e., the $\mathrm{Diff}^\circ(S)$-orbit) $[A]$ of A. We express this by sometimes writing $\Gamma(S - [A])$ and $\mathcal{T}(S - [A])$ instead. Consider the disjoint union of the $\overline{\mathcal{T}}(S)$ of the Teichmüller spaces $\mathcal{T}(S - [A])$, where $[A]$ runs over all the admissible isotopy classes. The group $\Gamma(S)$ acts in this union and the $\Gamma(S)$-stabilizer of $\mathcal{T}(S - [A])$ maps to a subgroup of $\Gamma(S - [A])$ of finite index with kernel the (free) abelian group generated by the Dehn twists along the connected components of $[A]$. There is a natural $\Gamma(S)$-invariant topology on $\overline{\mathcal{T}}(S)$ which has the property that the closure of $\mathcal{T}(S - [A])$ meets $\mathcal{T}(S - [B])$ if and only if $[A]$ is represented by a union of connected components of B. In view of the preceding, the action of $\Gamma(S)$ on $\overline{\mathcal{T}}(S)$ is not proper (unless the only admissible A is the empty set, but this happens only when S is a thrice punctured sphere). Yet the orbit space $\overline{\mathcal{M}}(S)$ has a natural complex orbifold structure extending the one on $\mathcal{M}(S)$. With this structure, $\overline{\mathcal{M}}(S)$ is even projective. In particular, it is compact and this explains the noun in its name: the *Deligne-Mumford compactification* of $\mathcal{M}(S)$. The boundary $\overline{\mathcal{M}}(S) - \mathcal{M}(S)$ is a normal crossing divisor (in the orbifold sense) whose natural partition (which counts the number of branches passing through a given point) coincides with the partition inherited from $\overline{\mathcal{T}}(S)$. The stratum $\mathcal{T}(S - [A])$ has an algebro-geometric interpretation: it parametrises complex structures on the quotient of S obtained by contracting each connected component of A to a point such that this point becomes an ordinary double point.

A virtue of this approach is that it behaves in a straightforward manner under passage of subgroups of finite index: if $\Gamma \subset \Gamma(S)$ is of finite index, then $\mathcal{M}_\Gamma := \Gamma\backslash\mathcal{T}(S)$ will be a finite cover of $\mathcal{M}(S)$ and $\overline{\mathcal{M}}_\Gamma := \Gamma\backslash\overline{\mathcal{T}}(S)$ is a projective orbifold compactification of $\mathcal{M}_\Gamma$ with a normal crossing boundary in the orbifold sense such that the evident map $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}(S)$ is like a ramified cover (a finite, flat and surjective morphism). In particular, $\mathcal{M}_\Gamma$ is quasi-projective, so that $H^k(\Gamma) \cong H^k(\mathcal{M}_\Gamma)$ comes with a mixed Hodge structure whose weights are $\geq k$ and $\leq 2k$.

Purity of stable classes. We now take S connected. Precisely, given nonnegative integers g, n with $2g - 2 + n > 0$, we fix a compact connected oriented surface S_g of genus g and pairwise distinct points $x_1, \dots, x_n$ on S_g. Then $S_{g,n} := S_g - \{x_1, \dots, x_n\}$ is hyperbolic in the sense above. The connected component group of the group of orientation preserving diffeomorphisms of S_g which fix each x_i can be regarded as a normal subgroup of $\Gamma(S_{g,n})$ with factor group the permutation group of degree n. We denote this group $\Gamma_{g,n}$ and write $\mathcal{M}_{g,n}$ for $\mathcal{M}_{\Gamma_{g,n}}$. If we choose $x = x_{n+1} \in S_{g,n}$, then we have a forgetful map $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ (fill x back in) which may in some sense be thought of as a universal family of punctured Riemann surfaces. This naturally extends as a morphism $\overline{\pi} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ between their Deligne-Mumford compactifications whose restriction over $\mathcal{M}_{g,n}$, $\pi : \overline{\pi}^{-1}\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$ may, again in some sense, be understood as a universal family of *pointed* Riemann surfaces: the difference $\overline{\pi}^{-1}\mathcal{M}_{g,n} - \mathcal{M}_{g,n+1}$ consists of the images of pairwise disjoint sections $s_1, \dots, s_n$ of π. Each s_i extends to a section $\bar{s}_i$ of $\bar{\pi}$ that takes it values in the Deligne-Mumford boundary[1].

A complex structure on $S_{g,n}$ defines one on $T_x^* S_{g,n}$. This gives rise to a line bundle $\mathscr{L}$ on $\overline{\mathcal{M}}_{g,n+1}$, whose Chern class we denote by $\bar{\psi} \in H^2(\overline{\mathcal{M}}_{g,n+1})$. From this class we obtain *tautological classes* on $\overline{\mathcal{M}}_{g,n}$: $\bar{\psi}_i := s_i^*(\bar{\psi}) \in H^2(\overline{\mathcal{M}}_{g,n})$ of Hodge type $(1,1)$ $(i = 1, \dots, n)$ and $\bar{\kappa}_r := \bar{\pi}_*(\bar{\psi}^{r+1}) \in H^{2r}(\overline{\mathcal{M}}_{g,n})$ of Hodge type (r,r) $(r = 1, 2, \dots)$. Here $\bar{\pi}_*$ is 'integration along the fibers' (when we use Poincaré duality on both source and target to identify cohomology with homology, then this is just the induced map on homology). These classes define a homomorphism of graded $\mathbb{Q}$-algebras:

$$\mathbb{Q}[K_1, K_2, \dots] \otimes \mathbb{Q}[\Psi_1, \Psi_2, \dots, \Psi_n] \to H^\bullet(\overline{\mathcal{M}}_{g,n}).$$

The theorem of Madsen-Weiss tells us (when combined with the theorems of Harer) that its composite with the restriction map $H^\bullet(\overline{\mathcal{M}}_{g,n}) \to H^\bullet(\mathcal{M}_{g,n})$ is an isomorphism in degrees $< 2g/3$. Before this was proved, it was shown by Pikaart [5] that $H^\bullet(\overline{\mathcal{M}}_{g,n}) \to H^\bullet(\mathcal{M}_{g,n})$ is onto in this range (so that the mixed Hodge structure on this part of $H^\bullet(\mathcal{M}_{g,n})$ is 'pure'). A simple instance of Pikaart's argument appears in the following observation.

Suppose M is a nonsingular complex variety and $\overline{M}$ is a nonsingular compactification such that $D := \overline{M} - M$ is a normal crossing divisor. If the first Chern class of the normal bundle of $D_{\rm reg}$ in $\overline{M}$ is nonzero on every connected component of $D_{\rm reg}$, then the map $H^1(\overline{M}) \to H^1(M)$ is an isomorphism. In particular, when $H^1(M) \neq 0$, then there exists a nonzero holomorphic differential on $\overline{M}$.

As Putman noted [6], this applies to the orbifold $M = \mathcal{M}_\Gamma$ and its Deligne-Mumford compactification when $\Gamma \subset \Gamma_{g,n}$ of finite index and $g \geq 3$. This is of interest in view of the *Ivanov conjecture* which states that then $H^1(\Gamma) = 0$. So when the conjecture fails for Γ, then $\overline{\mathcal{M}}_\Gamma$ admits a nonzero holomorphic differential (as an orbifold). This leads us to ask:

Question 1. Let for $g \geq 2$, $\Delta_g \subset \Gamma_g$ be a subgroup of finite index that is 'sufficiently natural' in its dependence on g (for instance, the kernel of the Γ_g-action on $H^1(S_g; \mathbb{Z}/m)$ for a fixed m). Is then $H^k(\overline{\mathcal{M}}_{\Delta_g}) \to H^k(\mathcal{M}_{\Delta_g}) \cong H^k(\Delta_g)$ onto (or equivalently, is $H^k(\Delta_g)$ pure) when $g \gg 0$?

Representations of the mapping class group. The Birman exact sequence

$$1 \to \pi_1(S_{g,n}, x) \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1$$

can be regarded as the fundamental group sequence of the fibration $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. Let $K \subset \pi_1(S_{g,n}, x)$ be a subgroup of finite index that is normal in $\Gamma_{g,n+1}$ and put $G := \pi_1(S_{g,n}, x)/K$ (a finite group) and $N_K := \Gamma_{g,n+1}/K$. We have then a

[1]To be precise, $\bar{s}_i$ is induced from a section of $\overline{\mathcal{T}}(S_{g,n+1}) \to \overline{\mathcal{T}}(S_{g,n})$ that is given by an embedded closed disk $D \subset S_g - \{x_1, \dots, \widehat{x}_i, \dots, x_n\}$ whose interior $\mathring{D}$ has been endowed with a complex structure and contains x_i and x: compose the isomorphism $\overline{\mathcal{T}}(S_{g,n}) \cong \overline{\mathcal{T}}(S_{g,n} \smallsetminus D)$ defined by the natural isotopy class of diffeomorphisms $S_{g,n} \to S_{g,n} \smallsetminus D$ with the embedding $\overline{\mathcal{T}}(S_{g,n} \smallsetminus D) \hookrightarrow \overline{\mathcal{T}}(S_{g,n+1} - \partial D) \subset \mathcal{T}(S_{g,n+1})$ defined by the complex structure on $\mathring{D}$.

G-covering $S_K \to S_{g,n}$ and N_K can be understood as the group of mapping classes of S_K that lift one of $S_{g,n}$. Such lifts are unique up to a covering transformation and the exact sequence $1 \to G \to N_K \to \Gamma_{g,n} \to 1$ expresses this fact. The group N_K acts on G by group automorphisms; the kernel $N_K^\flat$ of this action is the centraliser of G in N_K and is clearly of finite index in N_K. The covering $S_K \to S_{g,n}$ extends naturally to a ramified covering $\overline{S}_K \to S_g$ (with $\overline{S}_K$ a compact oriented surface) so that $H_K := H^1(\overline{S}_K)$ defines a symplectic representation $\rho_K : N_K \to \mathrm{Sp}(H_K)$ with $\rho_K(N_K^\flat) \subset \mathrm{Sp}_G(H_K)$. This makes $\rho(N_K^\flat)$ appear as the monodromy of a family $\mathcal{C}_K \to \mathcal{M}_\Gamma$ with fiber $\overline{S}_K$, where $\Gamma \subset \Gamma_{g,n}$ is the image of $N_K^\flat$. According to Deligne such a representation is semisimple. Hence the Zariski closure $\mathcal{G}_K \subset \mathrm{Sp}(H_K)$ of $\rho(N_K)$ is also semisimple with its identity component $\mathcal{G}_K^\circ$ mapping to $\mathrm{Sp}_G(H_K)$. We pose the following questions without offering a conjectured answer.

Question 2. Is $H_K^{\mathcal{G}_K^\circ} = \{0\}$, when $g \geq 3$? This is a reformulation of a question asked by Putman-Wieland [7]; they showed that a yes answer implies that the Ivanov conjecture holds in genus ≥ 4. This question is also of interest to algebraic geometers because this property is detectable infinitesimally via the period map: a theorem of Deligne [2] implies that the Hodge structure that we get on $H_K^{\mathcal{G}_K^\circ}$ when we give S_g a complex structure is *independent of that complex structure*. As Avila-Matheus-Yoccoz observed (personal communication), the answer is *no* for $g = 2$: an example is provided by the cyclic cover of degree 6 of the Riemann sphere which totally ramifies in 6 distinct points; such a cover factors through the degree 2 cover, which is in fact the general genus 2 curve (this example appears in the work of Deligne-Mostow).

Question 3. Do we have $\mathcal{G}_K^\circ = \mathrm{Sp}_G(H_K)$?

Question 4. Is $\rho(N_K)$ arithmetic in $\mathcal{G}_K$?

Note that a 'yes' to Q3 implies also a 'yes' to Q2 and so the Ivanov conjecture would then follow for $g \geq 4$. Some time ago [4] I proved that the answer is yes for both Q3 and Q4 when $n = 0$ and G abelian. The recent work of Grünewald-Larsen-Lubotsky-Malestein [3] should furnish many examples with $n = 0$ and G non-commutative.

Potential quantum representations. Let $\alpha \subset S_{g,n}$ be an oriented embedded circle and denote by τ_α the associated Dehn twist. Every connected component $\tilde{\alpha}$ of $p^{-1}\alpha$ has the same degree m_α over α and hence $\tilde{\tau}_\alpha := \prod_{\tilde{\alpha}/\alpha} \tau_{\tilde{\alpha}}$ is lift of $\tau_\alpha^{m_\alpha}$ which lies in $N_K^\flat$. The action of this lift on H_K is a unipotent transformation given by the Picard-Lefschetz formula. Its associated 1-parameter subgroup of $\mathcal{G}_K$ is

$$T_\alpha : \mathbb{G}_a \hookrightarrow \mathcal{G}_K, \quad T_\alpha(\lambda) : v \mapsto v + \lambda \sum_{\tilde{\alpha}/\alpha} ([\tilde{\alpha}] \cdot v)[\tilde{\alpha}].$$

The subgroup $\mathcal{D}_K \subset \mathcal{G}_K$ generated by such 1-parameter groups is a normal connected subgroup of $\mathcal{G}_K$ that is defined over $\mathbb{Q}$. Hence $\mathcal{D}_K$ is also semisimple. Marco Boggi and I [1] observed that the subspace $P_K \subset H_K$ spanned by the classes $[\tilde{\alpha}]$ that are obtainable as above (so with also α varying) is the symplectic perp of $H_K^{\mathcal{D}_K}$. By Deligne's semisimplicity theorem, P_K must then be nondegenerate for the symplectic form on H_K so that we have a symplectic decomposition $H_K = P_K \oplus H_K^{\mathcal{D}_K}$ with the second summand containing the (possibly trivial) $H_K^{\mathcal{G}_K^\circ}$.

In view of the dependence of Q3 on the Ivanov conjecture, it is more reasonable to first address the following

Conjecture. For $g \geq 3$, $\mathcal{D}_K = \mathrm{Sp}_G(P_K)$ and $\rho(N_K) \cap \mathcal{D}_K$ is arithmetic.

Presumably the results of [3] imply that for $n = 0$ this conjecture has a positive answer when asked for the part of P_K on which G acts through a quotient with at most $g-1$ generators.

Notice that the group $\mathcal{G}_K$ acts on $H_K^{\mathcal{D}_K}$ via the semisimple $\mathbb{Q}$-group $\mathcal{G}_K/\mathcal{D}_K$. Since $\Gamma_{g,n}$ is generated Dehn twists, N_K is generated by lifts of Dehn twists and as we just saw, any lift of $\tau_\alpha \in \Gamma_{g,n}$ in N_K maps to a torsion element of $\mathcal{G}_K/\mathcal{D}_K$ of order divisible by m_α. On the other hand, the image of N_K in $\mathcal{G}_K/\mathcal{D}_K$ is also Zariski dense and so we ask:

Question 5. Are there any examples for which $H_K^{\mathcal{D}_K} \neq 0$ and if so, with $\mathcal{G}_K^\circ/\mathcal{D}_K$ acting nontrivially on it?

The cases covered by [4] and [3] do not produce such examples, for we then have $P_K = H_K$ so that $H_K^{\mathcal{D}_K} = 0$ (and Q3 is equivalent to our conjecture). I have no idea what the situation is in general[2], but I would in fact be pleased if the answer to the stronger version of Q5 were yes. For we then obtain representions of N_K (some of which could define a projective representation of $\Gamma_{g,n}$) in $H_K^{\mathcal{D}_K}$ with the property that the lifts of Dehn twists act with finite order and since the quantum representations also have this property, it is then natural to ask:

Question 6. Is any quantum representation of the mapping class group (i.e., one arising from the theory of conformal blocks) obtained as a *complex* subrepresentation of this type?

The quantum representations are also conjectured to be unitary and indeed, we would then expect this inner product to come from the intersection pairing on $H_K^{\mathcal{G}_K^\circ}$ (yielding a compact factor of $\mathcal{G}_K^\circ/\mathcal{D}_K(\mathbb{R})$).

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[2] After my talk Julien Marché referred me to his Mathoverflow posting of January 2012 (`http://mathoverflow.net/questions/86894/`), where he asks (in the somewhat more restricted context of no ramification) whether $P_K = H_K$. Boggi and I had been wondering about that, too. Note that in view of the example mentioned above, we have to assume in our setting (where we allow ramification) that $g > 2$. It is a bit of a scandal that the answer is not known.

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Arithmetic quotients of the mapping class group

Alexander Lubotzky, Justin Malestein
(joint work with Fritz Grunewald, Michael Larsen)

It is a classical theorem that there exists a surjective homomorphism $\mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ arising from the action of the mapping class group, $\mathrm{Mod}(\Sigma_g)$, on $\mathrm{H}_1(\Sigma_g, \mathbb{Z})$. We show that this is only one example in a rich collection of virtual arithmetic quotients of $\mathrm{Mod}(\Sigma_g)$. Specifically, we produce a "machine" which, for any finite group H with fewer than g generators and an irreducible $\mathbb{Q}$ representation r, produces an arithmetic group $\mathscr{G}(\mathscr{O})$, a finite index subgroup $\Gamma < \mathrm{Mod}(\Sigma_g)$, and a representation $\rho_{H,r}\colon \Gamma \to \mathscr{G}(\mathscr{O})$ whose image is of finite index. The classical representation $\mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ corresponds to the choice of the trivial representation of H.

The algebraic group $\mathscr{G}$ is produced, for nontrivial representations r, as follows. The group ring $\mathbb{Q}[H]$ is isomorphic to a product $\mathbb{Q} \times \prod_{i=1}^{\ell} A_i$ of simple $\mathbb{Q}$-algebras where the factors are in one-to-one correspondence with irreducible $\mathbb{Q}$ representations r_i. We equip the module A_i^{2g-2} with an A_i-valued skew-Hermitian sesquilinear form such that on an A_i basis $x_1, \ldots, x_{g-1}, y_1, \ldots, y_{g-1}$ all pairings are zero except that $\langle x_j, y_j \rangle = -\langle y_j, x_j \rangle = 1$ for all j. The algebraic group $\mathscr{G}$ is $\mathrm{Aut}_{A_i}(A_i^{2g-2}, \langle -, - \rangle)$. (More details about $\mathscr{G}$ are available in [1]). By inputting various pairs H, r into our machine, we produce the following types of virtual arithmetic quotients.

Corollary 1 ([1])**.** *For all* $g \geq 1$ *and* $m \in \mathbb{N}$*, there exist virtual surjections of* $\mathrm{Mod}(\Sigma_g)$ *onto arithmetic groups of type* $\mathrm{Sp}(2m(g-1)), \mathrm{SU}(m(g-1), m(g-1))$*, and* $\mathrm{SO}(2m(g-1), 2m(g-1))$.

The representations $\rho_{H,r}$ are easy to define but difficult to understand. Given a surjective map $p\colon \pi_1(\Sigma_g) \to H$ where H is a finite group, there is a corresponding regular (Galois) cover $\hat{\Sigma} \to \Sigma_g$ with deck group H. Any mapping class preserving $\ker(p)$ and acting trivially as an automorphism on H can be lifted to a mapping class of $\hat{\Sigma}$ which lies in the centralizer of H. Modulo some technical details, there is a well-defined homomorphism on a finite index subgroup $\Gamma \to \mathrm{Cent}_{\mathrm{Mod}(\hat{\Sigma})}(H)$, and we can compose to get a representation $\rho_{H,p}\colon \Gamma \to \mathrm{Cent}_{\mathrm{Sp}(2\hat{g}, \mathbb{Q})}(H)$. By a theorem of Chevalley and Weil, it follows that $\mathrm{H}_1(\hat{\Sigma}, \mathbb{Q}) \cong \mathbb{Q}^2 \oplus \mathbb{Q}[H]^{2g-2}$ as a $\mathbb{Q}[H]$-module. With some work, one can show that the target group decomposes as a product $\mathrm{Sp}(2g, \mathbb{Q}) \times \prod_{i=1}^{\ell} \mathrm{Aut}_{A_i}(A_i^{2g-2}, \langle -, - \rangle)$ for an appropriate sesquilinear skew-Hermitian form $\langle -, - \rangle$.

Theorem 2 ([1]). *Let $g \geq 3$ and suppose p is ϕ-redundant. Then the map $\rho_{H,p,i}$ which is the projection of $\rho_{H,p}$ onto the factor $\mathscr{G} = \mathrm{Aut}_{A_i}(A_i^{2g-2}, \langle -, - \rangle)$, is a virtual epimorphism onto $\mathscr{G}(\mathscr{O})$.*

Here, "ϕ-redundant" means that p factors through a surjective map $\phi\colon \pi_1(\Sigma_g) \to F_g$ to $p'\colon F_g \to H$ where p' kills some generator of F_g. Going back to our machine, given an H with fewer than g generators and an r, one can find a ϕ-redundant $p\colon \pi_1(\Sigma_g) \to H$, and then $\rho_{H,r} = \rho_{H,p,i}$ where i is the index of the factor A_i corresponding to r. In an earlier paper, Looijenga showed, for the special case of abelian H (without requiring p to be ϕ-redundant), that $\rho_{H,p}$ is a virtual epimorphism [4].

One difficulty in establishing surjectivity is that it is unclear what lifts of (powers of) Dehn twists generate. In addition to analyzing a few Dehn twists, one key step in our proof uses the ϕ-redundancy of p to relate the problem to one on $\mathrm{Aut}(F_g)$. In [2], Grunewald and Lubotzky study virtual representations of $\mathrm{Aut}(F_g)$ to $\mathscr{G}(\mathscr{O})$ where $\mathscr{G} = \mathrm{GL}_{g-1}(A_i^{\mathrm{op}})$. These representations are defined in a similar manner as for $\mathrm{Mod}(\Sigma_g)$, and they show (in the case of redundant maps $F_g \to H$) that the image contains a finite index subgroup of the elements of reduced norm 1 in $\mathscr{G}(\mathscr{O})$. After some work, one can use this to deduce that the image of $\rho_{H,p}$ (for $\mathrm{Mod}(\Sigma_g)$) contains a large number of parabolic elements. Combining this with the image of some Dehn twists and some further arguments, one can deduce arithmeticity. (See [1] for details.)

Our theorem can be used to recover an earlier result of Masbaum and Reid that every finite group is the quotient of a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ for all $g \geq 1$ [7]. In earlier works, the representations $\rho_{H,p}$ were used to prove that $\mathrm{Mod}(\Sigma_2)$ virtually maps onto a free group [8, 3] and that pseudo-Anosov classes are generic in the Torelli subgroup of $\mathrm{Mod}(\Sigma_g)$ [6, 5].

In general, it remains open whether or not $\rho_{H,p}$ is virtually surjective (onto each component of) $\mathrm{Cent}_{\mathrm{Sp}(2\hat{g},\mathbb{Z})}(H)$ for arbitrary H, p. While it is unclear what the answer should be, an affirmative answer (that includes surfaces with a single boundary component) would, via a result of Putman and Wieland [9], prove a conjecture of Ivanov that mapping class groups (for $g \geq 3$) do not virtually surject onto $\mathbb{Z}$.

As $\rho_{H,p}$ is a generalization of the classical $\mathrm{Sp}(2g, \mathbb{Z})$ representation, so is each kernel a generalized "Torelli" group. Moreover, as established in [1], these are different from the terms of the Johnson filtration. Thus, we may now investigate all these new "Torelli" groups as we have investigated the original Torelli group. E.g. are they finitely generated? Can we compute their first homology groups or cohomological dimension, etc.?

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Introduction to quantum representations of mapping class groups

Julien Marché

I reviewed some motivations and examples of constructions of finite-dimensional unitary and projective representations of the mapping class group of a surface of genus g.

The geometric origin of such representations comes from the "quantization of character varieties". Given a compact Lie group G, the character variety is the space of conjugacy classes of representations of the fundamental group of the surface into G. This space is a singular symplectic manifold on which the mapping class group acts naturally. There are various procedures for quantizing the space at level k and the action, which produce the so-called quantum representations. I gave the example of the group U(1), which can be written down explicitly using theta functions and produces the discrete metaplectic representations. Then I presented a procedure called "universal construction" which constructs a representation of the mapping class group from an invariant of closed 3-manifolds. With some luck, this gives quantum representations. I developed two examples, one being related to the case of a finite group G, the other one – more involved – using spin structures and the Rochlin invariant of 3-manifolds. These representations already present general features of quantum representations; that is, Dehn twists have finite order and the matrices have entries in some cyclotomic ring of integers.

Then I explained one possible construction of the quantum representations for SU_2 at level k which rely on the combinatorics of the Kauffman bracket at a root of unity of order $4k$. One can find an explicit basis for this vector space whose dimension is given by the Verlinde formula, which I recalled. Finally, I listed the main properties of these representations and finished my talk with three open questions, the detection of Thurston's classification of mapping classes, the arithmeticity of the image of quantum representations and the τ-property for the family of quantum representations.

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Constructing hyperbolic four-manifolds

Bruno Martelli

(joint work with Alexander Kolpakov)

As a consequence of Margulis's Lemma, every finite volume complete hyperbolic n-manifold M is diffeomorphic to the interior of a compact n-manifold N with (possibly empty) boundary $\partial N = X_1 \sqcup \ldots \sqcup X_k$. Each X_i has an induced flat metric (unique up to rescaling), and the end of M near X_i is a *cusp*, that is a portion isometric to $X_i \times [0, +\infty)$ where the slice $X_i \times t$ is just the flat X_i rescaled by e^{-2t}.

In particular, each X_i admits a flat metric and hence by Bieberbach's Theorem it is finitely covered by an $(n-1)$-torus. Using Selberg's Lemma, up to passing to a finite cover we can ensure that every X_i is indeed an $(n-1)$-torus.

Complete hyperbolic manifolds – with or without cusps – exist in every dimension $n \geq 2$. Moreover, in every dimension $n \geq 4$ there are only finitely many $\rho_n(V)$ of them with volume bounded by V. It is then natural to estimate the growth of the function $\rho_n(V)$: the following result was proved in [1].

Theorem 1. *For every $n \geq 4$ there are constants $0 < C_1 < C_2$ such that*

$$C_1^{V \ln V} < \rho_n(V) < C_2^{V \ln V}$$

for all sufficiently big V.

As we mentioned above, the theorem is also valid if we restrict our attention either to closed, or to cusped hyperbolic n-manifolds only.

Thanks to this theorem, we know that there are plenty of cusped hyperbolic n-manifolds in every dimension n, and we now would like to refine this investigation by controlling the number of cusps.

Let $\rho_n^c(V)$ be the number of complete hyperbolic n-manifolds with exactly $c > 0$ cusps with volume smaller than V. What do we know about the function $\rho_n^c(V)$? We can contribute in [2] with an answer in dimension $n = 4$.

Theorem 2. *For every $c \geq 1$ there are constants $0 < C_1 < C_2$ such that*

$$C_1^{V \ln V} < \rho_4^c(V) < C_2^{V \ln V}$$

for all sufficiently big V.

This shows in particular that there exist hyperbolic 4-manifolds with any given number $c > 0$ of cusps, that they are actually infinitely many, and that their number grows with V as much as possible.

The question of estimating $\rho_n^c(V)$ in higher dimension remains widely open: for instance, no example of a single-cusped n-manifold is apparently known in any dimension $n \geq 5$. One should compare with the following striking result of Stover [4]:

Theorem 3. *For every $c > 0$ there are only finitely many commensurability classes of arithmetic hyperbolic manifolds (of any dimension!) with at most c cusps.*

This shows in particular that there are no c-cusped arithmetic hyperbolic manifolds above a certain dimension n that depends only on c, for every $c > 0$.

The proof of Theorem 2 uses a peculiar and beautiful four-dimensional right-angled polytope C, the *ideal 24-cell*. This regular hyperbolic Coxeter polytope has already been used successfully to provide the largest census of hyperbolic four-manifolds known in the literature [3]: the Ratcliffe–Tschantz paper contains a table of 1171 hyperbolic cusped manifolds, all obtained by pairing the facets of C. All the manifolds in this table have either 5 or 6 cusps.

To build plenty of hyperbolic four-manifolds with any given number $c > 0$ of cusps, we note that C has 24 octahedral facets, naturally distributed into three sets of eight pairwise disjoint ones, which we color respectively in blue, red, and green. We can pick any regular 8-valence graph G as a pattern and glue some copies of C along the blue facets via this pattern. Then we double the boundary of the resulting object along the red facets, and then we double again along the green ones. The result is a cusped hyperbolic four-manifold, whose number c of cusps can be controlled combinatorially by choosing carefully the initial gluings of the blue facets.

There are more than $C^{v \ln v}$ regular 8-valence graphs with v vertices, hence via this method we can construct more than $C^{V \ln V}$ manifolds with volume at most V. To prove that different graphs G give rise to non-isometric manifolds, we prove that G is intrinsically determined by the canonical Epstein–Penner decomposition of the manifold.

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Maximal representations of complex hyperbolic lattices

Maria Beatrice Pozzetti

In the talk I discussed the results of [8] about maximal representations of a complex hyperbolic lattice Γ with values in $\mathrm{SU}(m,n)$. It is well known that lattices in $\mathrm{SU}(1,p) = \mathrm{Isom}(\mathbb{H}^p_{\mathbb{C}})$ are in general not superrigid, for example some complex hyperbolic lattices surject onto nonabelian free groups, and hence their representation theory is extremely wild. In general not much is known about linear representations of these groups.

A Lie group H is *Hermitian* if its associated symmetric space $\mathscr{X}$ admits a complex structure. If the target of a representation ρ of Γ has this property, it is possible to use the Kähler form $\omega_{\mathscr{X}}$ of $\mathscr{X}$ to define a characteristic invariant on the representation variety $\mathrm{Hom}(\Gamma, H)$, the socalled *Toledo invariant* $T(\rho)$. In the case of cocompact lattices the definition of $T(\rho)$ is straightforward:

$$T(\rho) = \frac{1}{p!} \int_{\Gamma\backslash\mathbb{H}^p_{\mathbb{C}}} f^*\omega_{\mathscr{X}} \wedge \omega^{p-1}_{\mathbb{H}^p_{\mathbb{C}}}$$

for any ρ-equivariant map $f : \mathbb{H}^p_{\mathbb{C}} \to \mathscr{X}$ (it is easy to check that, since $\mathscr{X}$ is contractible, the definition of T does not depend on the choice of f). The situation is more delicate in the case of nonuniform lattices, but Burger and Iozzi managed to show [1], with the aid of bounded cohomology, that in all cases $T(\rho)$ is well defined and satisfies the Milnor–Wood inequality $|T(\rho)| \leq \mathrm{rk}(\mathscr{X})\mathrm{vol}(\Gamma\backslash\mathbb{H}^p_{\mathbb{C}})$. Representations for which the equality holds are particularly interesting and are referred to as *maximal representations.*

Maximal representations of lattices in $\mathrm{SU}(1,p)$ are conjecturally rigid if $p > 1$, for target of rank one this was proven independently by Koziarz and Maubon [6] and by Burger Iozzi [2], extending previous results of Goldman and Millson [4] and Corlette [3]. In the case of cocompact lattices Koziarz and Maubon were also able in [7] to prove rigidity for maximal representations in classical Hermitian Lie groups of rank 2. In [8] I confirmed, under some mild nondegeneracy hypothesis, that maximal representations of complex hyperbolic lattices are superrigid:

Theorem 1. *Let Γ be a lattice in $\mathrm{SU}(1,p)$ with $p > 1$. If m is different from n, then every Zariski dense maximal representation of Γ into $\mathrm{SU}(m,n)$ is the restriction of a representation of $\mathrm{SU}(1,p)$.*

In particular this implies that the only Zariski dense maximal representation of a complex hyperbolic lattice is the lattice embedding in $\mathrm{SU}(1,p)$, and in general all maximal representations in $\mathrm{SU}(m,n)$ are diagonal embeddings up to factors of tube type in the Zariski closure and homomorphisms in the compact centralizer of the image. A notable corollary of Theorem 1 is the following local rigidity theorem, which is a generalization of a result of Klingler [5]:

Corollary 2. *Let Γ be a lattice in $\mathrm{SU}(1,p)$, with $p > 1$, and let ρ be the restriction to Γ of the diagonal embedding of m copies of $\mathrm{SU}(1,p)$ in $\mathrm{SU}(m, pm+k)$. Then ρ is locally rigid.*

The strategy of my proof of Theorem 1 is similar Margulis' proof of superrigidity for higher rank lattices: in order to show that a representation $\rho : \Gamma \to H$ extends to the group G in which Γ sits as a lattice, it is enough to exhibit a ρ-equivariant algebraic map $\phi : G/P \to H/L$ for some parabolic subgroups P of G and L of H. The only compact $\mathrm{SU}(1,p)$-space is the boundary $\partial\mathbb{H}^p_{\mathbb{C}}$ consisting of isotropic lines for the Hermitian form defining $\mathrm{SU}(1,p)$, in the case of $\mathrm{SU}(m,n)$ the compact space that turns out to be relevant to this problem is the socalled *Shilov boundary* $\mathcal{S}_{m,n}$ which identifies with the set of maximal isotropic subspaces of $\mathbb{C}^{m+n}$. If $p > 1$ and $n > m$ these two spaces carry an interesting incidence structure: given two (transversal) points x, y in such a space the *line* containing them is the subset of isotropic subspaces contained in the linear span $\langle x, y\rangle$. In the case of $\partial\mathbb{H}^p_{\mathbb{C}}$ this incidence structure was first studied by E.Cartan, who called these lines *chains*.

Exploiting work of Clerc and Ørsted and properties of bounded cohomology, Burger and Iozzi proved that a measurable boundary map that is equivariant with respect to a maximal representation needs to preserve this incidence geometry and hence the main step in my proof of Theorem 1 is the proof of the following theorem which seems, at first glance, unrelated.

Theorem 3. *Let $p > 1$, $1 < m < n$ and let $\phi : \partial\mathbb{H}^p \to \mathcal{S}_{m,n}$ be a measurable map whose essential image is Zariski dense. Assume that for almost every triple with $\dim\langle x, y, z\rangle = 2$, it holds $\dim\langle \phi(x), \phi(y), \phi(z)\rangle = 2m$. Then ϕ coincides almost everywhere with a rational map.*

In the talk I also explained one main idea in the proof of Theorem 3 for which I here refer to [8].

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The high dimensional cohomology of $\mathrm{SL}_n\mathcal{O}$

Andrew Putman
(joint work with Thomas Church, Benson Farb)

Let $\mathcal{O}$ be a number ring and let ν be the virtual cohomological dimension of $\mathrm{SL}_n\mathcal{O}$. We discussed the following two contrasting theorems, which appear in our paper [1] (joint with Church and Farb). Denote by $\mathrm{cl}\,\mathcal{O}$ the class number of $\mathcal{O}$.

Theorem 1. $\dim_{\mathbb{Q}} \mathrm{H}^{\nu}(\mathrm{SL}_n\mathcal{O}; \mathbb{Q}) \geq (\mathrm{cl}\,\mathcal{O} - 1)^{n-1}$.

Remark 2. Theorem 1 is classical and not particularly difficult for $n = 2$. Difficulties arise for $n \geq 3$ because of the complicated topology of the boundary of the Borel–Serre compactification of the associated locally symmetric space.

Theorem 3. *Assume that* $\mathrm{cl}\,\mathcal{O} = 1$. *Also, assume either that* $\mathcal{O} \subset \mathbb{R}$ *or that* $\mathcal{O}$ *is Euclidean. Then* $\mathrm{H}^{\nu}(\mathrm{SL}_n\mathcal{O}; \mathbb{Q}) = 0$.

Remark 4. Theorem 3 was originally proved by Lee–Szczarba [2] when $\mathcal{O}$ is Euclidean. However, our proof is quite different from their proof; indeed, our proof is also able to show vanishing for certain nontrivial coefficient systems.

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4-manifolds can be surface bundles over surfaces in many ways

Nick Salter

1. A brief tour of surface bundles

Definition 1. *A* surface bundle *is a fiber bundle*

$$\Sigma_g \to E \to B$$

with fiber Σ_g*, a surface of genus* g*. For the purposes of the talk,* B *(and hence* E*) will be a smooth manifold, and also* $g \geq 2$.

The fundamental invariant associated to a surface bundle is its *monodromy representation.*

Definition 2. *Let* $\Sigma_g \to E \to B$ *be a surface bundle. The* monodromy representation *is the homomorphism*

$$\rho : \pi_1 B \to \mathrm{Mod}(\Sigma_g)$$

that records the isotopy class of the diffeomorphism obtained by parallel transporting the fiber over the basepoint around loops in B *(here* $\mathrm{Mod}(\Sigma_g)$ *denotes the mapping class group of* Σ_g*).*

Surface bundles appear in various contexts in 4-dimensional topology.

Theorem 3 (Chern, Hirzebruch, Serre). *Suppose*

$$M \to E \to N$$

is a fiber bundle with M, E, N all closed oriented manifolds. Suppose the action of $\pi_1 N$ on $H^*(M)$ is trivial. Then the signature $\sigma(E)$ is computed as

$$\sigma(E) = \sigma(M)\sigma(N).$$

(This formula is valid even when the dimensions of M, E, N are not all divisible by 4, under the stipulation that $\sigma = 0$ in these cases).

An example of Atiyah and Kodaira shows that the assumption on the action of $\pi_1 N$ on H^*M is necessary.

Theorem 4 (Atiyah, Kodaira). *There exists a 4-manifold E that is the total space of a surface bundle over a surface, for which $\sigma(E) \neq 0$.*

The construction proceeds by taking a suitable fiberwise-branched covering $E \to \Sigma_g \times \Sigma_g$ of a product. The key is to equip Σ_g with a free involution τ, so that for each $z \in \Sigma_g$, the points $z, \tau(z)$ are distinct. As z varies in Σ_g, this parametrizes different points of branching, giving rise to a nontrivial surface bundle. In fact, one can check that the projection onto either factor in $\Sigma_g \times \Sigma_g$ yields *two distinct* surface bundle structures on E.

2. The monodromy-topology dictionary

The classifying space $\mathrm{BDiff}(\Sigma_g)$ for surface bundles has a remarkable property.

Theorem 5. $\mathrm{BDiff}(\Sigma_g)$ *is an Eilenberg-MacLane space* $K(\mathrm{Mod}(\Sigma_g, 1)$.

Corollary 6. *Σ_g-bundles over B are in $1-1$ correspondence with monodromy representations* $\rho\colon \pi_1 B \to \mathrm{Mod}(\Sigma_g)$.

This raises the question of translating between geometric and topological properties of surface bundles on the one hand, and algebraic/geometric/dynamical properties of the monodromy representations on the other. The main question of the talk is in this spirit: *what properties of the monodromy representation are related to the existence of more than one surface bundle structure on the total space?*

3. Multiple fiberings in dimensions 3 and 4

After Thurston, it is known that if M^3 fibers as a Σ_g-bundle over S^1, and if $b_1(M) \geq 2$, then M in fact has infinitely many distinct surface bundle structures. The situation in 4 dimensions is quite different.

Theorem 7 (F.E.A. Johnson). *Every 4-manifold E fibers as a surface bundle over a surface in finitely many ways.*[1]

[1] I am glossing over when I consider two fiberings to be equivalent. Fiberwise-diffeomorphism is suitable for these purposes.

Johnson's argument provides an upper bound on the number of fiberings that is superexponential as a function of $d = \chi(E)$, the Euler characteristic.

4. Obstructions

The following theorems provide obstructions for a surface bundle over a surface to admit additional fiberings; these criteria are decidable using only the datum of the monodromy representation.

Theorem 8 (S—). *Let $\Sigma_g \to E \to \Sigma_h$ be a surface bundle with monodromy ρ. Suppose the invariant fiber cohomology $(\mathrm{H}^1\Sigma_g)^\rho = \{0\}$. Then E has no other fiberings.*

Theorem 9 (S—). *Suppose $\Sigma_g \to E \to \Sigma_h$ has monodromy contained in the group $\mathscr{K}_g$ generated by twists about separating curves. Then either $E \cong \Sigma_g \times \Sigma_h$, or else E has no other fiberings.*

5. Constructions

To complement the non-existence results of the previous section, the following theorem provides examples of 4-manifolds that have arbitrarily many fiberings as surface bundles.

Theorem 10 (S—). *For any n, there are distinct integers $g_1, \dots g_n$, and a 4-manifold E_n such that E_n fibers over surfaces $B_1, \dots B_n$ with fibers of genera $g_1, \dots, g_n$ respectively.*

The construction methods can be done "efficiently" with respect to the Euler characteristic of $\chi(E_n)$, in the sense that the above theorem furnishes an exponential lower bound on the number of fiberings, to complement F.E.A. Johnson's superexponential upper bound.

The construction method is similar in spirit to that of Atiyah and Kodaira. In this setting, one performs a fiberwise *connect-sum* of two product bundles $\Sigma_g \times \Sigma_g$, varying the points of connection as in the Atiyah–Kodaira example. The two projections $\Sigma_g \times \Sigma_g \to \Sigma_g$ on each half of the bundle can be coherently amalgamated, leading to 4 distinct surface bundle structures. This construction can be suitably generalized to prove Theorem 10.

On arithmetically defined groups and their cohomology – some examples, and some results

Joachim Schwermer

It was the aim of this talk to give a reasonably detailed account of a specific bundle of investigations and results pertaining to arithmetic groups, the geometry of the corresponding generalized locally symmetric space X/Γ attached to a given arithmetic subgroup $\Gamma \subset G$ of a reductive algebraic group G defined over some algebraic number field k and its cohomology groups $\mathrm{H}^*(X/\Gamma, E)$, E a rational finite-dimensional representation of G. One way to gain insight into the cohomology of an arithmetic group is the study of Lefschetz numbers of homomorphisms in cohomology induced be automorphisms of finite order of the underlying algebraic group G/k. Of course, the computation of Euler characteristics is the most prominent example [attached to the identity] but in most cases the Euler characteristic vanishes; this is a consequence of the fact that in many cases the real rank of the real Lie group G_∞ differs from the one of its maximal subgroups. The idea to use Lefschetz numbers in such cases goes back to Harder [1] in the case SL_2/k; it was put on firm grounds in the general framework by Rohlfs [5], [6], [7], later on pursued in [8], [9], [10] in different directions. The method has had various applications, ranging from the existence of cuspidal automorphic representations in the case $\mathrm{SL}_3/\mathbb{Q}$ in 1983 [4] to a new proof of the rationality of the values of the zeta-function ζ_F, F a totally real number field, at negative integers, a classical result due to Siegel and Klingen.

As most recent examples for this approach to the cohomology of arithmetic groups we discussed two results, they regard:

- Involutions of symplectic type on inner forms of the special linear group over k
- the growth of the first Betti number of arithmetic hyperbolic 3-manifolds.

In the first case, dealt with by Kionke in [2], given a quaternion algebra D defined over k, the matrix algebra $A = M_n(D)$ is naturally endowed with an involution τ the first kind, defined by the assignment $(a_{ij}) \mapsto (\tau_c(a_{ij}))^t$ where τ_c denotes the canonical involution on D. On the associated reduced norm-one group G, an algebraic k–group, composition of τ with taking the inverse gives rise to an automorphism τ^* of order two on G. The real Lie group, given as the group of real points of the $\mathbb{Q}$–group $\mathrm{Res}_{k/\mathbb{Q}}(G)$ obtained by restriction of scalars, is of the form

$$G_\infty = \mathrm{SL}_{2n}(\mathbb{R})^s \times \mathrm{SL}_n(\mathbb{H})^r \times \mathrm{SL}_{2n}(\mathbb{C})^t$$

where s resp. r denotes the number of archimedean places where D splits resp. ramifies. The number of complex places is denoted by t. We note $[k : \mathbb{Q}] = s+r+t$. Let $\Gamma(\mathfrak{a})$ be a torsion-free principal congruence subgroup in G originating with an integral ideal $\mathfrak{a}$ in the ring of integers of k. If the rational representation E of G is equipped with a compatible τ^*-action, the Lefschetz number of τ^* on the cohomology is defined. In [2], Kionke obtained the following results regarding this integral number:

– $L(\tau^*, \Gamma(\mathfrak{a}), E) = 0$ if k has at least one complex place, that is, $t \neq 0$
– If k is a totally real number field, then

$$L(\tau^*, \Gamma(\mathfrak{a}), E) = 2^{-r}\mathrm{N}(\mathfrak{a})^{n(2n+1)}\Delta_{\mathrm{red}}(D)^{\frac{n(n+1)}{2}}\,\mathrm{tr}(\tau^*_{|E})\prod_{j=1}^{n} M(j, \mathfrak{a}, D)$$

where $\Delta_{\mathrm{red}}(D)$ denotes the signed reduced discriminant of D and $M(j, \mathfrak{a}, D)$ is a product of $\zeta_k(1-2j)$ with Euler factors depending on the prime ideals dividing $\mathfrak{a}$ and the non-archimedean primes not dividing $\mathfrak{a}$ which ramify in D. Moreover, $L(\tau^*, \Gamma(\mathfrak{a}), E) = 0$ if and only if $\mathrm{tr}(\tau^*_{|E}) = 0$.

This result has various applications, one of them gives a precise formula for the genus of a compact Riemann surface of the type $\mathfrak{H}/\Gamma(\mathfrak{a})$. Another one is the result by Siegel and Klingen alluded to above.

In the second case, dealt with in [3], we are concerned with arithmetically defined hyperbolic 3-manifolds and corresponding Kleinian groups which originate with orders in division quaternion algebras defined over some algebraic number field E. More precisely, let F be a totally real algebraic number field, and let E be a quadratic extension field of F so that E has exactly one complex place. Let Γ be an arithmetic subgroup in the algebraic group $\mathrm{SL}_1(D)$ where D is a quaternion division algebra over E whose finite set of places ramified in D contains the set of real places of E. We suppose that the norm $N_{E/F}(D)$, a central simple algebra of degree 4 over F, has order 1 viewed as an element in the Brauer group $\mathrm{Br}(F)$. Thus, the F-algebra $N_{E/F}(D)$ is isomorphic to the matrix algebra $M_4(F)$, that is, it splits. By a result of Albert and Riehm, $N_{E/F}(D)$ splits if and only if there is an involution of the second kind on D which fixes F elementwise. Let τ denote this involution of the second kind. By definition of this notion, the restriction of τ to the centre of D is of order 2, hence $\tau_{|E}$ coincides with the non-trivial Galois automorphism σ of the extension E/F . As Albert has proved, an involution of the second kind on a quaternion algebra has a particular type. There exists a unique quaternion F–subalgebra $D_0 \subset D$ such that $D = D_0 \otimes_F E$ and τ is of the form $\tau = \gamma_0 \otimes \sigma$ where γ_0 is the canonical involution (also called quaternion conjugation) on D_0. The algebra D_0 is uniquely determined by these conditions.

Then the main result in [3] reads as follows: there are a positive number $\kappa > 0$ and a nested sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ of torsion-free, finite index congruence subgroups $\Gamma_i \subset \Gamma$ (whose intersection is the identity) such that the first Betti number of the compact hyperbolic 3-manifold $\mathbb{H}^3/\Gamma_i$ corresponding to Γ_i satisfies the inequality

$$b_1(\Gamma_i) \geq \kappa[\Gamma : \Gamma_i]^{1/2}$$

for all indices $i \in \mathbb{N}$. Further, Γ_i is normal in Γ_1 for all $i \in \mathbb{N}$.

We actually prove more than the existence of such sequences, we explicitly construct them using principal congruence subgroups. The proof of this result relies on an approach via Lefschetz numbers: The non-trivial Galois automorphism σ of the extension E/F induces an orientation-reversing involution on the hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$, whenever Γ is σ-stable. In the case the extension E/F is unramified over 2 one can determine the Lefschetz number $L(\sigma, \Gamma)$ of the induced homomorphism in the cohomology of $\mathbb{H}^3/\Gamma$ where Γ is a suitable congruence subgroup in $\mathrm{SL}_1(D)$. In the general case, one gets the analogous value as a lower bound for $L(\sigma, \Gamma)$. This bound is given up to sign and some power of two as

$$\pi^{-2d}\zeta_F(2)|\mathrm{disc}_F|^{3/2}\Delta(D_0) \times [K_0 : K_0(\mathfrak{a})],$$

where $\zeta_F(2)$ denotes the value of the zeta-function of F at 2, $|\mathrm{disc}_F|$ denotes the absolute value of the discriminant of F, $[K_0 : K_0(\mathfrak{a})]$ denotes a global index attached to the congruence subgroup of level $\mathfrak{a} \subseteq \mathcal{O}_F$, and

$$\Delta(D_0) = \prod_{\mathfrak{p}_0 \in \mathrm{Ram}_f(D_0)} (N_{F/\mathbb{Q}}(\mathfrak{p}_0) - 1)$$

depends on the set of finite places of F in which the quaternion division algebra D_0 ramifies. In turn, this bound can be used to give a lower bound for the first Betti number of the hyperbolic 3-manifold in question.

In the classical case of Bianchi groups, that is, these are non-cocompact arithmetic subgroups of $\mathrm{SL}_2(\mathbb{C})$ originating with orders in the ring of integers of an imaginary quadratic number field, a similar result can be obtained but with higher order of growth, see [3, Thm. 6.1].

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Geometric dimension of mapping class groups and lattices in classical simple groups

Juan Souto

(joint work with Javier Aramayona, Dieter Degreijse, Conchita Martinez)

Let Γ be an infinite discrete group. A Γ-CW-complex X is said to be a model for $\underline{E}\Gamma$, or a *classifying space for proper actions*, if the stabilizers of the action of Γ on X are finite, and if for every finite subgroup H of Γ the fixed point space X^H is contractible. The *proper geometric dimension* $\underline{\mathrm{gdim}}(\Gamma)$ of Γ is by definition the smallest possible dimension of a model of $\underline{E}\Gamma$. We refer the reader to the survey paper [9] for more details and terminology about these spaces.

Our aim is to compare the geometric dimension $\underline{\mathrm{gdim}}(\Gamma)$ of certain virtually torsion-free groups Γ with their *virtual cohomological dimension* $\mathrm{vcd}(\Gamma)$. Recall that $\mathrm{vcd}(\Gamma)$ is the cohomological dimension of a torsion-free finite index subgroup of Γ. Due to a result by Serre, this definition does not depend of the choice of finite index subgroup [5].

In general one has $\mathrm{vcd}(\Gamma) \leq \underline{\mathrm{gdim}}(\Gamma)$ but this inequality may be strict. Indeed, in [8] Leary and Nucinkis constructed examples of groups Γ for which $\underline{\mathrm{gdim}}(\Gamma)$ is finite but strictly greater than $\mathrm{vcd}(\Gamma)$. In fact, they show that the gap can be arbitrarily large.

On the other hand, one has $\mathrm{vcd}(\Gamma) = \underline{\mathrm{gdim}}(\Gamma)$ for many important classes of virtually torsion-free groups. For instance, equality holds for elementary amenable groups of type FP_∞ [7], $\mathrm{SL}(n,\mathbb{Z})$ [3, 9], outer automorphism groups of free groups [9, 11], mapping class groups [2], and for groups that act properly and chamber transitively on a building, such as Coxeter groups and graph products of finite groups [6]. We add lattices in classical simple Lie groups to this list. The classical simple Lie groups are the complex Lie groups

$$\mathrm{SL}(n,\mathbb{C}),\ \mathrm{SO}(n,\mathbb{C}),\ \mathrm{Sp}(n,\mathbb{C})$$

and their real forms

$$\mathrm{SL}(n,\mathbb{R}),\ \mathrm{SL}(n,\mathbb{H}),\ \mathrm{SO}(p,q),\ \mathrm{SU}(p,q),\ \mathrm{Sp}(p,q),\ \mathrm{Sp}(2n,\mathbb{R}),\ \mathrm{SO}^*(2n)$$

with conditions on n and $p+q$ to ensure simplicity:

Theorem 1 ([1]). *If Γ is a lattice in a classical simple Lie group, then $\underline{\mathrm{gdim}}(\Gamma) = \mathrm{vcd}(\Gamma)$.*

The idea of the proof is as follows. Suppose that Γ is virtually torsion free, that it has a cocompact model X for $\underline{E}\Gamma$, and suppose for the sake of concreteness that the action of Γ on X is free. The virtual cohomological dimension of Γ can be computed as follows

$$\mathrm{vcd}(\Gamma) = \max\{n \in \mathbb{N} \mid \mathrm{H}^n_c(X) \neq 0\},$$

where H^*_c stands for cohomology with compact support. Similarly, combining results by Lück–Meintrup [10] and Degreijse–Martinez [6] we get that

$$\underline{\mathrm{gdim}}(\Gamma) = \max\{n \in \mathbb{N} \mid \exists K \subset G \text{ finite and s.t. } \mathrm{H}^n_c(X^K, X^K_{\mathrm{sing}}) \neq 0\}$$

as long as for instance $\mathrm{vcd}(\Gamma) \geq 3$. Here X^K_{sing} is the subset of X^K consiting of points in X whose stablizer is strictly larger than K.

These results can be applied for lattices Γ in semi-simple Lie groups if we denote by X the Borel–Serre bordification of the corresponding symmetric space. It thus follows that in order to prove that the geometric dimension and the virtual cohomological dimension agree it suffices to show that

(1) $\mathrm{H}^k_c(X_{\mathrm{sing}}) = 0$ for all $k > \mathrm{vcd}(\Gamma)$, and that
(2) $\mathrm{H}^n_c(X) \to \mathrm{H}^n_c(X_{\mathrm{sing}})$ is surjective for $n = \mathrm{vcd}(\Gamma)$,

where X_{sing} is the subset of X consiting of points with non-trivial stabilizer.

To explain a simple instance of the proof of the theorem above, suppose that Γ is a lattice in $\mathrm{SL}_n\,\mathbb{C}$ for $n \geq 3$ and let X be the associated Borel–Serre bordification of the symmetric space $\mathrm{SL}_n\,\mathbb{C}/\,\mathrm{SU}_n$. It follows from the work of Borel-Serre [4] that Γ has virtual cohomological dimension

$$\mathrm{vcd}(\Gamma) = \dim(\mathrm{SL}_n\mathbb{C}/\,\mathrm{SU}_n) - \mathrm{rank}_{\mathbb{Q}}\,\Gamma \geq \dim(\mathrm{SL}_n\mathbb{C}/\,\mathrm{SU}_n) - \mathrm{rank}_{\mathbb{R}}\,\mathrm{SL}_n\mathbb{C} = n^2 - n$$

On the other hand, a linear algebra computation shows that for any $A \in \mathrm{SL}_n\mathbb{C}$ we have that $\dim X^A \leq (n-1)^2$ which means that automatically we also have

$$\dim X_{\mathrm{sing}} \leq (n-1)^2$$

Since $(n-1)^2 < n^2 - n$ it follows that $\mathrm{H}^k_c(X_{\mathrm{sing}}) = 0$ for all $k \leq \mathrm{vcd}(\Gamma)$ and hence (1) and (2) hold. We have just proved the theorem above for lattices in $\mathrm{SL}_n\mathbb{C}$.

The basic idea we just explained can also be applied to give a different proof of the following result by Aramayona–Martinez [2]:

Theorem 2 (Aramayona–Martinez)**.** *For every surface* Σ *we have* $\underline{\mathrm{gdim}}(\mathrm{Map}(\Sigma))$ $= \mathrm{vcd}(\mathrm{Map}(\Sigma))$*, where* $\mathrm{Map}(\Sigma)$ *is the mapping class group of* Σ*.*

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Geometric compactifications of locally symmetric spaces of infinite co-volume

Anna Wienhard
(joint work with François Guéritaud, Olivier Guichard, Fanny Kassel)

Let G be a non-compact semisimple Lie group, $K < G$ a maximal compact subgroup and $X = G/K$ the Riemannian symmetric spaces associated to G. Then any discrete subgroup $\Gamma < G$ acts on X properly discontinuous by isometries, and the locally symmetric space $M_\Gamma = \Gamma\backslash G/K$ has the structure of a Riemannian orbifold. If $\Gamma < G$ is a uniform lattice, the space M_Γ is compact. If Γ is a non-uniform, M_Γ is of finite volume. In all other cases M_Γ is of infinite volume. When M_Γ is of finite volume, the problem of compactifying M_Γ has been well studied and several compactifications exist (see [BJ05] for an overview). Much less is known in the case when M_Γ is of infinite volume, in particular when G is a Lie group of higher rank.

We address the question of constructing a geometric compactification $\overline{M_\Gamma}$ of M_Γ, by which we mean that the compactification $\overline{M_\Gamma}$ is modeled on a compactification $\overline{X}$ of the symmetric space X. We provide such a compactification for a special class of discrete subgroups of semisimple Lie groups which have nice structural properties, namely for discrete groups which arise as images of Anosov representations [Lab06, GW12, GGKW1], or equivalently, which are τ_{mod}-asymptotically embedded discrete subgroups of G in the terminology of [KLPa, KLPb, KLPc].

The class of Anosov representations include many interesting examples:

Example 1. (1) *Convex cocompact subgroups:* A discrete subgroup $\Lambda < G$ is convex cocompact if there exists a convex Λ-invariant subset $C \subset X$ on which Λ acts properly discontinuously and cocompactly. If G is of real rank one, e.g. $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, then a representation $\rho\colon \Gamma \to G$ is Anosov if and only if ρ is faithful with finite kernel and such that $\rho(\Gamma) < G$ is a convex cocompact subgroup. When G is of higher rank Anosov representations provide a meaningful generalization of the notion of convex cocompact subgroups.

(2) *Openness*: The set of Anosov representations is open in $\mathrm{Hom}(\Gamma, G)$. Thus a small deformation of an Anosov representation is again an Anosov representation. This applies in particular when we compose an Anosov representation $\rho\colon \Gamma \to H$ into a Lie group of rank one with an embedding of H into a Lie group G of higher rank. This provides a wealth of examples.

(3) *Hitchin representations:* When $\Gamma = \pi_1(\Sigma)$ is the fundamental group of a closed surface of negative Euler characteristic, and G is a split real simple Lie group, the Hitchin component $\mathrm{Hit}(\Sigma, G)$ is a connected component of the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, which is homeomorphic to a ball. All representations in the Hitchin component are Anosov [Lab06, FG06].

(4) *Maximal representations:* When $\Gamma = \pi_1(\Sigma)$ is the fundamental group of a closed surface of negative Euler characteristic, and G is a Lie group of Hermitian type, then the Toledo number allows us to define the set of maximal representations $\mathrm{Max}(\Sigma, G)$, which is a union of connected components of the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. Maximal representations are Anosov [BIW10, BILW05, GW12] (compare to B. Pozzetti's report).

(5) *Schottky groups/ping pong pairs:* Representations arising from Schottky groups, or more generally from ping pong on flag varieties give rise to Anosov representation of free groups (compare to E. Fuchs's report).

We prove the following result for Anosov representations.

Theorem 2. *Let Γ be a finitely generated word hyperbolic group. Let G be a semisimple Lie group and $\rho\colon \Gamma \to G$ an Anosov representation. Then there exists a generalized Satake compactification $\overline{X}$ of $X = G/K$, and a closed subset $C < \overline{X} \setminus X$ such that:*

(1) The action of $\rho(\Gamma)$ on $\Omega = \overline{X} \setminus C$ is properly discontinuous and cocompact.
(2) The quotient $\overline{M} = \rho(\Gamma)\backslash\Omega$ is a geometric compactification of $M = \rho(\Gamma)\backslash X$.
(3) If Γ is torsion free, then $\overline{M}$ is homeomorphic to a manifold with boundary.
(4) M is the interior of $\overline{M}$. In particular, M is topologically tame.

We introduce the notion of *generalized* Satake compactifications. These are constructed in the same way as Satake compactifcations by taking a faithful representation of G into $\mathrm{PSL}(n, \mathbb{C})$, and embedding G/K as a subset of the space $\mathbb{P}(\mathscr{H}_n)$ of projective classes of Hermitian $n \times n$-matrices. However, contrary to Satake's original definition, which requires the representation of G into $\mathrm{PSL}(n, \mathbb{C})$ to be irreducible, we allow for reducible representations as well. The class of generalized Satake compactifications has following nice property. Let $Y \subset X$ a totally geodesic subsymmetric space. Then the closure of Y in a generalized Satake compactification of X is a generalized Satake compactification of Y. Note that this is not true within the class of Satake compactifications – the closure of a totally geodesic subsymmetric space of X in a Satake compactification of X is in general only a generalized Satake compactification.

Remark 3. (1) Note that the Anosov property of a representation $\rho\colon \Gamma \to G$ is usually defined with respect to a parabolic subgroup P, and by an Anosov representation we mean a representation which is Anosov with respect to some proper parabolic subgroup $P < G$. For special parabolic subgroups (e.g. minimal parabolic subgroups or maximal parabolic subgroups of a specific type) Theorem 2 can be strengthened to the extent that the compactification $\overline{X}$ can be chosen to be a genuine Satake compactification. A proof of Theorem 2 together with other results and a discussion of consequences will appear in [GGKW3].

(2) For special Anosov representations into the symplectic group the compactifications $\overline{M_\Gamma}$ we construct have been first constructed by Guichard and Wienhard [GW12].

(3) Using very different method, Kapovich and Leeb [KL] recently constructed a geometric compactification for Anosov representations with respect to minimal parabolic subgroups. They use that the maximal Satake compactification can be realized as the horofunction compactification of a Finsler metric on X. Such horofunction compactifications of Finsler metrics and ideas that Satake compactifications can be realized in this way have also be discussed by Schilling in [S].
(4) Geometric compactifications of the homogeneous spaces $\Gamma\backslash G$ have been constructed in [GGKW2].

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Character varieties and representations of the Kauffman bracket skein algebra

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(joint work with Francis Bonahon)

In the early 1990s, the Kauffman bracket skein module was introduced [13, 11] to generalize the Jones polynomial in the 3-sphere to arbitrary oriented 3-manifolds. In the special case of a connected, oriented surface Σ, the Kauffman bracket skein module of $\Sigma \times [0,1]$ has the structure of an algebra, which we denote by $\mathcal{S}^A(\Sigma)$. The skein algebra of a surface plays an important role in the skein theoretic version of the Witten-Reshetikhin-Turaev topological quantum field theory, [1]. For example, when Σ is closed and bounds a handlebody H, its action on the skein module of H underpins the quantum representation of the mapping class group mentioned in other talks in this workshop, e.g. in ones by Julien Marché and Pat Gilmer.

Although the skein algebra originated in quantum topology, it is distinguished in that it has a relatively well-understood connection with hyperbolic geometry. In this talk, we assume that Σ is a hyperbolic surface of finite topological type. We focus on the relationship between the skein algebra $\mathcal{S}^A(\Sigma)$ and the $\mathrm{SL}_2\mathbb{C}$-character variety, $\mathscr{R}_{\mathrm{SL}_2\mathbb{C}} = \{r : \pi_1\Sigma \to \mathrm{SL}_2\mathbb{C}\}//\mathrm{SL}_2\mathbb{C}$.

When $A = -1$, the combined works of Bullock–Frohman–Kania-Bartoszyńska [7, 8], and Przytycki–Sikora [12] among others show that $\mathcal{S}^{-1}(\Sigma)$ is isomorphic to the algebra of $\mathbb{C}$-valued regular functions on $\mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$. For generic values of A, Turaev [14] used Goldman's description of the Weil–Petersson–Atiyah–Bott–Goldman Poisson bracket to interpret $\mathcal{S}^A(\Sigma)$ as a quantization of $\mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$. When A is a certain root of unity, the relationship is even more explicit: there is an identification of a character in $\mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$ to any representation of $\mathcal{S}^A(\Sigma)$, as in the following theorem.

Theorem 1. *Let A be a primitive $2N$-th root of 1 for N odd. Every finite-dimensional, irreducible representation $\rho\colon \mathcal{S}^A(\Sigma) \to \mathrm{End}\, V$ is associated to a unique $r_\rho \in \mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$ and numbers $p_i \in \mathbb{C}$ for each boundary component of Σ which are compatible with r_ρ.*

For details, see [3]. We give a very brief sketch here. A skein in $\mathcal{S}^A(\Sigma)$ which is "threaded" by the Chebyshev polynomial T_N is central in $\mathcal{S}^A(\Sigma)$. The character r_ρ (which we call the *classical shadow* of ρ) is obtained by application of Schur's Lemma. The *puncture invariants* p_i are similarly obtained from application of Schur's lemma on small loops around the ith puncture. The compatibility condition mentioned in the statement of the theorem comes from threading the puncture loops by T_N. For brevity, we suppressed the technical definition, which can be found in [3].

Very loosely, Theorem 1 says that there is a fibering of the set of finite-dimensional, irreducible representations of $\mathcal{S}^A(\Sigma)$ over the $\mathrm{SL}_2\mathbb{C}$-character variety. The next theorem says that there exists a section.

Theorem 2. *Let A be a primitive $2N$-th root of 1 for N odd. For every choice of $r \in \mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$ and compatible puncture invariants p_i, there exists a finite-dimensional, irreducible representation $\rho\colon \mathcal{S}^A(\Sigma) \to \mathrm{End}\, V$ whose classical shadow is r and whose puncture invariants are the p_i.*

See [5, 4] for details of the construction, which uses the quantum Teichmüller space of Σ of [9, 10, 2, 6]. We conjecture that there is a Zariski dense open subset of $\mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$ on which the correspondence described in Theorems 1 and 2 is one-to-one.

We end with a quick outline of how a (possibly non-trivial) representation of the mapping class group can be constructed using the representation ρ_{Id} which Theorem 2 associates to the trivial character which maps all of $\mathscr{R}_{\mathrm{SL}_2\mathbb{C}}$ to the identity matrix.

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