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Partial Differential Equations

Organised by
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Abstract. The workshop dealt with nonlinear partial differential equations and some applications in geometry, touching several different topics such as minimal surfaces and harmonic maps, equations in conformal geometry, geometric flows, extremal eigenvalue problems and optimal transport.

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Introduction by the Organisers

The workshop *Partial differential equations*, organised by Alice Chang (Princeton), Camillo De Lellis (Zürich), and Peter Topping (Warwick) was held August 2-8, 2015. The meeting was well attended by 51 participants, including 7 females, with broad geographic representation. The program consisted of 21 talks and left sufficient time for discussions.

There were several contributions to regularity of solutions of elliptic partial differential equations and geometric flows, in particular concerning minimal surfaces, harmonic maps, Ricci flow, mean curvature flow and nonlinear wave equations. Other talks dealt with the underlying variational structure of some geometric problems, such as the existence of Riemannian manifolds with extremal eigenvalues, the construction of hypersurfaces with constant mean curvature and the existence of solutions to Toda-type systems.

A number of experts in conformal geometry have attended the workshop. Here new results were presented in the existence of conformal Willmore tori, in the study of conformal invariance in conformally compact Einstein manifolds and generalized Perelman's functional, the mean-field equation and the Q-curvature.

Finally, a group of talks dealt with the theory of optimal transport maps and its applications, ranging from metric geometry to random matrices.

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Workshop: Partial Differential Equations

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Abstracts

An extremal eigenvalue problem for surfaces

Richard Schoen

In this lecture we introduce the variational problem of finding a Riemannian metric g on a compact surface M which maximizes $\lambda_1 A$, where λ_1 is the first nonzero eigenvalue of g and A is the area of g. We describe the connection of this problem with the problem of finding minimal immersions of M into a sphere by first eigenfunctions. We summarize the known results on the problem which include the cases when M is S^2, $\mathbb{RP}^2$, T^2, and the Klein bottle. We then describe upper and lower bounds in terms of the genus for orientable surfaces. We go on to describe our recent theorem which shows that the supremum value $\lambda^*(\gamma)$ for a surface of genus γ satisfies $\lambda^*(\gamma+1) > \lambda^*(\gamma)$ provided that $\lambda^*(\gamma)$ is achieved. We give two corollaries of the theorem. First we show how this, together with known results, implies that $\lambda^*(\gamma)$ is achieved for all $\gamma \geq 1$ since it implies that the conformal structures do not degenerate for a maximizing sequence of metrics. Secondly we show that the upper bound given by P. Yang and S.T. Yau is not achieved for odd values of γ. Finally we give a sketch of the proof which involves carefully choosing the geometry of a handle which is added to a surface of genus γ to obtain a surface of genus $\gamma+1$ so that the change in area and first eigenvalue can be controlled.

A transportation approach to random matrices

Alessio Figalli

Large random matrices appear in many different fields, including quantum mechanics, quantum chaos, telecommunications, finance, and statistics. One of the main question in the field consists in understanding how the asymptotic properties of the spectrum depend on the fine details of the model.

The most classical model of random matrices are the so-called *Wigner matrices.* These are $N \times N$ Hermitian matrices with independent identically distributed real or complex entries, with zero mean and covariance $1/N$. Let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of such a matrix.

If the entries are Gaussian, then the eigenvalues' distribution is given by

$$d\mathbb{P}^N_\gamma(\lambda_1, \ldots, \lambda_N) := \frac{1}{Z^N_\gamma} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta e^{-N \sum_{i=1}^N \lambda_i^2} \, d\lambda_1 \ldots d\lambda_N,$$

where

$$\beta = \begin{cases} 1 & \text{for real entries,} \\ 2 & \text{for complex entries.} \end{cases}$$

The main question is: what is the distribution of λ_i as $N \to \infty$?

A natural step to begin is to look at the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$. Notice that, since we have random matrices, this measure is random too. The following result holds [6]:

Theorem 1.
$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \rightharpoonup \rho_{sc}(x)\,dx \qquad a.s.,$$
where
$$\rho_{sc}(x) := \frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}(x) \qquad \text{is the semicircle law.}$$

This result tells us that, with very high probability, the eigenvalues are all contained in the interval $[-2,2]$. Since there are N eigenvalues, at least in the interior of this interval it is natural to expect that the average distance between two consecutive eigenvalues should be of order $1/N$.

To formalize this, let us order the eigenvalues so that $\lambda_1 \le \ldots \le \lambda_N$. Then the following holds:
- At the *edge* (that is, when $i \le C$), the random variable $N^{2/3}(\lambda_{i+1}-\lambda_i)$ is of order 1, and in the limit as $N \to \infty$ it follows the *Tracy-Widom law.*
- In the *bulk* (that is, when $i \in [\epsilon N, (1-\epsilon)N]$ for some $\epsilon > 0$), the random variable $N(\lambda_{i+1}-\lambda_i)$ is of order 1, and in the limit as $N \to \infty$ it follows the *sine-Kernel law.*

These results have been first proven for Gaussian matrices, and then extended to general Wigner matrices by many authors (see for instance [1, 4] for more references).

Our goal is to extend these results to general β-ensembles: this corresponds to replace the quadratic potential $t^2/2$ appearing in the Gaussian distribution $e^{-t^2/2}$ by a general potential $V : \mathbb{R} \to \mathbb{R}$. In other words, we assume that the eigenvalues are distributed as
$$d\mathbb{P}_V^N(\lambda_1,\ldots,\lambda_N) := \frac{1}{Z_V^N}\prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)}\,d\lambda_1\ldots d\lambda_N,$$
with $V : \mathbb{R} \to \mathbb{R}$ *smooth* and *uniformly convex*, and we want to understand what survives of the previous results.

Concerning the empirical measure, one can prove that this roughly behaves as in the Gaussian case:
$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \rightharpoonup \rho_V(x)\,dx \qquad \text{a.s.},$$
$$\rho_V(x) = S_V(x)\sqrt{(x-a)(b-x)}\,\chi_{[a,b]}(x), \qquad S_V \ge c > 0.$$
Hence the main question becomes understanding the distribution
$$N(\lambda_{i+1}-\lambda_i) \quad \text{in the bulk,} \qquad N^{2/3}(\lambda_{i+1}-\lambda_i) \quad \text{at the edge.}$$
The basic idea is to try to find a "nice" change of variable to parameterize the eigenvalues distributed accordingly to $\mathbb{P}_V^N$ in terms of the ones distributed accordingly to $\mathbb{P}_\gamma^N$.

To state our result we first introduce some notation: we denote by $T_0 : \mathbb{R} \to \mathbb{R}$ the monotone rearrangement of ρ_{sc} onto ρ_V, that is

$$\int_{-\infty}^{x} \rho_{sc}(y)\,dy = \int_{-\infty}^{T_0(x)} \rho_V(y)\,dy \qquad \forall\, x \in \mathbb{R}. \tag{1}$$

Also, for $\alpha \in (0,1)$ we define $\gamma_\alpha \in (-2,2)$ at the unique number satisfying

$$\int_{-\infty}^{\gamma_\alpha} \rho_{sc}(y)\,dy = \alpha.$$

Then, we can prove that the following result holds [1]:

Theorem 2. *Let $i/N \to \alpha \in (0,1)$ and $f : \mathbb{R}^m \to \mathbb{R}$ be a Lipschitz function, with $m \le N^{1/2}$. Then, for any $\eta \in (0,1/6)$,*

$$\begin{aligned} \Bigg| \int f\Big(N(\lambda_{i+1}-\lambda_i), \ldots, N(\lambda_{i+m}-\lambda_i)\Big)\, d\mathbb{P}_V^N \\ - \int f\Big(T_0'(\gamma_\alpha)\, N(\lambda_{i+1}-\lambda_i), \ldots, T_0'(\gamma_\alpha)\, N(\lambda_{i+m}-\lambda_i)\Big)\, d\mathbb{P}_\gamma^N \Bigg| \\ \le C_\eta\, \frac{(m+N^\eta)\,\log N}{N^{1/2}} \Big(\|f\|_\infty + \|\nabla f\|_\infty\Big). \end{aligned}$$

Notice that, choosing $m = 1$, we discover that asymptotically the law of $N(\lambda_{i+1}-\lambda_i)$ is the same (up to a dilation) under the two measures $\mathbb{P}_V^N$ and $\mathbb{P}_\gamma^N$. Hence, since $N(\lambda_{i+1}-\lambda_i)$ follows the sine-Kernel law for the Gaussian model we deduce that the same holds for any potential V. In other words, the fluctuations of the eigenvalues are independent of the model (this phenomenon is known as *universality*).

We mention that the above result holds also at the edge with a bound of the form $C_{m,\eta}\, N^{-1/3+\eta}$ [1]. Also, we note that similar results have also been obtained independently by Bourgade-Erdös-Yau and Shcherbina [2, 3, 5]. However, an advantage of our method is that it is extremely robust, and applies also to several-matrix models. For instance, as a corollary of the results in [4], the following holds:

Theorem 3. *Let $\{X_i\}_{i=1}^d$ be independent $N \times N$-Gaussian matrices and set*

$$Y_i = X_i + \epsilon\, P_i(X_1, \ldots, X_d),$$

where $P_1, \ldots, P_d$ are self-adjoint polynomials.

There exists $\epsilon_0 > 0$ such that, for $\epsilon \in [-\epsilon_0, \epsilon_0]$, as $N \to \infty$ the eigenvalues of the matrices $\{Y_i\}_{1\le i\le d}$ fluctuate in the bulk and at the edge as when $\epsilon = 0$, up to rescaling.

References

[1] F. Bekerman, A. Figalli, A. Guionnet. Transport maps for β-matrix models and universality. *Comm. Math. Phys.*, 338 (2015), no. 2, 589-619.

[2] P. Bourgade, L. Erdös, H.-T. Yau. Universality of general ?-ensembles. *Duke Math. J.*, 163 (2014), no. 6, 1127–1190.

[3] P. Bourgade, L. Erdös, H.-T. Yau. Edge universality of beta ensembles. *Comm. Math. Phys.*, 332 (2014), no. 1, 261–353.
[4] A. Figalli, A. Guionnet. Universality in several-matrix models via approximate transport maps. Preprint, 2014.
[5] M. Shcherbina. Change of variables as a method to study general β-models: bulk universality. arXiv:1310.7835, 2013. *J. Math. Phys.*, 55 (2014), 043504, 23 pp.
[6] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals Math.*, 62 (1955), 548–564.

Variational theory for $SU(3)$ Toda Systems

Andrea Malchiodi
(joint work with Luca Battaglia, Aleks Jevnikar, Sadok Kallel, Cheikh Ndiaye, David Ruiz)

The following Toda system of coupled Liouville equations has been extensively studied because of its role in Chern-Simons models of superconductivity, see [12], [13] (we also refer to the references in the bibliography for a more complete account on results on the subject). Given a boundary-less Riemannian surfaces (Σ, g) we consider

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right). \end{cases} \tag{1}$$

Here h_1, h_2 are smooth positive functions on Σ and ρ_1, ρ_2 are real parameters. Flat tori might model for example periodic physical systems in the plane. The above system also appears in the description of holomorphic curves in projective spaces.

Problem (1) has variational structure, and the corresponding Euler functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ has the expression

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i dV_g - \log \int_\Sigma h_i e^{u_i} dV_g \right), \tag{2}$$

where $\rho = (\rho_1, \rho_2)$, and where $Q(u_1, u_2)$ is the positive-definite quadratic form

$$Q(u_1, u_2) = \frac{1}{3} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2 \right). \tag{3}$$

It is well-known that $H^1(\Sigma)$ embeds into any L^p space, and that indeed the embedding can be pushed to exponential class via the Moser-Trudinger inequality. Concerning the functional J_ρ a sharp inequality has been found in [8].

Theorem 1. *([8]) For $\rho = (\rho_1, \rho_2)$ the functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma)$ is bounded from below if and only if both ρ_1 and ρ_2 satisfy $\rho_i \le 4\pi$.*

By the latter theorem we have that when both $\rho_1, \rho_2 < 4\pi$ the functional J_ρ is coercive, and solutions can be found by minimization via the direct methods of the calculus of variations. When one of the ρ_i's exceeds 4π the energy becomes unbounded from below, and solutions have to be found as saddle points. One result in this direction is the following one.

Theorem 2. *([9], [10]) Suppose m is a positive integer, and let $h_1, h_2 : \Sigma \to \mathbb{R}$ be smooth positive functions. Then for $\rho_1 \in (4\pi m, 4\pi(m+1))$ and for $\rho_2 < 4\pi$ problem* (1) *is solvable.*

The above theorem was proved in [9] for $m = 1$ when Σ has positive genus: it was then extended in [10] to the remaining cases of the theorem. It turns out that, under the above conditions, when $J_\rho(u)$ is large negative then e^{u_1} concentrates near at most m points of Σ. Using improved versions of the Moser-Trudinge rinequality, this was noticed in [5], where the *prescribed Q-curvature problem* in four dimension is studied, and induces to consider the family Σ_m of formal sums

$$\Sigma_m = \left\{ \sum_{i=1}^{m} t_i \delta_{x_i} \, : \, \sum_{i=1}^{m} t_i = 1, t_i \geq 0, x_i \in \Sigma, \forall\, i = 1, \dots, m \right\}. \tag{4}$$

called the *barycentric sets of* Σ of order m. Using the fact that this set is never contractible, one can then employ min-max arguments on J_ρ (using the compactness results from [9] and [4]).

We are interested here in the situation when both the ρ_i's exceed the threshold coercivity value 4π. Using again improved inequalities, it is possible to prove that if $\rho_1 < 4(m+1)\pi$, $\rho_2 < 4(n+1)\pi$, $m, n \in \mathbb{N}$, and if $J_\rho(u_1, u_2)$ is sufficiently low, then either e^{u_1} is close to Σ_m or e^{u_2} is close to Σ_n in the distributional sense. This (non-mutually exclusive) alternative can be expressed in term of the *topological join* of Σ_m and Σ_n. Recall that, given two topological spaces A and B, their join $A * B$ is defined as the family of elements of the form (see [6])

$$A * B = \frac{\{(a, b, s) : \; a \in A, \; b \in B, \; s \in [0, 1]\}}{E}, \tag{5}$$

where E is an equivalence relation given by

$$(a_1, b, 1) \overset{E}{\sim} (a_2, b, 1) \quad \forall a_1, a_2 \in A, b \in B$$

and

$$(a, b_1, 0) \overset{E}{\sim} (a, b_2, 0) \quad \forall a \in A, b_1, b_2 \in B.$$

This construction allows to map low sublevels of J_ρ into $\Sigma_m * \Sigma_n$, with the join parameter s expressing whether distributionally e^{u_1} is closer to Σ_m or whether e^{u_2} is closer to Σ_n. This construction allowed to prove the following theorem.

Theorem 3. *([1]) Suppose $\rho_i \notin 4\pi\mathbb{N}$ for both $i = 1, 2$ and that Σ has positive genus. Then system* (1) *is solvable.*

The assumption on the genus is used to construct suitable test functions with low energy whose components concentrate on two distinct curves on Σ, and such that there exist global retractions of Σ onto these curves. This condition is used to minimize the interaction term in the quadratic form Q, that penalizes concentration of both components at the same time. To analyse this phenomenon a couple of new, scaling invariant, inequalities have been derives, allowing to prove the following existence result for surfaces of any genus.

Theorem 4. *([11], [7]) Let h_1, h_2 be two positive smooth functions and let Σ be any compact surface. Suppose that $\rho_1 \in (4m\pi, 4(m+1)\pi), m \in \mathbb{N}$ and $\rho_2 \in (4\pi, 8\pi)$. Then problem (1) has a solution.*

It would be interesting to remove the upper bound on the second parameter ρ_2, as well as to understand the case when some parameter ρ_i is a multiple of 4π (and compactness fails). It would be also interesting to consider the problem (1) in presence of singularities, modelling vortex or ramification points. Some partial progress, but only in special situations, is available in [2], [3].

References

[1] L. Battaglia, A. Jevnikar, A. Malchiodi, D. Ruiz, A general existence result for the Toda System on compact surfaces, preprint, 2013.

[2] L.Battaglia, Existence and multiplicity result for the singular Toda system, J. Math. Anal. Appl. 424 (2015), no. 1, 49–85.

[3] L.Battaglia and A.Malchiodi, Existence and non existence results for the SU(3) singular Toda system on compact surfaces, preprint, 2015.

[4] L.Battaglia and G.Mancini, A note on compactness properties of the singular Toda system, Rend. Lincei, to appear.

[5] Z. Djadli and A. Malchiodi, Existence of conformal metrics with constant Q-curvature, Annals of Math., 168 (2008), no. 3, 813–858.

[6] A. Hatcher, Algebraic Topology, Cambridge University Press 2002.

[7] A.Jevnikar, S.Kallel and A.Malchiodi, A topological join construction and the Toda system on compact surfaces of arbitrary genus, preprint, 2015.

[8] J. Jost and G. Wang, Analytic aspects of the Toda system I. A Moser-Trudinger inequality, Comm. Pure Appl. Math. 54 (2001), 1289–1319.

[9] J. Jost, C. S. Lin and G. Wang, Analytic aspects of the Toda system II. Bubbling behavior and existence of solutions, Comm. Pure Appl. Math. 59 (2006), 526–558.

[10] A. Malchiodi and C. B. Ndiaye, Some existence results for the Toda system on closed surfaces, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 18 (2007), no. 4, 391–412.

[11] A. Malchiodi and D. Ruiz, A variational Analysis of the Toda System on Compact Surfaces, Comm. Pure Appl. Math., 66 (2013), no. 3, 332–371.

[12] G. Tarantello, Self-Dual Gauge Field Vortices: An Analytical Approach, PNLDE 72, Birkhäuser Boston, Inc., Boston, MA, 2007.

[13] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer-Verlag, 2001.

From Ginzburg-Landau Equations to n-harmonic maps

Yuxin Ge

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Given $g : \partial\Omega \to \mathbb{S}^{n-1}$ a smooth prescribed map with the degree $d = \deg(g, \partial\Omega, \mathbb{S}^{n-1})$ we consider the functional

$$\mathbf{E}_\varepsilon(u, \Omega) = \int_\Omega \left[\frac{|\nabla u|^n}{n} + \frac{1}{4\varepsilon^n} \left(1 - |u|^2\right)^2 \right] \mathrm{d}x \tag{1}$$

for $u \in W_g^{1,n}(\Omega, \mathbb{R}^n) = \left\{w \in W^{1,n}(\Omega, \mathbb{R}^n) : w|_{\partial\Omega} = g\right\}$. The critical points satisfy the so called generalized Ginzburg-Landau system

$$\text{(2)} \qquad \begin{cases} -div\left(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon\right) &= \frac{1}{\varepsilon^n}\left(1-|u_\varepsilon|^2\right)u_\varepsilon & \text{in } \Omega \\ u_\varepsilon &= g & \text{on } \partial\Omega\,. \end{cases}$$

In the case of $n = 2$, the minimizers and critical points of this functional were studied by F.Bethuel, H.Brezis and F.Hélein [1] and many authors after them.

Several authors have studied the sequences of minimizers of $\mathbf{E}_\varepsilon$ in the case $n \geq 3$, namely P.Strzelecki [8], M-C.Hong [5] and Z-C.Han and Y-Y.Li [3]. We define a constant

$$\kappa_n = \frac{1}{n}(n-1)^{\frac{n}{2}}\omega_n \tag{3}$$

where $\omega_n = |\mathbb{S}^{n-1}|$. Let us recall the main results in [3].

Theorem (HL). *Assume $d > 0$, $n \geq 3$. For any sequence $\varepsilon_k \to 0$, let $\{u_k\} \subset W_g^{1,n}(\Omega, \mathbb{R}^n)$ be the corresponding sequence of minimizer for $\mathbf{E}_{\varepsilon_k}$. Then there exists a subsequence $\{u_{k'}\}$, a collection of d distinct points $\{a_1, a_2, \cdots, a_d\} \subset \Omega$, and an n-harmonic map $u_* : \Omega \setminus \cup_i \{a_i\} \to \mathbb{S}^{n-1}$ such that*

$$u_{k'} \to u_* \quad \textit{strongly in} \quad \mathbf{W}^{1,n}_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n), \tag{4}$$

$$u_{k'} \to u_* \quad \textit{in} \quad \mathbf{C}^0_{loc}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n), \tag{5}$$

$$u_{k'} \to u_* \quad \textit{strongly in} \quad \mathbf{W}^{1,p}(\Omega; \mathbb{R}^n)\, \textit{for all}\, 1 \leq p < n. \tag{6}$$

Furthermore, $\deg(u_*, \partial B_\sigma, \mathbb{S}^{n-1}) = 1$ for all $1 \leq j \leq d$ and $\sigma > 0$ small enough.
When $d = 0$, $u_{k'}$ converges to u_* strongly in $\mathbf{W}^{1,n} \cap \mathbf{C}^0$.

Our first result proves that the singularities of u_* minimize a renormalized energy. This renormalized energy was actually introduced by R.Hardt, F-H.Lin and C-Y.Wang [4] as follows. Given d distinct points in Ω denoted $a = \{a_1, a_2, \cdots, a_d\}$, and for $\delta > 0$, let

$$\Omega_{a,\delta} = \Omega \setminus \cup_{i=1}^d B_\delta(a_i).$$

Then define for any δ small enough

$$\mathcal{W}_{a,\delta} = \left\{w \in W^{1,n}(\Omega_{a,\delta}; \mathbb{S}^{n-1}) : w|\partial\Omega = g, \deg(w, \partial B_\delta(a_i)) = 1 \,\text{for all}\, i\right\}.$$

The renormalized energy of $a = \{a_1, a_2, \cdots, a_d\}$ is defined to be

$$W_g(a) := \lim_{\delta \to 0}\left(\min_{w \in \mathcal{W}_{a,\delta}} E_n(w, \Omega_{a,\delta}) - d\kappa_n |\ln \delta|\right), \tag{7}$$

where

$$E_n(w, \Omega_{a,\delta}) = \int_{\Omega_{a,\delta}} \frac{|\nabla w|^n}{n}\, \mathrm{d}x.$$

From now on we will assume the dimension $n \geq 3$. We have the following result.

Theorem 1. *Let $a = \{a_i\}_{i=1}^d$ be the limit singular points of Theorem (HL), then*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) = d\kappa_n |\ln \varepsilon| + W_g(a) + d\gamma + o(1) \; as\, \varepsilon \to 0,$$

where γ is a constant. Moreover, the configuration $\{a_i\}_{i=1}^d$ minimizes W_g.

The results above deal only with sequences of energy-minimizers. The ones below deal with limits of solutions to the system (2), whose energy is in the same range as that of minimizers.

Theorem 2. *Assume that for each $\varepsilon > 0$ the map u_ε, is a critical point of $\mathbf{E}_\varepsilon$ that for some $M > 0$ independent of ε it holds that*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \le d\kappa_n \, |\ln \varepsilon| + M. \tag{8}$$

Then there exists a subsequence $\{\varepsilon\}$ tending to zero, a collection of d distinct points $\{a_1, a_2, \cdots, a_d\} \subset \Omega$, a finite subset U of $\bar{\Omega}$, and a stationary n-harmonic map $u_0 : \Omega_0 := \Omega \setminus \{a_1, a_2, \cdots, a_d\} \to \mathbb{S}^{n-1}$, such that

$$u_\varepsilon \to u_0 \quad strongly\ in \quad \mathbf{W}^{1,n}_{loc}(\Omega_0 \setminus U, \mathbb{R}^n)$$

and for any $1 \le p < n$

$$u_\varepsilon \rightharpoonup u_0 \quad weakly\ in \quad \mathbf{W}^{1,p}(\Omega, \mathbb{R}^n).$$

Furthermore, $\deg(u_0, \partial B_\sigma(a_j), \mathbb{S}^{n-1}) = 1$, for $1 \le j \le d$ and any small enough $\sigma > 0$.

It was proved by R. Jerrard in [6] that the upper bound condition (8) is sufficient to guarantee the local weak convergence in Ω_0 of a subsequence. Here we improve this to strong convergence for solutions of the system (2). However, contrary to the case $n = 2$ we need to remove a finite set S corresponding to the bubbling-off of nontrivial finite energy n-harmonic maps from $\mathbb{R}^n$ to $\mathbb{S}^{n-1}$ which do not exist when $n = 2$.

In the case $n = 3$ an example of such a map is the Hopf fibration, and recently T.Rivire in [7] showed that there exists in fact many of them. This multiplicity arises in particular from a richer topology, due to the non-trivial fundamental group $\pi_3(\mathbb{S}^2)$, for which the Hopf map is a generator. This hints at the fact that the moduli space of critical points of the generalized Ginzburg-Landau equations for small parameter ε could be quite rich too. For $n > 3$ the same situation is expected because of homotopy groups of the spheres, for example, $\pi_7(\mathbb{S}^4), \pi_{15}(\mathbb{S}^8)$, or other topological invariants.

Theorem 2 contains a criticality condition satisfied by the points $\{a_1, a_2, \cdots, a_d\}$ hidden in the word "stationary n-harmonic map" that we now define.

Definition 3. *Let $u : \Omega_0 \to \mathbb{S}^{n-1}$ be an n-harmonic map, where $\Omega_0 = \Omega \setminus \{a_1, a_2, \cdots, a_d\}$. We say u is a* stationary n-harmonic map *if its stress-energy tensor*

$$T_{i,j} := |\nabla u|^{n-2} \langle \partial_i u, \partial_j u \rangle - \frac{1}{n} |\nabla u|^n \delta_{i,j}$$

satisfies

$$\sum_i \partial_i T_{i,j} = 0$$

in Ω_0*, and if for any* $1 \leq k \leq d$ *and* $\rho > 0$ *such that* $\partial B_\rho(a_k) \subset \Omega_0$ *it holds that*

$$\int_{\partial B_\rho(a_k)} \sum_i T_{i,j}\nu_i = 0, \tag{9}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ *is the outward-pointing normal to* $\partial B_\rho(a_k)$*.*

When both conditions are satisfied we say that T_{ij} *is* divergence free in Ω_0.

The following proposition links the property of being a stationary n-harmonic map with the vanishing gradient property. Unfortunately it is not clear yet whether its assumptions are satisfied for the stationary n-harmonic maps arising as limits of critical points of the Ginzburg-Landau functional in dimension n.

Proposition 4. *Assume* $u : \Omega_0 \subset \mathbb{R}^n \to \mathbb{S}^{n-1}$ *is a stationary* $n-$*harmonic map in the above sense, where* $\Omega_0 = \Omega \setminus (\{a_1, \cdots, a_d\}$*, and that* $\deg(u, a_k) = 1$ *. Assume that around each singular point* a_k*, one has the asymptotic expansion*

$$u(x) = \mathrm{e}^{B_k(x)} \frac{x - a_k}{|x - a_k|}$$

where $B_k(x) \in so(n)$ *is an antisymmetric matrix satisfying* $B_k(a_k) = 0$ *such that* $x \to B_k(x)$ *is* C^1 *in a neighborhood of* a_k*. Then*

$$\sum_{i=1}^n \partial_i B_k(a_k) e_i = 0, \tag{10}$$

where $(e_1, \cdots, e_n)$ *is the canonical basis in* $\mathbb{R}^n$*. Moreover, we have the expansion*

$$u(x) = \frac{x - a_k}{|x - a_k|} + \frac{Q_k(x - a_k)}{|x - a_k|} + O(|x - a_k|^2), \tag{11}$$

where $Q_k(x)$ *is a harmonic polynomial of degree 2. In particular, when* $n = 2$*, we have* $B_k(x) = O(|x - a_k|^2)$*.*

Finally we will construct an example of a sequence of non-minimizing critical points.

Theorem 5. *Let* $n = 3$*. There exists a domain* $\Omega \subset \mathbb{R}^3$*, a boundary map* $g : \partial\Omega \to \mathbb{S}^{n-1}$*, and for every small enough* $\varepsilon > 0$ *a non minimizing critical point* u_ε *of the functional* $\mathbf{E}_\varepsilon(u, \Omega)$ *such that the energy bound (8) holds.*

The above results are contained in the preprint [2].

References

[1] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, **13**, Birkhäuser Boston Inc., Boston, MA (1994).

[2] Y. Ge, E. Sandier and P. Zhang, *Generalized Ginzburg-Landau Equations in high dimensions*, preprint.

[3] Z. Han and Y. Li, *Degenerate elliptic systems and applications to Ginzburg-Landau type equations. I*, Calc. Var. Partial Differential Equations **4** (1996),171–202.
[4] R. Hardt, F. Lin and C. Wang, *Singularities of p-energy minimizing maps*, Comm. Pure Appl. Math. **50** (1997), 399–447.
[5] M. Hong, *Asymptotic behavior for minimizers of a Ginzburg-Landau-type functional in higher dimensions associated with n-harmonic maps*, Advances in Differential Equations **1** (1996), 611–634.
[6] R. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), 721–746.
[7] T. Rivière, *Minimizing Fibrations and p-Harmonic maps in Homotopy Classes from S^3 into S^2*, Comm. Anal. Geom. **6** (1998), 427–483.
[8] P. Strzelecki, *Asymptotics for the minimization of a Ginzburg-Landau energy in n dimensions*, Colloquium Mathematicae **70** (1996), 271–289.

Analyzing the rate of convergence of geometric flows

Alessandro Carlotto
(joint work with Otis Chodosh and Yanir Rubinstein)

Over the last three decades there has been huge interest in studying various kind of properties of geometric flows, by which we shall mean here parabolic evolution equations for the Riemannian metric on a compact manifold M (of dimension $n \geq 3$) without boundary. In this lecture, we shall be concerned with the question whether, in case the flow in question is known to exist for all (positive) times and to converge, in a suitably strong sense, to a limit metric g_∞ one can in fact produce a systematic analysis of the corresponding rate of convergence. We can provide a rather complete answer to this question under the assumption that the flow is a gradient flow: we characterize the rate of convergence of the flow in terms of Morse theoretic properties of the limiting metric g_∞. We shall take the Yamabe flow as our model to work with, even though some of our results do have a rather direct counterpart for other relevant flows, for instance for the Calabi flow.

Therefore, given (M, g_0) and denoted by R_g the scalar curvature of a metric g on M and with r_g its mean value, we shall deal with the geometric evolution problem

$$\begin{cases} \frac{\partial g}{\partial t} = -(R_g - r_g)g \\ g(0) = g_0. \end{cases}$$

This describes the evolution of Riemannian metrics inside a volume-normalized conformal class on M. The flow was introduced by R. Hamilton to solve the Yamabe problem [14, 7]: given (M, g_0) as above is there $g \in [g_0]$ having constant scalar curvature? Such problem was solved by variational methods through the combined effort of Aubin [2], Trudinger [13] and Schoen [9], yet the question of convergence for such a flow turned out to be surprisingly delicate and is still partially open. Indeed, while long-time existence had been settled by Hamilton in the early 80s, unconditional convergence results have been obtained only in 2005 by Brendle for $3 \leq n \leq 5$ in [3] with earlier significant contributions by Chow [6], Ye [15] and Schwetlich-Struwe [11] among others. A rather general convergence result also holds when $n \geq 6$ and a technical assumption on the conformal class of g_0 is made, see [4]. The problem of analyzing the rate of convergence for the (volume-normalized) Yamabe flow was explicitly posed by R. Ye in 1994.

In the setting above, g_∞ is a critical point of the Yamabe functional, which is defined by

$$\mathcal{Y}(g) := \mathrm{Vol}(M,g)^{-\frac{2}{N}} \int_M R_g \, dV_g, \text{ for } N = \frac{2n}{n-2}.$$

We consider the restriction of $\mathcal{Y}(\cdot)$ to the *unit volume conformal class* $[g_\infty]_1$: if $g = w^{N-2} g_\infty$, we have that the first variation of $\mathcal{Y} = \mathcal{Y}(w)$ is given by

$$\begin{aligned}\frac{1}{2} D\mathcal{Y}(w)[v] &= \int_M \left[-(N+2)\Delta_{g_\infty} w + R_{g_\infty} w - r_{w^{N-2}g_\infty} w^{N-1}\right] v \, dV_{g_\infty} \\ &= \int_M (R_{w^{N-2}g_\infty} - r_{w^{N-2}g_\infty}) w^{N-1} v \, dV_{g_\infty}.\end{aligned}$$

and the second variation is described by the self-adjoint operator $\mathcal{L}_\infty$ that is defined by means of the formula

$$-(N-2) \int_M w \mathcal{L}_\infty v \, dV_{g_\infty} := \frac{1}{2} D^2 \mathcal{Y}(g_\infty)[v, w]$$

for $v \in C^2(M)$ so that (via a simple computation)

$$\mathcal{L}_\infty v = (n-1)\Delta_{g_\infty} v + R_{g_\infty} v.$$

We define $\Lambda_0 := \ker \mathcal{L}_\infty \subset L^2(M, g_\infty)$. Spectral theory shows that Λ_0 is finite dimensional (it is the eigenspace of the Laplacian for the eigenvalue $\frac{R_{g_\infty}}{n-1}$). We will write $\Lambda_0^\perp$ for the $L^2(M, g_\infty)$-orthogonal complement. Let us denote by CSC_1 the set of unit volume, constant scalar curvature metrics in the normalized conformal class $[g_\infty]_1$. We shall recall the following two definitions:

- For $g_\infty \in CSC_1$, we say that g_∞ is *degenerate* if Λ_0 is not trivial;
- For $g_\infty \in CSC_1$, we say that g_∞ is *integrable* if for all $v \in \Lambda_0$, there is a path $w(t) \in C^2((-\epsilon, \epsilon) \times M, g_\infty)$ so that $w(t)^{N-2} g_\infty \in CSC_1$ and $w(0) = 1$, $w'(0) = v$.

Our first theorem in [5] ensures exponential convergence to integrable critical points (in particular, this holds for non-degenerate g_∞) and that the slowest the convergence occurs is polynomially.

Theorem 1. *Assume that $g(t)$ is a Yamabe flow that is converging in $C^{2,\alpha}(M, g_\infty)$ to g_∞ as $t \to \infty$ for some $\alpha \in (0,1)$. Then, there is $\delta > 0$ depending only on g_∞ so that*

(1) *If g_∞ is an integrable critical point, then the convergence occurs at an exponential rate*

$$\|g(t) - g_\infty\|_{C^{2,\alpha}(M, g_\infty)} \leq C e^{-\delta t},$$

for some constant $C > 0$ depending on $g(0)$.

(2) *In general, the convergence cannot be worse than a polynomial rate*

$$\|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq C(1+t)^{-\delta},$$

for some constant $C > 0$ depending on $g(0)$.

Our proof relies on the Łojasiewicz inequality (see [8] for the finite-dimensional version) together with a preliminary Lyapunov-Schmidt reduction (in the analytic category) of the Yamabe functional $\mathcal{Y}$ at g_∞. So, there is $\epsilon > 0$ and $\Phi : \Lambda_0 \cap \{v : \|v\|_{L^2} < \epsilon\} \to C^{2,\alpha}(M, g_\infty) \cap \Lambda_0^\perp$ so that $\Phi(0) = 0$, $D\Phi(0) = 0$ and defining $\Psi(v) = 1 + v + \Phi(v)$, we have $\mathrm{Vol}(M, \Psi(v)^{N-2} g_\infty) = 1$

$$\begin{aligned}\mathrm{proj}_{\Lambda_0^\perp}[D\mathcal{Y}(\Psi(v))] =& \mathrm{proj}_{\Lambda_0^\perp}\left[\left(R_{\Psi(v)^{N-2}g_\infty} - r_{\Psi(v)^{N-2}g_\infty}\right)\Psi(v)^{N-1}\right] = 0\\ \mathrm{proj}_{\Lambda_0}[D\mathcal{Y}(\Psi(v))] =& \mathrm{proj}_{\Lambda_0}\left[\left(R_{\Psi(v)^{N-2}g_\infty} - r_{\Psi(v)^{N-2}g_\infty}\right)\Psi(v)^{N-1}\right] = DF,\end{aligned}$$

where $F : \Lambda_0 \cap \{v : \|v\|_{L^2} \leq \epsilon\} \to \mathbb{R}$ is defined by $F(v) = \mathcal{Y}(\Psi(v))$. The intersection of CSC_1 with a small $C^{2,\alpha}(M, g_\infty)$-neighborhood of 1 is

$$\mathcal{S}_0 := \{\Psi(v) : v \in \Lambda_0, \|v\|_{L^2} < \epsilon, DF(v) = 0\}.$$

It is readily seen that the function F is locally constant if and only if g_∞ is integrable: if that is not the case we shall consider the power-series expansion $F(v) = F(0) + \sum_{j \geq p} F_j(v)$ and call p *order of integrability* of g_∞ (a direct check ensures that $p \geq 3$). Following the works [12, 1] on the analysis of isolated singularities of minimal subvarieties and harmonic maps, we say that g_∞ satisfies the *Adams–Simon positivity condition*, AS_p for short, if it is non-integrable and one has that $F_p|_{S^k}$ attains a positive maximum for some $\hat{v} \in S^k \subset \Lambda_0$. We can then provide an abstract existence result for slowly converging Yamabe flows:

Theorem 2. *Assume that g_∞ is a non-integrable critical point of the Yamabe energy with order of integrability $p \geq 3$. If g_∞ satisfies the Adams–Simon positivity condition AS_p, then there exists a metric $g(0)$ conformal to g_∞ so that the Yamabe flow $g(t)$ starting from $g(0)$ exists for all time and converges in $C^\infty(M, g_\infty)$ to g_∞ as $t \to \infty$. The convergence occurs "**slowly**" in the sense that*

$$C^{-1}(1+t)^{-\frac{1}{p-2}} \leq \|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq C(1+t)^{-\frac{1}{p-2}},$$

for some constant $C > 0$.

The proof of this theorem is quite long and technical and relies on a separate analysis of the coupled system of two flows: the *kernel-projected flow* and the *kernel orthogonal-projected flow*. To leading order, the former can be solved explicitly, and then the residual kernel-projected flow is a *system of ODEs* in a diagonalizing basis for $D^2 F_p(\hat{v})$ while the kernel-orthogonal projected flow is a perturbed linear parabolic equation, with the RHS terms that are coupled and complicated, but for which one can derive good a priori estimates. Then the conclusion comes by a suitable contraction mapping argument in parabolic setting.

In order to ensure the actual existence of slowly converging Yamabe flows we need to be able to identify critical points of the Yamabe functional $\mathcal{Y}$ that satisfy the following three conditions:

(1) *degeneration*
(2) *non-integrability*
(3) *Adams-Simon positivity* of order $p \geq 3$.

The first of these conditions is already non-generic so we need to detect some sporadic pheonomena for the Yamabe functional. We construct two classes of examples, the former satisfying AS_3 and the second AS_p for some $p \geq 4$ even.

Proposition 3. *Fix integers $n, m > 1$ and a closed m-dimensional Riemannian manifold (M^m, g_M) with constant scalar curvature $R_{g_M} \equiv 4(n+1)(m+n-1)$. We denote the complex projective space equipped with the Fubini–Study metric by $(\mathbb{C}P^n, g_{FS})$, where the normalization of g_{FS} is fixed so that the fibration $S^{2n+1}(1) \to (\mathbb{C}P^n, g_{FS})$ is a Riemannian submersion. Then, the product metric $(M^m \times \mathbb{C}P^n, g_M \oplus g_{FS})$ is a degenerate critical point satisfying AS_3.*

The second example is produced by following ideas of Schoen [10]: we choose the radius of the first factor below in a way that the corresponding conformal class does have the product metric as unique (hence isolated), but degenerate solution of the Yamabe problem.

Proposition 4. *Let $n > 2$. The product metric on $S^1\left(\frac{1}{\sqrt{n-2}}\right) \times S^{n-1}(1)$ is a non-integrable critical point satisfying AS_p for some $p \geq 4$.*

Of course, we expect *most* trajectories converging to a non-integrable costant scalar curvature metric to converge rapidly (i. e. exponentially) anyway: what the main theorems in [5] ensure is that slowly converging flow exist and that the polynomial estimates above are effective.

References

[1] D. Adams and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. **37** (1988), no. 2, 225–254.

[2] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296.

[3] S. Brendle, *Convergence of the Yamabe flow for arbitrary initial energy*, J. Differential Geom. **69** (2005), no. 2, 217–278.

[4] S. Brendle, *Convergence of the Yamabe flow in dimension 6 and higher*, Invent. Math. **170** (2007), no. 3, 541–576.

[5] A. Carlotto, O. Chodosh, Y. Rubinstein *Slowly converging Yamabe flows*, Geom. Top. **19** (2015), no. 3, 1523-1568.

[6] B. Chow, *The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature*, Comm. Pure Appl. Math. **45** (1992), no. 8, 1003–1014.

[7] J. M. Lee and T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91.

[8] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Les Équations aux Dérivées Partielles (Paris, 1962), Éditions du Centre National de la Recherche Scientifique, Paris, 1963, pp. 87–89.

[9] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495.

[10] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in calculus of variations (Montecatini Terme, 1987), Lecture Notes in Math., vol. 1365, Springer, Berlin, 1989, pp. 120–154.

[11] H. Schwetlick and M. Struwe, *Convergence of the Yamabe flow for "large" energies*, J. Reine Angew. Math. **562** (2003), 59–100.
[12] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571.
[13] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.
[14] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37.
[15] R. Ye, *Global existence and convergence of Yamabe flow*, J. Differential Geom. **39** (1994), no. 1, 35–50.

Area bounds for minimal surfaces

Brian White

Consider a sequence of m-dimensional minimal varieties M_i in a Riemannian manifold N (not necessarily complete) such that the measures of the boundaries are bounded (independent of i) on compact subsets of N. Let Z be the area blowup set for the sequence M_i:

$$Z = \{p \in N : \limsup_i \operatorname{area}(M_i \cap \mathbf{B}(p,r)) = \infty \quad \text{for all } r > 0\}.$$

We prove that Z behaves in some ways like a minimal variety without boundary. In particular, it satisfies the same maximum and barrier principles that are satisfied by a smooth, m-dimensional, minimal submanifold without boundary. For example, if $f : N \to \mathbb{R}$ is a C^2 function and if the restriction of f to Z has a local maximum at $p \in Z$, then

$$\operatorname{Trace}_m(\mathrm{D}^2 f(p)) \leq 0$$

where $\operatorname{Trace}_m(\mathrm{D}^2 f(p))$ is the sum of the m lowest eigenvalues of the Hessian of f at p. For suitable open subsets W of N, this allows one to show that if the areas of the M_i are uniformly bounded on compact subsets of W, then the areas are in fact uniformly bounded on all compact subsets of N.

As an application, we prove a form of Allard's Boundary Regularity Theorem that does not assume any area bounds. According to Allard's theorem, the following holds:

Theorem 1. *Suppose that $M \subset N$ is a smooth, embedded, connected m-dimensional submanfold with smooth, nonempty boundary. Suppose $M_1, M_2, \dots$ is a sequence m-dimensional minimal varieties in N such that*

(1) *M_i is supported in $\{x \in N : \operatorname{dist}(x, M) < \epsilon_i\}$, where $\epsilon_i \to 0$,*
(2) *∂M_i is smooth (with multiplicity 1) and converges smoothly to ∂M, and*
(3) *The Radon measures μ_{M_i} converge weakly to μ_M.*

Then the M_i converge smoothly to M. In other words, if $U \subset\subset N$ and if ∂U is transverse to M, then $M_i \cap U$ is smooth for all sufficiently large i and converges smoothly to $M \cap U$ as $i \to \infty$.

Our theorem about area blow up sets allows us to prove theorem 1 without assuming any area bounds. In particular, the assumption (3) is not necessary.

References

[1] W.K. Allard, *On the first variation of a varifold: boundary behavior*, Ann. of Math. (2) **101** (1975), 418–446.
[2] B. White, *Controlling area blow-up in minimal or bounded mean curvature varieties*, To appear in J. Differential Geometry (2013); preprint at arXiv:1207.3448

Super-Ricci flow for metric measure spaces

Karl-Theodor Sturm

A super-Ricci flow of smooth Riemannian manifolds is a time-dependent family $(M, g_t)_{t\in I}$ s.t.

$$\mathrm{Ric}_{\mathrm{g_t}} \geq -\frac{1}{2}\partial_{\mathrm{t}}\mathrm{g} \quad \forall \mathrm{t} \in \mathrm{I}.$$

This includes all static manifolds of nonnegative Ricci curvature as well as all solutions to the Ricci flow equation. We extend this concept to the setting of metric measure spaces. This new approach builds upon the theory of metric measure spaces with synthetic lower Ricci bounds in the sense of Lott-Sturm-Villani. Our main results states that the following are equivalent:

(i) the Boltzmann entropy $\mathrm{Ent}(\cdot|\mathrm{m_t})$ is dynamically convex on $(\mathcal{P}(X), W_t)$;

(ii) the backward heat flow is contracting

$$W_s(P_t^s\mu, P_t^s\nu) \leq W_t(\mu,\nu) \quad \forall s < t;$$

(iii)

$$\nabla_t|P_t^s u|^2 \leq P_t^s|\nabla_s u|^2 \quad \forall s < t;$$

(iv) Bakry-Energy

$$\Gamma_{2,t}(u) \geq \frac{1}{2}\partial_t\Gamma_t(u).$$

A formal Riemannian structure on conformal classes and the inverse Gauss curvature flow

Matthew J. Gursky
(joint work with Jeffrey Streets)

In this talk I presented some aspects of an ongoing research project with J. Streets (UC-Irvine), in which we define a formal Riemannian metric on the set of metrics in a conformal class with positive (or negative) curvature. Namely, let (M, g_0) be a compact Riemannian surface with positive Gauss curvature $K_0 > 0$, and let $[g_0]$ denote the conformal class of g_0. Define

$$\Gamma_1^+ = \{g_u = e^{2u}g_0 \in [g_0] \ : \ K_u = K_{g_u} > 0\}, \tag{1}$$

the space conformal metrics with positive Gauss curvature. Formally, the tangent space to $[g_0]$ at any metric $g_u \in [g_0]$ is given by $C^\infty(M)$. Let K_u denote the Gauss curvature of $g_u \in \Gamma_1^+$. We define for $\phi, \psi \in C^\infty(M)$,

$$\langle \phi, \psi \rangle_u = \int_M \phi\psi K_u dA_u, \tag{2}$$

where dA_u is the area form of g_u. In other words, we weight the standard L^2 metric with the Gauss curvature of the given conformal metric. If the Gauss curvature of g_0 is negative, we define

$$\Gamma_1^- = \{g_u = e^{2u} g_0 \in [g_0] \ : \ K_u = K_{g_u} < 0\}, \tag{3}$$

and the metric associated to this space is given by

$$\langle \phi, \psi \rangle_u = \int_M \phi\psi(-K_u) dA_u. \tag{4}$$

This definition is loosely inspired by the Mabuchi-Semmes-Donaldson metric [3, 5, 2] of Kähler geometry, wherein a formal Riemann metric is put on a Kähler class by imposing on the tangent space to a given Kähler potential the L^2 metric with respect to the associated Kähler metric. As observed in [3], this metric enjoys many nice formal properties, for instance nonpositive sectional curvature. Moreover, it has a profound relationship to natural functionals in Kähler geometry such as the Mabuchi K-energy and the Calabi energy, as well as their gradient flow, the Calabi flow. Based on these excellent formal properties Donaldson proposed a series of conjectures on the existence of geodesics, geodesic rays, as well as the existence properties of the Calabi flow.

There is a tight analogy in many respects between the Mabuchi metric and the metric defined in (2). Formal calculations derived using either the path derivative or variations of the length functional yield that a one-parameter family of conformal factors $u : [a, b] \to \Gamma_1^+$ is a geodesic if and only if

$$\frac{\partial^2 u}{\partial t^2} + \frac{|\nabla_0 \frac{\partial u}{\partial t}|^2}{K_0 - \Delta_0 u} = 0. \tag{5}$$

This is a degenerate elliptic fully nonlinear equation, and existence and regularity are therefore delicate. We remark that one-parameter families of conformal transformations are automatically geodesics.

After establishing the existence of $C^{1,1}$ geodesics, we show that the length of the unique regularizable geodesic connecting any two points does indeed define a metric space structure (Γ_1^+, d), and that this metric space is nonpositively curved in the sense of Alexandrov:

Theorem 1. *Let (M^2, g_0) be a compact Riemann surface. Then $(\Gamma_1^\pm, d)$ is a length space, with any two points connected by a unique regularizable $C^{1,1}$ geodesic. Moreover, it is nonpositively curved in the sense of Alexandrov.*

Furthering the analogy with the Kähler setting, the metric (2) is closely associated with the gradient flow of the normalized Liouville energy F. Previously Osgood-Phillips-Sarnack [4] studied the negative gradient flow, but with respect to the L^2 metric, yielding an equation which is similar to Ricci flow. With the ambient geometry given by the weighted L^2 metric on Γ_1^+, we arrive at a different evolution equation, expressed in terms of the conformal factor as

$$\frac{\partial u}{\partial t} = -1 + \frac{\bar{K_u}}{K_u},$$

where $\bar{K}$ is the average Gauss curvature. This is a fully nonlinear parabolic equation for u. On Γ_1^- we arrive at

$$\frac{\partial u}{\partial t} = 1 - \frac{\bar{K_u}}{K_u}.$$

Generically we will refer to these as *inverse Gauss curvature flow.* Our primary results are as follows:

Theorem 2. *Fix (M^2, g) a compact Riemann surface and $u \in \Gamma_1^{\pm}$.*

(1) *The solution to IGCF with initial condition u exists on $[0, \infty)$.*
(2) *The normalized Liouville energy is convex in time along the flow line, i.e.*

$$\frac{d^2}{dt^2} F[u(t)] \geq 0.$$

(3) *Given $v(x,t)$ another solution to IGCF, the distance between flow lines is nonincreasing, i.e.*

$$\frac{d}{dt} d(u(t), v(t)) \leq 0.$$

(4) *If $u \in \Gamma_1^-$, then the solution converges as $t \to \infty$ in the C^∞ topology to the unique conformal metric of constant scalar curvature.*
(5) *If $u \in \Gamma_1^+$ and $(M^2, g) \cong (S^2, g_{S^2})$, then the solution converges weakly in the distance topology to a minimizer for F in the completion $(\bar{\Gamma}_1^+, \bar{d})$.*

Properties (2) and (3) are directly analogous to results relating the K-energy, Mabuchi metric, and Calabi flow ([1]). We emphasize that the point of the hypothesis $(M^2, g) \cong (S^2, g_{S^2})$ is that we are NOT yet able to use the IGCF to provide an a priori proof of the Uniformization Theorem. We require the existence of a constant scalar curvature metric to ensure the convergence of the flow in the distance topology.

Although our results are in the setting of two dimensions, this is actually a special case of a more general construction on even dimensional manifolds. In dimensions $n \geq 4$, one can define a Riemannian structure on subsets of conformal classes satisfying an admissibility condition which naturally arises in the study of the $\sigma_{\frac{n}{2}}$-Yamabe problem. As in the case of surfaces, the underlying metric is closely associated to a functional whose critical points 'uniformize' the conformal class.

References

[1] E. Calabi, X.X. Chen, *The space of Kähler metrics II*, J. Diff. Geom. **61** (2002), 173-193.

[2] S. K. Donaldson, *Symmetric spaces, Kähler geometry and Hamiltonion dynamics*, in Northern California Symplectic Geometry Seminar, Amer. Math. Soc. Transl. Ser. 2, **196**, Amer. Math. Soc., Providence, 1999, 13-33.

[3] T. Mabuchi, *Some symplectic geometry on compact Kähler manifolds (1)*, Osaka J. Math **24** (1987), 227-252.

[4] B. Osgood, R. Phillips, P. Sarnack, *Extremals of determinants of Laplacians*, J. Func. Anal. **80**, 148-211 (1988).

[5] S. Semmes, *Complex Monge-Ampere equations and symplectic manifolds*, Amer. J. Math. **114** (1992), 495-550.

Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds

Andrea Mondino

(joint work with Fabio Cavalletti)

The isoperimetric problem, having its roots in myths of more than 2000 years ago, is one of the most classical and beautiful problems in mathematics. It amounts to answer the following natural questions:

(1) Given a space X what is the minimal amount of area needed to enclose a fixed volume v?
(2) Does an optimal shape exist?
(3) In the affermative case, can we describe the optimal shape?

There are not many examples of spaces where the answer to all the three questions above is known. If the space X is the euclidean N-dimensional space $\mathbb{R}^N$ then it is well known that the only optimal shapes, called from now on isoperimetric regions, are the round balls; if X is the round N-dimensional sphere $\mathbb{S}^N$ then the only isoperimetric regions are metric balls, etc. To the best of our knowledge, the spaces for which one can fully answer all the three questions above either have a *very strong symmetry* or they are perturbations of spaces with a very strong symmetry. For an updated list of geometries admitting an isoperimetric description we refer to [24, Appendix H]. Let us also mention that the isoperimetric problem has already been studied in presence of (mild) singularities of the space: mostly for conical manifolds [37, 41] and polytopes [40].

Besides the euclidean one, the most famous isoperimetric inequality is probably the Lévy-Gromov inequality [31, Appendix C], which states that if E is a (sufficiently regular) subset of a Riemannian manifold (M^N, g) with dimension N and Ricci bounded below by $K > 0$, then

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S|}, \tag{1}$$

where B is a spherical cap in the model sphere, i.e. the N-dimensional sphere with constant Ricci curvature equal to K, and $|M|, |S|, |\partial E|, |\partial B|$ denote the appropriate N or $N-1$ dimensional volume, and where B is chosen so that $|E|/|M| = |B|/|S|$. In other words, the Lévy-Gromov isoperimetric inequality states that isoperimetry in (M, g) is at least as strong as in the model space S.

Let us observe next that the isoperimetric problem makes sense in the larger class of metric measure spaces. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$, m.m.s. for short, is a metric space[1] (X, d) endowed with a Borel probability measure $\mathfrak{m}$. In the standard situation where the metric space is a compact Riemannian manifold, $\mathfrak{m}$ is nothing but the normalized volume measure. Notice that in the Lévy-Gromov inequality (1) one considers exactly this normalized volume measure.
Regarding the m.m.s. setting, it is clear that the volume of a Borel set is replaced by its $\mathfrak{m}$-measure, $\mathfrak{m}(E)$; the boundary area of the smooth framework instead can be replaced by the Minkowski content

$$\mathfrak{m}^+(E) := \liminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon}, \tag{2}$$

where $E^\varepsilon := \{x \in X \,:\, \exists y \in E \text{ such that } \mathsf{d}(x,y) < \varepsilon\}$ is the ε-neighborhood of E with respect to the metric d. So the isoperimetric problem for a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ amounts to finding the largest function $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})} : [0,1] \to \mathbb{R}^+$ such that for every Borel subset $E \subset X$ it holds $\mathfrak{m}^+(E) \geq \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mathfrak{m}(E))$.

The main goal of the talk is to prove that the Lévy-Gromov isoperimetric inequality holds in the general framework of metric measure spaces. In order the problem to make sense, we also need a notion of "Ricci curvature bounded below by K and dimension bounded above by N" for m.m.s..

1.1. **Ricci curvature lower bounds for metric measure spaces.** The investigation about the topic began with the seminal papers of Lott-Villani [34] and Sturm [46, 47], though has been adapted considerably since the work of Bacher-Sturm [7] and Ambrosio-Gigli-Savaré [3, 4]. The crucial property of any such definition is the compatibility with the smooth Riemannian case and the stability with respect to measured Gromov-Hausdorff convergence. While a great deal of progress has been made in this latter general framework from both the analytic, geometric and structural points of view, see for instance [1, 2, 3, 4, 5, 6, 7, 11, 12, 15, 25, 26, 28, 30, 29, 22, 38, 48], the isoperimetric problem has remained elusive.

The notion of lower Ricci curvature bound on a general metric-measure space comes with two subtleties. The first is that of *dimension*, and has been well understood since the work of Bakry-Emery [8] and Bakry-Ledoux [9]: in both the geometry and analysis of spaces with lower Ricci curvature bounds, it has become clear the correct statement is not that "X has Ricci curvature bounded from below by K", but that "X has N-dimensional Ricci curvature bounded from below by K". Such spaces are said to satisfy the (K,N)-*Curvature Dimension* condition,

[1] We assume (X, d) to be complete, separable and proper

$\mathsf{CD}(K,N)$ for short; a variant of this is that of *reduced* curvature dimension bound, $\mathsf{CD}^*(K,N)$. See [7, 8, 9, 47] for more on this.

The second subtle point is that the classical definition of a metric-measure space with lower Ricci curvature bounds allows for Finsler structures (see the last theorem in [48]), which after the aforementioned works of Cheeger-Colding are known not to appear as limits of smooth manifolds with Ricci curvature lower bounds. To address this issue, Ambrosio-Gigli-Savaré [4] introduced a more restrictive condition which rules out Finsler geometries while retaining the stability properties under measured Gromov-Hausdorff convergence, see also [1] for the present simplified axiomatization. In short, one studies the Sobolev space $W^{1,2}(X)$ of functions on X. This space is always a Banach space, and the imposed extra condition is that $W^{1,2}(X)$ is a Hilbert space. Equivalently, the Laplace operator on X is linear. The notion of a lower Ricci curvature bound compatible with this last Hilbertian condition is called *Riemannian Curvature Dimension* bound, RCD for short. Refinements of this have led to the notion of $\mathsf{RCD}^*(K,N)$-spaces, which is the key object of study in this talk.

1.2. **Main results.** Our main result is that the Lévy-Gromov isoperimetric inequality holds for m.m.s. satisfying N-Ricci curvature lower bounds:

Theorem 1 (Lévy-Gromov in $\mathsf{RCD}^*(K,N)$-spaces)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}^*(K,N)$ space for some $N \in \mathbb{N}$ and $K > 0$. Then for every Borel subset $E \subset X$ it holds*

$$\mathfrak{m}^+(E) \geq \frac{|\partial B|}{|S|},$$

where B is a spherical cap in the model sphere (the N-dimensional sphere with constant Ricci curvature equal to K) chosen so that $|B|/|S| = \mathfrak{m}(E)$.

Remark 1.
- *Theorem 1 is a particular case of a more general statement* [13] *including any lower bound $K \in \mathbb{R}$ on the Ricci curvature and any upper bound $N \in [1,\infty)$ on the dimension. In order to state the result one needs some model space to compare with: the same role that the round sphere played for the Lévy-Gromov inequality. The model spaces for general K,N have been discovered by E. Milman* [36] *who extended the Lévy-Gromov isoperimetric inequality to smooth manifolds with densities, i.e. smooth Riemannian manifold whose volume measure has been multiplied by a smooth non negative integrable density function. Milman detected a model isoperimetric profile $\mathcal{I}_{K,N,D}$ such that if a Riemannian manifold with density has diameter at most $D > 0$, generalized Ricci curvature at least $K \in \mathbb{R}$ and generalized dimension at most $N \geq 1$ then the isoperimetric profile function of the weighted manifold is bounded below by $\mathcal{I}_{K,N,D}$. In* [13] *we extend such Lévy-Gromov-Milman inequality to* $\mathsf{RCD}^*(K,N)$ *spaces with diameter at most $D > 0$.*
- *Theorem 1 holds in the more general framework of essentially non branching $\mathsf{CD}^*(K,N)$-spaces. but we decided to state it in this form so to give a unified presentation also with the rigidity statement below.*

- *A first natural question is rigidity: if for some $v \in (0,1)$ it holds $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) = \mathcal{I}_{K,N,\infty}(v)$, does it imply that X has a special structure? The answer is YES, indeed in case $K > 0$, if for some $v \in (0,1)$ the lower bound in Theorem 1 is attained then the space X must be a spherical suspension. Moreover in this case we classify completely the isoperimetric regions: they are the "spherical caps centered at the poles of the spherical suspension".*
- *A last question is the almost rigidity: if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}^*(K,N)$ space such that $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v)$ is close to $\mathcal{I}_{K,N,\infty}(v)$ for some $v \in (0,1)$, does this force X to be close to a spherical suspension? The answer is again YES, for a precise statemetent we again refer to* [13].

Remark 2 (Notable examples of spaces fitting in the assumptions of the main theorems)**.** *The class of $\mathsf{RCD}^*(K,N)$ spaces include many remarkable family of spaces, among them:*

- Measured Gromov Hausdorff limits of Riemannian N-dimensional manifolds satisfying Ricci $\geq K$. *Despite the fine structural properties of such spaces discovered in a series of works by Cheeger-Colding* [17, 18, 19] *and Colding-Naber* [20]*, the validity of the Lévy-Gromov isoperimetric inequality (and the above generalizations and rigidity statements) has remained elusive. We believe this is one of the most striking applications of our results. For Ricci limit spaces let us also mention the recent work by Honda* [33] *where a lower bound on the Cheeger constant is given, thanks to a stability argument on the first eigenvalue of the p-Laplacian for $p = 1$.*
- Alexandrov spaces with curvature bounded from below. *Petrunin* [44] *proved that the lower curvature bound in the sense of comparison angles is compatible with the optimal transport type lower bound on the Ricci curvature given by Lott-Sturm-Villani (see also* [49]*). Moreover it is well known that the Laplace operator on an Alexandrov space is linear. It follows that Alexandrov spaces with curvature bounded from below are examples of* $\mathsf{RCD}^*(K,N)$ *and therefore our results apply as well. Let us note that in the framework of Alexandrov spaces the best result regarding isoperimetry is a sketch of a proof by Petrunin* [43] *of the Lévy-Gromov inequality for Alexandrov spaces with (sectional) curvature bounded below by 1.*

A last class of spaces where Theorem 1 apply is the one of smooth Finsler manifolds where the norm on the tangent spaces is strongly convex, and which satisfy lower Ricci curvature bounds. More precisely we consider a C^∞-manifold M, endowed with a function $F : TM \to [0,\infty]$ such that $F|_{TM\setminus\{0\}}$ is C^∞ and for each $p \in M$ it holds that $F_p := T_pM \to [0,\infty]$ is a strongly-convex norm, i.e.

$$g^p_{ij}(v) := \frac{\partial^2(F_p^2)}{\partial v^i \partial v^j}(v) \quad \text{is a positive definite matrix at every } v \in T_pM \setminus \{0\}.$$

Under these conditions, it is known that one can write the geodesic equations and geodesics do not branch; in other words these spaces are non-branching. We also assume (M,F) to be geodesically complete and endowed with a C^∞ probability *measure* $\mathfrak{m}$ *in a such a way that the associated m.m.s.* $(X, F, \mathfrak{m})$ *satisfies the* $\mathsf{CD}^*(K,N)$ *condition. This class of spaces has been investigated by Ohta* [45] *who established the equivalence between the Curvature Dimension condition and a Finsler-version of Bakry-Emery* N*-Ricci tensor bounded from below. Recalling Remark 1, these spaces fit in the assumptions of Theorem 1, and to our knowledge the Lévy-Gromov inequality (and its generalizations) is new also in this framework.* □

1.3. **Outline of the argument.** The main reason why the Lévy-Gromov type inequalities have remained elusive in non smooth metric measure spaces is because the known proofs heavily rely on the existence and sharp regularity properties of isoperimetric regions ensured by Geometric Measure Theory (see for instance [31, 39]). Clearly such tools are available if the ambient space is a smooth Riemannian manifold (possibly endowed with a weighted measure, with smooth and strictly positive weight), but are out of disposal for general metric measure spaces.

In order to overcome this huge difficulty we have been inspired by a paper of Klartag [23] where the author gave a proof of the Lévy-Gromov isoperimetric inequality still in the framework of smooth Riemannian manifolds, but via an optimal transportation argument involving L^1-transportation and ideas of convex geometry. In particular he used a localization technique, having its roots in a work of Payne-Weinberger [42] and developed by Gromov-Milman [32], Lovász-Simonovits [35] and Kannan-Lovász-Simonovits [21], which consists in reducing an n-dimensional problem, via tools of convex geometry, to one-dimensional problems that one can handle.

Let us stress even if the approach by Klartag [23] does not rely on the regularity of the isoperimetric regions, still heavily makes use of the smoothness of the ambient space in order to establish sharp properties of the geodesics in terms of Jacobi fields and estimates on the second fundamental forms of suitable level sets, all objects that are still not enough understood in general m.m.s. in order to repeat the same arguments.

To overcome this difficulty we use the structural properties of geodesics and of L^1-optimal transport implied by the $\updownarrow^*(K,N)$ condition. Such results have their roots in previous works of Bianchini-Cavalletti [10] and the first author [11, 12]. The first key point is to understand the structure of d-monotone sets, in particular we prove that under the curvature condition one can decompose the space, up to a set of measure zero, in equivalence classes called rays where the L^1-transport is performed. A second key point, which is the technical novelty of the present work with respect to the aforementioned papers [10, 11, 12], is that on almost every ray the conditional measure satisfies a precise curvature inequality. This last technical novelty is exactly the key to reduce the problem on the original m.m.s. to a one dimensional problem.

Let us end by mentioning that the techniques presented in the seminar have been recently used by the authors in a subsequent paper [14] to prove functional inequalities like spectral gap, Poincaré and log-Sobolev inequalities, the Payne-Weinberger/Yang-Zhong inequality, among others. Some of these inequalities are consequences of the four functions theorem of Kannan, Lovász and Simonovits that we will establish as well. Some of these inequalities were open problems proposed in the celebrated Optimal Transport book of Villani [48].

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure, *to appear in Trans. Amer. Math. Soc.*, arxiv:1207.4924.

[2] L. Ambrosio, N Gigli, G. Savaré, Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. *Annals of Probab.*, to appear.

[3] ———, Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, Arxiv preprint, arXiv:1106.2090, (2011), *To appear on Invent. Math.*

[4] ———, Metric measure spaces with Riemannian Ricci curvature bounded from below, *Duke Math. J.*, 163, 1405–1490, (2014).

[5] L. Ambrosio, A. Mondino and G. Savaré, Nonlinear diffusion equations and curvature conditions in metric measure spaces, *in preparation.*

[6] ———, On the Bakry-Émery condition, the gradient estimates and the Local-to-Global property of $RCD^*(K,N)$ metric measure spaces, *Journal of Geometric Analysis* (in press) DOI: 10.1007/s12220-014-9537-7, Arxiv preprint 1309.4664.

[7] K. Bacher and K.-T. Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, *J. Funct. Anal.,* **259** (2010), 28–56.

[8] D. Bakry and M. Emery, Diffusions hypercontractives *Seminaire de Probabilites XIX, Lecture Notes in Math.*, Springer-Verlag, New York. 1123 (1985), 177–206.

[9] D. Bakry and M. Ledoux, A logarithmic Sobolev form of the Li-Yau parabolic inequality, *Rev. Mat. Iberoam.,* **22**, 2, (2006), 683–702.

[10] S. Bianchini and F. Cavalletti, The Monge problem for distance cost in geodesic spaces. *Commun. Math. Phys.*, 318, 615 – 673 (2013).

[11] F. Cavalletti, Monge problem in metric measure spaces with Riemannian curvature-dimension condition. *Nonlinear Anal.* **99** (2014), 136–151.

[12] ———, Decomposition of geodesics in the Wasserstein space and the globalization property. *Geom. Funct. Anal.*, **24** (2014) 493 – 551.

[13] F. Cavalletti and A. Mondino, Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. *preprint arXiv:1502.06465*, (2015).

[14] ———, Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds. *preprint arXiv:1505.02061*, (2015).

[15] F. Cavalletti and K.-T. Sturm. Local curvature-dimension condition implies measure-contraction property. *J. Funct. Anal.*, **262**, 5110 – 5127, 2012.

[16] G. Cheeger and T.H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, *Annals of Math.*, **144**, 1, (1996), 189–237.

[17] ———, On the structure of spaces with Ricci curvature bounded below. I. *J. Diff. Geom.*, **45** (1997), 406 – 480.

[18] ———, On the structure of spaces with Ricci curvature bounded below. II. *J. Diff. Geom.*, **54**, (2000), 13–35.

[19] ———, On the structure of spaces with Ricci curvature bounded below. III. *J. Diff. Geom.*, **54**, (2000), 37 – 74.

[20] T. Colding, A. Naber, Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications *Annals of Math.*, **176**, (2012).

[21] R. Kannan, L. Lovász and M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, *Discrete Comput. Geom.,* **13**, 3-4, (1995), 541–559.

[22] K.. Ketterer, Cones over metric measure spaces and the maximal diameter theorem. Journ. Math. Pures et Appl. (to appear), arXiv:1311.1307v3.

[23] B. Klartag, Needle decomposition in Riemannian geometry, *preprint*, arXiv:1408.6322.

[24] M. Eichmair and J. Metzger, *Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions,* Invent. Math., **194**, (2013), 591–630.

[25] M Erbar, Kuwada and K.T. Sturm, On the Equivalence of the Entropic Curvature-Dimension Condition and Bochner's Inequality on Metric Measure Space, *Invent. Math.* (to appear) arXiv:1303.4382.

[26] N. Garofalo and A. Mondino, Li-Yau and Harnack type inequalities in $\mathsf{RCD}^*(K,N)$ metric measure spaces, *Nonlinear Analysis: Theory, Methods & Applications* **95**, (2014), 721–734.

[27] N. Gigli, Optimal maps in non branching spaces with Ricci curvature bounded from below, *Geom. Funct. Anal.*, 22 (2012) no. 4, 990–999.

[28] ———, The splitting theorem in non-smooth context, *preprint arXiv:1302.5555*, (2013).

[29] N. Gigli, A. Mondino and G. Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, *preprint arXiv:1311.4907*, (2013).

[30] N. Gigli, A. Mondino and T. Rajala, Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below *Journal fur die Reine und Ang. Math. (Crelle's journal)* (in press) DOI 10.1515/ crelle-2013-0052, preprint arXiv:1304.5359.

[31] M. Gromov, Metric structures for Riemannian and non Riemannian spaces, *Modern Birkhäuser Classics*, (2007).

[32] M. Gromov and V. Milman, Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces. *Compositio Math.*, **62**, 3, (1987), 263–282.

[33] S. Honda, Cheeger constant, p-Laplacian, and Gromov-Hausdorff convergence, *Preprint Arxiv: 1310.0304v3*, (2014).

[34] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* (2) **169** (2009), 903–991.

[35] L. Lovász and M. Simonovits, Random walks in a convex body and an improved volume algorithm, *Random Structures Algorithms,* **4**, 4, (1993), 359–412.

[36] E. Milman, Sharp Isoperimetric Inequalities and Model Spaces for Curvature-Dimension-Diameter Condition, *Journ. Europ. Math. Soc.* to appear.

[37] E. Milman and L. Rotem, Complemented Brunn-Minkowski Inequalities and Isoperimetry for Homogeneous and Non-Homogeneous Measures, *Advances in Math.*, **262**, 867–908, (2014).

[38] A. Mondino and A. Naber, Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds I, *preprint*, arXiv:1405.2222.

[39] F. Morgan, Geometric Measure Theory (A Beginner's Guide), *Elsevier/Academic Press,* Amsterdam, Fourth edition, (2009).

[40] ———, In polytopes, small balls about some vertex minimize perimeter. J. Geom. Anal. **17**, 97–106, (2007).

[41] F. Morgan and M. Ritoré, Isoperimetric regions in cones, *Trans. Amer. Math. Soc.*, **354**, (2002), 2327–2339.

[42] L.E. Payne and H.F. Weinberger, An optimal Poincaré inequality for convex domains *Arch. Rational Mech. Anal.*, **5**, (1960), 286–292.

[43] A. Petrunin, Harmonic functions on Alexandrov spaces and their applications, *Electronic Res. Announc. AMS*, **9**, (2003), 135—141.

[44] ———, Alexandrov meets Lott-Sturm-Villani, *Münster J. Math.*, **4**, (2011), 53–64.

[45] S.I.. Ohta, Finsler interpolation inequalities, *Calc. Var. Partial Differential Equations*, **36**, (2009), 211–249.
[46] K.T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.* **196** (2006), 65–131.
[47] K.T. Sturm, On the geometry of metric measure spaces. II, *Acta Math.* **196** (2006), 133–177.
[48] C. Villani, Optimal transport. Old and new, *Grundlehren der Mathematischen Wissenschaften*, **338**, Springer-Verlag, Berlin, (2009).
[49] H.C. Zhang and X.P. Zhu, Ricci curvature on Alexandrov spaces and rigidity theorems, *Comm. Anal. Geom.* **18(3)** (2010), 503–553.

BV estimates in optimal transportation and applications

Guido De Philippis
(joint work with Alpár Mészáros, Filippo Santambrogio and Bozhidar Velichkov)

In several applications it is interesting to establish regularity properties for minimizers of the following variational problem:

$$\min_{\nu\in\mathcal{W}_2(\mathbb{R}^N)} \frac{1}{2}W_2^2(\mu,\nu)+F(\nu). \tag{1}$$

Here $\mathcal{W}_2(\mathbb{R}^N)$ is the set of probability measures on $\mathbb{R}^N$ with finite second moment, μ is a given element in $\mathcal{W}_2(\mathbb{R}^N)$, $W_2(\mu,\nu)$ is the *Wassertstein distance* between μ and ν and $F:\mathcal{W}_2(\mathbb{R}^N)\to\mathbb{R}\cup\{+\infty\}$ is defined as

$$F(\nu)=\begin{cases}\int h(\varrho(x))dx & \text{if } \nu=\varrho dx\\ +\infty & \text{otherwise}\end{cases}$$

for some *convex* and *superlinear* function $h:\mathbb{R}_+\to\mathbb{R}\cup\{+\infty\}$.

Let us mention some applications related to (1):

• Solutions of (1) can be used to build a discrete in time approximation to solutions of the following degenerate parabolic PDE

$$\partial_t\varrho_t-\nabla\cdot(\varrho_t h''(\varrho_t)\nabla\varrho_t)=0,$$

see [1, 2, 4, 7].

• The choice of

$$h(\varrho)=\begin{cases}0 & \text{if } 0\le\varrho\le 1\\ +\infty & \text{otherwise}\end{cases}$$

reduces (1) to

$$\min_{\nu\in K_1} W_2^2(\mu,\nu), \tag{2}$$

where

$$K_1:=\left\{\nu\in\mathcal{W}_2(\mathbb{R}^N):\nu=\varrho dx \text{ and } 0\le\varrho\le 1\right\}$$

is the set of probability measures with density bounded by 1. Hence the solution $\bar\nu$ of (2) is the Wasserstein projection of μ on K_1. Besiede its own interest, the Wasserstein projection problem naturally arises in some continuous models of crowd motion, see [5, 6].

More in general, given a function $f : \mathbb{R}^N \to \mathbb{R}_+$ with $\int f > 1$, one can also consider the more general projection problem

$$\min_{\nu \in K_f} W_2^2(\mu, \nu), \tag{3}$$

where

$$K_f := \{\nu \in \mathcal{W}_2(\mathbb{R}^N) : \nu = \varrho dx \text{ and } 0 \leq \varrho(x) \leq f(x) \text{ for a.e. } x\}$$

is the set of probability measures with density bounded by f.

The main result in [3] is the validity of the following *a-priori BV* bound for solutions of (1):

Let $\mu \in BV(\mathbb{R}^N) \cap \mathcal{W}_2(\mathbb{R}^N)$, i.e. $\mu = \hat{\varrho} dx$ with $\hat{\varrho} \in BV(\mathbb{R}^N)$. Then for any minimizer $\bar{\nu}$ of (1) (the minimizer actually turns out to be unique), $\bar{\nu} = \bar{\varrho} dx$, $\bar{\varrho} \in BV(\mathbb{R}^N)$ and one has the estimate

$$\int |\nabla \bar{\varrho}| \leq \int |\nabla \hat{\varrho}|. \tag{4}$$

Note that the above estimate does not depend on the regularity of h and thus can be extended also to the projection problem (2). In the case of the projection problem with a general density ,(3), one instead obtains

$$\int |\nabla \bar{\varrho}| \leq \int |\nabla \hat{\varrho}| + 2 \int |\nabla f|. \tag{5}$$

The key ingredient of the proof of (4) and (5) is the following general lemma, see [3, Lemma 3.1]:

Lemma 1. *Suppose that $\varrho_1, \varrho_2 \in L^1$ are smooth compactly supported probability densities, which are bounded away from 0 and infinity and let H be a convex function. Then we have the following inequality*

$$\int \Big(\varrho_1 \nabla \cdot \big[\nabla H(\nabla \varphi_1)\big] - \varrho_2 \nabla \cdot \big[\nabla H(-\nabla \varphi_2)\big] \Big) \leq 0, \tag{6}$$

where φ_1 (resp φ_2) is the Kantorovich potential associated to the optimal transport problem from $\varrho_1 dx$ to $\varrho_2 dx$ (resp from $\varrho_2 dx$ to $\varrho_1 dx$).

The optimality conditions naturally associated to (1),

$$\nabla \varphi + h''(\bar{\varrho}) \nabla \bar{\varrho} = 0$$

where φ is the Kantorovich potential associated to the optimal transport problem from the minimizer $\bar{\nu} = \bar{\varrho} dx$ to μ and the choice of $H(z) = |z|$ in (6) then formally give the bound (4). A similar argument (together with several approximations) provides the proof of (5).

Let us conclude mentioning that the validity of (6) has nothing to do with the minimization problem (1). It would be interesting to find some other applications of (6) as well as a more “transport” proof of it, the proof in [3] being based just on some quite “miraculous” computations related to the (-1)-concavity of

Kantorovitch potentials. To this end let us note that in the case $H(z) = |z|^2$, inequality (6) is actually equivalent to the convexity of the entropy functional

$$\mathcal{E}(\varrho) = \int \varrho \log \varrho$$

along Wasserstein geodesics.

References

[1] L. Ambrosio, Movimenti minimizzanti, *Rend. Accad. Naz. Sci. XL Mem. Mat. Sci. Fis. Natur.*, **113** (1995), 191–246.

[2] L. Ambrosio, N. Gigli, G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Math., ETH Zürich, (2005).

[3] G. De Philippis, A. R. Mészáros, F. Santambrogio, B. Velichkov, *BV estimates in optimal transportation and applications*, To appear in Arch. Rati. Mech. An., Available at http://cvgmt.sns.it/paper/2559/.

[4] R. Jordan, D. Kinderlehrer, F. Otto, *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal. **29** (1998), no. 1, 1–17.

[5] B. Maury, A. Roudneff-Chupin, F. Santambrogio *A macroscopic crowd motion model of gradient flow type*, Math. Models and Methods in Appl. Sciences, **Vol. 20** (2010), No. 10, 1787–1821.

[6] A. R. Mészáros, F. Santambrogio, *A diffusive model for macroscopic crowd motion with density constraints*, preprint available at http://cvgmt.sns.it/paper/2644/

[7] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Commun. in PDE, **26** (2001), No. 1–2, 101–174.

Singular set of harmonic maps and minimal surfaces

Aaron Naber

The talk focused on the regularity of stationary and minimizing harmonic maps $f : B_2(p) \subseteq M \to N$ between Riemannian manifolds. More specifically, recall that $S^k(f) \equiv \{x \in M : \text{ no tangent map at } x \text{ is } k+1\text{-symmetric}\}$ is k^{th}-stratum of the singular set of f. In this case it is well known that $\dim S^k \leq k$, however little else about the structure of $S^k(f)$ is understood in any generality. The first result we discussed is for a general stationary harmonic map, where we prove that $S^k(f)$ is k-rectifiable. In fact, we prove more and show that for k-a.e. point $x \in S^k(f)$ that there exists a unique k-plane $V^k \subseteq T_xM$ such that *every* tangent map at x is k-symmetric with respect to V. This is a slightly subtle point in that the tangent map may not be unique, but the plane of symmetry is.

We also discussed the case of minimizing harmonic maps, and showed that the singular set $S(f)$, which is well known to satisfy $\dim S(f) \leq n-3$, is in fact $n-3$-rectifiable with uniformly *finite* $n-3$-measure. We even discussed packing content estimates for the set. An effective version of this allows us to prove that $|\nabla f|$ has estimates in L^3_{weak}, an estimate which is sharp as $|\nabla f|$ may not live in L^3.

The above results are in fact just the main applications of a new class of estimates we prove on the *quantitative* stratifications $S^k_{\epsilon,r}(f)$ and $S^k_\epsilon(f) \equiv S^k_{\epsilon,0}(f)$.

Roughly, $S^k_\epsilon \subseteq M$ is the collection of points $x \in S^k_\epsilon$ for which no ball $B_r(x)$ is ϵ-close to being $k+1$-symmetric. We show that S^k_ϵ is k-rectifiable and satisfies the Minkowski estimate $Vol(B_r\, S^k_\epsilon) \leq Cr^{n-k}$. It turns out that using the equality $S^k = \bigcup S^k_\epsilon$ one can recover the rectifiable statement for S^k from this, and using an ϵ-regularity one can recover the main results on minimizing maps as well.

The proofs require a new L^2-subspace approximation theorem for stationary harmonic maps, as well as new $W^{1,p}$-Reifenberg and rectifiable-Reifenberg type theorems, which we spent some time discussing. These results are generalizations of the classical Reifenberg, and give checkable criteria to determine when a set is k-rectifiable with uniform measure estimates. The new Reifenberg type theorems may be of some independent interest. The L^2-subspace approximation theorem we prove is then used to help break down the quantitative stratifications into pieces which satisfy these criteria.

Some integral curvature estimates for the Ricci flow in four dimensions

Miles Simon

Abstract

We consider solutions $(M^4, g(t)), 0 \leq t < T$, to Ricci flow on compact, four dimensional manifolds without boundary. We prove integral curvature estimates which are valid for any such solution. In the case that the scalar curvature is bounded and $T < \infty$, we show that these estimates imply that the (spatial) integral of the square of the norm of the Riemannian curvature is bounded by a constant independent of time t for all $0 \leq t < T$ and that the space time integral over $M \times [0,T)$ of the fourth power of the norm of the Ricci curvature is bounded.

1. Introduction

The results of this talk may be found in [3] and [4]. The first part of the talk is concerned with proving general integral curvature estimates for any smooth Ricci flow $(M^4, g(t))_{t\in[0,T)}$, $T < \infty$ on a closed four dimensional manifold. The second part of the talk considers the special case that the absolute value of the scalar curvature R is bounded on $[0,T)$, that is $\sup_{M\times[0,T)} |\mathrm{R}|(\cdot,\cdot) < \infty$. In this case we obtain estimates on the regular region and the singular region. $x \in M$ is in the regular region if there exists an $r = r(p) > 0$ such that

$$\int_{{}^tB_r(p)} |\operatorname{Riem}|^2(\cdot,t)\, d\mu_{g(t)} \leq \epsilon_0$$

for all times $t \in [0,T)$, where $\epsilon_0 > 0$ is a small fixed constant depending on $(M^4, g(0))$ and T. The singular region is the complement of the regular region.

Using the estimates on the singular and regular region, we show that $(M, d(g(t)))$ approaches an C^0-Riemannian orbifold (X,h) with finitely many orbifold points, and that it is possible to flow this orbifold by the orbifold Ricci flow.

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2. Report

We consider smooth closed solutions $(M^4, g(t))_{t\in[0,T)}$ to Ricci flow, $\frac{\partial}{\partial t}g = -2\mathrm{Rc}(g)$ on a four manifold with $T < \infty$. The Ricci flow was introduced and first studied by R. Hamilton, [2]. By scaling the solution once, we may assume without loss of generality, that $\mathrm{R}(\cdot,\cdot) > -\frac{1}{2}$. This enables us to consider the evolution equation of the smooth function $f := \frac{|\mathrm{Rc}|^2}{\mathrm{R}+1}$. Using the generalised Gauss-Bonnet Theorem,

$$\int_{M^4} |\operatorname{Riem}|^2 - 4|\mathrm{Rc}|^2 + \mathrm{R}^2 = 32\pi^2\chi,$$

the evolution equations (see [2])

$$\begin{aligned} &\frac{\partial}{\partial t}|\mathrm{Rc}|^2 = \Delta|\mathrm{Rc}|^2 - 2|\nabla\mathrm{Rc}|^2 + 4\,\mathrm{Rm}^{ikjl}\,\mathrm{Rc}_{ij}\mathrm{Rc}_{kl} \\ &\frac{\partial}{\partial t}\mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Rc}|^2, \end{aligned} \tag{1}$$

and Young's inequality, we show that

$$\frac{d}{dt}(e^{-64t}\int_M f d\mu_g) \leq e^{-64t}2^8\pi^2\chi + e^{-64t}\int_M (-f^2 + 2^{10}\mathrm{R}^2) \;\; d\mu_g.$$

Integrating in time from 0 to S, we get

$$\begin{aligned} &\int_M \frac{|\mathrm{Rc}|^2(\cdot,S)}{(\mathrm{R}(\cdot,S)+1)}d\mu_{g(S)} + \int_0^S\int_M \frac{|\mathrm{Rc}|^4(\cdot,t)}{(\mathrm{R}(\cdot,t)+1)^2}d\mu_{g(t)}dt \\ &\quad\leq 2^2\pi^2\chi(e^{64S}-1) + e^{64S}\int_M \frac{|\mathrm{Rc}|^2(\cdot,0)}{(\mathrm{R}(\cdot,0)+1)}d\mu_{g(0)} \\ &\quad\quad +2^{10}e^{64S}\int_0^S\int_M \mathrm{R}^2(\cdot,t)d\mu_{g(t)}dt \\ &\quad =: c_0(M,g(0),S) + 2^{10}e^{64S}\int_0^S\int_M \mathrm{R}^2(\cdot,t)d\mu_{g(t)}dt, \end{aligned} \tag{2}$$

for all $S < T$. In the case that $|\mathrm{R}| \leq c_1$ on $M\times[0,T)$, (2) implies

$$\int_M |\operatorname{Riem}|^2 \leq c_1(M,g_0,T) \tag{3}$$

for all $t\in[0,T)$ and

$$\int_0^T\int_M |\mathrm{Rc}|^4 \leq c_1(M,g_0,T). \tag{4}$$

A result of the type (3) was independently onbtained in [1] using different methods. A point $p\in M$ is a *regular point in* M (or $p\in M$ is *regular*) if there exists an

$r = r(p) > 0$ such that

$$\int_{{}^tB_r(p)} |\operatorname{Riem}|^2(\cdot, t) d\mu_{g(t)} \leq \epsilon_0$$

for all times $t \in [0, T)$, where $\epsilon_0 > 0$ is a small fixed constant depending on $(M^4, g(0))$ and T.
A point $p \in M$ is a *singular point in M* (or $p \in M$ is *singular*) if $p \in M$ is not a regular point.
Let $t \in (0, T)$. We say $p \in \operatorname{Reg}_t(M)$ if

$$\int_{{}^tB_{R\sqrt{T-t}}(p)} |\operatorname{Riem}|^2(\cdot, t) d\mu_{g(t)} \leq \epsilon_0,$$

where ϵ_0 is as above, and R is a large constant depending only on $(M, g(0))$ and T.
A point $t \in [0, T)$ is a good time if $\int_M |\mathrm{Rc}|(\cdot, t) \leq \frac{1}{T-t}$. From (4), we see that for every given $0 < \tilde{t} < T$ near enough T, there is a *nearby* good time t with $\tilde{t} < t < \tilde{t} + \frac{(T-\tilde{t})}{2}$.

We obtain the following, and give some of the proof ideas in the talk:

(i) If $p_0 \in \operatorname{Reg}_t(M)$ and t is a *good time* near enough to T, then $({}^tB_{(R/2)\sqrt{T-t}}(p_0), g(s))$ satisfies uniform estimates for all $s > t$, and $({}^tB_{(R/2)\sqrt{T-t}}(p_0), g(s))$ approaches some smooth limit as $s \nearrow T$.

(ii) Singular regions concentrate at finitely many points and don't move around too much : there are fixed constants $J_0 < J_1 < J_2$ such that the following holds. For all good times t near enough to T, we find points $p_1(t), p_2(t), \ldots, p_L(t)$, L is independent of t, such that $\operatorname{Sing}(M) \subseteq \cup_{j=1}^L {}^tB_{\sqrt{T-t}J_1}(p_j(t))$ and ${}^rB_{\sqrt{T-t}J_0}(p_j(t)) \subseteq {}^tB_{\sqrt{T-t}J_1}(p_j(t)) \subseteq {}^sB_{\sqrt{T-t}J_2}(p_j(t))$ for all $t \leq r, s < T$ for all $j = 1, \ldots, L$.

(iii) There is a Gromov-Hausdorff limit $(X, h) := \lim_{GH t \nearrow T}(M, d(g(t)))$ where (X, h) is a C^0 Riemannian orbifold with at most finitely many orbifold points. The limit (and the limiting process) is smooth away from the orbifold points.

(iv) It is possible to flow (X, h) for a short time using the orbifold Ricci flow, the solution becomes smooth instantaneously.

Results related to (i) and (iii) were obtained independently using different methods by Q. Zhang and R. Bamler in [1].

References

[1] R. Bamler, Q. Zhang *Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature* , Arxiv Preprint, arXiv:1501.01291 (2015)

[2] R. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. Volume 17, Number 2, 255-306. (1982),

[3] M. Simon, *Some integral curvature estimates for the Ricci flow in four dimensions*, Arxiv Preprint, arXiv:1504.02623 (2015)

[4] M. Simon, *Extending four dimensional Ricci flows with bounded scalar curvature* Arxiv Preprint, arXiv:1504.02910 (2015)

A fully nonlinear flow for two-convex hypersurfaces

Gerhard Huisken
(joint work with Simon Brendle)

Consider a closed embedded hypersurface $F_0 : M^n \to (N^{n+1}, \bar{g})$ in a smooth Riemannian manifold without boundary, where $n \geq 3$. We say that the hypersurface is *2-convex* if the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of the second fundamental form satisfy $\lambda_1 + \lambda_2 > 0$. We then solve the evolution system

$$\frac{d}{dt}F = -G\nu,$$

where ν is the (outer) unit normal to the hypersurface and the normal velocity G is the harmonic mean of the 2-sums of the principal curvatures:

$$G = \Big(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j} \Big)^{-1}.$$

This flow is a fully nonlinear, parabolic system on 2-convex hypersurfaces and has a smooth solution at least for short time in this class. It has the property that 2-convex hypersurfaces remain 2-convex provided that the ambient curvature tensor satisfies $\bar{R}_{ikik} + \bar{R}_{jkjk} \geq 0$ in any orthonormal frame. This distinguishes the flow in a crucial way from mean curvature flow, where 2-convexity is only preserved in locally symmetric spaces. Huisken-Sinestrari [5] have shown that for 2-convex hypersurfaces in Euclidean space there existss a mean curvature flow modified by finitely many surgeries that becomes extinct in finite time. We show that the fully nonlinear flow above has this property in general Riemannian manifolds satisfying the curvature condition mentioned above:

Theorem 1. [4] *Let $M_0 = \partial\Omega_0$ be a closed, embedded, 2-convex hypersurface in a compact Riemannian manifold. Given any $T > 0$, there exists a surgically modified flow with velocity G which starts from M_0 and is defined on the time interval $[0, T)$. Moreover, if the ambient manifold satisfies $\bar{R}_{ikik} + \bar{R}_{jkjk} \geq 0$ at each point in Ω_0, then the flow becomes extinct in finite time.*

In the proof we follow Andrews [1] to establish a lower bound for G and the ratio $(\lambda_i + \lambda_j)/H$. We then prove convexity estimates and cylindrical estimates using the Michael-Simon Sobolev inequality and Stampacchia iteration. In a next step we prove a non-collapsing estimate for the inscribed radius, exploiting the embeddedness of the initial surface. In the Euclidean case such a non-collapsing estimate was shown by Andrews-Langford-McCoy [2] and we are able to overcome the error terms occurring in the general Riemannian setting.

The most difficult a priori estimate necessary for the application of the surgery technique in [5] is a gradient estimate for the second fundamenrtal form: We show that the quantity $G^{-2}|\nabla A| + G^{-3}|\nabla^2 A|$ is uniformly bounded from above at all points where the curvature is sufficiently large. The corresponding estimate for mean curvature flow was first established by White [6] in the mean cinvex case and by Huisken-Sinestrari [5] with an argument using the maximum principle in the

2-convex case. For our fully nonlinear flow we combine the non-collapsing estimate with the convexity estimate, the cylindrical estimate and regularity estimates for radial graphs while using a point picking argument of Perelman to derive the desired gradient estimates for the curvature, see [4] for the details.

We finally remark that these results can be extended to Riemannian manifolds admitting some negative curvature: If the ambient manifold satisfies $\bar{R}_{ikik} + \bar{R}_{jkjk} \geq -2\kappa^2, \kappa \geq 0$, everywhere and if $\lambda_1 + \lambda_2 > 2\kappa$ holds everywhere on the initial hypersurface, then the results of the theorem can be established for the flow with velocity

$$G_\kappa = \Big(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j - 2\kappa} \Big)^{-1}.$$

References

[1] B. Andrews, *contraction of convex hypersurfaces in Riemannian spaces*, J. Diff. Geom.**39** (1994), 407-431

[2] B. Andrews, M. Langford, and J. McCoy, *Non-collapsing in fully nonlinear curvature flows*, Ann. Inst. H. Poincare Anal. Non Lineaire **30** (2013), 23-32

[3] S. Brendle and G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in R^3*, Invent. Math. Online (2015), DOI 10.1007/s00222-015-0599-3.

[4] S. Brendle and G. Huisken, *A fully nonlinear flow for two-convex hypersurfaces*, preprint, arXiv:1507.04651.

[5] G. Huisken and C. Sinestrari, *Mean curvature flow with surgeries of two-convex hy- persurfaces*, Invent. Math. **175** (2009), 137-221

[6] B. White, *The nature of singularities in mean curvature flow of mean convex sets*, J. Amer. Math. Soc. **16**, (2003), 123-138.

Conformal Willmore Tori in $\mathbb{R}^4$

Tobias Lamm
(joint work with Reiner M. Schätzle)

For an immersion $f : \Sigma \to \mathbb{R}^n$ of a Riemann surface Σ the Willmore energy is defined to be

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |H|^2 \mathrm{d}\mu,$$

where H and $\mathrm{d}\mu$ are the mean curvature resp. the induced area element of the immersion. Critical points of $\mathcal{W}$ are called Willmore immersions. The global minimum value of $\mathcal{W}$ among all immmersions from a closed surface is 4π and it is attained by round spheres. The minimum of the Willmore energy among all immersions $f : T^2 \to \mathbb{R}^3$ from a two-dimensional torus is equal to $2\pi^2$ and it is attained by the stereographic image of the Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset S^3$. This is the famous Willmore conjecture which was recently proved by Marques and Neves [5]. Here we are interested in whether the infimum of the Willmore energy is also attained in every conformal class of tori, or even more generally, if the infimum is attained for every closed Riemann surface Σ of genus $g \geq 1$. The immersions minimizing the Willmore energy in a fixed conformal class are called conformally constrained Willmore minimizers. Various existence results for conformally constrained Willmore minimizers were obtained in [2], [3], [6] and [7]. In this talk we focused on an existence result for conformal Willmore tori in higher codimension $n \geq 4$. Any torus is conformally equivalent to a quotient $T^2_\omega := \mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$ with the euclidean metric g_{euc} and

$$\omega \in \mathcal{M} := \{a+ib \mid b>0, 0\leq a\leq 1/2, a^2+b^2\geq 1\},$$

and we put

$$\mathcal{M}_n(\omega) := \mathcal{M}_{n,1}(\omega) := \inf\{\mathcal{W}(f) \mid f: T^2_\omega \to \mathbb{R}^n \text{ conformal }\} \tag{1}$$

for $\omega \in \mathcal{M}$. Our first main result is an existence statement for conformal Willmore tori in every conformal class with a prescribed energy value.

Theorem 1. *For any conformal class* $\omega \in \mathcal{M}$ *and* $k \in \mathbb{N}_0, k \geq 3$, *there exist conformal Willmore immersions* $f_{\omega,k}: T^2_\omega \to \mathbb{R}^4$ *with exactly one point of density* k *and*

$$\mathcal{W}(f_{\omega,k}) = 4k\pi \quad \textit{for } k \geq 3.$$

We complement this result with a non-existence statement for conformal Willmore tori with at least one double point and Willmore energy 8π.

Theorem 2. *For every torus* T^2 *there is no immersion* $f_0: T^2 \to \mathbb{R}^4$ *which has at least one double point and for which* $\mathcal{W}(f_0) = 8\pi$.

The particular implication of Theorem 1 that for any conformal class $\omega \in \mathcal{M}$ there exists a conformal Willmore immersion $T^2_\omega \to \mathbb{R}^4$ is already known, as Bryant showed in [1] that any closed Riemann surface Σ admits a conformal minimal, even a superminimal, immersion $\Sigma \to S^4$, which then is Willmore as well. Moreover, he constructed the immersions as a Twistor projection $T: \mathbb{C}P^3 \to S^4$ of a holomorphic horizontal curve $\Phi: \Sigma \to \mathbb{C}P^3$ and he showed that the Willmore energy of the superminimal immersion has to be a multiple of 4π. Note however, that Bryant only obtained the existence of one such surface, whereas our result shows the existence of infinitely many Willmore immersions on every torus.

Our construction of the conformal Willmore immersions works via an inversion of suitable conformal minimal immersions in $\mathbb{R}^4$ with ends of multiplicity one. More precisely, we construct these immersions via a pair of meromorphic functions $(f,h): T^2_\omega \to \mathbb{R}^4$ with exactly $k \geq 3$ simple poles and no common branch points. The existence of these functions follows basically from the Riemann-Roch theorem. It then remains to show that by inverting the immersion (f,h) one obtains an immersion as claimed in the theorem.

We also classify all branched conformal immersions from T^2_ω, for every $\omega \in \mathcal{M}$, into $\mathbb{R}^4$ with at least one branch point and Willmore energy 8π. We show that modulo Möbius transformations these immersions are given by a branched double cover $T^2_\omega \to S^2 \times \{0\}$.

In our third main result we derive an estimate from above for $\mathcal{M}_n$.

Theorem 3. *For any conformal class* $\omega \in \mathcal{M}$ *, we have*

$$\mathcal{M}_4(\omega) \leq 8\pi.$$

We show this result by using a perturbation argument which slighly resembles the constructions of counterexamples to rigidity results in [4]. More precisely, close to the branch points of the branched double cover we add a small multiple of a suitably localized holomorphic function in the second component and the new immersion is conformal everywhere. The drawback of this construction is that this might change the conformal class of the torus and we cope with this problem by showing that the induced Teichmüller class is surjective if the perturbation is small enough. Altogether, this yields a sequence of conformal immersions from every torus whose Willmore energy converges to 8π from above.

References

[1] R. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, Journal of Differential Geometry **17** (1982), 455–474.

[2] E. Kuwert and Y. Li, *$W^{2,2}$-conformal immersions of a closed Riemann surface into $\mathbb{R}^n$*, Communications in Analysis and Geometry **20** (2012), 313–340.

[3] E. Kuwert and R. Schätzle, *Minimizers of the Willmore functional under fixed conformal class*, Journal of Differential Geometry **93** (2013), 471–530.

[4] T. Lamm and R. M. Schätzle, *Rigidity and non-rigidity results for conformal immersions*, Advances in Mathematics **281** (2015), 1178–1201.

[5] F. Marques and A. Neves, *Min-Max theory and the Willmore conjecture*, Annals of Mathematics **179** (2014), 683–782.

[6] T. Rivière, *Variational Principles for immersed Surfaces with L^2-bounded Second Fundamental Form*, Journal für die Reine und Angewandte Mathematik **695** (2014), 41–98.

[7] T. Rivière, *Critical weak immersed surfaces within sub-manifolds of the Teichmüller space*, Advances in Mathematics **283** (2015), 232–274.

Compactness of the space of minimal hypersurfaces with bounded index

Ben Sharp

(joint work with L. Ambrozio and A. Carlotto)

Given a closed Riemannian manifold (N^{n+1}, g) of dimension less than eight, we present compactness results for the space of closed, embedded minimal hypersurfaces satisfying a volume bound and; either an index bound or a uniform lower bound on the $p-$th Jacobi eigenvalue for $p \geq 1$. All the results stated below can be found in [2].

The type of compactness we consider is with respect to smooth graphical convergence away from a finite set $\mathcal{Y} \subset M$, where M is the limit hypersurface (which is always a smooth minimal hypersurface). If the multiplicity of the convergence is one and then $\mathcal{Y} = \emptyset$ and we say the convergence is smooth in the C^k topology for all k.

Such compactness results can be seen as generalisations of classical compactness where one assumes a bound on the volume and the total curvature, or more specifically a strong compactness result of Choi-Schoen (see [3]).

These results are relevant to us since we know the class $\mathfrak{M}^n(N)$ (of connected, closed, smooth and embedded minimal hypersurfaces $M \subset N$) is not empty whenever $2 \le n \le 6$ due to work of Almgren-Pitts [5] (and Schoen-Simon [6]). Moreover when $Ric_N > 0$ we now know that $\mathfrak{M}^n(N)$ contains infinitely many distinct elements due to Marques and Neves [4].

The general form of result goes as follows

Theorem 1. *Let $2 \le n \le 6$ and N^{n+1} a smooth, closed Riemannian manifold. Denote by $\mathfrak{M}^n(N)$ the class of closed, smooth and embedded minimal hypersurfaces $M \subset N$. Let $\lambda_p(M)$ denote the p-th eigenvalue of the Jacobi operator for $M \in \mathfrak{M}^n(N)$. Given any $0 < \Lambda < \infty$ and $0 \le \mu < \infty$, define the class*

$$\mathcal{M}_p(\Lambda, \mu) := \{M \in \mathfrak{M}^n(N) : \mathcal{H}^n(M) \le \Lambda,\ \lambda_p(M) \ge -\mu\}.$$

Given a sequence $\{M_k\} \subset \mathcal{M}_p(\Lambda, \mu)$ there exists $M \in \mathcal{M}_p(\Lambda, \mu)$ such that $M_k \to M$ in the varifold sense and furthermore:

(1) *if $p = 1$ then $M_k \to M$ locally in the sense of smooth graphs;*
(2) *if $p \ge 2$ then there exists a finite set $\mathcal{Y} = \{y_i\}_{i=1}^P$ with $P \le p-1$ such that the convergence $M_k \to M$ is smooth and graphical for all $x \in M \setminus \mathcal{Y}$; if the number of leaves of the convergence is one then $\mathcal{Y} = \emptyset$.*

The above theorem allows one to recover the results in [7], moreover we have the following strong compactness results which follow easily from the above result.

Corollary 2. *see also* [3, when $n = 2$], *and* [7, in the case of bounded index] *Suppose $Ric_N > 0$. Then given any $0 < \Lambda < \infty$ and $0 \le \mu < \infty$ and $p \ge 1$ the class $\mathcal{M}_p(\Lambda, \mu)$ is compact in the C^k topology for all $k \ge 2$.*

Corollary 3. *Suppose that there are no one-sided minimal hypersurfaces $M \subset N$ (e.g. if N is simply connected with an arbitrary metric). Then $\mathcal{M}_1(\Lambda, \mu)$ is compact in the C^k topology for all $k \ge 2$.*

Finally, we are able to characterise how the spectrum behaves along certain sequences of smooth hypersurfaces with bounded index, giving analytical information that is hitherto very difficult to calculate.

Corollary 4. *Let $\{M_k\} \subset \mathfrak{M}^n(N)$ be a sequence satisfying a uniform volume bound, so that $M_k \to M$ for some stationary integral varifold M in N - see* [1].

- *If M is not smooth, then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \ge 1$ provided $Ric_N > 0$.*
- *If the convergence to M is of multiplicity ≥ 2 then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \ge 1$.*
- *If M is smooth, let $\mathcal{Y} \subset M$ denote the set where the convergence is not smooth and graphical. Then $\lambda_p(M_k) \to -\infty$. for all $1 \le p \le |\mathcal{Y}|$. In particular if $|\mathcal{Y}| = \infty$ then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \ge 1$.*

In the above results we are always assuming a bound on volume of our minimal hypersurfaces. However in the case that we have some control on the spectrum of the Jacobi operator (e.g. a bound on the index) and the ambient manifold is positively curved, we have the following conjecture posed to the author by André Neves and Fernando Codá Marques

Conjecture 1. *Let $M \in \mathfrak{M}^n(N)$ and suppose index$(M) = I$. If we assume that the sectional curvatures $\sigma_N > 0$ of N are strictly positive then there exists some $C = C(N, I)$ such that*

$$\mathcal{H}^n(M) \leq C.$$

Thus given a sequence $\{M_k\} \subset \mathfrak{M}^n(N)$ with index$(M_k) = I_k \leq I$ then we must have that $\mathcal{H}^n(M_k)$ is uniformly bounded above independently of k.

Moreover, if we assume N to be as above and we are given a sequence $\{M_k\} \subset \mathfrak{M}^n(N)$ with $\mathcal{H}^n(M_k) \to \infty$, then index$(M_k) \to \infty$.

References

[1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. **95**(3) (1972), 417–491.

[2] L. Ambrozio, A. Carlotto and B. Sharp, *Compactness of the space of minimal hypersurfaces with bounded volume and p-th Jacobi eigenvalue*, Preprint (2015) arXiv:1505.06652, accepted in J. Geom. Anal.

[3] H. I. Choi and R. Schoen, *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math. **81** (1985), 387–394.

[4] F. C. Marques and A. Neves, *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*, Preprint arXiv:1311.6501, 2013.

[5] J. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo. Mathematical Notes **27** (1981).

[6] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math. **34** (1981), 741–797.

[7] B. Sharp, *Compactness of minimal hypersurfaces with bounded index*, Preprint (2015) arXiv:1501.02703

The singular set of two-dimensional almost minimal integral currents

Emanuele Spadaro

(joint work with Camillo De Lellis, Luca Spolaor)

In this talk we discuss the optimal interior regularity of the integer rectifiable currents T_i for $i = 1, 2, 3$ satisfying one of the following conditions (we adopt classical notation and terminology in Geometric Measure Theory, cp. [16]):

(a) T_1 is two-dimensional and *locally mass minimizing* in a Riemannian manifold Σ^{2+n};

(b) T_2 is two-dimensional and *semi-calibrated* in a Riemannian manifold Σ^{2+n}, i.e. there exists a two-dimensional differential form ω on Σ with comass $\|\omega(x)\|_c \leq 1$ for every $x \in \Sigma$ such that

$$\langle \vec{T}(x), \omega(x) \rangle = 1 \text{ for } \|T\|\text{-a.e. } x \in \Sigma;$$

(c) T_3 is a *mass minimizing three-dimensional cone* in $\mathbb{R}^{2+n}$.

We denote by $\mathrm{Reg}(T)$ the set of regular points of a current T: namely, $x \in \mathrm{Reg}(T)$ if there exists $r > 0$ such that $B_r(x) \cap \mathrm{spt}(T)$ is an embedded submanifold of class C^2 (where $B_r(x)$ is the metric ball centered in x and radius r); correspondingly

$$\mathrm{Sing}(T) := \mathrm{spt}(T) \setminus \big(\mathrm{Reg}(T) \cup \mathrm{spt}(\partial T)\big).$$

The main theorem, proved in a series of papers [12, 13, 14, 15] in collaboration with C. De Lellis and L. Spolaor, is the following.

Theorem 1 (De Lellis, Spolaor and S., 2015)**.** *Assume that the Riemannian manifold Σ is of class C^{3,ε_0} and the differential form ω of class C^{2,ε_0} for some $\varepsilon_0 > 0$. Then,*

$$\mathcal{H}^0(\mathrm{Sing}(T_i) \cap K) < +\infty \quad \forall\, K \subset\subset \Sigma \setminus \mathrm{spt}(\partial T_i) \quad i = 1, 2,$$
$$\mathcal{H}^1(\mathrm{Sing}(T_3) \cap K) < +\infty \quad \forall\, K \subset\subset \mathbb{R}^n \setminus \mathrm{spt}(\partial T_3).$$

This regularity result is optimal in the case of higher co-dimension $n \geq 2$, since there are explicit examples of currents T_1 and T_2 as in (a) and (b) with an arbitrary finite number of interior singularities (see, for instance, the currents induced by holomorphic varieties, cp. [16, 5.4.19]), as well as three-dimensional cones T_3 with arbitrarily many lines of singularities (consider for example the Cartesian products of the union of intersecting complex planes with a line).

The Main Theorem establishes an unified approach to the regularity of two-dimensional integer rectifiable currents which solve the above variational problems (note that the case of the three-dimensional cones reduces indeed to the study of their traces on a sphere).

The result for mass minimizing currents (a) has been established by Chang [6] under a more restrictive regularity assumption on Σ, building upon the pioneering work by Almgren [2]. However, Chang's arguments are not entirely complete. Indeed, a substantial part of the proof, namely the approximation of the current over the so-called branched center manifold, is only sketched in an appendix of [6], while it requires a careful and intricate analysis which goes beyond Almgren's monograph (cp. [14]). In our papers we establish the estimates which are needed for such approximations in the three cases considered in the Main Theorem, thus providing a complete proof of the regularity of mass minimizing currents.

The case (b) settles a question which has been considered recently in the literature in connection to several problems in differential geometry, see, e.g., the introductions to the papers by Rivière and Tian [21] and Pumberger and Rivière [19]. The notion of semi-calibrations extends that of *calibration* to the generic case of forms ω which are not necessarily closed. In particular, contrary to calibrated currents, the generic semi-calibrated current fails to be mass minimizing in its homological class.

Due to their geometric relevance, the question of the regularity for calibrated and semi-calibrated currents has been considered independently from the Almgren-Chang theory. Alternative proofs of the regularity were already known for some classes of calibrated currents, e.g. positive (1-1)-currents in Kähler manifolds, cp. the work by King [18], or have been given recently, see for instance the works by Taubes [22] and Rivière and Tian [20, 21] on almost complex two-dimensional currents in a manifold satisfying the locally symplectic property (see also the papers by Harvey and Shiffman [17] and Alexander [1] for the integrable case). For two-dimensional semi-calibrated currents which are not mass minimizing, Pumberger and Rivière [19] proved the uniqueness of tangent cones at every point (recently also reproved by Bellettini [3, 4]), and Bellettini and Rivière [5] the structure of the singular set in the case of special Legendrian cycles in $\mathbb{S}^5$. Our Main Theorem provides a general interior regularity result for semi-calibrated two-dimensional currents which in particular implies all the known cases in the literature.

Finally, case (c) establishes the first optimal regularity result for a class of three-dimension mass minimizing currents which goes beyond Almgren's estimate of the Hausdorff dimension. Note that the the special Legendrian cycles considered by Bellettini and Rivière [5] arise also as spherical cross-sections of 3-dimensional special Lagrangian cones, and are therefore also included in our class (c).

In the proof of the Main Theorem we follow the approach pioneered by Almgren and Chang, as revisited in the first two authors' previous works [7, 8, 9, 10, 11]. Indeed, the Main Theorem is established through a suitable "blow-up argument" which requires the theory of multiple valued functions (cp. [7, 8]).

The proof of the Main Theorem is given in four papers, where several additional results are established:

(1) *Uniqueness of tangent cones for 2-dimensional almost minimizing currents* [12];
(2) *Regularity for 2-dimensional almost minimal currents I: Lipschitz approximation* [13];
(3) *Regularity for 2-dimensional almost minimal currents II: branched center manifold* [14];
(4) *Regularity for 2-dimensional almost minimal currents III: blowup* [15].

In [12] we show the uniqueness of the tangent cones at the interior points of a large class of almost minimizing two-dimensional integer rectifiable currents, namely any current T for which there exists constants $r_0, \alpha, C > 0$ such that

$$\|T\|(B_r(x)) \leq \|T + \partial S\|(B_r(x)) + C\, r^{2+\alpha}$$

for all $x \notin \operatorname{spt}(T)$, for all $0 < r < r_0$ and for all integral 3-dimensional integer rectifiable currents S supported in $B_r(x)$. The uniqueness of tangent cone is by now a classical theorem of White [23] for area-minimizing 2-dimensional currents in the Euclidean space, extended by Chang [6] for currents in a Riemannian manifold, and by Pumberger and Rivière [19] to semi-calibrated currents, as already mentioned. Our paper [12] gives a unified proof to these existing results and establishes also the uniqueness of blowups for the cross-sections of three-dimensional mass minimizing cones.

In [13] we prove a strong approximation (with errors given by superlinear powers of the excess) for integer rectifiable currents T of any dimension m satisfying a

Ω-minimizing condition:

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S) + \Omega\,\mathbf{M}(S)$$

for every $(m+1)$-dimensional integer rectifiable current S with compact support. This result extends the strong approximation for mass minimizing currents by Almgren as proved in [9].

Finally, [14, 15] contain the main part of the proof and deal exclusively with the classes of currents in the statement of the Main Theorem. In particular, [14] concerns the construction of a *branched center manifold* and [15] the blowup analysis.

References

[1] H. Alexander, *Holomorphic chains and the support hypothesis conjecture*, J. Amer. Math. Soc. **10** (1997), no. 1, 123–138.

[2] F. J. Almgren, Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, 1, World Scientific Publishing Co. Inc., River Edge, NJ, (2000) xvi+955 pp.

[3] C. Bellettini, *Tangent cones to positive-(1,1) De Rham currents*, J. Reine Angew. Math., to appear.

[4] C. Bellettini, *Semi-calibrated 2-currents are pseudo holomorphic, with applications*, Bull. Lond. Math. Soc. **46** (2014), no. 4, 881–888.

[5] C. Bellettini, T. Rivière, *The regularity of special Legendrian integral cycles*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), no. 1, 61–142.

[6] S. X. Chang, *Two-dimensional area minimizing integral currents are classical minimal surfaces*, J. Amer. Math. Soc. **1** (1988), no. 4, 699–778.

[7] C. De Lellis, E. Spadaro, *Q-valued functions revisited*, Mem. Amer. Math. Soc. **211**(2011), no. 991, vi+79 pp.

[8] C. De Lellis, E. Spadaro, *Multiple valued functions and integral currents*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), to appear.

[9] C. De Lellis, E. Spadaro, *Regularity of area-minimizing currents I: gradient L^p estimates*, Geom. Funct. Analysis **24** (2014), no. 6, 1831–1884.

[10] C. De Lellis, E. Spadaro, *Regularity of area-minimizing currents II: center manifold*, Preprint (2013) arXiv:1306.1191.

[11] C. De Lellis, E. Spadaro, *Regularity of area-minimizing currents III: blow-up*, Preprint (2013), arXiv:1306.1194.

[12] C. De Lellis, E. Spadaro, L. Spolaor, *Uniqueness of tangent cones for 2-dimensional almost minimizing currents*, Preprint (2015), arXiv:1508.05266.

[13] C. De Lellis, E. Spadaro, L. Spolaor, *Regularity for 2-dimensional almost minimal currents I: Lipschitz approximation*, Preprint (2015), arXiv:1508.05507

[14] C. De Lellis, E. Spadaro, L. Spolaor, *Regularity for 2-dimensional almost minimal currents II: branched center manifold*, Preprint (2015), arXiv:1508.05509.

[15] C. De Lellis, E. Spadaro, L. Spolaor, *Regularity for 2-dimensional almost minimal currents III: blowup*, Preprint (2015), arXiv:1508.05510.

[16] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, (1969), xiv+676 pp.

[17] R. Harvey, B. Shiffman, *A characterization of holomorphic chains*, Ann. of Math. (2) **99** (1974), 553–587

[18] J. R. King, *The currents defined by analytic varieties*, Acta Math. **127** (1971), no. 3-4, 185–220

[19] D. Pumberger, T. Rivière, *Uniqueness of tangent cones for semicalibrated integral 2-cycles*, Duke Math. J. **152** (2010), no. 3, 441–480

[20] T. Rivière, G. Tian, *The singular set of J-holomorphic maps into projective algebraic varieties*, J. Reine Angew. Math. **570** (2004), 47–87

[21] T. Rivière, G. Tian, *The singular set of 1-1 integral currents*, Ann. of Math. (2) **169** (2009), no. 3, 741–794

[22] C. H. Taubes, *Seiberg Witten and Gromov invariants for symplectic* 4*-manifolds*, First International Press Lecture Series, 2. International Press, Somerville, MA, (2000), vi+401 pp.

[23] B. White, *Tangent cones to two-dimensional area-minimizing integral currents are unique*, Duke Math. J. **50** (1983), no. 1, 143–160.

A notion of the weighted σ_k-curvature for manifolds with density

Jeffrey Case

One of Perelman's contributions to the Ricci flow was his introduction of the $\mathcal{W}$-functional [7], defined on a Riemannian manifold (M^n, g) by

$$\mathcal{W}_1(g,\phi,\tau) := \int_M \left(\tau\left(R + 2\Delta\phi - |\nabla\phi|^2\right) + \phi - n\right)(4\pi\tau)^{-\frac{n}{2}}\, e^{-\phi}\mathrm{dvol}$$

for all $\phi \in C^\infty(M)$ and all $\tau \in (0,\infty)$. Consider

$$\widetilde{R}_\phi := R + 2\Delta\phi - |\nabla\phi|^2 + \frac{1}{\tau}(\phi - n)$$

as the weighted scalar curvature of the manifold with density $(M^n, g, e^{-\phi}\mathrm{dvol})$. Critical points of $\mathcal{W}_1$ in the class

$$\mathcal{C}_1(g) := \left\{(\phi,\tau) \in C^\infty(M)\times(0,\infty) \colon \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\phi}\mathrm{dvol} = 1\right\}$$

are such that $\widetilde{R}_\phi$ is constant; critical points within $\bigcup_g \mathcal{C}_1(g)$ are shrinking gradient Ricci solitons, namely solutions of

$$\mathrm{Ric}_\phi := \mathrm{Ric} + \nabla^2\phi = \frac{1}{2\tau}g;$$

shrinking gradient Ricci solitons are local minimizers of $\mathcal{W}_1$; and if the ν-entropy $\inf_{\mathcal{C}_1}\mathcal{W}_1$ is finite, then (M^n, g) is κ-noncollapsed. Moreover, the gradient flow of $\mathcal{W}_1$ is the Ricci flow [7].

The aforementioned properties of Perelman's $\mathcal{W}$-functional are all analogous to properties of the total scalar curvature functional: Critical points within a conformal class $[g]_1$ of metrics with fixed volume have constant scalar curvature; critical points within $\bigcup_g [g]_1$ are Einstein metrics, namely solutions of $\mathrm{Ric} = \lambda g$; Einstein metrics with positive scalar curvature locally minimize the total scalar curvature functional; and the positivity of the Yamabe constant implies that the manifold is noncollapsed. Viaclovsky proposed studying the σ_k-curvatures, namely the k-th elementary symmetric functions of the eigenvalues of the Schouten tensor, as analogues of the scalar curvature [9]. Indeed, the behavior of the σ_k-curvatures within a conformal class is controlled by a fully nonlinear second-order PDE in the conformal factor, and it is known that the total σ_k-curvature functional shares all of the same properties as the total scalar curvature functional [5, 8, 9], with the important caveat that the former is variational if and only if $k \in \{1,2\}$ or the conformal class is locally conformally flat [1].

Formally, the weighted scalar curvature can be identified as the limit

$$\widetilde{R}_\phi = \lim_{m\to\infty} \sigma_1 \left(M^n \times S^m(1/m\tau), g \oplus e^{-\frac{2\phi}{m}} h\right)$$

where $(F^m(1/m\tau), h)$ is the simply-connected spaceform of sectional curvature $1/m\tau$. Similarly, given $(M^n, g, e^{-\phi}\mathrm{dvol})$ and $\lambda \in \mathbb{R}$, we define the weighted scalar curvature $\widetilde{\sigma}_{k,\phi}$ by

$$\widetilde{\sigma}_{k,\phi} := \lim_{m\to\infty} \sigma_k \left(M^n \times F^m(\lambda/m), g \oplus e^{-\frac{2\phi}{m}} h\right).$$

A more rigorous definition is given in [3]. The most important special cases are

$$\widetilde{\sigma}_{1,\phi} = \frac{1}{2}\left(R + 2\Delta\phi - |\nabla\phi|^2 + 2\lambda(\phi - n)\right),$$
$$\widetilde{\sigma}_{2,\phi} = \frac{1}{2}\left((\widetilde{\sigma}_{1,\phi})^2 - |\mathrm{Ric} + \nabla^2\phi - \lambda g|^2\right).$$

With the analogue of a conformal change of metric being a change of the measure $(4\pi\tau)^{-\frac{n}{2}} e^{-\phi}\mathrm{dvol}$, the weighted σ_k-curvature has all of the same properties as the σ_k-curvature, thus generalizing Perelman's observations for $\widetilde{\sigma}_{1,\phi}$. In particular, the variational status of the weighted σ_k-curvatures is completely understood:

Theorem 1 ([3]). *Let (M^n, g) be a Riemannian manifold and fix $\lambda \in \mathbb{R}$ and a positive integer $k \le n$. Then the weighted σ_k-curvature $\widetilde{\sigma}_{k,\phi}$ is variational if and only if $k \in \{1,2\}$ or the Riemann curvature tensor of g vanishes identically.*

This makes it meaningful to study the functional

$$\mathcal{W}_2(g,\phi,\tau) := \int_M \tau^2 \widetilde{\sigma}_{2,\phi} \, (4\pi\tau)^{-\frac{n}{2}} \, e^{-\phi}\mathrm{dvol}_g;$$

Critical points of $\mathcal{W}_2$ within $\mathcal{C}_1(g)$ are such that $\widetilde{\sigma}_{2,\phi} - \frac{1}{2\tau}\widetilde{\sigma}_{1,\phi}$ is constant, and shrinking gradient Ricci solitons are critical points of $\mathcal{W}_2$ within $\bigcup_g \mathcal{C}_1(g)$. Moreover, shrinking gradient Ricci solitons locally maximize $\mathcal{W}_2$ within $\mathcal{C}_1(g)$:

Theorem 2 ([3]). *Let $(M^n, g, e^{-\phi}\mathrm{dvol})$ be a shrinking gradient Ricci soliton with $\mathrm{Ric}_\phi = \frac{1}{2\tau} g$ and let $\{(\phi_t, \tau_t)\}_{t\in(-\varepsilon,\varepsilon)} \subset \mathcal{C}_1(g)$ be a smooth variation of (ϕ, τ). Then*

$$\frac{d^2}{dt^2}\mathcal{W}_2(g,\phi_t,\tau_t)\bigg|_{t=0} \le 0. \tag{1}$$

Moreover, equality holds in (1) for nontrivial variations $(\phi_t, \tau_t) \in \mathcal{C}_1(g)$ if and only if $(M^n, g, e^{-\phi}\mathrm{dvol})$ factors as an isometric product with a Gaussian.

In fact, critical points of $\mathcal{W}_2$ are completely classified on Euclidean space in terms of shrinking Gaussians:

Theorem 3 ([3])**.** *Suppose that $(\phi,\tau)\in C^\infty(\mathbb{R}^n)\times(0,\infty)$ is a critical point of the functional $\mathcal{W}_2\colon \mathcal{C}_1(dx^2)\to\mathbb{R}$. Suppose additionally that*

$$\widetilde{\sigma}_{1,\phi} < \frac{1}{2\tau}, \tag{2}$$

$$\widetilde{\sigma}_{2,\phi} > \frac{1}{2\tau}\widetilde{\sigma}_{1,\phi} - \frac{1}{8\tau^2}, \tag{3}$$

$$1 = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\phi} dx. \tag{4}$$

Then there is a point $x_0\in\mathbb{R}^n$ such that

$$\phi(x) = \frac{|x-x_0|^2}{4\tau}.$$

The cone $\Gamma_2^\infty \subset \mathcal{C}_1(g)$ consisting of pairs satisfying (2) and (3) is important because the equation $\widetilde{\sigma}_{2,\phi}=f$ is elliptic within this cone. The assumption (4) is used to obtain estimates at infinity on ϕ; it is not clear if this assumption can be removed (cf. [2]). The key ingredient in the proof of Theorem 3 is the divergence structure coming from the variational structure of $\widetilde{\sigma}_{2,\phi}$.

One expected topological consequence of the weighted σ_2-curvature is the validity of the Hitchin–Thorpe inequality.

Conjecture 1. *Let (M^n,g) be a compact Riemannian manifold. If $\sup_{\Gamma_2^\infty}\mathcal{W}_2<\infty$, then there is a metric $\widehat{g}\in[g]$ such that $\widehat{R},\widehat{\sigma_2}>0$. In particular, if (M^4,g) is a compact gradient Ricci soliton, then*

$$\chi(M^4) > \frac{3}{2}\left|\tau(M^4)\right|.$$

Conjecture 1 is based on the validity of the similar relationship between the ν-entropy and the Yamabe constant (cf. [4]). The Hitchin–Thorpe inequality follows immediately from the existence of a metric $\widehat{g}\in[g]$ such that $\widehat{\sigma_2}>0$ (cf. [6]).

While I expect $\mathcal{W}_2$ to be relevant to the study of certain geometric flows, it is not clear what form that should take. I expect it either gives rise to a new monotone quantity along the Ricci flow or it gives rise to a new and interesting geometric flow in its own right.

Conjecture 2. *Let (M^n,g) be a compact Riemannian manifold. At least one of the following is true:*

(a) $\sup_{\Gamma_2^\infty}\mathcal{W}_2$ *is monotone along the Ricci flow; or*
(b) *The negative gradient flow of $\mathcal{W}_2$ is well-posed within Γ_2^∞.*

If the Obata-type argument used to prove Theorem 3 can be extended to general shrinking gradient Ricci solitons, it would yield the first claim of Conjecture 2. Unfortunately, it is not even known if the analogous Obata-type theorem for Einstein metrics and the σ_2-curvature.

References

[1] T. P. Branson, A. R. Gover, *Variational status of a class of fully nonlinear curvature prescription problems.*, Calc. Var. Partial Differential Equations **32** (2) (2008), 253–262

[2] L. A. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth.*, Comm. Pure Appl. Math. **42** (3) (1989), 271–297

[3] J. S. Case, *A notion of the weighted σ_k-curvature for manifolds with density*, arXiv:1409.4455, preprint.

[4] J. S. Case, *A Yamabe-type problem on smooth metric measure spaces*, J. Differential Geom. , to appear.

[5] P. Guan, G. Wang, *Geometric inequalities on locally conformally flat manifolds*, Duke Math. J. **124** (1) (2004), 177–212

[6] M. J. Gursky, *The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE*, Comm. Math. Phys. **207** (1) (1999), 131–143

[7] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:0211159, preprint.

[8] W.-M. Sheng, N. S. Trudinger, and X.-J. Wang, *The Yamabe problem for higher order curvatures*, J. Differential Geom. **77** (3) (2007), 515–553

[9] J. A. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J. **101** (2) (2000), 283–316

On conformally compact Einstein manifolds with conformal infinities of large Yamabe constant

Jie Qing

In this paper we first obtain a gap theorem for a class of conformally compact Einstein manifolds with a renormalized volume that is close to the maximum. We also use the blow-up method to derive curvature estimates for conformally compact Einstein manifolds with large renormalized volume. The major part of this paper is on the study of how a property of the conformal infinity influences the geometry of the interior of a conformally compact Einstein manifold. Specifically we are interested in conformally compact Einstein manifolds with conformal infinities of large Yamabe constants. Based on the approach initiated in the work of Shi and Tian, and Dutta and Javaheri, we present the complete proof of the relative volume inequality on conformally compact Einstein manifolds. This leads to not only the complete proof of the rigidity theorem for conformally compact Einstein manifolds in general dimensions with no spin structure assumption but also the new curvature pinch estimates for conformally compact Einstein manifolds with conformal infinities of Yamabe constants that are close to the maximum. In particular it implies the all conformally compact Einstein manifolds including the ones constructed by Graham and Lee for conformal infinities that are perturbations of the round sphere are all negatively curved. We also derive the curvature estimates for conformally compact Einstein manifolds with conformal infinities of large Yamabe constants.

Gluing constructions for constant mean curvature hypersurfaces

Christine Breiner
(joint work with Nicos Kapouleas)

We outline a generalized gluing construction for constant mean curvature (CMC) hypersurfaces. The work builds on the analogous result for surfaces, [5].

CMC hypersurfaces $\Sigma^n \subset \mathbb{R}^{n+1}$ are critical points for the area functional subject to an enclosed volume constraint. The variational condition corresponds to a pointwise condition of the form

$$nH = \sum_{i=1}^{n} \kappa_i$$

where the κ_i's are the principle curvatures of the hypersurface Σ. Until the late 1980's, the only known examples in Euclidean space were the round sphere, the Wente torus [2], and the surfaces of Delaunay [1]. In 1990, Kapouleas [6] determined a very general gluing construction for CMC surfaces that produced infinitely many new examples. The work of [5] refines and simplifies the previous construction and guarantees embeddedness for a much larger class of surfaces. In the current work, we extend the results of [5] to produce infinitely many new examples of CMC hypersurfaces.

The gluing outline proceeds as follows. First, we consider a "background structure" Γ consisting of a collection of vertices, edges, and rays, where each edge and ray comes equipped with a parameter $\tau_i \neq 0$. We construct an initial hypersurface M by positioning a round hypersphere at each vertex and a Delaunay piece of the appropriate parameter along each edge and ray. We transition between these hypersurfaces with a smooth function and call the resulting hypersurface M. Because we hope to find a CMC hypersurface near M and because we know the singular behavior of the Delaunay pieces, we expect the structure Γ to satisfy a few criteria. First, the singular behavior of the Delaunay model implies that we want all of our edges to have even integer lengths. Second, CMC hypersurfaces satisfy a force balancing condition. The condition translates to the structure in a convenient way because of our choice of immersion for the Delaunay pieces. More specifically, given any CMC hypersurface Σ^n, let $C \subset \Sigma$ be an $(n-1)$-chain and $K^n \subset \mathbb{R}^{n+1}$ such that $\partial K = C$. Then

$$\text{Force}(C) := \int_C \eta \, ds - n \int_K N_K \, dV$$

is a homological invariant. Here η is the conormal to C and N_K is the normal to K. For a Delaunay piece with parameter τ, one can easily calculate the force along any C by considering $C = \{r_{min}\} \times \mathbb{S}^{n-1}$ where r_{min} denotes the smallest radius. If the Delaunay piece is positioned with axis in the direction of $\mathbf{e}_1$, then

$$\text{Force}(C) = \omega_{n-1} \tau \mathbf{e}_1.$$

(We choose a parameterization so that the two terms that appear in the force calculation add up conveniently.)

The force calculation relates to the structure of Γ in the following way. Consider any vertex in Γ and let $\mathbf{v}_i$ denote the unit directions of the edges and rays emanating from this vertex. Let τ_i denote the corresponding τ-parameters associated to each edge or ray. The force contribution for each Delaunay piece positioned around the hypersphere will then be $\omega_{n-1}\tau_i\mathbf{v}_i$. The homological invariance of the force for CMC hypersurfaces implies that if we want the initial surface M to satisfy the force condition, then at each vertex of Γ we must satisfy the balancing condition

$$\sum_i \tau_i \mathbf{v}_i = 0.$$

The final condition on the structure Γ is more technical and comes because of the nature of our construction. We solve the problem not on the hypersurface M but instead on a hypersurface nearby M. Roughly, we require that Γ satisfies enough flexibility to admit a large class of nearby hypersurfaces.

Given Γ satisfying the conditions described, we are able to find a CMC hypersurface. The rough idea comes from the fact that for $f \in C^{2,\alpha}(M)$, we may calculate

$$H_{M_f} = H_M + \mathcal{L}_M f + Q_f$$

where H_M is the mean curvature of M, $\mathcal{L}_M$ is the stability operator on M, M_f is the normal graph over M by f, and Q_f are quadratic and higher terms in f and its derivatives. Therefore, we hope to find f such that

$$\mathcal{L}_M f = 1 - H_M - Q_f.$$

Unfortunately, we cannot solve this problem directly. Both the dependence on f on the right hand side and the obstructions to invertibility of $\mathcal{L}_M$ cause problems. Therefore, rather than trying to solve the problem directly on M, we show that there exists a hypersurface in a family of hypersurfaces near M and a function defined on that hypersurface that solves the problem.

Let $\mathcal{F}(M)$ denote a family of hypersurfaces near M, where every member of this family will arise by the choice of two parameters that completely describe the modification of M. (In this abstract, we will not be any more precise in the definition of "near".) We then show that there exists a space of functions $\mathcal{K} \subset C^{2,\alpha}(M)$ such that for all $\widetilde{M} \in \mathcal{F}(M)$ and $E \in C^{0,\alpha}(\widetilde{M})$ there exist $w \in \mathcal{K}$ and $\phi \in C^{2,\alpha}(\widetilde{M})$ such that $\mathcal{L}_{\widetilde{M}}\phi = E + w$. Note that w is defined on M, but as M and $\widetilde{M}$ have the same domain, we may think of w as being defined on every hypersurface in $\mathcal{F}(M)$.

We then show that, given $w \in \mathcal{K}$, there exists $\widetilde{M} \in \mathcal{F}(M)$ and $\Phi_w \in C^{2,\alpha}(M)$ such that

$$\mathcal{L}_{\widetilde{M}}\Phi_w = 1 - H_{\widetilde{M}} + w.$$

Of course, we do not show this exactly but that we can solve this problem for some $\bar{w}$ sufficiently close to w.

We define a Banach space $B \subset C^{2,\alpha}(M) \times \mathcal{K}$ with an appropriate weighted norm on M and show that there exists a map $J : B \to B$ that satisfies the criteria to invoke Schauder's fixed point theorem. Specifically, $J(u, w) := (u', w')$ where $v = \Phi_w - u$ and $\mathcal{L}_{\widetilde{M}} u' = Q_v + w'$. Therefore, at a fixed point we have that

$$\mathcal{L}_{\widetilde{M}} v = \mathcal{L}_{\widetilde{M}} \Phi_w - \mathcal{L}_{\widetilde{M}} u = 1 - H_{\widetilde{M}} + w - Q_v - w = 1 - H_{\widetilde{M}} - Q_v.$$

We mention just a few of the differences from the surface case. First, the operator $\mathcal{L}_M$ no longer scales well under conformal change. Therefore, solving the linear problem requires some new techniques. We simplify how we solve the linear problem on the Delaunay pieces by realizing that we can project the linear operator onto the eigenspaces for $\mathbb{S}^{n-1}$ and thus treat the problem as an ODE on each eigenspace.

Going forward, there are a few questions one might consider. First, can we extend such a construction to more general manifolds? What requirements will be needed for Γ and what restrictions will need to be placed on the ambient manifold? Second, can one extend the hypersurface construction to include hypersurfaces with infinite topology? (This question is fairly straightforward and could be done by a graduate student.) Third, are the hypersurfaces that we construct non-degenerate? An affirmative answer would let us appeal to [4] to conclude that there exists a neighborhood in the moduli space that is a smooth manifold. One then might attempt to produce stronger fixed point estimates (contraction mappings) so that modifications to Γ result in surfaces that vary along a curve in the moduli space.

References

[1] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constant*, J. Math. Pures Appl. **6** (1841), 309–320.

[2] H. Wente, *Counterexample to a conjecture of H. Hopf*, Pac. J. Math. **121** (1986), 193–243.

[3] M. Jleli, *End-to-end gluing of constant mean curvature hypersurfaces*, Ann. Fac. Sci. Toulouse Math. **18** (2009), 717–737.

[4] M. Jleli, *Moduli space theory of constant mean curvature hypersurfaces*, Adv. Nonlinear Stud. **9** (2009), 29–68.

[5] C. Breiner and N. Kapouleas, *Embedded constant mean curvature surfaces in Euclidean three-space*, Math. Ann. **360** (2014), 1041–1108.

[6] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. Math. (2) **131** (1990), 239–330.

Finite total Q-curvature in conformal geometry and the CR geometry

Yi Wang

(joint work with Paul Yang)

1. Background

The study of non-compact complete surfaces with finite total Gaussian curvature dates back to the early 1930s. The works of Fiala, Huber, Cohn-Vossen demonstrate that the integral of the Gaussian curvature has rigid geometric consequences. The famous Fiala-Huber isoperimetric inequality states that if the integral of the positive part of Gaussian curvature is less than that on the half cylinder, then the isoperimetric inequality is valid on this complete surface. In higher dimensions, PDE aspect of Q-curvature has been intensively studied. However, the geometric implication of the Q-curvature has always been a mystery. It was conjectured by Bonk, Heinonen, Saksman that on conformally flat manifolds if the integral of the Q-curvature is less than that on the half cylinder, then the manifold is bi-Lipschitz to the Euclidean space. A few years ago, I proved this conjecture, with some non-uniform isoperimetric constant. Later, I improved this result by showing the full analog of Fiala-Huber type isoperimetric inequality on higher dimensional manifolds.

2. Isoperimetric inequality and Q-curvature in conformal geometry

The main result that I have presented is the following.

Theorem 1. *(W. '13) Suppose $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ is a noncompact complete Riemannian manifold with normal metric. If its Q-curvature satisfies*

$$\alpha \overset{def}{=} \int_{M^n} Q_g^+ dv_g < c_n \tag{1}$$

and

$$\beta \overset{def}{=} \int_{M^n} Q_g^- dv_g < \infty, \tag{2}$$

then the manifold satisfies the isop inequality:

$$|\Omega|_g \leq C(\alpha, \beta, n)|\partial\Omega|_g^{n/(n-1)}. \tag{3}$$

In order to prove this theorem, I adopt some techniques from harmonic analysis. More precisely, the theory of A_p weights, especially the strong A_∞ weight is applied to solve the problem.

The Q-curvature is generally believed to have very rich geometric meanings. In a recently prepared preprint, I prove that under certain curvature conditions the integral of Q-curvature is quantized.

Theorem 2. *(W. '15) Suppose $(M^4, g) = (\mathbb{R}^4, e^{2u}|dx|^2)$ is a noncompact complete Riemannian manifold with normal metric. If M^4 embeds in $\mathbb{R}^5$ with*

$$\int_{M^4} |L|^4 dv_g < \infty, \tag{4}$$

with L being the second fundamental form, then

$$\int_{M^4} Q_g dv_g = 4\pi^2\mathbb{Z}.$$

Q-curvature is related to asymptotic behavior of the ends of local conformally flat manifolds. In the long run, it would be interesting to understand volume growth estimates, geodesic distance estimates, etc. using Q-curvature.

3. Q'-curvature in CR geometry

In CR geometry, if one considers conformally Heisenberg manifolds, it is known that the Q-curvature's integral is equal to zero. Therefore, the Q-curvature's integral is not the right quantity to study geometric properties. Recently, Case and Yang defined P_4' on general three dimensional CR manifolds. This is an operator which only acts on pluriharmonic functions. It satisfies the transformation law in a manner similar to that of the Q-curvature in the Riemannian setting, modular the space of pluriharmonic functions. Namely, for conformal change $\theta^u = e^u\theta_0$,

$$P_4'(f) = e^{2u}(P^u)_4'(f) \mod \mathcal{P}^+,$$

for all pluriharmonic functions f. The corresponding Q_4'-curvature satisfies

$$P_4'(u) + Q_4' = e^{2u}(Q^u)_4' \mod \mathcal{P}^+.$$

Theorem 3 (joint with Paul Yang, '15)**.** *Suppose the CR Q_4'-curvature of $(\mathbb{H}^1, e^u\theta)$ is nonnegative. The Webster scalar curvature is nonnegative at infinity, and u is a pluriharmonic funcion on $\mathbb{H}^1$. If*

$$\int_{\mathbb{H}^1} Q_4' e^{2u} dx < c_1, \tag{5}$$

then e^{2u} is an A_1 weight. Thus on such a conformal Heisenberg group, the isoperimetric inequality is valid. Moreover, the isoperimetric constant depends only on the integral of the Q_4'-curvature.

In our theorem, we provide further evidence that P_4' operator is the correct analogue on CR manifolds. First, analogous isoperimetric inequality holds when Q_4' is nonnegative. Also, nonnegativity of Webster scalar curvature at infinity implies the metric is normal.

References

[1] Y. Wang, *Isoperimetric inequality, Q-curvature and A_p weights*, Adv. Math. **281** (2015), 823-844

[2] Y. Wang, *Finite total Q-curvature on locally conformally flat manifolds*, preprint (2015)

[3] M. Muster, *Computing certain invariants of topological spaces of dimension three*, Topology **32** (1990), 100–120.

[4] Y. Wang, P. Yang, *On the isoperimetric inequality for conformally Heisenberg manifolds*, preprint (2015)

Scattering for a critical nonlinear wave equation in two space dimensions

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abstract

In joint work with Martin Sack we show that the solutions to the Cauchy problem for a wave equation with critical exponential nonlinearity in 2 space dimensions scatter for arbitrary smooth, compactly supported initial data.

1. Introduction

Consider the initial value problem for the equation

$$u_{tt} - \Delta u + u\big(e^{u^2} - 1 - u^2\big) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 . \tag{1}$$

with smooth Cauchy data

$$(u, u_t)_{|t=0} = (u_0, u_1) \in C_c^\infty(\mathbb{R}^2). \tag{2}$$

Observe that for a classical solution u of (1), (2) the energy

$$E(u(t)) = \frac{1}{2} \int_{\{t\}\times\mathbb{R}^2} \big(|u_t|^2 + |\nabla u|^2 + F(u)\big)dx \tag{3}$$

is conserved, where $F(u) = e^{u^2} - 1 - u^2 - u^4/2$ (up to a factor 2) is a primitive of the nonlinear term $f(u) = u\big(e^{u^2} - 1 - u^2\big)$.

For the related problem when $f(u)$ is replaced by the nonlinearity $n(u) = ue^{u^2}$ Ibrahim, Majdoub, and Masmoudi in [3] showed that whenever the corresponding initial energy is at most 2π the Cauchy problem (1), (2) admits a global smooth solution. Together with Nakanishi, in [5] the same authors also showed that when $f(u)$ is replaced by $l(u) = u(e^{u^2} - u^2)$ the solution scatters, again assuming the associated initial energy to be bounded by 2π. The constant 2π is related to the best constant in the Moser-Trudinger inequality [6], [12], which defines the limit case of Sobolev's embedding of the space $H^1(\mathbb{R}^2)$. It was conjectured in [5] that this number also marks an energy threshold for the onset of "super-critical" behavior in (1) and its variants. This conjecture was partially confirmed through the examples given in [4], showing that the solutions no longer depend in a locally uniformly continuous fashion on the data when the initial energy exceeds the value 2π.

In contrast with these expectations, however, Struwe [10] showed that the initial value problem for equation (1) has a global smooth solution for smooth Cauchy data (u_0, u_1) with arbitrarily large energy. This result was originally demonstrated when $f(u)$ is replaced by the nonlinearity $n(u) = ue^{u^2}$ but the proof is valid also for all the above variants of equation (1).

Moreover, by building on the techniques developed in [10], Sack [7] was able to show scattering for any solution u of (1), (2) for arbitrarily large smooth, compactly supported data with rotational symmetry. Here, by definition, a solution u to (1) scatters if for the solution v to the homogeneous linear wave equation

$$v_{tt} - \Delta v = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2 \tag{4}$$

for suitable "scattering data"

$$(v, v_t)_{|t=0} = (v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^2) \tag{5}$$

there holds

$$||Du(t) - Dv(t)||_{L^2(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \infty, \tag{6}$$

where $Du = (u_t, \nabla u)$ is the space-time differential of u.

Combining the insights of [7] and [10], in joint work with Martin Sack [8] we establish scattering in the general (non-symmetric) case.

Theorem 1. *For any* $u_0, u_1 \in C_c^\infty(\mathbb{R}^2)$ *there exist* $(v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^2)$ *such that the solution* u *to* (1), (2) *scatters to the solution* v *of* (4), (5) *in the sense of* (6).

For the proof of Theorem 1, as in [7] it suffices to show finiteness of the scattering norm

$$\|u_{tt} - \Delta u\|_{L^{1,2}_{t,x}} = \|f(u)\|_{L^{1,2}_{t,x}} = \int_0^\infty \|f(u(t))\|_{L^2(\mathbb{R}^2)} dt$$

of the solution u to (1), (2) for given data. In [7] this already was partially achieved by applying the techniques of [10] to the function U obtained from u through conformal inversion, which satisfies an equation similar to (1). Conformal inversion also is a key element in the proof of Theorem 1 in the present paper, and we crucially exploit the fact that the wave operator and nonlinear terms of degree 5 and higher are well-behaved under this transformation. Even though our proof therefore cannot be extended to the case when $f(u)$ is replaced by the nonlinearity $l(u) = u(e^{u^2} - u^2)$, it is to be expected that the analogue of Theorem 1 also holds in this case, since scattering properties should only improve in the presence of a mass term. However, it is not clear if scattering holds when $f(u)$ is replaced by the nonlinearity $n(u) = ue^{u^2}$ since the cubic term seems difficult to treat even in the small energy regime.

Note that also when $f(u)$ is replaced by either $l(u)$ or $n(u)$, by [10] the solutions to the Cauchy problem (1), (2) for smooth data always are globally regular.

2. Higher dimensions

Theorem 1 provides the analog of scattering results for critical nonlinear wave equations

$$u_{tt} - \Delta u + f(u) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^n \tag{7}$$

with smooth, compactly supported Cauchy data and power-type nonlinearities $f(u) = u|u|^{p-2}$, $2 < p \leq 2^* := 2n/(n-2)$ in dimensions $n \geq 3$; see Grillakis [1] for the case $n = 3$, or Tao [11] for a survey of well-posedness results for the general case.

It is a highly challenging and largely open problem if the Cauchy problem for equation (7) is well-posed also for more general nonlinearities f that are defocusing in the sense that $f(u)u \geq 0$ for all $u \in \mathbb{R}$.

References

[1] M. G. Grillakis, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. (2) **132** (1990), no. 3, 485–509.

[2] S. Ibrahim, R. Jrad, *Strichartz type estimates and the wellposedness of an energy critical 2d wave equation in a bounded domain,* J. Differential Equations **250** (2011), 3740–3771.

[3] S. Ibrahim, M. Majdoub, N. Masmoudi, *Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity*, Comm. Pure Appl. Math. **59** (2006), no. 11, 1639–1658.

[4] S. Ibrahim, M. Majdoub, N. Masmoudi, *Well and ill-posedness issues for energy supercritical waves*, preprint (2009).

[5] S. Ibrahim, M. Majdoub, N. Masmoudi, K. Nakanishi, *Scattering for the two-dimensional energy-critical wave equation*, Duke Math. J. **150.2** (2009), 287–329.

[6] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[7] M. Sack, *Scattering for a massless critical nonlinear wave equation in* 2 *space dimensions*, Analysis and PDE (2014).

[8] M. Sack, M. Struwe, *Scattering for a critical nonlinear wave equation in two space dimensions*, Mathematische Annalen (2015), (in print).

[9] J. Shatah, M. Struwe, *Geometric wave equations*, Courant Lecture Notes **2**, New York University (1998), 2nd edition: AMS (2000).

[10] M. Struwe, *The critical nonlinear wave equation in 2 space dimensions*, J. European Math. Soc. **15** (2013), 1805–1823.

[11] T. Tao, *Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions*, Dyn. Partial Differ. Equ. **3** (2006), no. 2, 93–110.

[12] N.S. Trudinger, *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

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