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Asymptotic Geometric Analysis

Organised by
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Abstract. The workshop was dedicated to new developments in Asymptotic Geometric Analysis, the study of high-dimensional geometric objects by analytic and probabilistic means, and its interplay with other fields such as functional analysis, convex geometry, probability and graph theory. A central aspect concerned volume inequalities for sections and projections of high-dimensional convex sets.

Mathematics Subject Classification (2010): 46B05, 46B20, 52A20, 52C17, 26B25.

Introduction by the Organisers

The workshop was organized by Shiri Artstein-Avidan (Tel Aviv), Hermann König (Kiel) and Alexander Koldobsky (Columbia, MO) and was attended by 25 experts in relevant fields. The aim was to bring together specialists in thriving areas related to Asymptotic Geometric Analysis, discuss exciting new results, open problems and the ramifications with other areas of mathematics.

The field is concerned with the asymptotic behavior of various quantitative parameters of geometric objects in high-dimensional spaces such as volume, isotropic constants or complexity parameters as their respective dimensions tend to infinity. The main results were often formulated as exact or asymptotically exact inequalities.

We desribe a few of these new results and inequalities in the following. P. Pivovarov proved isoperimetric inequalities for convex sets defined by random points in $\mathbb{R}^n$, e. g. volume minimization in the euclidean case and maximization in the polar setup. This also included Brunn-Minkowski type inequalities, which were also studied for more general measures by A. Zvavitch. Very useful in the Brunn-Minkowski theory is the Prekopa-Leindler inequality. D. Cordero-Erausquin presented a far-reaching generalization of this inequality to the multi-function setting.

The question how well the volume of lower dimensional projections or sections of convex bodies determine the volume of the body leads to many interesting questions which were discussed intensively. The Shephard- and the Busemann-Petty problem are solved for some time, but the slicing problem is still an open and important question which is equivalent to the boundedness of the isotropic constant of convex bodies. These questions were discussed by A. Giannopoulos who showed that estimates for the volume of projections by sections between two bodies enables inequalities for the volume of the two sets. E. Milman estimated the MM^*-parameter of convex bodies, which is of importance in Dvoretzky-type theorems on spherical sections, in the case when the bodies are in isotropic position; usually this is dome for the l-position. Special positions, after transformation by linear maps, are often required for quantitative volume estimates. B.-H. Vritsiou studied the thin-shell conjecture for the operator norm and, more generally, for Schatten classes. Possible counterexamples to the slicing question were discussed in terms of perturbations of Schatten classes (V. Milman), but this remains a difficult open problem. A. Giannopoulos proved a quantitative version of Helly's theorem by estimating the volume of the intersection of $(n+1)$ sets in $\mathbb{R}^n$ by asymptotically optimal constants times the volume of the intersection of all sets in a given family.

The question how well convex bodies can be approximated by polytopes (in volume or in the Hausdorff metric), e. g. by random choices of N points, was considered by several lecturers. S. Szarek talked on complexity issues related to this, E. Werner studied surface area deviations and R. Schneider explained results for approximations of spherically convex sets in S^{n-1} by random spherical polytopes. Here curvature influences the estimates. Random approximations by polytopes often require concentration inequalities for random variables. S. Bobkov explained new concentration inequalities and showed how the spread constants and the subgaussian constants for subset measures can be estimated by those for full measures. This has applications to concentration properties for certain non-Lipschitz functions.

Several new constructions or characterizations of known constructions for convex bodies were another topic of the meeting. V. Milman introduced harmonic means of two convex bodies via polarity and defined geometric means by an iteration technique. He studied surprising properties of these constructions, e. g. solving "quadratic" equations for convex bodies. L. Rotem characterized Minkowski additions and radial additions by very simple and natural properties. Intersection bodies of non-symmetric convex bodies are generally not convex. M. Meyer proved, however, that they can be reasonably well approximated by convex sets. O. Mordhorst presented the solution of a problem of Grünbaum on affine invariant points,

points like the center of gravity or the center of the maximal volume ellipsoid associated to a convex body. B. Klartag initiated the study of affine hemispheres of elliptic type.

Some lectures were concerned with the interplay of Asymptotic Geometric Analysis with graph theory. A. Litvak and N. Tomczak-Jaegermann estimated the probability that adjacency matrices of random graphs are invertible, building on extensive experience of studying singular numbers of random matrices. M. Rudelson presented estimates for perfect matchings in random graphs by estimating probabilities for permanents of adjacency matrices of these graphs.

Non-linear phenomena were discussed by G. Schechtman and G. Paouris. G. Schechtman considered bi-Lipschitz embeddings of l_2-sums of Hamming spaces in the l_q-norm into L_1, using combinatorial inequalities. G. Paouris gave quantitative estimates for the condition numbers of random polynomial systems, introduced to find the zeros of these systems by Newton's algorithm.

The Gaussian correlation problem, namely whether the Gaussian measure of the intersection of two symmetric convex sets in $\mathbb{R}^n$ dominates the product of the Gaussian measures of both sets, had attracted the attention of many experts in the field. Very recently only, the positive solution of this problem by Thomas Royen has become known, having been published already 2014 in a not too well-known journal. M. Rudelson gave a beautiful presentation of the proof of this exciting result, which was also attended by various participants of the parallel workshop "New Developments in Functional and Highly Multivariate Statistical Methodology". The participants benefited from many discussions and the stimulating atmosphere at the MFO.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of two junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". The organizers would like to thank the MFO for supporting two further participants through the Leibniz Graduate Students programme.

Workshop: Asymptotic Geometric Analysis

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Abstracts

Inequalities for measures of sections and projections of convex bodies

Apostolos A. Giannopoulos

We discuss lower dimensional versions of the slicing problem and of the Busemann-Petty problem, both in the classical setting and in the generalized setting of arbitrary measures in place of volume, which was put forward by Koldobsky for the slicing problem and by Zvavitch for the Busemann-Petty problem. We introduce an alternative approach which is based on the generalized Blaschke-Petkantschin formula and on asymptotic estimates for the dual affine quermassintegrals.

The lower dimensional slicing problem can be posed for a general measure as follows: let g be a locally integrable non-negative function on $\mathbb{R}^n$. For every Borel subset $B \subseteq \mathbb{R}^n$ we define $\mu(B) = \int_B g(x)dx$, where, if $B \subseteq F$ for some subspace $F \in G_{n,s}$, $1 \leqslant s \leqslant n-1$, integration is understood with respect to the s-dimensional Lebesgue measure on F. Then, for any $1 \leqslant k \leqslant n-1$ one may define $\alpha_{n,k}(\mu)$ as the smallest constant $\alpha > 0$ with the following property: For every centered convex body K in $\mathbb{R}^n$ one has

$$\mu(K) \leqslant \alpha^k \max_{F\in G_{n,n-k}} \mu(K\cap F)\,|K|^{\frac{k}{n}}.$$

Koldobsky proved in [7] and in [8] that if K is a symmetric convex body in $\mathbb{R}^n$ and if g is even and continuous on K then

$$\mu(K) \leqslant (c\sqrt{n})^k \max_{F\in G_{n,n-k}} \mu(K\cap F)\,|K|^{\frac{k}{n}}$$

for every $1 \leqslant k \leqslant n-1$. In other words, for the symmetric analogue $\alpha_{n,k}^{(s)}$ of $\alpha_{n,k}$ one has $\sup_\mu \alpha_{n,k}^{(s)}(\mu) \leqslant c\sqrt{n}$. We provide a different proof of this fact; our method allows us to drop the symmetry and continuity assumptions.

Theorem 1 (Chasapis-Giannopoulos-Liakopoulos). *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Let g be a bounded locally integrable non-negative function on $\mathbb{R}^n$ and let μ be the measure on $\mathbb{R}^n$ with density g. For every $1 \leqslant k \leqslant n-1$,*

$$\mu(K) \leqslant \left(c\sqrt{n-k}\right)^k \max_{F\in G_{n,n-k}} \mu(K\cap F)\cdot |K|^{\frac{k}{n}},$$

where $c > 0$ is an absolute constant. In particular, $\alpha_{n,k}(\mu) \leqslant c\sqrt{n-k}$.

The lower dimensional Busemann-Petty problem can be also posed for a general measure: for any $1 \leqslant k \leqslant n-1$ and any measure μ on $\mathbb{R}^n$ with a locally integrable non-negative density g one may define $\beta_{n,k}(\mu)$ as the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and D in $\mathbb{R}^n$ that satisfy $\mu(K\cap F) \leqslant \mu(D\cap F)$ for every $F \in G_{n,n-k}$, one has

$$\mu(K) \leqslant \beta^k \mu(D).$$

Similarly, one may define the "symmetric" constant $\beta^{(s)}_{n,k}(\mu)$. Koldobsky and Zvavitch [9] proved that $\beta^{(s)}_{n,1}(\mu) \leqslant \sqrt{n}$ for every measure μ with an even continuous non-negative density. In fact, the study of these questions in the setting of general measures was initiated by Zvavitch in [10], where he proved that the classical Busemann-Petty problem for general measures has an affirmative answer if $n \leqslant 4$ and a negative one if $n \geqslant 5$. We study the lower dimensional question and provide a general estimate in the case where μ has an even log-concave density.

Theorem 2 (Chasapis-Giannopoulos-Liakopoulos)**.** *Let μ be a measure on $\mathbb{R}^n$ with an even log-concave density g and let $1 \leqslant k \leqslant n-1$. Let K be a symmetric convex body in $\mathbb{R}^n$ and let D be a compact subset of $\mathbb{R}^n$ such that $\mu(K \cap F) \leqslant \mu(D \cap F)$ for all $F \in G_{n,n-k}$. Then,*

$$\mu(K) \leqslant (ckL_{n-k})^k \, \mu(D),$$

where $c > 0$ is an absolute constant and L_s is the maximal isotropic constant in $\mathbb{R}^s$.

We also discuss a variant of the classical Busemann-Petty and Shephard problems, proposed by V. Milman at the Oberwolfach meeting on Convex Geometry and its Applications (December 2015): Assume that K and D are origin-symmetric convex bodies in $\mathbb{R}^n$ and satisfy

$$|P_{\xi^\perp}(K)| \leqslant |D \cap \xi^\perp|$$

for all $\xi \in S^{n-1}$. Does it follow that $|K| \leqslant |D|$? We show that the answer to this question is affirmative. In fact, the lower dimensional analogue of the problem has an affirmative answer. Moreover, one can drop the symmetry assumptions and even the assumption of convexity for D.

Theorem 3 (Giannopoulos-Koldobsky)**.** *Let K be a convex body in $\mathbb{R}^n$ and let D be a compact subset of $\mathbb{R}^n$ such that, for some $1 \leqslant k \leqslant n-1$,*

$$|P_F(K)| \leqslant |D \cap F|$$

for all $F \in G_{n,n-k}$. Then,

$$|K| \leqslant |D|.$$

In the third part of the talk we provide general inequalities that compare the surface area $S(K)$ of a convex body K in $\mathbb{R}^n$ to the minimal, average or maximal surface area of its hyperplane or lower dimensional projections. We discuss the same questions for all the quermassintegrals of K. We examine separately the dependence of the constants on the dimension in the case where K is in some of the classical positions or K is a projection body. Our results are in the spirit of the hyperplane problem, with sections replaced by projections and volume by surface area.

The starting point are two inequalities of Koldobsky about the surface area of hyperplane projections of projection bodies. We provide analogues for an arbitrary convex body. For example, we have:

Theorem 4 (Giannopoulos-Koldobsky-Valettas). *There exists an absolute constant $c_1 > 0$ such that, for every convex body K in $\mathbb{R}^n$,*

$$|K|^{\frac{1}{n}} \min_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \leqslant \frac{c_1 \partial_K}{\sqrt{n}} S(K) \leqslant c\sqrt{n}\, S(K),$$

where ∂_K is the minimal surface area parameter of K.

Conversely, assuming that K is in the minimal surface area position we have:

$$|K|^{\frac{1}{n}} \min_{\xi \in S^{n-1}} S(P_{\xi^\perp}(K)) \geqslant \frac{c}{\sqrt{n}} S(K).$$

This estimate is sharp: we provide an example in which the two quantities are of the same order.

Replacing $\min S(P_{\xi^\perp}(K))$ by the expectation of $S(P_{\xi^\perp}(K))$ on the sphere we get:

$$|K| \int_{S^{n-1}} S(P_{\xi^\perp}(K))\, d\sigma(\xi) \leqslant \frac{c}{\sqrt{n}} S(K)^2.$$

It follows that if K is in some of the *classical positions* (minimal surface area, isotropic or John's position, or it is symmetric and in Löwner's position) then

$$|K|^{\frac{1}{n}} \int_{S^{n-1}} S(P_{\xi^\perp}(K))\, d\sigma(\xi) \leqslant c\sqrt{n}\, S(K).$$

The reason is that, in all these cases, the surface area of K satisfies an inequality of the form $S(K) \leqslant cn|K|^{\frac{n-1}{n}}$. Passing to lower bounds, we have

$$\int_{S^{n-1}} S(P_{\xi^\perp}(K))\, d\sigma(\xi) \geqslant c\, S(K)^{\frac{n-2}{n-1}}.$$

A consequence is that if K is in the minimal surface area, minimal mean width, isotropic, John or Löwner position, then

$$|K|^{\frac{1}{n}} \int_{S^{n-1}} S(P_{\xi^\perp}(K))\, d\sigma(\xi) \geqslant c\, S(K).$$

Our main tools are a result from [3] stating that

$$(*) \qquad \frac{S(P_{\xi^\perp}(K))}{|P_{\xi^\perp}(K)|} \leqslant \frac{2(n-1)}{n} \frac{S(K)}{|K|}$$

for every convex body K in $\mathbb{R}^n$ and any $\xi \in S^{n-1}$, estimates from [4] for the volume of the projection body of a convex body in terms of its minimal surface area parameter, and Aleksandrov's inequalities. For the study of the surface area and the quermassintegrals of lower dimensional projections, the main additional ingredient is a generalization of $(*)$ to subspaces of arbitrary dimension and quermassintegrals of any order, proved in [2]: If K is a convex body in $\mathbb{R}^n$ and $0 \leqslant p \leqslant k \leqslant n$, then, for every $F \in G_{n,k}$,

$$\frac{V_{n-p}(K)}{|K|} \geqslant \frac{1}{\binom{n-k+p}{n-k}} \frac{V_{k-p}(P_F(K))}{|P_F(K)|},$$

where $V_{n-k}(K)$ denotes the mixed volume $V((K, n-k), (B_2^n, k))$.

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A few solutions to many problems on convex bodies, mainly from the volumetric point of view

Mathieu Meyer

(joint work with Matthieu Fradelizi and Vlad Yaskin)

Let K be a convex body in $\mathbb{R}^n$. According to a result of Grünbaum, if the centroid of K is at the origin, then for every $u \in S^{n-1}$ we have

$$|K \cap u^+| \geq e^{-1}|K|,$$

where $u^+ = \{x \in \mathbb{R}^n : \langle u, x\rangle \geq 0\}$. It is a natural question whether there is a similar result for sections of K. In other words, does there exist an absolute constant $c > 0$ such that

$$|K \cap v^\perp \cap u^+| \geq c|K \cap v^\perp|,$$

for every $u, v \in S^{n-1}$ such that $u \neq \pm v$? Here, $v^\perp = \{x \in \mathbb{R}^n : \langle v, x\rangle = 0\}$. The centroid of $K \cap v^\perp$ may not be at 0 and so we cannot apply Grünbaum's result. We show that the answer to the latter question is affirmative. More generally, there is an absolute constant $c > 0$ such that for every convex body $K \subset \mathbb{R}^n$, every $(n-k)$-dimensional subspace V, and any $u \in S^{n-1} \cap V$ we have

$$|K \cap V \cap u^+|_{n-k} \geq \frac{c}{k^2}\left(1 + \frac{k}{n-k}\right)^{-n+k+2} |K \cap V|_{n-k}.$$

The results are actually proved in a more general setting, than described above:

Theorem 1. *Let $n, p, k \geq 1$ be integers such that $p \leq k \leq n$. Let F be an $(n-k)$-dimensional subspace of $\mathbb{R}^n$ and $C \subset F^\perp$ be a cone. Let $G = \operatorname{span}(C)$, $p = \dim(G)$, and assume that $|C \cap B_2^n|_p > 0$. Let K be a convex body in $\mathbb{R}^n$ with centroid at the origin.*
(1) Then

$$\frac{|K \cap (F - C)|_{n-k+p}}{|K \cap (F + C)|_{n-k+p}} \leq k^p \left(1 + \frac{k}{n+1-k}\right)^{n-k} \binom{n+p-k}{p} \binom{n+1}{k+1}^{-\frac{p}{k+1}}.$$

(2) If, moreover, K is in isotropic position, then

$$b^{-1} \frac{|B_2^n \cap (F + C)|_{n-k+p}}{|B_2^n \cap (F + G)|_{n-k+p}} \leq \frac{|K \cap (F + C)|_{n-k+p}}{|K \cap (F + G)|_{n-k+p}} \leq b \frac{|B_2^n \cap (F + C)|_{n-k+p}}{|B_2^n \cap (F + G)|_{n-k+p}},$$

where a is the absolute constant and

$$b = \min \left\{ n^p, a^{kp} \left(1 + \frac{k}{n+1-k}\right)^{n-k} \binom{n+p-k}{p} \binom{n+1}{k+1}^{-\frac{p}{k+1}} \right\},$$

Corollary 1. *There is an absolute constant $c > 0$ such that for any integers $n \geq k \geq 1$, any convex body K in $\mathbb{R}^n$ whose centroid is at the origin, any $(n-k)$-dimensional subspace F of $\mathbb{R}^n$, and any $\theta \in S^{n-1} \cap F^\perp$, we have*

$$\frac{|K \cap (F + \mathbb{R}_+ \theta)|_{n-k+1}}{|K \cap (F + \mathbb{R}_- \theta)|_{n-k+1}} \leq c k^2 \left(1 + \frac{k}{n-k+1}\right)^{n-k-1}.$$

Corollary 2. *There exists a constant $c > 0$ such that for any convex body K in $\mathbb{R}^n$, $n \geq 2$, with centroid at 0, and every $u, v \in S^{n-1}$ such that $v \neq \pm u$, one has*

$$\frac{1}{c} \leq \frac{\big|\{x \in K : \langle x, u \rangle = 0,\ \langle x, v \rangle \geq 0\}\big|_{n-1}}{\big|\{x \in K : \langle x, u \rangle = 0,\ \langle x, v \rangle \leq 0\}\big|_{n-1}} \leq c.$$

Corollary 3. *Let K be an isotropic convex body in $\mathbb{R}^n$. If $u_1, \ldots, u_k \in S^{n-1}$ are pairwise orthogonal, then, for some absolute constant $c > 0$, one has*

$$\frac{1}{\min\{n^k, c^{k^2}\}} |K| \leq |K \cap \bigcap_{i=1}^k \{x \in \mathbb{R}^n : \langle x, u_i \rangle \geq 0\}| \leq \min\{n^k, c^{k^2}\} |K|.$$

Our results also allow us to give a positive answer to a problem posed in [13] by Meyer and Reisner. A *star body* K in $\mathbb{R}^n$ is a compact set such that $[0, x] \subset K$ for every $x \in K$, and whose *radial function*, defined by $r_K(u) = \max\{a \geq 0 : au \in K\}$, $\quad u \in S^{n-1}$, is positive and continuous. The *intersection body of K*, introduced by Lutwak [12], is a star body $I(K)$ whose radial function is

$$r_{I(K)}(u) = |K \cap u^\perp|, \text{ for all } u \in S^{n-1}.$$

By Busemann's theorem, if K is origin-symmetric and convex, $I(K)$ is also convex. Without symmetry, this statement is not true. To rectify the situation, Meyer and Reisner [13] suggested a new construction, which allowed to extend Busemann's theorem to non-symmetric bodies. Let K be a convex body whose centroid is at 0. Define the *convex intersection body* $CI(K)$ of K by its radial function:

$$r_{CI(K)}(u) = \min_{z \in u^\perp} |(P_u K^*)^{*z}|, \quad u \in S^{n-1},$$

where K^* is the polar body of K with respect to 0, P_u is the orthogonal projection onto the hyperplane $u^\perp$, and for $C \subset u^\perp$ and $z \in u^\perp$, C^{*z} is the polar body of C in $u^\perp$ with respect of z: $C^{*z} = \{y \in u^\perp : \langle y - z, x - z\rangle \leq 1 \text{ for all } x \in C\}$. It was proved in [13] that $CI(K)$ is always convex, whenever K is convex. Moreover, $CI(K) \subset I(K)$, with equality if and only if K is origin-symmetric.

Recall that K is said to be in isotropic position if the centroid of K is at 0, $|K| = 1$ and $\int_K \langle x, u\rangle^2\, dx$ is independent of $u \in S^{n-1}$. It is a well-known result (see Hensley [10], Ball [1], Schütt [15], Fradelizi [7]) that for a convex body K in isotropic position, $I(K)$ is "almost" a ball,: there is a universal constant c such that

$$c^{-1} \leq \frac{r_{I(K)}(u_1)}{r_{I(K)}(u_2)} \leq c,$$

for any u_1, $u_2 \in S^{n-1}$. In [13] it was asked whether the same would be true for $CI(K)$. As an application of our results, we obtain a positive answer to this conjecture:

Theorem 2. *There exists an absolute constant $c > 0$ such that for every $n \geq 2$ and every convex body K in $\mathbb{R}^n$ with centroid at the origin, one has*

$$cI(K) \subset CI(K) \subset I(K).$$

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On adjacency matrices of d-regular graphs and almost constant vectors

Nicole Tomczak-Jaegermann
(joint work with Alexander Litvak, Anna Lytova, Konstantin Tikhomirov, Pierre Youssef)

In this talk we describe structure of adjacency matrices of regular d-graphs which is responsible for non-singularity of these matrices.

Our results depend in an essential way on (old and new) properties of regular graphs, however in the talk we shall not develop this direction, and we rather concentrate on properties of a corresponding set of matrices.

Let $1 \leq d < n$ be integers. Let M be an $n \times n$ matrix with 0/1 entries such that the sum over each column and each row is equal to d. By $\mathcal{M}_{n,d}$ we shall denote the set of all such matrices.We also consider the uniform probability measure $\mathbb{P}$ on $\mathcal{M}_{n,d}$ defined by

$$\mathbb{P}(\mathcal{F}) = \frac{|\mathcal{F}|}{|\mathcal{M}_{n,d}|},$$

for any subset $\mathcal{F} \subset \mathcal{M}_{n,d}$.

Our main theorem states

$$\mathbb{P}\{M \in \mathcal{M}_{n,d} : M \text{ is nonsingular}\} > 0. \tag{1.1}$$

Moreover, the probability above is a function of n and d, with the limit equal to 1 whenever $n \to \infty$ and $d \to \infty$.

One of a general idea in proving (1.1) is to identify some special sets of vectors which have empty intersections with kernels of matrices taken from a set of probability close to 1. We illustrate this method by introducing the concept of *almost constant* vectors, which is of interest on its own.

We say that a non-zero vector is "almost constant" if for some $0 < p < 1/2$ at least $(1-p)n$ of its coordinates are equal to each other. Formally, for $0 < p < 1/2$ consider the following set of vectors

$$AC(p) = \{x \in \mathbb{R}^n \setminus \{0\} \, : \, \exists \lambda_x \in \mathbb{R} \quad |\{i \, : \, x_i = \lambda_x\}| \geq (1-p)n\}. \tag{1.2}$$

In this section we estimate the probability of the event

$$\mathcal{E}^{AC}(p) := \{M \in \mathcal{M}_{n,d} \, : \, \forall x \in AC(p) \quad Mx \neq 0 \quad \text{and} \quad x^T M \neq 0\}, \tag{1.3}$$

which relates almost constant vectors to null vectors of M. We show that that this probability is close to one, in other words we show that with high probability a matrix $M \in \mathcal{M}_{n,d}$ cannot have almost constant null vectors. This will be used in the proof of the main theorem allowing to restrict the proof to the event $\mathcal{E}^{AC}(p)$. More precisely, we prove the following theorem.

Theorem 1. *There are absolute positive constants C, c such that for $C \le d \le cn$ and $p \le c/\ln d$ one has*

$$\mathbb{P}\big(\mathcal{E}^{AC}(p)\big) \ge 1 - \left(\frac{Cd}{n}\right)^{cd}. \tag{1.4}$$

We will split $AC(p)$ into four sets: the set of vectors having less than n/d non-zero coordinates, the set of vectors having significant jump in absolute values of the first largest n/d coordinates, the set of vectors having a part of coordinates far enough from λ_x of the order n/d, and the rest of vectors. The most difficult is the second case above and the rest is relatively easy and all them together are leading to a proof of Theorem 1.

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Concentration properties of restricted measures with applications to non-Lipschitz functions

Sergey G. Bobkov

(joint work with Piotr Nayar and Prasad Tetali)

Given a metric probability space (M, d, μ), we consider two characteristics quantifying the concentration property: The spread constant

$$s^2(\mu) = \sup \mathrm{Var}_\mu(f) = \sup \int (f-m)^2\, d\mu,$$

where $m = \int f\, d\mu$ and the sup is running over all Lipschitz functions f on M with $\|f\|_{\mathrm{Lip}} \le 1$, and the subgaussian constant, which is an optimal value $\sigma^2 = \sigma^2(\mu)$ such that

$$\int e^{tf}\, d\mu \le e^{\sigma^2 t^2/2} \qquad (t \in \mathbf{R}),$$

for any f on M with $m = 0$ and $\|f\|_{\mathrm{Lip}} \le 1$. The latter is equivalent to the maximal ψ_2-norm over all Lipschitz mean zero functions on M.

These characteristics were explicitly introduced in [A-B-S] and [B-H-T], cf. also [B-G], [L]. For example, $s^2 = \sigma^2 = 1$ for the standard Gaussian measure on $M = \mathbf{R}^n$ with the Euclidean distance, and $s^2 = \sigma^2 = \frac{1}{n-1}$ for the uniform measure on the unit sphere $M = S^{n-1}$ equipped with the geodesic distance.

Given an arbitrary Borel subset $A \subset M$ with $p = \mu(A) > 0$, we consider these quantities for the normalized restricted measure

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}, \qquad B \subset M,$$

and prove the following:

Theorem 1. *Up to a numerical constant $c > 0$,*

$$\sigma^2(\mu_A) \ \leq \ c \log\left(\frac{e}{p}\right) \sigma^2(\mu).$$

In addition, if the Poincaré constant $\lambda_1 = \lambda_1(\mu)$ is positive, we also have

$$s^2(\mu_A) \ \leq \ c \log^2\left(\frac{e}{p}\right) \frac{1}{\lambda_1}.$$

In case of the standard Gaussian measure on $\mathbf{R}^n$ and convex sets A, a result of D. Bakry and M. Ledoux [B-L] yields a uniform bound $\sigma^2(\mu_A) \leq 1$. However, in general, the p-dependent factors in the above bounds are asymptotically optimal for $p \to 0$. The statement about the spread constant may further be sharpened using exponential bounds going back to the work of M. Gromov and V. D. Milman [G-M].

These bounds may be used to study large deviations of not necessarily Lipschitz functions on M. Given a locally Lipschitz function f on M, consider the sublevel sets

$$A_L = \{x \in M : |\nabla f(x)| \leq L\}, \qquad L > 0.$$

Corollary 2. *Suppose that f has Lipschitz semi-norms at most L on the sets A_L. If $\int e^{|\nabla f|^2}\, d\mu \leq 2$, then with some absolute constant $c > 0$,*

$$\mu\big\{|f - m| \geq r\big\} \ \leq \ 2\, e^{-r/c\sigma(\mu)} \qquad (r > 0).$$

Equivalently (up to an absolute factor), we have a Sobolev-type inequality

$$\|f - m\|_{\psi_1} \leq c\sigma(\mu)\, \|\nabla f\|_{\psi_2}$$

for the Orlicz norms generated by the Young functions $\psi_\alpha(t) = e^{|t|^\alpha} - 1$ ($\alpha = 1, 2$).

Theorem 1 may also be used to generalize a theorem due to M. Talagrand [T1-2] about Gaussian deviations of Lipschitz convex functions on $[-1,1]^n$ with respect to arbitrary product measures $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ on the cube. For such measures, we have:

Corollary 3. *If $\mu\{|\nabla f| \geq L_0\} \leq \frac{1}{2}$, then for all $r > 0$,*

$$(\mu \otimes \mu)\big\{|f(x) - f(y)| \geq r\big\} \ \leq \ 2 \inf_{L \geq L_0} \Big[e^{-r^2/cL^2} + \mu\big\{|\nabla f| > L\big\}\Big].$$

In particular, for any convex f on $\mathbf{R}^n$ with μ-mean zero,

$$\|f\|_{\psi_1} \ \leq \ c\, \|\nabla f\|_{\psi_2}.$$

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Some extensions of the Prékopa-Leindler inequality using Borell's stochastic approach

Dario Cordero-Erausquin

(joint work with Bernard Maurey)

There is a long story of functional generalizations of the Brunn-Minkowski inequality. A somewhat definitive form is given by the following theorem (see also [2]).

Theorem 1 (Prékopa–Leindler inequality). *Let $t \in [0,1]$ and let $f_0, f_1, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be Borel functions such that, for every $x_0, x_1 \in \mathbb{R}^n$,*

$$g\left((1-t)x_0 + tx_1\right) \le (1-t)f_0(x_0) + tf_1(x_1).$$

Then

$$\int_{\mathbb{R}^n} e^{-g(x)}\,dx \ge \left(\int_{\mathbb{R}^n} e^{-f_0(x)}\,dx\right)^{1-t} \left(\int_{\mathbb{R}^n} e^{-f_1(x)}\,dx\right)^{t}.$$

Accepting the value $+\infty$ enables us to reach directly indicator functions $\mathbf{1}_E = e^{-f_E}$, by letting f_E be 0 on E and $+\infty$, and thus to reproduce the Brunn-Minkowski inequality $|(1-t)A + tB| \ge |A|^{1-t}|B|^t$.

One can reasonably argue that the interest of the Prékopa–Leindler inequality resides not only in its consequences but also in the emphasis it has put on log-concavity, and in the related techniques of proof it has originated, such as mass transportation or semi-group techniques, and more recently L^2-methods (as in [4]). Here, we will concentrate on Borell's stochastic approach [3] to the inequality above. It somehow reduces the inequalities under study to the convexity of $|\cdot|^2$, the square of the Euclidean norm on $\mathbb{R}^n$. It will allow us to obtain some unexpected inequalities, for instance that of the following proposition.

Proposition 2. *Let f_0, f_1, g_0, g_1 be four Borel functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ such that, for every $x_0, x_1 \in \mathbb{R}^n$,*

$$g_0(2x_0/3 + x_1/3) + g_1(x_0/3 + 2x_1/3) \le f_0(x_0) + f_1(x_1). \tag{1.1}$$

Then

$$\left(\int_{\mathbb{R}^n} e^{-g_0(x)}\,dx\right)\left(\int_{\mathbb{R}^n} e^{-g_1(x)}\,dx\right) \ge \left(\int_{\mathbb{R}^n} e^{-f_0(x)}\,dx\right)\left(\int_{\mathbb{R}^n} e^{-f_1(x)}\,dx\right).$$

We will see that it is rather natural to arrive to this type of inequality using Borell's stochastic approach, whereas it seems not to be the case with other methods, for instance those based on transportation methods. Note however that this functional inequality is interesting only for functions (the game is to distribute the values between the four functions): it does not give anything new when applied to the case where the functions e^{-f_i}'s are indicators of sets. The previous proposition and its proof suggest actually more general inequalities.

Assume we are given two measure spaces $X_1 = (\Omega_1, \Sigma_1, \mu_1)$ and $X_2 = (\Omega_2, \Sigma_2, \mu_2)$, where Σ_i is a σ-algebra of subsets of Ω_i, $i = 1, 2$, and where μ_1 and μ_2 have the same finite mass, $\mu_1(\Omega_1) = \mu_2(\Omega_2) < +\infty$. We are also given an integer $n \ge 1$ and a continuous linear operator

$$A : L^2(X_1, \mathbb{R}^n) \to L^2(X_2, \mathbb{R}^n),$$

where the L^2-norms of the $\mathbb{R}^n$-valued functions are computed with respect to the Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ and the measures μ_1 and μ_2, respectively. We assume that

(1) the operator A satisfies the norm condition $\|A\| \le 1$,
(2) the operator A acts as the identity on the constant vector valued functions, *i.e.*, for any $v_0 \in \mathbb{R}^n$, the constant function $\Omega_1 \ni s \to v_0$ is sent by A to the constant function $\Omega_2 \ni t \to v_0$.

Theorem 3. *Let $\{f_s\}_{s\in\Omega_1}$ and $\{g_t\}_{t\in\Omega_2}$ be two families of Borel functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ for which we make some appropriate integrability assumptions (see our paper). Then, if for every $\alpha \in L^2(X_1, \mathbb{R}^n)$ we have*

$$\int_{\Omega_2} g_t((A\alpha)(t))\,d\mu_2(t) \le \int_{\Omega_1} f_s(\alpha(s))\,d\mu_1(s), \tag{1.2}$$

it follows that

$$\int_{\Omega_2} -\log\left(\int_{\mathbb{R}^n} e^{-g_t(x)}\,dx\right) d\mu_2(t) \le \int_{\Omega_1} -\log\left(\int_{\mathbb{R}^n} e^{-f_s(x)}\,dx\right) d\mu_1(s). \tag{1.3}$$

With first establish this inequality when integration on $\mathbb{R}^n$ is with respect to an (isotropic) Gaussian measure (that is, the law of of a standard Brownian motion at some time $T > 0$), using Borell's stochastic approach. The condition $\mu(X_1) = \mu(X_2)$ is then only used to pass to the Lebesgue measure using homogeneity and letting the variance go to $+\infty$. As a matter of fact, many of the functional Brunn-Minkowski type inequalities are genuinely Gaussian inequalities: the Lebesgue measure is here only a "flat" Gaussian measure.

In many applications, X_1 and X_2 are finite probability spaces, and we are then working with finite families of objects parametrized by Ω_1 and Ω_2, or rather by the supports $\mathrm{supp}(\mu_1)$ and $\mathrm{supp}(\mu_2)$; in particular, the mappings $\alpha : \Omega_1 \to \mathbb{R}^n$ are families of $|\Omega_1|$ vectors of $\mathbb{R}^n$ and the linear operator

$$A : (\mathbb{R}^n)^{|\Omega_1|} \to (\mathbb{R}^n)^{|\Omega_2|}$$

is norm-one for the operator norm associated to the ℓ^2-norms weighted by the μ_i's, with the property that the vector $(x, \dots, x) \in (\mathbb{R}^n)^{|\Omega_1|}$ is sent to $(x, \dots, x) \in (\mathbb{R}^n)^{|\Omega_2|}$, for every $x \in \mathbb{R}^n$. In this case, there is no further integrability assumptions in the Theorem above.

In the case $|\Omega_1| = 1$ and $|\Omega_2| = 2$, we find the Hölder inequality. In the case $|\Omega_1| = 2$ and $|\Omega_2| = 1$ we get the Prékopa-Leindler inequality; indeed, for $X_1 = (\{0\}, \delta_0)$ and $X_2 = (\{0, 1\}, (1 - t)\delta_0 + t\delta_1)$, the assumptions are satisfied for the map $A(x_0, x_1) = (1-t)x_0 + tx_1$. New situations appear when $|\Omega_1| = |\Omega_2| = 2$. For instance if $X_1 = X_2 = (\{0, 1\}, \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)$ with $A(x_0, x_1) = (\frac{2}{3}x_0 + \frac{1}{3}x_1, \frac{1}{3}x_0 + \frac{2}{3}x_1)$, we get the Proposition above, since $A(x, x) = (x, x)$ and $|2x_0/3 + x_1/3|^2 + |x_0/3 + 2x_1/3|^2 \leq |x_0|^2 + |x_1|^2$.

We can also work with functions leaving in different dimensions, for instance on $\mathbb{R}^n$ on one side and on $\mathbb{R}^m$ on the other side. The only difference is that we have to adapt the "preserving constant" assumption, using projections (or rather adjoints of partial isometries). Then, we can reproduce and extend the Brascamp–Lieb inequality (in geometric form) and its reverse form devised by Franck Barthe [1].

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M, M* and Concentration of Isotropic Convex Bodies

Emanuel Milman

Let K denote an origin-symmetric convex body (i.e. compact set with non-empty interior) in $\mathbb{R}^n$. The norm on $\mathbb{R}^n$ having unit-ball K is denoted by $\|\cdot\|_K$, and the dual norm by $\|\cdot\|_K^*$. Let:

$$M(K) = \int_{S^{n-1}} \|\theta\|_K \, d\sigma_{S^{n-1}}(\theta) \ , \ M^*(K) = \int_{S^{n-1}} \|\theta\|_K^* \, d\sigma_{S^{n-1}}(\theta),$$

denote the mean-norm and (half) mean-width of K (here $\sigma_{S^{n-1}}$ denotes the Haar probability measure on S^{n-1}). It is well-known by the Jensen and Urysohn inequalities that:

$$\frac{1}{M(K)} \leq \text{v.rad}(K) := \left(\frac{|K|}{|B_2^n|}\right)^{1/n} \leq M^*(K), \tag{1.1}$$

where $|\cdot|$ denotes Lebesgue measure. In our talk, we describe new best-known reverse inequalities when K is in isotropic position.

Recall that K is called isotropic if $|K| = 1$, its barycenter is at the origin and its covariance matrix $Cov(K) := \left(\int_K x_i x_j dx\right)_{i,j}$ is a multiple of the identity $L_K^2\text{Id}$. The constant $L_K > 0$ is called the isotropic constant of K. It is well-known that any body K has an affine image which is isotropic (and which is unique modulo orthogonal transformations). The Slicing Problem, posed by Bourgain in the 80's, asked whether it is possible to upper bound L_K for all convex bodies by a universal constant, independent of dimension. Note that $\text{v.rad}(K) \simeq \sqrt{n}$ when $|K| = 1$.

In the first part of the talk, based on our work [2], we obtain the bound:

$$M^*(K) \leq C\sqrt{n}\log(n)^2 L_K \ ,$$

where $C > 0$ is a universal constant. This improves the previous best-known estimate $M^*(K) \leq Cn^{3/4}L_K$. Up to the power of the $\log(n)$ term and the L_K one, the improved bound is best possible, and implies that the isotropic position is (up to the L_K term) an almost 2-regular M-position. The bound extends to any arbitrary position, depending on a certain weighted average of the eigenvalues of the covariance matrix $Cov(K)$. Furthermore, the bound applies to the mean-width of L_p-centroid bodies, extending a sharp upper bound of Paouris [4] for $1 \leq p \leq \sqrt{n}$ to an almost-sharp bound for an arbitrary $p \geq \sqrt{n}$. The question of whether it is possible to remove the L_K term from the new bound is essentially equivalent to the Slicing Problem, to within logarithmic factors in n. The proof is based on an application of a refinement of Dudley's entropy bound due to V. Milman and G. Pisier [3].

In the second part of the talk, based on a joint work with Apostolos Giannopoulos [1], we show that if $K \supseteq rB_2^n$ then:

$$\sqrt{n}M(K) \leqslant C\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\min\left(\frac{1}{r}, \frac{n}{k}\log\left(e+\frac{n}{k}\right)\frac{1}{v_k^-(K)}\right),$$

where $v_k^-(K)$ denotes the minimal volume-radius of a k-dimensional orthogonal projection of K. We apply this result to the study of $M(K)$ and its L_p-centroid bodies when K is isotropic. In particular, we obtain the following best-known estimate:

$$\frac{1}{M(K)} \geq \frac{\sqrt[10]{n}L_K}{C\log^{2/5}(e+n)}.$$

Better estimates on $M(K)$ and on the dual covering estimates on K are obtained in the case that K's marginals (of arbitrary dimension) have bounded isotropic constant.

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Pythagorean powers of hypercubes

Gideon Schechtman

(joint work with Assaf Naor)

For $1 \le p, q \le 2$, $\ell_p(\ell_q)$ isomorphically embeds into $L_1 = L_1(0,1)$ if and only if $p \le q$. The best proof of this with the right estimates for the distance of $\ell_p^n(\ell_q^m)$ from a subspace of L_1 follows from an inequality of Kwapień and Schütt.

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_1 \lesssim \frac{1}{n!}\sum_{\pi\in S_n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{j\pi(j)}\Big\|_1,$$

for all n and all $\{z_{jk}\}_{j,k=1}^n \subseteq L_1$, where S_n is the group of all permutations of $\{1,\dots,n\}$.

If $\{z_{jk}\}_{j,k}^n$ is the natural basis of $\ell_p^n(\ell_q^n)$,

$$\frac{1}{n}\sum_{j=1}^{n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\| = n^{1/q}$$

and

$$\frac{1}{n!}\sum_{\pi\in S_n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{j\pi(j)}\Big\| = n^{1/p}.$$

So $n^{\frac{1}{p}-\frac{1}{q}} \lesssim d(\ell_p^n(\ell_q^n), SL_1)$.

Recall that a metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exist $s \in (0,\infty)$, $D \in [1,\infty)$ and a mapping $f : X \to Y$ such that

$$\forall\, x, y \in X, \qquad s d_X(x,y) \le d_Y(f(x), f(y)) \le D s d_X(x,y).$$

When this happens we say that (X, d_X) embeds into (Y, d_Y) with distortion at most D. We denote by $c_Y(X)$ the infimum over such $D \in [1,\infty]$. When $Y = L_p$ we use the shorter notation $c_{L_p}(X) = c_p(X)$.

We will be interested in lower bounding the distortion of embedding the ℓ_p^n sum of the discrete cube $\mathbb{F}_2^n$ with the ℓ_q norm into L_1. We shall restrict ourselves here to the case $p = 2$ and $q = 1$. So we're interested in $c_1(\ell_2^n(\mathbb{F}_2^n))$ where $\mathbb{F}_2^n$ is the

n-dimensional discrete hypercube, endowed with the metric inherited from ℓ_1^n via the identification $\mathbb{F}_2^n = \{0,1\}^n \subset \mathbb{R}^n$.

By general principles (ultraproduct, w^*-Gâteaux differentiation), the above stated result of Kwapień and Schütt formally implies that

$$\lim_{n\to\infty} c_1(\ell_2^n(\mathbb{F}_2^n)) = \infty,$$

but such arguments don't give quantitative results. Our main result is

Theorem 1. *$c_1(\ell_2^n(\mathbb{F}_2^n)) \asymp \sqrt{n}$.*

More generally

Theorem 2. *For all $1 \le p < q$*

$$c_1(\ell_q^n(\mathbb{F}_2^n, \|\cdot\|_p)) \asymp n^{\frac{1}{p}-\frac{1}{q}}.$$

To prove Theorem 1 it is tempting to try and prove the inequality

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\left|f\Big(x+\sum_{k=1}^{n}e_{jk}\Big)-f(x)\right| \le \frac{K}{n!}\sum_{\pi\in S_n}\sum_{x\in M_n(\mathbb{F}_2)}\left|f\Big(x+\sum_{j=1}^{n}e_{j\pi(j)}\Big)-f(x)\right|$$

for every $n \in \mathbb{N}$ and every $f : M_n(\mathbb{F}_2) \to \mathbb{R}$. This follows a paradigm set out by Enflo in the 70-s. For $f(x) = \sum_{i=1}^n \sum_{k=1}^n (-1)^{x_{ik}} z_{ik}$ we recover the linear inequality. However, it turns out that the inequality we propose never holds. Instead, the inequality

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{k=1}^{n}\varepsilon_k z_{jk}\Big\|_1 \le \frac{C}{n^n}\sum_{k\in\{1,\dots,n\}^n}\sum_{\varepsilon\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\varepsilon_j z_{jk_j}\Big\|_1 \tag{1.1}$$

for every $n \in \mathbb{N}$ and every $\{z_{jk}\}_{j,k=1}^n \subset L_1$, is, provided it holds, as good to prove that $c_1(\ell_2^n(\ell_1^n)) \gtrsim \sqrt{n}$.

It turns out that this inequality holds, generalizes to an appropriate metric inequality, and the proof of the metric inequality is even simpler than that of the original KS inequality.

Theorem 3. *For all $n \in 2\mathbb{N}$ and every $f : M_n(\mathbb{F}_2) \to L_1$ we have*

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{k=1}^{n}e_{jk}\Big)-f(x)\Big\|_1 \le \frac{2e^2}{e^2-1}\frac{1}{n^n}\sum_{k\in\{1,\dots,n\}^n}\sum_{x\in M_n(\mathbb{F}_2)}\Big\|f\Big(x+\sum_{j=1}^{n}e_{jk_j}\Big)-f(x)\Big\|_1.$$

Theorem 1 as well as (1.1) follow easily.

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Affine hemispheres of elliptic type

Bo'az Klartag

Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected hypersurface which is locally strongly-convex, i.e., the second fundamental form is a definite symmetric bilinear form at any point $y \in M$. For $y \in M$ we write T_yM for the tangent space to M at y, viewed as an affine subspace of $\mathbb{R}^{n+1}$. When the origin does not belong to this affine subspace, we may define a vector $\nu_y \in \mathbb{R}^{n+1}$ via the requirements that

$$\langle \nu_y, y \rangle = 1 \qquad \text{and} \qquad \nu_y \perp T_yM.$$

When ν_y is well-defined for any $y \in M$, we refer to $\nu : M \to \mathbb{R}^{n+1}$ as the polarity map. In this case the polar hypersurface M^* is defined as

$$M^* := \nu(M) = \{\nu_y \, ; \, y \in M\}.$$

It is well-known that M^* is always a smooth, connected, locally strongly-convex hypersurface, that the polarity map $\nu : M \to M^*$ is a diffeomorphism, and its inverse is the polarity map associated with M^*. In particular, $(M^*)^* = M$. We define the cone measure μ_M on a smooth hypersurface $M \subseteq \mathbb{R}^{n+1}$ via the requirement that for any Borel subset $S \subseteq M$ that does not contain two distinct points on the same ray from the origin,

$$\mu_M(S) = \mathrm{Vol_{n+1}}\left(\{\mathrm{tx} \, ; \, 0 \leq \mathrm{t} \leq 1, \, \mathrm{x} \in \mathrm{S}\}\right).$$

Definition 1. *Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly-convex hypersurface. We say that M is "affinely-spherical with center at the origin" if the polarity map $\nu : M \to M^*$ is well-defined, and it pushes forward the cone measure μ_M to a measure proportional to the cone measure μ_{M^*}.*

An *affine sphere* is an affinely-spherical hypersurface which is *complete*, i.e., it is a closed subset of $\mathbb{R}^{n+1}$. This definition is clearly affinely-invariant, hence the term "affine sphere". Affine spheres were introduced by the Romanian geometer Tzitzéica in 1909. All convex quadratic hypersurfaces in $\mathbb{R}^{n+1}$ are affine spheres, as well as the hypersurface

$$M = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \, ; \, \forall i, x_i > 0, \prod_{i=1}^{n} x_i = 1 \right\}. \tag{1.1}$$

Any affinely spherical hypersurface M with center at the origin is either of elliptic type or of hyperbolic type: We say that M is of elliptic type if for any $y \in M$, the ray from y to the origin passes through the convex side of M, while it is of hyperbolic type if the ray goes through the concave side of M.

Ellipsoids in $\mathbb{R}^{n+1}$ are elliptic affine spheres. There are no other examples of complete affine spheres of elliptic type. This non-trivial theorem is the culmination of the works of Blaschke, Deicke, Calabi and Cheng-Yau. See the survey by Loftin [3] for details. While affine spheres of elliptic or parabolic type are quite rare, there are many hyperbolic affine spheres in $\mathbb{R}^{n+1}$. From the works of Calabi and Cheng-Yau we learn that for any non-empty, open, convex cone $C \subseteq \mathbb{R}^{n+1}$ that does not contain a full line, there exists a hyperbolic affine sphere which is asymptotic to the cone. This hyperbolic affine sphere is determined by the cone C up to homothety, and all hyperbolic affine spheres in $\mathbb{R}^{n+1}$ arise this way. The hyperbolic affine sphere in (1.1), for example, is asymptotic to the positive orthant. Why are there so few elliptic affine spheres, compared to the abundance of hyperbolic affine spheres? Perhaps completeness is too strong a requirement in the elliptic case. We propose the following:

Definition 2. *Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly-convex hypersurface. We say that M is an "affine hemisphere" if*

(1) *There exist compact, convex sets $K, \tilde{K} \subseteq \mathbb{R}^{n+1}$, with* $\dim(K) = n$ *and* $\dim(\tilde{K}) = n + 1$, *such that M does not intersect the affine hyperplane spanned by K and*
$$K \cup M = \partial \tilde{K}.$$
(2) *The hypersurface M is affinely-spherical with center at the relative interior of K.*

We say that K is the "anchor" of the affine hemisphere M.

In Definition 2, the dimension $\dim(K)$ is the maximal number N such that K contains $N + 1$ affinely-independent vectors. Note that when $M \subseteq \mathbb{R}^{n+1}$ is an affine hemisphere, its anchor K is the compact, convex set enclosed by $\overline{M} \setminus M$, where $\overline{M}$ is the closure of M. In particular, $K = \mathrm{Conv}(\overline{M} \setminus M)$ where Conv denotes convex hull. It is clear that an affine hemisphere is always of elliptic type. Our main result is the following:

Theorem 3. *Let $K \subseteq \mathbb{R}^{n+1}$ be an n-dimensional, compact, convex set. Then there exists an affine hemisphere $M \subseteq \mathbb{R}^{n+1}$ with anchor K, uniquely determined up to affine transformations. The affine hemisphere M is centered at the Santaló point of K.*

Thus, with any n-dimensional, compact, convex set $K \subseteq \mathbb{R}^{n+1}$ we associate an $(n+1)$-dimensional, compact, convex set $\tilde{K} \subseteq \mathbb{R}^{n+1}$ whose boundary consists of two parts: the convex set K itself is a facet, and the rest of the boundary is an affine hemisphere M centered at the Santaló point of K.

There is a rich geometric structure associated with an affine sphere M. One may associate a natural Riemannian metric to M, which is simply the second fundamental form divided by the Gauss curvature to the power of $1/(n+2)$. As it turns out, the Ricci curvature is positive for affinely spherical hypersurfaces of elliptic type and it is non-positive for affine spheres of hyperbolic type. The first eigenvalue of the Dirichlet Laplacian on an affine hemisphere equals $-n$, with the

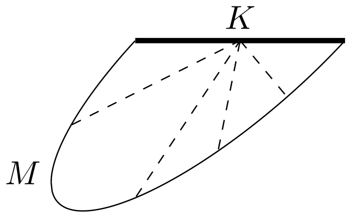

FIGURE 1. Half of an ellipse, which is an affine one-dimensional hemisphere in $\mathbb{R}^2$.

eigenfunction being a coordinate function. As for the proof of Theorem 3, it relies upon analysis of the equation with the constraint

$$\begin{cases} \det \nabla^2 \varphi = C/\varphi^{n+2} & \text{in } \mathbb{R}^n \\ \nabla\varphi(\mathbb{R}^n) = K^\circ \end{cases} \tag{1.2}$$

where $\varphi : \mathbb{R}^n \to (0, \infty)$ is a smooth, convex function and $K \subseteq \mathbb{R}^n$ is a convex body whose Santaló point is at the origin. In fact, the affine hemisphere with anchor K is $M = J(\text{graph}(\varphi))$ where

$$\text{graph}(\varphi) = \{(x,t)\,;\, x \in \mathbb{R}^n, t = \varphi(x)\} \qquad \text{and} \qquad J(x,t) = \left(\frac{x}{t}, \frac{1}{t}\right).$$

The equation (1.2) is similar to the moment measures of Berman and Berndtsson [1] and Cordero-Erausquin and Klartag [2]. The existence and the uniqueness of the solution are proven via a variational problem. Namely, we analyze the subgradient of the functional

$$\mathcal{I}(\psi) = \left(\int_{\mathbb{R}^n} \frac{dx}{(\psi^*(x))^{n+1}} \right)^{-1},$$

where ψ^* is the Legendre transform of ψ. The convexity of the functional $\mathcal{I}$, which follows from the Borell-Brascamp-Lieb inequality, is the key for the proof.

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The Surface Area Deviation of the Euclidean Ball and a Polytope

Elisabeth Werner
(joint work with Steven Hoehner and Carsten Schütt)

How well can a convex body be approximated by a polytope? This is a fundamental question not only in convex geometry, but also in view of applications in stochastic geometry, complexity, geometric algorithms and many more (e.g., [6, 7, 8, 9, 10, 14, 15, 22, 24]).

Accuracy of approximation is often measured in the symmetric difference metric, which reflects the volume deviation of the approximating and approximated objects. Approximation of a convex body K by inscribed or circumscribed polytopes with respect to this metric has been studied extensively and many of the major questions have been resolved. We refer to, e.g., the surveys and books by Gruber [13, 16, 17] and the references there and to, e.g., [1, 2, 4, 11, 19, 23, 25, 27, 28].

Sometimes it is more advantageous to consider the surface area deviation Δ_s [3, 4, 12] instead of the volume deviation Δ_v. It is especially desirable because if best approximation of convex bodies is replaced by random approximation, then we have essentially the same amount of information for volume, surface area, and mean width ([4],[5]), which are three of the quermassintegrals of a convex body (see, e.g., [26, 8]).

For convex bodies K and L in $\mathbb{R}^n$ with boundaries ∂K and ∂L, the symmetric surface area deviation is defined as

$$\Delta_s(K,L) = \mathrm{vol}_{n-1}\left(\partial(K\cup L)\right) - \mathrm{vol}_{n-1}\left(\partial(K\cap L)\right). \tag{1.1}$$

Typically, approximation by polytopes often involves side conditions, like a prescribed number of vertices, or, more generally, k-dimensional faces [2]. Most often in the literature, it is required that the body contains the approximating polytope or vice versa. This is too restrictive as a requirement and it needs to be dropped. Here, we do exactly that and prove upper and lower bounds for approximation of convex bodies by arbitrarily positioned polytopes in the symmetric surface area deviation. This addresses questions asked by Gruber [17].

Theorem 1. *There exists an absolute constant $c>0$ such that for every integer $n\geq 3$, there is an N_n such that for every $N \geq N_n$ there is a polytope P_N in $\mathbb{R}^n$ with N vertices such that*

$$\Delta_s(B_2^n, P_N) \leq c\,\frac{\mathrm{vol}_{n-1}\left(\partial B_2^n\right)}{N^{\frac{2}{n-1}}}.$$

Here, B_2^n is the n-dimensional Euclidean unit ball with boundary $S^{n-1} = \partial B_2^n$. Moreover, throughout the paper a, b, c, c_1, c_2 will denote positive absolute constants that may change from line to line.

The proof of Theorem 1 is based on a random construction. A crucial step in its proof is a result by J. Müller [21] on the surface deviation of a polytope *contained* in the unit ball. It describes the asymptotic behavior of the surface deviation of a random polytope P_N, the convex hull of N randomly (with respect to the uniform measure) and independently chosen points on the boundary of the unit ball as the number of vertices increases. It says that

$$(1.2)\quad \lim_{N\to\infty} \frac{\mathrm{vol}_{n-1}(S^{n-1}) - \mathbb{E}(\partial P_N)}{N^{-\frac{2}{n-1}}} = \frac{n-1}{n+1}\,\frac{\Gamma\left(n+\frac{2}{n-1}\right)}{2(n-2)!}\,\frac{\left(\mathrm{vol}_{n-1}\left(\partial B_2^n\right)\right)^{\frac{n+1}{n-1}}}{\left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{\frac{2}{n-1}}},$$

where $\mathbb{E}(\partial P_N)$ denotes the expected surface area of P_N.

The right hand side of (1.2) is of order $c\,n\mathrm{vol}_{n-1}(\partial B_2^n)$. Thus, dropping the restriction that P_N is contained in B_2^n improves the estimate by a factor of dimension. The same phenomenon was observed for the volume deviation in [20].

A lower bound can be formulated as follows.

Proposition 2. *For all $\delta > 0$ there is an $N_1(\delta)$ such that for all polytopes P_N with $N \geq N_1(\delta)$ vertices, we have*

$$\Delta_s(B_2^n, P_N) \geq n(1-\delta)\frac{\mathrm{ldel}_{n-1}}{2}\frac{\left(\mathrm{vol}_{n-1}\left(\partial B_2^n\right)\right)^{\frac{n+1}{n-1}}}{N^{\frac{2}{n-1}}}.$$

Here, ldel_{n-1} is a constant that depends only on the dimension, $\frac{c_1}{n} \leq \mathrm{ldel}_{n-1} \leq c_2$ where c_1 and c_2 are absolute constants. The left estimate for ldel_{n-1} is due to [2], the right to [20]. As $\left(\mathrm{vol}_{n-1}\left(\partial B_2^n\right)\right)^{\frac{2}{n-1}} \geq \frac{\pi e}{n}$, we thus get that

$$\Delta_s(B_2^n, P_N) \geq n(1-\delta)\frac{\pi e}{n}\,\frac{\mathrm{ldel}_{n-1}}{2}\,\frac{\mathrm{vol}_{n-1}\left(\partial B_2^n\right)}{N^{\frac{2}{n-1}}} \geq \frac{c}{n}\,\frac{\mathrm{vol}_{n-1}\left(\partial B_2^n\right)}{N^{\frac{2}{n-1}}}.$$

Thus, there is a gap of the order of dimension between upper and lower bounds. A similar gap, in the vertex situation, also remains for the volume deviation comparing results in [2] and [20].

For all proofs and more details we refer to [18]

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Some new results on affine invariant points

Olaf Mordhorst

This talk is based on the preprint [8]. We denote by $\mathcal{K}_d$ the set of d-dimensional convex bodies in $\mathbb{R}^d$ and we equip this set with the Hausdorff distance d_H:

$$d_H(C_1, C_2) = \inf\{\lambda > 0 : C_1 \subseteq C_2 + \lambda B_2^d \wedge C_2 \subseteq C_1 + \lambda B_2^n\} \quad .$$

An affine invariant point is a map $p : \mathcal{K}_d \to \mathbb{R}^d$ which is continuous with respect to the Hausdorff distance and the euclidean norm and such that for every $C \in \mathcal{K}_d$ and every T invertible, affine map we have $p(T(C)) = T(p(C))$. This notion was introduced by B. Grünbaum in his seminal paper on measures of symmetry of convex bodies [1]. Well-known examples of affine invariant points are the centroid, the Santaló point, the John point and the Löwner point. Let us denote by $\mathfrak{P}_d$ the set of affine invariant points on $\mathcal{K}_d$, by

$$\mathfrak{P}_d(C) = \{p(C) : C \in \mathfrak{P}_d\}$$

and by

$$\mathfrak{F}_d(C) = \{x \in \mathbb{R}^d : Tx = x \text{ for every } T \text{ affine map with } T(C) = C\}$$

where $C \in \mathcal{K}_d$. Note that the three sets $\mathfrak{P}_d \subseteq C(\mathcal{K}_d, \mathbb{R}^d)$, $\mathfrak{P}_d(C) \subseteq \mathbb{R}^d$ and $\mathfrak{F}_d(C) \subseteq \mathbb{R}^d$ are affine subspaces. The set $\mathfrak{F}_d(C)$ provides symmetry information for the convex body C in the sense that C is considered to be more symmetric the smaller the affine dimension of $\mathfrak{F}_d(C)$ is. Examples of convex bodies where $\mathfrak{F}_d(C)$ is only a singleton are centrally symmetric bodies and simplices. We present a proof of the following theorem which was conjectured by Grünbaum in [1]:

Conjecture 1. *For every $K \in \mathcal{K}_d$ we have $\mathfrak{P}_d(K) = \mathfrak{F}_d(K)$.*

The inclusion $\mathfrak{P}_d(K) \subseteq \mathfrak{F}_d(K)$ follows from an elementary computation and the hard part is therefore to show that there are enough affine invariant points to ensure $\mathfrak{P}_d(C) \supseteq \mathfrak{F}_d(C)$. Partial answers to this conjecture were obtained by P. Kuchment (see [3] and also [4] for an English translation), by M.Meyer, C. Schütt and E. Werner (see [6]) and by I. Iurchenko (see [2]). The proof of the conjecture is based on ideas of [3] where the problem was solved for points which are invariant under affine T where the linearity part is an element of a fixed compact subgroup of $GL(d)$. There is also a proof of the conjecture for similarity invariant points by reducing the problem to the setting of compact groups. Our approach is somewhat similar: Denote by $\mathcal{K}_d^J$ the set of convex bodies in John position then we have the following lemma:

Lemma 2. *Let $\tilde{p} : \mathcal{K}_d^J \to \mathbb{R}^d$ be a continuous map such that for every $L \in O(d)$ and $C \in \mathcal{K}_d^J$ we have $\tilde{p}(L(C)) = L\tilde{p}(C)$. Then there is an affine invariant point p with $p|_{K_d^J} = \tilde{p}$.*

With this extension lemma one can reduce the conjecture to just considering the compact group $O(d)$. Thereafter, the proof of the conjectur can be carried by an averaging argument over the group $O(d)$.
We discuss two applications of Theorem 1:

1. $\mathfrak{P}_d$ is an affine subspace of the vector space of continuous $\mathbb{R}^d$-valued functions on $\mathcal{K}_d$. Grünbaum asked whether the affine dimension of $\mathfrak{P}_d$ is finite. A negative answer to this conjecture was given recently by M. Meyer, C. Schütt and E Werner (see [6]). We will give an alternative proof of the infinite dimensionality of $\mathfrak{P}_d$ which turns out to be just a simple corollary of Theorem 1.

2. In [7] the notion of dual affine invariant points was introduced. An affine invariant point p is proper if for every $C \in \mathcal{K}_d$ we have $p(C) \in \text{int}(C)$. We define:

Definition 3. *Let $p, q \in \mathfrak{P}_d$ be proper. We say that q is dual to p if for every $C \in \mathcal{K}_d$ we have $q((C - p(C))^o) = 0_d$.*

This notion generalizes two classical examples: We know that the Löwner point is dual to the John point and that the Santaló point is dual to the centroid. We have that q is dual to p if and only if p is dual to q (see [7]) and this explains why this notions is called duality. In [7] an example of a proper affine invariant point with no dual is given. The question arises naturally whether there are more proper points with a dual or more proper points without a dual in some sense. We present a topological answer to this question which tells us morally that there are way more proper points without a dual. For this purpose we consider the following metric on $\mathfrak{P}_d$ which was first introduced in [6]:

$$\mathrm{dist}(p_1, p_2) = \sup_{B_2^d \subseteq C \subseteq dB_2^d} \|p_1(C) - p_2(C)\|_2$$

This metric is a natural choice because it generates the topology of uniform convergence on compacta on $\mathfrak{P}_d$. The result we present is:

Theorem 4. *There exists an open and dense set $W \subseteq \mathfrak{P}_d$ of affine invariant points with no dual.*

A key to the proof is the following lemma of [7]:

Lemma 5. *Let $p \in \mathfrak{P}_d$ be proper then p has a dual if and only if for every $C \in \mathcal{K}_d$ we have $p((C-z)^o) = 0_d$ for at most one $z \in \mathrm{int}(C)$*

A slight extension of Theorem 1 allows us to produce plenty of examples of affine invariant points where there is a C such that $p((C-z)^0) = 0_d$ for at least two $z \in \mathrm{int}(C)$, i.e. p with out a dual. This will turn out to be the major step to prove that the set of affine invariant points with no dual is of second category.

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1. “Irrational” Convexity, 2. Solutions of some Basic Operator Equations

Vitali D. Milman

The main goal of the talk is to show how some classical constructions in Geometry and Analysis appear, in a unique way, from elementary and very simple properties.

For example, the polarity relation is a very important and well-known construction in Convex Geometry. However, its properties to be an involution and to reverse the partial order of inclusion essentially define it in a uniquely. Similar results are true in the case of convex or log-concave functions, see [AM1], [AM2], [BS]. In the talk we use these geometric results in two ways.

1. As an introduction to similar results in Analysis.
2. To develop a new theory of ” irrational ” constructions of convex bodies.

An example of such construction is Molchanov’s Theorem ([M]): Let K be an arbitrary compact convex body containing the euclidean unit ball $\mathcal{D}_n$. Then there is a convex body Z which solves the equation

$$Z^\circ = Z + K \ ,$$

and this solution Z is unique, an additional fact observed by L. Rotem. The idea behind this result is the useful analogy, following the above ” ideology ” , that Z° should in some sense be considered as ” $\frac{1}{Z}$ ” because $r \to \frac{1}{r}$ is the involution reversing order on $\mathbb{R}^+$. And the solution of this equation follows the same basic line as the quadratic equation $\frac{1}{x} = x+1$, using continued fractions. Another remarkable construction is the one of the ” geometric mean ” $g(K,T)$ of two compact convex bodies K and T containing 0 in their interior. It starts by introducing the ” harmonic mean ” of K and T by

$$H(K,T) = \left(\frac{K^\circ + T^\circ}{2} \right)^\circ ,$$

which was already considered by Firey [F]. Then an iteration procedure is applied by constructing

$$A_1 := A_1(K,T) = \frac{K+T}{2} \ , \ H_1 := H_1(K,T) = H(K,T) \ ,$$
$$A_2 := A_2(K,T) = \frac{A_1+H_1}{2} \ , \ H_2 := H_2(K,T) = H(A_1,H_1) \ ,$$
$$\dots$$
$$A_n := A_n(K,T) = \frac{A_{n-1}+H_{n-1}}{2} \ , \ H_n := H_n(K,T) = H(A_{n-1},H_{n-1}) \ .$$

Note that $\lim_n A_n = \lim_n H_n =: g(K,T)$ exists. Applying the procedure to numbers $x>0$ and $y>0$ it is well-known that it converges to $g(x,y) = \sqrt{xy}$. See [MR] for details and discussion. Since the role of 1 is played by $\mathcal{D}_n$, the set $g(\mathcal{D}_n, K)$ may be seen as ” $\sqrt{K}$ ” .

We show a few more similar constructions and note that in almost all cases we cannot visualize the new bodies which we recover from the classical ones, as we cannot visualize irrational numbers we usually create solving equations with rational coefficients.

We start the second part, the Analysis part, by characterizing the Fourier transform (on the Schwartz class in $\mathbb{R}^n$) as essentially the only bijective map which transforms the product to the convolution, see [AAFM], [AFM]. Then we show that the Chain Rule Operator Equation characterize the derivation operation in a very strong sense, see [AKM]. A series of very surprising results on the rigidity of the chain rule are found in [KM1], [KM2]. There are more results of a similar nature in the lecture which indicate further characterizations of operations in Analysis in this line of research.

The results of the geometric part are mostly joint with Shiri Artstein-Avidan and Liran Rotem. The ones in the analytic part are joint work with Shiri Artstein-Avidan and Hermann König.

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Approximation of convex bodies by polytopes and the complexity of entanglement detection

Stanisław J. Szarek
(joint work with Guillaume Aubrun)

In this talk we present a link between Dvoretzky's theorem [1] in its tangible version due to Milman [7] and the problem of entanglement detection in quantum information theory. Specifically, we use the inequality of Figiel-Lindenstrauss-Milman (1977) giving bounds on the number of vertices/faces of polytopes, which we interpret as a result on approximating of convex bodies by polytopes. We also identify some peculiarities in so approximating various sets that appear naturally in the non-commutative context.

This inequality, essentially contained in [2], asserts that, for any n-dimensional convex body K,

$$\dim_F(K) \cdot \dim_V(K) \cdot a(K)^2 = \Omega(n^2) \tag{1.1}$$

where the parameters $\dim_F(\cdot)$, $\dim_V(\cdot)$ and $a(\cdot)$, defined below, are three different ways to quantify the complexity of a convex body. If we know that two of these parameters are small, the third has to be large: complexity must not entirely disappear. Our main result, Theorem 1, is proved by applying this strategy to K equal to the set of separable ($\Leftrightarrow$ unentangled) states.

Here are the precise definitions of the parameters appearing in (1.1). The first two, dual to each other, are the *verticial dimension* and the *facial dimension*, respectively defined as

$$\dim_V(K) = \log\inf\{N \;:\; \text{there is a translate } \tilde{K} \text{ of } K \text{ and a polytope } P \text{ with } N \text{ vertices such that } \tilde{K} \subset P \subset 4\tilde{K}\}$$

and

$$\dim_F(K) = \log\inf\{N \;:\; \text{there is a polytope } Q \text{ with } N \text{ facets with } \tilde{K} \subset Q \subset 4\tilde{K}\},$$

where by facets we mean faces of dimension $n-1$. The third parameter is the *asphericity* and measures how much the convex body differs from a Euclidean ball. Since one may consider – as Plato did – the sphere to be the ideal of simplicity, the asphericity is another way to quantify complexity. It is defined as

$$a(K) = \inf\left\{\frac{R}{r} \;:\; \text{there is a 0-symmetric ellipsoid } \mathcal{E} \text{ with } r\mathcal{E} \subset \tilde{K} \subset R\mathcal{E}\right\}.$$

We use ellipsoids instead of spheres to obtain quantities invariant under linear or affine transformations. As an illustration, Table 1 contains estimates of these parameters for a selection of (families of) convex bodies. The first three examples (the ball, the cube and the simplex) are well-known, easy to establish, and included only to provide a perspective. The main technical part of our work are the bounds from the last two rows of the table that concern the set of states (i.e., positive operators of trace 1) and that of separable states (denoted Sep) two objects which appear naturally in quantum theory. The arguments appeal to inequality (1.1) and use other fairly standard considerations. However, we do encounter some surprises, primarily due to the lack of central symmetry of the sets

	dimension	$a(K)$	$\dim_V(K)$	$\dim_F(K)$
Euclidean ball B_2^n	n	1	$\Theta(n)$	$\Theta(n)$
Cube $[-1,1]^n$	n	$\sqrt{n}$	$\Theta(n)$	$\Theta(\log n)$
Simplex in $\mathbb{R}^n$	n	n	$\Theta(\log n)$	$\Theta(\log n)$
Set of states on $\mathbb{C}^m$	m^2-1	$m-1$	$\Theta(m)$	$\Theta(m)$
$\mathrm{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$	d^4-1	d^2-1	$\Theta(d\log d)$	$\Omega(d^3/\log d)$

TABLE 1. Parameters appearing in (1.1) for some families of convex bodies. Simlar bounds hold if, in our definitions of $\dim_F$ and $\dim_V$, we use in place of 4 any other number greater than 1.

A fundamental problem in quantum theory is to decide whether a given state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$ is entangled or not. It is known that this question is inherently difficult, specifically NP-hard [4, 6, 3]. However, such complexity results usually focused on "boundary effects," i.e., on states located very close to the entanglement/separability border. In order to ignore such effects, we restrict ourselves to *robustly entangled* states, i.e., the states ρ with the property that the noisy mixture $\frac{1}{2}(\rho + \rho_*)$ is still entangled, where ρ_* denotes the maximally mixed state. In this setting, the lower bound implicit in the last entry of the last row in Table 1 implies that any family of linear functional that separate an arbitrary robustly entangled state from Sep must be of cardinality at least $\exp(cd^3/\log d)$, where $c > 0$ is a universal constant.

Our main result gives an identical lower bound for a more sophisticated scheme of detecting entanglement, based on the well-known Horodecki criterion [5]: a state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$ is entangled if and only if there exists a positive map $\Phi : \mathsf{M}_d \to \mathsf{M}_d$ (a.k.a. entanglement witness) such that the operator $(\Phi \otimes \mathrm{Id})(\rho)$ is not positive. For $d = 2$ this approach leads to the elegant fact, the Peres-Horodecki partial transposition criterion: the transposition is a universal witness, i.e., detects all entangled states. However this statement has no higher-dimensional analogue and fails dramatically for large d. We have

Theorem 1. *There is a universal constant $c > 0$ such that the following holds. Suppose that $(\Phi_i)_{1\le i\le N}$ is a family of positive maps on M_d which has the property that for any robustly entangled state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$ there exists $1 \le i \le N$ such that $(\Phi_i \otimes \mathrm{Id})(\rho)$ is not positive. Then $N \ge \exp(cd^3/\log d)$.*

Since implementations of the quantum information/computing protocols are likely to involve substantial noise, the robust setting is arguably more to the point than the one involving boundary effects.

Once the entries from the last two rows of Table 1 are determined, the path towards Theorem 1 is relatively straightforward. By the bound on the facial dimension of the set of states on $\mathbb{C}^d$, each positive map $\Phi : \mathsf{M}_d \to \mathsf{M}_d$ may introduce only $\exp(O(d^2))$ facets; since $\exp(\Omega(d^3/\log d))$ facets (linear functionals) are required to approximate the set of separable states, many maps are needed.

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On the condition number of random polynomial systems

Grigoris Paouris

(joint work with A. Ergür and M. Rojas)

In a series of three papers ([1], [2], [3]) F. Cucker, T. Krick, G. Malajovich and M. Wschebor, introduced an algorithm that counts the number of real solutions of a system $\mathbf{p} := (p_1, \cdots, p_n)$ of real polynomial equations. Their work builds on the "alpha-theory" introduced by the work on Smale on Newton iteration. In particular the complexity of the algorithm depends on the condition number of the polynomial system a notion that has been introduced in [2] as the real-case analogue of the condition number of complex polynomial systems introduced by Shub and Smale in [6].

In [3] the authors proved an upper estimate for the expectation of the logarithm of the condition number of a random homogeneous polynomial system, i.e. the coefficients of the monomial x^α of the polynomials are independent centered Gaussian with variance $\binom{d_i}{\alpha}$. (α is a multi-index and d_i is the the degree of the i-th polynomial.

In this work we consider the problem of estimating the expectation of the logarithm of the condition number of a random polynomial system under more general assumptions on the randomness. Our methods are inspired by the recent work in non-asymptotic theory of random matrices and in particular of the work of Rudelson and Vershynin [5]. Let $\mathbf{p} := (p_1, \cdots, p_{n-1}$ be a random polynomial system of $n-1$ polynomials of degrees d_i in n variables, where

$$p_i := \sum_{|\alpha|=d_i} c_\alpha^{(i)} \sqrt{\binom{d_i}{\alpha}} x^\alpha,$$

where $C_i := (c_\alpha^{(i)})_{|\alpha|=d_i}$ is a random vector in R^{D_i} $(D_i := \binom{n+d_i-1}{d_i})$ that satisfies the following

(1) C_i are "subgaussian" with constant K, i.e. $\mathbb{P}\left(|\langle C_i, \theta\rangle| \geq t\right) \leq 2e^{-\frac{t^2}{K^2}},\ t > 0$,
(2) C_i satisfy a "small ball probability estimate" with constant c_0, i.e. $\mathbb{P}\left(|\langle C_i, \theta\rangle| \leq \varepsilon\right) \leq c_0\varepsilon,\ \varepsilon > 0$.

Under the above assumptions we have proved the following

Theorem 1. [4] *Let* $\mathbf{p} := (p_1, \cdots, p_{n-1})$ *be a random polynomial system with degrees* $d_1, \cdots, d_{n-1}$. *Let* $D_i := \binom{n+d_i-1}{d_i}$, $N := \sum_{i=1}^{n-1} D_i$ *and* $k(p)$ *be the condition number of* $\mathbf{p}$. *Then*

$$\mathbb{E}(\log k(p)) \leq \log M + c$$

where $M := \sqrt{N} M (c_0 K d^2 \log d)^{n-2}$ *and* $c > 0$ *an absolute constant.*

The above estimate is up to universal constant the same with the estimate that it was proved in [3]. However the above result can be extended to the case of overdetermined systems. More importantly the above theorem can be extended to the case of "sparse" polynomial systems.

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Quantitative versions of Helly's theorem and related questions

Apostolos A. Giannopoulos

We present new quantitative versions of Helly's theorem, due to Silouanos Brazitikos. Recall that the classical result asserts that if $\mathcal{F} = \{F_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if any $n+1$ members of $\mathcal{F}$ have non-empty intersection then $\bigcap_{i\in I} F_i \neq \emptyset$. Variants of this statement have found important applications in discrete and computational geometry.

Quantitative Helly-type results were first obtained by Bárány, Katchalski and Pach. In particular, they proved the following volumetric result:

Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\left|\bigcap_{i\in I} P_i\right| > 0$. There exist $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant n^{2n^2} \left|\bigcap_{i\in I} P_i\right|.$$

The example of the cube $[-1,1]^n$ in $\mathbb{R}^n$, expressed as an intersection of exactly $2n$ closed half-spaces, shows that one cannot replace $2n$ by $2n-1$ in the statement above. Naszódi has recently proved a volume version of Helly's theorem with a constant $\leqslant (cn)^{2n}$, where $c > 0$ is an absolute constant. In fact, a slight modification of Naszódi's argument leads to the exponent $\frac{3n}{2}$ instead of $2n$. In [5], relaxing the requirement that $s \leqslant 2n$ to the weaker one that $s = O(n)$, Brazitikos has improved the exponent to n:

Theorem 1 (Brazitikos). *There exists an absolute constant $\alpha > 1$ with the following property: for every family $\{P_i : i \in I\}$ of closed convex sets in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \ldots, i_s \in I$ such that*

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (cn)^n\, |P|,$$

where $c > 0$ is an absolute constant.

The proof of Theorem 1 involves a theorem of Srivastava and the following approximate geometric Brascamp-Lieb inequality (we state below its reverse counterpart too).

Theorem 2 (Brazitikos). *Let $\gamma > 1$. Assume that $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m > 0$ satisfy*

$$I_n \preceq A := \sum_{j=1}^{s} c_j u_j \otimes u_j \preceq \gamma I_n$$

and set $\kappa_j = c_j \langle A^{-1}u_j, u_j\rangle > 0$, $1 \leqslant j \leqslant m$. If $f_1, \ldots, f_m : \mathbb{R} \longrightarrow [0, +\infty)$ are integrable functions then

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{\kappa_j}(\langle x, u_j\rangle)dx \leqslant \gamma^{\frac{n}{2}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j}.$$

Also, if $w, h_1, \ldots, h_m : \mathbb{R} \longrightarrow [0,\infty)$ *are integrable functions and* $w(x) \geqslant \sup\left\{ \prod_{j=1}^m h_j^{\kappa_j}(\theta_j) : \theta_j \in \mathbb{R}\ ,\ x = \sum_{j=1}^m \theta_j c_j u_j \right\}$*, then*

$$\int_{\mathbb{R}^n} w(x)dx \geqslant \gamma^{-\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j(t)\, dt \right)^{\kappa_j}.$$

Note that a continuous version of Theorem 2 can be also obtained. We say that a Borel measure ν on S^{n-1} is a *γ-approximation of an isotropic measure* (for some $\gamma > 1$) if

$$I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u\, d\nu(u) \preceq \gamma I_n.$$

Following Barthe's argument for the isotropic case and a generalization of the so-called Ball-Barthe lemma (proved by Lutwak, Yang and Zhang for isotropic measures on the sphere) one can obtain a continuous Brascamp-Lieb inequality and its reverse form for a γ-approximation of an isotropic measure.

Theorem 3 (Brazitikos-Giannopoulos)**.** *Let ν be a γ-approximation of an isotropic Borel measure on S^{n-1} and let (f_u), $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \longrightarrow [0,+\infty)$ that satisfy natural continuity consitions. Then,*

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x,u\rangle) \langle T_\nu^{-1}u,u\rangle d\nu(u) \right) dx \leqslant \gamma^{\frac{n}{2}} \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u \right) \langle T_\nu^{-1}u,u\rangle\, d\nu(u) \right).$$

Also, if h is a measurable function such that

$$h\left(\int_{S^{n-1}} \theta(u) u\, d\nu(u) \right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u)) \langle T_\nu^{-1}u,u\rangle\, d\nu(u) \right)$$

for every integrable function θ, then

$$\gamma^{\frac{n}{2}} \int_{\mathbb{R}^n} h(y)\, dy \geqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u \right) \langle T_\nu^{-1}u,u\rangle\, d\nu(u) \right).$$

Bárány, Katchalski and Pach proved a quantitative Helly-type theorem for the diameter in place of volume:

Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\mathrm{diam}\left(\bigcap_{i\in I} P_i \right) = 1$. *There exist $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that*

$$\mathrm{diam}\,(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant (cn)^{n/2},$$

where $c > 0$ is an absolute constant.

In the same work the authors conjecture that the bound should be polynomial in n; in fact they ask if $(cn)^{n/2}$ can be replaced by $c\sqrt{n}$. Relaxing the requirement that $s \leqslant 2n$, and using a similar strategy as in [5], Brazitikos proved in [6] the following:

Theorem 4 (Brazitikos)**.** *There exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$*, then there exist $z \in \mathbb{R}^n$, $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq cn^{3/2}\left(z + \bigcap_{i\in I} P_i\right),$$

where $c > 0$ is an absolute constant.

It is clear that Theorem 4 implies polynomial estimates for the diameter:

Theorem 5 (Brazitikos)**.** *There exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$*, then there exist $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{3/2},$$

where $c > 0$ is an absolute constant.

The proof of Theorem 4 is based on the following non-symmetric version of a lemma of Barvinok: There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there is a subset $X \subset \mathrm{bd}(K) \cap S^{n-1}$ of cardinality $\mathrm{card}(X) \leqslant \alpha n$ such that

$$B_2^n \subseteq cn^{3/2}\mathrm{conv}(X),$$

where $c > 0$ is an absolute constant. The fact that X is a set of contact points of K and its Löwner ellipsoid is crucial for the application in Theorem 4. Note that a random analogue can be obtained with a better dependence on the dimension (see [8]).

Theorem 6 (Brazitikos-Chasapis-Hioni)**.** *There exists an absolute constant $\beta > 1$ with the following property: if K is a convex body in $\mathbb{R}^n$ whose center of mass is at the origin, if $N = \lceil \beta n \rceil$ and if $x_1, \dots, x_N$ are independent random points uniformly distributed in K then, with probability greater than $1 - e^{-n}$ we have*

$$K \subseteq c_1 n\, \mathrm{conv}(\{x_1, \dots, x_N\}),$$

where $c_1 > 0$ is an absolute constant.

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Random points in halfspheres

Rolf Schneider
(joint work with Imre Bárány, Daniel Hug, Matthias Reitzner)

Let K be a d-dimensional convex body in $\mathbb{R}^d$, let $X_1, \dots, X_n$ be stochastically independent random points in K with uniform distribution, and let P_n be their convex hull. The random polytope P_n has been thoroughly investigated (see, e.g., the surveys [2], [5], [6]). For basic functionals φ of polytopes, such as f_j, the number of j-faces, or V_j, the jth intrinsic volume, the expectations $\mathbb{E}\,\varphi(P_n)$ have been studied, mainly for $n \to \infty$. The asymptotic behaviour depends strongly on the boundary structure of K. For example, if K is a polytope, then

$$\mathbb{E} f_j(P_n) \sim c_1(d,j,K) \log^{d-1} n,$$

and if K is a body of class C^2_+, then

$$\mathbb{E} f_j(P_n) \sim c_2(d,j,K) n^{\frac{d-1}{d+1}},$$

as proved by Reitzner [4]. For K of class C^2_+,

$$V_j(K) - \mathbb{E} V_j(P_n) \sim c_3(d,j,K) n^{-\frac{2}{d+1}},$$

see Reitzner [3]. (The dependence of the constants on K has been made more precise.) The crucial observation to be made here is that the asymptotic order does not depend on j, but is very different for polytopes and for smooth bodies (also for V_j, in the known cases).

This was the motivation to study in [1] the analogous question in d-dimensional spherical space (with ‘convex’ replaced by ‘spherically convex’), for the special convex body K given by a closed halfsphere. Since this body is at the same time a (spherical) polytope and smooth, new phenomena were to be expected.

Indeed, the asymptotic behaviour is decidedly different. A first surprise is that the expected face numbers remain bounded. For the number of facets we obtain (with κ_d = volume of the d-dimensional unit ball, and $\omega_d = d\kappa_d$)

$$\mathbb{E} f_{d-1}(P_n) = \frac{2\omega_d}{\omega_{d+1}} \binom{n}{d} \int_0^\pi \left(1 - \frac{\alpha}{\pi}\right)^{n-d} \sin^{d-1} \alpha \, \mathrm{d}\alpha$$

and hence

$$\lim_{n\to\infty} \mathbb{E} f_{d-1}(P_n) = 2^{-d} d! \kappa_d^2.$$

Similarly precise limit relations for $\mathbb{E} f_j(P_n)$, $j < d-1$, are not known, but we were able to show the existence of the limit

$$\lim_{n\to\infty} \mathbb{E} f_0(P_n)$$

and express it by some multiple integral (with explicit values in dimensions two and three).

As counterparts to the Euclidean intrinsic volumes V_j we may consider the spherical quermassintegrals. For a spherically convex body M lying in some open hemisphere, they are defined by

$$U_j(M) = \frac{1}{2} \int_{G(d+1,d+1-j)} \mathbf{1}(L \cap M \neq \emptyset)\, \nu_{d+1-j}(\mathrm{d}L)$$

for $j = 1, \dots, d$. Here we consider the underlying sphere as the unit sphere of $\mathbb{R}^{d+1}$, the space $G(d+1,k)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^{d+1}$, and ν_k is the Haar probability measure on $G(d+1,k)$. Then U_d is a constant multiple of the spherical volume of M, which we denote by $\lambda(M)$, and $U_{d-1}(M)$ is a constant multiple of the spherical surface area of M, denoted by $\sigma(M)$. The functional U_1 is called the spherical mean width. These functionals can be extended to more general sets, in particular, $\lambda(K) = \omega_{d+1}/2$, $\sigma(K) = \omega_d$, $U_1(K) = 1/2$ (where K is still the closed halfsphere). For these three functionals, we have the following results.

For the spherical mean width:

$$U_1(K) - \mathbb{E} U_1(P_n) = \frac{\omega_d}{\omega_{d+1}} \int_0^\pi \left(1 - \frac{\alpha}{\pi}\right)^n \sin^{d-1} \alpha \, \mathrm{d}\alpha$$

and hence

$$U_1(K) - \mathbb{E} U_1(P_n) \sim \frac{\omega_d}{\omega_{d+1}} (d-1)! \pi^d n^{-d}.$$

For the surface area, there is a similar formula, yielding the asymptotic result

$$\sigma(K) - \mathbb{E}\sigma(P_n) \sim \omega_d \binom{d+1}{3} \pi^2 n^{-2}.$$

For the volume:

$$\lambda(K) - \mathbb{E}\lambda(P_n) \sim C(d) \pi^{d+1} \left(\frac{2}{\omega_{d+1}}\right)^d \omega_d n^{-1},$$

with $C(d)$ expressed by a multiple integral.

The interesting fact in the preceding results is that, in contrast to the case of convex bodies in Euclidean space, the asymptotic orders are different in each case. This motivates the following conjecture.

Conjecture. For U_j with $j \in \{1, \dots, d\}$, the expectation $\mathbb{E}(U_j(K) - U_j(P_n))$ is of order $n^{-(d+1-j)}$, as $n \to \infty$.

If one is interested in the approximation of K by P_n, consideration of the Hausdorff metric is natural. Let δ_s denote the Hausdorff metric on nonempty compact subsets of the sphere that is induced by the usual spherical distance. From our result on the expected volume of P_n it is easy to deduce that there are two constants c_1, c_2, depending only on the dimension, such that

$$c_1 n^{-1} \leq \mathbb{E}\, \delta_s(P_n, K) \leq c_2 n^{-1}.$$

For slightly differently defined sequences of random polytopes, we can provide information on the almost sure behaviour of the Hausdorff distance. First, we assume that $X_1, X_2, \dots$ is a sequence of i.i.d. uniform random points in K, and we define P_n as the spherical convex hull of $X_1, \dots, X_n$. Then we can show that there is a constant c, depending only on the dimension, such that

$$\mathbb{P}\left(\delta_s(P_n, K) \leq cn^{-1} \log n \text{ for almost all } n\right) = 1.$$

In the other direction, we can only show that, for any $\gamma > 2$,

$$\mathbb{P}\left(\delta_s(P_n, K) \geq cn^{-\gamma} \text{ for almost all } n\right) = 1.$$

Second, we assume that for each $n \in \mathbb{N}$, the points $X_1^{(n)}, \dots, X_n^{(n)}$ are i.i.d. uniform random in K, that P_n is their spherical convex hull, and that the sequence $P_1, P_2, \dots$ is independent. Under this assumption, there is a constant c, depending only on the dimension, such that

$$\mathbb{P}\left(\delta_s(P_n, K) \geq cn^{-1} \log n \text{ for infinitely many } n\right) = 1,$$

and there is a number $0 < \varepsilon < 1$, depending only on the dimension, such that

$$\mathbb{P}\left(\delta_s(P_n, K) \geq cn^{-(1+\varepsilon)} \text{ for infinitely many } n\right) = 1.$$

The fact that the orders do not match suggests some open questions.

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The number of perfect matchings in deterministic graphs via random matrices

Mark Rudelson
(joint work with Alex Samorodnitsky and Ofer Zeitouni)

This is a report on papers [2, 3].

Let $G = (V, E)$ be a graph with an even number of vertices. A perfect matching in the graph G is a partition of the set of vertices into pairs such that the vertices in each pair are connected by an edge. The existence of a perfect matching in a given graph can be efficiently checked. However, the computation of the number of perfect matchings is believed to be a hard problem. More precisely, Valiant proved that this computation is #-P hard even for a bipartite graph.

Algebraically, the number of perfect matchings of a bipartite graph is represented by the *permanent* of its adjacency matrix. Let A be a $\{0, 1\}$-matrix with $A_{i,j} = 1$ whenever the left vertex i is connected to the right vertex j. Then

$$\#\text{perfect matchings} = \text{per}(A) = \sum_{\sigma \in \mathcal{S}_n} \prod_{j=1}^{m} A_{j,\,\sigma(j)}$$

Since exact calculation of this quantity is hard, one considers an approximate evaluation. There is an extensive literature on the approximate evaluation of the permanent. The best deterministic result belongs to Linial, Samorodnitsky, and Wigderson (LSW), who devised a completely polynomial algorithm allowing to rescale the given matrix to a doubly stochastic form. Since the permanent scales accordingly, it reduces the general problem to estimating permanents of doubly stochastic matrices. The known combinatorial bounds in combination with the LSW algorithm allow to estimate the number of perfect matchings with the multiplicative error at most e^n. An alternative probabilistic method developed by Jerrum, Sinclair, and Vigoda, estimates the permanent with a constant error with high probability. However, its running time is $O(n^{10})$ which is prohibitively long.

Based on the previous construction of Godsil and Gutman, Barvinok [1] suggested an new estimator of the permanent (and so of the number of perfect matchings) relying on random matrices. Let V be an $n \times n$ matrix with independent $N(0, 1)$ entries $V_{i,j}$ if the left vertex i is connected to the right vertex j and 0 otherwise. It is easy to see that

$$\#\text{perfect matchings} = \text{per}(A) = \mathbb{E} \det^2(V).$$

Therefore, sampling V and calculation of the determinant provides an unbiased estimator for the number of perfect matchings. The later can be performed efficiently making this estimator the fastest known algorithm. Barvinok proved that with high probability, the multiplicative error of it does not exceed 3.6^n for real Gaussian matrices and 1.8^n for complex ones. These bounds cannot be improved in general. However, the error of Barvinok's estimator is subpolynomial for the complete bipartite graph. This suggests that for some "nice" graphs the error can be subexponential. Proving it requires establishing measure concentration bounds for random determinants. In [2], the authors were able to describe the class of graphs for which the error of Barvinok's estimator is subexponential. Namely, if for any fixed $\delta > 0$, the minimal degree of the bipartite graph is at least δn and the graph has a certain expansion property, then the error of the Barvinok estimator does not exceed $\exp(C\sqrt{n \log n})$ with high probability.

In paper [3], the authors address a more difficult problem of estimating the number of perfect matchings in a general graph with an even number of vertices. For such graphs, many deterministic and probabilistic methods developed for bipartite graphs fail to provide an estimate for the number of perfect matchings. This leaves the Barvinok estimator essentially the only tool capable of obtaining such bounds. The difficulty of calculation of the number of perfect matchings stems from the fact that it is no longer the permanent of the adjacency matrix, but a more complicated quantity called hafnian:

$$\mathrm{haf}(A) = \frac{1}{(n/2)! \cdot 2^{n/2}} \sum_{\sigma \in \mathcal{S}_n} \prod_{j=1}^{n/2} A_{\sigma(2j-1),\sigma(2j)}.$$

Correspondingly, the random matrix in Barvinok's estimator should be different [1]. It is an $n \times n$ *skew-symmetric* matrix W with independent $N(0,1)$ entries $W_{i,j}$ above the main diagonal wherever the vertices i and j are connected by an edge and 0 otherwise. Similarly to the previous case, it is easy to show that

$$\#\text{perfect matchings} = \mathrm{haf}(A) = \mathbb{E}\det(W).$$

Thus, providing a probabilistic guarantee for the Barvinok estimator in case of the general graph boils down to establishing the concentration of a random determinant as in the previous case. However, the approach to this problem is entirely different as the spectral properties of a random skew-symmetric matrix differ significantly from the case of independent entries.

We prove a subexponential error guarantee with high probability for graphs with minimal degree at least δn possessing the strong expander property. This property is a strengthening of the vertex expansion property $|\partial_V A| \geq \kappa |A|$. Instead of this property, we require that

$$|\partial_V A| - |\mathrm{Comp}(A)| \geq \kappa |A|,$$

where $\mathrm{Comp}(A)$ is the set of connected components of A. Intuitively, this condition means that highly disconnected sets should expand faster. We also show that this property is essentially optimal, in particular, it cannot be replaced by the standard vertex expansion.

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On the Brunn-Minkowski inequality for general measures

Artem Zvavitch
(joint work with Galyna Livshyts, Arnaud Marsiglietti, Piotr Nayar)

In this talk we presented a joint work with Galyna Livshyts (Georgia Institute of Technology), Arnaud Marsiglietti (University of Minnesota) and Piotr Nayar (University of Pennsylvania)

The classical Brunn-Minkowski inequality states that for any two non-empty compact sets $A, B \subset \mathbb{R}^n$ and any $\lambda \in [0, 1]$ we have

$$\text{vol}_n(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda \text{vol}_n(A)^{1/n} + (1-\lambda)\text{vol}_n(B)^{1/n}. \tag{1.1}$$

Here vol_n stands for the Lebesgue measure on $\mathbb{R}^n$ and $A+B = \{a+b : a \in A, b \in B\}$ is the Minkowski sum of A and B. The Brunn-Minkowski inequality turns out to be a powerful tool. In particular, it implies the classical isoperimetric inequality, Brunns theorem and many other useful facts. Using the inequality between means one gets an a priori weaker dimension free form of (1.1), namely

$$\text{vol}_n(\lambda A + (1-\lambda)B) \geq \text{vol}_n(A)^{\lambda}\text{vol}_n(B)^{1-\lambda}. \tag{1.2}$$

In fact (1.2) and (1.1) are equivalent! This amazing phenomenon is a consequence of homogeneity of the Lebesgue measure.

A measure μ is called log-concave if for any compact sets $A, B \subset \mathbb{R}^n$ we have

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}. \tag{1.3}$$

We say that the support of a measure μ is non-degenerate if it is not contained in any affine subspace of $\mathbb{R}^n$ of dimension less than n. It was proved by Borell that a measure μ, with non-degenerate support, is log-concave if and only if it has a log-concave density, i.e. a density of the form $\varphi = e^{-V}$, where V is convex (and may attain value $+\infty$). Inequality (1.2) says that the Lebesgue measure is log-concave.

We say that the measure μ is $1/n$ concave if for any compact sets $A, B \subset \mathbb{R}^n$ we have

$$\mu(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda\mu(A)^{1/n} + (1-\lambda)\mu(B)^{1/n}. \tag{1.4}$$

In general log-concavity does not imply $1/n$-concavity. Indeed, consider the standard Gaussian measure γ_n on $\mathbb{R}^n$, i.e., the measure with density $(2\pi)^{-n/2}\exp(-|x|^2/2)$. This density is clearly log-concave and therefore γ_n satisfies (1.3). To see that γ_n does not satisfy (1.4) it suffices to take $B = \{x\}$ and send $x \to \infty$.

One might therefore ask whether (1.4) holds true for γ_n if we restrict ourselves to some special class of subsets of $\mathbb{R}^n$. A few years ago Richard Gardner and the speaker conjectured that

$$\gamma_n(\lambda A+(1-\lambda)B)^{1/n} \geq \lambda\gamma_n(A)^{1/n}+(1-\lambda)\gamma_n(B)^{1/n} \tag{1.5}$$

holds true for any closed convex sets with $0 \in A\cap B$ and $\lambda \in [0,1]$ and verified this conjecture in the number of cases including when A and B are products of intervals containing the origin; when $A=[-a_1,a_2]\times\mathbb{R}^{n-1}$, where $a_1,a_2>0$ and B is arbitrary and when $A=aK$ and $B=bK$ where $a,b>0$ and K is a convex set, symmetric with respect to the origin.

It is interesting to note that the last case is related to the B-conjecture for Gaussian measures proposed by Banaszczyk and solved by Cordero-Erausquin, Fradelizi, and Maurey. It states that for any convex symmetric set K the function $t\mapsto\gamma_n(e^tK)$ is log-concave. The B-conjecture is asking the same question for the general class of the even log-concave measures. Cordero-Erausquin, Fradelizi, and Maurey proved that the conjecture is true for the case of unconditional log-concave measures and unconditional sets (see the definition below). Moreover, the conjecture has an affirmative answer for $n=2$ due to the works of Livne Bar-on and of Saroglou. Saroglou's proof is done by linking the problem to the new log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang and Zhang.

P. Nayar and T. Tkocz showed that in general (1.5) is false under the assumption $0\in A\cap B$, and conjectured that (1.5) should be true for origin-symmetric convex bodies A,B.

One of the most important Brunn-Minkowski type inequalities for the Gaussian measure is Ehrhard's inequality, which states that for any two non-empty compact sets $A,B\subset\mathbb{R}^n$ and any $\lambda\in[0,1]$ we have

$$\Phi^{-1}(\gamma_n(\lambda A+(1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma_n(A))+(1-\lambda)\Phi^{-1}(\gamma_n(B)), \tag{1.6}$$

where $\Phi(t)=\gamma_1((-\infty,t])$. This inequality has been considered for the first time by Ehrhard, where the author proved it assuming that both A and B are convex. Then Latała generalized Ehrhard's result to the case of arbitrary A and convex B. In its full generality, the inequality (1.6) has been established by Borell. Note that (1.5) is an inequality of the same type, with $\Phi(t)$ replaced with t^n, but none of them is a direct consequence of the other. The crucial property of Ehrhard's inequality is that it gives the Gaussian isoperimetry as a simple consequence.

In this talk we discussed inequality (1.4), for a different classes of measures. We first presented the following theorem

Theorem 1. *Let μ be an unconditional product measure with non-increasing density. Then μ satisfies (1.4) inequality in the class of unconditional convex bodies in $\mathbb{R}^n$.*

We remind that a function $f:\mathbb{R}^n\to\mathbb{R}$ is unconditional if for any choice of signs $\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}$ and any $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ we have $f(\varepsilon_1x_1,\dots,\varepsilon_nx_n)=f(x)$. We say that an unconditional function is non-increasing if for any $1\leq i\leq n$ and any real numbers $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n\geq 0$ the function $t\mapsto$

$f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ is non-increasing on $[0, \infty)$. Finally, a set $A \subseteq \mathbb{R}^n$ is called unconditional if the characteristic function of A is unconditional.

Next we discussed a link between the Brunn-Minkowski inequality and the log-Brunn-Minkowski inequality. To state our observation we need two remind the definition of 0-sum (log -sum) of two convex symmetric bodies:

$$\lambda A +_0 (1-\lambda)B = \{x \in \mathbb{R}^n : \langle x, u \rangle \le h_A^\lambda(u) h_B^{1-\lambda}(u), \ \forall u \in S^{n-1}\}.$$

Here h_A is the support function of A, i.e., $h_A(u) = \sup_{x \in A}\langle x, u\rangle$. We say that a Borel measure μ on $\mathbb{R}^n$ satisfies the log-Brunn-Minkowski inequality if for any convex, symmetric bodies $A, B \subset \mathbb{R}^n$ and for any $\lambda \in [0,1]$ we have $\mu(A +_0 \lambda B) \ge \mu(A)^\lambda \mu(B)^{1-\lambda}$.

We also proved the following proposition.

Proposition 1. *Suppose that a Borel measure μ with a radially non-increasing density f satisfies the log-Brunn-Minkowski inequality for a certain class of convex symmetric bodies, then μ satisfies the Brunn-Minkowski inequality in the same class of bodies.*

Böröczky, Lutwak, Yang and Zhang, proved the log-Brunn-Minkowski inequality for the Lebesgue measure and symmetric convex bodies on $\mathbb{R}^2$. Saroglou generalized the inequality to the case of measures with even log-concave densities on $\mathbb{R}^2$. Thus, as a consequence of Proposition 1 we get the following theorem.

Theorem 2. *Let μ be a measure on $\mathbb{R}^2$ with an even log-concave density. Then μ satisfies the Brunn-Minkowski inequality in the class of all symmetric convex sets in $\mathbb{R}^2$.*

Moreover, Cordero-Erausquin, Fradelizi, and Maurey proved an analog of log-Brunn-Minkowski inequality for unconditional log-concave measures and unconditional convex bodies, this gives us

Theorem 3. *Let μ be an unconditional log-concave measure on $\mathbb{R}^n$. Then μ satisfies the Brunn-Minkowski inequality in the class unconditional convex bodies in $\mathbb{R}^n$.*

Characterizing the Minkowski and radial additions

Liran Rotem

(joint work with Vitali Milman)

The content of this report is based on two papers – [6] and [5].

Denote by $\mathcal{K}_0^n$ the class of closed convex sets in $\mathbb{R}^n$. By an addition on $\mathcal{K}_0^n$ we mean a map $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ which:

- Is *associative*: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ for all $A, B, C \in \mathcal{K}_0^n$
- Has a two-sided *identity element*: There exists $K_0 \in \mathcal{K}_0^n$ such that $A \oplus K_0 = K_0 \oplus A = A$ for all $A \in \mathcal{K}_0^n$.

We say that an addition $\oplus$ is *monotone* if $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ implies $A_1 \oplus A_2 \subseteq B_1 \oplus B_2$. A family of examples of monotone additions on $\mathcal{K}_0^n$ is given by the p-sums (for $1 \le p \le \infty$) which were introduced by Firey in [2]. They are defined by the relation

$$h_{A+_pB}(\theta)^p = h_A(\theta)^p + h_B(\theta)^p$$

where h_A denotes the support function of A. Of course, for $p = 1$ we have the standard Minkowski addition. A second family of examples is given by the p-polar sums, defined as $A +_{-p} B = (A^\circ +_p B^\circ)^\circ$, where A° denotes the polar body of A (see [1]). Notice that these are indeed additions according to our definition, with $\mathbb{R}^n$ being the identity element.

Let us also denote by $\mathcal{S}_0^n$ the class of "star sets", which for us simply means sets $A \subseteq \mathbb{R}^n$ such that $0 \in A$ and for every line ℓ through the origin the intersection $A \cap \ell$ is closed and connected. A monotone addition on $\mathcal{S}_0^n$ is defined in the obvious way. A natural family of examples is given by the p-radial sums, defined by

$$r_{A\tilde{+}_pB}(\theta)^p = r_A(\theta)^p + r_B(\theta)^p.$$

Here r_A denotes the radial function of A, and the definition makes sense for every $p \in [-\infty, \infty] \setminus \{0\}$.

The class $\mathcal{K}_0^n$ equipped with the Minkowski addition $+$ behaves in many ways like the class $\mathcal{S}_0^n$ equipped with the radial addition $\tilde{+}$. For example, the Alexandrov-Fenchel inequality for mixed volumes is a deep result in convexity (see, e.g. [7]). Its analogue for $\mathcal{S}_0^n$ is the dual Alexandrov-Fenchel inequality for dual mixed volumes, proved by Lutwak in [4]. While the proofs of such dual statements are usually much easier than the proofs of their convex counterparts, it is not clear why such a dual theory exists in the first place. One of the goals of this project was to express in a formal way the similarity between the class $\mathcal{K}_0^n$ equipped with $+$ and the class $\mathcal{S}_0^n$ equipped with $\tilde{+}$.

The simplest formulation of our theorems involves the induced homothety operation. Given an addition $\oplus$ on $\mathcal{K}_0^n$ or $\mathcal{S}_0^n$, the induced homothety $\odot$ is defined by

$$m \odot A = \underbrace{A \oplus A \oplus \cdots \oplus A}_{m \text{ times}}.$$

Theorem. *Let $\oplus$ be a monotone addition on $\mathcal{K}_0^n$ such that $m \odot A = mA = \{ma : a \in A\}$. Then $\oplus = +$.*

Theorem. *Let $\oplus$ be a monotone addition on $\mathcal{S}_0^n$ such that $m \odot A = mA = \{ma : a \in A\}$. Then $\oplus = \tilde{+}$.*

In fact, these theorems are corollaries of more general theorems characterizing all p-additions and p-radial additions. For brevity, we only state the theorem for $\mathcal{K}_0^n$:

Theorem. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a monotone addition such that $m \odot A = f(m)A$ for some function $f : \mathbb{N} \to \mathbb{R}_+$. Then:*

- *If $f(2) > 1$ then $\oplus = +_p$ for some $1 \leq p < \infty$.*
- *If $f(2) < 1$ then $\oplus = +_{-p}$ for some $1 \leq p < \infty$.*
- *If $f(2) = 1$ and the identity element of $\oplus$ is $\{0\}$ then $\oplus = +_\infty$.*
- *If $f(2) = 1$ and the identity element of $\oplus$ is $\mathbb{R}^n$ then $\oplus = +_{-\infty}$.*

Characterization theorems for the p-additions and p-radial additions appeared also in the work of Gardner, Hug and Weil ([3]). However, in their theorems the conditions in each case were essentially different – the p-additions were assumed to be projection covariant, while the p-radial additions were assumed to be section covariant (in addition to other assumptions such as continuity). In our theorem exactly the same conditions characterize both cases.

The above theorems assume the homothety $\odot$ has a specific form. One can also state a more general theorems, where $\odot$ is only assumed to satisfy several natural properties. Specifically, we say that a monotone addition $\oplus$ is:

- *Strongly monotone* if $m \odot A \subseteq m \odot B$ implies $A \subseteq B$.
- *Divisible* if for every A and every $m \in \mathbb{N}$ there exists a B such that $m \odot B = A$.
- *Subspace preserving* if $V \oplus V = V$ for every linear subspace V of $\mathbb{R}^n$.

Theorem. *Let $\oplus : \mathcal{K}_0^n \times \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a monotone addition with $\{0\}$ as the identity element. Assume that $\oplus$ is strongly monotone, divisible and subspace preserving. Then $\oplus = +_p$ for some $1 \leq p \leq \infty$.*

The same theorem cannot hold for $\mathcal{S}_0^n$. For example, one may choose an arbitrary function $p : S^{n-1} \to (0, \infty)$ and define

$$r_{A \oplus B}(\theta)^{p(\theta)} = r_A(\theta)^{p(\theta)} + r_B(\theta)^{p(\theta)}$$

(this example appeared already in [3]). Therefore one needs an extra assumption on $\oplus$, "relating the different directions". For example:

Theorem. *Let $\oplus : \mathcal{S}_0^n \times \mathcal{S}_0^n \to \mathcal{S}_0^n$ be a monotone addition with $\{0\}$ as the identity element. Assume that $\oplus$ is strongly monotone, divisible and subspace preserving. Assume further that $m \odot A$ is convex whenever A is convex. Then $\oplus = \tilde{+}_p$ for some $0 < p \leq \infty$.*

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The thin-shell conjecture for the operator norm

Beatrice-Helen Vritsiou
(joint work with Jordan Radke)

We verify the thin-shell conjecture for the operator norm, or, more precisely, for the unit balls of spaces of square matrices endowed with the operator norm of the matrices. This amounts to showing that most of the volume of these unit balls, or rather of their homothetic copies of volume 1, is found within an annulus of radius the average distance of an element in them from the origin (this average distance will be of the order of the square root of the dimension) and of almost constant width (namely, which may depend on the dimension only logarithmically).

More generally, we can ask the same question for the unit balls of all Schatten classes, which are defined as follows. Let S_p^n denote the space of $n \times n$ real, complex or quaternionic matrices endowed with the norm that sends each matrix T to the ℓ_p norm of its singular-values-vector, or in other words to the norm

$$\|s(T)\|_p := \left(\sum_{i=1}^{n} |s_i(T)|^p\right)^{1/p}$$

of the vector $s(T) = (s_1(T), \dots, s_n(T))$ of the eigenvalues of $\sqrt{T^*T}$ (ordered in an non-increasing way). As usual, when $p = \infty$, $\|T\|_{S_\infty^n} = \|s(T)\|_\infty$ is just the maximum singular value of T, which we call the spectral or operator norm of T.

The unit balls K_p of S_p^n, $p \in [1, \infty]$, have been studied in the past with respect to other important conjectures or questions in Convex Geometry as well:

- in [4] König, Meyer and Pajor established the hyperplane conjecture for them;
- in [3] Guédon and Paouris studied the behaviour of the Schatten classes with respect to concentration of volume, and showed that all but an exponentially small (in the dimension) fraction of the unit balls K_p of S_p^n is found in a Euclidean ball of radius twice the average distance of an element in K_p from the origin.

Not long after [3], Paouris [5] resolved the latter question in the affirmative for all convex bodies in isotropic position (as are the unit balls of the Schatten classes of real, complex or quaternionic matrices); however, as should be expected perhaps, the method he used was quite different from the methods of [3] and of [4], which are very specific to the Schatten classes.

We use a refinement of the latter methods: one of the key ideas that we as well employ is to reduce estimates about moments of the Euclidean norm with respect to the uniform measure on the balls K_p, which are convex bodies of an n^2, $2n^2$ or $4n^2$-dimensional real space, to estimates about moments of the Euclidean norm with respect to a density f_p on $\mathbb{R}^n$ now; the new density is no longer a uniform, or even a log-concave, density, but it is invariant under permutations of the coordinates of vectors in $\mathbb{R}^n$. We then exploit fully the symmetry properties of the new density f_p to get precise recursive identities that involve the variance of the Euclidean norm, which is what, in the case of the thin-shell conjecture, one has to bound.

We are able, through these identities, to get tight estimates for the variance of the Euclidean norm when p is really large, that is, when p is at least as large as the dimension of the balls K_p: more specifically, we show that this variance is of the order of 1 when $p \gtrsim n^2 \log n$, which establishes the thin-shell conjecture in these ranges, and in particular in the case of the operator norm ($p = \infty$).

The same arguments also work, again for the same p, in the case of unit balls of Hermitian matrices, of anti-symmetric Hermitian matrices, and of complex symmetric matrices, although now it may happen that these unit balls are not in isotropic position.

It is a natural question of course to ask what happens for the remaining p. In [6] we also give a necessary condition for the thin-shell conjecture to hold for any of the Schatten classes S_p^n, $p \geqslant 1$, however most probably this condition will not turn out to be a sufficient one as well (or rather, one would need quite precise quantitative versions of it to deduce anything better than the currently known bounds for the S_p^n when $p \lesssim n \log n$, see [1] and [2]).

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Invertibility of adjacency matrices of random digraphs

Alexander E. Litvak

(joint work with A. Lytova, K. Tikhomirov, N. Tomczak-Jaegermann, P.Youssef)

Consider the set $\mathcal{M}_{n,d}$ of adjacency matrices of random d-regular directed graphs, that is $n \times n$ matrices with 0/1-entries such that every row and column has exactly d ones. By probability we always mean normalized counting measure.

Our work is motivated by related questions on singular probability. One conjecture on symmetric adjacency matrices was mentioned by Vu in his survey [10, Problem 8.4] (see also 2014 ICM talks by Frieze and by Vu). The corresponding question for non-symmetric adjacency matrices was formulated by Cook in [4]:

Is it true that for every $3 \le d \le n-3$*, one has*

$$(*) \qquad p_{n,d} := \mathbb{P}\{M \in \mathcal{M}_{n,d} : M \text{ is singular}\} \longrightarrow 0 \text{ as } n \to \infty?$$

Note that for $d = 1$, M is a permutation matrix, so $p_{n,1} = 0$; for $d = 2$, $p_{n,2} \to 1$ ([10, 4]); and that by interchanging zero and ones, $p_{n,d} = p_{n,n-d}$.

Singularity of random square matrices is a subject with a long history and many results. A fundamental role is played by what is nowadays called *the Littlewood-Offord (LO) theory.* In its classical form, established by Erdös, the LO inequality states that for every $z \in \mathbb{R}$, every $a \in \mathbb{R}^n$ with non-zero coordinates a_k's and for independent random signs r_k's, the probability $\mathbb{P}\{\sum_{k=1}^n r_k a_k = z\} \le n^{-1/2}$. This result has been substantially strengthened and generalized in subsequent years, leading to a much better understanding of interrelationship between the law of the sum $\sum_{k=1}^n r_k a_k$ and the arithmetic structure of the vector a.

The LO theory is used as follows: given an $n \times n$ matrix A with i.i.d. entries, A is non-singular if and only if the inner product of a normal vector to the span of any subset of $n-1$ columns of A with the remaining column is non-zero. Thus, knowing the typical arithmetic structure of the random normal vectors and conditioning on their realization, one can estimate the probability that A is singular. For more details we refer to [9, Section 3], [8, Section 4], and references therein.

The main difficulty in singularity questions such as $(*)$ stems from the restrictions on row/column-sums, and from possible symmetry constraints for the entries. Note that for a random matrix on $\mathcal{M}_{n,d}$ every two entries/rows/columns are dependent; moreover, the first $n-1$ columns uniquely define the last column. This makes a straightforward application of the LO theory (as illustrated above) impossible and an extension of the theory covering this probabilistic model is needed.

Below c, C, denote absolute positive constants, whose actual values can change from line to line, and $f \ge \omega(a_n)$ means $f/a_n \to \infty$ as $n \to \infty$.

Main result. The question $(*)$ has been recently addressed in [4] by Cook who obtained the bound $p_{n,d} \le d^{-1/18}$ for d satisfying $\omega(\ln^2 n) \le d \le n - \omega(\ln^2 n)$. Our main result shows that one can drop the condition $d \ge \omega(\ln^2 n)$.

Theorem 1. *For* $3 \le d \le cn/((\ln n)\ln\ln n)$ *one has* $p_{n,d} \le C \ln^3 d/\sqrt{d}$*.*

Thus we proved that $p_{n,d} \to 0$ as $d \to \infty$, which in particular verifies $(*)$ whenever $d = \omega(1)$. Moreover, the probability bound is better than in [4].

It is natural to compare our model to the Erdös-Rényi model, when edges of a random graph appear independently with probability $p = d/n$. Recently, the invertibility of adjacency matrices in the Erdös-Rényi model for $d = \omega(\ln n)$ was shown in [1]. Note that in this model for $d \le \ln n$ a zero row appears with probability at least 1/2, which is not the case for d-regular graphs.

Strategy of proof. The proof is naturally split into two distinct parts. First we establish certain (extension) properties of random d-regular directed graphs and their adjacency matrices. Then we use these results to deal with the singularity.

We start with two properties of d-regular directed graphs, which can be called "no large intersections" and "no large zero minors." Such properties are known for random undirected graphs (see e.g. [5]). A key ingredient in the proofs of these results is the *simple switching*, introduced for general graphs by Senior, and applied by McKay for d-regular graphs. The following lemma shows that the support of the sum of any $k \leq cn/d$ rows has almost maximal possible cardinality.

Lemma 2. *For every $\varepsilon \in (\sqrt{\ln d/d}, 1)$ and $k \leq c\varepsilon n/d$, the union of supports of any k rows (or columns) of a random matrix on $\mathcal{M}_{n,d}$ has cardinality exceeding $(1-\varepsilon)dk$ with probability at least $1 - \exp(-c\varepsilon^2 d \ln(c\varepsilon n/d))$.*

The next lemma deals with large zero minors.

Lemma 3. *For $Cn \ln d/d \leq \ell \leq r \leq n/4$ a random matrix on $\mathcal{M}_{n,d}$ has no $\ell \times r$ zero minors with probability at least $1 - \exp(-cr\ell d/n)$.*

Lemmas 2 and 3 show that a random graph has good "regularity" properties. Analogous statements in the Erdös-Rényi model follow from standard Bernstein type inequalities. For random d-regular directed graphs, in the paper [3], which serves as a basis for the main theorem of [4], rather strong concentration properties were established. However, the results there are valid only for $d \geq \omega(\ln n)$. The proof in [4] is based on the method of exchangeable pairs introduced by Stein and developed for concentration inequalities by Chatterjee ([2]). Contrary, our proof is simpler, self-contained, and works for $d \geq C$.

Now we turn to the proof of Theorem 1. We follow the scheme and expand on some of the techniques developed in [4] adding new crucial ingredients to remove logarithmic lower bound on d. In this scheme, at the first step, one shows that a random matrix does not have any (left or right) null vectors with many (more than Cnd^{-c}) equal coordinates, provided that $d \geq \omega(\ln^2 n)$. Then one shows that, conditioned on this event, a random matrix is not singular.

Lemmas 2 and 3, applied on the second step, allow us to modify this scheme so that at first step it is enough to consider a much smaller class of *almost constant* vectors. Using that lemmas we obtain a new crucial anti-concentration property, which in turn allows us to show that for every $C \leq d \leq cn$ a random matrix does not have null vectors having $n - n/\ln d$ equal coordinates. This step essentially uses a new delicate approximation argument dealing with tails of appropriately rescaled vectors in $\mathbb{R}^n$. Note that a logarithmic lower bound on d is not required.

Then, conditioning on the event that M does not have almost constant null vectors, we show that a random matrix M is non-singular with high probability. In [4], a sophisticated approach based on "shuffling" of two rows was developed to treat this case. The shuffling consists in a random perturbation of two rows of a fixed matrix $M \in \mathcal{M}_{n,d}$ in such a way that the sum of the rows remains unchanged. Then one uses a variant of the classical Erdös anti-concentration inequality to show that the number of "bad" perturbations is small. To apply this we need that the supports of these two rows have a small intersection, which is given by Lemma 2 with $k = 2$. As shuffling involves supports of only two rows, we get that probability tends to zero with d and not with n (and this is the only such step – in all our other statements the probability converges with n). We developed further the shuffling technique to simplify the proof and to obtain better probability estimates.

Finally, we explain how to complete the proof. Using that there are no almost constant vectors and that there are no large zero minors, we show that for singular matrices with high probability the minor $M^{1,2}$ obtained by removing the first two rows has largest possible rank, that is, either $\operatorname{rk} M^{1,2} = \operatorname{rk} M$ when $\operatorname{rk} M \leq n-2$ or $\operatorname{rk} M^{1,2} = n-2$ when $\operatorname{rk} M = n-1$. We consider the equivalence classes of matrices with the same minor $M^{1,2}$. Noticing that fixing such a minor determines the support of the first two rows, we use the shuffling procedure for the first two rows and show that the set of matrices of rank $\leq n-2$ (resp. $= n-1$) is small inside the set of matrices of rank $\leq n-1$ (resp. $= n$). This implies the bound on the probability that M is singular.

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Randomized isoperimetric inequalities

Peter Pivovarov
(joint work with Grigoris Paouris)

I discussed randomized versions of isoperimetric inequalities for convex sets. For example, recall the Brunn-Minkowski inequality for the volume V_n of convex bodies $K, L \subseteq \mathbb{R}^n$,

$$V_n(K+L)^{1/n} \geqslant V_n(K)^{1/n} + V_n(L)^{1/n}. \tag{1.1}$$

Equality holds in (1.1) if K and L are homothetic so the latter can equivalently be stated in isoperimetric form,

$$V_n(K+L) \geqslant V_n(r_K B + r_L B), \tag{1.2}$$

where B is the Euclidean unit ball and r_K, r_L denote the radii of Euclidean balls with the same volume as K, L, respectively, i.e., $r_K = (V_n(K)/V_n(B))^{1/n}$. Inequality (1.2) admits a stronger empirical version associated with random convex sets. Specifically, let $x_1, \ldots, x_N$ be independent random vectors distributed according to the uniform density on a convex body $K \subseteq \mathbb{R}^n$, say $f_K = \frac{1}{V_n(K)}\mathbb{1}_K$, i.e., $\mathbb{P}(x_i \in A) = \int_A f_K(x)dx$ for Borel sets $A \subseteq \mathbb{R}^n$. For each such K and $N > n$, we associate a random polytope

$$K_N = \operatorname{conv}\{x_1, \ldots, x_N\},$$

where conv denotes convex hull. Then the following stochastic dominance holds for the random polytopes K_{N_1}, L_{N_2} and $(r_K B)_{N_1}$, $(r_L B)_{N_2}$ associated with the bodies in (1.2): for all $\alpha \geqslant 0$,

$$\mathbb{P}\left(V_n(K_{N_1} + L_{N_2}) > \alpha\right) \geqslant \mathbb{P}\left(V_n((r_K B)_{N_1} + (r_L B)_{N_2}) > \alpha\right). \tag{1.3}$$

Integrating in α gives

$$\mathbb{E}V_n(K_{N_1} + L_{N_2}) \geqslant \mathbb{E}V_n((r_K B)_{N_1} + (r_L B)_{N_2}), \tag{1.4}$$

where $\mathbb{E}$ denotes expecation. By the law of large numbers, when $N_1, N_2 \to \infty$, the latter convex hulls converge to their ambient bodies and this leads to (1.2). Thus (1.1) is a global inequality which can be proved by a random approximation procedure in which stochastic dominance holds at each stage.

Inequalities for the volume of random convex hulls in stochastic geometry have a rich history starting with Blaschke's resolution of Sylvester's famous four-point problem in the plane. I reviewed fundamental related inequalities of Busemann, Groemer and Bourgain-Meyer-Milman-Pajor on random simplices, general convex hulls and Minkowski sums, respectively. The stochastic form of the Brunn-Minkowski inequality (1.3) intertwines two operations - convex hulls and Minkowski sums. It turns out that (1.3) is a just a special case of a general result due to the authors [2]. For vectors $x_1, \ldots, x_N$ in $\mathbb{R}^n$, we form the $n \times N$ matrix $[x_1, \ldots, x_N]$ and view it as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. If $C \subseteq \mathbb{R}^N$, then

$$[x_1, \ldots, x_N]C = \left\{ \sum_{i=1}^N c_i x_i : c = (c_i) \in C \right\}. \tag{1.5}$$

Let $e_1, \dots, e_N$ denote the standard unit vector basis for $\mathbb{R}^N$. If $C = \mathrm{conv}\{e_1, \dots, e_N\}$, then

$$[x_1, \dots, x_N]\mathrm{conv}\{e_1, \dots, e_N\} = \mathrm{conv}\{x_1, \dots, x_N\}. \tag{1.6}$$

If $C = [-1, 1]^N$, then one obtains Minkowski sums,

$$[x_1, \dots, x_N]B_\infty^N = \sum_{i=1}^{N}[-x_i, x_i]. \tag{1.7}$$

Assume we have the following sequences of independent random vectors defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $X_1, X_2, \dots$, sampled according to densities $f_1, f_2, \dots$.
2. $X_1^*, X_2^*, \dots$, sampled according to $f_1^*, f_2^*, \dots$.

Write $\mathbf{X} = [X_1 \cdots X_N]$ and $\mathbf{X}^* = [X_1^* \cdots X_N^*]$. With the above notation, the theorem is formulated as follows.

Theorem 1.1. *Let C be a compact convex set in $\mathbb{R}^N$ and $1 \leqslant j \leqslant n$. Then for each $\alpha \geqslant 0$,*

$$\mathbb{P}(V_j(\mathbf{X}C) > \alpha) \geqslant \mathbb{P}(V_j(\mathbf{X}^*C) > \alpha). \tag{1.8}$$

Assume that $f_i = \frac{1}{V_n(K)}\mathbb{1}_K$ for $i = 1, \dots, N_1$ and $f_{N_1+i} = \frac{1}{V_n(L)}\mathbb{1}_L$ for $i = 1, \dots, N_2$. Note that $f_i^* = \frac{1}{V_n(K)}\mathbb{1}_{r_K B}$, $i = 1, \dots, N_1$ and $f_{N_1+i}^* = \frac{1}{V_n(L)}\mathbb{1}_{r_L B}$ for $i = 1, \dots, N_2$. To see that (1.3) holds, set

$$C_1 = \mathrm{conv}\{e_1, \dots, e_{N_1}\}, \quad C_2 = \mathrm{conv}\{e_{N_1+1}, \dots, e_{N_1+N_2}\},$$

both considered as subsets of $\mathbb{R}^{N_1+N_2}$. Then

$$\begin{aligned} K_{N_1} + L_{N_2} &= [X_1, \dots, X_{N_1}]C_1 + [X_{N_1+1}, \dots, X_{N_1+N_2}]C_2 \\ &= [X_1, \dots, X_{N_1}, X_{N_1+1}, \dots, X_{N_1+N_2}](C_1 + C_2). \end{aligned}$$

More generally, one can replace usual Minkowski addition in the latter by M-addition as introduced by Gardner, Hug, and Weil. Let M be an arbitrary subset of $\mathbb{R}^m$ and define the *M-combination* $K \oplus_M L$ of sets K and L in $\mathbb{R}^n$ by

$$K \oplus_M L = \{a_1x + a_2y : (a_1, a_2) \in M, x \in K, y \in L\}.$$

With this notion of addition, we have the following stochastic form when M is contained in the positive orthant and K and L are convex bodies

$$\mathbb{P}\left(V_j(K_{N_1} \oplus_M L_{N_2}) > \alpha\right) \geqslant \mathbb{P}\left(V_j((r_K B)_{N_1} \oplus_M (r_L B)_{N_2}) > \alpha\right). \tag{1.9}$$

This follows from Theorem 1.1 and the identity

$$\begin{aligned} K_{N_1} \oplus_M L_{N_2} &= [X_1, \dots, X_{N_1}]C_1 \oplus_M [X_{N_1+1}, \dots, X_{N_1+N_2}]C_2 \\ &= [X_1, \dots, X_{N_1}, X_{N_1+1}, \dots, X_{N_1+N_2}](C_1 \oplus_M C_2). \end{aligned}$$

A dual version of Theorem 1.1 is due D. Cordero-Erausquin, M. Fradelizi and the authors [1].

Theorem 1.2. *Let C be a symmetric convex body in $\mathbb{R}^N$. Let ν be a radial measure on $\mathbb{R}^n$ with a density ψ which is $-1/(n+1)$-concave on $\mathbb{R}^n$. Then for each $\alpha \geqslant 0$,*

$$\mathbb{P}(\nu((\mathbf{X}C)^\circ) > \alpha) \leqslant \mathbb{P}(\nu((\mathbf{X}^*C)^\circ) > \alpha). \tag{1.10}$$

I also discussed key ingredients in the proof, namely the rearrangement inequality of Rogers and Brascamp-Lieb-Luttinger. I recalled a result of Kanter on stochastic dominance for products of unimodal densities. I explained how one can interpret the Rogers/Brascamp-Lieb-Luttinger inequality as a result about stochastic dominace. I also concluded with applications to small deviations for operator norms of random matrices. The reader is invited to see [3] for further information.

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