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Moduli spaces and Modular forms

Organised by
Jan Hendrik Bruinier, Darmstadt
Gerard van der Geer, Amsterdam
Valery Gritsenko, Villeneuve d'Ascq

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Abstract. The roots of both moduli spaces and modular forms go back to the theory of elliptic curves in the 19th century. Both topics have seen an enormous growth in the second half of the 20th century, but the interaction between the two remained limited. Recently there have been new developments that led to new points of contact between the two topics. One is the theory of K3 surfaces that is rapidly gaining a lot of new interest. Here the link with modular forms on orthogonal groups has led to progress on the Kodaira dimension of the moduli spaces of K3 surfaces. Another new development has been the use of moduli spaces of curves to gather new information about Siegel modular forms. The workshop intended to bring representatives from both the theory of moduli and the theory of modular forms together to further the interaction between the two topics as the time seemed ripe to do this.

Mathematics Subject Classification (2010): 11xx, 14xx.

Introduction by the Organisers

The workshop *Moduli Spaces and Modular Forms*, organized by Jan Bruinier (Darmstadt), Gerard van der Geer (Amsterdam) and Valéry Gritsenko (Lille) was held 25-29 April, 2016 and was attended by 52 participants from all over the world. The attendance ranged from senior leaders in the field to young postdocs and advanced Ph.D. students. The program consisted of 21 talks of one hour or 50 minutes. The lectures and the simple fact that people from different fields were brought together initiated lots of discussions and forged new contacts between participants. The program highlighted the diversity of the interactions between

'Moduli' and 'Modular Forms'. Topics ranged from the sphere packing problem to moduli of supersingular K3 surfaces and Enriques surfaces in characteristic 2.

Three main themes of the workshop were 'Moduli of K3 surfaces and Modular Forms on Orthogonal Groups,' 'Moduli of Curves and Siegel Modular Forms' and 'Modular Forms on Ball Quotients.'

In recent years there has been a strong revival of interest in moduli of K3 surfaces. One development was the determination of the Kodaira dimension for moduli of K3 surfaces of not too small degree, which was the last open problem in A. Weil's program on K3 surfaces. This progress used modular forms on orthogonal groups and Borcherds' automorphic products in an essential way. Another development was the proof of the conjectures of Artin and Tate for K3 surfaces over finite fields for characteristic not 2 last year. Also this proof uses modular forms. Besides this there are interesting developments concerning the compactification of moduli of K3 surfaces. Also the moduli of polarized hyperkähler varieties and Enriques surfaces are attracting new interest in algebraic and differential geometry. All these topics are related to modular forms on orthogonal groups. Apart from this there are interesting links between moduli of K3 surfaces and moduli of curves in a number of papers by Kondo, Allcock and others and modular forms on ball quotients. The modular forms on ball quotients belong to the theory of authomorphic forms on unitary groups, but there has been almost no interaction between these two disciplines.

Siegel Modular forms occur in the cohomology of local systems on moduli spaces of abelian varieties. Sometimes these moduli spaces are strongly related to moduli of curves. For example, for genus ≤ 3 the moduli space of principally polarized abelian varieties is very close to the moduli space of curves. This fact and the fact that one can extract information about cohomology by using Frobenius over finite fields have been used very effectively to obtain a lot of new information about Siegel modular forms of genus ≤ 3 and also for Picard modular forms. The link between the two topics that is thus obtained is an extremely useful tool. An example of an application is the disproval of the Gorenstein conjecture for the tautological ring of the moduli space $\mathcal{M}_{2,n}$ of n-pointed curves of genus 2.

Modular forms on ball quotients have not attracted much attention. Ball quotients are associated to moduli of abelian varieties associated to groups of type $U(n,1)$. But there are interesting links between various other moduli spaces in algebraic geometry and these Shimura varieties of type $U(n,1)$. For example, moduli of K3 surfaces and moduli of curves are linked in a number of papers by Kondo, Allcock and others to ball quotients and to modular forms on these ball quotients. The modular forms on ball quotients belong to the theory of automorphic forms on unitary groups, but there has been almost no interaction between the geometric aspects and the automorphic aspects.

Recently there has been a lot of activity on Kudla's program for unitary groups. Kudla and Rapoport defined special cycles on integral models of unitary Shimura varieties of type $GU(n,1)$ as the locus of abelian varieties (with additional data) whose endomorphism ring contains certain special endomorphisms. They computed some of their arithmetic intersection numbers and related them to coefficients of derivatives of Eisenstein series. The height pairing of such Kudla-Raport divisors with CM cycles has been expressed as the derivative of the central value of a Rankin type L-function.

All these themes were well represented among the talks on this workshop. The great variety of topics treated became already visible on the first day. The workshop started with a beautiful survey talk of Farkas on his joint work with Alexeev, Donagi, Izadi and Ortega on the uniformization of the moduli space $\mathcal{A}_6$ of principally polarized abelian varieties of dimension 6. It was followed by a talk by S. Kondo who discussed Enriques surfaces in characteristic 2, an exceptional but intriguing case where the moduli space is reducible. He considered the question whether there exist Enriques surfaces in characteristic 2 with finite automorphism group and with a prescribed dual graph of the configuration of all smooth rational curves. He gave a 1-dimensional family of Enriques surfaces with a configuration of type VII constructed using Rudakov-Shafarevich derivations on K3 surfaces. The other two talks of the day showed the same diversity of topics. There was talk of Maryna Viazovska on her sensational work on the sphere packing problem: E_8 and the Leech lattice provide the densest possible sphere packings in dimensions 8 and 24. And Taïbi presented recent impressive advances on Siegel modular forms, obtained by using Arthur's multiplicity formula; he is able to derive explicit dimension formulae for spaces of vector-valued Siegel modular forms of level 1.

The diversity of the first day was continued on the following days as illustrated by the abstracts of the talks that follow hereafter. Some talks highlighted the algebraic geometry aspect of the topic, others concentrated on Arakelov geometry and there were talks dealing mostly with the modular forms side. The talks showed a wide range of topics but also presented quite a number of unexpected relations. The variation in the program was appreciated very much by the participants. It led to a very lively atmosphere with many discussions and a very fruitful workshop.

The organizers thank the staff of Oberwolfach for creating excellent working conditions during the workshop.

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Workshop: Moduli spaces and Modular forms

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Abstracts

The uniformization of the moduli space of principally polarized abelian 6-folds

Gavril Farkas
(joint work with Valery Alexeev, Ron Donagi, Elham Izadi, Angela Ortega)

It is a classical idea that general principally polarized abelian varieties (ppavs) and their moduli spaces are hard to understand, and that one can use algebraic curves to study some special classes, such as Jacobians and Prym varieties. This works particularly well in small dimension, where in this way one reduces the study of all abelian varieties to the rich and concrete theory of curves. For $g \leq 3$, a general ppav is a Jacobian, and the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ between the moduli spaces of curves and ppavs respectively, is birational. For $g \leq 5$, a general ppav is a Prym variety by a classical result of Wirtinger. In particular, for $g = 5$, this gives a uniformization of $\mathcal{A}_5$ given by the degree 27 Prym map

$$P : \mathcal{R}_6 \to \mathcal{A}_5.$$

The purpose of this paper is to prove a similar uniformization result for the moduli space $\mathcal{A}_6$ of principally polarized abelian varieties of dimension 6. The idea of this construction is due to Kanev and it uses the rich geometry of the 27 lines on a cubic surface. Suppose $\pi : C \to \mathbb{P}^1$ is a cover of degree 27 whose monodromy group equals the Weyl group $W(E_6) \subset S_{27}$ of the E_6 lattice. In particular, each smooth fibre of π can be identified with the set of 27 lines on an abstract cubic surface and, by monodromy, this identification carries over from one fibre to another. Assume furthermore that π is branched over 24 points and that over each of them the local monodromy of π is given by a reflection in $W(E_6)$. A prominent example of such a covering $\pi : C \to \mathbb{P}^1$ is given by the *curve of lines* in the cubic surfaces of a Lefschetz pencil of hyperplane sections of a cubic threefold. Each such a pencil contains precisely 24 singular cubic surfaces, each having exactly one node.

By the Hurwitz formula, we find that each such E_6-cover C has genus 46. Furthermore, C is endowed with a symmetric correspondence D of degree 10, compatible with the covering π and defined using the intersection form on a cubic surface. Precisely, a pair $(x, y) \in C \times C$ with $x \neq y$ and $\pi(x) = \pi(y)$ belongs to D if and only if the lines corresponding to the points x and y are incident. The correspondence D is disjoint from the diagonal of $C \times C$. The associated endomorphism $D : JC \to JC$ of the Jacobian of C satisfies the quadratic relation $(D-1)(D+5) = 0$. Using this, Kanev showed that the associated *Prym-Tyurin-Kanev variety*

$$PT(C, D) := \mathrm{Im}(D - 1) \subset JC$$

of this pair is a 6-dimensional ppav of exponent 6. Thus, if Θ_C denotes the Riemann theta divisor on JC, then $\Theta_{C|P(C,D)} \equiv 6 \cdot \Xi$, where Ξ is a principal polarization on $P(C, D)$.

Since the map π has 24 branch points corresponding to choosing 24 roots in E_6 specifying the local monodromy at each branch point, the Hurwitz scheme Hur parameterizing E_6-covers $\pi : C \to \mathbb{P}^1$ as above is 21-dimensional (and also irreducible). The geometric construction described above induces the *Prym-Tyurin-Kanev* map

$$PT : \mathrm{Hur} \to \mathcal{A}_6$$

between two moduli spaces of the same dimension.

Theorem 1. *The Prym-Tyurin-Kanev map* $PT : \mathrm{Hur} \to \mathcal{A}_6$ *is generically finite. It follows that the general principally polarized abelian variety of dimension* 6 *is a Prym-Tyurin variety of exponent* 6 *corresponding to a* E_6*-cover* $C \to \mathbb{P}^1$.

In the course of proving this result, we establish numerous facts concerning the geometry of the E_6-Hurwitz space. One of them is a surprising link between the splitting of the rank 46 Hodge bundle $\mathbb{E}$ on the Hurwitz space into Hodge eigenbundles and the Brill-Noether theory of E_6-covers. For a point

$$[\pi : C \to \mathbb{P}^1] \in \mathrm{Hur},$$

we denote by $D : H^0(C, K_C) \to H^0(C, K_C)$ the map induced at the level of cotangent spaces by the Kanev endomorphism of JC and by

$$H^0(C, K_C) = H^0(C, K_C)^{(+1)} \oplus H^0(C, K_C)^{(-5)},$$

the decomposition into the $(+1)$ and the (-5)-eigenspaces of holomorphic differentials respectively. Denoting by L the degree 27 pencil on C, we show that the following canonical identifications hold:

$$H^0(C, K_C)^{(+1)} = H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee)$$

and

$$H^0(C, K_C)^{(-5)} = \left(\frac{H^0(C, L^{\otimes 2})}{\mathrm{Sym}^2 H^0(C, L)}\right)^\vee \otimes \bigwedge^2 H^0(C, L).$$

In particular, the $(+1)$-Hodge eigenbundle is fibrewise isomorphic to the image of the Petri map $\mu(L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(C, K_C)$, whenever the Petri map is injective.

We are also able to describe the ramification divisor of the Prym-Tyurin-Kanev map in terms of the geometry of the *Abel-Prym-Tyurin curve*

$$\varphi_{(-5)} = \varphi_{H^0(K_C)^{(-5)}} : C \to \mathbb{P}^5$$

given by the linear system of (-5)-invariant holomorphic forms on C.

Theorem 2. *An* E_6*-cover* $[\pi : C \to \mathbb{P}^1]$ *lies in the ramification divisor of the map* $PT : \mathrm{Hur} \to \mathcal{A}_6$ *if an only if the Abel-Prym-Tyurin curve* $\varphi_{(-5)}(C) \subset \mathbb{P}^5$ *lies on a quadric.*

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Enriques surfaces with finite automorphism group in characteristic 2

Shigeyuki Kondō

(joint work with Toshiyuki Katsura)

Recall that, over the complex numbers, a generic Enriques surface has an infinite group of automorphisms (Barth and Peters [1]). On the other hand, Fano [5] first gave an Enriques surface with a finite group of automorphisms. Later Dolgachev [3] gave another example of such Enriques surfaces. Then Nikulin [8] classified the periods of such Enriques surfaces. Finally Kondo [7] classified all complex Enriques surfaces with a finite group of automorphisms and gave their explicit constructions. There are seven types $\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}$ of such Enriques surfaces. The Enriques surfaces of type I or II form an irreducible 1-dimensional family, and each of the remaining types consists of a unique Enriques surface. The first two types contain exactly 12 nonsingular rational curves, on the other hand, the remaining five types contain exactly 20 nonsingular rational curves. The Enriques surface of type I (resp. of type VII) is the example given by Dolgachev (resp. by Fano). We call the dual graphs of all nonsingular rational curves on the Enriques surface of type K the dual graph of type K ($K = \mathrm{I}, \mathrm{II}, ..., \mathrm{VII}$). For example, the dual graph of type VII is given in the Figure 1. For other dual grpahs, we refer the reader to [7].

In positive characteristics, the classification problem of such Enriques surfaces is still open. The most interesting case is in characteristic 2. In the paper [2], Bombieri and Mumford classified Enriques surfaces in characteristic 2 into three classes, namely, classical, singular and supersingular Enriques surfaces. As in the case of characteristic 0, an Enriques surface X in characteristic 2 has a canonical double cover $\pi : Y \to X$, which is a purely inseparable μ_2-cover, $\mathbf{Z}/2\mathbf{Z}$-cover or a purely inseparable α_2-cover according to X being classical, singular, or supersingular. Here Y might have singularities, and moreover non-normal case occurs, but it is a $K3$-like surface, that is, the dualizing sheaf of Y is trivial.

In this talk we consider the following problem: *does there exist an Enriques surface in characteristic* 2 *with a finite group of automorphisms whose dual graph of all nonsingular rational curves is of type* $\mathrm{I}, \mathrm{II}, ..., \mathrm{VI}$ *or* VII ? Table 1 gives the answer to this problem.

In Table 1, ◯ means the existence and × means the non-existence of an Enriques surface with the dual graph of type I, ..., VII. In case of singular Enriques surfaces with the dual graph of type $\mathrm{I}, \mathrm{II}, \mathrm{VI}$, the construction of such Enriques surfaces over the complex numbers works well in characteristic 2. On the other hand, the non-exsistence follows from some properties of genus one fibrations on Enriques surfaces. The most difficult and interesting case is of type VII. The main result

Type	I	II	III	IV	V	VI	VII
singular	○	○	×	×	×	○	×
classical	×	×	×	×	×	×	○
supersingular	×	×	×	×	×	×	○

Table 1

is to give a 1-dimensional family of classical and supersingular Enriques surfaces with the finite automorphism group $\mathrm{Aut}(X)(\cong \mathfrak{S}_5)$ whose dual graph of all nonsingular rational curves is of type VII. We use Rudakov and Shafarevich's theory of derivations [9] to construct such Enriques surfaces. The following is the dual graph of all nonsingular rational curves of type VII:

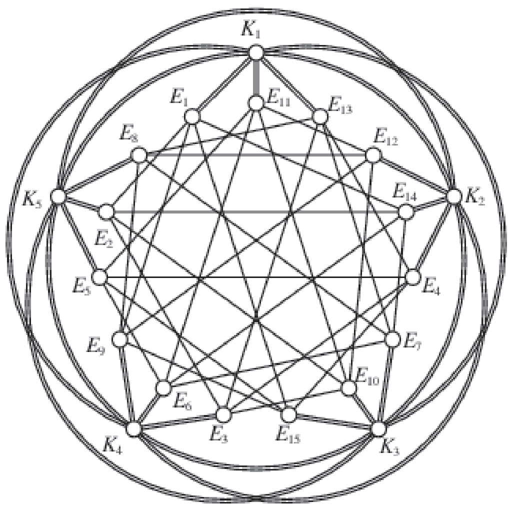

Figure 1

It is known that there exist Enriques surfaces in characteristic 2 with a finite group of automorphisms whose dual graphs of all nonsingular rational curves do not appear in the case of complex surfaces (Ekedahl and Shepherd-Barron[4], Salomonsson[10]). For example, there exists an Enriques surface X which has a genus one fibration with a multiple singular fiber of type $\tilde{E}_8$ and with a bi-section. We have ten nonsingular rational curves on X, that is, nine components of the reducible singular fiber and the bi-section, whose dual graph is given in Figure 2.

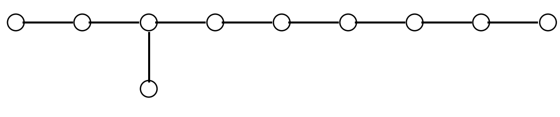

Figure 2

The *remaining problem* of the classification of Enriques surfaces in characteristic 2 with a finite group of automorphisms is to determine such Enriques surfaces appeared only in characteristic 2.

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The sphere packing problem in dimensions 8 and 24

Maryna Viazovska
(joint work with H. Cohn, A. Kumar, S. D. Miller, D. Radchenko)

In this talk we present a solution of the following two theorems.

Theorem 1 (Viazovska [3])**.** *No packing of unit balls in Euclidean space* $\mathbb{R}^8$ *has density greater than that of the* E_8*-lattice packing.*

Theorem 2 (Cohn, Kumar, Miller, Radchenko, Viazovska [2])**.** *No packing of unit balls in Euclidean space* $\mathbb{R}^{24}$ *has density greater than that of the Leech lattice packing.*

Our proof is based on a linear programming method developed by H. Cohn and N. Elkies. Let us briefly explain this method. The *Fourier transform* of an L^1 function $f:\mathbb{R}^d\to\mathbb{C}$ is defined as

$$\widehat{f}(y) := \int\limits_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot y}\, dx, \quad y\in\mathbb{R}^d$$

where $x\cdot y=\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\frac{1}{2}\|x-y\|^2$ is the standard scalar product in $\mathbb{R}^d$.

Theorem 3 (Cohn and Elkies [1]). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Schwartz function and r a positive real number such that $f(0) = \widehat{f}(0) = 1$, $f(x) \leq 0$ for $|x| \geq r$, and $\widehat{f}(y) \geq 0$ for all y. Then the sphere packing density in $\mathbb{R}^n$ is at most*

$$\frac{\pi^{n/2}}{\Gamma(n/2+1)}\left(\frac{r}{2}\right)^n.$$

The main step in our proof of Theorem 1 is the explicit construction of a function with the following properties.

Theorem 4. *There exists a radial Schwartz function $f : \mathbb{R}^8 \to \mathbb{R}$ which satisfies:*

$$f(x) \leq 0 \text{ for } \|x\| \geq \sqrt{2} \tag{1}$$

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^8 \tag{2}$$

$$f(0) = \widehat{f}(0) = 1. \tag{3}$$

The proof of Theorem 2 follows from

Theorem 5. *There exists a radial Schwartz function $f : \mathbb{R}^{24} \to \mathbb{R}$ which satisfies:*

$$f(x) \leq 0 \text{ for } \|x\| \geq 2 \tag{4}$$

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^{24} \tag{5}$$

$$f(0) = \widehat{f}(0) = 1. \tag{6}$$

Let us briefly explain our strategy for the proof of Theorems 4 and 5. First, we observe that conditions (1)–(3) imply additional properties of the function f. Suppose that there exists a Schwartz function f such that the conditions (1)–(3) hold. The E_8 lattice $\Lambda_8 \subset \mathbb{R}^8$ is even and unimodular. Therefore, the Poisson summation formula implies

$$\sum_{\ell\in\Lambda_8} f(\ell) = \sum_{\ell\in\Lambda_8} \widehat{f}(\ell). \tag{7}$$

Since $\|\ell\| \geq \sqrt{2}$ for all $\ell \in \Lambda_8\backslash\{0\}$ then conditions (1) and (3) imply

$$\sum_{\ell\in\Lambda_8} f(\ell) \leq f(0) = 1. \tag{8}$$

On the other hand, conditions (2) and (3) imply

$$\sum_{\ell\in\Lambda_8} f(\ell) \geq f(0) = 1. \tag{9}$$

Therefore, we deduce that $f(\ell) = \widehat{f}(\ell) = 0$ for all $\ell \in \Lambda_8\backslash\{0\}$. Moreover, the first derivatives $\frac{d}{dr}f(r)$ and $\frac{d}{dr}\widehat{f}(r)$ also vanish at all Λ_8-lattice points of length bigger then $\sqrt{2}$. We will say that f and $\widehat{f}$ have double zeroes at these points. This property gives us a hint how to construct function f explicitly.

We illustrate the idea of the proof of Theorems 4 and 5 with the following easier construction. For simplicity, we concentrate on the dimension 8 case. We show

how to construct Fourier eigenfunctions with simple zeroes at vectors of integer norm.

Proposition 6. *Fix $\varepsilon \in \pm 1$ and $d \in 8\mathbb{N}$. Let $g \in M^!_{2-d/2}(\Gamma_\theta, \chi_\varepsilon)$ be a weakly holomorphic modular form. Suppose that the only pole of g is at the cusp $i\infty$ and at this cusp g has the Fourier expansion*

$$g(\tau) = \sum_{n \gg -\infty} c(n) e^{\pi i n \tau}.$$

Consider the following radial function in $\mathbb{R}^d$

$$f(r) := \int\limits_C g(\tau)\, e^{\pi i r^2 \tau}\, d\tau,$$

where C is any contour in the upper half-plane connecting -1 and 1. Then

$$\widehat{f}(y) = -\,\varepsilon\, f \tag{10}$$

$$f(\sqrt{n}) = c(-n) \text{ for } n \in \mathbb{Z}_{>0}. \tag{11}$$

Proof. Firstly, we prove (10). We have

$$\widehat{f}(r) = \int\limits_{-1}^{1} g(\tau)\, \tau^{-d/2}\, e^{\pi i r^2(-1/\tau)}\, d\tau.$$

After the change of variables $\tau = \frac{-1}{w}$ we obtain

$$\mathcal{F}(f)(r) = \int\limits_{1}^{-1} g\left(\frac{-1}{w}\right)\, w^{d/2}\, e^{\pi i r^2 w}\, w^{-2}\, dw.$$

Using $g\left(\frac{-1}{w}\right) = \varepsilon\, w^{2-d/2}\, g(w)$ be arrive at

$$\mathcal{F}(f)(r) = -\varepsilon \int\limits_{-1}^{1} g(w)\, e^{\pi i r^2 w}\, dw = -\varepsilon\, f(r).$$

Now we prove (11). Note that $g(\tau)\, e^{\pi i n \tau}$ is periodic with period 2 for $n \in \mathbb{Z}$. Therefore, by the residue theorem

$$f(\sqrt{n}) = \int\limits_{-1}^{1} g(\tau)\, e^{\pi i n \tau}\, d\tau = c(-n).$$

□

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Computing with Siegel modular forms using Arthur's endoscopic classification

Olivier Taïbi

For a genus $g \geq 1$, let $\mathcal{H}_g$ be the Siegel upper-half space, $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathcal{A}_g = \Gamma_g \backslash \mathcal{H}_g$ the moduli space of principally polarized abelian varieties. Any tuple $\underline{k} = (k_1, \ldots, k_g)$ of integers satisfying $k_1 \geq \cdots \geq k_g$ defines an irreducible algebraic representation of GL_g, from which one defines the finite-dimensional $\mathbb{C}$-vector space $S_{\underline{k}}(\Gamma_g)$ of Siegel cusp forms of weight $\underline{k}$. See [9] for precise definitions. The space $S_{\underline{k}}(\Gamma_g)$ is naturally endowed with a semisimple action of the Hecke algebra $\mathbf{T} = \bigotimes'_{p \text{ prime}} \mathbf{T}_p$, where each $\mathbf{T}_p$ is a commutative $\mathbb{C}$-algebra and $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbf{T}_p, \mathbb{C})$ is in bijection with the set of semisimple conjugacy classes in $\mathrm{GSpin}_{2g+1}(\mathbb{C})$, via the Satake isomorphism. The goal of this report is to present recent advances, by others and myself, in explicitly computing these spaces along with the action of $\mathbf{T}$, using Arthur's multiplicity formula, that is a precise formulation of the global Langlands correspondence for arbitrary connected reductive groups over number fields.

This problem can be reformulated in terms of automorphic representations as follows. Assume that $k_n \geq n+1$, then $S_{\underline{k}}(\Gamma_g)$ is isomorphic a subspace of the space of discrete automorphic forms of "level one" for Sp_{2g}:

$$\mathcal{A}_{\mathrm{disc}}(\mathrm{Sp}_{2g}(\mathbb{Q}) \backslash \mathrm{Sp}_{2g}(\mathbb{A}) / \mathrm{Sp}_{2g}(\widehat{\mathbf{Z}})).$$

This subspace is defined by a condition of extremality for the action of the Lie algebra $\mathfrak{sp}_{2g}$, as in the theory of Verma modules, and this identification is equivariant for the action of the Hecke algebra $\mathbf{T}^0 = \bigotimes'_{p \text{ prime}} \mathbf{T}^0_p \subset \mathbf{T}$. Note that $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathbf{T}^0_p, \mathbb{C})$ is in bijection with the set of semisimple conjugacy classes in $\mathrm{SO}_{2g+1}(\mathbb{C})$, so that some information is lost in this reformulation using Sp_{2g} instead of GSp_{2g}. Decomposing the space of discrete automorphic forms in terms of automorphic representations, we obtain a $\mathbf{T}^0$-equivariant isomorphism

$$S_{\underline{k}}(\Gamma_g) \simeq \bigoplus_{\pi} \left(\pi_f^{\mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})} \right)^{\oplus m(\pi)}$$

where the sum is over the set of discrete automorphic representations $\pi = \pi_\infty \otimes \pi_f$ for Sp_{2g} such that π_∞ is the holomorphic discrete series representation $\sigma_{\underline{k}}^{\mathrm{hol}}$ of $\mathrm{Sp}_{2g}(\mathbb{R})$ having infinitesimal character $(\pm(k_1 - 1), \ldots, \pm(k_g - g), 0)$, seen as a semisimple conjugacy class in the Lie algebra $\mathfrak{so}_{2g+1}(\mathbb{C})$ dual to $\mathfrak{sp}_{2g}$. The integer $m(\pi) \geq 1$ is the multiplicity of π, and $\pi_f^{\mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})}$ is either 0 or an irreducible one-dimensional representation of $\mathbf{T}^0$. The natural approach to compute these representations of $\mathbf{T}^0$ consists in applying some version of the Arthur-Selberg trace formula, using a pseudo-coefficient for $\sigma_{\underline{k}}^{\mathrm{hol}}$ to select the relevant automorphic representations. In [10], I used Arthur's L^2-Lefschetz trace formula [4] instead, to derive dimension formulae for the spaces $S_{\underline{k}}(\Gamma_g)$.

Theorem 1. *For $m \geq 1$ denote $\zeta_m = \exp(2i\pi/m)$. There exists a finite family $(m_a, P_a, \Lambda_a)_{a\in A}$, where for any $a \in A$*

- *$m_a \geq 1$ is an integer,*
- *$P_a \in \mathbb{Q}(\zeta_{m_a})[X_1, \ldots, X_g]$,*
- *$\Lambda_a : (\mathbb{Z}/m_a\mathbb{Z})^g \to \mathbb{Z}/m_a\mathbb{Z}$ is a surjective group morphism,*

such that for any $k_1 \geq k_2 \geq \cdots \geq k_g > g+1$, we have

$$\dim S_{\underline{k}}(\Gamma_g) = \sum_{a\in A} \mathrm{tr}_{\mathbb{Q}(\zeta_{m_a})/\mathbb{Q}} \left(P_a(k_1, \ldots, k_g)\zeta_{m_a}^{\Lambda_a(k_1,\ldots,k_g)} \right). \tag{1}$$

Moreover there is an algorithm to compute the family $(m_a, P_a, \Lambda_a)_{a\in A}$, and they have been computed for all $g \leq 7$. There are similar formulae for $k_g = g+1$, but they are not the specialisation of (1).

This version of the trace formula uses a *stable* pseudo-coefficient of discrete series, which has a non-vanishing trace in each of the 2^g representations in the L-packet of discrete series containing $\sigma_{\underline{k}}^{\mathrm{hol}}$. The reason for using this version of the trace formula is that the geometric side simplifies, involving only semisimple $\mathbb{R}$-elliptic conjugacy classes in Levi subgroups of Sp_{2g}. Nevertheless, the main difficulty in computing the geometric side is the evaluation of local orbital integrals, because the level $\mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})$ is not neat. The price to pay for this simplification of the geometric side is that the spectral side does not distinguish between the elements of the L-packet containing $\sigma_{\underline{k}}^{\mathrm{hol}}$. This is where Arthur's endoscopic classification [3] comes into play.

Arthur proved that for any automorphic discrete representation $\pi = \pi_\infty \otimes \pi_f$ for Sp_{2g}, there is an associated "formal Arthur-Langlands parameter" $\psi = \boxplus_{i\in I}\pi_i[d_i]$. Here each π_i is a self-dual automorphic cuspidal representation of GL_{n_i}, and $d_i \geq 1$ are integers. They satisfy several additional conditions, among which we have $\sum_{i\in I} d_i n_i = 2g+1$. In level one, the relation between π and ψ is that each π_i is everywhere unramified, and for any prime p the eigenvalues of the Satake parameter of π_p (in $\mathrm{SO}_{2g+1}(\mathbb{C})$) are $\left(\alpha_{i,j}p^{(d_i-1)/2}, \ldots, \alpha_{i,j}p^{(1-d_i)/2}\right)_{i\in I, j\in J_i}$ where $(\alpha_{i,j})_{j\in J_i}$ denote the eigenvalues of the Satake parameter of $(\pi_i)_p$. Furthermore, Arthur proved a multiplicity formula, which given such a ψ characterises the representations π_∞ of $\mathrm{Sp}_{2g}[\mathbb{R})$ such that $\pi_\infty \otimes \pi_f$ is automorphic, where $\pi_f = \otimes'_p \pi_p$ is determined from ψ as above. He also gave a formula for the integers $m(\pi)$. Arthur's characterisation is quite abstract, but recently Arancibia, Moeglin and Renard [2] have shown that, in all cases relevant to Arthur's L^2-Lefschetz trace formula, Arthur's packets of representations of $\mathrm{Sp}_{2n}(\mathbb{R})$ coincide with those constructed in a more explicit manner by Adams and Johnson in [1]. Finally, using an inductive procedure, which also involves the families of split reductive groups SO_{2n+1} and SO_{4n}, the multiplicity of each π_∞ in the discrete automorphic spectrum for Sp_{2g} in level one can be recovered. The reason for the appearance of these two additional families of reductive groups is that for each π_i as above, the Satake parameters associated come naturally from a Langlands dual group which can be $\mathrm{SO}_{n_i}(\mathbb{C})$ (if $n_i \neq 2 \mod 4$) or $\mathrm{Sp}_{n_i}(\mathrm{C})$ (if n_i is even).

In theory one could use the same method to explicitly compute the trace of an arbitrary element of $\mathbf{T}^0$ on $S_{\underline{k}}(\Gamma_g)$, but the computation of the geometric side would be considerably more involved. In some cases, this difficulty can be circumvented by the use of *definite inner forms*. When $m = -1, 0, 1 \mod 8$, there exists a non-degenerate quadratic form q on $\mathbb{Q}^m$ such that the associated special orthogonal group $G_m = \mathrm{SO}(q)$ is definite (i.e. $G_m(\mathbb{R})$ is compact) and split over $\mathbb{Q}_p$ for all primes p. Unfortunately, symplectic groups do not admit definite inner forms split at all finite places. Thanks to the compactness of $G_m(\mathbb{R})$, level one automorphic forms for G_m can be seen as functions on the set of even unimodular (if m is even) or 2-modular (if m is odd) lattices in $(\mathbb{Q}^n, q)$ taking values in an irreducible representation V of $G_m(\mathbb{R})$. Since $G_m(\mathbb{Q})$ has finitely many orbits on the set of such lattices, spaces of automorphic forms for G_m are algebraic in nature and very explicit. In particular, the trace formula for G_m is much more elementary than the general case of arbitrary reductive groups. This allowed Chenevier and Renard [7] to compute dimension formulae for $m \in \{7, 8, 9\}$ and recently Mégarbané [8] to compute traces for a number of Hecke operators. Note that Chenevier and Renard's computations precede mine. The groups G_m are inner forms of split special orthogonal groups, which as explained above occur naturally in the inductive endoscopic analysis of the spectral side of the trace formula for Sp_{2n}. Similarly, Sp_6 is an endoscopic group for G_8, the relation between the dual groups being the natural embedding $\mathrm{SO}_7(\mathbb{C}) \hookrightarrow \mathrm{SO}_8(\mathbb{C})$. As a result one can easily derive, via the Satake isomorphism, the action of $\mathbf{T}_2^0$ on $S_{\underline{k}}(\Gamma_3)$ for $14 \geq k_1 \geq k_2 \geq k_3 \geq 4$, as well as traces of some Hecke operators for $3 \leq p \leq 23$. Mégarbané's computations seem to be in accordance with the conjectural values computed by Bergström, Faber and van der Geer [5]. The fact that Arthur's multiplicity formula is valid for inner forms such as G_m is a consequence of [11], which came after [7], but of course Chenevier and Renard already interpreted their computations using endoscopy.

Finally, I would like to point out that Chenevier and Lannes' study of lattices and Kneser neighbours (see [6]) using automorphic forms, which motivated the following works mentioned in this report, contains other interesting applications to Siegel modular forms.

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From exceptional groups to del Pezzo surfaces, via principal bundles over elliptic curves

Nick I. Shepherd-Barron
(joint work with I. Grojnowski)

Brieskorn and Tyurina showed that deformations of simple singularities possess simultaneous resolutions; Brieskorn, Grothendieck, Slodowy and Springer then showed that this phenomenon could be realized inside the corresponding split simply connected simple algebraic group G. Later it was observed that deformations of simply elliptic singularities possess simultaneous log resolutions; this can be seen in terms of type II degenerations of K3 surfaces. In this talk I explained how these simultaneous log resolutions could be realized inside the stack of G-bundles over an elliptic curve, by considering the spaces that parametrize the reductions of such a bundle to a Borel subgroup. In particular we see del Pezzo surfaces arising directly from the group; for example, cubic surfaces appear from the simply connected split group E_6. This inverts, in a geometrical way, the classical passage from del Pezzo surface to group by way of root data constructed from the configuration of lines on the del Pezzo.

Constructing Antisymmetric Paramodular Forms

Cris Poor
(joint work with Valery Gritsenko, David S.Yuen)

There are a couple of reasons to compute paramodular cusp forms of low weight. Weight 3 forms give canonical divisors on $\mathcal{A}_2(1, N)$, the moduli space of abelian surfaces of polarization type $(1, N)$. Weight 2 forms occur in the Paramodular

Conjecture of Brumer and Kramer, a conjecture definitively describing the modularity of abelian surfaces. There are also reasons to single out antisymmetric paramodular cusp forms. In 1995, Gritsenko [3] asked for ways of constructing canonical divisors that were antisymmetric. Antisymmetric weight 2 paramodular cusp forms conjecturally correspond to abelian surfaces over $\mathbb{Q}$ whose group of rational points has odd rank.

Let $N \in \mathbb{N}$. We set $K(N) = \mathrm{Stab}_{\mathrm{Sp}_2(\mathbb{Q})}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z})$, where we view elements of $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus N\mathbb{Z}$ as column vectors. The geometric significance of the paramodular groups is that $\mathcal{A}_2(1, N) \cong K(N)\backslash\mathcal{H}_2$.

Here we use Borcherds Products to construct examples of antisymmetric paramodular cusp forms of weights 2 and 3 with applications to both geometry and modularity. Our method relies heavily on the *theta blocks* introduced by Gritsenko, Skoruppa and Zagier in [5]. We use the theta blocks

$$\mathrm{TB}_k[d_1, \ldots, d_\ell] = \eta^{2k} \prod_i \left(\frac{\vartheta_{d_i}}{\eta} \right),$$

for some list $T = [d_1, \ldots, d_\ell]$ of natural numbers. The product $\mathrm{BTB}_T(\zeta) = \prod_i \left(\zeta^{d_i/2} - \zeta^{-d_i/2}\right)$ is called the *baby* theta block. The basic examples of Jacobi forms we make use of are the Dedekind Eta function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}}(1 - q^n)$ and the odd Jacobi theta function:

$$\vartheta(\tau, z) = q^{\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{j\in\mathbb{N}} (1 - q^j\zeta)(1 - q^j\zeta^{-1})(1 - q^j).$$

We have $\vartheta \in J^{\mathrm{cusp}}_{\frac{1}{2},\frac{1}{2}}(\epsilon^3 v_H)$, $\eta \in J^{\mathrm{cusp}}_{\frac{1}{2},0}(\epsilon)$ and $\vartheta_\ell \in J^{\mathrm{cusp}}_{\frac{1}{2},\frac{1}{2}\ell^2}(\epsilon^3 v_H^\ell)$, where $\vartheta_\ell(\tau, z) = \vartheta(\tau, \ell z)$ and $\ell \in \mathbb{N}$, compare [4].

Suppose that we have a Borcherds product $f \in S_k\left(K(p)\right)^\epsilon$ with Fourier-Jacobi expansion

$$f = \phi_p\, \xi^p + \phi_{2p}\, \xi^{2p} + \cdots$$

We assume that f is antisymmetric, that is, $(-1)^k\epsilon = -1$. The antisymmetry of f implies that ϕ_p vanishes to order two, $\phi_p \in J^{\mathrm{cusp}}_{k,p}(2)$. If f is a Borcherds product then ϕ_p is a theta block; indeed, $J^{\mathrm{cusp}}_{2,587}(2)$ is spanned by the single theta block

$$\phi_{587} = \mathrm{TB}_2(1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14).$$

As a Borcherds product, f is determined by its first two Fourier-Jacobi coefficients ϕ_p and ϕ_{2p}. If we write $\phi_{2p} = -\phi_p|V_2 + \Xi$, this forces $\Xi \in J^{\mathrm{cusp}}_{k,2p}(2)$. The involution condition $c(n, r; \phi_{mp}) = (-1)^k\epsilon c(m, r; \phi_{np})$ tells us the q^1 and q^2 coefficients of Ξ. A direct computation reveals that the q^4-coefficient of ϕ_{587} is a baby theta block: $\mathrm{Coeff}(q^4, \phi_{587}) = \mathrm{Coeff}(q^2, \Xi) =$

$$\mathrm{BTB}(1, 10, 2, 2, 18, 3, 3, 4, 4, 15, 5, 6, 6, 7, 8, 16, 9, 10, 22, 12, 13, 14).$$

So we set

$$\Xi = \mathrm{TB}_2(1, 10, 2, 2, 18, 3, 3, 4, 4, 15, 5, 6, 6, 7, 8, 16, 9, 10, 22, 12, 13, 14)$$

and check that $\Xi \in J^{\text{cusp}}_{2,1174}(2)$. The list for this theta block Ξ has been written so that each entry is visibly an integral multiple of the corresponding entry in the theta block ϕ_{587}.

We now combine two methods for constructing weight zero weakly holomorphic Jacobi forms with integral Fourier coefficients and define

$$\psi = \frac{\phi_{587}|V_2 - \Xi}{\phi_{587}} \in J^{\text{w.h.}}_{0,587}(\mathbb{Z}).$$

Here V_2 is the level raising Hecke operator on Jacobi forms defined in [2]. Many Fourier coefficients of $\psi = \sum_{n,r} c(n,r;\psi) q^n \zeta^r$ have been computed, and we see that the associated Borcherds product is holomorphic because the singular Fourier coefficients are positive.

$$f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = q^2 \zeta^{72} \xi^{587} \prod_{\substack{n,r,m\in\mathbb{Z}:\, m\geq 0,\text{ if } m=0 \text{ then } n\geq 0 \\ \text{and if } m=n=0 \text{ then } r<0.}} \left(1 - q^n \zeta^r \xi^{587m}\right)^{c(nm,r;\psi)}$$

This Borcherds product representation of the eigenform $f \in S_2\left(K(587)\right)^-$ will assist the computation of further eigenvalues.

Using this example as a model, we translated the search for meromorphic antisymmetric Borcherds products into a Diophantine problem. We wrote a computer program that found two infinite families of integral solutions, $\mathcal{F}_1$ and $\mathcal{F}_2$.

Definition 1. *Take $c \in \mathbb{N}^{24}$ with $\prod_{j=1}^{24} c_j = 1080$. Define the following algebraic set $A_c = \{d \in \mathbb{C}^{24} : \text{equation (1) holds}\}$.*

$$\text{(1)} \quad \exp\left(\sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n)!} \zeta(1-2n) \sum_{j=1}^{24} (1 - c_j^{2n}) d_j^{2n} z^{2n} \right) = 1 +$$
$$\frac{1}{540} \sum_{n\in\mathbb{N}} \frac{(-1)^n}{(2n)!} \left(\sum_{1\leq i<j\leq 24} \left[(d_i + d_j)^{2n} + (d_i - d_j)^{2n} \right] - \sum_{j=1}^{24} d_j^{2n} \right) z^{2n}$$

Note that A_c is defined by a countable set of homogeneous polynomials, one for each positive even degree.

Theorem 2. *Take $c \in \mathbb{N}^{24}$ with $\prod_{j=1}^{24} c_j = 1080$. Every nontrivial integral point $d \in A_c$ corresponds to an antisymmetric meromorphic paramodular Borcherds product as follows. Let k be the number of zero entries in d, and set $\epsilon = (-1)^{k+1}$. The number $N = \frac{1}{2} \sum_{j=1}^{\ell} d_j^2$ is intergal. Set $m = \prod_{j:d_j=0} c_j$. We have* $\operatorname{Borch}(\psi) \in M_k^{\text{mero}}\left(K(N)\right)^\epsilon$, *where $\psi = \dfrac{\phi|V_2 - m\Xi}{\phi} \in J^{\text{w.h.}}_{0,N}(\mathbb{Z})$, $\phi = \eta^{2k} \prod_{j:d_j\neq 0} (\vartheta_{d_j}/\eta) \in J^{\text{weak}}_{k,N}(2)$, and $\Xi = \eta^{2k} \prod_{j:d_j\neq 0} (\vartheta_{c_j d_j}/\eta) \in J^{\text{weak}}_{k,2N}(2)$. The leading Fourier-Jacobi coefficient of* $\operatorname{Borch}(\psi)$ *is ϕ.*

We are especially interested in holomorphic Borcherds products. A direct search through the two infinite families has located the following finite number of *holomorphic* antisymmetric paramodular Borcherds products.

<u>$k = 2$</u>

N= 713 and N= 893 on $\mathcal{F}_1$. These, like 587 (which however is not on either family) are conjecturally modular with respect to (known) $A/\mathbb{Q}$ with rank 1 and conductor N, as in the Paramodular Conjecture.

<u>$k = 3$</u>

N=167 on $\mathcal{F}_2$, lowest weight 3 antisymmetric *newform.*
N=173 on $\mathcal{F}_1$ and $\mathcal{F}_2$,.
N=197 on $\mathcal{F}_1$.
Also N=122 on $\mathcal{F}_1$ (but this is an oldform), and 213 on $\mathcal{F}_1$ and 285 on $\mathcal{F}_1$.

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Tautological classes with twisted coefficients

Dan Petersen

(joint work with Mehdi Tavakol, Qizheng Yin)

Let M_g be the moduli space of smooth curves of genus $g \geq 2$, and $\pi\colon C_g \to M_g$ the universal curve. Let C_g^n denote the n-fold fibered power of C_g with itself over M_g. If C_g^n is thought of as the space of curves with n ordered marked points, then we may consider the n line bundles on C_g^n whose fibers are given by the cotangent space of the curve at the respective markings; the first Chern classes of these bundles are denoted $\psi_1, \ldots, \psi_n$. The *kappa classes* on M_g are defined by $\kappa_d = \pi_* \psi_1^{d+1}$. We denote by the same symbol the pullbacks of the kappa classes to C_g^n. Finally for $1 \leq i, j \leq n$ we let Δ_{ij} be the locus where the ith and jth point coincide. The *tautological ring* $RH^\bullet(C_g^n)$ is defined to be the subalgebra of $H^\bullet(C_g^n, \mathbf{Q})$ generated by the psi-, kappa- and diagonal classes. (The usual tautological ring is defined as a subalgebra of $\mathrm{CH}^\bullet_{\mathbf{Q}}(C_g^n)$, and everything we say below is in fact valid also in Chow.)

The generators for the tautological rings satisfy the following relations:

$$\Delta_{ij}\Delta_{ik} = \Delta_{ij}\Delta_{jk} \qquad \Delta_{ij}(\psi_i - \psi_j) = 0 \qquad \Delta_{ij}^2 = -\psi_i \Delta_{ij}.$$

The first two are geometrically obvious, and the third is a consequence of the excess intersection formula. The celebrated Madsen–Weiss theorem [5] (formerly Mumford's conjecture), combined with work of Looijenga [4], implies that for any fixed N and $g \gg 0$, the map $RH^\bullet(C_g^n) \to H^\bullet(C_g^n)$ is an isomorphism in degrees up to N, and that the above three relations are the *only* relations between the generators for the tautological ring in degrees up to N. We may in particular think of the tautological ring as the image of the stable cohomology of C_g^n in the unstable cohomology.

If $f\colon C_g^n \to M_g$ is the forgetful map, then by Deligne's degeneration theorem (since f is smooth and proper) there is an isomorphism of $\mathbf{Q}$-vector spaces

$$H^k(C_g^n) \cong \bigoplus_{p+q=k} H^p(M_g, R^q f_* \mathbf{Q}).$$

Moreover, by the relative Künneth formula, there is also an isomorphism $Rf_*\mathbf{Q} \cong (R\pi_*\mathbf{Q})^{\otimes n}$, where π again denotes the projection from the universal curve. Since the local systems $R^0\pi_*\mathbf{Q}$ and $R^2\pi_*\mathbf{Q}$ are trivial, the cohomology of C_g^n is thus completely determined by the cohomology of the local system $\mathbb{V} \overset{\text{def}}{=} R^1\pi_*\mathbf{Q}$, and its tensor powers, on M_g. The local system $\mathbb{V}$ is of rank $2g$ and underlies a polarized variation of Hodge structure of weight 1. The tensor powers of $\mathbb{V}$ may in turn be decomposed in terms of the irreducible representations of $\mathrm{Sp}(2g)$. We denote by $\mathbb{V}_\lambda$ the local system on M_g corresponding to the irreducible representation with highest weight λ.

The upshot of the discussion in the previous paragraph is that the cohomology groups $H^\bullet(C_g^n)$ (where n varies) are determined by, and completely determine, the cohomology groups $H^\bullet(M_g, \mathbb{V}_\lambda)$ (where λ varies). But the cohomology of the local systems "packages" the same information in a much more efficient way. Our first result is that an analogous statement holds for tautological cohomology groups:

Theorem A (informally stated). *Under the correspondence between $H^\bullet(C_g^n)$ and $H^\bullet(M_g, \mathbb{V}_\lambda)$ sketched above, the subspace $RH^\bullet(C_g^n)$ corresponds to a well defined subspace $RH^\bullet(M_g, \mathbb{V}_\lambda)$. Thus the tautological groups $RH^\bullet(C_g^n)$ for varying n are determined by, and completely determine, the tautological groups $RH^\bullet(M_g, \mathbb{V}_\lambda)$ with twisted coefficients, for varying λ.*

In low genus, we are able to determine $RH^\bullet(M_g, \mathbb{V}_\lambda)$ completely, for all λ.

Theorem B. *For $g = 2$,*

$$RH^0(M_2, \mathbb{V}_0) \cong \mathbf{Q}$$

and all other tautological cohomology groups of all $\mathbb{V}_\lambda$ vanish. For $g = 3$,

$$RH^0(M_3, \mathbb{V}_0) \cong RH^2(M_3, \mathbb{V}_0) \cong \mathbf{Q},$$

$$RH^1(M_3, \mathbb{V}_{111}) \cong \mathbf{Q}$$

and all other tautological cohomology groups of all $\mathbb{V}_\lambda$ vanish. For $g = 4$,

$$RH^0(M_4, \mathbb{V}_0) \cong RH^2(M_4, \mathbb{V}_0) \cong RH^4(M_4, \mathbb{V}_0) \cong \mathbf{Q},$$

$$RH^1(M_4, \mathbb{V}_{111}) \cong RH^3(M_4, \mathbb{V}_{111}) \cong \mathbf{Q},$$

$$RH^2(M_4, \mathbb{V}_{11}) \cong \mathbf{Q}$$

and all other tautological cohomology groups of all $\mathbb{V}_\lambda$ *vanish.*

The proof uses Pixton's extension of the "FZ-relations" to C_g^n [6, 2], and some representation theory for the symplectic group.

Theorem B implies quite complete descriptions of the tautological rings of C_2^n, C_3^n and C_4^n for all n. When $g = 2$ our Theorem B recovers results of Tavakol [10] in a different way; the results for $g = 3$ and $g = 4$ are completely new. One also obtains descriptions of the tautological rings of the spaces $M_{g,n}^{rt}$ of n-pointed stable curves with rational tails for $g \leq 4$, since the tautological rings of C_g^n and $M_{g,n}^{rt}$ determine each other [8].

We also have explicit cycles which are generators for the above tautological cohomology groups. For instance, $RH^1(M_g, \mathbb{V}_{111})$ is spanned by the *Gross–Schoen cycle*: the class

$$\Delta_{12}\Delta_{13} - \frac{1}{2g-2}\left(\psi_1\Delta_{23} + \psi_2\Delta_{13} + \psi_3\Delta_{12}\right) + \frac{1}{(2g-2)^2}(\psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3)$$

in $RH^4(C_g^3)$ lies in the first step of the Leray filtration, and turns out to define a class in $RH^1(M_g, \mathbb{V}_{111})$.

The above results are particularly interesting for the study of the *Faber conjectures.* In the 1990's, Faber conjectured that the tautological rings C_g^n should always enjoy Poincaré duality [1], based on extensive computer experiments. More precisely, Looijenga has proved [3] that $RH^{2g-4+2n}(C_g^n) \cong \mathbf{Q}$ and that the tautological ring vanishes above this degree; the conjecture says that the pairing into this degree is perfect. However, since then Faber, Pixton, Yin and others have carried out even further computer experiments and found likely counterexamples to the Faber conjectures. Also, analogous versions of the Faber conjecture on $M_{g,n}^{ct}$ and $\overline{M}_{g,n}$ are now known to be false [9, 7]. Our results imply that one can reformulate the Faber conjecture equivalently in terms of local systems:

Theorem C. *The Faber conjecture holds for* C_g^n *for all* n*, if and only if the pairing*

$$RH^k(M_g, \mathbb{V}_\lambda) \otimes RH^{2g-4-k}(M_g, \mathbb{V}_\lambda) \to RH^{2g-4}(M_g, \mathbb{V}_0) \cong \mathbf{Q}$$

is perfect for all k *and* λ*.*

A corollary of Theorems B and C is that the Faber conjectures are satisfied for C_g^n for all n and all $g \leq 4$, since one can verify the above perfect pairing property for all the above local systems easily enough. This explains the "symmetry" in our expressions for $RH^\bullet(M_g, \mathbb{V}_\lambda)$: for all $g \geq 4$ and all λ, the tautological cohomology groups are symmetric around degree $g - 2$.

For all $g \geq 5$, numerical experimentation suggests that the Faber conjecture *fails*: for example, it appears that $RH^2(M_5, \mathbb{V}_{2222}) = 0$ and $RH^4(M_5, \mathbb{V}_{2222}) \cong \mathbf{Q}$, which would contradict the Faber conjecture. This approach to constructing a counterexample seems more tractable than previous ones: it would now be enough to prove that the single cohomology group $RH^4(M_5, \mathbb{V}_{2222})$ is *nonzero* to find a counterexample, which ought to be easier than explicitly calculating the dimensions of the (very large) tautological groups of C_5^8.

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Height of CM points on orthogonal Shimura varieties

Fabrizio Andreatta

(joint work with Eyal Goren, Ben Howard, Keerthi Madapusi Pera)

Let K be a number filed and let A be an abelian variety over K of relative dimension g. Assume that A has good reduction everywhere, i.e., that the Néron model $\mathcal{A}$ of A over $\mathcal{O}_K$ is an abelian scheme. We recall the definition of Faltings' height of A. Given a non-zero section $s \in \mathrm{H}^0\big(A, \Omega^g_{A_K/K}\big)$ define

$$h^{\mathrm{Fal}}_\infty(A,s) := -\frac{1}{2[K:\mathbb{Q}]}\sum_{\sigma\colon K\to\mathbb{C}} \log\Big|\int_{A^\sigma(\mathbb{C})} s^\sigma \wedge \overline{s^\sigma}\Big|$$

and

$$h^{\mathrm{Fal}}_f(A,s) := \frac{1}{[K:\mathbb{Q}]}\sum_{\mathcal{P}\subset\mathcal{O}_K} \mathrm{ord}_{\mathcal{P}}(s)\log \mathrm{Nm}(\mathcal{P}),$$

here the sum is over all non-zero prime ideals of $\mathcal{O}_K$ and $\mathrm{ord}_{\mathcal{P}}(s)$ is the order of s viewed as a rational section of the invertible $\mathcal{O}_K$-module $\mathrm{H}^0\big(\mathcal{A}, \Omega^g_{\mathcal{A}/\mathcal{O}_K}\big)$

One defines the Falting's height $h^{\mathrm{Fal}}(A)$ of A as the sum $h^{\mathrm{Fal}}(A) := h^{\mathrm{Fal}}_\infty(A,s) + h^{\mathrm{Fal}}_f(A,s)$. The definition is independent of the choice of the section s and the number field K.

In particular if E is a CM field and A has complex multiplication by its ring of integers $\mathcal{O}_E$, then A has potentially good reduction and its height is defined using the formulas above. A theorem of Colmez [2] shows that that $h^{\mathrm{Fal}}(A)$ depends *only* on the field E and the CM type $\Phi \subset \mathrm{Hom}(E,\mathbb{C})$ and not on A itself. We denote such quantity by $h^{\mathrm{Falt}}_{(E,\Phi)}$. Colmez has also provided a conjectural formula for the value of $h^{\mathrm{Falt}}_{(E,\Phi)}$ in terms of the logarithmic derivatives at $s = 0$ of certain Artin L-functions. The first main application of our results is an averaged version of his formula, namely

$$\frac{1}{2^d}\sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{1}{2}\cdot\frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{4}\cdot\log\left|\frac{D_E}{D_F}\right| - \frac{d}{2}\log(2\pi).$$

Here $2d$ is the degree of E, $F \subset E$ is the totally real subfield, χ is the quadratic Hecke character associated to the extension $F \subset E$, the sum on the left is taken over all CM types Φ and D_E and D_F are the discriminants of E and F respectively.

Remarks: Shortly after our announcement X. Yuan and S.-W. Zhang announced a proof of the same result, but using different methods.

Recently J. Tsimerman has proved that the averaged Colmez's conjecture implies the André-Oort conjecture for all Siegel modular varieties, without using GRH.

The result is obtained studying the arithmetic intersection between big CM cycles and certain arithmetic Heegner divisors on orthogonal type Shimura varieties, following a strategy due to T. Yang [3] in the case $d = 2$.

GSpin Shimura varieties and Heegner divisors

Choose an element $\lambda \in F$ such that λ is negative for one real embedding ι_0 of F and positive for all the others, we can associate a quadratic space $V(Q)$ of signature $(2d-2,2)$ as follows. We put $V := E$ and $Q(x) = \mathrm{Tr}_{E/\mathbf{Q}}(\lambda x\bar{x})$. In this way (V,Q) is a quadratic space over $\mathbb{Q}$ of signature $(n,2)$ with $n = 2d-2$.

If we fix a maximal lattice $L \subset V$, we obtain a Shmura variety M defined over $\mathbb{Q}$. The underlying algebraic group is $G = \mathrm{GSpin}(V)$, the compact open subgroup $K \subset G(\mathbb{A}_f)$ is defined using L, and the hermitian symmetric space is

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z,z] = 0,\ [z,\overline{z}] < 0\}/\mathbb{C}^{\times}.$$

The complex points of M coincide with the n-dimensional complex manifold

$$M(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D}\times G(\mathbb{A}_f)/K.$$

Thanks to results of A. Vasiu, M. Kisin for primes not dividing the order D_L of $L^{\vee}/L$, K. Madapusi Pera for primes $p \geq 3$ and W. Kim and K. Madapusi Pera for $p = 2$ it admits a flat and normal integral model $\mathcal{M}$ over $\mathbb{Z}$, which is smooth over $\mathbb{Z}[1/2D_L]$. Furthermore it is endowed with an arithmetic line bundle $\hat{\omega}$ (called the *tautological bundle*): over $\mathbb{C}$ the fiber over a point $[z] \in \mathcal{D}$ is the line $\mathbb{C}z$ with metric $-[z,\overline{z}]$.

From now on we *assume* that $D_L = 1$ to void extra difficulties. The model $\mathcal{M}$ carries a universal abelian scheme $\mathcal{A}$, arising from the Kuga-Satake construction. Denote by $\mathbb{H}(\mathcal{A})$ the *motive* associated to $\mathcal{A}$: the underlying variation of $\mathbb{Z}$-Hodge structures on $\mathcal{M}(\mathbb{C})$, the relative de Rham homology over $\mathcal{M}$, the ℓ-adic Tate module over $\mathbb{Z}[\ell^{-1}]$, the Dieudonné crystal in characteristic p. Then over $\mathcal{M}$ we have a natural sub-motive $\mathbb{V} \subset \mathrm{End}(\mathbb{H}(\mathcal{A}))$ (in each of the realizations mentioned above).

For every $m \in \mathbb{N}$ one defines $\mathcal{Z}(m) \to \mathcal{M}$ as the divisor classifying the endomorphisms f of $\mathcal{A}$ whose realizations lie in $\mathbb{V} \subset \mathrm{End}(\mathbb{H}(\mathcal{A}))$ and such that $f \circ f$ is multiplication by m. Thanks to work of Borcherds, Bruinier, Bruinier and Funke one can endow the invertible sheaf $\mathcal{O}_{\mathcal{M}}(\mathcal{Z}(m))$ with a natural metric at infinity obtaining a metrized line bundle that we denote as $\widehat{\mathcal{Z}}(m)$.

Big CM cycles and the Bruinier-Kudla-Yang theorem

Associated to our field E we also have a $T_E := \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ that acts via isometries on V: given $\alpha \in E^*$ we let it act on $V = E$ via multiplication by $x\bar{x}^{-1}$. One can lift such a map to a morphism of algebraic groups $T_E \to G$. The image $T \subset G$ is a torus. The choice of an embedding ι_0 of E over the given one of F and the choice of a suitable open compact subgroup of $T(\mathbb{A}_f)$ provide a 0-dimensional Shimura variety Y and a morphism of Shimura varieties $Y \to M$. The image consists of special points called *big CM points* in [1]. Upon taking integral models we get an arithmetic curve $\mathcal{Y}$ and a morphism $\mathcal{Y} \to \mathcal{M}$. Given a metrized line bundle $\widehat{\mathcal{L}}$ over $\mathcal{M}$ we define its arithmetic degree $[\mathcal{Y} : \widehat{\mathcal{L}}]$ along $\mathcal{Y}$ to be the arithmetic degree of its pull–back to $\mathcal{Y}$.

In [1] the contribution $[\mathcal{Y} : \widehat{\mathcal{Z}}(m)]_\infty$ at infinity to the arithmetic degree of $\widehat{\mathcal{Z}}(m)$ along $\mathcal{Y}$ is computed and the contribution $[\mathcal{Y} : \widehat{\mathcal{Z}}(m)]_f$ at finite places is conjectured. This fits in the general framework of the Kudla's program as the conjecture predicts that the generating series $\sum_m [\mathcal{Y} : \widehat{\mathcal{Z}}(m)]_f q^m$ is obtained, except for the constant term, by diagonal restriction of the derivative of an incoherent Hilbert Eisenstein series for the field F, of parallel weight 1. Our main result is the proof of such conjecture up to rational multiples of $\log p$ for p primes in a finite collection $D_{\mathrm{bad},L}$ depending on λ and L.

Application to Colmez's conjecture

One proves that the pullback to $\mathcal{Y}$ of the metrized line bundle $\widehat{\omega}$ computes Faltings heights of CM abelian varieties so that

$$\frac{[\widehat{\omega} : \mathcal{Y}]}{\deg_{\mathbb{C}}(Y)} \approx_L \frac{1}{2^{d-2}} \sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} + 2d \cdot \log(2\pi).$$

On the other hand $\widehat{\omega}$ is a rational combination of the Heegner divisors $\widehat{\mathcal{Z}}(m)$. using teh formulas for $[\mathcal{Y} : \widehat{\mathcal{Z}}(m)]$, we find that

$$\frac{1}{2^d} \sum_{\Phi} h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{1}{2} \cdot \frac{L'(0,\chi)}{L(0,\chi)} - \frac{1}{4} \cdot \log \left| \frac{D_E}{D_F} \right| - \frac{d}{2} \log(2\pi) + \sum_p b_E(p) \log(p)$$

for some rational numbers $b_E(p)$, with $b_E(p) = 0$ for all primes p not dividing $D_{\mathrm{bad},L}$.

The last step consists in showing that for every prime p one can choose L such that p does not divide $D_{\mathrm{bad},L}$. This implies that $b_E(p) = 0$, concluding the proof of the averaged Colmez's conjecture.

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Holomorphic torsion invariants for $K3$ surfaces with involution and Borcherds products

Ken-Ichi Yoshikawa

(joint work with Shouhei Ma)

1. Introduction

Let $\mathfrak{H}$ be the complex upper-half plane. For $\tau \in \mathfrak{H}$, let E_τ be an elliptic curve with period $(1, \tau)$, equipped with the flat Kähler metric g_τ with area 1. Let $\square_\tau$ be the Laplacian of (E_τ, g_τ) acting on $C^\infty(E_\tau)$. Let $\zeta_\tau(s)$ be the spectral zeta function of $\square_\tau$. Up to a trivial factor, $\zeta_\tau(s)$ is given by the real analytic Eisenstein series. By Kronecker's limit formula, we have

$$e^{\zeta'_\tau(0)} = (2\|\eta(\tau)\|^4)^{-1}, \tag{1.1}$$

where $\eta(\tau)$ is the Dedekind η-function, $\|\eta(\tau)\|$ is its Petersson norm, and the quantity $e^{\zeta'_\tau(0)}$ is known as the (holomorphic) analytic torsion. In this note, we report a recent progress on a generalization of (1.1) for $K3$ surfaces with involution.

2. 2-elementary $K3$ surfaces and their moduli space

The pair (X, ι) consisting of a $K3$ surface X and an anti-symplectic involution $\iota\colon X \to X$ is called a 2-elementary $K3$ surface. Let $\mathbb{L}_{K3}$ be the $K3$-lattice, i.e., a fixed even unimodular lattice of signature $(3, 19)$. Let $M \subset \mathbb{L}_{K3}$ be a sublattice such that $M \cong H^2(X, \mathbf{Z})^+$, where $H^2(X, \mathbf{Z})^+ \subset H^2(X, \mathbf{Z})$ is the invariant sublattice of $H^2(X, \mathbf{Z})$ with respect to the ι-action. Then M is a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$, whose isometry class determines the deformation type of (X, ι). There exist 75 distinct deformation equivalence classes of 2-elementary $K3$ surfaces, each of which is labeled by (an isometry class of) a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$. (See [1].)

We set $\Lambda := M^{\perp_{\mathbb{L}_{K3}}}$. Then Λ is a 2-elementary lattice of signature $(2, 20-r)$ isometric to $H^2(X, \mathbf{Z})^-$, the anti-invariant sublattice of $H^2(X, \mathbf{Z})$, where $r := \mathrm{rk}_{\mathbf{Z}} M$. Let $\Omega_\Lambda := \{[\eta] \in \mathbf{P}(\Lambda \otimes \mathbf{C});\ \langle \eta, \eta \rangle = 0,\ \langle \eta, \bar{\eta} \rangle > 0\}$ be the Hermitian domain of type IV attached to Λ. By the global Torelli theorem and the surjectivity of the period mapping for $K3$ surfaces, the moduli space of 2-elementary $K3$ surfaces of type M is given by the modular variety of dimension $20 - r$

$$\mathcal{M}^0_\Lambda := O(\Lambda) \backslash (\Omega_\Lambda - \mathcal{D}_\Lambda),$$

where $\mathcal{D}_\Lambda := \bigcup_{d\in\Lambda,\, d^2=-2} d^\perp$ is the discriminant divisor of Ω_Λ (cf. [10]). There is another distinguished divisor $\mathcal{H}_\Lambda$, called the characteristic Heegner divisor, associated to the norm $(12-r(\Lambda))/2$ characteristic vectors of $\Lambda^\vee$.

3. Holomorphic torsion invariants for 2-elementary $K3$ surfaces

Let (X,ι) be a 2-elementary $K3$ surface of type M. In what follows, we set $\Lambda = M^\perp$ and $r = \mathrm{rk}_{\mathbf{Z}}M$. Let X^ι be the set of fixed points of ι. Then either X^ι is empty or the disjoint union of smooth curves. Let γ be an ι-invariant Ricci-flat Kähler metric on X. By [2], we have a spectral invariant $\tau_{\mathbf{Z}_2}(X,\gamma)(\iota)$ of (X,γ,ι) called the equivariant analytic torsion. Similarly, by [3], for a theta characteristics Σ on X^ι, we have a spectral invariant $\tau(X^\iota,\Sigma;\gamma|_{X^\iota})$ of $(X^\iota,\Sigma;\gamma|_{X^\iota})$ called the analytic torsion. Here $\tau(X^\iota,\Sigma;\gamma|_{X^\iota})$ is understood to be multiplicative with respect to the irreducible decomposition of X^ι. We define

$$\tau_M^{\mathrm{spin}}(X,\iota) := \prod_{\Sigma^2=K_{X^\iota},\, h^0(\Sigma)=0} \mathrm{Vol}(X,\gamma)^{\frac{14-r}{4}}\tau_{\mathbf{Z}_2}(X,\gamma)(\iota)\,\tau(X^\iota,\Sigma;\gamma|_{X^\iota})^{-2},$$

where Σ runs over all ineffective even theta characteristics on X^ι. Thanks to the theory of (equivariant) Quillen metrics [2], [3], [8], $\tau_M^{\mathrm{spin}}(X,\iota)$ is independent of the choice of γ. Hence τ_M^{spin} is viewed as a function on $\mathcal{M}_\Lambda^0$. In general, τ_M^{spin} is a smooth function on $\mathcal{M}_\Lambda^0 \setminus \mathcal{H}_\Lambda$. However, if $\mathcal{H}_\Lambda \neq \emptyset$, τ_M^{spin} is discontinuous along $\mathcal{H}_\Lambda$, because of the jumping of $\tau(X^\iota,\Sigma;\gamma|_{X^\iota})$ as X^ι approaches to a curve with extra effective even theta characteristics. Thus $\mathcal{H}_\Lambda$ is the jumping locus of τ_M^{spin}.

4. An explicit formula for τ_M^{spin}

To express τ_M^{spin} as a Borcherds product, we use a vector-valued modular form constructed as follows. For a primitive 2-elementary sublattice $\Lambda \subset \mathbb{L}_{K3}$ of signature $(2,n)$, we set $\phi_\Lambda(\tau) := \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\theta_{\mathbb{A}_1^+}(\tau)^{10-n}$, where $\theta_{\mathbb{A}_1^+}(\tau)$ is the theta series of the A_1-lattice $\langle 2\rangle$. Let $\rho_\Lambda\colon \mathrm{Mp}_2(\mathbf{Z}) \to \mathrm{GL}(\mathbf{C}[\Lambda^\vee/\Lambda])$ be the Weil representation attached to Λ. Let $\{\mathfrak{e}_\gamma\}_{\gamma\in\Lambda^\vee/\Lambda}$ be the standard basis of the group ring $\mathbf{C}[\Lambda^\vee/\Lambda]$. We set

$$F_\Lambda(\tau) := \sum_{\gamma\in\widetilde{\Gamma}_0(4)\backslash\mathrm{Mp}_2(\mathbf{Z})} \phi_\Lambda|_\gamma(\tau)\,\rho_\Lambda(\gamma^{-1})\mathfrak{e}_0,$$

where $|_\gamma$ denotes the Petersson slash operator. The $\mathbf{C}[\Lambda^\vee/\Lambda]$-valued function F_Λ is a modular form for $\mathrm{Mp}_2(\mathbf{Z})$ of type ρ_Λ of weight $1-\frac{n}{2}$. If $r(\Lambda)\leq 16$, F_Λ has integral Fourier coefficients. If g is the total genus of the fixed curve of a 2-elementary $K3$ surface of type $\Lambda^\perp$, $2^{g-1}F_\Lambda$ has integral Fourier coefficients.

For a modular form φ of type ρ_Λ of weight $1-\frac{n}{2}$ with integral Fourier coefficients, let $\Psi_\Lambda(\cdot,\varphi)$ denote the Borcherds lift of φ (cf. [4]). For an even 2-elementary lattice M, the parity of the discriminant form on the discriminant group $M^\vee/M$ is denoted by $\delta(M)\in\{0,1\}$. Now, our main result is stated as follows [7].

Main Theorem. *The following equality of functions on $\mathcal{M}_\Lambda^0 \setminus \mathcal{H}_\Lambda$ holds*

$$\tau_M^{\mathrm{spin}} = C_M\,\|\Psi_\Lambda(\cdot, 2^{g-1}F_\Lambda + f_\Lambda)\|^{-1/2}, \tag{4.1}$$

where C_M is a constant depending only on the lattice M and f_Λ is given as follows:
(1) *If $(r,\delta) \neq (2,0)$, then $f_\Lambda = \delta_{r,10}\, F_\Lambda$.*
(2) *If $(r,\delta) = (2,0)$, then either $\Lambda \cong \mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2}$ or $\Lambda \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8^{\oplus 2}$ and*

$$f_\Lambda(\tau) = \theta_{\mathbb{E}_8^+}(\tau)/\eta(\tau)^{24} = E_4(\tau)/\eta(\tau)^{24} \qquad (\Lambda \cong \mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2})$$

$$f_\Lambda(\tau) = 8 \sum_{\gamma \in \Lambda^\vee/\Lambda} \{\eta(\tau/2)^{-8}\eta(\tau)^{-8} + (-1)^{\gamma^2}\eta(\frac{\tau+1}{2})^{-8}\eta(\tau+1)^{-8}\}\mathfrak{e}_\gamma + \eta(\tau)^{-8}\eta(2\tau)^{-8}\mathfrak{e}_0 \qquad (\Lambda \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8^{\oplus 2}).$$

We remark that there is a non-twisted holomorphic torsion invariant τ_M for 2-elementary $K3$ surfaces [10]. By the spin-1/2 bosonization formula, τ_M and τ_M^{spin} are equivalent invariants. See [7] for an explicit formula for τ_M.

Problem 1. Determine the universal constant C_M in Main Theorem.

Problem 2. Understand a geometric meaning of the modular form F_Λ. In [5], [6], for some M, F_Λ appears as the elliptic genus of some bundle over a $K3$ surface. Is a similar understanding possible for all F_Λ or $2^{g-1}F_\Lambda + f_\Lambda$?

Problem 3. When $\mathcal{H}_\Lambda \neq \emptyset$, $\|\Psi_\Lambda(\cdot, 2^{g-1}F_\Lambda + f_\Lambda)\|$ is a discontinuous function on $\mathcal{M}_\Lambda^0$ (cf. [9]). As an equality of discontinuous functions on $\mathcal{M}_\Lambda^0$, does (4.1) remain valid?

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Birational geometry of the moduli space of quartic $K3$ surfaces

Radu Laza
(joint work with Kieran O'Grady)

An important problem in algebraic geometry is to construct a geometric compactification for the moduli space of polarized degree d $K3$ surfaces $\mathscr{F}_d$. By Global Torelli, $\mathscr{F}_d$ is isomorphic to a locally symmetric variety, and hence it has natural compactifications, such as Baily-Borel compactification $\mathscr{F}_d^*$, Mumford's toroidal compactifications, and more generally Looijenga's semitoric compactifications. However, a priori, none of these compactifications have geometric meaning. In order to attach some geometric meaning to them, it is natural to compare these compactifications, especially the Baily-Borel one, with GIT compactifications. Looijenga ([9, 10]) has proposed a framework to compare locally symmetric varieties (associated to type IV or $I_{1,n}$ Hermitian symmetric domains) with GIT quotients. Looijenga's approach was successfully applied in the case of moduli of degree 2 $K3$ surfaces ([8], [13]), cubic fourfolds ([11], [7]), and a few other related examples (e.g. cubic threefolds, see [1], [12], del Pezzo surfaces, etc.). One case where the Looijenga's framework does not apply is the moduli space of degree 4 $K3$ surfaces ([14]). While attempting to study this case, we uncovered a rich and intriguing picture.

The starting point of our investigation are two limitations in Looijenga's construction. First of all, a certain technical assumption for Looijenga's construction is false for quartic $K3$s, while, in contrast, for the degree 2 case this assumption is satisfied. Namely, for arithmetic reasons, the combinatorics of the hyperplane arrangement involved in Looijenga's construction [10] is much simpler for degree 2 $K3$ surfaces (and similarly cubic fourfolds) than for degree 4 $K3$ surfaces. Secondly, and more seriously, there exists a plethora of GIT models. In the low degree cases considered here and in the literature, there might be a "natural" choice for GIT, but this is misleading (see [3] for a hint of what would happen already in degree 6). The solution that we propose to handle these two issues is to give flexibility to Looijenga's construction by considering a continuous variation of models. More precisely, we recall that for a locally symmetric variety $\mathscr{F} = \mathscr{D}/\Gamma$, Baily-Borel have shown that the eponymous compactification $\mathscr{F}^*$ is the Proj of the ring of automorphic functions, i.e. $\mathscr{F}^* = \operatorname{Proj} R(\mathscr{F}, \lambda)$, where λ is the Hodge bundle. Looijenga's deep insight was to observe that in certain situations of geometric interest, a certain GIT quotient $\overline{\mathcal{M}}$ is nothing but the Proj of the ring of meromorphic automorphic forms with poles on a (geometrically meaningful) Heegner (or Noether–Lefschetz) divisor Δ, and thus $\overline{\mathcal{M}} = \operatorname{Proj} R(\mathscr{F}, \lambda + \Delta)$. Furthermore, Looijenga has shown that under a certain assumption on Δ (which fails for quartics), $\operatorname{Proj} R(\mathscr{F}, \lambda + \Delta)$ has an explicit combinatorial/arithmetic description. Our approach is to continuously interpolate between the two models by controlling the order of poles for the meromorphic automorphic function, i.e. to consider $\operatorname{Proj} R(\mathscr{F}, \lambda + \beta\Delta)$ where $\beta \in [0, 1]$. This allows to understand the case of quartics and more importantly to capture more GIT quotients.

The variational approach to studying birational maps is the natural one from the perspective of the modern MMP ([2]), and its power in the context of moduli is clear ever since the VGIT theory of Thaddeus [15]. An inspiration for our work is the so called Hassett-Keel program for moduli of curves (which studies the variation of log canonical models $\operatorname{Proj} R(\overline{M}_g, K+\alpha\Delta)$ of the Deligne-Mumford compactification $\overline{M}_g$). Experts in the field have speculated on analogue of Hassett-Keel program (see [5, 6]) for (special) surfaces; for instance, one can hope to understand the elusive KSBA compactification by starting with a GIT compactification and interpolating (see [4]). Our study can be viewed as a first example of a Hassett-Keel program for surfaces. Indeed, beyond the obvious analogy (i.e. $\lambda+\beta\Delta$ can be easily rewritten as $K_{\mathscr{F}}+\alpha\Delta'$), the modular behavior is also similar: for instance, the first birational wall crossing for quartic $K3$s is associated to Dolgachev (or triangle) singularities in a manner similar to the case of curves with cusp singularities (which gives the first birational modification for $\operatorname{Proj} R(\overline{M}_g, K+\alpha\Delta)$ at $\alpha=\frac{9}{11}$). The main point that we want to emphasize here is that the birational transformations that occur are controlled by the arithmetic and combinatorics of the hyperplane arrangement associated to Δ. While the picture that we discover for quartic $K3$ surfaces is more complicated and subtle than that for $K3$ surfaces of degree 2, the fundamental insight of Looijenga that arithmetic controls the birational models of $\mathscr{F}_d^*$ still holds true. We view our work as a quantitative and qualitative refinement of Looijenga's seminal work [10].

While a variation of models $\operatorname{Proj} R(\mathscr{F}, \lambda+\beta\Delta)$ makes sense in large generality for Type IV locally symmetric varieties (and also ball quotients), we focus here on the so called D-tower of locally symmetric varieties, i.e. Type IV locally symmetric varieties associated to lattices $U^2 \oplus D_n$. More precisely, we let $\mathscr{F}(N)$ be the N-dimensional locally symmetric variety corresponding to the lattice $\Lambda_N := U^2 \oplus D_{N-2}$ (so that $\dim \mathscr{F}(N) = N$), and an arithmetic group Γ_N, which is intermediate between the orthogonal group $O^+(\Lambda_N)$, and the stable orthogonal subgroup $\widetilde{O}^+(\Lambda_N)$. Then $\mathscr{F}(19)$ is the period space for quartic $K3$ surfaces, $\mathscr{F}(20)$ is the period space for double EPW sextics modulo the duality involution (i.e. non-polarized desingularized EPW sextics), and $\mathscr{F}(18)$ is the period space for hyperelliptic quartic surfaces. The salient point in the D tower is that $\mathscr{F}(N-1)$ is isomorphic to a natural "hyperelliptic" Heegner divisor $H_h(N)$ of $\mathscr{F}(N)$, and this leads to an inductive behavior. From the perspective of comparing to GIT we are led to study the variation of models under $\lambda(N)+\beta\Delta(N)/2$, where $\lambda(N)$ is the automorphic (or Hodge) orbiline bundle on $\mathscr{F}(N)$, and $\Delta(N)$ is $H_h(N)$ except when $N \equiv 3, 4 \pmod 8$, in which case $\Delta(N) = H_h(N) + H_u(N)$ where $H_u(N)$ is the "unigonal" divisor. *Our main result is to predict the critical values of β where birational modifications of the models occur, and to identify the centers of these birational modifications.* Of course, the same type of arguments apply in general for predicting the behavior of the interpolation between Baily-Borel with say Chow GIT of 1-embedded $K3$ surfaces (and again, due to Mayer's Theorem, the essential actors are hyperelliptic H_h and unigonal divisor H_u), but the actual predictions depend very much on the arithmetic of the primitive lattice of a very general polarized $K3$ of degree d. For instance, in the degree 2 case, for arithmetic reasons, the situation is very simple leading to Looijenga/Shah result (while in $d = 2$ our analysis would be essentially empty, our perspective might give deeper information, e.g. about the structure of the moving cone, than what is currently known). In fact, the hyperelliptic divisor is the one leading to the failure of the technical condition in Looijenga [10], and thus essentially the analysis of the quartic surface case is the key case for quartic surfaces (complementing [10]).

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The geometric theta lift for non-compact quotients of the complex n-ball

Jens Funke

(joint work with John Millson)

The theory of cohomology of arithmetic groups is one critical aspect of automorphic forms. One very fruitful line of investigation in this direction is to employ the theta correspondence.

Let V be a Hermitian space of signature $(n,1)$ over an imaginary quadratic field, and let D be the associated symmetric space, which in our case can be realized as the complex n-ball. We let Γ be a suitable congruence subgroup, so that $X = \Gamma\backslash D$ is a quasi-projective algebraic variety. Following the classical work of Hirzebruch-Zagier for Hilbert modular surfaces, one can define a family of special (non-compact) divisors Z_N in X parameterized by non-negative integers N. Then Kudla and Millson [5] showed that the generating series of these cycles

$$P(\tau) := \sum_{N\geq 0}^{\infty} [Z_N]q^N \in M_{n+1}(\Gamma') \otimes H^2(X)$$

defines a holomorphic modular form of weight $n+1$ with respect to a congruence subgroup Γ' of $\mathrm{SL}_2(\mathbb{Z})$ with values in the second cohomology group of X. (Here $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, the upper half plane). One can then obtain scalar-valued modular forms by pairing P with homology classes in $H_2(X)$ or with classes in $H_c^{2(n-1)}(X)$, the cohomology of compact supports. In [5], $P(\tau)$ is realized as a certain theta series $\theta_V(\tau,z)$ associated to the indefinite space V. More precisely, $\theta_V(\tau,z)$ as a function of $\tau \in \mathbb{H}$ transforms as a non-holomorphic modular form of weight $n+1$ with respect to Γ', while as a function of $z \in X$ it is a closed differential $(1,1)$-form on X. Then the resulting cohomology class $[\theta_V(\tau,z)] \in H^2(X)$ is equal to $P(\tau)$ via Poincaré duality. The work in [5] actually deals with the situation in *much* greater generality for cycles of arbitrary codimension for arithmetic quotients for orthogonal and unitary groups of any signature (p,q).

Millson and I have been pursuing a program to systematically extend the theta lift defined by θ_V to (co)homology groups which capture the boundary of X. For several cases we have succeeded to do so, namely for Hilbert modular surfaces providing a different prespective on the celebrated work of Hirzebruch-Zagier [4], and also for cycles in modular curves, extending the Shintani-lift to Eisenstein series in a cohomological way [3]. The work described below should be seen in this context.

We let ℓ be a rational isotropic line in V. Then $W := \ell^{\perp}/\ell$ is a positive definite Hermitian space of dimension $n-1$. Let P be the real parabolic stabilizing ℓ. Its nilpotent N subgroup is a Heisenberg: $N = W_{\mathbb{R}} \ltimes \mathbb{R}$. For convenience we assume that X has only one cusp. We set $X_\infty = X_\ell = (\Gamma \cap N)\backslash N$. This is a circle bundle over a complex torus T_W associated to W. Note we have a natural map $H^*(T_W) \to H^*(X_\infty)$. We obtain the Borel-Serre compactification $\overline{X} = X \cup X_\infty$ which gives a real manifold with boundary. Also note that the Baily-Borel compactification $\widetilde{X}$ is obtained by collapsing the center of the Heisenberg N. Then the boundary divisor $\widetilde{X}_\infty$ is the torus T_W.

Theorem 1.

(i) The theta series $\theta_V(\tau)$ extends to a differential 2-form on $\overline{X}$, and the restriction of $\theta_V(\tau)$ to the boundary stratum is a non-holomorphic theta series of weight $n+1$ associated to the positive definite space W with values in the image of $H^2(T_W)$ in $H^2(X_\infty)$.

(ii) Assume $n = 2$, so that X is a Picard modular surface. Then the restriction to the boundary is an *exact* differential form on X_∞. Moreover, a primitive can be explicitly constructed as a non-holomorphic theta series $\theta_W(\tau)$ of weight 3 for W with values in the differential 1-forms on X_∞:

$$\theta_V|_{X_\infty} = d\theta_W.$$

We now consider the mapping cone associated to the embedding of X_∞ into $\overline{X}$. Then its cohomology realizes the cohomology of compact supports of X. In this realization a class in degree 2 is represented by a pair $[\alpha, \beta]$ with α a 2-form on $\overline{X}$ and β a 1-form on the boundary such that $\alpha|_{\overline{X}} = d\beta$. This (and further analysis) yields

Theorem 2. Let $n = 2$.

(i) The pair $[\theta_V(\tau), \theta_W(\tau)]$ defines a class in the compactly supported cohomology $H^2_c(X)$ of X, which as a modular form of weight 3 is *holomorphic.* (Both $\theta_V(\tau)$ and $\theta_W(\tau)$ are individually non-holomorphic in τ). Thus

$$[\theta_V, \theta_W] \in M_3(\Gamma') \otimes H^2_c(X).$$

In particular, this class $[\theta_V(\tau), \theta_W(\tau)]$ can be cohomologically paired with any class in $H^2(X)$ to obtain scalar-valued holomorphic modular forms.

(ii) Furthermore,

$$[\theta_V, \theta_W] = \sum_{N \geq 0} [Z^c_N] q^n.$$

for some (appropriately) capped cycles Z^c_N on $\overline{X}$. (The intersection of the cycles Z_N with X_∞ yields 1-chains which are boundaries in X_∞).

By considering the Baily-Borel compactification $\widetilde{X}$ we also obtain a result for $[\theta_V, \theta_W]$ in a more algebraic-geometric setting. The cohomology of compact supports for X is then replaced by the piece of the cohomology of $H^2(\widetilde{X})$ which is orthogonal to the boundary divisor with respect to the intersection product on $H^2(\widetilde{X})$. This recovers work of Cogdell [2].

For $n > 2$, the restriction of θ_V is no longer exact; the obstruction can be explicitly described. We are currently working on a modification of our construction which again should yield holomorphic modular forms of weight $n + 1$ with an interesting geometric interpretation. In fact, using the theory of singular theta lifts going back to Borcherds, Bruinier-Howard-Yang [1] obtained such a result (in a slightly stronger setting). However, one aspect is that we expect that our constructions also work for cycles of higher codimension r, where the associated Kudla-Millson generating series now define Hermitian modular forms for $\mathrm{U}(r, r)$. In this situation we do not have a theory of singular theta lifts, so the methods of [1] are not available (yet).

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On the Kodaira dimension of orthogonal modular varieties

Shouhei Ma

Let L be an integral lattice of signature $(2, n)$ and $\mathrm{O}(L)$ be its full orthogonal group. Let $\mathcal{D}_L$ be the Hermitian symmetric domain attached to L and $\mathrm{O}^+(L) < \mathrm{O}(L)$ be the index ≤ 2 subgroup preserving $\mathcal{D}_L$. We are interested in the birational type of the quotient space

$$\mathcal{F}_L = \mathrm{O}^+(L)\backslash\mathcal{D}_L,$$

which has the structure of a quasi-projective variety of dimension n. Note that this space is invariant by scaling L.

Theorem 1. *There are up to scaling only finitely many integral lattices L of signature $(2, n)$ with $n \geq 21$ or $n = 17$ such that $\mathcal{F}_L$ is not of general type.*

In particular, when n is sufficiently large, $\mathcal{F}_L$ is always of general type, and this covers the general case of arithmetic group in that range.

Corollary 2. *Let V be a rational quadratic space of signature $(2, n)$ with n sufficiently large (e.g. $n \geq 300$) and Γ be an arithmetic subgroup of $\mathrm{O}^+(V)$. Then the quotient space $\Gamma\backslash\mathcal{D}_V$ is always of general type.*

In order to construct sufficiently many pluricanonical forms, we use a generalization of the approach proposed by Gritsenko-Hulek-Sankaran [4], which is a combination of the Gritsenko-Borcherds additive lifting ([2], [1]) and estimate of the Hirzebruch-Mumford proportionally constants ([6], [3]).

The main idea of the proof is to show, in a quantitative way, that $\mathcal{F}_L$ can be of non-general type only when the lattice L is "small". Here we use the dimension n and the exponent of the discriminant group $L^{\vee}/L$ (i.e., the maximal order of elements) as the measure of "size". The proof is effective, so it is in principal possible to list all *potential* non-general type L (but this should contain large redundancy). The proof also has the following consequences:

Corollary 3. *Let $n \geq 22$ be an even number. Then for every lattice L of signature $(2, n)$ the quotient $\mathrm{SO}^+(L)\backslash\mathcal{D}_L$ is of general type.*

Corollary 4. *Let $N > 0$ be a fixed natural number. Then there are up to scaling only finitely many lattices L with $n \geq 4$ which carries a reflective modular form of vanishing order $\leq N$.*

The last corollary (especially the $N = 1$ case) was conjectured by Gritsenko-Nikulin in [5].

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Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections

Viacheslav V. Nikulin
(joint work with Valery Gritsenko)

One of the most known examples of Lorentzian Kac–Moody algebras is the Fake Monster Lie algebra defined by R. Borcherds (see [3]–[4]) in his solution of Moonshine Conjecture. Lorentzian Kac–Moody (Lie, super) algebras are automorphic corrections of hyperbolic Kac–Moody algebras. In our papers [32], [33], [34], [12]–[19], we developed a general theory of Lorentzian Kac–Moody algebras (see [16] and [17] for the most complete exposition) based on the results by Kac [22]–[24], Moody, and Borcherds [1]–[4]. In these our papers (especially see [16] and [17]), we constructed and classified some of these algebras for the rank 3.

Here, we construct and classify some of Lorentzian Kac–Moody algebras for all ranks ≥ 3. In our papers above and here, we mainly consider and classify Lorentzian Kac–Moody algebras with Weyl groups W of 2-reflections. They are groups generated by reflections in elements with square 2 of hyperbolic (that is of signature $(n, 1)$) lattices (that is integral symmetric bilinear forms) S of rkS = n+1.

For an (automorphic) Lorentzian Kac–Moody Lie algebra with a hyperbolic *root* lattice S, the Weyl group W must have the fundamental chamber $\mathcal{M}$ of finite (*elliptic case*) or almost finite (*parabolic case*) volume in the hyperbolic space $\mathcal{L}(S) = V^+(S)/\mathbb{R}_{++}$ where $V^+(S)$ is a half of the cone $V(S) \subset S \otimes \mathbb{R}$ of elements $x \in S \otimes \mathbb{R}$ with $x^2 < 0$. For parabolic case, there exists a point $c = \mathbb{R}_{++}r \in \mathcal{M}$,

$r \in S$, $r \neq 0$ and $r^2 = 0$, at infinity of $\mathcal{M}$ such that $\mathcal{M}$ is finite at any cone in $\mathcal{L}(S)$ with the vertex at c.

We denote by $P = P(\mathcal{M}) \subset S$ the set of *simple real roots* or all elements of S with square 2 which are perpendicular to faces of codimension one of $\mathcal{M}$ and directed outwards. For a Lorentzian Kac–Moody algebra, $P = P(\mathcal{M})$ must have *the lattice Weyl vector* $\rho \in S \otimes \mathbb{Q}$ such that $(\rho, \alpha) = -\alpha^2/2 = -1$ for all $\alpha \in P = P(\mathcal{M})$. For elliptic case, $\rho^2 = (\rho, \rho) < 0$, and $\rho^2 = 0$ for parabolic case where $\mathbb{R}_{++}\rho = c$. For elliptic case, W has finite index in $O(S)$, then S is called *elliptically* 2*-reflective*. For parabolic case, $O^+(S)/W$ is $\mathbb{Z}^m$, up to finite index, for some $m > 0$. We want to construct Lorentzian Kac–Moody algebras with the root lattice S, the set of simple real roots $P = P(\mathcal{M}) \subset S$ and the Weyl group W.

Here, we consider the basic case of this problem when the Weyl group W is the full group $W = W^{(2)}(S)$ generated by all reflections in vectors with square 2 of a hyperbolic even lattice S.

All elliptically 2-reflective hyperbolic lattices S when the group $W^{(2)}(S)$ has finite index in $O(S)$ were classified by the second author in [27] and [30] for rkS $\neq 4$, and by E.B. Vinberg [35] for rkS = 4. See also [31]. Their total number is finite and rkS ≤ 19. They classify algebraic K3 surfaces over $\mathbb{C}$ with finite automorphism groups. The number of parabolically 2 reflective hyperbolic lattices S for $W = W^{(2)}(S)$ is also finite by [33], but their full classification is unknown. Many of them were found in [27].

We use the list of elliptically 2-reflective even hyperbolic lattices S from [27], [30] and [35], to find those of them which have the lattice Weyl vector ρ for $P = P(\mathcal{M})$ of $W^{(2)}(S)$. **There are 59 such lattices.** 15 of them are of rank 3 and 44 of rank ≥ 4, and the maximal rank is equal to 18. For all these lattices S, we calculate the set $P = P(\mathcal{M}) \subset S$ of simple real roots and its Dynkin diagram which is equivalent to the generalized Cartan matrix

$$A = ((\alpha_1, \alpha_2)), \quad \alpha_1, \alpha_2 \in P = P(\mathcal{M}). \tag{1}$$

This matrix defines the usual hyperbolic Kac–Moody algebra $\mathfrak{g}(A)$, see [22]. We calculate the lattice Weyl vector ρ for $P = P(\mathcal{M})$ for all these cases.

For an extended lattice $T = U(m) \oplus S$ of signature $(n + 1, 2)$ where U is the even unimodular lattice of signature $(1, 1)$, $U(m)$ means that we multiply the pairing of the lattice U by some $m \in \mathbb{N}$, and $\oplus$ is the orthogonal sum of lattices, we consider the Hermitian symmetric domain $\Omega(T) \cong S \otimes \mathbb{R} + iV^+(S)$. For all 59 lattices S of Theorem 3, we conjecture existence for some m of so called 2-reflective holomorphic automorphic form $\Phi(z) \in M_k(\Gamma)$ on $\Omega(T)$ of weight $k > 0$ with integral Fourier coefficients, where $\Gamma \subset O(T)$ is of finite index, whose divisor is union of rational quadratic divisors with multiplicity one orthogonal to elements with square 2 of T. *The Fourier coefficients of* $\Phi(z)$ *at a* 0*-dimensional cusp define additional sequence of simple imaginary roots* ${P'}^{im} \subset S$ *with non-positive squares.* The sequences of the simple real roots P and the imaginary simple roots ${P'}^{im}$ define Lorentzian Kac–Moody–Borcherds Lie superalgebra $\mathfrak{g}(P(\mathcal{M}), \Phi)$ by exact generators and defining relations. This superalgebra is the *(automorphic)*

Lorentzian Kac–Moody super-algebra which we want to construct. The Lorentzian Kac–Moody (Lie super) algebra $\mathfrak{g}(P(\mathcal{M}), \Phi)$ is graded by S. *The dimensions* $\dim \mathfrak{g}_\alpha(P(\mathcal{M}), \Phi)$, $\alpha \in S$, *of this grading (equivalently, the multiplicities of all roots of the algebra) are defined by the Borcherds product expansion of the automorphic form* $\Phi(z)$ *at a zero dimensional cusp.* See our papers above for the exact definitions and details of the automorphic correction.

We determine automorphic corrections for 36 of 59 lattices of Theorem 3 but we consider here more than 70 reflective modular forms. We are planing to construct automorphic corrections for the rest 10 of 2-reflective lattices of rank 4 and 5 from Theorem 3 in a separate publication. Some of these functions will be modular with respect to congruence subgroups similar to [17].

We remark that the denominator functions of the corresponding Lorentzian Kac–Moody algebras are automorphic discriminants of moduli spaces of some $K3$ surfaces with a condition on Picard lattices and they realise the arithmetic mirror symmetry for some of such $K3$ surfaces and some of $K3$ surfaces with finite automorphism groups (see [14], [18] and [19]).

See details and some other results and remarks in our recent preprint [20].

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Riemann–Roch isometries in the non-compact orbifold setting

Anna-Maria von Pippich
(joint work with Gerard Freixas i Montplet)

1. Introduction

A fundamental result in intersection theory is the arithmetic Riemann–Roch theorem for arithmetic varieties by Gillet and Soulé [6]. This theorem developed from previous versions by Faltings [3] and Deligne [2], who treated the case of arithmetic surfaces. Deligne's isometry and the arithmetic Riemann–Roch theorem both require the vector bundles to be endowed with smooth hermitian metrics. However, many cases of arithmetic interest do not satisfy this assumption, for example, the case of a modular curve, when considering the trivial bundle and the dualizing sheaf endowed with the Poincaré metric. Already in this case the metric is singular at the cusps and the elliptic fixed points, and the results of Deligne and Gillet–Soulé do not apply to this setting. In presence only of cusps, hence excluding elliptic fixed points, Freixas proved a version of the arithmetic Riemann–Roch theorem for the trivial sheaf on a modular curve [4]. His method of proof has the drawback that it cannot be adapted to the presence of elliptic fixed points and does not carry over to more general bundles or to higher dimensions. Therefore, one needs to develop new ideas that are better suited to these more general settings.

2. Statement of the main theorem

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind. The quotient space $\Gamma\backslash\mathbb{H}$ admits a canonical structure of a Riemann surface. The points with non-trivial automorphisms are called elliptic fixed points. By adding a finite number of cusps, the Riemann surface $\Gamma\backslash\mathbb{H}$ can be completed into a compact Riemann surface X. We denote the set of elliptic fixed points and cusps by $p_1, \ldots, p_n$, and assign to them multiplicities $m_1, \ldots, m_n \in \mathbb{N} \cup \{\infty\}$. The multiplicity of a cusp is ∞, while for an elliptic fixed point it is the order of its automorphism group. We set $m := \prod_{m_i<\infty} m_i$. The hyperbolic metric on $\mathbb{H}$ is given by

$$ds^2_{\mathrm{hyp}} = \frac{dx^2 + dy^2}{y^2},$$

where $x+iy$ is the usual parametrization of $\mathbb{H}$. As a metric on X, it has singularities at the cusps and the elliptic fixed points.

In the recent preprint [5], we generalize the work of Deligne and Gillet–Soulé to the case of the trivial sheaf on X, equipped with the singular hyperbolic metric. Our main theorem relates the determinant of cohomology of the trivial sheaf, with an explicit Quillen type metric in terms of the Selberg zeta function of Γ, to a metrized version of the ψ line bundle of the theory of moduli spaces of pointed orbicurves, and the self-intersection bundle of a suitable twist of the canonical sheaf ω_X.

To be more precise, we consider the hermitian line bundle

$$\psi_W^{\otimes m^2} = \bigotimes_i (\omega_{X,p_i}, \| \cdot \|_{W,p_i})^{m^2(1-m_i^{-2})},$$

carrying the Wolpert metric. The underlying $\mathbb{Q}$-line bundle is denoted by ψ.

Furthermore, the singular hyperbolic metric on X induces a singular hermitian metric on the $\mathbb{Q}$-line bundle

$$\omega_X(D), \quad D := \sum_i \left(1 - \frac{1}{m_i}\right) p_i.$$

By $\omega_X(D)_{\mathrm{hyp}}$ we denote the resulting $\mathbb{Q}$-hermitian line bundle over X. It still fits the L_1^2 formalism of Bost [1], which implies that the metrized Deligne pairing

$$\langle \omega_X(D)_{\mathrm{hyp}}, \omega_X(D)_{\mathrm{hyp}} \rangle$$

is defined. This is a $\mathbb{Q}$-hermitian line bundle over $\operatorname{Spec} \mathbb{C}$.

Finally, the determinant of cohomology of $\mathcal{O}_X$ is the complex line

$$\det H^\bullet(X, \mathcal{O}_X) = \det H^0(X, \mathcal{O}_X) \otimes \det H^1(X, \mathcal{O}_X)^{-1}.$$

We define a Quillen metric on it by rescaling the L^2 metric as follows

$$\| \cdot \|_Q = (C(\Gamma) Z'(1, \Gamma))^{-1/2} \| \cdot \|_{L^2}.$$

Here, $C(\Gamma)$ is a real positive constant, which can be explicitly expressed in terms of the multiplicities m_i, special values of the Riemann zeta function $\zeta(s)$, the genus of X, and the Euler–Mascheroni constant γ (see [5]). Furthermore, $Z(s, \Gamma)$ is the Selberg zeta function of Γ; it is defined, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, by

$$Z(s, \Gamma) = \prod_\gamma \prod_{k=0}^\infty (1 - e^{-(s+k)\ell_{\mathrm{hyp}}(\gamma)})^2,$$

where γ runs over the non-oriented primitive closed geodesics in $\Gamma \backslash \mathbb{H}$, and $\ell_{\mathrm{hyp}}(\gamma)$ denotes the hyperbolic length of γ. The determinant of cohomology together with this Quillen metric will be denoted $\det H^\bullet(X, \mathcal{O}_X)_Q$.

In [5], we prove the following Riemann–Roch isometry.

Theorem 1. There is a canonical isometry of $\mathbb{Q}$-hermitian line bundles

$$\det H^\bullet(X, \mathcal{O}_X)_Q^{\otimes 12} \otimes \psi_W \xrightarrow{\sim} \langle \omega_X(D)_{\mathrm{hyp}}, \omega_X(D)_{\mathrm{hyp}} \rangle. \tag{1}$$

For the proof of Theorem 1 we refer to [5]. The proof makes use of surgery techniques and Mayer–Vietoris type formulae for determinants of Laplacians, Bost's L_1^2-formalism of arithmetic intersection theory, the Selberg trace formula, and exact evaluations of determinants of Laplacians on models of cusps and cones. The explicit computations of the regularized determinants of Laplacians on models of cusps and cones for the singular hyperbolic metric are of independent interest.

3. Arithmetic applications

The advantage of the Riemann–Roch isometry (1) is that it easily leads to arithmetic versions of the Riemann–Roch formula, in the sense of Arakelov geometry. Let K be a number field and $\mathcal{X} \to \mathcal{S} = \operatorname{Spec} \mathcal{O}_K$ a flat and projective regular arithmetic surface. We suppose given sections $\sigma_1, \ldots, \sigma_n$, that are generically disjoint. We also assume that for every complex embedding $\tau\colon K \hookrightarrow \mathbb{C}$, the compact Riemann surface $\mathcal{X}_\tau(\mathbb{C})$ arises as the compactification of a quotient $\Gamma_\tau \backslash \mathbb{H}$, and that the set of elliptic fixed points and cusps is precisely given by the sections. We construct $\mathbb{Q}$-hermitian line bundles over $\mathcal{S}$, with classes in the arithmetic Picard group (up to torsion) $\widehat{\operatorname{Pic}}(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We use similar notations as in the complex case. A straightforward application of Theorem 1 then yields the following arithmetic Riemann–Roch formula.

Theorem 2. We have the equality

$$\begin{aligned} 12\,\widehat{\deg}\, H^\bullet(\mathcal{X}, \mathcal{O}_{\mathcal{X}})_Q - \delta + \widehat{\deg}\, \psi_W =& (\omega_{\mathcal{X}/\mathcal{S}}(D)_{\mathrm{hyp}}, \omega_{\mathcal{X}/\mathcal{S}}(D)_{\mathrm{hyp}}) \\ & - \sum_{i \neq j} \left(1 - \frac{1}{m_i}\right)\left(1 - \frac{1}{m_j}\right)(\sigma_i, \sigma_j)_{\mathrm{fin}}, \end{aligned}$$

where δ measures the bad reduction of $\mathcal{X} \to \mathcal{S}$, and the right most intersection numbers account for the intersections of the sections happening at finite places.

Because our results cover arbitrary Fuchsian groups, we can apply Theorem 2 to cases of arithmetic interest, for example to the case of $\mathbb{P}^1_{\mathbb{Z}}$, seen as an integral model of $\mathrm{PSL}_2(\mathbb{Z})\backslash \mathbb{H} \cup \{\infty\}$. This leads to an explicit evaluation (carried out in [5]) of the special value $\log Z'(1, \mathrm{PSL}_2(\mathbb{Z}))$ as a rational expression in

$$\frac{L'(0,\chi_i)}{L(0,\chi_i)}, \frac{L'(0,\chi_\rho)}{L(0,\chi_\rho)}, \frac{\zeta'(0)}{\zeta(0)}, \frac{\zeta'(-1)}{\zeta(-1)}, \gamma, \log 2, \log 3;$$

here, χ_i resp. χ_ρ (with $\rho := e^{2\pi i/3}$) is the quadratic character of $\mathbb{Q}(i)$ and $\mathbb{Q}(\rho)$, respectively. This result can be considered as the analog of the analytic class number formula for $\mathrm{PSL}_2(\mathbb{Z})$.

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Quasimodular forms and counting torus coverings

Martin Moeller
(joint work with D. Chen, D. Zagier)

Quasimodular forms were first studied systematically in [4] and later in [3] in the context of counting torus coverings. Torus coverings of degree d are, on the other hand, the $\frac{1}{d}$-integral points for the natural period coordinates on the moduli space of flat surfaces $\Omega\mathcal{M}_g$. This space is stratified into the strata $\Omega\mathcal{M}_g(m_1,\ldots,m_n)$ where $(m_1,\ldots,m_n)$ is a partition of $2g-2$. In terms of the counting problem, this stratification corresponds to prescribing a ramification profile. The period coordinates give rise to the natural Masur-Veech volume form on strata of flat surfaces. A conjecture of Eskin and Zorich predicts the large genus asymptotics of this strata, namely.

$$\mathrm{vol}(\Omega\mathcal{M}_g(m_1,\ldots,m_n))) \sim \frac{4}{(m_1+1)(m_2+1)\cdots(m_n+1)} + o(1)$$

as $\sum m_i = 2g-2$ tends to infinity.

The methods presented here give a proof of this conjecture for the principal stratum, i.e. for all m_i equal to one (see [2]), and an outline for the general case. First, the counting problem (for covers without unramified components) can be reduced via Burnside's formula and inclusion-exclusion to the computation of certains q-brackets

$$\langle f\rangle_q = \frac{\sum_{\lambda\in\mathbf{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in\mathbf{P}} q^{|\lambda|}}.$$

If the function f on partitions is a shifted symmetric function, i.e. a polynomial in the power sum functions $p_\ell(\lambda)$, then Bloch and Okounkov have shown ([1]) that its q-bracket is a quasimodular form, i.e. a polynomial in the Eisenstein series E_2, E_4 and E_6. The second step is the observation that the homomorphism given by $\mathrm{Ev}(E_2) = X+12$, $\mathrm{Ev}(E_4) = X^2$ and $\mathrm{Ev}(E_6) = X^3$ fully records the growth of the coefficients of a quasimodular form.

The main step, both for computational and theoretical considerations, is to obtain a managable expression of the partition function

$$\Phi(\mathbf{u})_q = \Big\langle \exp\Big(\sum_{\ell\ge 1} p_\ell\, u_\ell\Big)\Big\rangle_q = \sum_{\mathbf{n}\ge 0} \langle \underbrace{p_1\cdots p_1}_{n_1}\cdot\underbrace{p_2\cdots p_2}_{n_2}\cdots\rangle_q\, \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}$$

of q-brackets. For the q-brackets themselves we are not aware of a good formula, but for their Ev-images we have the following result.

Theorem 1 ([2]). *The* Ev-*image* $\Phi(\mathbf{u})_X = \mathrm{Ev}[\Phi(\mathbf{u})_q]$ *of the partition function can be expressed as the formal Gaussian integral*

$$\Phi(\mathbf{u})_X = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2+\mathcal{B}(\mathbf{u},iy,X)}\,dy\,.$$

where we use the coefficients of $\sum_{k\geq 0}\beta_k z^k = \frac{z/2}{\sinh(z/2)}$ *to define*

$$\mathcal{B}(\mathbf{u}, y, X) \;=\; \sum_{\substack{\mathbf{a}>0\\ r\geq 0}} (a_1 + 2a_2 + 3a_3 + \cdots)!\; \beta_{2-r+w(\mathbf{a})} \sqrt{X}^{2-r+w(\mathbf{a})}\, \frac{\mathbf{u}^{\mathbf{a}}}{\mathbf{a}!}\frac{y^r}{r!}\,.$$

For volume computations the counting functions for connected coverings is of primary interest. This involves the passage from q-brackets to their cumlants (in the sense of [3]). From the theorem above we deduce a formula for the generating series of cumulants in terms of a fixed point of the y-derivative of $\mathcal{B}$. If define the renormalizations

$$v_n \;=\; \frac{2^{n-1}(2n)!}{n!}\,\mathrm{vol}(\Omega\mathcal{M}_g(m_1,\ldots,m_n))) \quad (n>0), \quad v_{-2} = v_0 = -\frac{1}{24},\, v_{-1} = 0$$

of the volumes, then the whole process gives the following amusing closed formula for the generating series of renormalized volumes.

Theorem 2. *Define a Laurent series*

$$\mathfrak{B}_{1/2}(X) \;=\; X^{1/2} + \frac{X^{-3/2}}{96} - \frac{7X^{-7/2}}{6144} + \frac{31X^{-11/2}}{65536} - \cdots$$

in $X^{-1/2}$ *as the unique solution in* $X^{-1/2}\mathbb{Q}[[1/X]]$ *of the functional equation*

$$\mathfrak{B}_{1/2}(X+\tfrac{1}{2}) \;-\; \mathfrak{B}_{1/2}(X-\tfrac{1}{2}) \;=\; \frac{X^{-1/2}}{2}\,.$$

Then the v_n *are given by the inversion formula*

$$Y = \mathfrak{B}_{1/2}(X) \quad\Longleftrightarrow\quad X \;=\; \sum_{n=-2}^{\infty} \frac{2n+1}{2^{2n+1}}\, v_n\, Y^{-2n-2}\,.$$

We provide a framework for computing the asymptotics of rapidly divergent power series that allows to prove the Eskin-Zorich conjecture for the principal stratum from this statement.

The preprint [2] that contains the results mentioned above also contains quasi-modularity results for a weighted version of counting torus coverings, motivated by the computation of Siegel-Veech constants for flat surfaces.

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Cohomology of local systems on M_3

Carel Faber
(joint work with Jonas Bergström and Gerard van der Geer)

We study the cohomology of irreducible symplectic local systems $\mathbb{V}_\lambda$ on the moduli space A_g of principally polarized abelian varieties of dimension g or their pullbacks via the Torelli morphism to the moduli space M_g of smooth curves of genus $g \geq 2$. Here $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$ indexes an irreducible representation of Sp_{2g}. One can go back and forth between the Euler characteristics (of compactly supported cohomology) $e_c(M_g, \mathbb{V}_\lambda)$, $e_c^{\mathbb{S}_n}(M_{g,n})$, and $e_c^{\mathbb{S}_n}(\overline{M}_{g,n})$ in certain ranges (e.g., between the latter two when $g \leq G$ and $2g + n \leq 2G + N$). For $g \leq 2$, these Euler charcteristics are known in $K_0(\mathrm{HS})$ or $K_0(\mathrm{Gal})$ (ℓ-adic Galois representations). So $g = 3$ is the case of current interest.

In a recent paper [2], we found an explicit conjectural formula for $e_c(A_3, \mathbb{V}_\lambda)$. It implies a dimension formula for spaces of Siegel cusp forms of genus 3, which has been confirmed by Taïbi (cf. his talk and [5]) in very many cases. The answer for A_3 determines the answer for M_3 when $|\lambda| = \lambda_1 + \cdots + \lambda_g$ is even, but gives nothing when $|\lambda|$ is odd. For $|\lambda| \leq 7$, $e_c(M_3, \mathbb{V}_\lambda)$ is known (Bergström [1]). For $9 \leq |\lambda| \leq 19$, we have (with 3 exceptions) conjectural formulas (for which there is much evidence); for $|\lambda| = 17$ or 19, these formulas involve 'motives' of odd weight $6 + |\lambda|$ and rank 6 or 8 (with associated group SO(7) or SO(9)), whose existence was first established (in the automorphic world) by Chenevier and Renard [3] and whose Hecke traces were recently computed by Mégarbané [4]; they match perfectly with the Frobenius traces found by us for $q \leq 17$. At the end of the talk, I discussed the relationship with vector valued Teichmüller modular forms.

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Harmonic weak Siegel Maaß forms

Martin Westerholt-Raum

Harmonic weak (elliptic) Maaß forms are functions on the Poincaré upper half space $\mathbb{H}$ that transform like elliptic modular forms, vanish under the suitably normalized hyperbolic Laplace operator Δ_k, and satisfy an exponential growth condition towards the cusps. Early on in the theory of harmonic weak Maaß forms, it was discovered that there is an exact sequence

$$0 \longrightarrow \mathrm{M}^!_k \longrightarrow \mathbb{M}_k \longrightarrow \mathrm{M}^!_{2-k} \longrightarrow 0\,,$$

where the left and right spaces consist of weakly holomorphic modular forms (i.e. modular forms with relax growth condition towards the cusps) and the middle one is the space of harmonic weak Maaß forms.

The above exact sequence corresponds to an exact sequence of Harish-Chandra modules that arises from each harmonic weak Maaß form. By passing from from a function f on $\mathbb{H}$ to one on the group $\mathrm{SL}_2(\mathbb{R})$, and then to the $(\mathfrak{g}, \mathrm{K})$-module generated by it, we obtain a Harish-Chandra module $\varpi(f)$. The differential equation $\Delta_k\, f = 0$ can be encoded in a composition series for $\varpi(f)$. In the case of integral weight k, we obtain

$$0 \longrightarrow \varpi_k^{\overline{\mathrm{hol}}} \oplus \varpi_k^{\mathrm{hol}} \longrightarrow \varpi(f) \longrightarrow \varpi_k^{\mathrm{finite}} \longrightarrow 0\,.$$

Composition factors to the left are the holomorphic and anti-holomorphic (limits of) discrete series (of Harish-Chandra parameter $k - 1$). The composition factor to the right is a finite dimensional representation. For completeness, note that in the case of half integral weight we obtain a similar short exact sequence

$$0 \longrightarrow \varpi_k^{\overline{\mathrm{hol}}} \longrightarrow \varpi(f) \longrightarrow \varpi_k^{\theta} \longrightarrow 0\,.$$

The composition factor of $\varpi(f)$ that is an antiholomorphic (limit of) discrete series corresponds to the classical ξ-operator mapping f to a holomorphic modular form. The composition factor of $\varpi(f)$ that is a holomorphic (limit of) discrete series corresponds to Bol's Identity for harmonic weak Maaß forms of integral weight. It also reflects the fact that harmonic weak Maaß forms have a holomorphic part—called a mock modular form. In order to obtain Siegel mock modular forms, one then should look for geometric realizations of extensions of Harish-Chandra modules for $\mathrm{Sp}_2(\mathbb{R})$ that contain a holomorphic Harish-Chandra module.

The easiest case of holomorphic Harish-Chandra modules arises from principal series attached to the Siegel parabolic. Note that except for the holomorphic discrete series with scalar minimal K-type, none of holomorphic subquotients $\omega_k^{(2)\,\mathrm{hol}}$ of principal series for the Siegel parabolic is square integral. The corresponding Langlands quotient $\omega_k^{(2)\,\mathrm{SK}}$ corresponds to a non-holomorphic Saito-Kurokawa lift. By Arthur's classification, non-holomorphic Saito Kurokawa lifts are in fact the only automorphic forms that have this Langlands quotient as their Harish-Chandra module.

Extending ideas of Bruinier and Funke to the higher dimensional case, we show that there are functions f on the Siegel modular 3-fold with meromorphic singularities whose attached Harish-Chandra module $\varpi(f)$ fits into the short exact sequence

$$0 \longrightarrow \varpi_k^{(2)\,\mathrm{SK}} \longrightarrow \varpi(f) \longrightarrow \varpi_k^{(2)\,\mathrm{hol}} \longrightarrow 0\,.$$

Harmonic weak Siegel Maaß forms that arise this way are mapped by the vector-valued lowering operator to non-holomorphic Saito-Kurokawa lifts. They have a holomorphic part, which we call a mock Siegel modular form.

In earlier joint work with Bringmann and Richter, the speaker studied Fourier coefficients of the analogous harmonic Siegel Maaß forms. In that situation, the Kohnen limit process allows us to pass to harmonic Maaß Jacobi forms, which are connected to usual harmonic Maaß forms (i.e. holomorphic modular forms and harmonic Eisenstein series) via the theta-decomposition. Extending the Kohnen limit process to harmonic weak Siegel Maaß forms, we obtain harmonic weak Maaß Jacobi forms with meromorphic singularities. For harmonic weak Siegel Maaß forms that map to suitable Saito-Kurokawa lifts via the lowering operator, these Fourier Jacobi coefficients should enjoy a connection to central derivatives of twisted L-values.

Regularized inner products

Kathrin Bringmann

(joint work with N. Diamantis, S. Ehlen, B. Kane, and A. von Pippich)

Many of the important application of modular forms come from its Hilbert space structure. We [1, 2] introduce a new regularization which works for all meromorphic modular forms. For simplicity we restrict in this talk to the modular forms of even integral weight for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, in [1] we use vector-valued modular forms.

Recall that for $f \in M_k$, the space of weight k holomorphic modular forms and $g \in S_k$, the corresponding subspace of cusps forms, the *Petersson inner product* is defined as $(z = x + iy)$

$$\langle f, g\rangle := \int_{\mathcal{F}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2},$$

where $\mathcal{F}$ is the classical fundamental domain. Borcherds generalized this for $f, g \in M_k^!$, the space of weight k weakly holomorphic modular forms by defining

$$\langle f, g\rangle := \mathrm{CT}_{s=0} \lim_{T\to\infty} \int_{\mathcal{F}_T} f(z)\overline{g(z)}y^{k-s} \frac{dxdy}{y^2}.$$

where by $\mathrm{CT}_{s=0}\, F(s)$ we mean the constant term of the analytic continuation of the function F at $s = 0$ and $\mathcal{F}_T$ denotes the standard fundamental domain cut off at height T. One can show that $\langle f, g\rangle$ exists for example if $f \in M_k$ and $g \in M_k^!$. However it does not, for example, exist if $f = g \in M_k^! \setminus M_k$. Petersson considered inner products for the space $\mathbb{S}_k$ of *meromorphic cusp forms of weight* k (i.e., meromorphic modular forms which behave like cusp forms towards $i\infty$) by removing hyperbolic balls around the poles of the functions that are integrated and letting their radii shrink to 0. This inner product however again does not always exist.

We next turn to the new regularizations. The idea is simple. We multiply the integrands with a function that forces convergence and then analytically continue. We start with weakly holomorphic forms. Observe that for $\mathrm{Re}(w) \gg 0$, the integral

$$I(f,g;w,s) := \int_{\mathcal{F}} f(z)\overline{g(z)}y^{k-s}e^{-wy}\frac{dxdy}{y^2}$$

converges and is analytic. For every $\varphi \in (\pi/2, 3\pi/2) \setminus \{\pi\}$ it has an analytic continuation $I_\varphi(f,g;w,s)$ to $U_\varphi \times \mathbb{C}$ with $U_\varphi \subset \mathbb{C}$ a certain open set. Then define

$$\langle f,g\rangle_\varphi := \mathrm{CT}_{s=0}\, I_\varphi(f,g;0,s) - i\sum_{n>0} c_f(-n)\overline{c_g(-n)}\,\mathrm{Im}\left(E_{2-k,\varphi}(-4\pi n)\right)$$

where c_F denote the Fourier coefficients of a function F and $E_{r,\varphi}$ is the generalized exponential integral defined with branch cut on the ray $\{xe^{i\varphi} : x \in \mathbb{R}^+\}$.

Theorem 1 (B.-Diamantis-Ehlen)**.** *For $f, g \in M_k^!$, $\langle f,g\rangle$ exists and is independent of the choice of φ. It equals Borcherds regularization if his exists.*

We next turn to meromorphic cusp forms. Let $s = (s_1, \ldots, s_r)$ and define

$$\langle f,g\rangle := \mathrm{CT}_{s=0}\left(\int_{\mathcal{F}} f(z)H_s(z)\overline{g(z)}y^k\frac{dxdy}{y^2}\right),$$

where $H_s(z) := \prod_{\ell=1}^{r} h_{s_\ell,z_\ell}(z)$ with $z_1, \ldots, z_\ell$ are all of the poles of f and g in $\mathcal{F}$. Here $h_{s_\ell,z_\ell}(z) := r_{z_\ell}^{2s_\ell}(Mz)$ with $r_{\mathfrak{z}}(z) := |\frac{z-\mathfrak{z}}{z-\overline{\mathfrak{z}}}|$ where $M \in \mathrm{SL}_2(\mathbb{Z})$ is chosen such that $Mz \in \mathcal{F}$.

Theorem 2 (B-Kane-von Pippich)**.** *The regularized inner product $\langle f,g\rangle$ always exists for $f, g \in \mathbb{S}_k$ and it equals Petersson's regularization whenever his exists.*

Let me now describe 3 applications of our inner product:

1) Duke, Imamoḡlu, and Tóth used regularized inner products to obtain interesting arithmetic information about elements of $M_0^!$. For $m \in \mathbb{N}$, let f_m be the unique modular function for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$f_m(z) = q^{-m} + \sum_{n\geq 0} c_m(n)q^n$$

and $c_m(0) = 24\sum_{d|m} d$. The $\{f_m\}_{m\in\mathbb{N}}$ together with the function 1 form a basis of $M_0^!$. Duke, Imamoḡlu, and Tóth showed that, for $m \neq n$,

$$\langle f_m, f_n\rangle = -8\pi^2\sqrt{mn}\sum_{c\geq 1}\frac{K(m,n;c)}{c}F\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{1}$$

Here $K(m,n;c)$ is a Kloosterman sum and F a certain Bessel function. They left it as an open problem to regularize $\langle f_m, f_n\rangle$ for $m = n$ and to find a closed formula.

Theorem 3 (B-Diamantis-Ehlen)**.** *Equation* (1) *also holds for $m = n$.*

2) A second application concerns weight 3/2 modular forms discussed again by Duke, Imamoḡlu, and Tóth. Although this is not covered by the restricted definition of regularized inner products, the vector-valued setting does contain it. This is contained in the vector-valued setting does contain it. For every $d \in \mathbb{N}$ satisfying $d \equiv 0, 1 \pmod 4$ there exists a unique form $g_d \in M^!_{3/2}$, the space of weakly holomorphic weight 3/2 modular forms in Kohnen's plus space, with Fourier expansion

$$g_d(\tau) = q^{-d} + \sum_{\substack{n \geq 0 \\ n \equiv 0,3 \pmod 4}} B_d(n) q^n.$$

The coefficients $B_d(n)$ are integers given by (twisted) traces of singular moduli. Duke, Imamoḡlu, and Tóth proved that for positive fundamental discriminants $d \neq d'$, the regularized inner product $\langle g_d, g_{d'} \rangle$ can be expressed in terms of certain cycle integrals of the j-invariant.

Theorem 4 (B-Diamantis-Ehlen). *We have*

$$\langle g_1, g_1 \rangle = -\frac{3}{4\pi} \operatorname{Re} \left(\int_i^{i+1} J(\tau) \psi(\tau)\, d\tau \right),$$

where $\psi(\tau) := \Gamma'(\tau)/\Gamma(\tau)$ denotes the Digamma function and $J := f_1 - 24$.

Remarks. (1) *One can also state Theorem 4 as $\langle g_1, g_1 \rangle \doteq L_J^*(0)$ for a certain regularized L-series for f_1.*

(2) *One could also obtain a similar identity for the Petersson norm of g_d.*

3) Our next application concerns cycle type integrals for meromorphic modular forms. Zagier encountered the following cusp forms of weight $2k$ while investigating the Doi-Naganuma lift ($k \in \mathbb{N}_{\geq 2}$, $\delta \in \mathbb{N}$)

$$f_{k,\delta}(z) := \sum_{Q \in \mathcal{Q}_\delta} Q(z, 1)^{-k}. \tag{2}$$

Here $\mathcal{Q}_\delta$ is the set of integral binary quadratic forms of discriminant δ. Choosing the discriminant to be negative instead, Bengoechea constructed meromorphic modular forms with poles at CM-points which have been shown to be images of theta lifts [1]. The singularities at the cusps are turned into singularities in the upper half-plane under this lift. We have an analogous theorem for harmonic Maass forms for the lift in the dual weight. Define for an equivalence class $[Q]$ of quadratic forms

$$\mathcal{G}_Q(z) := D^{\frac{1-k}{2}} \sum_{\mathcal{Q} \in [Q]} \mathcal{Q}(z, 1)^{k-1} \int_0^{\operatorname{artanh}\left(\frac{\sqrt{D}}{\mathcal{Q}_z}\right)} \sinh^{2k-2}(\theta)\, d\theta.$$

Note that $\mathcal{G}_Q(z)$ comes from a higher Green's function $G_k(z, \mathfrak{z})$ by applying raising operators and then evaluating at the root τ_Q of the quadratic form Q. It turns out that these are *polar harmonic Maass forms* which are harmonic Maass forms, annihilated by the weight $2 - 2k$ Laplacian, but may have poles in the upper half-plane. Moreover they are preimages of f_Q (restricting the sum in (2) to forms in $[Q]$) under $\xi_{2-2k} := 2iy^{2-2k}\overline{\frac{\partial}{\partial \overline{z}}}$ and $D^{2k-1} := (\frac{1}{2\pi i}\frac{\partial}{\partial z})^{2k-1}$.

The functions $\mathcal{G}_Q$ reappear when taking inner products of f_Q against meromorphic cusp forms. In the simplest case we have

Theorem 5 (B-Kane-von Pippich)**.** *Suppose that f has simple poles in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ only at distinct $[z_1], \ldots, [z_r]$ and $z_j \neq \tau_Q$. Then*

$$\langle f, f_Q \rangle = 2\pi i \sum_{\ell=1}^{r} \frac{1}{\omega_{z_\ell}} \mathcal{G}_Q(z_\ell) \mathrm{Res}_{z=z_\ell} f(z).$$

Remark 6. *Our theorem also allows poles at τ_Q and higher order poles.*

Moreover, the Petersson inner product of two f_Q functions gives evaluation of this Green's function.

Theorem 7 (B-Kane-von Pippich)**.** *Suppose that $-D_1, -D_2 < 0$ are two discriminants. For $Q \in \mathcal{Q}_{-D_1}$ and $\mathcal{Q} \in Q_{-D_2}$ with $[\tau_Q] \neq [\tau_{\mathcal{Q}}]$, we have*

$$\langle f_{\mathcal{Q}}, f_Q \rangle \doteq \frac{G_k\left(\tau_{\mathcal{Q}}, \tau_Q\right)}{\omega_{\tau_{\mathcal{Q}}}\omega_{\tau_Q}}.$$

Note that using work of Zhang one can also relate this result to certain height pairings.

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Moduli of Supersingular K3 Crystals

Christian Liedtke

Supersingular K3 surfaces. A K3 surface X over an algebraically closed field k of positive characteristic p is called *supersingular*, if the following two equivalent conditions are fulfilled

(1) X has Picard rank $\rho(X) = 22$,
(2) the height of the formal Brauer group $\widehat{\mathrm{Br}}(X)$ is infinite.

Concerning the equivalences: whereas (1) $\Rightarrow$ (2) is relatively easy to show (via the Igusa-Mazur inequality), the implication (2) $\Rightarrow$ (1) follows from the Tate-conjecture for K3 surface, which is now a theorem of Charles, Nygaard, Maulik, Madapusi-Pera, and Ogus. Moreover, the discriminant of the Néron–Severi group satisfies

$$\mathrm{disc}\,\mathrm{NS}(X) \;=\; -p^{2\sigma_0} \quad \text{for some integer} \quad 1 \leq \sigma_0 \leq 10$$

that is called the *Artin invariant* and that was introduced by Artin [1]. In fact, σ_0 determines the Néron–Severi $\mathrm{NS}(X)$ up to isometry and such lattices are called *supersingular K3 lattices.*

Moving torsor families. If X is a supersingular K3 surface with Artin invariant $\sigma_0 \leq 9$ in characteristic p, then it admits a genus-one fibration $X \to \mathbb{P}^1_k$ with a section [2]. Associated to this data, there exists a smooth family of supersingular K3 surfaces, a family of *moving torsors*

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \to & \mathrm{Spec}\, k[[t]] \end{array}$$

such that specialization induces a short exact sequence

$$0 \to \mathrm{Pic}(\mathcal{X}_{\overline{\eta}}) \to \mathrm{Pic}(X) \to \mathbb{Z}/p\mathbb{Z} \to 0,$$

whose cokernel is generated by the class of the zero-section of the fibration $X \to \mathbb{P}^1$. In particular, $\sigma_0(\mathcal{X}_{\overline{\eta}}) = \sigma_0(X) + 1$, and thus, this family has non-trivial moduli. Moreover, this family can be algebraized and spread out from $\mathrm{Spec} k[[t]]$ to a family of smooth supersingular K3 surfaces over a curve that is of finite type over k. We refer to [2] for details.

Moduli of supersingular K3 crystals. The second crystalline cohomology group of a supersingular K3 surface in positive characteristic p is an F-crystal, which is of slope 1 and weight 2. Moreover, Poincaré duality equips it with a perfect pairing. In [3], Ogus classified such crystals together with markings by a supersingular K3 lattice N, so called *N-marked supersingular K3 crystals.* In [3], he also constructed a moduli space $\mathcal{M}_N \to \mathrm{Spec}\, \mathbb{F}_p$ for such crystals.

Theorem (Ogus)**.** *Let N be a supersingular K3 lattice in odd characteristic p. Then, $\mathcal{M}_N$ is smooth, proper, and of dimension $(\sigma_0(N) - 1)$ over $\mathbb{F}_p$, and it has two geometric components.*

Given a K3 surface X together with a genus-one fibration $X \to \mathbb{P}^1_k$ and a section, the divisor classes of a fiber and the divisor class of the section span a hyperbolic plane U inside the Néron-Severi lattice $\mathrm{NS}(X)$. This said, the moving torsor families from above manifest themselves on the level of Ogus' moduli spaces $\mathcal{M}_N$ as follows.

Theorem. *Let N and N_+ be supersingular K3 lattices in odd characteristic such that $\sigma_0(N_+) = \sigma_0(N) + 1$. Then, a choice of hyperbolic plane $U \subset N$ gives rise to a morphism*

$$\mathcal{M}_{N_+} \to \mathcal{M}_N.$$

This morphism is a fibration, whose geometric fibers are rational curves with at worst unibranch singularities ("cusps").

Examples (Ogus)**.** *Let N be a supersingular K3 lattice in characteristic $p \geq 3$.*

(1) If $\sigma_0(N) = 1$, then $\mathcal{M}_N \cong \mathrm{Spec}\, \mathbb{F}_{p^2}$.
(2) If $\sigma_0(N) = 2$, then $\mathcal{M}_N \cong \mathbb{P}^1_{\mathbb{F}_{p^2}}$.

(3) If $\sigma_0(N) = 3$, then $\mathcal{M}_N \times_{\operatorname{Spec}\mathbb{F}_p} \operatorname{Spec}\overline{\mathbb{F}}_p$ is the disjoint union of two Fermat surfaces of degree $(p+1)$.

Shioda [4] showed that the Fermat surface $F_n := \{x_0^n + ... + x_3^n = 0\} \subset \mathbb{P}^3$, $n \geq 4$, in characteristic p with $\gcd(n,p) = 1$ is unirational if and only if there exists an integer $\nu \geq 1$ such that $p^\nu \equiv -1 \mod n$. In particular, the Fermat surfaces F_{p+1} are unirational in characteristic p. Together with Ogus' examples, we obtain a new proof of Shioda's theorem in the following special case.

Corollary (Shioda)**.** *For every prime p, the Fermat surface $F_{p+1} \subset \mathbb{P}^3$ is unirational in characteristic p.*

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Complete moduli of cubic threefolds and their intermediate Jacobians

Samuel Grushevsky
(joint work with S. Casalaina-Martin, K. Hulek, R. Laza)

We work over $\mathbb{C}$. To a smooth cubic threefold X one can associate its intermediate Jacobian $IJ(X) := H^{2,1}(X,\mathbb{C})^*/H_3(X,\mathbb{Z})$, which is a principally polarized abelian variety of dimension 5. The intermediate Jacobians of cubic threefolds were studied by Clemens and Griffiths [4], who used them to show that any smooth cubic threefold is not rational.

We denote $\mathcal{M}$ the moduli space of smooth cubic threefolds, and denote $\mathcal{A}_5$ the moduli space of principally polarized abelian fivefolds. We thus view the intermediate Jacobian as a map $IJ : \mathcal{M} \to \mathcal{A}_5$. The theta divisor of an intermediate Jacobian of a smooth cubic threefold has a unique singular point, which has multiplicity three. The projectivized tangent cone to the theta divisor at this singular point is the cubic threefolds itself. Thus the Torelli theorem for cubic threefolds holds: the map IJ is injective.

We study degenerations of cubic threefolds and the extension of the intermediate Jacobian map IJ. The moduli space of cubic threefolds admits a natural GIT compactification $\overline{\mathcal{M}}$, described by Allcock [2]. There does not exist a universal family over the GIT compactification. The most singular point $\Xi \in \overline{\mathcal{M}}$ corresponds to the chordal cubic — the secant variety of the rational normal curve. From a different viewpoint, Allcock, Carlson, and Toledo [3] and Looijenga and Swierstra [6] showed that $\mathcal{M}$ is an open dense subset of a suitable ball quotient. They furthermore showed that there exists a suitable common resolution $\widehat{\mathcal{M}}$ both of $\overline{\mathcal{M}}$ and of the Baily-Borel compactification of the ball quotient. The space

$\widehat{\mathcal{M}}$ is obtained from $\overline{\mathcal{M}}$ as a Kirwan blowup of Ξ. The boundary $\partial\widehat{\mathcal{M}}$ is essentially a hyperplane arrangement, and thus by a wonderful blowup construction of de Concini-Procesi one can construct the so-called wonderful compactification $\widetilde{\mathcal{M}} \to \widehat{\mathcal{M}}$, whose boundary is now a normal crossing divisor. The wonderful compactification was first considered by Casalaina-Martin and Laza [5]. Our main result is the following

Theorem. The intermediate Jacobian map extends to a morphism $\widetilde{IJ} : \widetilde{\mathcal{M}} \to \overline{\mathcal{A}}_5^{Vor}$ from the wonderful compactification of the moduli space of cubic threefolds to the second Voronoi toroidal compactification of the moduli space of principally polarized abelian fivefolds.

Our proof of this theorem proceeds by studying the monodromy cones (for everything except the blowup of the moduli point corresponding to the chordal cubic, where the argument proceeds differently). We first recall that the intermediate Jacobian of a smooth cubic threefold can also be described as the Prym variety of an étale double cover of a smooth plane quintic. We then extend this description to degenerating families of cubic threefolds, which acquire isolated singularities — showing that these correspond to degenerating families of plane quintics, acquiring the same type of singularities, while the double cover remains étale including over the singularities. We thus reduce the extension question for the map $\widetilde{IJ}$ to the problem of extending the Prym map.

The indeterminacy locus of the Prym map was described explicitly by Alexeev, Birkenhake, Hulek [1] and Vologodsky [7] — it is equal to the closure of the locus of so-called Friedman-Smith covers. By considering carefully the stable reductions of étale (including over the singularities!) double covers of plane quintics with arbitrary isolated singularities we are able to show that the corresponding covers never lie in the Friedman-Smith loci. Thus the Prym map extends to all étale double covers of singular plane quintics, and thus the intermediate Jacobian map extends to all of $\widetilde{\mathcal{M}}$.

By further studying the details of this construction, we can further see that the image of $\widetilde{IJ}$ is contained in the so-called matroidal locus, which is the largest set of cones that the perfect cone and second Voronoi toroidal compactification have in common. Our approach also gives a way to geometrically describe the image of the locus of cubic threefolds with a given singularity profile under the extended intermediate Jacobian map.

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On the (intersection) cohomology of $\mathcal{A}_g$ and its compactifications

Klaus Hulek
(joint work with Sam Grushevsky)

1. Introduction

The moduli space of principally polarized abelian varieties is, over the complex numbers, the quotient

$$\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$$

where $\mathbb{H}_g$ denotes the Siegel upper half space. A well known result of Borel says that the cohomology of $\mathcal{A}_g$ stabilizes. More precisely, let $\mathbb{E}$ be the Hodge vector bundle on $\mathcal{A}_g$, and denote its Chern classes by $\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\mathcal{A}_g, \mathbb{Q})$. It is well known result due to Mumford that these classes fulfill the relation

$$(1) \qquad (1 - \lambda_1 + \lambda_2 - \ldots + (-1)^g \lambda_g)(1 + \lambda_1 + \lambda_2 + \ldots + \lambda_g) = 1.$$

Theorem 3 (Borel 74). *The cohomolgy groups $H^k(\mathcal{A}_g, \mathbb{Q})$ are independent of k in the range $k < g$. In this range the cohomology is freely generated by the odd λ-classes.*

This result was generalized to the Baily-Borel-Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ by Charney and Lee, who also proved that the classes λ_i can be extended to $\mathcal{A}_g^{\mathrm{Sat}}$.

Theorem 4 (Charney, Lee 83). *The cohomology groups $H^k(\mathcal{A}_g^{\mathrm{Sat}}, \mathbb{Q})$ do not depend on g for $k < g$, and the stable cohomology ring is freely generated by the odd λ-classes $\lambda_1, \lambda_3, \lambda_5, \ldots$, together with classes $\alpha_3, \alpha_5, \alpha_7, \ldots$ where the class α_j is in degree $2j$.*

The situation for toroidal compactifications is more subtle. For (stack) smooth toroidal compactifications $\tilde{\mathcal{A}}_g^{\mathrm{tor}}$ with normal crossing boundary the pullback of the λ-classes from $\mathcal{A}_g^{\mathrm{Sat}}$ can be understood as the Chern classes of the extended Hodge bundle. These classes still satisfy relation (1) and the *tautological ring* $R_g \subset H^\bullet(\tilde{\mathcal{A}}_g^{\mathrm{tor}})$ is defined as the ring generated by the λ-classes.

The stabilization of cohomology for toroidal compactifications depends on the fan, and hence the specific compactification chosen. For the perfect cone **toroidal** compactification $\mathcal{A}_g^{\mathrm{Perf}}$ we showed the following stabilization result in [5]

Theorem 5. *The cohomology of the perfect cone compactification stabilizes in close to the top degree, i.e. the groups $H^{g(g+1)-k}(\mathcal{A}_g^{\mathrm{Perf}}, \mathbb{Q})$ are independent of g for $k < g$.*

In contrast to this, the cohomology of the second Voronoi toroidal compactification $\mathcal{A}_g^{\mathrm{Vor}}$ does not stabilize. The stack $\mathcal{A}_g^{\mathrm{Perf}}$ is singular and Poincaré duality fails. For this reason the above theorem cannot be reformulated to give a stabilization result for $H^k(\mathcal{A}_g^{\mathrm{Perf}}, \mathbb{Q})$.

The failure of Poincaré duality for singular varieties is one of the motivations for studying intersection cohomology, in particular also for compactifications of $\mathcal{A}_g$. We first recall that the tautological ring R_g is naturally contained in the intersection cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$. The latter stabilizes, in fact one has the

Theorem 6. *The intersection cohomology $IH^k(\mathcal{A}_g^{\mathrm{Sat}}, \mathbb{Q})$ is independent of g in the range $k < g$ where it coincides with the tautological ring.*

This follows from combining results of Borel on the one hand and the proof of the Zucker conjecture by Saper and Stern and Looijenga on the other hand. Indeed, Borel showed in [2] that in the stable range $H^k(\mathcal{A}_g, \mathbb{C}) \cong H^k_{(2)}(\mathcal{A}_g)$, where the latter denotes L_2 cohomology, whereas by Saper-Stern [9] and Looijenga [8] one has that $H^k_{(2)}(\mathcal{A}_g) \cong H^k(\mathcal{A}_g^{\mathrm{Sat}})$.

2. Main result

The main purpose of our work is to compute the intersection cohomology of $\mathcal{A}_g^{\mathrm{Sat}}$, not just in the stable range, for small values of g. We obtain

Theorem 7. *For $g \leq 4$ there is an isomorphism of graded vector spaces*

$$IH^\bullet(\mathcal{A}_g^{\mathrm{Sat}}) \cong R_g^\bullet$$

between the intersection cohomology of the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ and the tautological ring – except that possibly $IH^{10}(\mathcal{A}_4^{\mathrm{Sat}}) \supsetneq R_4^{10}$.

The main tool we use for proving this is the decomposition theorem for intersection cohomology for projective morphisms $f : X \to Y$. This theorem is due to Beilinson, Bernstein, Deligne and Gabber. It was reproved by de Cataldo and Migliorini [3] and it is their approach which we use. We shall apply this theorem to the projective morphism $\varphi : \mathcal{A}_g^{\mathrm{Vor}} \to \mathcal{A}_g^{\mathrm{Sat}}$. The Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ is naturally stratified as

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_0 \tag{2}$$

and over the strata $\mathcal{A}_k$ the map φ is a topological fibration. In genus $g \leq 4$ the Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ is smooth and hence intersection cohomology and cohomology coincide. We thus obtain the following decomposition where $i \in [-r(\varphi), r(\varphi)]$ and $r(\varphi)$ is the defect of the morphism φ:

(3)

$$H^m(\mathcal{A}_g^{\mathrm{Vor}}, \mathbb{Q}) \cong IH^m(\mathcal{A}_g^{\mathrm{Sat}}, \mathbb{Q}) \oplus \bigoplus_{k<g,i,\beta} IH^{m-g(g+1)/2+k(k+1)/2+i}(\mathcal{A}_k^{\mathrm{Sat}}, \mathcal{L}_{i,k,\beta})$$

for suitable local systems $\mathcal{L}_{i,k,\beta}$ on $\mathcal{A}_k$. The left hand side was computed in [5], apart from the case $g = 4$ and $m = 10$. Also from [5] one knows the topology of the fibres of the map φ. Moreover, we know that the tautological ring R_g is contained in $IH^\bullet(\mathcal{A}_g^{\mathrm{Sat}})$. The combination of this information, together with the calculation of the link cohomology of the stratum $\mathcal{A}_{g-1}$, allows us to find sufficiently many local systems $\mathcal{L}_{i,k,\beta}$ to account for the entire topology of $\mathcal{A}_g^{\mathrm{Vor}}$ and thus to conclude that $IH^\bullet(\mathcal{A}_g^{\mathrm{Sat}}) = R_g$ in the range stated.

Applying the decomposition theorem to the map $\varphi' : \mathcal{A}_4^{\mathrm{Vor}} \to \mathcal{A}_4^{\mathrm{Perf}}$ we also obtain the

Proposition 8. *All the odd degree intersection Betti numbers of* $\mathcal{A}_4^{\mathrm{Perf}}$ *are zero, while the even ones* $ib_j := \dim IH^j(\mathcal{A}_4^{\mathrm{Perf}})$ *are as follows:*

(4)

j	0	2	4	6	8	10	12	14	16	18	20
ib_j	1	2	4	9	14	$*$	14	9	4	2	1

where we know that $* = \dim IH^{10}(\mathcal{A}_4^{\mathrm{Perf}}) \geq 16$.

Our method also gives us considerable information on the cohomology of the link bundles.

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Reporter: Stephan Ehlen

Participants

Prof. Dr. Fabrizio Andreatta
Dipartimento di Matematica
"Federigo Enriques"
Universita di Milano
Via Saldini 50
20133 Milano
ITALY

Dr. Jitendra Bajpai
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Jonas Bergström
Matematiska Institutionen
Stockholms Universitet
10691 Stockholm
SWEDEN

Prof. Dr. Kathrin Bringmann
Mathematisches Institut
Universität zu Köln
50923 Köln
GERMANY

Prof. Dr. Jan Hendrik Bruinier
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Fabien Cléry
Fakultät III
Universität Siegen
LST für Finanzwissenschaft
57068 Siegen
GERMANY

Prof. Dr. Anton Deitmar
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Stephan Ehlen
Department of Mathematics & Statistics
McGill University
805, Sherbrooke Street West
Montreal, QC H3A OB9
CANADA

Prof. Dr. Carel F. Faber
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
3508 TA Utrecht
NETHERLANDS

Prof. Dr. Gavril Farkas
Institut für Mathematik
Humboldt-Universität
Unter den Linden 6
10099 Berlin
GERMANY

Dr. Alessio Fiorentino
U. F. R. Mathématiques
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Prof. Dr. Jens Funke
Department of Mathematical Sciences
Durham University
Science Laboratories
South Road
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Sergey Galkin
Higher School of Economics
Faculty of Mathematics and
Laboratory of Algebraic Geometry
National Research University
Vavilova Str. 7
Moscow 117 312
RUSSIAN FEDERATION

Dr. Alice Garbagnati
Dipartimento di Matematica
Universita di Milano
Via C. Saldini, 50
20133 Milano
ITALY

Prof. Dr. Valery Gritsenko
U.F.R. de Mathématiques
Université de Lille I
USTL; Bat. M 2
59655 Villeneuve d'Ascq. Cedex
FRANCE

Prof. Dr. Samuel Grushevsky
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Dr. Eric Hofmann
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Klaus Hulek
Institut für Algebraische Geometrie
Leibniz Universität Hannover
30167 Hannover
GERMANY

Prof. Dr. Toshiyuki Katsura
Faculty of Science & Engineering
Hosei University
3-7-2 Kajino-cho, Koganei
Tokyo 184-8584
JAPAN

Prof. Dr. Shigeyuki Kondo
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
JAPAN

Prof. Dr. Jürg Kramer
Institut für Mathematik
Humboldt-Universität Berlin
10099 Berlin
GERMANY

Prof. Dr. Stephen S. Kudla
Department of Mathematics
University of Toronto
Toronto ON M5S 2E4
CANADA

Prof. Dr. Radu Laza
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Dr. Yingkun Li
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Christian Liedtke
Zentrum Mathematik
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Prof. Dr. Eduard J. N. Looijenga
Department of Mathematical Sciences
Tsinghua University
Beijing 100 084
CHINA

Prof. Dr. Shouhei Ma
Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
2-12-1, Ookayama, Meguro
Tokyo 152-8551
JAPAN

Prof. Dr. Martin Möller
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-10
60325 Frankfurt am Main
GERMANY

Dr. Michael Neururer
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Viacheslav V. Nikulin
Department of Pure Mathematics
University of Liverpool
Liverpool L69 3BX
UNITED KINGDOM

Sebastian Opitz
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Dr. Dan Petersen
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen
DENMARK

Prof. Dr. Cris Poor
Department of Mathematics
Fordham University
441 East Fordham Road
Bronx, NY 10458
UNITED STATES

Prof. Dr. Riccardo Salvati Manni
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Prof. Dr. Gregory Sankaran
Department of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Prof. Dr. Nils Scheithauer
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Matthias Schütt
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Markus Schwagenscheidt
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Nick I. Shepherd-Barron
Department of Mathematics
Kings College
Strand
London WC2R 2LS
UNITED KINGDOM

Prof. Dr. Jerry Shurman
Department of Mathematics
Reed College
3202 SE Woodstock Blvd.
Portland, OR 97202-8199
UNITED STATES

Prof. Dr. Nils-Peter Skoruppa
Department Mathematik
Universität Siegen
Emmy-Noether-Campus
Walter-Flex-Strasse 3
57068 Siegen
GERMANY

Dr. Olivier N. Taibi
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Gerard van der Geer
Korteweg-de Vries Instituut
Universiteit van Amsterdam
Postbus 94248
1090 GE Amsterdam
NETHERLANDS

Dr. Maryna Viazovska
Institut für Mathematik
Humboldt-Universität zu Berlin
Raum 1.001
Unter den Linden 6
10117 Berlin
GERMANY

Dr. Anna von Pippich
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Dr. Martin Westerholt-Raum
Department of Mathematics
Chalmers University of Technology
L 2130
Chalmers Tvärgata 3
412 96 Göteborg
SWEDEN

Michael C. Woodbury
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Prof. Dr. Tonghai Yang
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Dr. Ken-Ichi Yoshikawa
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. David S. Yuen
Mathematics & Computer Science
Department
Lake Forest College
555 N. Sheridan Road
Lake Forest, IL 60045-2399
UNITED STATES

Prof. Dr. Noriko Yui
Department of Mathematics & Statistics
Queen's University
Jeffery Hall
Kingston, Ont. K7L 3N6
CANADA

Dr. Shaul Zemel
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
Jerusalem 91904
ISRAEL